Supplementary Material

I. NONLINEAR AVERAGED EQUATIONS FOR FOURIER COMPONENTS

The Fokker-Planck Eq. (9ms) can be rewritten as an infinite set of differential equations for the Fourier amplitudes ($m \geq 1$):

$$\begin{aligned}
\dot{a}_{m|\zeta} &= \frac{m}{2\tau}\left(iD_{m|\zeta} - \sigma^2 N_{m|\zeta}\right), \\
D_{m|\zeta} &= 2(A_\zeta + 1)a_{m|\zeta} + (A_\zeta - 1)(a_{m-1|\zeta} + a_{m+1|\zeta}), \\
N_{m|\zeta} &= 3ma_{m|\zeta} + (2m-1)\,a_{m-1|\zeta} + (2m+1)\,a_{m+1|\zeta} \\
&\quad + \frac{1}{2}\left[(m-1)a_{m-2|\zeta} + (m+1)a_{m+2|\zeta}\right].
\end{aligned} \tag{1}$$

Here, $D_{m|\zeta}$ and $N_{m|\zeta}$ denote the deterministic and noise terms, respectively. The auxiliary variable A_ζ is defined as

$$A_\zeta = \mu + \Delta\zeta - J\tau S. \tag{2}$$

To express the mean firing rate and the mean membrane potential in terms of Fourier amplitudes, it is convenient to introduce a complex variable

$$W_\zeta(t) = \pi\tau R_\zeta(t) - iV_\zeta(t). \tag{3}$$

It can be shown that

$$W_\zeta(t) = 1 + 2\sum_{m=1}^{\infty}(-1)^m a_{m|\zeta}(t). \tag{4}$$

Denoting the averaged value of $W_\zeta(t)$ over ζ as

$$W(t) = \int_{-\infty}^{+\infty} d\zeta\; f(\zeta)W_\zeta(t), \tag{5}$$

for the global mean firing rate and membrane potential, we obtain the following expressions:

$$R(t) = \frac{1}{\pi\tau}\,\mathrm{Re}[W(t)], \tag{6a}$$

$$V(t) = -\,\mathrm{Im}[W(t)]. \tag{6b}$$

To evaluate the accuracy and applicability of the above mean-field equations based on the two-cumulant approximation, we compare their solutions with those of a more accurate system of differential equations involving a large number of Fourier amplitudes. In the case of a rational distribution function $f(\zeta)$, an exact infinite set of differential equations valid for arbitrary noise amplitudes can be derived from the Eqs. (1):

$$\begin{aligned}
\dot{a}_{m|j} &= \frac{m}{2\tau}\left(iD_{m|j} - \sigma^2 N_{m|j}\right), \\
D_{m|j} &= 2(A_j + 1)a_{m|j} + (A_j - 1)(a_{m-1|j} + a_{m+1|j}), \\
N_{m|j} &= 3ma_{m|j} + (2m-1)\,a_{m-1|j} + (2m+1)\,a_{m+1|j} \\
&\quad + \frac{1}{2}\left[(m-1)a_{m-2|j} + (m+1)a_{m+2|j}\right]
\end{aligned} \tag{7}$$

for $m = 1, \ldots, \infty$ and $j = 1, \ldots, n$. In numerical simulations, we truncate this system at some large $m = M$, that is, we assume that $a_{m|j} \equiv 0$ for $m > M$. We usually take $M = 300$. Here the auxiliary variable A_j is equal to A_ζ at $\zeta = \zeta_j$ (see Eq. 2):

$$A_j = \mu + \Delta\zeta_j - J\tau S. \tag{8}$$

The value of $W(\zeta, t)$ averaged over ζ is given by

$$W(t) = \sum_{j=1}^{n} b_j W_j(t). \tag{9}$$

The Eq. (9) for the function $W(t)$ remains valid, but the expression for $W_j(t)$ obtained from the Eq. (4) is now as follows:

$$W_j(t) = 1 + 2\sum_{m=1}^{M}(-1)^m a_{m|j}(t). \tag{10}$$

In order to close the equations (1) and (7) we need to find the synaptic activation $S(t)$ that satisfies the relaxation equation

$$\tau_s \dot{S} = R(t) - S, \tag{11}$$

II. FINDING FIXED POINTS FOR AVERAGED EQUATIONS FOR FOURIER COMPONENTS

In order to perform the linear analysis of the equations for Fourier components, we need the coordinates of the fixed points. The fixed point, by definition, can be found from algebraic equations, after setting the time derivatives in Eqs.(7),(11) equal to zero:

$$F^{(j)}(a^{(j)}, S, p_j) = 0, \quad j = 1, .., n, \tag{12a}$$

$$R - S = 0. \tag{12b}$$

R is found from Eqs.(6a) and (9),(10). Here $a^{(j)}$ is a column vector defined as

$$a^{(j)} = [a_{j|1}, a_{j|2}, .., a_{j|m}, .., a_{j|M}]^T. \tag{13}$$

In Eq.(12a), the nonlinear function $F^{(j)}$ is a column vector with elements defined by r.h.s. of Eq.(7), with integer m overrunning $m = 1, .., M$. The vector $F^{(j)}$ is only a part of the whole r.h.s. of the equations for Fourier components. The latter is composed of such parts:

$$F = [F^{(1)T}, F^{(1)T}, .., F^{(j)T}, .., F^{(n)T}, F_S]^T. \tag{14}$$

Here F_S is a r.h.s. of Eq.(11).

However, solving the system (12) for every set of parameters p_j is a hard task. We invoke a more simple procedure. Here we are specifically interested in finding

of Hopf curves on the parameters plane. These curves constitute the boundary between areas of parameters in which there exists either a stable limit cycle or a stable fixed point. As soon as the fixed point becomes a limit cycle, the Hopf curve is crossed.

The procedure of numerical search for the Hopf curve on the parameters plane is described in detail in IV.

III. DERIVING LINEARIZED EQUATIONS FOR FOURIER COMPONENTS

The deviation of the mean firing rate from the fixed point value is given by

$$\delta R(t) = \frac{1}{\pi\tau}\,\mathrm{Re}[\delta W(t)] = \frac{1}{2\pi\tau}[\delta W(t) + \delta W^*(t)]. \quad (15)$$

The corresponding deviation of the variable $W(t)$ reads:

$$\delta W(t) = 2\sum_{j=1}^{n}\sum_{m=1}^{M}(-1)^m b_j \delta a_{j|m}(t). \quad (16)$$

As a result, we have

$$\delta R(t) = \sum_{j=1}^{n}[B^{(j)}\delta a^{(j)} + B^{(j)*}\delta a^{(j)*}]. \quad (17)$$

For each pole number j we define an M-dimensional vector

$$B^{(j)} = [B_{j|1}, B_{j|2}, .., B_{j|m}, .., B_{j|M}]. \quad (18)$$

Each element of the vector (18) reads:

$$B_{j|m} = \frac{1}{\pi}(-1)^m b_j. \quad (19)$$

Here b_j $(j = 1, .., n)$ are the expansion coefficients in Eq.(31ms), they are found from Eq.(32ms).

The deviation of the vector of harmonics for each pole j is given by

$$\delta a^{(j)} = [\delta a_{j|1}, \delta a_{j|2}, .., \delta a_{j|m}, .., \delta a_{j|M}]^T. \quad (20)$$

Now we can write the linearized equation for the simplest case of one pole (for $n = 1$):

$$\tau\frac{d}{dt}\begin{pmatrix}\delta a^{(1)}\\ \delta S\end{pmatrix} = \begin{pmatrix}L^{(1)} & l^{(1)}\\ B^{(1)} & -\tau/\tau_s\end{pmatrix}\begin{pmatrix}\delta a^{(1)}\\ \delta S\end{pmatrix} + + \begin{pmatrix}0 & 0\\ B^{(1)*} & 0\end{pmatrix}\begin{pmatrix}\delta a^{(1)*}\\ \delta S\end{pmatrix}. \quad (21)$$

Next, the $M \times M$ matrix $L^{(j)}$ is a vector derivative of the vector of functions $F^{(j)}(a^{(j)}, S, p_j)$ w.r.t. the vector $a^{(j)}$:

$$L^{(j)} \equiv D_{a^{(j)}}F^{(j)}(a^{(j)}, S, p_j). \quad (22)$$

The column vector $l^{(j)}$ for each pole is defined as a derivative of the functions on the r.h.s. of the equations for Fourier components w.r.t. the variable S:

$$l^{(j)} \equiv D_S F(a^{(j)}, S, p_j). \quad (23)$$

The vector $l^{(j)}$ for each pole j reads:

$$l^{(j)} = [l_{j|1}, l_{j|2}, .., l_{j|m}, .., l_{j|M}]^T. \quad (24)$$

Each element in this vector is defined as

$$l_{j|m} = -im\tau J\left(a_{j|m} + \frac{1}{2}a_{j|m+1} + \frac{1}{2}a_{j|m-1}\right). \quad (25)$$

If $m + 1 > M$, then $a_{j|m+1}$ is set equal to zero, and, if $m - 1 = 0$, then we set $a_{j|m-1} = a_{j|0} = 1$.

Next, we define the $M \times M$ matrix $L^{(j)}$ for each pole j:

$$L^{(j)} = \{L^{(j)}_{mk}\}, \quad m, k = 1, .., M. \quad (26)$$

The elements of m-th row of this matrix are given as

$$L^{(j)}_{m,m-2} = -\sigma^2\frac{1}{4}m(m-1), \quad (27a)$$

$$L^{(j)}_{m,m-1} = m\left[\frac{i}{2}(A_j - 1) - \sigma^2\left(m - \frac{1}{2}\right)\right], \quad (27b)$$

$$L^{(j)}_{m,m} = m\left[i(A_j + 1) - \sigma^2\frac{3}{2}m\right], \quad (27c)$$

$$L^{(j)}_{m,m+1} = m\left[\frac{i}{2}(A_j - 1) - \sigma^2\left(m + \frac{1}{2}\right)\right], \quad (27d)$$

$$L^{(j)}_{m,m+2} = -\sigma^2\frac{1}{4}m(m+1). \quad (27e)$$

The other elements in the m-th row are zeros. Note that the numbers of indices of columns $k = m - 2, .., m + 2$ should be inside of the interval $k = 1, .., n$, unless the corresponding matrix elements are not involved. In (27) we used notation A_j given above (see Eq.(8)).

Next, we separate the real and imaginary parts in the Eq.(21), thus deriving the linearized spectral equation for one pole, $n = 1$:

$$\tau\frac{d}{dt}\begin{pmatrix}\delta a^{(1)'}\\ \delta a^{(1)''}\\ \delta S\end{pmatrix} = \begin{pmatrix}L^{(1)'} & -L^{(1)''} & l^{(1)'}\\ L^{(1)''} & L^{(1)'} & l^{(1)''}\\ 2B^{(1)'} & -2B^{(1)''} & -\tau/\tau_s\end{pmatrix}\begin{pmatrix}\delta a^{(1)'}\\ \delta a^{(1)''}\\ \delta S\end{pmatrix} \quad (28)$$

Here, all complex-valued quantities have real parts with one prime, and imaginary parts with two primes, i.e. $z' \equiv \mathrm{Re}(z)$ and $z'' \equiv \mathrm{Im}(z)$.

Next, after making analogous operations, we can derive the linearized spectral equations for two poles, i.e. with $n = 2$:

$$\tau\frac{d}{dt}\begin{pmatrix}\delta a^{(1)'}\\ \delta a^{(2)'}\\ \delta a^{(1)''}\\ \delta a^{(2)''}\\ \delta S\end{pmatrix} = \Phi^{(2)}\begin{pmatrix}\delta a^{(1)'}\\ \delta a^{(2)'}\\ \delta a^{(1)''}\\ \delta a^{(2)''}\\ \delta S\end{pmatrix}. \quad (29)$$

Here the Jacobi matrix reads:

$$\Phi^{(2)} = \begin{pmatrix} L^{(1)'} & 0 & -L^{(1)''} & 0 & l^{(1)'} \\ 0 & L^{(2)'} & 0 & -L^{(2)''} & l^{(2)'} \\ L^{(1)''} & 0 & L^{(1)'} & 0 & l^{(1)''} \\ 0 & L^{(2)''} & 0 & L^{(2)'} & l^{(2)''} \\ 2B^{(1)'} & 2B^{(2)'} & -2B^{(1)''} & -2B^{(2)''} & -\tau/\tau_s \end{pmatrix} \quad (30)$$

Here the zeros denote the $M \times M$ zero matrices. The generalization for an arbitrary number of poles n is straightforward.

IV. FINDING HOPF BIFURCATION CURVES FOR EQUATIONS OF THE FOURIER COMPONENTS

In order to perform the linear analysis of the equations for Fourier components, we need the coordinates of the fixed points. The fixed point, by definition, can be found from algebraic equations (after setting the time derivatives in Eqs.(7), (11) equal to zero). The fixed point satisfies Eqs.(12), where $R = \frac{1}{\pi\tau}\,\mathrm{Re}[W]$ and W is found from Eqs. (9), (10) at the coordinates $a_{j|m}$ of the fixed point.

However, solving the system (12) for every set of parameters p is a hard task. Here we invoke a more simple procedure. We are specifically interested in founding of Hopf curves on the parameters plane. These curves constitute the boundary between areas of parameters in which there exists either a stable limit cycle or an unstable fixed point. As soon as the fixed point becomes a limit cycle, the Hopf curve is crossed.

To start, we integrate the nonlinear system (7), (11) in the area of fixed points for a long time, until it settles on the stable fixed point. At the end, we record the coordinates of the fixed point.

Next, we slightly shift the value of the vector of parameters, $p \leftarrow p + \delta p$. The slightly shifted set of parameters will correspond to the new coordinates of the fixed point. Since the shift of parameters is small, the new coordinates of the fixed point will be also near the previous coordinates, which are already known.

Starting with the previous coordinates of fixed point, we use the Newton iterations to find the correct coordinates corresponding to the shifted parameters.

We can start the whole procedure with the new coordinates of fixed point corresponding to the new parameters, and repeat the whole procedure with a new small shift of parameters.

We are thus enabled to "travel" over the whole parameters plane by making small shifts of the parameters, and find the coordinates of the fixed points and their eigenvalues. From the eigenvalues of the Jacobi matrix we extract the information, whether the fixed point is stable or unstable. The points of parameter plane, at which the real parts of the leading eigenvalue become zero (and the corresponding imaginary parts are non-zeros), belong to the Hopf curve. Note that due to the pre-factor τ on the l.h.s. of the Eq.(29), the correct eigenvalues of the fixed point are obtained by division of eigenvalues of the matrix $\Phi^{(n)}$ by τ.

V. NOTATION OF VARIABLES IN THE PROGRAM

I	II	III
τ	tau	-
M	M	-
τ_s	taus	-
n	n	-
μ	muv	-
σ^2	sig2	-
J	J	-
ζ_j	zeta	zeta(np)
η_j	eta_k	eta_k(np)
A_j	A	A(np)
$L^{(j)}$	LM1	LM1_v(amt,St,np)
$B^{(j)}$	Bv	Bv1_v(np)
$l^{(j)}$	Lv1	Lv1_v(amt)
S	St	-
$a^{(j)}$	amt	amt(:,1)
$\Phi^{(n)}$	Phin	Phin(:,:)

In the above table we list the variables contained in this Supplementary material (I-st column), the corresponding notations in the program code (II-nd column), and their notations as they are called in the code (III-rd column) if they are arrays or functions.