, 28.7 LANGUAGE FUNCTION

Prove that the expression (18.112) yields the nth Laguerre polynomia.

Evaluating the nth derivative in (18.112) using Leibnitz' theorem we find

$$L_n(x) = \frac{e^x}{n!}\sum_{r=0}^{n} {}^nC_r \frac{d^r x^n}{dx^r}\frac{d^{n-r}e^-x}{dx^n-r}$$

$$= \frac{e^x}{n!}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\frac{n!}{(n-r)!}x^{n-r}(-1)^{n-r}e^{-x}$$

$$= \sum_{r=0}^{n}(-1)^{n-r}\frac{n!}{r!(n-r)!(n-r)!}x^{n-r}.4$$

Relabeling the summation using the index m = n - r, we obtain

$$L_n(x) = \sum_{m=0}^{n}(-1)^m\frac{n!}{(m!)^2(n-m)!}x^m,$$

which is precisely the expression (18.111) for the nth Languerre polynomal. ¡
Matual orthogonality
In sectin 17.4, we noted than Laguarre's aquation could be put into Sturm-Liouville from whith $p = xe^{-x}$, $q = 0$, $\lambda = v$ and $\rho = e^{-x}$, and its natural interval is thus [0,∞]$.SincetheLaguerrepolynomials$L$_n(x)$ ara solutions of the equation and are regular at the end-points, they must be mutually orthogonal over this interval with respect tothe weight function $\rho = e^{-x}$ i.e

$$\int_0^\infty L_n(x)e^{-x}dx = 0$$

if

$$n \neq k$$

This result may also be proved directly using the Rodrigues' formula (18.112).
Indeed, the normalisation, when $k = n$, is most easly found using this method.

Show that

$$I = \int_0^\infty L_n(x)L_n(x)e{-}xdx = 1$$