write $(x^2-1) = (x+1)(x-1)$ and use Leibnitz'theorom to evaluate the derevative, which yields

$$P_l^m(x) = \frac{1}{2^l * (l!)} * (1-x^2)^{m/2} * \sum_{r=0}^{l+m} \frac{(l+m)!}{r!(l+m+r)!} \frac{d^r(x+1)^l}{dx^r} \frac{d^{l+m-r}(x-1)^l}{dx^{l+m-r}}.$$

Considering the two derivative factors in a term in the summation, we note that the first is non-zero only for $r \le l$ and the second is non-zero for $l+m-r \le l$. Combining these condition yields $m \le r \le l$. Performing the derivatives, we thus obtain

$$P_l^m(x) = \frac{1}{2^l * (l!)} * (1-x^2)^{m/2} \sum_{r=m}^{l} \frac{(l+m)!}{r!(l+m-r)!} \frac{l!(x+1)^{l-r}}{(l-r)!} \frac{l!(x-1)^{r-m}}{(r-m)!} = (-1)^{m/2} \frac{l!(l+m)!}{2^l} \sum_{r=m}^{l} \frac{(x+1)^{l-r+\frac{m}{2}}(x-1)^{r-\frac{m}{2}}}{r!(l+m-r)!(l-r)!(r-m)!}. \quad (18.34)$$

Repearing the above calculation for $P_l^{-m}(x)$ and identifying once more those terms in the sum that are non-zero, we find

$$P_l^{-m}(x) = (-1)^{-m/2} \frac{l!(l-m)!}{2^l} \sum_{r=0}^{l-m} \frac{(x+1)^{l-r-\frac{m}{2}}(x-1)^{r+\frac{m}{2}}}{r!(l-m-r)!(l-r)!(r+m)!} = (-1)^{-m/2} \frac{l!(l-m)!}{2^l} \sum_{\overline{r}=m}^{l} \frac{(x+1)^{l-\overline{r}+\frac{m}{2}}(x-1)^{\overline{r}-\frac{m}{2}}}{(\overline{r}-m)!(l-\overline{r})!(l+m-\overline{r})!\overline{r}!}, \quad (18.35)$$

where, in the second equality, we have rewritten the summation in terms of the new index $\overline{r} = r+m$. Comparing (18.34) and (18.35), we immediately arrive at the required result (18.33). ◀

Since $P_l(x)$ is a polynomial of order l, we have $P_l^m(x) = 0$ for $|m| > l$. From its definition, it is clear that $P_l^m(x)$ is also a polynomial of order l if m is even, but contains the factor $(1-x^2)$ to a fractional power if m is odd. In either case, $P_l^m(x)$ is regular at $x = \pm 1$. The first few associated Legendre functions of the first kind are easily constructed and are given by (omitting the m=0 cases)

$P_1^1(x) = (1-x^2)^1/2,$

$P_2^2(x) = 3(1-x^2),$

$P_2^3(x) = 15x(1-x^2),$

$P_1^2(x) = 3x(1-x^2)^1/2,$

$P_1^3(x) = \frac{3}{2}(5x^2-1)(1-x^2)^1/2,$

$P_3^3(x) = 15(1-x^2)^3/2.$

Finally, we note that the associated Legendre functions of the second kind $Q_l^m(x)$, like $Q_l(x)$, are singular at $x = \pm 1$.

18.2.2 Properties of associated Legendre functions $P_l^m(x)$

When encountered in physical problems, the variable x in the associated Legendre equation (as in the ordinary Legendre equation) is usually the cosine of the polar angle θ in spherical polar coordinates, and we then require the solution $y(x)$ to be regular at $x = \pm 1$ (corresponding to $\theta = 0$ or $\theta = \pi$). For this occur, we require l to be an integer and the coefficient c_2 of the function $Q_l^m(x)$ in (18.31)