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IT1030 – Mathematics for Computing
Lecture 01 – Number Systems

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Number Systems

- Mathematical notation/symbols for representing values (numbers).
- In different systems, different symbols are used.

0123456789

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〇一二三四五六七八九

Positional Number System

- We are used to dealing with numbers in the **decimal** system.
- This is probably a result of having ten fingers.
- It is a is a **positional** number system.
  - Roman number system is not such a system.
- The position of the symbol denotes the magnitude.
- A positional number system uses a **base** (aka **radix**).
- A number system with base *b* is a system that uses distinct symbols for *b* digits.

Positional Number System (cont'd.)

- The most common base used in everyday activities is 10 (Decimal System).
- Different bases are used in other situations.
- The base can be written as a subscript to the number for easy identification.
  - Example: 1265_{10} & 010000001_{2}
- 4 types of positional number systems are discussed.
  - Decimal (Base = 10, {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}).
  - Binary (Base = 2, {0, 1}).
  - Octal (Base = 8, {0, 1, 2, 3, 4, 5, 6, 7, 8}).
  - Hexadecimal (Base = 16, {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}).

Positional Number System (cont'd.)

- To determine the quantity, it is necessary to multiply each digit by an integer power of the base and then form the sum of all weighted digits.

Example (7241_{10}):-

7		2		4		1
×		×		×		×
10^3		10^2		10^1		10^0
=		=		=		=
7000	+	200	+	40	+	1

7241

Binary Number System

- Mainly used in computers and computer-based devices.
- A computer contains electronic components that uses voltages.
- Therefore numbers are represented in a computer with a base of 2.
- All other information is represented using a binary code as well.
  - Letters of the alphabet and punctuation marks
  - Microprocessor instruction
  - Graphics/Video
  - Sound

Number Base Conversion

- Should be able to convert a number in one base to another base.
- Examples will be discussed in converting,
  - Decimal ↔ Binary
  - Decimal ↔ Octal
  - Decimal ↔ Hexadecimal
  - Binary ↔ Octal

Repeated Division Method

- Can be easily used to convert a decimal number to another base.

1. Divide the number successively by the base.
2. After each division record the remainder.
3. The result is read from the last remainder upwards.

Repeated Division Method

- Convert 123_{10} to a binary representation.
  - $123/2 \Rightarrow q = 61 \ \& \ r = 1$
  - $61/2 \Rightarrow q = 30 \ \& \ r = 1$
  - $30/2 \Rightarrow q = 15 \ \& \ r = 0$
  - $15/2 \Rightarrow q = 7 \ \& \ r = 1$
  - $7/2 \Rightarrow q = 4 \ \& \ r = 1$
  - $4/2 \Rightarrow q = 2 \ \& \ r = 0$
  - $2/2 \Rightarrow q = 1 \ \& \ r = 0$
- $123_{10} = 0011011_2$

Repeated Subtraction Method

- Can be used to convert a decimal number to binary.

1. Starting with the 1s place, write down all of the binary place values in order until you get to the first binary place value that is GREATER THAN the decimal number you are trying to convert.
2. Mark out the largest place value (it just tells us how many place values we need).
3. Subtract the largest place value from the decimal number. Place a “1” under that place value.

Repeated Subtraction Method

4. For the rest of the place values, try to subtract each one from the previous result.
   - If you can, place a “1” under that place value.
   - If you can’t, place a “0” under that place value.
5. Repeat Step 4 until all of the place values have been processed.

- Convert 123_{10} to binary using the repeated subtraction method.

Other Base Conversions (cont'd.)

- Binary/Octal/Hexadecimal to Decimal

1. Take the left most none zero bit,
2. Multiply by the base and add it to the bit on its right.
3. Now take this result, multiply by the base it and add it to the next bit on the right.
4. Continue in this way until the right-most bit has been added in.

The fundamental setup of positional number systems can be used as well.

Other Base Conversions (cont'd.)

- Binary to Octal/Hexadecimal

1. Form the bits into groups of **three/four** starting at the right and move leftwards.
2. Replace each group of three bits with the corresponding octal/hexadecimal digit.

- Octal/Hexadecimal to Binary

The opposite of the above process is used.

Conversion of Fractions

- Decimal Fractions to Binary Fractions

1. Begin with the decimal fraction and multiply by 2. The whole number part of the result is the first binary digit to the right of the point.
2. Disregard the whole number part of the previous result and multiply by 2 once again. The whole number part of this new result is the second binary digit to the right of the point.
3. Continue this process until we get a zero as our decimal part or until we recognize an infinite repeating pattern.

Conversion of Fractions (Example)

- Convert 0.625_{10} to binary.
- Convert 0.1_{10} to binary.
- Convert 1.625_{10} to binary.

Conversion of Fractions (cont'd.)

- Binary Fractions to Decimal Fractions

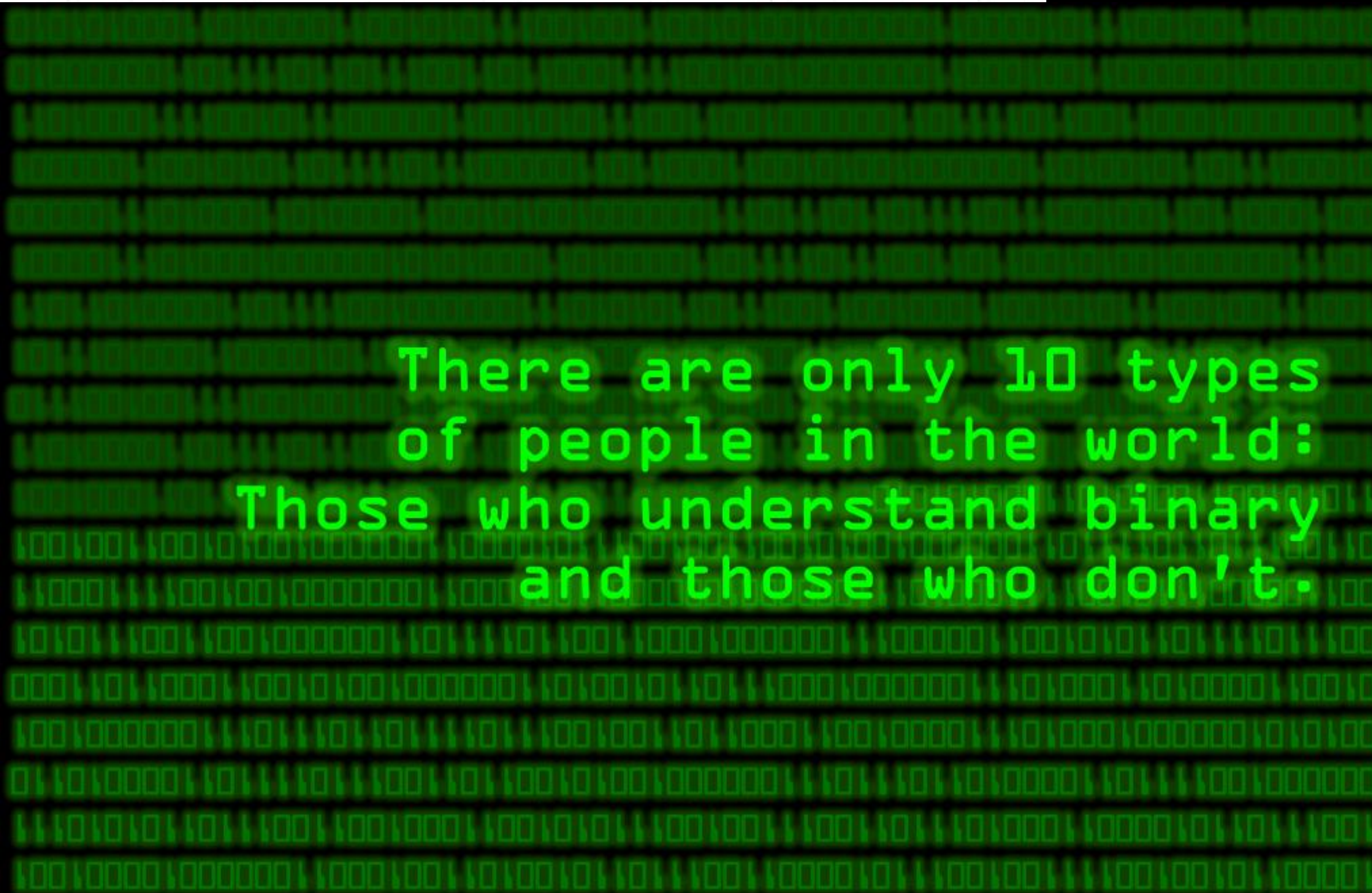
The fundamental setup of positional number systems used in converting binary integers to decimals can be used here.

- Represent 10.01101_2 as a decimal number.

Summary

- Students should be able to,
  - Understand the numerical system.
  - Explain why computer designers chose to use the binary system for representing information in computers.
  - Explain different number systems.
  - Translate numbers between number systems.

- Understanding the pattern in each set of conversions will make it easier to remember the methods.


There are only 10 types
of people in the world:
Those who understand binary
and those who don't.

End of Lecture 01

Next Lecture:-
Computer Arithmetic

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Lecture 1.1. – Computer Arithmetic

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Introduction

Recap:

- Binary numbers are a number system with base 2.
- Information represented inside a computer takes binary values.
- Previous lecture dealt with the conversions between different number systems.
- This lecture deals with basic mathematical operations (such as addition, subtraction, multiplication and division) for binary numbers.

Binary Addition

- Addition in the decimal number system.
- Add values in the rightmost position (least significant).
- If this addition is grater than 10, 1 is carried to the 2^{nd} position and added.
- This process is carried for all the positions.
- Binary addition follows the same set of rules.
- If the addition is greater that 2, 1 is carried to the 2^{nd} next position.

Examples

- Evaluate the following.
  - $101_2 + 101_2$
  - $00011010_2 + 00001100_2$
  - $10001 \ + \ 11101$
  - $1110 \ + \ 1111$
  - $101101 \ + \ 11001$
  - $10111 \ + \ 110101$
  - $1011001 \ + \ 111010$
  - $11011 \ + \ 1001010$

Binary Subtraction

- Similar to subtraction in the decimal number system.
- Inverse of addition.
- If the values cannot be subtracted, borrow from the next position.
- Subtraction table,
  - $0 - 0 = 0$
  - $1 - 0 = 1$
  - $1 - 1 = 0$
  - $0 - 1 = 1$ With a borrow of 1

Examples

- Evaluate the following.
  - $10110 - 10010$
  - $1011011 - 10010$
  - $100010110 - 1111010$
  - $1010110 - 101010$
  - $101101 - 100111$
  - $1000101 - 101100$
  - $1110110 - 1010111$

Binary Multiplication

- Similar to multiplication and division in the decimal number system.
- Rules of binary multiplication,
    - $0 \times 0 = 0$
    - $0 \times 1 = 0$
    - $1 \times 0 = 0$
    - $1 \times 1 = 1.$

Examples

- Evaluate the following.
  - 1100×1010
  - 1111×101
  - 0011×11
  - 1100110×1000

Binary Division

- Rules of binary division,
  - $0 \div 1 = 0$
  - $1 \div 1 = 1.$

- Examples
  - $1000 \div 10$
  - $1010 \div 11$
  - $1111 \div 111$

Complementary Arithmetic

- Complements are used in digital computers for simplifying,
  - the subtraction operation
  - the logical manipulation.
- Two types of compliments for each base system.
  - $r's$ compliment
  - $(r-1)'s$ compliment

Example: For binary numbers, 2's complement and 1's complement.

Complementary Arithmetic

- Given a number N in base r having n digits,

$(r-1)$'s complement of N,

$$N' = (r^n - 1) - N$$

- Given a number N in base r having n digits,

r's complement of N,

$$N'' = r^n - N \; for \; \mathrm{N} \neq 0; 0 \; otherwise$$

Examples

- Obtain 9's complement and 10's compliment of 246700.
- Obtain 1's and 2's compliments of the following binary numbers.
  - 1100011
  - 0001111
  - 1010100
  - 1111011


BINARY
IT'S AS EASY AS
01.10.11

End of Lecture 1.2

Next Lecture : Boolean Algebra

Mathematics for Computing - IT1030

Lecture 02
Boolean Algebra

Boolean Algebra

- A variable used in an **algebraic formula** so far, is assumed to take *a set of numerical values*.
- All variables in **boolean equations** can take only one of *two possible values*.
- Used symbols for the two values are **0** and **1**.
- Rules first defined for logic by George Boole (1854), were adapted for the use in designing electronic circuits.
- The circuits in computers and other electronic devices have inputs, each of which is either a 0 or a 1.

Boolean Algebra (cont'd.)

- One major advantage in using these rules is to simplify an electronic circuit.
- Boolean algebra provides the operations and the rules for working with boolean variables.
- Three (3) boolean operators are discussed.
  - Complement
  - Boolean sum
  - Boolean product
- Ten (10) rules are also discussed (aka Boolean Identities).

Boolean Operators

- **Complement**
  - Defined as the opposite of the value that a boolean variable takes.
  - Denoted with a bar (E.g.: $\overline{\mathrm{A}}$).
  - $\overline{0} = 1$ and $\overline{1} = 0$.
- **Boolean Sum**
  - Defined as the output to be 1 if at least one variable is 1.
  - Denoted with the symbol $+$ or by OR.
  - $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$ and $1 + 1 = 1$.

Boolean Operators (cont'd.)

- **Boolean Product**
  - Defined as the output to be $\mathbf{0}$ if at least one variable is $\mathbf{0}$.
  - Denoted with the symbol ($\cdot$) or by $\mathbf{AND}$.
  - $\mathbf{0 \cdot 0 = 0}$, $\mathbf{0 \cdot 1 = 0}$, $\mathbf{1 \cdot 0 = 0}$ and $\mathbf{1 \cdot 1 = 1}$.
  - When there is no danger of confusion, the symbol $\cdot$ can be omitted.

- Order of boolean operators,
  1. Complement.
  2. Boolean products.
  3. Boolean sums.

Boolean Identities

1. Law of Double Complement
   - $\overline{\overline{A}} = A$
2. Idempotent Laws
   - $A + A = A$
   - $A \cdot A = A$
3. Identity Laws
   - $A + 0 = A$
   - $A \cdot 1 = A$
4. Domination/Null/Universal Bound Laws
   - $A + 1 = 1$
   - $A \cdot 0 = 0$

Boolean Identities (cont'd.)

5. Commutative Laws
   - $A + B = B + A$
   - $A \cdot B = B \cdot A$
6. Associative Laws
   - $A + (B + C) = (A + B) + C$
   - $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
7. Distributive Laws
   - $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$
   - $A + B \cdot C = (A + B) \cdot (A + C)$

Boolean Identities (cont'd.)

8. De Morgan's Laws
   - $\overline{(A \cdot B)} = \overline{A} + \overline{B}$
   - $\overline{(A + B)} = \overline{A} \cdot \overline{B}$
9. Absorption Laws
   - $A \cdot (A + B) = A$
   - $A + A \cdot B = A$
10. Inverse Laws / Unit & Zero Properties
    - $A + \overline{A} = 1$
    - $A \cdot \overline{A} = 0$

Examples

1. Find the values of the following expressions.
   i. $1 \cdot \bar{0}$
   ii. $1 + \bar{1}$
   iii. $\overline{(1 + 0)}$
2. Prove both variants of the absorption law using other boolean identities.
3. Simplify the following expressions.
   i. $A\bar{B}D + A\bar{B}\bar{D}$
   ii. $(\bar{A} + B)(A + B)$
   iii. $M = \bar{W}\bar{X}\bar{Y}\bar{Z} + \bar{W}X\bar{Y}\bar{Z} + WX\bar{Y}\bar{Z} + W\bar{X}\bar{Y}\bar{Z}$

Truth Tables

- To verify the above rules, a **truth table** can be used.
- It's also known as a **Table of Combinations**.
- It's a table displaying all possible values for the variables and the outcomes for a boolean expression.
- If there are $\mathbf{n}$ number of variables, there will be 2^{n} number of rows in the truth table.
- If the truth table for two boolean expressions shows the same outcomes for the same values for the variables, it can be concluded that the expressions are the same/equal.

Examples

1. Use a table to express the values of each of these Boolean functions.
   1. $\overline{A}B$
   2. $M = x\overline{y} + \overline{(xyz)}$
   3. $F(x, y, z) = \overline{y}(xz + \overline{x}\overline{z})$

2. Using a truth table, show that,

$$x\overline{y} + y\overline{z} + \overline{x}z = \overline{x}y + \overline{y}z + x\overline{z}$$

Sum of Products (SoP)

- In some cases, the truth table might be known and we might want to know the expression that gives the truth table.
- This can be done by representing as a **Sum of Products (SoP)** of the variables and their complements.
- Steps:-
  1. Select the rows in the truth table that gives **1** as the outcome.
  2. Write how we can obtain **1** for the first selected row by using the **product** of the variables.
  3. Repeat step two for all selected rows and use the **sum** to combine all results.

Example

Find the boolean expression for F from the given truth table.

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Product of Sums (PoS)

- Used for the same reason as a SoP.
- Product of Sums (PoS) has opposite steps of SoP.
- Steps:-
  1. Select the rows in the truth table that gives **0** as the outcome.
  2. Write how we can obtain **0** for the first selected row by using the **sum** of the variables.
  3. Repeat step two for all selected rows and use the **product** to combine all results.
- Conversion can be done between the two using De Morgan's rule.

Duality Principle

- In a boolean expression, if all the sums (+) and products (·) are exchanged as well as if 1's and 0's are exchanged, the resulting expression is the opposite of the initial expression.

- This property is observed between SoP and PoS.

- The duel of the complement of one form is equal to the expression in the other form.


WHAT DID THE BOOLEAN SAY
TO THE INTEGER?
YOU CAN'T HANDLE THE TRUTH

Summary

- Students should be able to,
  - Understand the boolean expressions.
  - Learn laws and rules of boolean algebra.
  - Simplify boolean expressions using boolean identities.
  - Use Sum of Products (SoP) and Product of Sums (PoS) to find boolean expressions.
  - Understand similarities and differences between boolean variables as opposed to regular variables.

End of Lecture 02

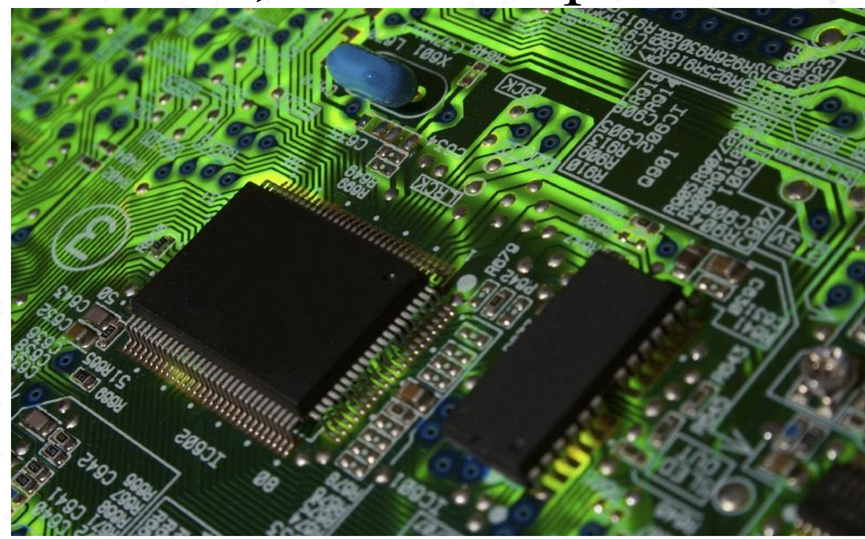
Next Lecture:-
Logic Gates

Mathematics for Computing (IT1030)

Lecture 2
Logic Gates

Logic Gates

- A computer, or other electronic device, is made up of a number of circuits.



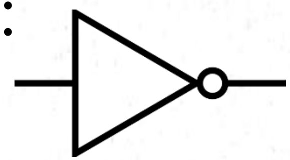
- The components in a logical circuit takes 0 and 1 as inputs.
- 1 is the state where there is a voltage on the input and 0 is the state where there is no voltage on the input.

Logic Gates (cont'd.)

- Therefore, boolean algebra is used to model the circuitry in electronic devices.
- The absence of a voltage is usually denoted as 0 (zero) and the presence of a voltage is denoted by 1 (one).
- As mentioned earlier, concepts of boolean algebra can be used to simplify logical circuitry.
- There are a set of components that matches the boolean operators discussed earlier.
- These components are called **Logic Gates**.

Basic Logic Gates (NOT Gate)

- **Complement** → NOT Gate.
- Also known as *inverter* or *complementer*.
- Consists of a single input and a single output.
- As in the complement, the input gets inverted.
- Symbol:

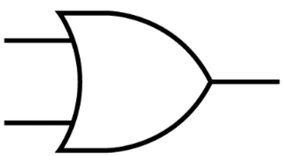


- Truth Table:

Input (A)	Output ($\bar{A}$)
0	1
1	0

Basic Logic Gates (OR Gate)

- **Boolean Sum** → OR Gate.
- Consists of two inputs and a single output.
- As in the sum, the output is 1 if at least one input is 1.
- Symbol

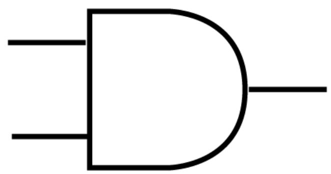


- Truth Table:

Input (A)	Input (B)	Output (A + B)
0	0	0
0	1	1
1	0	1
1	1	1

Basic Logic Gates (AND Gate)

- **Boolean Product** → AND Gate.
- Consists of two inputs and a single output.
- As in the product, the output is 0 if at least one input is 0.
- Symbol:

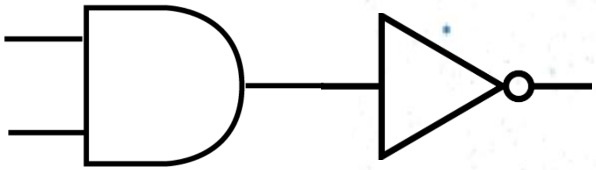


- Truth Table:

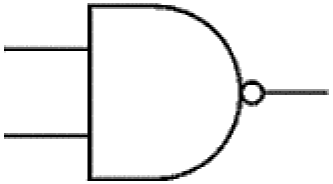
Input (A)	Input (B)	Output (A · B)
0	0	0
0	1	0
1	0	0
1	1	1

Derived Logic Gates (NAND Gate)

- Created by combining an AND gate with a NOT gate.



- Symbol:

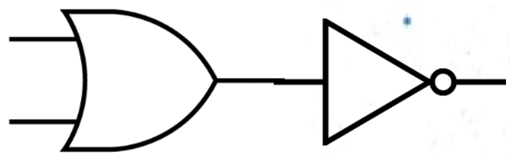


- Truth Table:

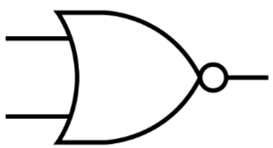
Input (A)	Input (B)	Output ($\overline{A \cdot B}$)
0	0	1
0	1	1
1	0	1
1	1	0

Derived Logic Gates (NOR Gate)

- Created by combining an OR gate with a NOT gate.



- Symbol

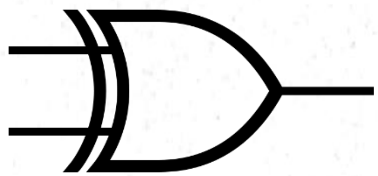


- Truth Table:

Input (A)	Input (B)	Output ($\overline{A+B}$)
0	0	1
0	1	0
1	0	0
1	1	0

Derived Logic Gates (XOR Gate)

- Similar to the OR Gate.
- Consists of two inputs and a single output.
- The output is 1 if **ONLY ONE** input is 1.
- Symbol:

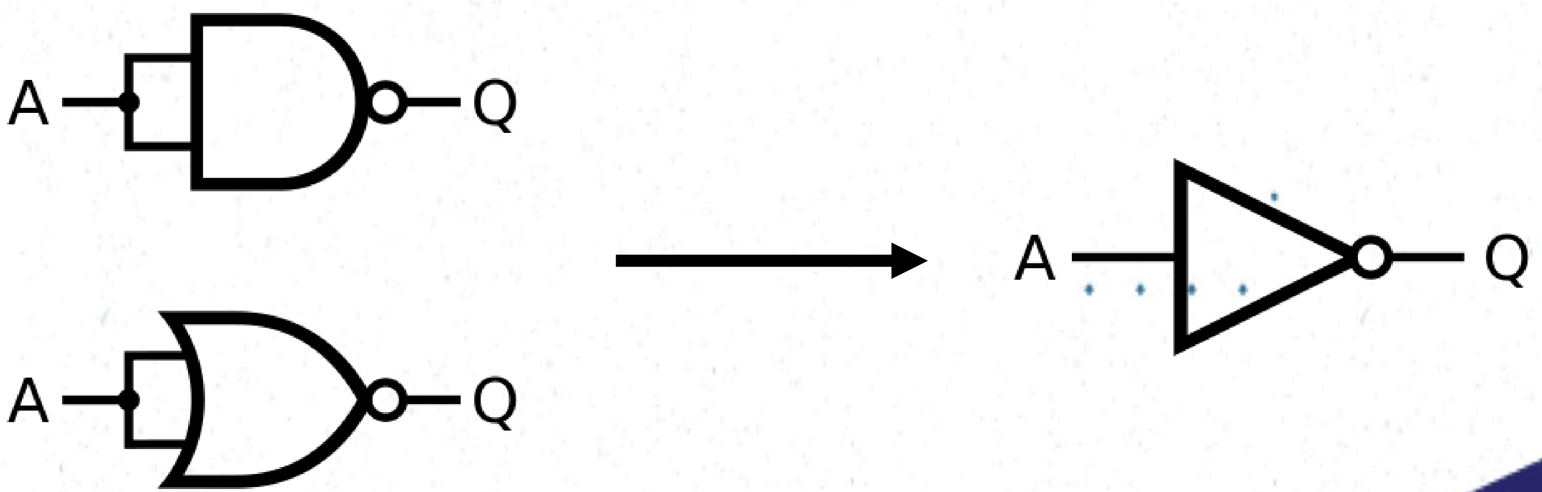


- Truth Table:

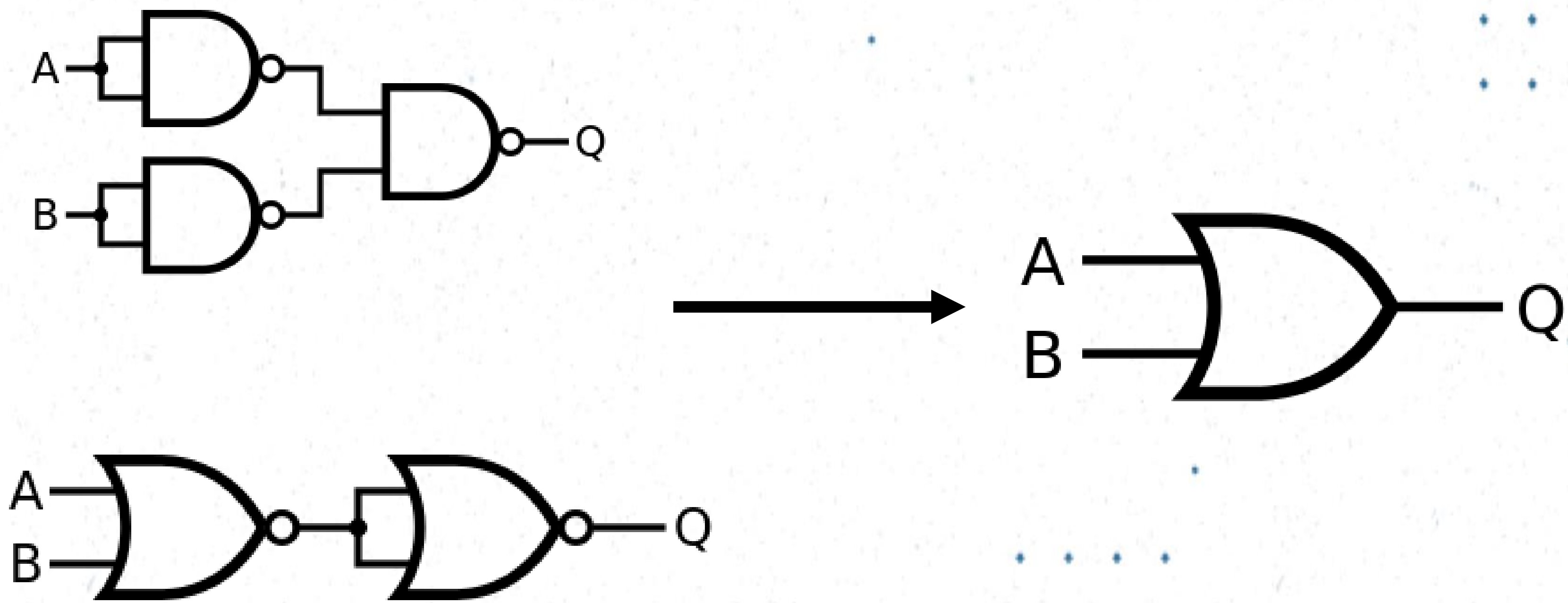
Input (A)	Input (B)	Output ($A \oplus B$)
0	0	0
0	1	1
1	0	1
1	1	0

Universality of NAND and NOR Gates

- Basic gates can be derived using the NAND and NOR gates.
- It has been identified that NAND can be replaced by NOR or otherwise



Universality of NAND and NOR Gates



Universality of NAND and NOR Gates

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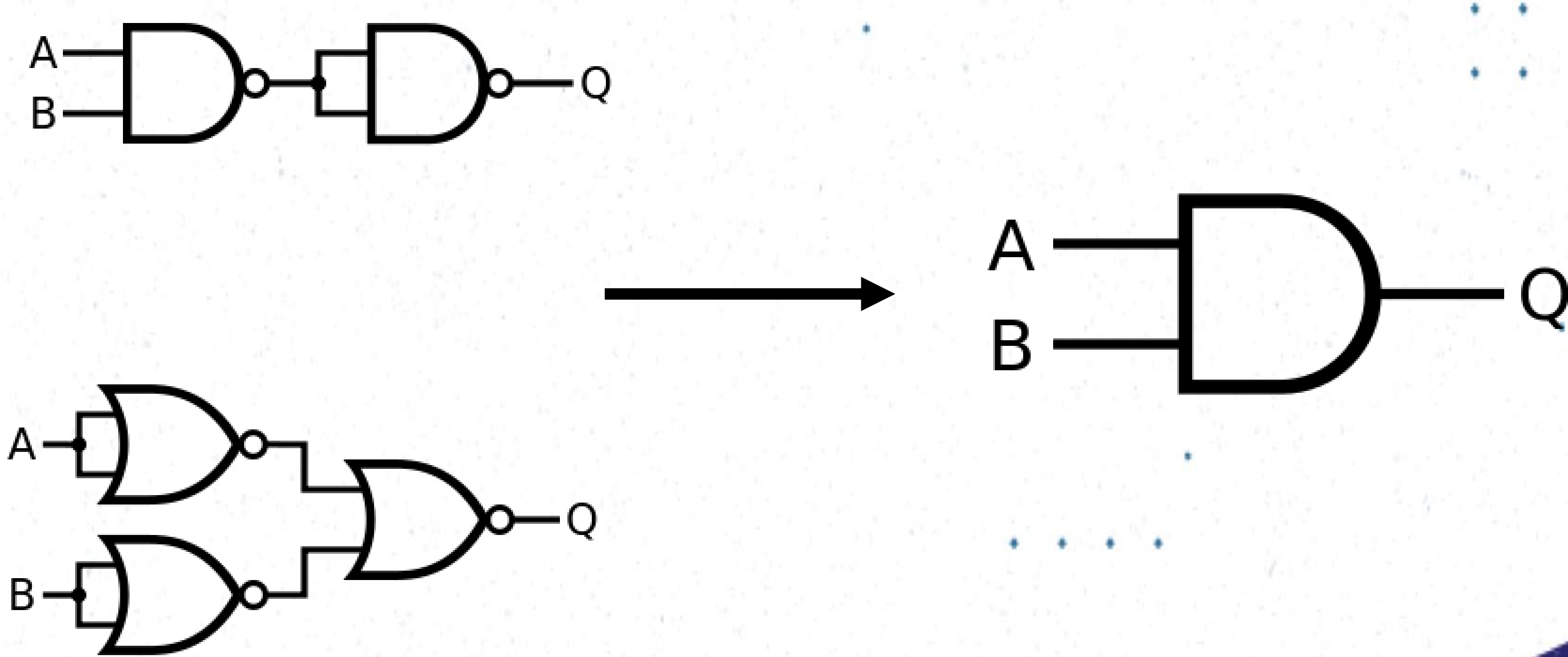
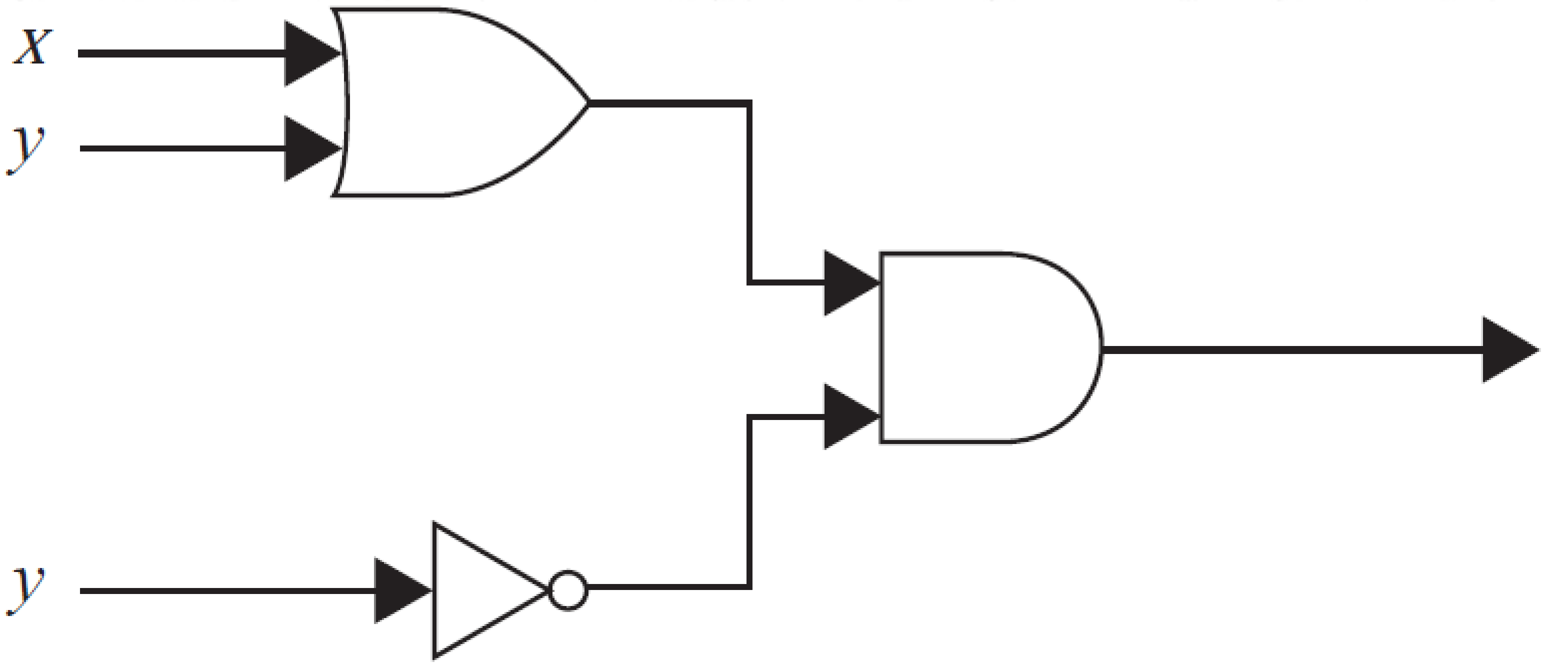


Exercises

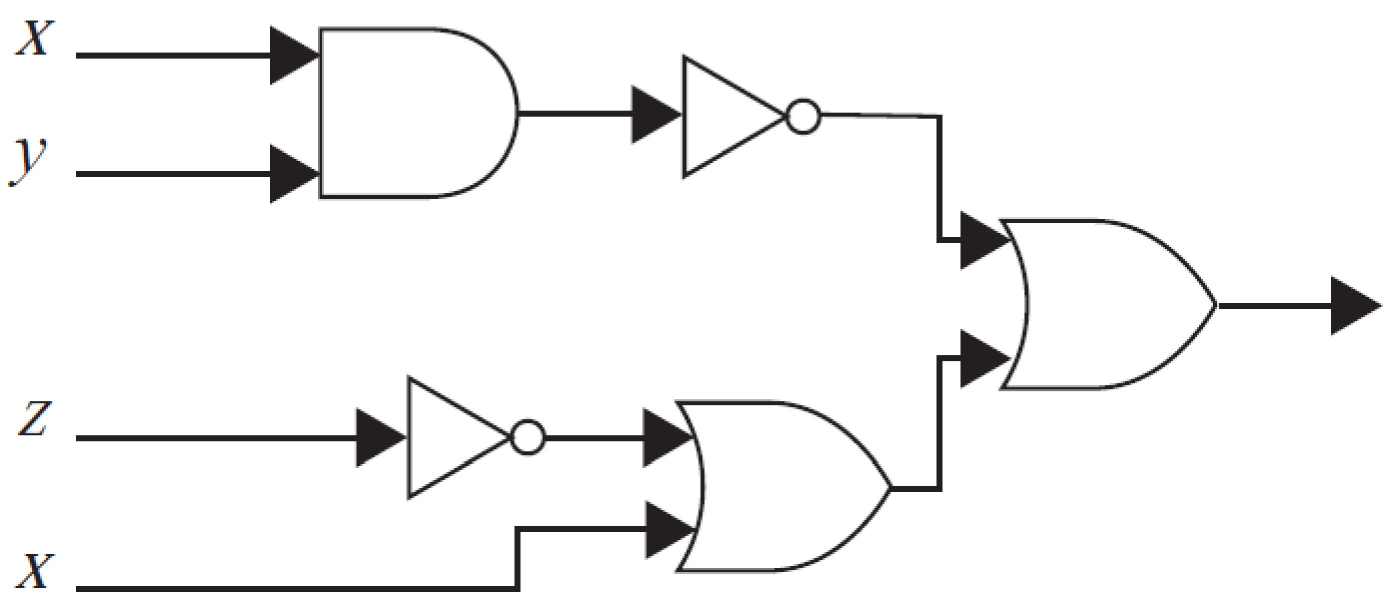
1. Find the outputs from the following logic circuits.

   i.



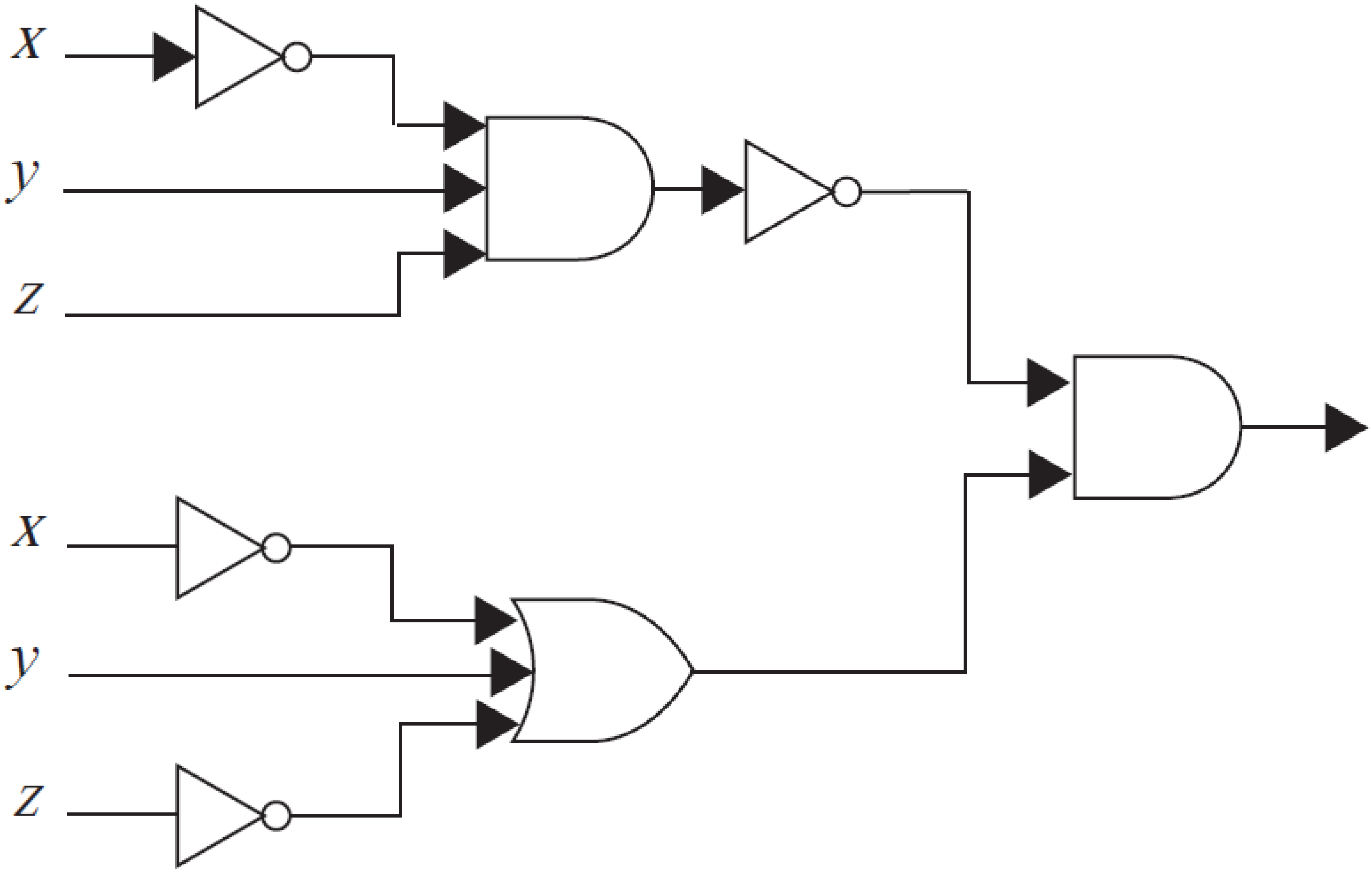
Exercises (cont'd.)

ii.



Exercises (cont'd.)

iii.



Exercises (cont'd.)

2. Construct circuits from inverters, AND gates, and OR gates to produce these outputs.
   - *i.* $\overline{A} + B$
   - *ii.* $\overline{(A + B)A}$
   - iii. $ABC + \overline{A}\,\overline{B}\,\overline{C}$
   - *iv.* $\overline{(\overline{x} + z)(y + \overline{z})}$
   - *v.* $\overline{W}\,\overline{X}\,\overline{Y}\,\overline{Z} + \overline{W}X\overline{Y}\,\overline{Z} + WX\overline{Y}\,\overline{Z} + W\overline{X}\,\overline{Y}\,\overline{Z}$
3. Design a circuit for a light fixture controlled by three switches, where flipping one of the switches turns the light on when it is off and turns it off when it is on.

[redacted]

Summary

- Students should be able to,
  - Get an understanding about the need and usage of logic gates.
  - Understand basic logic gates and the connection with boolean operators.
  - The functions obtained by the logic gates.
  - Draw truth tables for the logic gates.
  - Draw circuit diagrams for the logic gates using the standard symbols.
  - Drawing circuit diagrams from boolean expressions and vice versa.

End of Lecture 2

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Mathematics for Computing – IT1030

Lecture 04 - Differentiation

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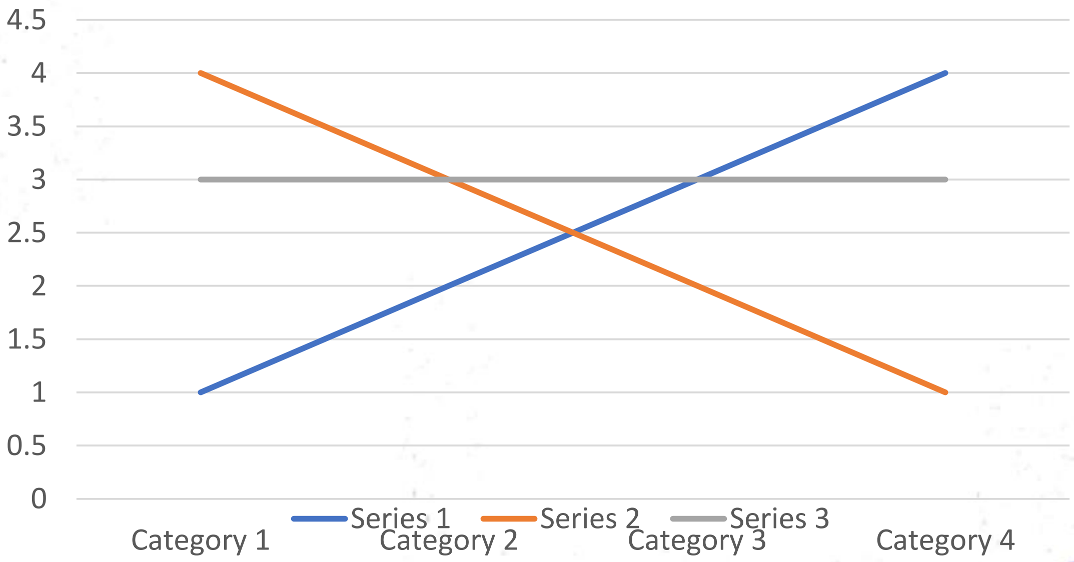
Introduction to Differentiation

- ➢ Slope of a line indicates the rate at which a line rises or falls
- ➢ For a line, this rate (or slope) is the same at every point on the line



Tangent Line to a Graph

- To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at the point.

- In simple terms, **the *tangent line* to the graph of a function at a point *P* is the line that best approximates the graph at that point**.

Slope of a Graph

- Because a tangent line approximates the graph at a point, the problem of finding the slope of a graph at a point becomes one of finding the slope of the tangent line at the point.

- **Example 1.1** Use the graph to approximate the slope of the graph of $f(x) = x^2$ at the point (1 ,1).
- From the graph of $f(x) = x^2$, you can see that the tangent line at (1,1) rises approximately two units for each unit change in *x*. Thus, the slope of the tangent line at (1,1) is given by

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} \approx \frac{2}{1} = 2.$$

The Derivative of a Function

- The derivative of $f(x)$ at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided this limit exists.

- A function is differentiable at x if its derivative exists at x.
- The process of finding derivatives is called differentiation.
- $f'(x)$ is also denoted by,

$$\frac{dy}{dx}, y', D_x[y], \frac{d\,[f(x)]}{dx}$$

Some Rules for Differentiation

- The Constant Rule
- The Power Rule
- The Constant Multiple Rule
- The Sum and Difference Rules
- The Product Rule
- The Quotient Rule
- The Chain Rule

The Constant Rule

The Constant Rule

The derivative of a constant function is zero. That is,

$$\frac{d}{dx}[c] = 0, \qquad c \text{ is a constant.}$$

Example 1.2

Find the derivative of $f(x) = 3$

$f'(x) = 0$

The Power Rule

The (Simple) Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}, \qquad n \text{ is any real number.}$$

- **Example 1.3**
- Find the derivative of $f(x) = x^3$
- $f'(x) = 3x^2$

The Constant Multiple Rule

The Constant Multiple Rule

If f is a differentiable function of x, and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x), \qquad c \text{ is a constant.}$$

- **Example 1.4**
- Find the derivative of $f(x) = 3x^2$
- $f'(x) = 6x$

The Sum and Difference Rules

If f and g are both differentiable, then

The sum rule $\quad \frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

The difference rule $\quad \frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

Example 1.5

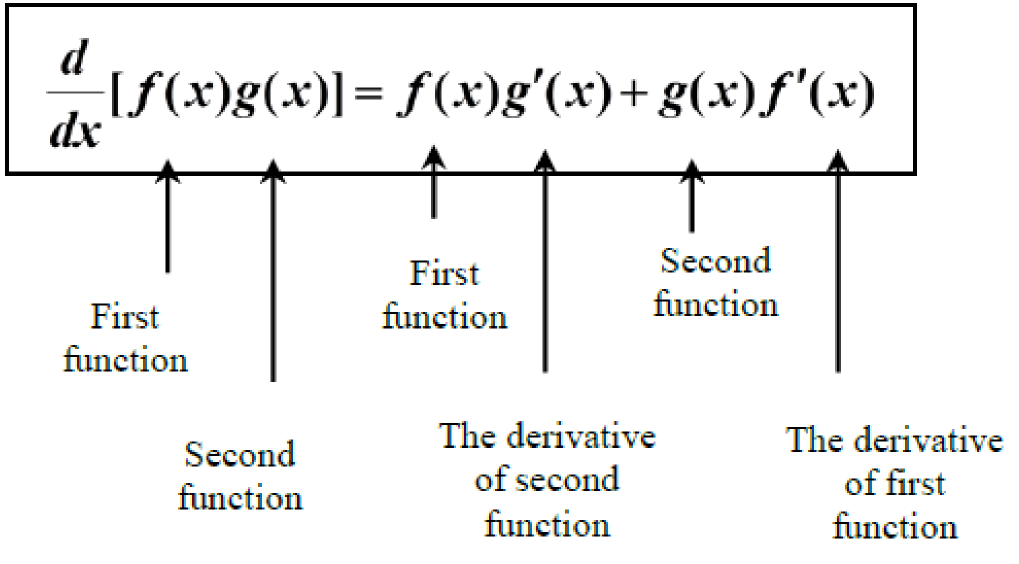
Find the derivative of $f(x) = 3x^2+3$

$f'(x) = 6x + 0 = 6x$

Find the derivative of $f(x) = 3x - 2x^2$

$f'(x) = 3 - 4x$

The Product Rule



- **Example 1.6**
- Find the derivative of $f(x) = 3x^2(2x\text{-}5)$
- $f'(x) = 3x^2 * 2 + (2x - 5) * 6x = 18x^2 - 30x$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2};g(x)\neq 0$$

Example 1.7

Find the derivative of $f(x) = \frac{2x}{(x-4)}$

$$f'(x) = \frac{(x-4)*2 - 2x*1}{(x-4)^2} = \frac{-8}{(x-4)^2}$$

The Chain Rule

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

- Example 1.8
- Find the derivative of $f(x) = (3x - 5)^2$
- $f'(x) = 2(3x - 5) * (3 * 1 - 0) = 6(3x - 5)$

[redacted]

End of Lecture 04

Next Lecture :
Integration

[redacted]

Mathematics for computing - IT1030

Lecture 05 – Integration and its Applications

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Indefinite Integration

Introduction

- Calculus involves two basic operations:
  - differentiation
  - integration (or anti differentiation)

- The two operations (integration & differentiation) are inverses of each other.

Definition of Anti derivative

- A function F is an anti derivative of a function f if for every x in the domain of f, it follows that

$$F'(x) = f(x)$$

- If $F(x)$ is an anti derivative of $f(x)$, then $F(x) + c$, where c is any constant, $F(x)$ is also an anti derivative of $f(x)$.

Notation for Anti derivatives and In[redacted] Integrals

$$\int f(x)\ dx = F\ (x) + C$$

Integrand

Anti derivative

Integral Sign

Finding Anti-Derivatives

- ➢ The inverse relationship between the operations of integration and differentiation can be shown symbolically, as follows.

$$\frac{d}{dx}\left[\int f(x)\right] = f(x)$$

$$\int f'(x)dx = f(x) + C$$

Basic Integration Rules

1. $\int k dx \;=\; kx + C \qquad ; \quad k \text{ is a constant}$
2. $\int kf(x)dx = k \int f(x)dx$
3. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)dx$
4. $\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)dx$
5. $\int x^n \, dx \;=\; \dfrac{x^{n+1}}{n+1} + C \qquad , n \neq -1$

Basic Integration Rules

Example **Finding Indefinite Integrals**

Find each indefinite integral.

a. $\int \frac{1}{2}\,dx$ **b.** $\int 1\,dx$ **c.** $\int -5\,dt$

SOLUTION

a. $\int \frac{1}{2}\,dx = \frac{1}{2}x + C$ **b.** $\int 1\,dx = x + C$ **c.** $\int -5\,dt = -5t + C$

Basic Integration Rules

Example **Finding an Indefinite Integral**

Find $\int 3x\,dx$

$$\int 3x\,dx = 3\int x\,dx \qquad \text{Constant Multiple Rule}$$

$$= 3\int x^1\,dx \qquad \text{Rewrite } x \text{ as } x^1.$$

$$= 3\left(\frac{x^2}{2}\right) + C \qquad \text{Simple Power Rule with } n = 1$$

$$= \frac{3}{2}x^2 + C \qquad \text{Simplify.}$$

Basic Integration Rules

Example **Finding an Indefinite Integral**

Original Integral

a. $\displaystyle\int \frac{1}{x^3}\,dx$

b. $\displaystyle\int \sqrt{x}\,dx$

Rewrite	*Integrate*	*Simplify*
$\displaystyle\int x^{-3}\,dx$	$\dfrac{x^{-2}}{-2} + C$	$-\dfrac{1}{2x^2} + C$
$\displaystyle\int x^{1/2}\,dx$	$\dfrac{x^{3/2}}{3/2} + C$	$\dfrac{2}{3}x^{3/2} + C$

Basic Integration Rules

Example **Integrating Polynomial Functions**

Find (a) $\int (x + 2)\, dx$ and (b) $\int (3x^4 - 5x^2 + x)\, dx$.

SOLUTION

a.
$$\int (x + 2)\, dx = \int x\, dx + \int 2\, dx$$
$$= \frac{x^2}{2} + C_1 + 2x + C_2$$
$$= \frac{x^2}{2} + 2x + C$$

b.
$$\int (3x^4 - 5x^2 + x)\, dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$$
$$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$$

Definite Integration

Definition of a Definite Integral

- Let f be nonnegative and continuous on the closed interval $[a, b]$. The area of the region bounded by the graph of f, the $x-$axis, and the lines $x = a$ and $x = b$ Is denoted by,

$$Area = \int_a^b f(x)$$

- The expression $\int_a^b f(x)$ is called the definite integral from a to b, where a is the lower limit of integration and b is the upper limit of integration.

The Fundamental Theorem of Calculus

- If is f is nonnegative and continuous on the closed interval [a, b], then,

$$\int_a^b f(x) = F(b) - F(a)$$

- where f is any function such that $\mathrm{F}'(x) = f(x)$ for all $x$in [a, b].

Properties of definite integrals

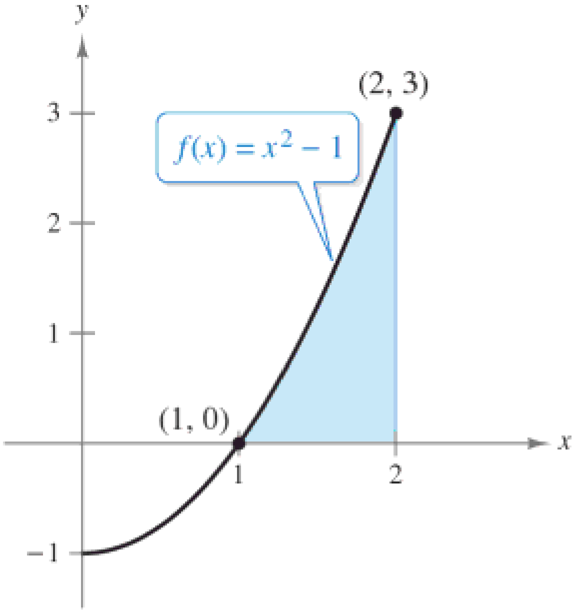
1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, k *is a constant.*
2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $a < c < b$
4. $\int_a^a f(x)\,dx = 0$
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Example

Example **Finding Area by the Fundamental Theorem**

Find the area of the region bounded by the x-axis and the graph of

$$f(x) = x^2 - 1, \quad 1 \leq x \leq 2.$$



$$\text{Area} = \int_1^2 (x^2 - 1)\, dx$$

Example

SOLUTION Note that $f(x) \geq 0$ on the interval $1 \leq x \leq 2$, as shown in Figure 5.9. So, you can represent the area of the region by a definite integral. To find the area, use the Fundamental Theorem of Calculus.

$$\text{Area} = \int_1^2 (x^2 - 1)\,dx \qquad \text{Definition of definite integral}$$

$$= \left[\frac{x^3}{3} - x\right]_1^2 \qquad \text{Find antiderivative.}$$

$$= \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right) \qquad \text{Apply Fundamental Theorem.}$$

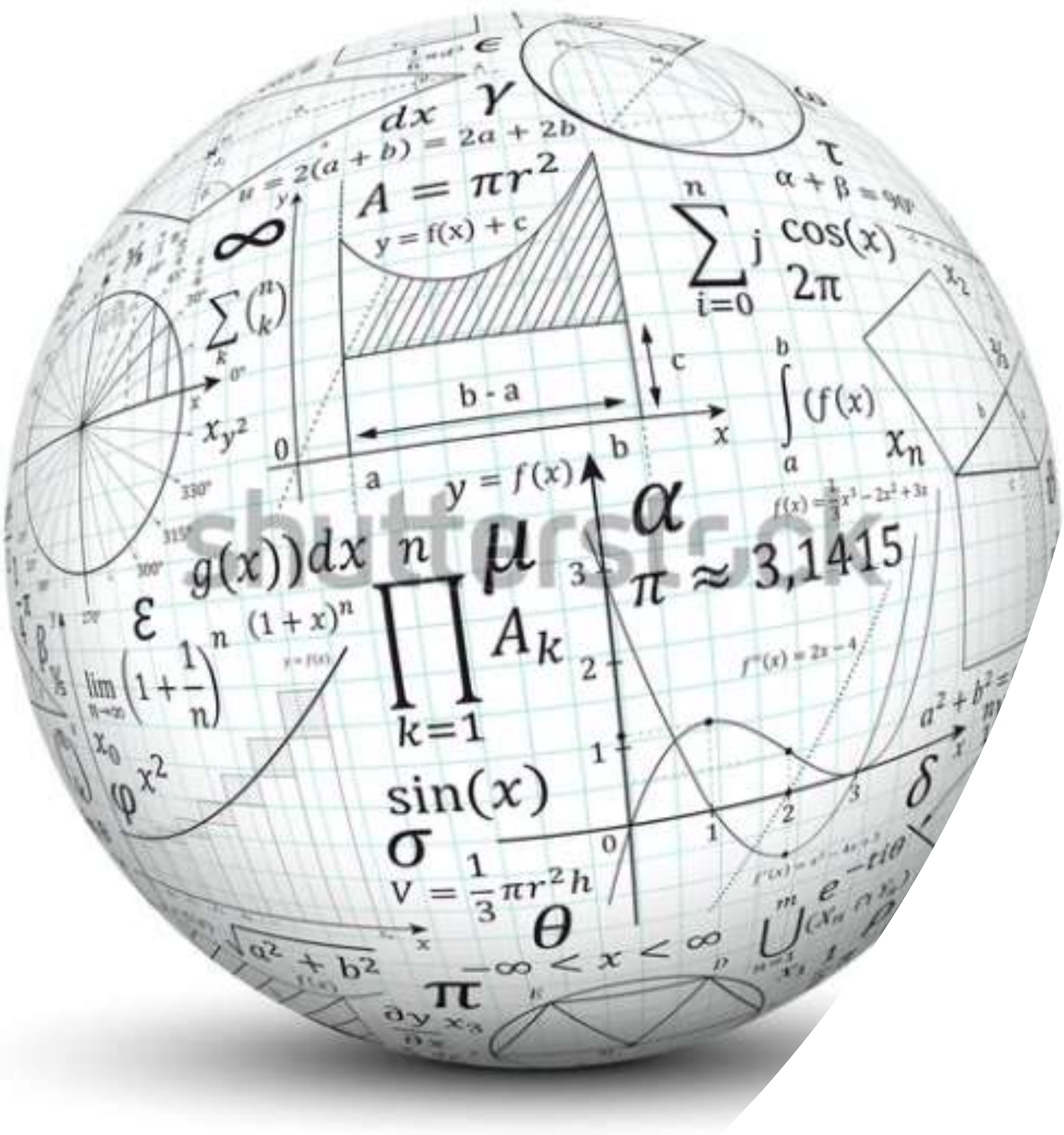
$$= \frac{2}{3} - \left(-\frac{2}{3}\right)$$

$$= \frac{4}{3} \qquad \text{Simplify.}$$

End of Lecture 05

Next Lecture :

Functions



Lecture 07

Functions

Mathematics for Computing
IT1030

Ruchira Kulasekara
Bsc.(Hons) (Special)
Mathematics & Statistics

Introduction

- Function is a relation between a set of inputs and a set of outputs with the property that each input maps to exactly one output.
- Typically functions are named with a single letter such as *f*.
- All functions are relations, but not all relations are functions.

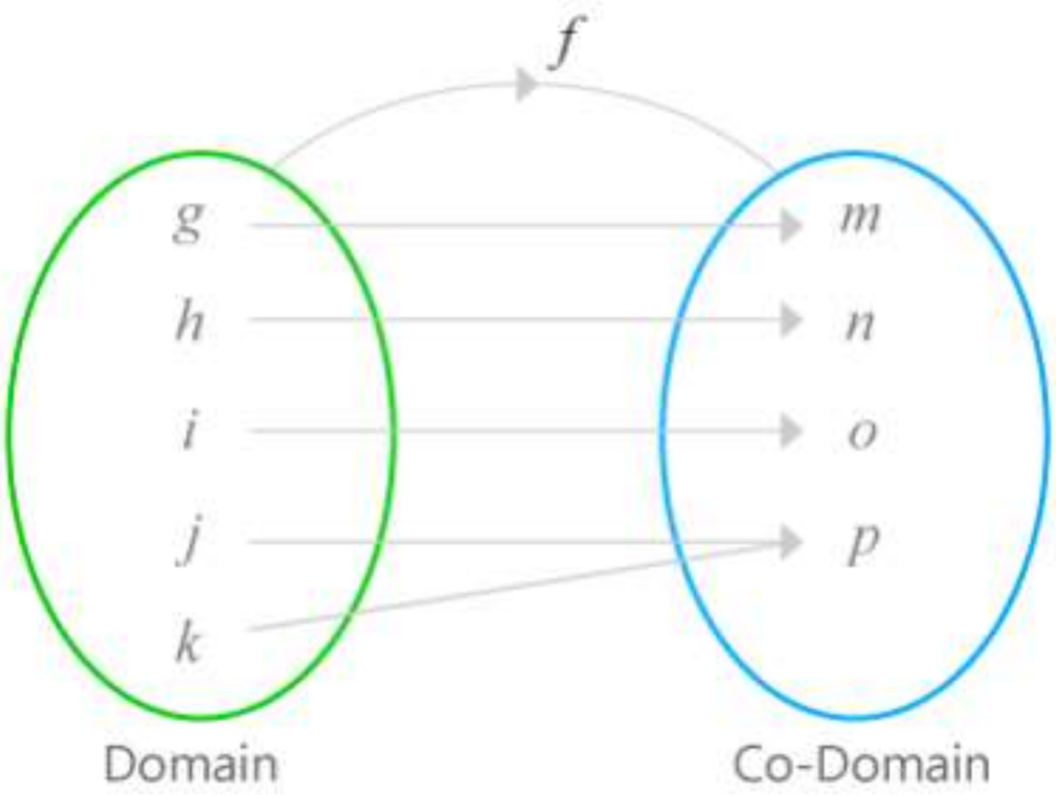
Key words:

- *Constant*: stand for a fixed number.
- *Variable*: quantity that may change within the context of a mathematical problem or experiment.

Definition

A function f from a set X to a set Y is a relationship between elements of X and elements of Y with the property that **each element of X** is related to **a exact one element of Y**.

f : X $\longrightarrow$ Y



Examples

Which of the following are functions ?

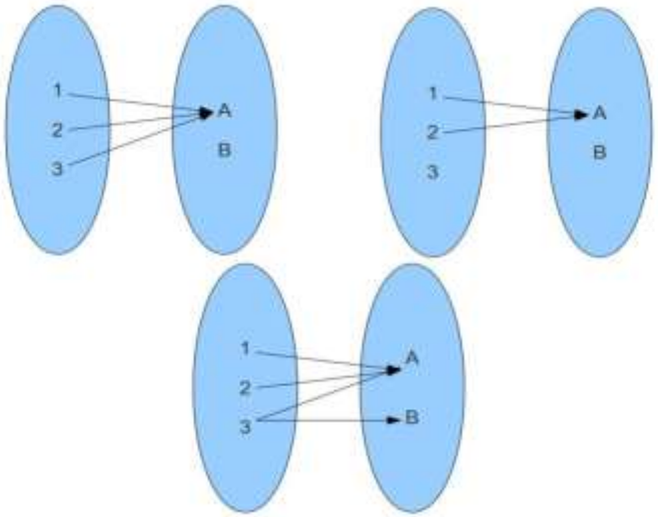


Figure 01

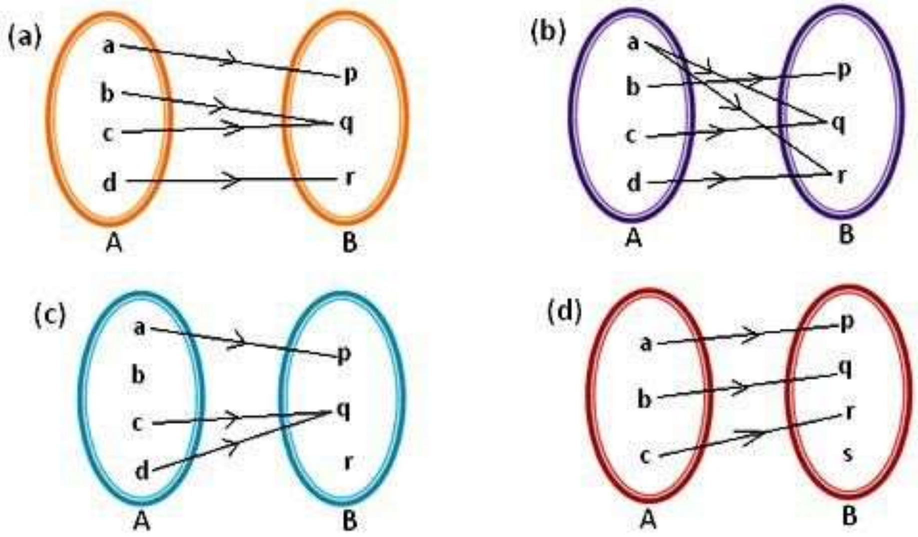


Figure 02

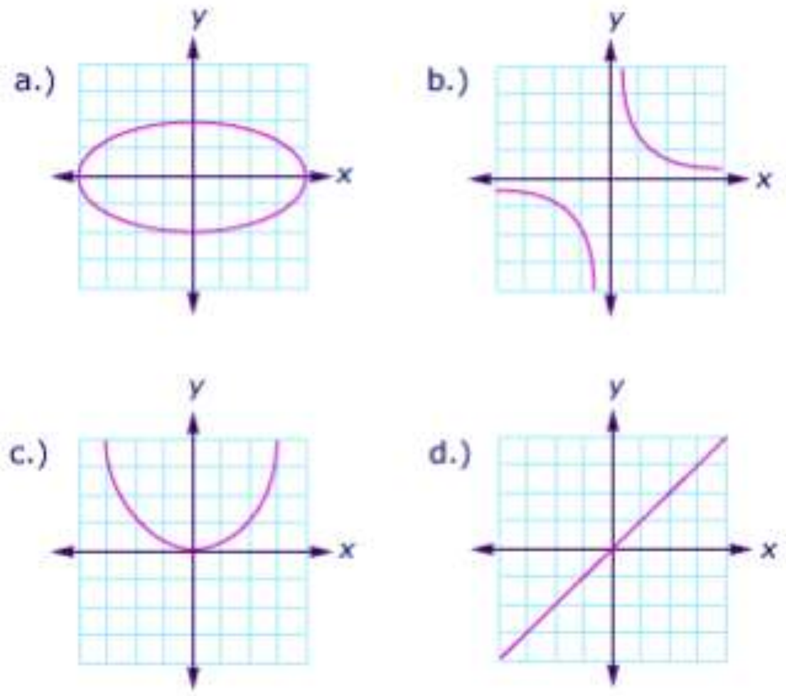


Figure 03

Domain

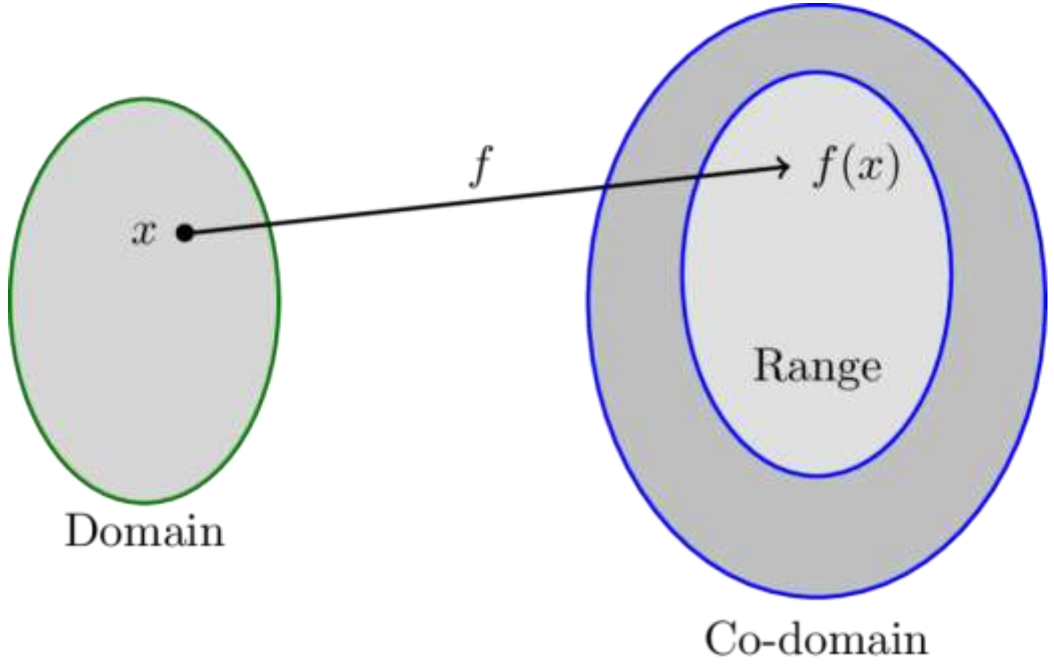
- The set of all inputs (set of X) for a function is called the **domain**.

Example :

X = {1,2}
Y = {1,2,3,4,5}

Where $y = x^2$;

The domain = {1,2}

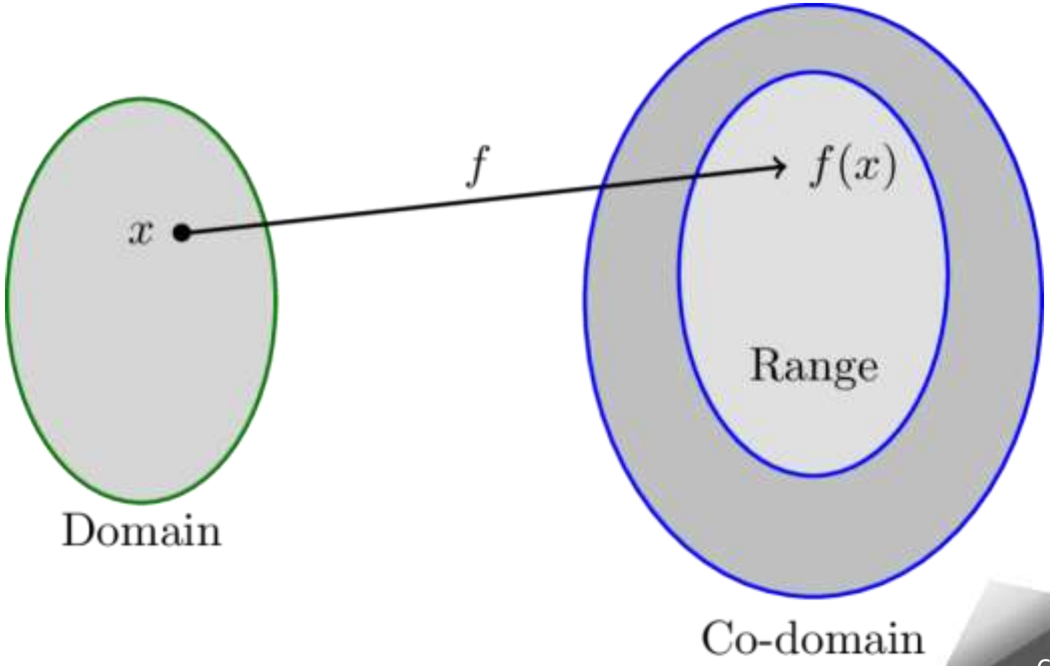


Range/Image

- The unique element y to which *f* sends x is denoted by *f*(x) and is called ***f* of x**, or **the value of *f* at x**, or **the image of x under *f***.

- The set of all values of *f* taken together is called the **range of *f*** or the **image of X under *f***.

range of *f*
= {y ϵ Y | y = *f*(x), for x in X }



Co-domain

- The set of all allowable outputs (set of Y) is called the **co-domain**.
- Range is clearly equal or a subset of co-domain.

Example :

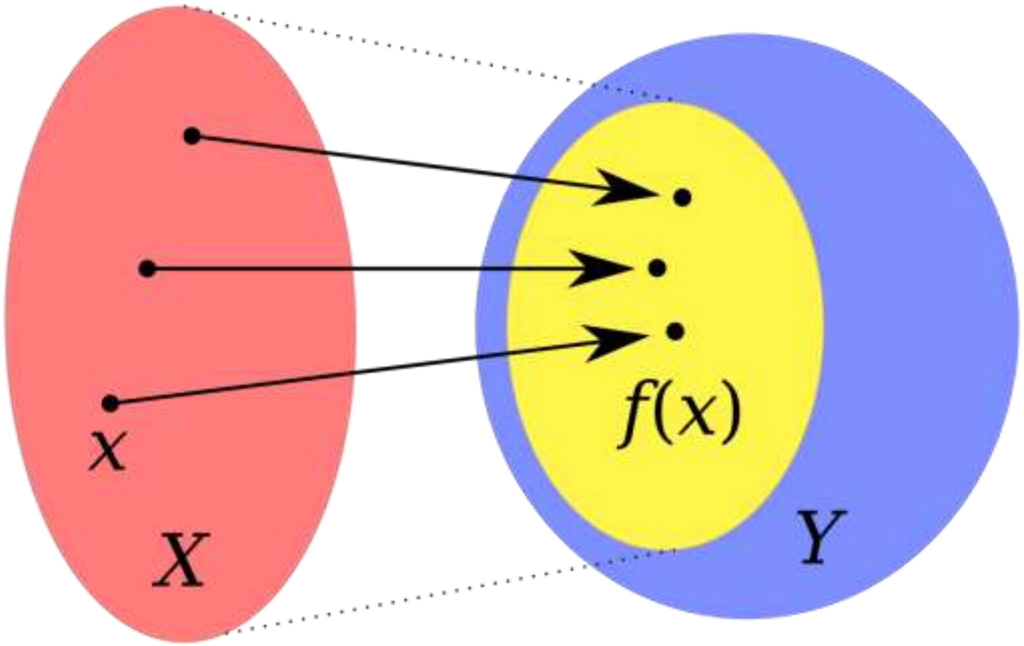
Let X = {1,2} and Y = {1,2,3,4,5}
Where $y = x^2$;

The domain (X) = {1,2}

The co-domain (Y) = {1,2,3,4,5}

As, $y = f(1) = 1^2 = 1$
and $y = f(2) = 2^2 = 4$

The range (*f*(x)) = {1,4}



$f : X \to Y$

Equality of Functions

Suppose f and g are functions from X to Y, Then f equals g, (written by $f = g$) if and only if,

$$f(x) = g(x) \text{ for all } x \in X.$$

Example :

Define f: R $\longrightarrow$ R and g: R $\longrightarrow$ R by the following formulas:

$f(x) = |\,x\,|$ for all x ϵ R,

$g(x) = \sqrt{x^2}$ for all x ϵ R.

Yes. Since the absolute value of a number equals the square root of its square,

$$|\,x\,| = \sqrt{x^2} \text{ for all } x \in R.$$

Hence $f = g$

One to One Functions

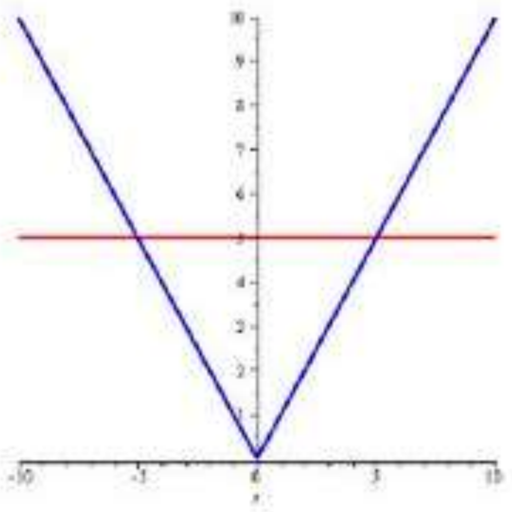
Let f be a function from a set X to a set Y. f is one to one (or injective) if, and only if, for all elements x_1 and x_2 in X,

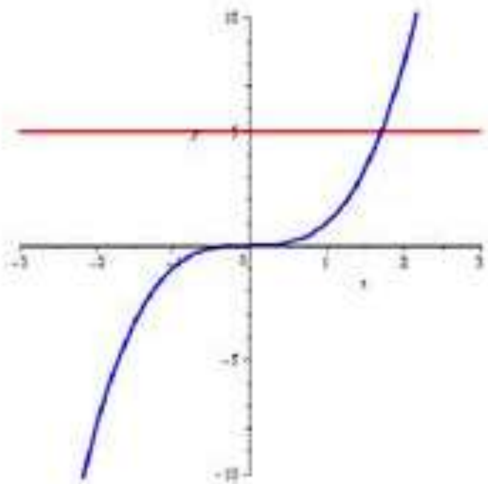
$$\text{if } f(x_1) = f(x_2)\text{, then } x_1 = x_2 .$$

Or, equivalently,

$$\text{if } x_1 \neq x_2\text{, then } f(x_1) \neq f(x_2) .$$

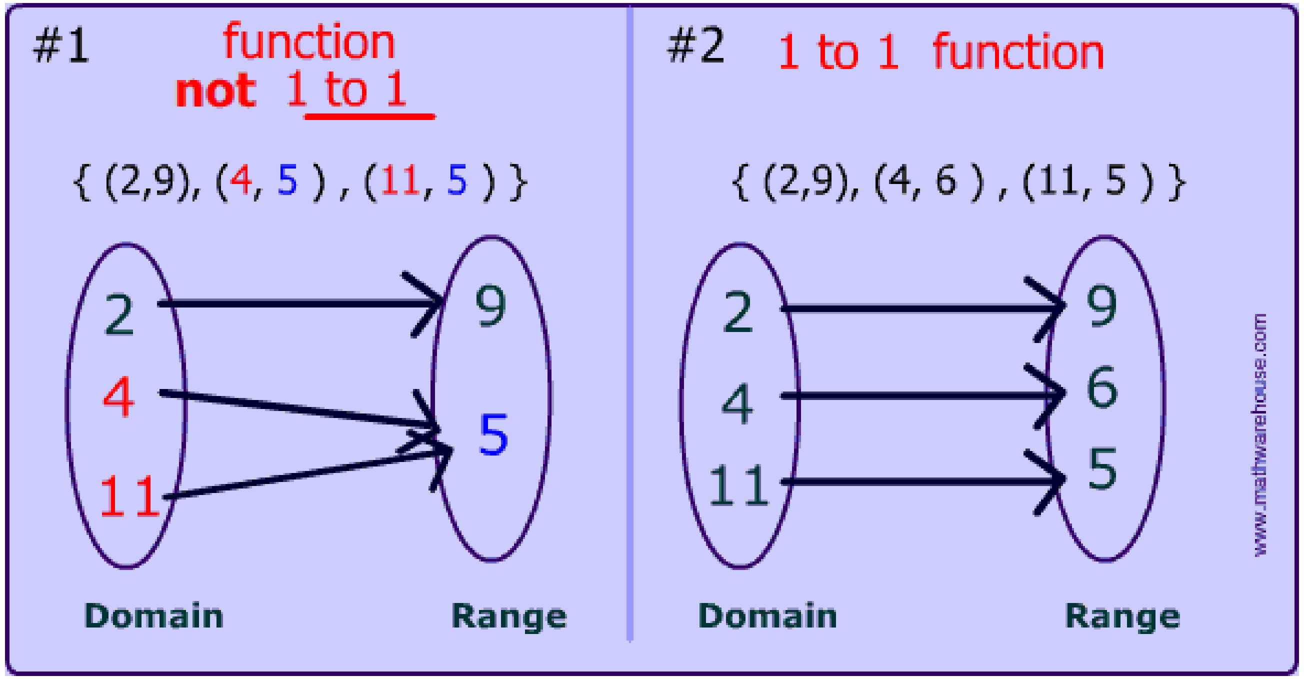
Example :





One to One Functions

Example :



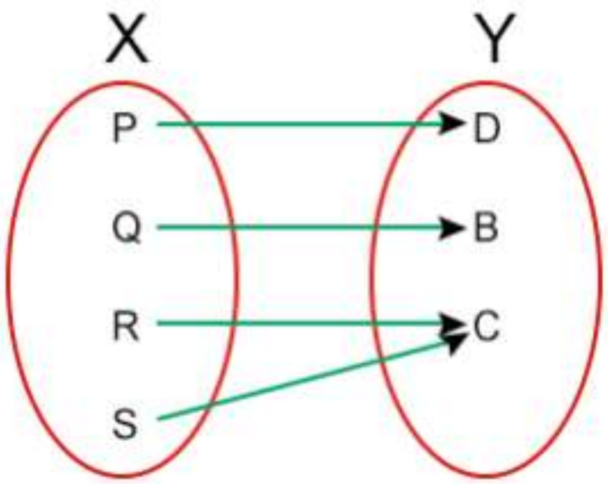
The function $f : R \longrightarrow R$ is defined by the formula $f(x) = 3x-5$ for all real numbers x.
Show that f is a one-to-one function.

Onto Function

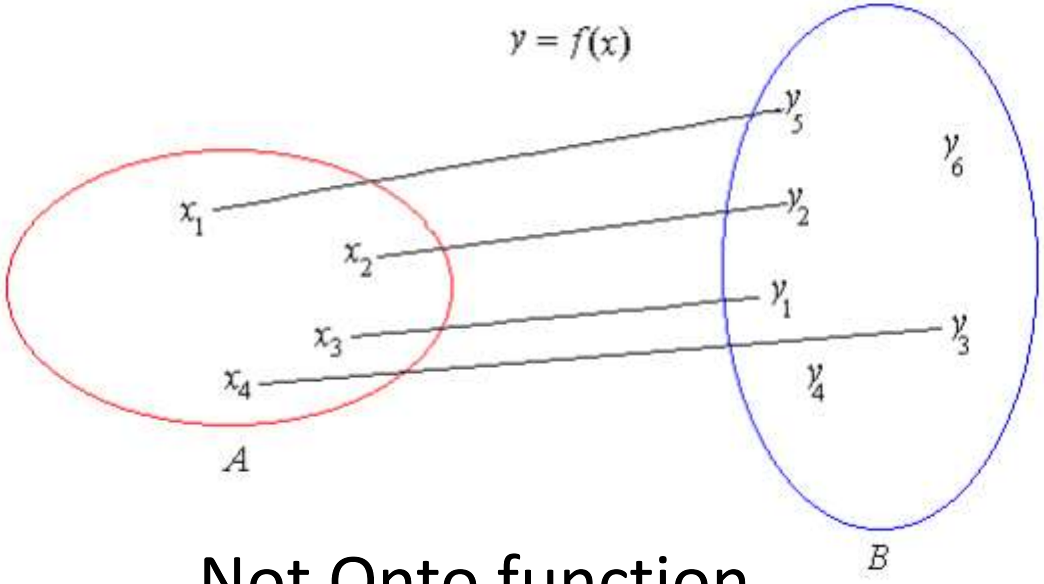
Let f be a function from a set X to a set Y. f is onto (or surjective) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that

$$y = f(x).$$

Example :



Onto function

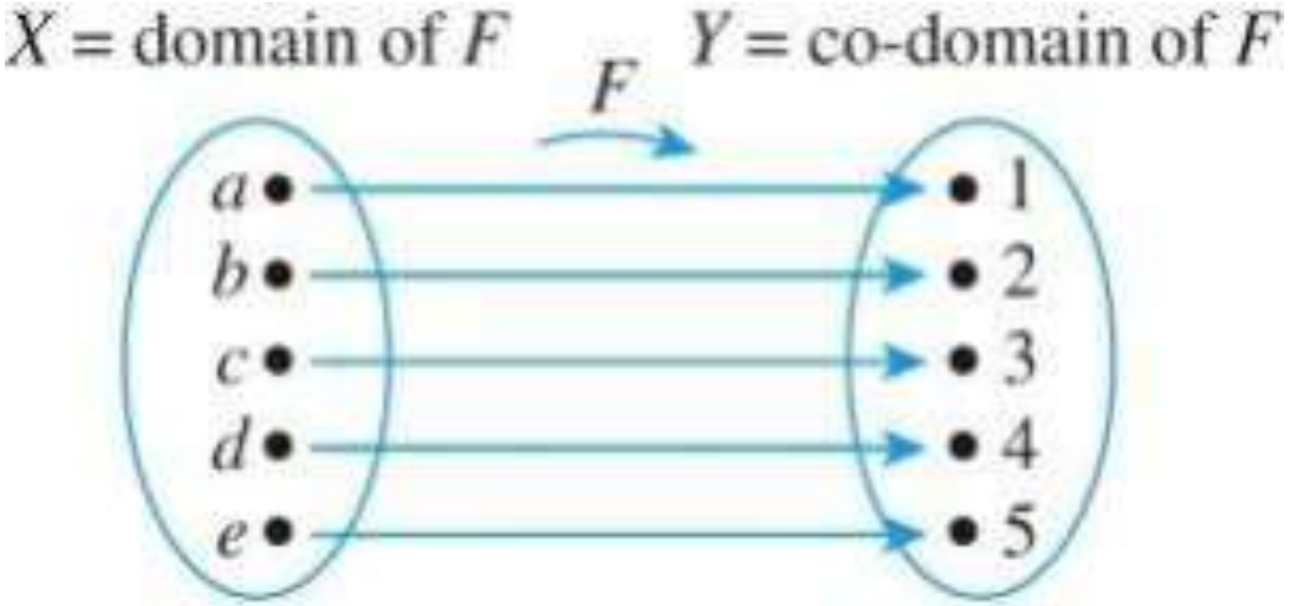


Not Onto function

One-to-one Correspondence

A one to one correspondence (or bijection) from a set X to a set Y is a function f: x $\longrightarrow$ y that is both one to one and onto.

Example :



Inverse Function

If f is one-to-one and onto then f^{-1} exists.

Suppose f: x $\longrightarrow$ y is a one to one correspondence; that is f is one to one and onto. Then, there is a function f^{-1} : y $\longrightarrow$ x

Given any element y in Y,

f^{-1}(y) = that unique element x in X such that f(x) equals y.

Example :

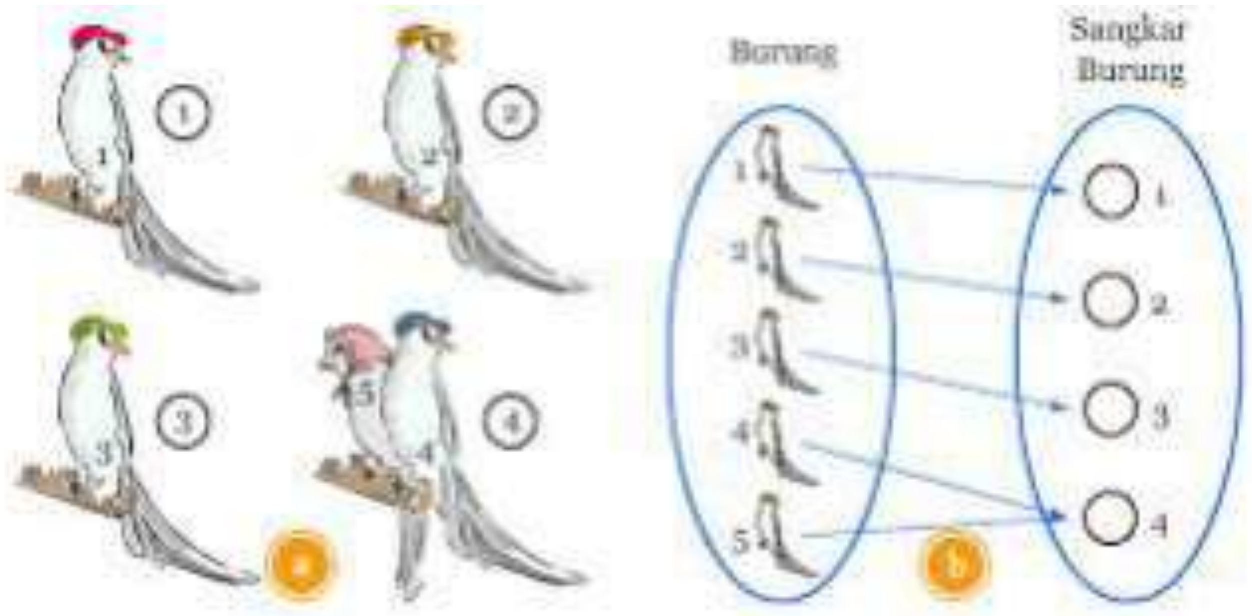
The function f : R $\longrightarrow$ R is defined by the formula f(x) = 4x -3 for all real numbers x.

Show that f is a one-to-one correspondence and find its inverse function.

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one.

There must be at least two elements in the domain that have the same image in the co-domain.



Exercise

1. Check whether the following equations are function of x or not .
   - $y = x^2 - 1$
   - $y^2 + x^2 = 1$
   - $y = \sqrt{\frac{1}{x+1}}$
   - $x = \frac{1-y^2}{y^2+1}$

2. The function h : R $\longrightarrow$ R is defined by the formula *h*(x) = 3x+7 for all real numbers x.

Show that f is a one-to-one correspondence and find its inverse function.

Summary

Discussed about,

- Equality of functions
- Identity function
- One-to-one functions
- Onto functions
- One-to-one correspondence
- Inverse function
- Pigeon-hole principle


THE END

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[redacted]

Counting

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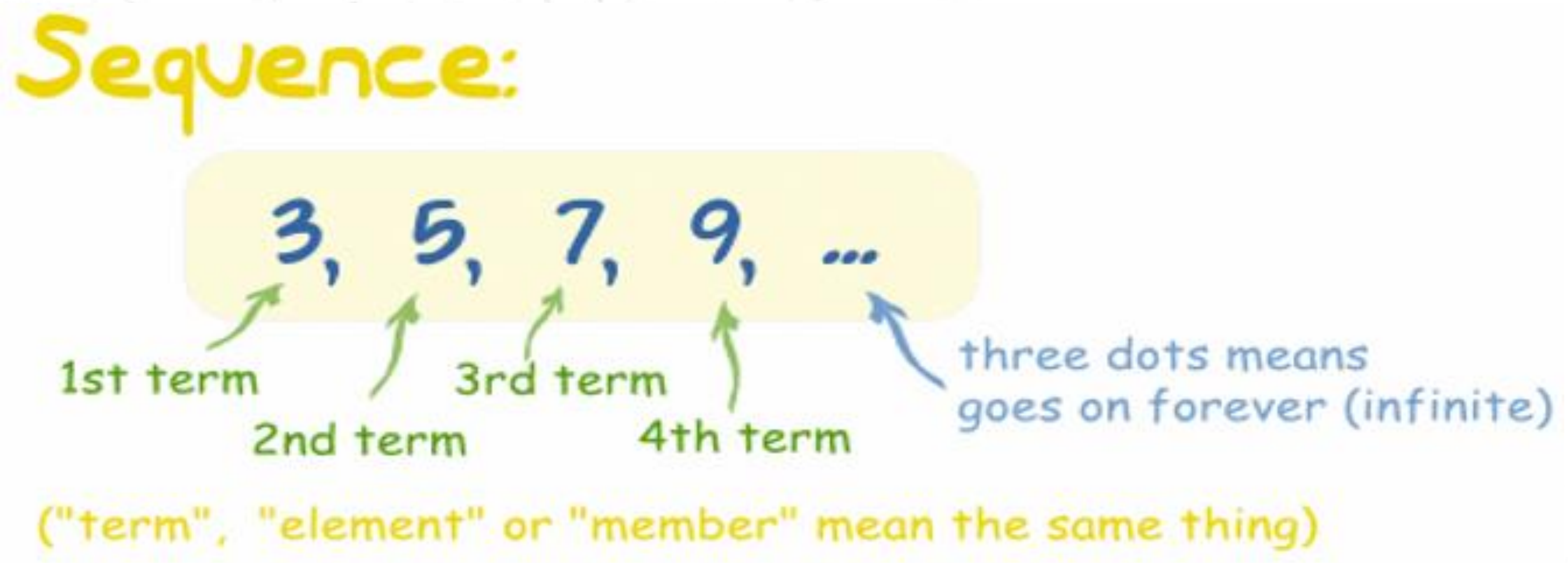
SEQUENCES

- A Sequence is a list of things (usually numbers) that are in order; Infinite or Finite



INFINITE OR FINITE SEQUENCES

- When the sequence goes on forever it is called an infinite sequence, otherwise it is a finite sequence.

Eg: {1,2,3,4,…} is an infinite sequence {2,4,6} is a finite sequence with 3 terms

SET VS. SEQUENCE

Set	Sequence
Terms need not to be in order	Terms must be in order
Values cannot repeat	Values can repeat

- Eg: {0, 1, 0, 1, 0, 1, ...} is the sequence of alternating 0s and 1s. The set is just {0,1}   or {1,0} .

ARITHMETIC SEQUENCE

- It has a common difference between successive terms.

Eg: 2, 4, 6, 8, …

Q: Find the 10th term and the sum of first 10 terms of the following sequence An. An : {3, 8, 13, 18, 23,…}

term position

$$a_n = a_1 + (n-1)d$$

n^{th} term

first term

common difference

Sum of nth terms =

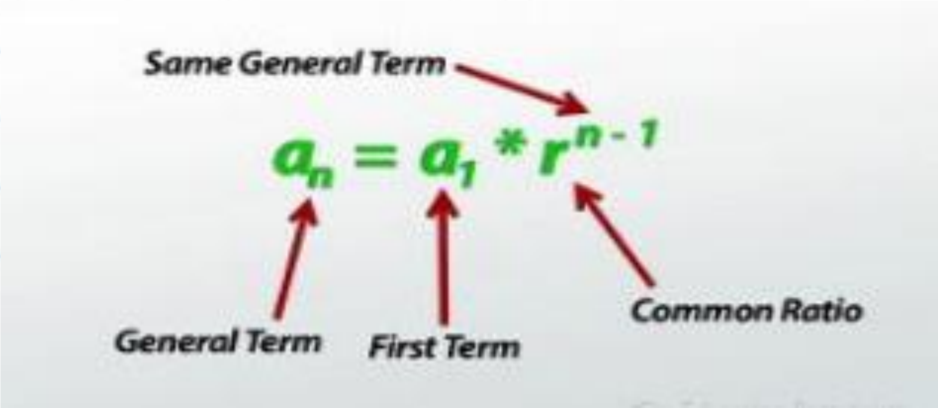
$$\frac{n}{2}(2a + (n-1)d)$$

GEOMETRIC SEQUENCE

- It has a common ratio between successive terms.

Eg: 2, 4, 8, 16, …

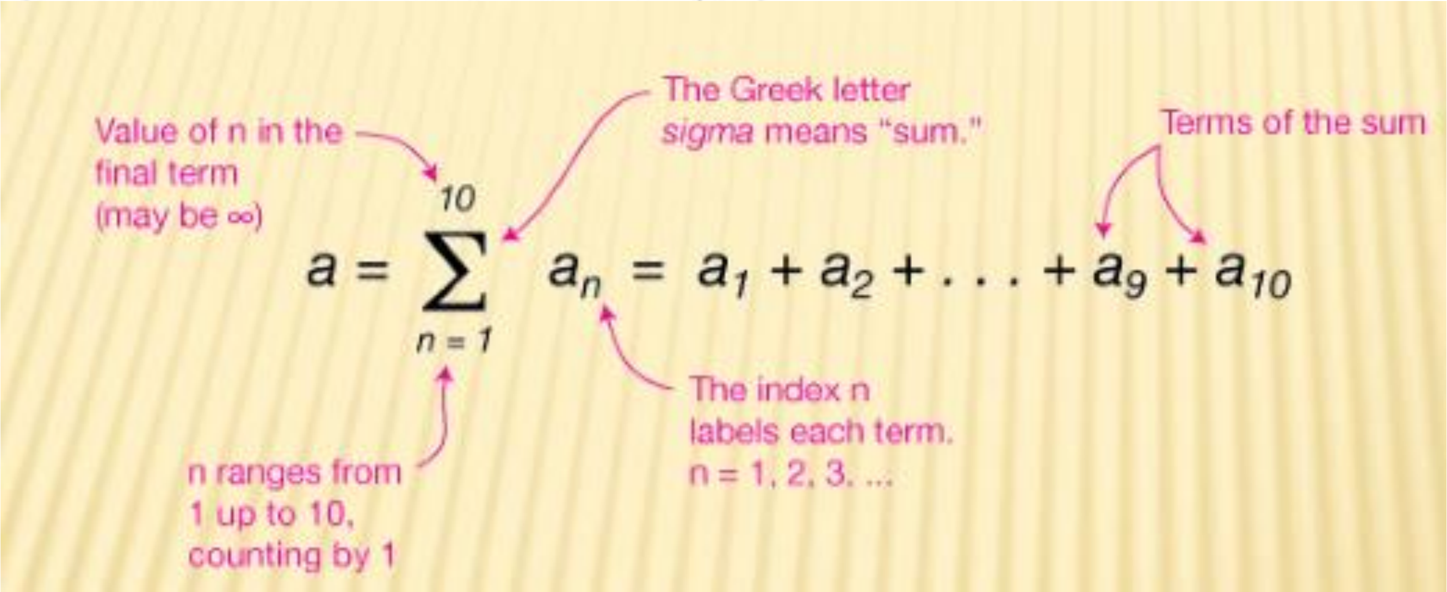
Q: Find the 11th term and the sum of first 11 terms of the following sequence An. An : {3, 6, 12, 24,…}



$$S_n = \frac{a_1(1-r^n)}{1-r}$$

SIGMA NOTATION

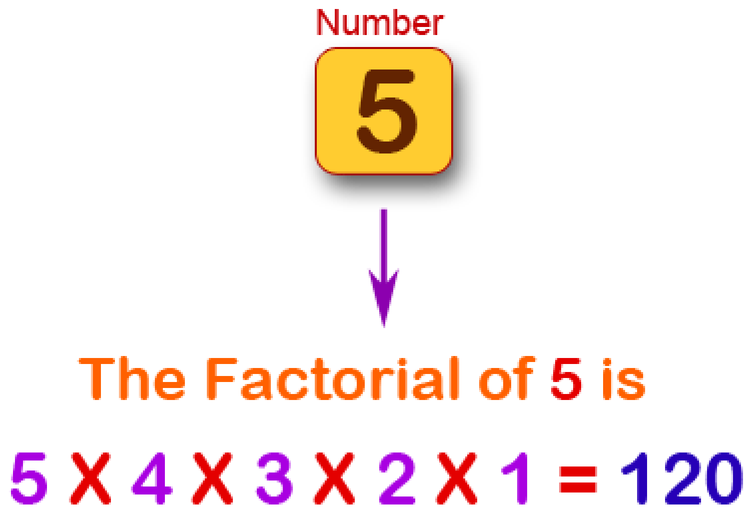
- It represents summation of many similar terms.



FACTORIAL of N

$$n! = 1 * 2 * 3 * \; \ldots \; n$$

$$0! = 1! = 1$$



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${}^{n}C_{r}$ *and* ${}^{n}P_{r}$ Notations

$$nCr = \frac{n!}{r!\ (n-r)!}$$

$$nPr = \frac{n!}{(n-r)!}$$

PERMUTATIONS

- A permutation is an arrangement of objects in specific order.
- The order of the arrangement is important.

Example: How many distinct, 3 letter words can be arranged using {a, b, c} ?? (6 arrangements)

- For any integer $n \geq 1$, the number of permutation of n elements is $n!$

EXAMPLE

(i) How many ways can the letters in the word COMPUTER be arranged in a row?

All the eight letters are in the word COMPUTER are distinct, so the number of ways,

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$

(ii) How many ways can the letters in the word COMPUTER be arranged if the letters “CO” must remain next to each other (in order) as a unit?

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$

PERMUTATIONS OF SELECTED ELEMENTS

- If n and r are integers and $1 \leq r \geq n$, then the number of r permutations of a set of n elements is given by the formula

Example: A license plate begins with three letters. If the possible letters are A, B, C, D and E, how many different permutations of these letters can be made if no letter is used more than once? (Ans: 60)

COMBINATIONS

- The number of combinations of n things taken r at a time is given by:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

- The order of the arrangement is not important. Example: In how many ways can a coach choose three swimmers from among five swimmers? (10 ways)

COMBINATIONS

- The number of combinations of n things taken r at a time is given by:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

- The order of the arrangement is not important. Example: In how many ways can a coach choose three swimmers from among five swimmers? (10 ways)

EXAMPLE

- (i) 16 teams enter a competition. They are divided up into four Pools (A, B, C and D) of four teams each.Every team plays one match against the other teams in its Pool.

After the Pool matches are completed:

- the winner of Pool A plays the second placed team of Pool B
- the winner of Pool B plays the second placed team of Pool A
- the winner of Pool C plays the second placed team of Pool D
- the winner of Pool D plays the second placed team of Pool C

The winners of these four matches then play semi-finals, and the winners of the semi-finals play in the final.

How many matches are played altogether?

QUESTION

- How many “Mahajana sampatha” Tickets can be printed in a single draw ?? (numbers are selected from 0 to 9 and it can repeat)

End of Lecture 07

Next Lecture:
Graph Theory

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in
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Mathematics for computing
Graph Theory

Basic Definition

- A graph consists of finite set of vertices V and edges E.
- Often we denote a graph by G and with the two sets of vertices and edges it is represented by V(G) and E(G).

  Example:

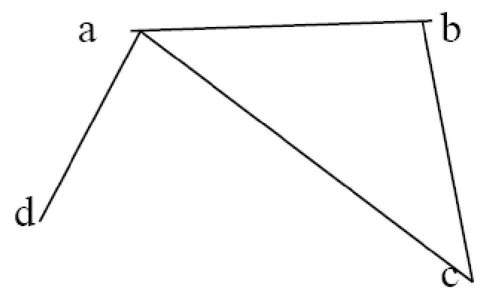
  Let V(G) = { a,b,c,d} and E(G) = {{a,b},{a,c}, {b,c},{a,d}}

- If {a,b}ε V(G) and a and b are the end points of the edge{a,b}
- A vertex 'a' and the edge {a, b} are said to be incident since the vertex is an end point of the edge.
- Two vertices u, v of a graph G are said to be adjacent if they are joined by an edge. When u and v are adjacent, we say they are neighbors.

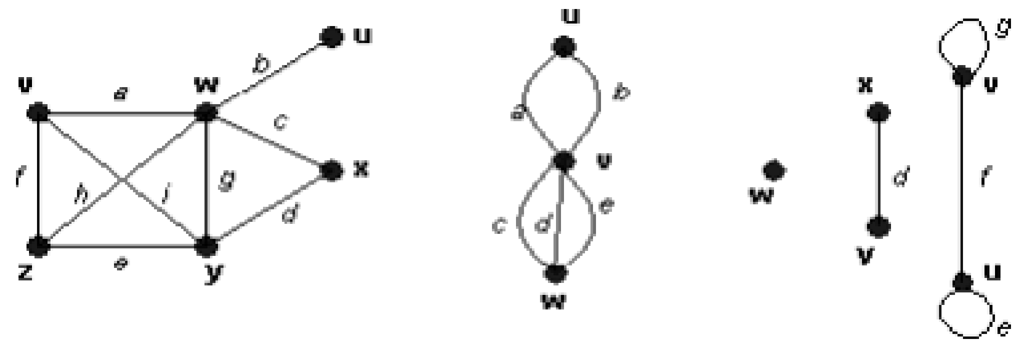
Definition Contd…

- The above graph can be shown pictorially as follows.

- V(G) = { a, b, c, d} and E(G) = {{a, b},{a, c), {b c},{a, d}}



Features of Graphs



(a) G_1 (b) G_2 (c) G_3

•In the graph G2, the edges a and b have the same endpoints, and so do the edges c, d and e.

•Such sets of edges are called <u>multiple edges or Parallel edges.</u>

•The edge e of the graph G3 in Fig.7.l(c) is an example for a <u>loop</u> (two endpoints of an edge are coincide)

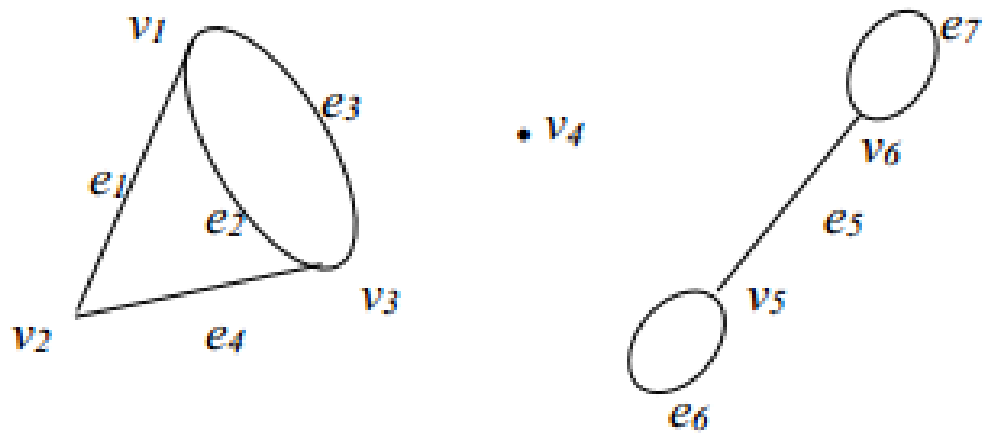
Multi-graphs and Simple graphs

- A graph that has loops or multiple edges is known as a multi-graph.
- A graph with neither loops nor multiple edges is called a simple graph.
- Therefore G_2 and G_3 are multi-graphs and G_1 is a simple graph.

Degree of a Vertex

- The degree of a vertex deg(v) is the number of edges incident with it.
- A loop will contribute 2 towards the degree.
- Thus in the graph G_1 deg(z) = 3, deg(w)= 5 and in G_2 , deg(v)=5 and in G_3, deg(u) =3 and deg(w)=0.
- W in G_3, is an isolated vertex.

Example



Write the vertex set and the edge set, and give a table showing the edge-endpoint function;

vertex set = {v1, v2, v3, v4, v5, v6}

edge set = {e1, e2, e3, e4, e5, e6, e7}

The end-point function table

Edge	End points
e_1	$\{V_1, V_3\}$
e_2	$\{V_2, V_4\}$
e_3	$\{V_1, V_2\}$
e_4	$\{V_1, V_2\}$
e_5	$\{V_3, V_4\}$
e_6	$\{V_7, V_6\}$
e_7	$\{V_6, V_5\}$
e_8	$\{V_7, V_5\}$
e_9	$\{V_6\}$

Q: Find all edges that are incident on v_1, all loops, all parallel edges, and all isolated vertices.

Theorem

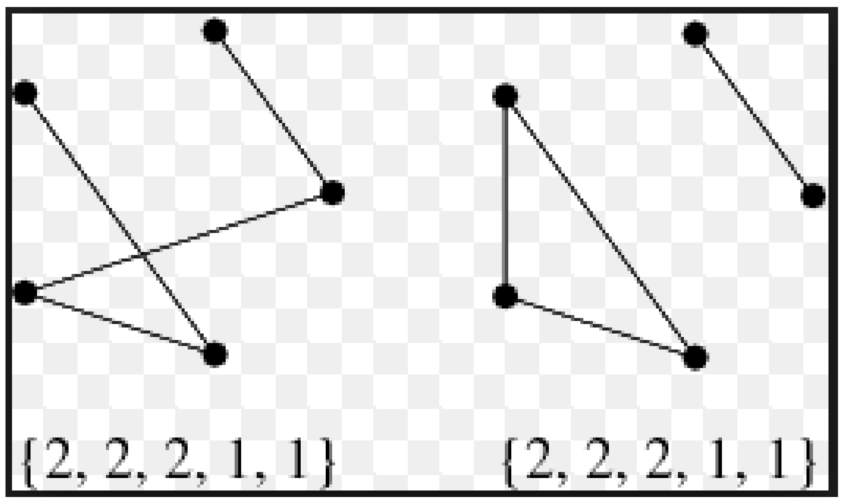
- Let g be a graph . Then the sum of degrees of the vertices of G is equal to twice the number of edges of V.

- Show that the theorem is valid for the above example.

Degree Sequence of a Graph

- The degree sequence of a graph G is the sequence of the degrees of its vertices in descending order of size.

  Example :



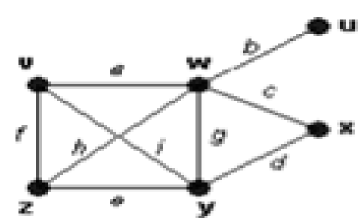
Paths, cycles and connectivity

- Path is an alternating sequence of vertices and edges of the form $v_1e_1v_2e_2\ldots e_{k-1}v_k$.

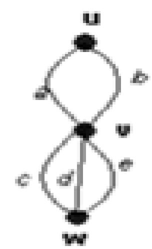
  (Edges can repeat but vertices cannot repeat)

- If G is a simple graph, path can be specified just by a sequence of distinct vertices $v_1v_2v_3\ldots v_k$.

Q: Find the paths from u to w in G_1 and G_2 ?



(a) G_1



(b) G_2

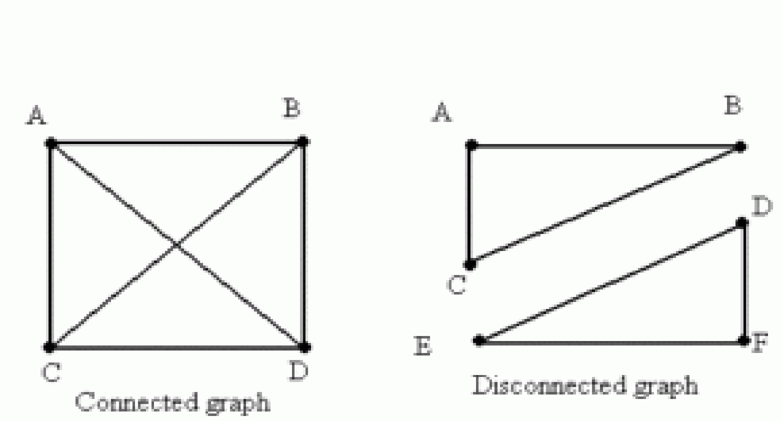
Paths, cycles and connectivity Contd…

- The length of a path is the number of EDGES in it. The path vfzey in G_1 has length 2.
- A cycle is a sequence of distinct vertices and edges that begins and ends at the same vertex.

  (xywx or wyxw are not 2 different cycles in G_1)
- Two cycles are different if they differ in at least one edge.

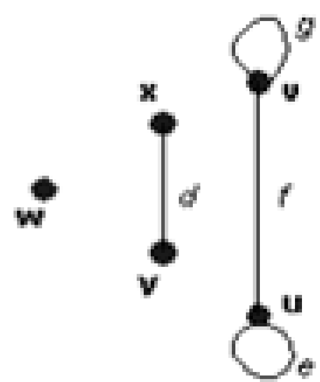
Paths, cycles and connectivity Contd…

- Two vertices u, v in a graph G are said to be connected if there is a path in G from u to v.
- The graph G is connected if every pair of vertices are connected; otherwise, G is said to be disconnected.



Components

- The separate parts of a disconnected graph are called its components. Thus the graph G3 has 3 components.
- every connected graph has just one component, the graph itself



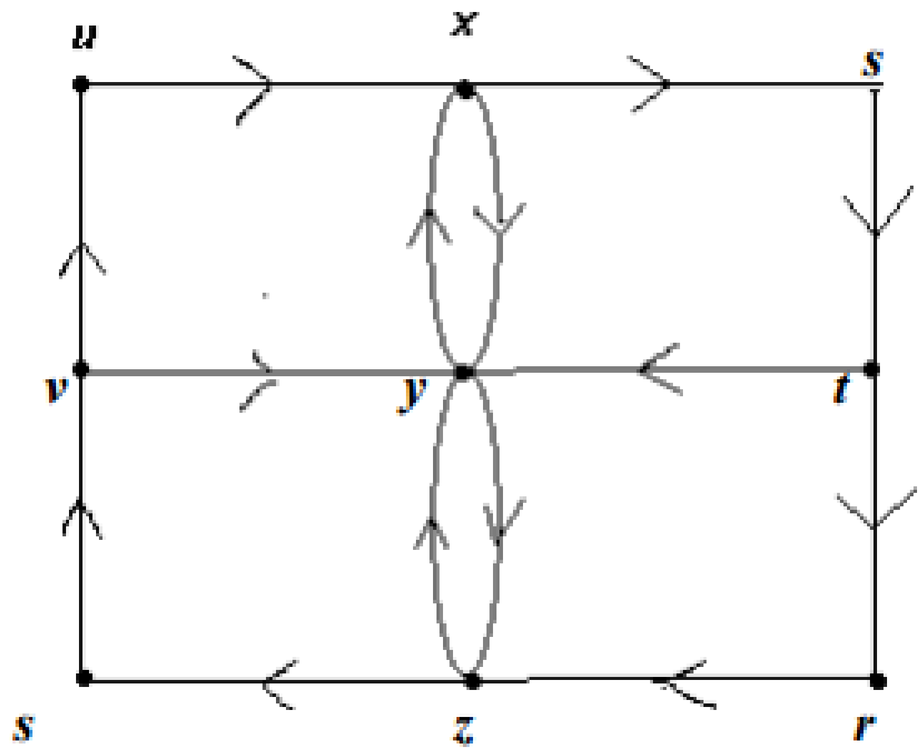
(c) G_3

Digraphs and relationship graphs

- A graph in which every edge has a direction assigned to it is called a digraph (an abbreviation of directed graph).
- The directed edges are often called arcs.
- In a digraph, we define the outdegree of vertex u, denoted by outdeg(u), as the number of arcs directed out of (away from) the vertex u.
- The indegree of vertex u, denoted by indeg(u), as the number of arcs directed into (towards) the vertex u.

Example

Q: Find the sum of the indegrees and the sum of the outdegrees of the vertices of the digraph shown in the following figure.



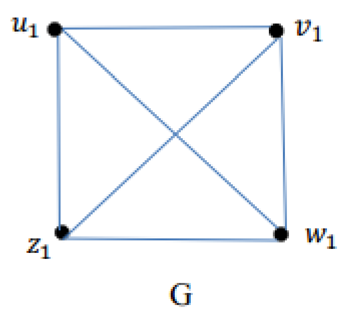
Example Contd…

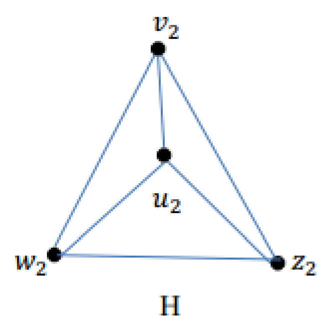
Answer:

vertex	indegrees	outdegrees
u	1	1
v	1	2
w	1	1
x	2	2
y	4	2
z	2	2
s	1	1
r	1	1
t	1	2
Total	14	14

Isomorphism of Graphs

- A graph G is determined by its vertex set V(G) and its edge set E(G).
- Given this information, two people might draw the graph differently.
- Example:





$V(G)=\{u_1, v_1, w_1, z_1\}$

$E(G)=\{u_1v_1, u_1z_1, v_1w_1, z_1w_1, u_1w_1, z_1v_1\}$

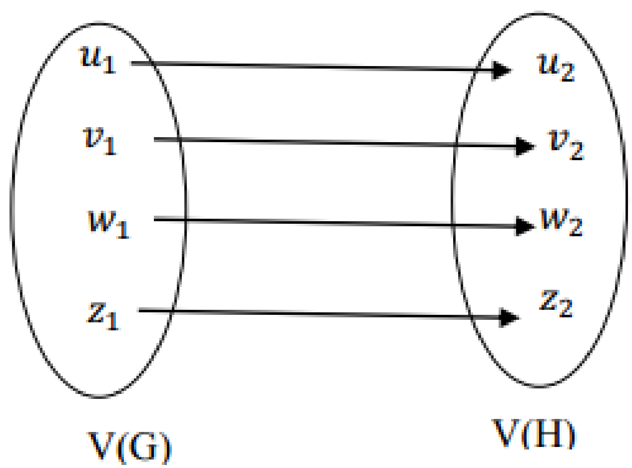
Isomorphism of Graphs Contd…

- G and H are isomorphic if we can label their vertices with the same set of labels in such a way that any pair of vertices u_1, v_1 are joined by the same number of edges in G as they are in H.

- In other words ,G and H are isomorphic if there is a one-to-one correspondence between V(G) and V(H).

Isomorphism of Graphs Contd…

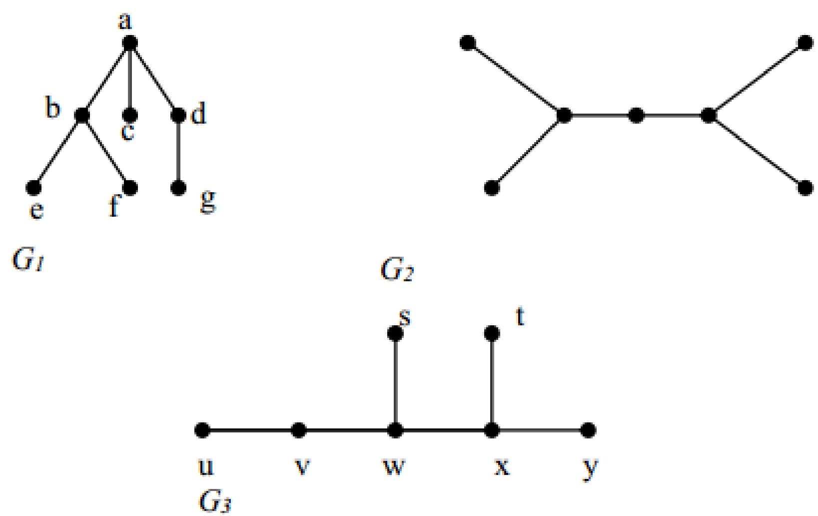
- G and H in the previous example are Isomorphic.



There is a 1-1 correspondence between V(G) and V(H)

Isomorphism of Graphs Contd…

- Show that G_1 and G_3 are Isomorphic and G_2 is not isomorphic to either G_1 or G_3.



note that in G_1 and G_3, the two vertices of degree 3 are adjacent, whereas in G_2 they are not

Isomorphism of Graphs Contd…

- When two graphs are isomorphic, their structural properties must be exactly the same.

- Let G and H be isomorphic graphs. Then G and H have
  1. the same number of components;
  2. the same number of vertices;
  3. the same number of edges;
  4. the same degree sequence;
  5. the same number of paths of any given length k;
  6. the same number of cycles of any given length k.

Note: The converse is not true.

Adjacency Matrix

- If we want to use a computer to analyze the properties of a graph, we must have some way of representing it in the computer.
- One method is using an Adjacency Matrix.

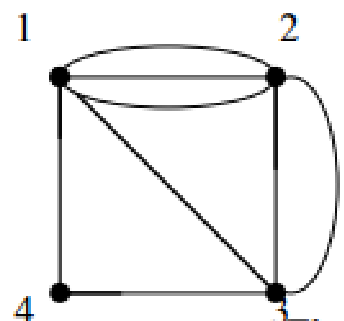


Fig 7.7

G

$$A(G) = \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Euler Path and Circuit

- An **Euler circuit** in a graph G is a simple circuit containing every edge of G. **Euler path** in G is a simple path containing every edge of G. (Edges cannot repeat)

  Theorem 1:

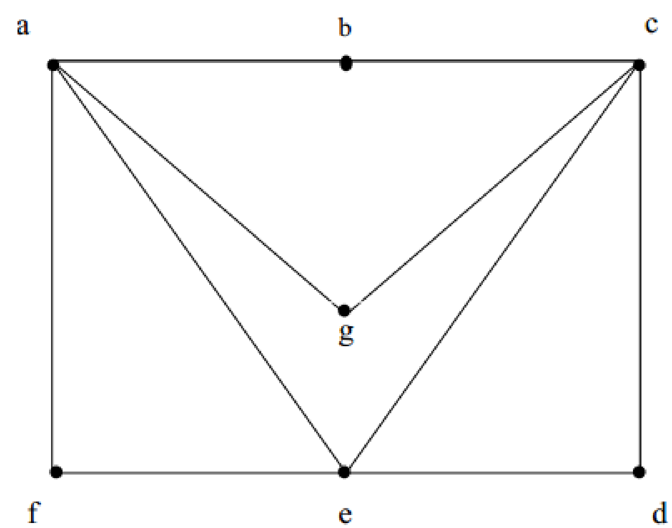
- A connected multi-graph has an Euler circuit if and only if each of its vertices has even degree.

  Theorem 2 :

- A connected multi-graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Euler Path and Circuit Contd…

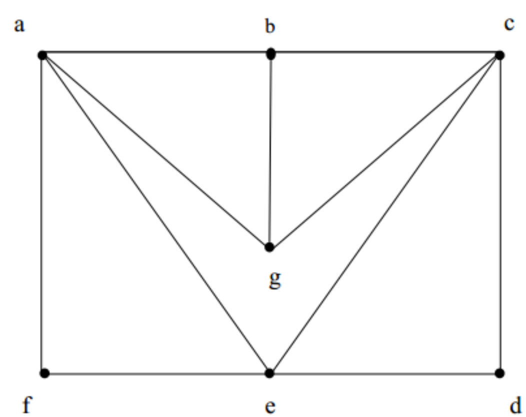
- Example:
  - Find an Euler Circuit of the following graph.



Euler Circuit: abcgaecdefa

Hamilton Path and Circuit

- a **Hamiltonian path** is a **path** in a graph that visits each vertex exactly once. (Edges and vertices cannot repeat)
- A **Hamiltonian** circuit is a **Hamiltonian path** that is a **circuit**.
- Example:
  - Find a Hamilton Circuit of the following graph.



Hamilton Circuit: **abgcdefa**

Hamilton Path and Circuit Contd…

- Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

- Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

The End

Matrices

MATHEMATICS FOR COMPUTING (IT 1030)

Matrix Definition and Notations

- A matrix is an array, which satisfies certain algebraic rules of operation.
- Capital letters are usually used to denote matrices

Eg: $\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}], \ (1 \le i \le m, \ 1 \le j \le n)$$

Vectors

- Matrices, which have either one row or one column, are known as **vectors**

- **Row vectors**

Eg: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

- **Column Vectors**

Eg: $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

Square Matrix

- A matrix in which the number of rows equals the number of columns is called a **square matrix**

Eg: $\begin{bmatrix} \mathbf{1} & \mathbf{8} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$

Rectangular Matrix

- A matrix in which the number of rows not equals the number of columns is called a **rectangular matrix**

Eg: $\begin{bmatrix} \mathbf{8} & \mathbf{-4} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Rules of Matrix Algebra

1. ***Equality***. Two matrices can only be equated if they are of the same *order,* that is, if they each have the same number of rows and the same number of columns. They are then said to be **equal** if the corresponding elements are equal. Thus if

   $A = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ and $B = \begin{bmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{bmatrix}$

   then ***A* = *B*** if and only if $a = e$, $b = f$, $c = g$, and $d = h$

Rules of Matrix Algebra

2. Multiplication by a constant. Let *k* be a constant or scalar. By the product *kA* we mean the matrix in which every element of *A* is **multiplied** by *k.* Thus, if

A = $\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$ and k = 10

k*A = $\begin{bmatrix} -2 * 10 & 1 * 10 & 0 * 10 \\ -1 * 10 & 3 * 10 & 4 * 10 \end{bmatrix} = \begin{bmatrix} -20 & 10 & 0 \\ -10 & 30 & 40 \end{bmatrix}$

Rules of Matrix Algebra

***3. Zero matrix**.* Any matrix in which every element is zero is called a **zero** or **null matrix**. If *A* is a zero matrix, we can simply write *A* = 0.

4. *Matrix sums and differences*. The sum of two matrices *A* and *B* is defined if *A* and *B* are of the same order, in which case *A* + *B* is defined as the matrix C whose elements are the sums of the corresponding elements in *A* and *B.* We write C = *A* + *B.*

Thus, if

$A = [a_{ij}]$ and $B = [b_{ij}]$ are both *m* x *n* matrices, then

$C = A + B = [a_{ij} + b_{ij}]$.

Rules of Matrix Algebra

5. Commutative property of matrix addition, that is,

$A + B = B + A$

6. **Associative law of additio**n

$A + (B+C) = (A+B) + C$

the order of addition of matrices is immaterial

7. Distributive law of addition

$A(B + C) = AB + AC$

8. Associative law of multiplication

$A(BC) = (AB)C$

Rules of Matrix Algebra

9. $k(A + B) = kA + kB$

$(k + l)A = kA + lA$. $\quad$ *k, l are real numbers*

Rules of Matrix Algebra

10. Matrix multiplication. Matrices can be multiplied only if the number of columns in *A* equals the number of rows in *B*. Let us look at the case where *A* is a 2x3 matrix, and *B* is a 3x2 matrix, which are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Rules of Matrix Algebra

$$AB = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix}} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix}} & \begin{matrix} b_{13} \\ b_{23} \\ b_{33} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}} & \boxed{a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

$$= C.$$

Rules of Matrix Algebra

- One conclusion which can be inferred from the previous example is that **matrix multiplication does not commute**; that is, in general,

$$AB \neq BA.$$

- *AB* can be a zero matrix without either *A* or *B* or *BA* being zero. Also, as a consequence, $A(B - C) = 0$ does not necessarily imply $B = C$.

Special Matrices

1. The transpose of any matrix is one in which the rows and columns are interchanged. Then, the first row becomes the first column, the second row the second column, and so on. We denote the transpose of A by A^T. Hence if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

The 3 x 2 matrix A becomes the 2 x 3 matrix A^T.

Properties of the transpose. Provided that the sum $A + B$ and product AB are defined for two matrices A and B, then

(a) $(A + B)^T = A^T + B^T$;

(b) $(AB)^T = B^T A^T$.

(c) $(A^T)^T = A$.

Special Matrices

2. Symmetric matrices. A *square* matrix is said to be **symmetric** if $A = A^T$. Since rows and columns are interchanged in the transpose, this is equivalent to $a_{ij} = a_{ji}$ for all elements if $A = [a_{ij}]$. Symmetric matrices are easy to recognize since their elements are reflected in the **leading diagonal.**

$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & 4 \\ -2 & 4 & -1 \end{bmatrix}$ is a 3 x 3 *symmetric matrix.*

Special Matrices

A square matrix A for which $A = -A^T$ is said to be **skew- symmetric**. The elements along the leading diagonal of a skew-symmetric matrix must all be zero.

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \text{ is skew-symmetric.}$$

Note: If A is any square matrix, then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

Special Matrices

3. Row and column vectors. As we defined them in Section 4.1, a row vector is a matrix with one row, and a column vector is one with one column. For vectors, we usually use bold-faced small letters and write, for example,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad \mathbf{b}^{\mathrm{T}} = [b_1 \;\; b_2 \;\; \dots \;\; b_n]$$

The transpose of a row vector is a column vector and vice versa. If A is an m x n matrix, then $A\mathbf{a}$ is a column vector with m rows.

Special Matrices

4. Diagonal matrices. A square matrix all of whose elements off the leading diagonal are zero is called a **diagonal matrix**. Thus, if $A = [a_{ij}]$ is an n x n matrix, then A is diagonal if $a_{ij} = 0$ for all $i \neq j$.

Hence

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ is an example of a 3 x 3 diagonal matrix.

A diagonal matrix is obviously symmetric. If A and B are diagonal matrices of the same order then $A + B$ and AB are also both diagonal

Special Matrices

5. *Identity matrix*. The diagonal matrix with all diagonal elements 1 is called the **Identity** or **unit matrix** I_n. Hence, the 3 x 3 identity is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. *Powers of matrices.* If A is a square matrix of order n x n, then we write AA as A^2, AA^2 as A^3 and so on.

7. *Inverse matrix*, for a square matrix A, the inverse is written A^{-1}. When A is multiplied by A^{-1} the result is the identity matrix.

$$A\,A^{-1} = A^{-1}A = I$$

Matrices – Part II

MATHEMATICS FOR COMPUTING (IT1030)

Determinant of a Square Matrix

The determinant of a square matrix A is a real number denoted by

det A or $|A|$

The determinant of a 2×2 matrix

Given the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

Then the determinant of A is denoted and defined by

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Minor

Minor

Minor of A is the determinant of a smaller matrix formed from deleting its rows and columns. M_{ij} is the minor of matrix A formed by eliminating row i and column j from A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

M_{11} is the minor of A formed by eliminating row 1 and column 1 .

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

Minor

M_{21} is the minor of A formed by eliminating row 2 and column1.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{21} = a_{12}a_{33} - a_{13}a_{32}$$

Cofactor

Cofactor

Cofactor of a matrix is denoted as C_{ij} and is equal to $(-1)^{i+j} M_{ij}$, Where M_{ij} is minor of matrix.

$$\boldsymbol{C_{ij} = (-1)^{i+j} M_{ij}}$$

So $$C_{11} = (-1)^{1+1} M_{11}$$
$$= M_{11}$$

$$C_{12} = (-1)^{1+2} M_{12}$$
$$= -M_{12}$$

Expanding to find the Determinant

This is the i^{th} row expansion

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

This is the j $^{\text{th}}$ column expansion

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Here are the steps to go through to find the determinant.

1. Pick any row or column in the matrix. It does not matter which row or which column you use, the answer will be the same for any row. There are some rows or columns that are easier than others, but we'll get to that later.
2. Multiply every element in that row or column by its cofactor and add. The result is the determinant.

Determinant by Expanding 1^{st} Row

Determinant of the matrix A by expanding 1^{st} row,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det A &= \sum_{j=1}^{3} a_{1j} C_{1j} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + a_{13}(-1)^{1+3} M_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Determinant by Expanding 1st Column

$$\det A = \sum_{i=1}^{3} a_{i1} C_{i1}$$

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= a_{11}(-1)^{1+1} M_{11} + a_{21}(-1)^{2+1} M_{21} + a_{31}(-1)^{3+1} M_{31}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}).$$

Determinant ctd..

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution

$$\det A = 1 \times \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= (3\text{-}2\ ) + (\ 2 + 2\ ) = 5$$

Determinant ctd…

$$\det A = \begin{vmatrix} 1 & 2 & k \\ 2 & -1 & 3 \\ -1 & 4 & -2 \end{vmatrix}$$

for any k. Find the value of k for which the determinant is zero.

Determinant ctd…

Expanding by the first row gives

$$\det A = \begin{vmatrix} 1 & 2 & k \\ 2 & -1 & 3 \\ -1 & 4 & -2 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= 1 \times (2 - 12) - 2 \times (-4 + 3) + k(8 - 1)$$
$$= -10 + 2 + 7k = -8 + 7k.$$

Hence det A = 0 if $k = \dfrac{8}{7}$

Properties of Determinants

***1. det A^T = det A**, where A^T is the transpose of A.* The determinant of a square matrix and its transpose are equal.

Eg:

Evaluate

$$\det A = \begin{vmatrix} 1 & 28 & -29 \\ 0 & 1 & -4 \\ 0 & -2 & 5 \end{vmatrix}.$$

Properties of Determinants

Solution

Since the determinant has two zeros in the first column, it is advantageous to use Rule 1. The determinant of the transpose of A is given by

$$\det A^{\mathrm{T}} = \begin{vmatrix} 1 & 0 & 0 \\ 28 & 1 & -2 \\ -29 & -4 & 5 \end{vmatrix}$$

Which now has two zeros in the first row. Hence the expansion by the top row becomes particularly easy:

$$\det \mathrm{A}^{\mathrm{T}} = 1 \times \begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 5 - 8 = -3$$

Properties of Determinants

2. If every element of any single row or column of the matrix A is multiplied by a scalar k, then the determinant of this matrix is k det A.

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}ka_{22}a_{33} - a_{11}ka_{32}a_{23} - a_{12}ka_{21}a_{33} + a_{12}ka_{31}a_{23} + a_{13}ka_{21}a_{32} - a_{13}ka_{31}a_{22}$$

$= k \det A$

By putting $k = 0$ in this result, note that any determinant must have zero value if all the elements of any row or column are zero.

Properties of Determinants

3. If B is obtained from A by interchanging two rows (or columns) then

det B = -det A.

5. If two rows (or columns) of A are identical, then det A = 0.

6. If the matrix B is constructed from A by adding k times one row (or column) to another row (column) then det B = det A*: in other words, any number of such operation on rows and on columns has no effect on the value of det A.*

Properties of Determinants

$$\det B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{11} & a_{32}+ka_{12} & a_{33}+ka_{13} \end{vmatrix}$$

$$= (a_{31}+ka_{11})C_{31} + (a_{32}+ka_{12})C_{32} + (a_{33}+ka_{13})C_{33} \quad \text{(Expanding by row 3)}$$

$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + k(a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33})$$

$$= \det A + \mathrm{k} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

$$= \det A$$

Properties of Determinants

7. If the matrix A and B are of equal size, $\det AB = \det A \times \det B$

Inverse Matrix

$A \rightarrow A^{-1}$

Requirements to have an inverse

1. Matrix should be a square matrix
2. Determinant of the matrix should be a non-zero value

Inverse Matrix

There are 2 methods to find the inverse matrix

- Method I

$$A^{-1} = \frac{adjoint\ A}{|A|}$$

- Method II

Gaussian Elimination Method

The Adjoint

The matrix formed by taking the transpose of the cofactor matrix of a given original matrix. The adjoint of matrix A is often written adj A.

Elements of the cofactor matrix are cofactors.

$$\text{Cofactor matrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Adjoint of a matrix A is transpose of cofactor matrix.

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

The Adjoint

Find the adjoint of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

The Adjoint

$$C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5$$

$$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4 \qquad C_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12$$

$$C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad C_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad C_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

The Adjoint

As a result the cofactor matrix of A is

$$\text{Cofactor matrix} = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Finally the adjoint of A is the transpose of the cofactor matrix:

$$adjA = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Inverse Matrix – Method I

$$A^{-1} = \frac{adjA}{\det A} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \frac{1}{\det A}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

We evaluate det A first. Thus

$$\det A = 1\times(-2-1) - 2\times(0+1) - 1\times(0-1) = -4$$

$$\begin{array}{lll} C_{11} = -3 & C_{12} = -1 & C_{13} = -1 \\ C_{21} = 5 & C_{22} = -1 & C_{23} = 3 \\ C_{31} = -1 & C_{32} = 1 & C_{33} = 1 \end{array}$$

Hence $A^{-1} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Inverse Matrix – Method II

The inverse matrix by Gaussian elimination

Use following elementary row operations to transform A into the identity I, and use the same operations to transform I into A^{-1}.

1. Any equation can be multiplied by a nonzero constant,
1. Any two equations can be interchanged,
2. Any equation can be replaced by the sum of, itself and any multiple of another equation.

Inverse Matrix – Method II

Write down the entries of the matrix A in a double - wide matrix.

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & & & & \\ 1 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 1 & & & & \\ 1 & 0 & 2 & 0 & & & & \end{array}\right].$$

In the other half of the double-wide matrix, write the identity matrix.

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Inverse Matrix – Method II

Then do matrix row operations to convert the left-hand side of the double wide in to the identity matrix. The right - hand side of the double wide is Inverse matrix of A.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 0 & 1 & \end{array}\right].$$

Inverse Matrix – Method II

Suppose that we require the inverse of

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

We reduce A to I_4 and perform the same row operations on I_4. Thus, we can write down the steps **in parallel** as follows:

Inverse Matrix – Method II

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad (r_1 \leftrightarrow r_2) \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(r_4^{'} = r_4 - r_1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right] (r_3^{'} = r_3 - r_2$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \quad (r_4^{'} = -r_4) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] (r_2^{'} = r_2 - 2r_4) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$(r_1^{'} = r - r_{31}) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$= [I_4 | A^{-1}]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Summary

Students Should be able to,

- Find the Determinant of a matrix
- Apply the properties of determinants
- Find the inverse matrix

The End

Solving Linear Equations

MATHEMATICS FOR COMPUTING (IT1030)

Linear System

In mathematics a system of linear equations (or linear system) is a collection of linear equations involving the same set of variables.

For example,

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
-2x_1 + 3x_2 - x_3 &= -7 \\
x_1 + 4x_2 - 2x_3 &= -7
\end{aligned}
$$

Different Types of Solutions

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

A linear system may behave in any one of three possible ways:

1. The system has infinitely many solutions.
2. The system has a single unique solution.
3. The system has no solution.

Solving Linear Equations – Method I

We can convert the linear system in to a matrix form that is $AX = b$.

i.e

$$AX = b.$$

$$A^{-1}AX = A^{-1}b$$

We know $\quad A^{-1}A = I$

$$I\,X = A^{-1}b$$

$$X = A^{-1}b$$

Where X means unknown variables, A^{-1} is inverse matrix and d is a constant matrix.

Consider now the case in which A is an arbitrary square matrix. If the inverse of A exists, then multiplication of on the left by A^{-1} leads to the solution vector

$$X = A^{-1}b = \frac{adjA}{\det A}b$$

Example

Solve the linear system.

$$x_1 + 2x_2 - x_3 = 1$$
$$x_2 - x_3 = -7$$
$$x_1 - x_2 - 2x_3 = -7$$

So

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix} \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{adjA}{\det A} b = A^{-1}\begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -31 \\ -1 \\ -29 \end{bmatrix}$$

Therefore $x_1 = \frac{31}{4}, x_2 = \frac{1}{4}, x_3 = \frac{29}{4}$

Solving Linear Equations – Method II (Cramer's Rule)

Suppose $AX = b$ is a square linear system in the variables $X = x_1, x_2, x_3 \dots x_n$

With the property that $\det A \neq 0$. Then the (unique) solution to the system is given by

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1,2,3,\dots,n$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector b.

Example

Solve the following system using Cramer's rule.

$$\begin{aligned} x_1 + x_2 - x_3 &= 6 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 \quad - 2x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 6 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

$$\begin{aligned}\det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^{1+1}M_{11} + 1(-1)^{1+2}M_{12} + (-1)(-1)^{1+3}M_{13} \\ &= \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ &= (2-0) - (-2-1) - (0+1) \\ &= 4\end{aligned}$$

$$\det A_1 = 16$$
$$\det A_2 = 16$$
$$\det A_3 = 8$$

Thus by Cramer's Rule

$$x_1 = \frac{\det A_1}{\det A} = \frac{16}{4} = 4$$
$$x_2 = \frac{\det A_2}{\det A} = \frac{16}{4} = 4$$
$$x_3 = \frac{\det A_3}{\det A} = \frac{8}{4} = 2$$

Solving Linear Equations – Method III (Gaussian Elimination Method)

$x + 2y + z = 1$

-$2x + 3y - z = -7$

$x + 4y - 2z = -7$

Obtain the Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ -2 & 3 & -1 & -7 \\ 1 & 4 & -2 & -7 \end{bmatrix},$$

which is known as the **augmented matrix** for the system of equations. The elementary operations referred to previously become **elementary row operations on the matrix**. We can reproduce the steps above by the following more compact procedure:

Example

$$\begin{bmatrix} \underline{1} & 2 & 1 & 1 \\ -2 & 3 & -1 & -7 \\ 1 & 4 & -2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & \underline{7} & 1 & -5 \\ 0 & 2 & -3 & -8 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 + 2r_1 \\ r_3' = r_3 - r_1 \end{pmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 7 & 1 & -5 \\ 0 & 0 & \dfrac{-23}{7} & \dfrac{-46}{7} \end{bmatrix} \left(r_3' = r_3 - \frac{2}{7} r_2 \right)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \dfrac{1}{7} & \dfrac{-5}{7} \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} r_3' = -\dfrac{7}{23} r_3 \\ r_2' = \dfrac{1}{7} r_2 \end{pmatrix}$$

$x_1 = 1$, $\quad x_2 = -1$, $\quad x_3 = 2.$

Where the arrow '$\rightarrow$' means 'is transformed into'. The final matrix is said to be in **echelon** form, that is, it has zeros below the diagonal elements starting from the top left. We can now solve the equations by back substitution as before.

Incompatible set of Equations (No Solution)

$x + y - z = 3$
$3x - y + 3z = 5$
$x - y + 2z = 2$

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 3 & -1 & 3 & 5 \\ 1 & -1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 6 & -4 \\ 0 & -2 & 3 & -1 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - r_1 \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left(r_3' = r_3 - \frac{1}{2} r_2 \right)$$

Which is the echelon form for this set of equations. However, row 3 is inconsistent since $0 \neq 1$. Hence these equations can have no solutions.

Compatible set of equations (Infinite Number of Solutions)

$x + y - z = 1,$

$3x - y + 3z = 5,$

$x - y + 2z = 2,$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & -1 & 3 & 5 \\ 1 & -1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 6 & 2 \\ 0 & -2 & 3 & 1 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - r_1 \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left(r_3' = r_3 - \frac{1}{2} r_2 \right)$$

Compatible set of equations (Infinite Number of Solutions)

Row 3 is now consistent, and row 2 is $-4y + 6z = 2$. Hence

$$y = -\frac{1}{4}(2-6z)$$

and, from row 1 ,

$$x = 1 - y + z = \frac{3}{2} - \frac{1}{2}z.$$

Thus z can take any value, say λ , so the full solution set is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}\lambda \\ -\frac{1}{2} + \frac{3}{2}\lambda \\ \lambda \end{bmatrix}$$

for any value of λ . It can be seen in this case that there exists an infinite number of solutions, a different one for each different value of λ .

The End

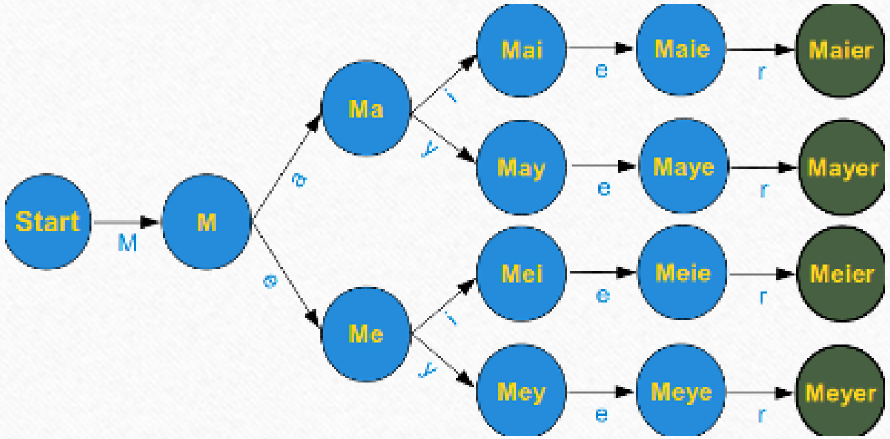
Finite –State Machines

Introduction

- There are several types of structures used in models of computation.
- namely, grammars, finite-state machines, and Turing machines
- Finite-state machines are used extensively in applications in computer science and data networking.

Application

- spell checking
- grammar checking
- indexing or searching large bodies of text
- recognizing speech,



Definition

1. Set I, called the **input alphabet**, of input symbols:

2. Set **S** of **states** the automation can be in

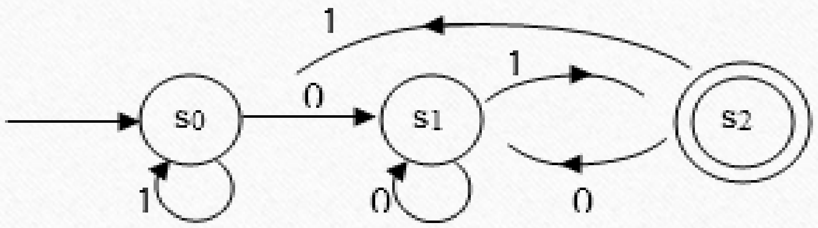
3. Designated state s_0 , called the **initial state**:

4. Designated set of states called the set of **accepting states**:

5. a **next-state function:** $S \times I \rightarrow S$ that associates a "next-state" to each ordered pair consisting of a "current-state" and a "current input".

Question 01

Consider the finite state automaton A defined by the transition diagram shown in figure given below



a) What are the states of A?
b) What are the input symbols of A?
c) What is the initial state of A?
d) What are the accepting states of A?
e) Find N(s1 , 1)
f) Find the annotated next state table for A.

Question 2

Consider the finite state automaton A defined by the following annotated next state table.

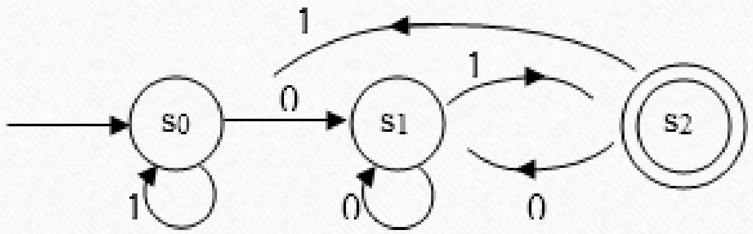
Input

State		a	b	c
→	U	Z	Y	Y
◎	V	V	V	V
	Y	Z	V	Y
◎	Z	Z	Z	Z

a) What are the states of A?
b) What are the input symbols of A?
c) What is the initial state of A?
d) What are the accepting states of A?
e) Find N(U , c).
f) Draw the transition diagram for A.

Question 3

Consider the finite state automaton A defined by the transition diagram shown in figure given below

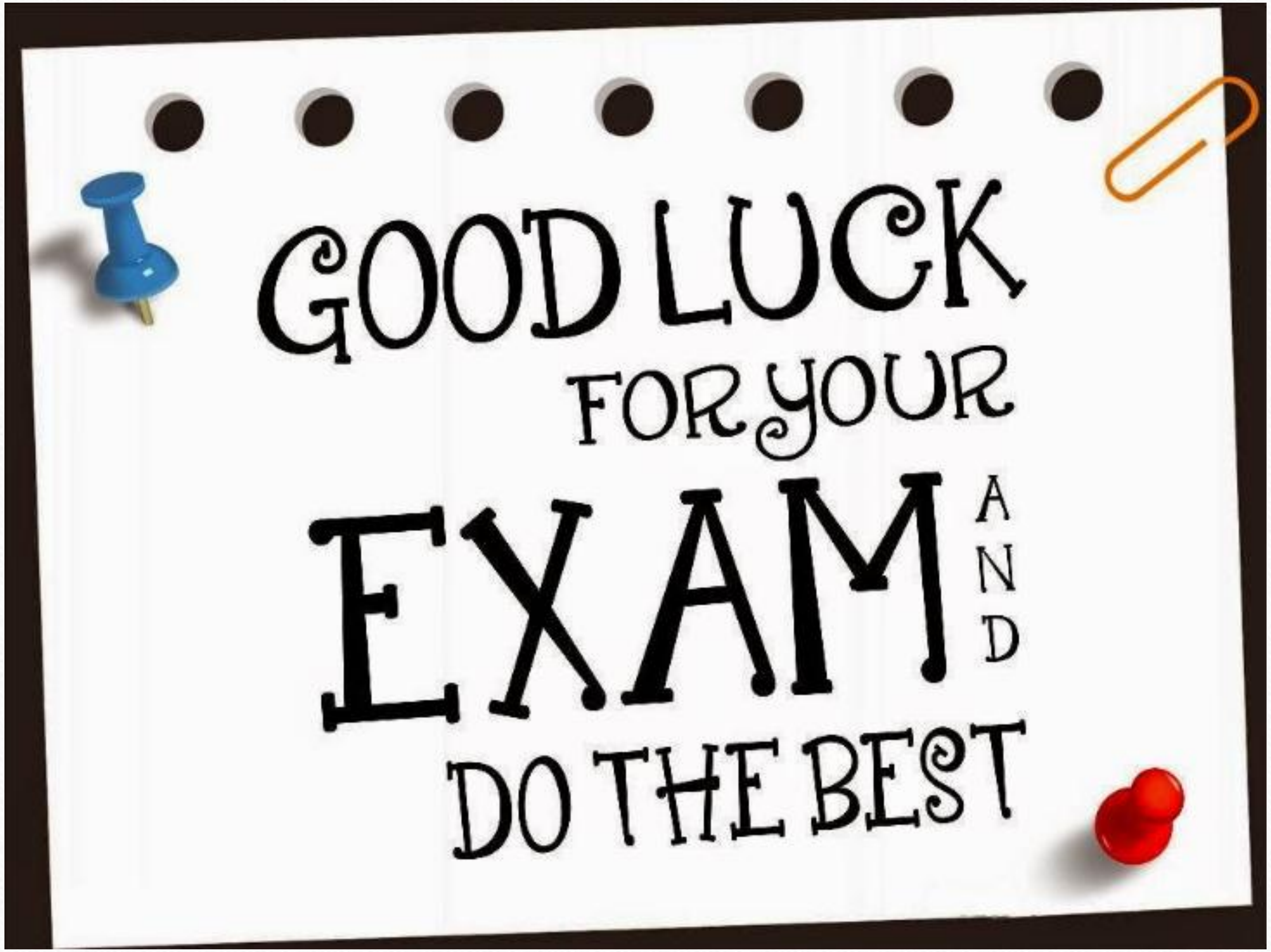


a) To what state does A go if the symbols of the following strings are input to A in sequence starting from the initial state?

1) 01 2) 011 3) 0101100 4) 10101

b) Which of the strings in part a) send A to an accepting state?

c) What is the language accepted by A?


GOOD LUCK
FOR YOUR
EXAM
AND
DO THE BEST