Łoś Ultraproduct Theorem and Ultracategories

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Abstract

In this paper, we give a categorical proof of a theorem from Mathematical Logic, known as the Łoś Ultraproduct Theorem. Our statement of the Łoś Ultraproduct Theorem is given in full generality, as to be able to use the tools of category theory. We then give some applications of Łoś Ultraproduct Theorem in conjunction with ultracategories, as defined by Jacob Lurie.

Contents

1 Background in First-Order Logic

Definition 1.0.1. A (single sorted, first-order) *language* $\mathcal{L}$ is a set of symbols divided into three groups:

- *Relation symbols* R which has a specified arity n,
- *Function symbols* f which has a specified arity n,
- *Constant symbols* c,

together with logical symbols (,), $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$, $\top$, $\bot$, $\Rightarrow$, $=$ and an infinite collection of variables $x_1, x_2, \ldots$.

Definition 1.0.2. The *terms* over $\mathcal{L}$ are defined inductively as follows:

- A variable x is a term.
- If $t_1, \ldots, t_n$ are terms and f is a function symbol of $\mathcal{L}$ of arity n, then $f(t_1, \ldots, t_n)$ is a term.

We write $t(\overline{x})$ to mean t is a term in which the variables $\overline{x} = x_1, \ldots, x_n$ appear in t.

Definition 1.0.3. The *formulae* ϕ over $\mathcal{L}$ with *free variables* $FV(\phi)$ are defined inductively as follows (Note: the free variables of a term $t(\overline{x})$ is $\overline{x} = x_1, \ldots, x_n$):

- If $t_1(\overline{x})$ and $t_2(\overline{x})$ are terms, then $t_1(\overline{x}) = t_2(\overline{x})$ is a formula with free variables $FV(t_1 = t_2) = \overline{x}$.
- If $t_1(\overline{x}_1), \ldots, t_n(\overline{x}_n)$ are terms and R is a relation symbol of arity n, then $R(t_1(\overline{x}_1), \ldots, t_n(\overline{x}_n))$ is a formula with $FV(R) = \{\overline{x}_1, \ldots, \overline{x}_n\}$.
- If $\phi(\overline{x})$ and $\psi(\overline{x})$ are formulae, then $(\phi \vee \psi)(\overline{x})$ is a formula with $FV(\phi \vee \psi) = FV(\phi) \cup FV(\psi)$.
- If $\phi(\overline{x})$ is a formula, then so is $\neg\phi(\overline{x})$ with $FV(\neg\phi) = FV(\phi)$.
- If $\psi(\overline{x}, y)$ is a formula and y is a variable, then $\exists y.\psi(\overline{x}, y)$ is a formula with with free variables $FV(\psi) \setminus \{y\}$.

A variable x that appears in a formula ϕ is called *bound* if it appears as part of a quantifier. For example, the variable x in $\exists x R(x, y)$ is bound while y is free. We say that a formula $\sigma(\overline{x})$ is a *sentence* if we have that $FV(\sigma) = \emptyset$, that is, all of the variables that appear in σ are bound.

We will use the common abbreviations for the following:

- $\phi(\overline{x}) \wedge \psi(\overline{x}) \stackrel{\text{def}}{=} \neg((\neg\phi(\overline{x})) \vee (\neg\psi(\overline{x})))$.
- $\phi(\overline{x}) \Rightarrow \psi(\overline{x}) \stackrel{\text{def}}{=} (\neg\phi(\overline{x})) \vee \psi(\overline{x})$.
- $\phi(\overline{x}) \Leftrightarrow \psi(\overline{x}) \stackrel{\text{def}}{=} (\phi(\overline{x}) \Rightarrow \psi(\overline{x})) \wedge (\psi(\overline{x}) \Rightarrow \phi(\overline{x}))$.
- $\forall y.\phi(\overline{x}, y) \stackrel{\text{def}}{=} \neg\exists y.\neg\phi(\overline{x}, y)$.
- $\top \stackrel{\text{def}}{=} \forall x.(x = x)$.
- $\bot \stackrel{\text{def}}{=} \exists x.(x \neq x)$.

Definition 1.0.4. A *theory* T is a set of sentences which we call the *axioms* of T.

Example 1.0.5. We define the *theory of partially ordered sets (posets)* as follows:

- $\mathcal{L} = \{\leq\}$, where $\leq$ is a 2-ary relation symbol.
- The theory consists of the following axioms:
  (i) *Reflexivity*: $\forall x.(x \leq x)$.
  (ii) *Asymmetry*: $\forall x, y.(x \leq y \wedge y \leq x \Rightarrow x = y)$.
  (iii) *Transitivity*: $\forall x, y, z.(x \leq y \wedge y \leq z \Rightarrow x \leq z)$.
- We obtain the *theory of directed posets* if we add the following axiom:
  (iv) *Directed*: $\forall x, y.\exists z.(x \leq z \wedge y \leq z)$.

We can define the theory of groups, rings, fields, number theory, etc. using first-order logic. However, we are not able to define the theory of topological spaces, since this would require second-order logic.

Definition 1.0.6. We define a $\mathcal{L}$*-structure* to be a function from our language to a *set* M such that:

- For each relation symbol R of arity n, we have an *interpretation* of R as a subset $M(R) \subseteq M^n$.
- For each function symbol f of arity n, we have an *interpretation* of f as a function $M(f) : M^n \to M$.
- For each constant symbol c, we have an *interpretation* of c as an element $M(c) \in M$.

Definition 1.0.7. The *value* of a term $t(\overline{x})$ at $\overline{a} = a_1, \ldots, a_n$ in a $\mathcal{L}$-structure M is defined as follows:

- If $t = x_i$ is a variable, then $t(\overline{a}) = a_i$.
- If $t = f(t_1, \ldots, t_n)$, f a function symbol and t_i are terms, then $t(\overline{a}) = M(f)(t_1(\overline{a}), \ldots, t_n(\overline{a}))$.

Definition 1.0.8. Given a $\mathcal{L}$-structure M, a formula $\phi(\overline{x})$, and a tuple of element $\overline{a} = a_1, \ldots, a_n \in M$, we define M *satisfies* ϕ with $\overline{a}$, denoted $M \vDash \phi(\overline{a})$, as follows:

- If $\phi(\overline{x}) := t_1(\overline{x}) = t_2(\overline{x})$, then $M \vDash \phi(\overline{a})$ iff $t_1(\overline{a}) = t_2(\overline{a})$.
- If $\phi(\overline{x}) := R(t_1, \ldots, t_n)$, then $M \vDash \phi(\overline{a})$ iff $t_1(\overline{a}), \ldots, t_n(\overline{a}) \in M(R)$.
- If $\phi(\overline{x}) := (\psi_1 \vee \psi_2)(\overline{x})$, then $M \vDash \phi(\overline{a})$ iff $M \vDash \psi_1(\overline{a})$ or $M \vDash \psi_2(\overline{a})$.
- If $\phi(\overline{x}) := (\neg\psi)(\overline{x})$, then $M \vDash \phi(\overline{a})$ iff not $M \vDash \psi(\overline{a})$.
- If $\phi(\overline{x}) := \exists y.\psi(\overline{x}, y)$, then $M \vDash \phi(\overline{a})$ iff there exists $b \in M$ such that $M \vDash \psi(\overline{a}, b)$.

Definition 1.0.9. If $T = \{\sigma_i\}_{i \in I}$ is a theory with axioms σ_i, then a $\mathcal{L}$-structure M is a *model* of T if $M \vDash T$, that is, $M \vDash \sigma_i$ for each $\sigma_i \in T$.

A poset $(P, \leq_P)$ is a model of the theory of posets with

$$x \leq_P y \Leftrightarrow (x, y) \in \leq_P \;\; (\subseteq P \times P).$$

Definition 1.0.10. Let M and N be models of a theory T. A function $f : M \to N$ is an *elementary embedding* if for every formula $\phi(\overline{x})$ and $\overline{a} \in M^n$, we have

$$M \vDash \phi(\overline{a}) \Leftrightarrow N \vDash \phi(f(\overline{a})).$$

Definition 1.0.11. An *ultrafilter* $\mathcal{U}$ on a set S is a subset of the powerset $\mathcal{P}(S)$ such that

- $S \in \mathcal{U}$.
- For each $S_0, S_1 \in \mathcal{U}$, we have $S_0 \cap S_1 \in \mathcal{U}$.
- If $S_0 \in \mathcal{U}$ and $S_1 \subseteq S_0$ then $S_1 \in \mathcal{U}$.
- For every $S_0 \subseteq S$, exactly one of S_0 and $S \setminus S_0$ is in $\mathcal{U}$.

If $s \in S$, then we have the *principal ultrafilter*

$$\mathcal{U}_s = \{S_0 \subseteq S : s \in S_0\}.$$

Now we will define the ultraproduct of sets.

Definition 1.0.12. Given a collection of (nonempty) sets $\{X_s\}_{s\in S}$ and an ultrafilter $\mathcal{U}$ on S, we define an equivalence relation $\sim_{\mathcal{U}}$ on $\prod_{s\in S} X_s$ by

$$\left(\{x_s\}_{s\in S} \sim_{\mathcal{U}} \{y_s\}_{s\in S}\right) \Leftrightarrow \{s \in S : x_s = y_s\} \in \mathcal{U}.$$

We let $x^{\mathcal{U}}$ denote the equivalence class of $\{x_s\}_{s\in S}$. Then the *ultraproduct of* $\{X_s\}_{s\in S}$ *w.r.t* $\mathcal{U}$, denoted

$$\left(\prod_{s\in S} X_s\right)/\mathcal{U} = \left\{x^{\mathcal{U}} : \{x_s\}_{s\in S} \in \prod_{s\in S} X_s\right\},$$

is the set of equivalence classes of the relation $\sim_{\mathcal{U}}$.

Definition 1.0.13. Let T be a theory and let $\{M_s\}_{s\in S}$ be a collection of models of T. Let $\mathcal{U}$ be an ultrafilter on S. Then the *ultraproduct of the models* $\{M_s\}_{s\in S}$ *w.r.t.* $\mathcal{U}$ is the $\mathcal{L}$-structure $\left(\prod_{s\in S} M_s\right)/\mathcal{U}$ defined by:

- For each relation symbol R of arity n and each tuple of equivalence classes $x_1^{\mathcal{U}}, \ldots, x_n^{\mathcal{U}}$,

$$\left(\left(\prod_{s\in S} M_s\right)/\mathcal{U}\right)(R)(x_1^{\mathcal{U}}, \ldots, x_n^{\mathcal{U}}) \Leftrightarrow \{s \in S : M_s(R)(x_{s,1}, \ldots, x_{s,n})\} \in \mathcal{U}.$$

- For each function symbol f of arity n each tuple of equivalence classes $x_1^{\mathcal{U}}, \ldots, x_n^{\mathcal{U}}$,

$$\left(\left(\prod_{s\in S} M_s\right)/\mathcal{U}\right)(f)(x_1^{\mathcal{U}}, \ldots, x_n^{\mathcal{U}}) = (M_s(f)(x_{s,1}, \ldots, x_{s,n}) : s \in S)^{\mathcal{U}}$$

- For each constant symbol c,

$$\left(\left(\prod_{s\in S} M_s\right)/\mathcal{U}\right)(c) = (M_s(c) : s \in S)^{\mathcal{U}} \in \left(\prod_{s\in S} M_s\right)/\mathcal{U}.$$

For example, we have that if $\{(P_s, \leq_{P_s})\}_{s\in S}$ are posets

$$\{x_s\}_{s\in S} \leq_{UP} \{y_s\}_{s\in S} \Leftrightarrow \{s \in S : x_s \leq_{P_s} y_s\} \in \mathcal{U}.$$

Łoś Ultraproduct Theorem. *Let $\{M_s\}_{s\in S}$ be a collection of models of a theory T and let $\mathcal{U}$ be an ultrafilter on S. Then the ultraproduct $\left(\prod_{s\in S} M_s\right)/\mathcal{U}$ is also a model of T.*

The ultraproduct of a collection of posets (groups, rings, etc.) is a poset (group, ring, etc.). We will prove a more general version of Łoś Ultraproduct Theorem using Category Theory.

2 Category Theory

2.1 Basic Notions

Definition 2.1.1. A *category* C consists of the following data:

- a collection C_0 of objects: $X, Y, Z, \ldots$
- a collection C_1 of morphisms (or arrows): $f, g, h, \ldots$

such that

- For each $f \in \mathsf{C}_1$ there are specified objects $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathsf{C}_0$ called the domain and the codomain; the notation $f : X \to Y$ represents that f is a morphism and $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$
- For each $X \in \mathsf{C}_0$, there is an identity morphism $1_X : X \to X$.
- For each pair $f, g \in \mathsf{C}_1$ such that $\mathrm{dom}(g) = \mathrm{cod}(f)$, there exist a composite morphism $g \circ f \in \mathsf{C}_1$ with $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.

This data is required to satisfy the following axioms:

- For any $f : X \to Y$, we have $1_Y \circ f = f = f \circ 1_X$, that is, composition is unital.
- For any composable triple $f, g, h \in \mathsf{C}_1$, we have that $h\circ(g\circ f) = (h\circ g)\circ f$, that is, composition is associative, and we denote this composition by $h \circ g \circ f$.

Given two objects $X, Y \in \mathsf{C}$, we will denote the collection of morphisms between X and Y by either $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ or $\mathsf{C}(X, Y)$. We call $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ the *hom-set* of X and Y. In general, the hom-set is not a set but a collection. In the case that $\mathrm{Hom}(X, Y)$ is a set for every pair $X, Y \in \mathsf{C}$, we say that C is *locally small*. Many examples of categories that we will encounter will be locally small.

Now we will provide an incomplete list some examples of categories that appear naturally throughout mathematics:

Example 2.1.2. 1. The category of sets, Set, has as objects sets with the morphisms being set functions.

2. The category of finite sets, Finset, has as objects finite sets with the morphisms being functions between finite sets.

3. The category of groups, Grp, has as objects groups with the morphisms being group homomorphisms. We have that the category of rings, Ring, and the category of fields Field, is defined similarly.

4. The category of topological spaces, Top, has as objects topological spaces with the morphisms being continuous functions.

5. Given a theory T, we have the category $\mathsf{Mod}(T)$ whose objects are models of T and whose morphisms are elementary maps (or elementary embeddings).

All of the examples above are of categories whose objects are some mathematical object with the morphisms the respective structure preserving maps. Many mathematical structures themselves are also categories:

Example 2.1.3. 1. A set X can be considered as a category. The elements of X are the objects of the category. The only morphisms are the identity morphisms. We say a category is *discrete* if every morphism is an identity.

2. Any partially ordered set $(P, \leq)$ can be regarded as a category. The objects of this category are the elements of P. We have that $p \leq q$ in P if and only if there is a unique arrow $p \to q$. The reflexivity and transitivity of $\leq$ give us the required unitality and associativity axioms.

3. Every group G can be considered as a category. The object of this category is G itself while the arrows are the elements of G. We will denote by **BG** to be G when considered as a category. We can also consider any monoid M as a category in a similar way.

4. There is the empty category $\mathbb{0}$ with no objects and no arrows. The one object category $\mathbb{1}$ has a single object and its only identity arrow. The category $\mathbb{2}$ is the category with two objects and one non-identity arrow, usually depicted as $0 \to 1$.

Given some category C, we can construct many other categories from C. We list a few examples below:

Example 2.1.4. 1. A *subcategory* D of a category C is defined as follows:

- the collection of objects D_0 of D is a subcollection of C_0.
- the collection of morphisms D_1 is a subcollection of C_1 with the requirement that for any $f : X \to Y$ in D_1, both $X, Y \in \mathsf{D}_0$.
- for every $X \in \mathsf{D}_0$, we have $1_X \in \mathsf{D}_1$.
- for each $f, g \in \mathsf{D}_1$ with $\mathrm{dom}(g) = \mathrm{cod}(f)$, we have $g \circ f \in \mathsf{D}_1$.

2. Given a category C and an object $X \in \mathsf{C}_0$, there is a category $\mathsf{C}/_X$ whose objects are morphisms $f : A \to X$ with $\mathrm{cod}(f) = X$ and whose morphisms between $f : A \to X$ and $g : B \to X$ is an arrow $h : A \to B$ such that $f = g \circ h$.

3. Given a category C, we can construct the *opposite category* by taking as the objects of C^{op} to be the objects as the objects of C and taking the arrows of C^{op} to be the arrows of C but reversed, that is, $f : x \to Y \in \mathsf{C}_1^{\mathrm{op}}$ if and only if $f : Y \to X \in \mathsf{C}_1$. We have that composition of arrows is reversed as well, that is, $f \circ g$ in C^{op} if and only if $g \circ f$ in C.

4. Let C and D be categories. Then the *product category* $\mathsf{C} \times \mathsf{D}$ is the category whose objects are ordered pairs (C, D), where $C \in \mathsf{C}_0$ and $D \in \mathsf{D}_0$, and whose arrows are ordered pairs $(f, g) : (C_0, D_0) \to (C_1, D_1)$, where $f \in \mathsf{C}_1$ and $g \in \mathsf{D}_1$. Composition of arrows is defined coordinate-wise.

The arrows within a category can have many properties, many of which appear naturally in Set.

Definition 2.1.5. Let C be a category and $f : X \to Y$ be an arrow in C. Then we say that f is:

- an *isomorphism* if there exists a unique arrow $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. We will say that X is *isomorphic to* Y, written $X \cong Y$, if there is an isomorphism between X and Y.

- a *monomorphism* if for every parallel pair of arrows $h, k : Z \rightrightarrows X$, $f \circ h = f \circ k$ implies that $h = k$. We sometimes say f is *monic* and use the notation $f : X \rightarrowtail Y$.

- an *epimorphism* if for every parallel pair of arrows $h, k : Y \rightrightarrows Z$, $h \circ f = k \circ f$ implies that $h = k$. We sometimes say f is *epic* and use the notation $f : X \twoheadrightarrow Y$.

In Set, we have that the isomorphisms, monomorphisms, and epimorphisms are the bijective, injective, and surjective functions, respectively. In a poset $(P, \leq)$ regarded as a category, due to $\leq$ being antisymmetric, the only arrows that are isomorphisms are the identity arrows.

As with most mathematical objects, we can define a morphism between two categories.

Definition 2.1.6. Let C and D be categories. A *(covariant) functor* $F : \mathsf{C} \to \mathsf{D}$ is a morphism consisting of the following data:

- for each object $X \in \mathsf{C}_0$ there is an object $F(C) \in \mathsf{D}_0$,

- for each morphism $f : X \to X' \in \mathsf{C}_1$, there is an arrow $F(f) : F(X) \to F(X') \in \mathsf{D}_1$.

The morphism F is required to satisfy the following axioms:

- For any composable pair $f, g \in \mathsf{C}_1$, we have $F(g) \circ F(f) = F(g \circ f)$,

- For each object $X \in \mathsf{C}_0$, we have $F(1_X) = 1_{F(X)}$.

Example 2.1.7.
1. For each category C we have the *identity functor* $\mathrm{Id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$.
2. If $Y \in \mathsf{D}_0$, we have the *constant functor* $\hat{Y} : \mathsf{C} \to \mathsf{F}$ with $\hat{Y}(X) = Y$ and $\hat{Y}(f) = \mathrm{id}_Y$.
3. There are *forgetful functors* $U : \mathsf{Grp} \to \mathsf{Set}$ and $U : \mathsf{Top} \to \mathsf{Set}$ that forgets the structure and sends an object to the underlying set.
4. The *fundamental group* is a functor $\pi_1 : \mathsf{Top} \to \mathsf{Grp}$.

Definition 2.1.8. A functor $F : \mathsf{C} \to \mathsf{D}$ is

- *full* if for each $X, Y \in \mathsf{C}$, the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is surjective.

- *faithfull* if for each $X, Y \in \mathsf{C}$, the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is injective.

The functor F is said to be *fully faithfull* if F is both full and faithfull.

Definition 2.1.9. A category C is a *full subcategory* of the category D if the functor $F : \mathsf{C} \to \mathsf{D}$ is fully faithfull and injective on objects.

Definition 2.1.10. Given categories C and D and functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of:

- an arrow $\alpha_X : F(X) \to G(X)$ in D for each object $X \in \mathsf{C}$, the collection of which define the *components* of the natural transformation,

so that, for any morphism $f : X \to Y$ in C, the following square of morphisms in D

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

commutes.

Definition 2.1.11. A *natural isomorphism* is a natural transformation $\alpha : F \Rightarrow G$ in which every component α_X is an isomorphism.

Example 2.1.12. Examples of natural transformations.

Definition 2.1.13. We say that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that we have natural isomorphism

$$G \circ F \cong \mathrm{Id}_{\mathsf{C}} \text{ and } F \circ G \cong \mathrm{Id}_{\mathsf{D}}.$$

Definition 2.1.14. Let C and D be categories. The define the *functor category* $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ to be the category whose objects are the functors $F : \mathsf{C} \to \mathsf{D}$ and whose arrows are natural transformation.

2.2 Limits

Definition 2.2.1. A *diagram* of *shape* J in a category C is a functor $D : \mathsf{J} \to \mathsf{C}$.

Given an object $C \in \mathsf{C}$, there is a constant functor $\Delta_{\mathsf{J}}(C) : \mathsf{J} \to \mathsf{C}$ which sends every object of J to C and every morphism in J to id_C. This constant functor defines the diagonal functor $\Delta_{\mathsf{J}} : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$, sending each $C \in \mathsf{C}$ to $\Delta_{\mathsf{J}}(C)$ and each $f : C \to C'$ to the constant natural transformation, where each component is f.

Note that in practice we will usually describe a diagram by the objects of our category in the shape that we will be working with.

Definition 2.2.2. Given a diagram $D : \mathsf{J} \to \mathsf{C}$, we define the *cone on D* with *vertex* $C \in \mathsf{C}$ to be a natural transformation $\lambda : \Delta_{\mathsf{J}}(C) \Rightarrow D$. Explicitly:

A cone on $D : \mathsf{J} \to \mathsf{C}$ with vertex $C \in \mathsf{C}$ is a collection of morphisms $\{\lambda_J : C \to D(J)\}_{J \in \mathsf{J}}$ such that for each morphism $f : J \to K$ in J, the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "\lambda_J"'] \arrow[dr, "\lambda_K"] & \\ D(J) \arrow[rr, "D(f)"] & & D(K) \end{tikzcd}$$

We well usually write $\lambda : C \to D$ (resp. $\lambda : D \to C$) for a cone (resp. cocone) on D with vertex C.

Definition 2.2.3. Let $D : \mathsf{J} \to \mathsf{C}$ be a diagram of shape J. A *limit of* D is a *universal cone* $\lambda : \lim_{\mathsf{J}} D \to D$, that is, given any other cone $\pi : C \to D$ there exists a unique map $\beta : C \to \lim_{\mathsf{J}} D$ with $\pi_J \circ \beta = \lambda_J$ for each $J \in \mathsf{J}$:

$$\begin{tikzcd}
C \arrow[rr, dashed, "\beta"] \arrow[dr, "\pi_J"'] \arrow[ddr, bend right, "\pi_K"'] & & \lim_{\mathsf{J}} D \arrow[dl, "\lambda_J"] \arrow[ddl, bend left, "\lambda_K"] \\
& D(J) \arrow[d, "D(f)"] & \\
& D(K) &
\end{tikzcd}$$

If $\lim_{\mathsf{J}} D$ is the limit of a diagram $D : \mathsf{J} \to \mathsf{C}$ and $f, g : C \rightrightarrows \lim_{\mathsf{J}} D$ are a parallel pair of arrows such that for all $J \in \mathsf{J}$, $\lambda_J \circ f = \lambda_J \circ g$, by the universality of the limit, we have that $f = g$.

We will now take a look at some specific examples of limits that may occur with in a category. The first of which is known as the product of two objects.

Definition 2.2.4. Let $X_0, X_1 \in \mathsf{C}_0$ be in the shape of the discrete category $\{0, 1\}$:

$$X_0 \qquad\qquad X_1.$$

Then a cone on this diagram looks as follows

$$X_0 \xleftarrow{f_0} Y \xrightarrow{f_1} X_1$$

The *product* of X_0 and X_1 is the limit of the discrete diagram above, that is, the universal cone $(X_0 \times X_1, \pi_0, \pi_1)$ such that for any other cone (Y, f_0, f_1) there exists a unique arrow $\langle f_0, f_1 \rangle : Y \to X_0 \times X_1$ such that the following diagram commutes:

$$\begin{tikzcd}
& Y \arrow[dl, "f_0"'] \arrow[d, dashed, "\langle f_0, f_1 \rangle"] \arrow[dr, "f_1"] & \\
X_0 & X_0 \times X_1 \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & X_1
\end{tikzcd}$$

Definition 2.2.4 can be extended to the case where the discrete diagram is of the form $\{0, 1, \ldots, n\}$.

Example 2.2.5. Many familiar categories have products:

1. In Set, the product of two sets is their Cartesian product.
2. In Top, the product of two topological spaces is their Cartesian product with the product topology.
3. In Grp, the product of two groups is their external product.
4. If we regard a poset $(P, \leq)$ as a category, then the product of two elements $x, y \in P$ (if it exists) is their infimum $x \vee y$.

Definition 2.2.6. Consider the empty diagram (or the diagram of shape $\mathbb{0}$):

then a cone of this diagram is just a single object of C:

$$X.$$

An object of C is a *terminal object*, denoted by 1, if it is the limit of the empty diagram, that is, for any other object $X \in \mathsf{C}_0$ there is a unique arrow $1_X : X \to 1$.

Example 2.2.7.

1. In Set, every singleton set $\{*\}$ is a terminal object.
2. The topological space with a single point is terminal in Top.
3. The trivial group $\{e\}$ is terminal in Grp.
4. In a poset $(P, \leq)$, the terminal object (if it exists) is the top element.

Definition 2.2.8. Let $f, g : X \rightrightarrows Y$ be a parallel pair of arrows in C as shown in the diagram below:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y. \end{tikzcd}$$

Then a cone on this diagram is such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "h"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y. \end{tikzcd}$$

The *equalizer* of f, g is the limit of the parallel pair diagram above, that is, it is the universal cone $(E, e : E \to X)$ such that for any other cone (Z, h), there is a unique arrow $k : Z \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[u, dashed, "k"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Example 2.2.9. In Set, if $f, g : X \to Y$ are set functions, there exists a subset $E \subseteq X$ defined by

$$E = \{x \in X : f(x) = g(x)\}.$$

This subset is the equalizer of f and g.

Our last example of a limit that we will look at is one known as a pullback. Pullbacks appear in many other areas of category theory and hence is almost all of mathematics. These limits will be critical in our study of Łoś Ultraproduct Theorem.

Definition 2.2.10. Let $f : X \to Z$, $g : Y \to Z$ be arrows in C. The *pullback of f and g* is the object $X \times_Z Y$ and the *projection maps* $g' : X \times_Z Y \to X$ and $f' : X \times_Z Y \to Y$ such that $f \circ g' = g \circ f'$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

with the universal property that for any other object W and arrows $h : W \to X$, $k : W \to Y$ such that $f \circ k = g \circ h$, there exists a unique arrow $\beta : W \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\ & X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z, \end{tikzcd}$$

One way to think about the construction of a pullback is that we are pulling f back along the arrow g to f' (or pulling g back to g' along f). We list some examples of how pullbacks appear in the category of sets in interesting ways.

Example 2.2.11. 1. In Set, the pullback of functions $f : X \to Z$ and $g : Y \to Z$ is the set $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\} \subseteq X \times Y$ with the usual projection maps.

2. In particular, if $f : X \to Y$ is a set function and $Y_0 \subseteq Y$ is a subset, then the *pullback of f along the inclusion function $i : Y_0 \hookrightarrow Y$* is the inverse image $f^{-1}(Y_0)$ with functions f and the inclusion $i' : f^{-1}(Y_0) \hookrightarrow X$, as in the following diagram:

$$\begin{tikzcd} f^{-1}(Y_0) \arrow[r, "f"] \arrow[d, hook, "i'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[d, hook, "i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

3. Let $f : X \to Y$ be a set function. Then the pullback of f along itself is the set

$$X \times_Y X = \big\{(x, x') \in X \times X : f(x) = f(x')\big\}$$

with the usual projection arrows $\pi, \pi' : X \times_Y X : X$. These projection arrows are called a *kernel pair*. This defines an equivalence relation $\sim_f$ on $X \times X$ by saying $x \sim_f x'$ if and only if $f(x) = f(x')$.

4. If we regard a poset $(P, \leq)$ as a category, then the pullback of $z \leq x$ and $z \leq y$ is the product of x and y, that is, their infimum.

The product, terminal object, equalizer, and pullback are all examples of finite limits. More specifically, if $D : \mathsf{J} \to \mathsf{C}$ is a diagram, then D is said to be *finite* if J is finite, that is, J has a finite number of morphisms. Then a *finite limit* is a limit on a finite diagram. A category C *has finite limits* if it has a limit for every finite diagram. We will require that the categories in consideration when studying categorical logic have finite limits.

To show that a category has all finite limits, we would be required to show for every possible finite diagram we can take the limit of that diagram. This may sound tedious, but it turns out that we do not need to look at all finite diagrams, just two of them.

Theorem 2.2.12. *Let C be a category with pullbacks for every pair of arrows and a terminal object 1. Then C has all finite limits.*

A category having at least pullbacks and a terminal object is enough to construct every other possible finite limit. The proof of Theorem 2.2.12 is to show that we can construct products and equalizers using pullbacks and terminal objects. The pullback of the arrows $1_X : X \to 1$, $1_Y : Y \to 1$ is the product of X and Y, i.e. the following commutative square is a pullback square:

$$\begin{tikzcd} X \times Y \arrow[r, "\pi'"] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "1_Y"] \\ X \arrow[r, "1_X"] & 1. \end{tikzcd}$$

The pullback of the arrows $f, g : X \rightrightarrows Y$ is the equalizer of f and g, there the projection arrows $i : E \to X$ of the pullback is the equalizer arrow, which can be seen in the following pullback square:

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

2.3 Colimits

Dually, we define the notions of cocone and colimit of a diagram. An intuitive way to think about a colimit in a category C is that the colimit is the limit in the opposite category C^{op}. For example, the *coproduct* of objects X and Y in C will be the product of X and Y in C^{op}. We can also define these explicitly as follows:

Definition 2.3.1. A *cocone on D* with vertex $C \in \mathsf{C}$ is natural transformation $\lambda : D \Rightarrow \Delta_{\mathsf{J}}(C)$. Explicitly:

A cocone on $D : \mathsf{J} \to \mathsf{C}$ with vertex $C \in \mathsf{C}$ is a collection of morphisms $\{\lambda_J : D(J) \to C)\}_{J \in \mathsf{J}}$ such that for each morphism $f : J \to K$ in J, the following diagram commutes:

$$\begin{tikzcd} D(J) \arrow[rr, "D(f)"] \arrow[dr, "\lambda_J"'] & & D(K) \arrow[dl, "\lambda_K"] \\ & C & \end{tikzcd}$$

Definition 2.3.2. Let $D : \mathsf{J} \to \mathsf{C}$ be a diagram of shape J. A *colimit of D* is a *universal cocone* $\lambda : D \to \operatorname{colim}_{\mathsf{J}} D$, that is, given any other cocone $\pi : D \to C$ there exists a unique map $\beta : \operatorname{colim}_{\mathsf{J}} D$ with $\beta \circ \pi_J = \lambda_J$ for each $J \in \mathsf{J}$:

$$\begin{tikzcd} \operatorname{colim}_{\mathsf{J}} D \arrow[rr, dashed, "\beta"] & & C \\ & D(J) \arrow[ul, "\lambda_J"'] \arrow[ur, "\pi_J"] \arrow[d, "D(f)"] & \\ & D(K) \arrow[uul, bend left, "\lambda_K"] \arrow[uur, bend right, "\pi_K"'] & \end{tikzcd}$$

If $\operatorname{colim}_{\mathsf{J}} D$ is the colimit of a diagram $D : \mathsf{J} \to \mathsf{C}$ and $f, g : \operatorname{colim}_{\mathsf{J}} D \rightrightarrows C$ are a parallel pair of arrows such that for all $J \in \mathsf{J}$, $f \circ \lambda_J = g \circ \lambda_J$, then by the universality of the colimit, we have that $f = g$.

Definition 2.3.3. Let $X_0, X_1 \in \mathsf{C}_0$ be in the shape of the discrete category $\{0, 1\}$:

$$X_0 \qquad X_1.$$

Then a cocone on this diagram looks as follows

$$X_0 \xrightarrow{f_0} Y \xleftarrow{f_1} X_1$$

The *coproduct* of X_0 and X_1 is the colimit of the discrete diagram, that is, the universal cocone $(X_0 + X_1, i_0, i_1)$ such that for any other cocone (Y, f_0, f_1) there exists a unique arrow $[f_0, f_1] : X_0 + X_1 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "i_0"] \arrow[dr, "f_0"'] & X_0 + X_1 \arrow[d, dashed, "{[f_0, f_1]}"] & X_1 \arrow[l, "i_1"'] \arrow[dl, "f_1"] \\ & Y & \end{tikzcd}$$

Example 2.3.4. 1. In Set, the coproduct of X and Y is their disjoint union.

2. If we regard a poset $(P, \leq)$ as a category, then the coproduct of x and y is their supremum.

Just like the terminal object was the limit of the empty diagram, we are able to define the *unital object* to be the colimit of the empty diagram. In Set, the initial object is the empty set $\emptyset$.

The next colimit, known as a *coequalizer*, will appear later and is important in our study of categorical logic.

Definition 2.3.5. Let $f, g : X \rightrightarrows Y$ be a parallel pair of arrows in C. The *coequalizer* of f and g is object Q and arrow $q : Y \to Q$ such that $q \circ f = q \circ g$, with the property that for any other object Z and arrow $h : Y \to Z$ such that $h \circ f = h \circ g$, there exists a unique arrow $k : Q \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "h"'] & Q \\ & & Z. \arrow[u, dashed, "k"'] \end{tikzcd}$$

Example 2.3.6. Given a set X and an equivalence relation $R \subseteq X \times X$, the set of equivalence classes X/R with the quotient map $q : X \twoheadrightarrow X/R$ is the coequalizer of the projections $\pi, \pi' : R \rightrightarrows X$.

2.4 Filtered Colimits

We will use a specific type of diagram to take the colimit of in order to construct ultraproducts in a categorical manner. The resulting colimit is known as a *filtered colomit* and is essentially the colimit of a product in which the colimit is taking over a directed set.

Definition 2.4.1. A category J is said to be *filtered* if every finite diagram has a cocone.

A filtered category is to be thought of as a categorification of a directed set. This can be easily seen by the following equivalent definition: A category C is filtered if it satisfies the following:

(i) There exists an object of C.

(ii) For every pair of objects $X_0, X_1 \in \mathsf{C}_0$, there exists an object $Y \in \mathsf{C}_0$ and a pair of morphisms

$$\begin{tikzcd} & Y & \\ X_0 \arrow[ur] & & X_1. \arrow[ul] \end{tikzcd}$$

(iii) For every parallel pair of morphism $f, g : X_0 \rightrightarrows X_1$ in C_1, there exists a morphism $h : X_1 \to X_2$ such that $h \circ f = h \circ g$, that is, the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_1 \arrow[r, "h"] & X_2. \end{tikzcd}$$

Definition 2.4.2. A diagram $F : \mathsf{J} \to \mathsf{C}$ is called a *filtered diagram* if J is filtered. A colimit of a filtered diagram is called a *filtered colimit*.

3 Regular, Coherent, Geometric Categories and Pretoposes

Definition 3.0.1. Let C be a category and let $X \in \mathsf{C}_0$. A *subobject* of X is an equivalence class of monomorphisms $i_0 : X_0 \rightarrowtail X$, where, if $i_1 : X_1 \rightarrowtail X$ is another subobject of X, $i_0 \sim i_1$ if and only there exists an isomorphism $e : X_0 \cong X_1$ such that $i_0 = i_1 \circ e$, that is, the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[rr, "e"] \arrow[dr, tail, "i_0"'] & & X_1 \arrow[dl, tail, "i_1"] \\ & X. & \end{tikzcd}$$

We will usually refer to a subobject of X by its domain and assume that he monomorphism is supplied. We denote by $\mathrm{Sub}(X)$ the set of subobjects of X.

We can define an partial order on $\mathrm{Sub}(X)$ ($X \in \mathsf{C}_0$) by, for $X_0, X_1 \in \mathrm{Sub}(X)$, $X_0 \leq X_1$ if and only if there is a unique morphism $e : X_0 \to X_1$ such that the diagram in Definition 3.0.1 commutes.

Definition 3.0.2. Let C be a category which admits finite limits and let X be an object of C. We say that a subobject $R \subseteq X \times X$ is an *equivalence relation* on X if, for every object $Y \in \mathsf{C}$, the image of the induced map

$$\mathrm{Hom}_{\mathsf{C}}(Y, R) \to \mathrm{Hom}_{\mathsf{C}}(Y, X \times X) \cong \mathrm{Hom}_{\mathsf{C}}(Y, X) \times \mathrm{Hom}_{\mathsf{C}}(Y, X)$$

is a equivalence relation on the set $\mathrm{Hom}_{\mathsf{C}}(Y, X)$, that is, if $a, b : R \rightrightarrows X$ is a parallel pair of morphisms of C, we have that:

(i) $(a, b) : R \to X \times X$ is mono.

(ii) The diagonal subobject $\Delta : X \rightarrowtail X \times X$ factors through (a, b) [i.e. (a, b) is *reflective*].

(iii) There exists a morphism $\tau : R \to R$ such that $b \circ \tau = a$ and $a \circ \tau = b$ [i.e. (a, b) is *symmetric*].

(iv) If

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "p"'] & R \arrow[d, "a"] \\ R \arrow[r, "b"'] & X \end{tikzcd}$$

is a pullback, then $(a \circ p, b \circ q) : T \to X \times X$ factors through (a, b) [i.e. (a, b) is *transitive*].

Definition 3.0.3. Let C be a category which admits fiber products, and suppose that we are given a morphism $f : X \to Y$ in C. Let $X \times_Y X$ denote the fiber product of X with itself over Y, and let $\pi, \pi' : X \times_Y X \to X$ denote the projection maps onto the two factors. We will say that f is an *effective* (or *regular*) *epimorphism* if it exhibits Y as a coequalizer of the maps $\pi.\pi' : X \times_Y X \rightrightarrows X$. In other words, f is an effective epimorphism if, for every object $Z \in \mathsf{C}$, composition with f induces a bijection

$$\mathrm{Hom}_{\mathsf{C}}(Y, Z) \cong \left\{ u \in \mathrm{Hom}_{\mathsf{C}}(X, Z) : u \circ \pi = u \circ \pi' \right\}.$$

Definition 3.0.4. Let C be a category. We say that C is a *regular category* if the following hold:

1. The category C has finite limits.

2. Every morphism $f : X \to Z$ in C can be written as a composition $X \overset{g}{\twoheadrightarrow} Y \overset{h}{\rightarrowtail} Z$, where g is an effective epimorphism and h is a monomorphism.

3. The collection of effective epimorphisms in C is closed under pullbacks, that is, if

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a pullback square in C where f is an effective epimorphism, then f' is also an effective epimorphism.

Definition 3.0.5. Let C and D be regular categories. A *regular functor* between C and D is a functor $F : \mathsf{C} \to \mathsf{D}$ between regular categories satisfying the following axioms:

(i) The functor F preserves finite limits.

(ii) The functor F carries effective epimorphism to effective epimorphisms.

Example 3.0.6. Examples of regular categories include

1. Set

Definition 3.0.7. A *coherent category* C is a regular category satisfying the additional axioms:

(i) For every object $X \in \mathsf{C}_0$, the poset $\mathrm{Sub}(X)$ is an upper semilattice, that is, it has a least element and for every pair of subobjects $X_0, X_1 \in \mathrm{Sub}(X)$ we have a least upper bound (join, union) $X_0 \vee X_1 \in \mathrm{Sub}(X)$.

(ii) For every morphism $f : X \to Y$ in C, the inverse map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a morphism of upper semilattices.

We actually have that in a coherent category C that, for each $X \in \mathsf{C}_0$, that $\mathrm{Sub}(X)$ is a distributive lattice.

Definition 3.0.8. Let C and D be coherent categories. A *coherent functor* between C and D is a regular functor $F : \mathsf{C} \to \mathsf{D}$ between coherent categories that satisfies the additional axiom:

(i) For every $X \in \mathsf{C}_0$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a morphism of upper semilattices.

Definition 3.0.9. Let C be a category which admits finite limits and let X be an object of C. We will say that an equivalence relation R on X is *effective* if there exists an effective epimorphism $f : X \twoheadrightarrow Y$ such that $R = X \times_Y X$ (as subobjects of $X \times X$).

Definition 3.0.10. Let C be a category which admits fiber products, and let $X, Y \in \mathsf{C}$ be objects which admit a coproduct $X \amalg Y$. We will say that $X \amalg Y$ is a *disjoint coproduct* of X and Y if the following pair of conditions is satisfied:

- Each of the maps $X \to (X \amalg Y) \leftarrow Y$ is a monomorphism.
- The fiber product $X \times_{X \amalg Y} Y$ is an initial object of C.

Definition 3.0.11. Let C be a category. We say that C is a *pretopos* if it satisfies the following axioms:

1. The category C admits finite limits.

2. Every equivalence relation if effective.

3. The category C admits finite coproducts, and coproducts are disjoint.

4. The collection of effective epimorphisms in C is closed under pullbacks. That is, if we are given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in C where f is an effective epimorphism, the morphism f' is also an effective epimorphism.

5. The formation of finite coproducts in C is preserved by pullbacks. More precisely, for every morphism $f : X \to Y$ in C, the pullback functor

$$f^* : \mathsf{C}_{/Y} \to \mathsf{C}_{/X} :: f^*(U) = U \times_Y X$$

preserves finite coproducts.

The Set is a pretopos.

Definition 3.0.12. Let C and D be pretoposes. We will say that a functor $F : \mathsf{C} \to \mathsf{D}$ is a *pretopos functor* if F preserves finite limits, finite coproducts, and carries effective epimorphisms in C to effective epimorphisms in D.

We let $\mathrm{Fun}^{\mathrm{Pretop}}(\mathsf{C}, \mathsf{D})$ denote the full subcategory of $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ spanned by the pretopos functors.

Definition 3.0.13. If C is a pretopos, then a *model of* C is a pretopos functor $M : \mathsf{C} \to \mathsf{Set}$. We let $\mathsf{Mod}(\mathsf{C})$ denote the full subcategory of $\mathrm{Fun}(\mathsf{C}, \mathsf{Set})$ spanned by the models of C.

4 Logic

In this section, we will provide all of the necessary definitions from mathematical logic that we will use. We will present these definitions in a general setting by defining our signature to be *many-sorted*. We will also consider *fragments* of first-order logic, that is classes of formulae, namely regular logic, coherent logic, and geometric logic.

Definition 4.0.1. A (first-order) *signature* Σ consists of the following data:

1. A set Σ-Sort of *sorts*.

2. A set Σ-Rel of *relation symbols*, together with a map assigning to each $R \in \Sigma$-Rel its *type*, which consists of a finite list of sorts. We write $R \rightarrowtail A_1 \times \cdots \times A_n$ to indicate that R has type $A_1, \ldots, A_n$, where $n > 0$ is the *arity* of R.

3. A set Σ-Fun of *function symbols*, together with a map assigning to each $f \in \Sigma$-Fun its *type*, which consists of a finite non-empty lists of sorts (with the last sort in the list having a distinguished status). We write $f : A_1 \times \cdots \times A_n \to B$ to indicate that f has type $A_1, \ldots, A_n, B$, where $n \geq 0$ is the *arity* of f. In the case that $n = 0$, we call f a *constant* of sort B.

Many, but not all, of the signatures we encounter in mathematics are single-sorted, that is Σ-Sort $= \{A\}$ and the relation and function symbols all have type A. This does not have any affect on the arity of these symbols, as each $A_i = A$ $(i \leq n)$.

Along with a signature Σ, we assume we have for each sort A of Σ a collection of *variables* of sort A. We will denote that x is a variable of sort A by the notation $x : A$.

Next we define the terms over a signature as follows:

Definition 4.0.2. The collection of *terms* over Σ is defined recursively by the following clauses below; simultaneously, we define the *sort* of each term and write $t : A$ to denote that t is a term of A.

1. Variables x of sort A are terms.

2. If $f : A_1 \times \cdots A_n \to B$ is a function symbol and $t_1 : A_1, \ldots, t_n : A_n$, then $f(t_1, \ldots, t_n) : B$. In the case that f is a constant, we will write f rather than $f()$ for the term obtained from the empty string of terms.

Next we will define the formulae over a signature Σ. The following definition is most likely more complicated than we are used to seeing in other texts of first-order logic. We present the definition this was as it is easier to then define the fragments of logic that we will be working with throughout the paper.

Definition 4.0.3. We define a class F of *formulae* over a signature Σ together with, for each formula ϕ, the (finite) set $FV(\phi)$ of *free variables* of ϕ recursively as follows:

(i) *Relations*: if $R \subseteq A_1 \times \cdots A_n$ is a relation symbol and $t_1 : A_1, \ldots, t_n : A_n$ are terms, then $R(t_1, \ldots, t_n)$ is in F. The free variables of this formula are all the variables occurring in some t_i.

(ii) *Equality*: if $s, t : A$, then $(s =_A t)$ is in F (we will drop the subscript when there is no confusion as to which sort the equality is taking place in). The free variables $FV(s =_A t)$ is the set of variables occurring in s or t or both.

(iii) *Truth*: $\top$ is in F and $FV(\top) = \emptyset$.

(iv) *Binary conjunction*: if ϕ and ψ are in F then $(\phi \wedge \psi)$ is in F and $FV(\phi \wedge \psi) = FV(\phi) \cup FV(\psi)$.

(v) *Falsity*: $\bot$ is in F and $FV(\bot) = \emptyset$.

(vi) *Binary disjunction*: if ϕ and ψ are in F then $(\phi \vee \psi)$ is in F and $FV(\phi \vee \psi) = FV(\phi) \cup FV(\psi)$.

(vii) *Negation*: if ϕ is in F then $(\neg\phi)$ is in F and $FV(\neg\phi) = FV(\phi)$.

(viii) *Implication*: if ϕ and ψ are in F then $(\phi \Rightarrow \psi)$ is in F and $FV(\phi \Rightarrow \psi) = FV(\phi) \cup FV(\psi)$.

(ix) *Existential quantification*: if ϕ is in F and $x : A$ is a variable of sort A then $(\exists x : A)\phi$ $((\exists x)\phi$ if there is no confusion of the sort) is in F and $FV((\exists x : A)\phi) = FV(\phi) \setminus \{x\}$.

(x) *Universal quantification*: if ϕ is in F and $x : A$ is a variable of sort A then $(\forall x : A)\phi$ (or $(\forall x)\phi$) is in F and $FV((\forall x : A)\phi) = FV(\phi) \setminus \{x\}$.

(xi) *Infinitary disjunction*: if for each $i \in I$ (I an index set) ϕ_i is in F and $FV(\bigvee_{i\in I} \phi_i) = \bigcup_{i\in I} FV(\phi_i)$ is finite, then $\bigvee_{i\in I} \phi_i$ is in F.

(xii) *Infinitary conjunction*: if for each $i \in I$ (I an index set) ϕ_i is in F and $FV(\bigwedge_{i\in I} \phi_i) = \bigcup_{i\in I} FV(\phi_i)$ is finite, then $\bigwedge_{i\in I} \phi_i$ is in F.

Now that we have defined what it means to be a formula, we can now describe some fragments of logic that will will make use of.

Definition 4.0.4. Let Σ be a signature. We define the following *fragments* of logic from the formulas defined in Definition 4.0.3 as follows:

(a) The set F^{at} of *atomic formulae* over Σ is the smallest set closed under the formulae of (i) and (ii).

(b) The set F^{reg} of *regular formulae* over Σ is the smallest set closed under the formulae of (i)-(iv) and (ix).

(c) The set F^{coh} of *coherent formulae* over Σ is the smallest set closed under the formulae of (i)-(vi) and (ix).

(d) The set F^{fo} of *first-order formulae* over Σ is the smallest set closed under the formulae of (i)-(x). Note that since we have negation (vii), we can use a smaller set of formulas, e.g. (i)-(iv), (viii), and (ix), and use the usual abbreviations.

(e) The class F^{geom} of *geometric formulae* over Σ is the smallest class closed under the formulae of (i)-(vi), (ix) and (xi).

(f) The class $F^{\text{inf-fo}}$ of *infinitary first-order formulae* over Σ is the smallest class closed under the formulae of (i)-(xii).

Notice that we have the following inclusions of the fragments listed above:

$$F^{\text{at}} \subseteq F^{\text{reg}} \subseteq F^{\text{coh}} \subseteq F^{\text{fo}} \subseteq F^{\text{inf-fo}} \text{ and } F^{\text{at}} \subseteq F^{\text{reg}} \subseteq F^{\text{coh}} \subseteq F^{\text{geom}} \subseteq F^{\text{inf-fo}}$$

We will see later how the fragment F^{reg} (resp. F^{coh}, F^{geom}) are related to regular (resp. coherent, geometric) categories (the fragment F^{fo} will be related to Heyting categories).

We call a variable $x : A$ in a formula ϕ *bound* if it is within the scope of come quantifier $(\exists x : A)$ or $(\forall x : A)$. A bound variable is not free, so in the notation of the previous sentence, $x \notin FV(\phi)$. We say a formula ϕ is a sentence if $FV(\phi) = \emptyset$, that is, every variable that occurs in ϕ is bound by some quantifier.

We will usually denote a formula by $\phi(\vec{x})$, where $\vec{x} = x_1, \ldots, x_n$ $(x_i : A_i)$ and for each $i \leq n$ we have $x_i \in FV(\phi(\vec{x}))$. We denote terms in a similar manner.

One of the most important topic in mathematical logic is the idea of *semantics*. In classical model theory, the structures that we encounter are sets that interpret a formula $\phi(\vec{x})$ in such a way to say what it means for $\phi(\vec{x})$ to be "true" in said structure. As we will soon see, classical model theory takes place in the category Set with our structures being Set-valued. We can extend this idea of interpretation to be in any category that has finite products.

Definition 4.0.5. Let C be a category with finite products and let Σ be a signature. A Σ-*structure* M *in* C is a function specified by the following data:

(i) For each sort $A \in \Sigma\text{-Sort}$, we have $M(A) \in \mathsf{C}_0$. If $A_1, \ldots, A_n \in \Sigma\text{-Sort}$, we have

$$M(A_1, \ldots, A_n) = M(A_1) \times \cdots \times M(A_n).$$

In the case that $n = 0$, we have that $M()$ is the terminal object of C.

(ii) For each $R \in \Sigma\text{-Rel}$ with $R \rightarrowtail A_1 \times \cdots \times A_n$, we have

$$M(R) \rightarrowtail M(A_1, \ldots, A_n)$$

(as a subobject) in C.

(iii) For each $f \in \Sigma\text{-Fun}$ with $f : A_1 \times \cdots \times A_n \to B$, we have

$$M(f) : M(A_1) \times \cdots \times M(A_n) \to M(B)$$

is an arrow in C.

When $\mathsf{C} = \mathsf{Set}$, when we have the classical definition of a Σ-structure, that is, a Σ-structure M in Set is a function such that

(i) If $A_1, \ldots, A_n \in \Sigma\text{-Sort}$, then $M(A_1, \ldots, A_n) = M(A_1) \times \cdots \times M(A_n)$ is a Cartesian product of sets and $M() = \emptyset$.

(ii) If $R \subseteq A_1 \times \cdots \times A_n$ then $M(R) \subseteq M(A_1, \ldots, A_n)$.

(iii) If $f : A_1 \times \cdots \times A_n \to B$, then $M(f) : M(A_1, \ldots, A_n) \to M(B)$ is a set function.

Just like almost every mathematical structure, we can define morphisms between Σ-structures.

Definition 4.0.6. Let Σ be a signature and let M and N be Σ-structures in a category C (with finite products). Then a Σ-*structure homomorphism* $h : M \to N$ is specified by a collection of morphisms $\{h_A : M(A) \to n(A)\}_{A \in \Sigma\text{-Sort}}$ in C satisfying the following conditions:

(i) For each $R \in \Sigma\text{-Rel}$ with $R \rightarrowtail A_1 \times \cdots \times A_n$, there is a commutative diagram in C of the form

$$\begin{tikzcd} M(R) \arrow[r, tail] \arrow[d] & M(A_1, \ldots, A_n) \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"] \\ N(R) \arrow[r, tail] & N(A_1, \ldots, A_n). \end{tikzcd}$$

(ii) For each $f \in \Sigma\text{-Fun}$ with $f : A_1 \times \cdots \times A_n \to B$, the following diagram commutes:

$$\begin{tikzcd} M(A_1, \ldots, A_n) \arrow[r, "M(f)"] \arrow[d, "h_{A_1} \times \cdots \times h_{A_n}"'] & M(B) \arrow[d, "h_B"] \\ N(A_1, \ldots, A_n) \arrow[r, "N(f)"] & N(B). \end{tikzcd}$$

If we have two Σ-structure homomorphisms $h : M \to N$ and $k : N \to L$ with collections $\{h_A : M(A) \to N(A)\}_{A \in \Sigma\text{-Sort}}$ and $\{k_A : N(A) \to L(A)\}_{A \in \Sigma\text{-Sort}}$ satisfying the conditions of Definition 4.0.6 respectively, then the composition $k \circ h : M \to L$ is specified by the collection $\{(k \circ h)_A : M(A) \to L(A)\}_{A \in \Sigma\text{-Sort}}$, where $(k \circ h)_A = k_A \circ h_A$. Also, the identity map $\mathrm{id}_M : M \to M$ is specified by the collection $\{(\mathrm{id}_M)_A : M(A) \to M(A)\}_{A \in \Sigma\text{-Sort}}$ It follows that the Σ-structures and their homomorphisms in a category C with finite products forms a category Σ -Str (C). The reader is invited to check that the morphisms of Σ -Str (C) satisfy the axioms of Definition 2.1.1.

Let Σ be a signature and consider two categories C and D, each of which having finite products. If $F : \mathsf{C} \to \mathsf{D}$ is a functor that preserves finite products and monomorphisms, then there is a functor $\Sigma\text{-}\mathsf{Str}(F) : \Sigma\text{-}\mathsf{Str}(\mathsf{C}) \to \Sigma\text{-}\mathsf{Str}(\mathsf{D})$ with, for each $A \in \Sigma$-Sort, $\Sigma\text{-}\mathsf{Str}(F)(M)(A) = F(M(A))$ (relation and function symbols being defined similarly) and each homomorphism $h : M \to N$ being specified by the collection $\{F(h_A) : F(M(A)) \to F(N(A))\}$, where $(\Sigma\text{-}\mathsf{Str}(F)(h))_A = F(h_A)$. Let $G : \mathsf{C} \to \mathsf{D}$ is another functor preserving finite products and monomorphisms. If $\alpha : F \Rightarrow G$ is a natural transformation, then α induces a natural transformation $\Sigma\text{-}\mathsf{Str}(F) \Rightarrow \Sigma\text{-}\mathsf{Str}(G)$.

Next we will define what it means to interpret terms and formulae in a Σ-structure.

Definition 4.0.7. Let M be a Σ-structure in a category C with (at least) finite products.

1. Let $t : B$ be a term and $\vec{x} = x_1, \ldots, x_n$ ($x_i : A_i$ for $i \le n$). Then the *interpretation of* $t(\vec{x})$ *in* M is a morphism
$$[\![t(\vec{x})]\!]_M : M(A_1, \ldots, A_n) \to M(B)$$
in C which is defined recursively as follows:
   (a) If t is a variable, then $t := x_i$ for a unique $i \le n$ and we have that $[\![t(\vec{x})]\!]_M = \pi_i$ is the ith projection map.
   (b) If $t := f(t_1, \ldots, t_m)$ with $t_i : C_i$ for $i \le n$, then $[\![t(\vec{x})]\!]_M$ is the composite map
$$M(A_1, \ldots, A_n) \xrightarrow{([\![t_1(\vec{x})]\!]_M, \ldots, [\![t_m(\vec{x})]\!]_M)} M(C_1, \ldots, C_m) \xrightarrow{M(f)} M(B).$$

2. Let $\phi(\vec{x})$ be a formula with free variables in $\vec{x} = x_1, \ldots, x_n$ ($x_i : A_i$). Then the *interpretation of* $\phi(\vec{x})$ *in* M is a subobject
$$[\![\phi(\vec{x})]\!]_M \rightarrowtail M(A_1, \ldots, A_n)$$
which is defined recursively as follows:
   (a) If $\phi : R(t_1, \ldots, t_m)$, where $R \rightarrowtail B_1 \times \cdots \times B_m$ is a relation symbol, them $[\![\phi(\vec{x})]\!]_M$ is the pullback
$$\begin{tikzcd} {[\![\phi(\vec{x})]\!]_M} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & M(R) \arrow[d, tail] \\ M(A_1, \ldots, A_n) \arrow[r, "{([\![t_1(\vec{x})]\!]_M, \ldots, [\![t_m(\vec{x})]\!]_M)}"] & M(B_1, \ldots, B_m). \end{tikzcd}$$
   (b) If $\phi := (s = t)$, where $s, t : B$ are terms, then $[\![\phi(\vec{x})]\!]_M$ is the equalizer of
$$\begin{tikzcd} M(A_1, \ldots, A_n) \arrow[r, shift left, "{[\![s(\vec{x})]\!]_M}"] \arrow[r, shift right, "{[\![t(\vec{x})]\!]_M}"'] & M(B). \end{tikzcd}$$
   (c) If $\phi := \top$, then $[\![\phi(\vec{x})]\!]_M$ is the top element of $\mathrm{Sub}(M(A_1, \ldots, A_n))$.

(d) If $\phi := (\psi \wedge \chi)$, then $[\![\phi(\vec{x})]\!]_M$ is the intersection, i.e. the pullback, of

$$\begin{array}{ccc} [\![\phi(\vec{x})]\!]_M & \rightarrowtail & [\![\psi(\vec{x})]\!]_M \\ \downarrow & & \downarrow \\ [\![\chi(\vec{x})]\!]_M & \rightarrowtail & M(A_1, \dots, A_n). \end{array}$$

(e) If $\phi := \bot$ and C is a coherent category, then $[\![\phi(\vec{x})]\!]_M$ is the bottom element of $\mathrm{Sub}(M(A_1, ..., A_n))$.

(f) If $\phi := (\psi \vee \chi)$ and C is a coherent category, then $[\![\phi(\vec{x})]\!]_M$ is the union of the subobjects $[\![\psi(\vec{x})]\!]_M$ and $[\![\chi(\vec{x})]\!]_M$.

(g) If $\phi : (\psi \Rightarrow \chi)$ and C is a Heyting Category, then $[\![\phi(\vec{x})]\!]_M$ is the implication $[\![\psi(\vec{x})]\!]_M \Rightarrow [\![\chi(\vec{x})]\!]_M$ in the Heyting algebra $\mathrm{Sub}(M(A_1, ..., A_n))$ defined in section A1.4.13 of [Joh02].

(h) If $\phi := \neg\psi$ and C is a Heyting category, then $[\![\phi(\vec{x})]\!]_M$ is the Heyting negation $\neg\, [\![\psi(\vec{x})]\!]_M$.

(i) If $\phi := (\exists y : B)\psi$ and C is a regular category, then $[\![\phi(\vec{x})]\!]_M$ is the image of the composition

$$[\![\psi(\vec{x}, y)]\!]_M \rightarrowtail M(A_1, ..., A_n, B) \xrightarrow{\pi} M(A_1, ..., A_n),$$

where π is the projection on the first n factors.

(j) If $\phi := (\forall y : B)\psi$ and C is a Heyting category, then $[\![\phi(\vec{x})]\!]_M$ is $\forall_\pi([\![\psi(\vec{x}, y)]\!]_M)$, where π is the projection on the first n factors.

(k) If $\phi := \bigvee_{i \in I} \psi_i$ and C is a geometric category, then $[\![\phi(\vec{x})]\!]_M$ is the union of the $[\![\psi_i(\vec{x})]\!]_M$ in $\mathrm{Sub}(M(A_1, ..., A_n))$.

(l) If $\phi := \bigwedge_{i \in I} \psi_i$ and C has arbitrary intersections of subobjects, then $[\![\phi(\vec{x})]\!]_M$ is the intersection of the $[\![\psi_i(\vec{x})]\!]_M$.

When $\mathsf{C} = \mathsf{Set}$, we can see that the interpretations defined in Definition 4.0.7 coincides with the classical interpretations. Another consequence of Definition 4.0.7 is that if C is a regular (resp. coherent, Heyting, geometric) category, then we are able to assign interpretations to all regular (resp. coherent, first-order, geometric) formulae over Σ.

We can also discuss the interaction between interpretations of formulae and Σ-structure homomorphisms. Not every homomorphism between Σ-structures preserves the interpretation of a formula. The ones that do will be our main concern.

Definition 4.0.8. Let Σ be a signature.

(a) If $h : M \to N$ is a Σ-structure homomorphism in a Heyting category C, then h is an *elementary morphism* if for each first-order formula $\phi(\vec{x})$ over Σ with free variables in $\vec{x}$ then we have a commutative square

$$\begin{array}{ccc} [\![\phi(\vec{x})]\!]_M & \rightarrowtail & M(A_1, ..., A_n) \\ \downarrow & & \downarrow{\scriptstyle h_{A_1} \times \cdots \times h_{A_n}} \\ [\![\phi(\vec{x})]\!]_N & \rightarrowtail & N(A_1, ..., A_n). \end{array}$$

(b) We call h an *elementary embedding* if the square from (a) is also a pullback for all first-order formulae $\phi(\vec{x})$ with free variables in $\vec{x}$.

(c) We call h an *embedding* if the square from (a) is a pullback for all atomic formulae.

We will say that two formulae ϕ and ψ are *α-equivalent* if they differ only in the names of the bound variables. Suppose $\phi(\vec{x})$ is a formula with free variables in $\vec{x} = x_1, \ldots, x_n$ and let $\vec{s} = s_1, \ldots, s_n$ be a list of terms. Then we denote by $\phi[\vec{s}/\vec{x}]$ to be the formula (well-defined up to α-equivalence) resulting from simultaneously *substituting* s_i for each free occurrence of x_i in ϕ, for all $i \leq n$, after first changing the names of the bound variables in ϕ if necessary to avoid capture of variables in $\vec{s}$ by quantifiers in ϕ. We define substitution for terms similarly.

Substitution Property. *1. Let $t(\vec{y})$ be a term all of whose variables are in $\vec{y}$ (where $t : C$ and $y_i : B_i$) and let $\vec{s} = s_1, \ldots, s_m$ be a string of terms whose type is the same as $\vec{y}$. Suppose for each $i \leq m$, $s_i(\vec{x})$ has all of its variables in $\vec{x}$. Then $[\![t[\vec{s}/\vec{y}](\vec{x})]\!]_M$ is the composite*

$$M(A_1, \ldots, A_n) \xrightarrow{([\![s_1(\vec{x})]\!]_M, \ldots, [\![s_m(\vec{x})]\!]_M)} M(B_1, \ldots, B_m) \xrightarrow{[\![t(\vec{y})]\!]_M} M(C).$$

2. Let $\phi(\vec{y})$ be a formula over Σ interpretable in a category C whose free variables are in $\vec{y} = y_1, \ldots, y_m$, let $\vec{s}$ be a string of terms with the same length and type as $\vec{y}$ such that for each $i \leq m$ $s_i(\vec{x})$ has all of its terms in $\vec{x}$. Then for any Σ-structure M in C, there is a pullback square

$$\begin{tikzcd}
{\llbracket \phi[\vec{s}/\vec{y}](\vec{x}) \rrbracket_M} \arrow[r] \arrow[d, tail] & {\llbracket \phi(\vec{y}) \rrbracket_M} \arrow[d, tail] \\
M(A_1, \ldots, A_n) \arrow[r, "{([\![s_1(\vec{x})]\!]_M, \ldots, [\![s_m(\vec{x})]\!]_M)}"] & M(B_1, \ldots, B_m),
\end{tikzcd}$$

where the A_i and B_j are the sorts of the variables x_i and y_i.

Definition 4.0.9. A *sequent* over a signature Σ is a formal expression of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are formulae over Σ and $\vec{x}$ is an enumeration of the free variables of both, intending to mean that ψ is a *logical consequence* of ϕ in $\vec{x}$. A sequent is *regular* (*coherent*, *first-order*, *geometric*) if both ϕ and ψ are regular (coherent, first-order, geometric) formulae.

Definition 4.0.10. A *theory* over a signature Σ is a set T of sequents over Σ. These sequents are called the *axioms* of T. We say that T is a *regular* (*coherent*, *first-order*, *geometric*) *theory* if all the sequents in T are regular (coherent, first-order, geometric).

Definition 4.0.11. Let M be a Σ-structure in a category C. If $\sigma = (\phi \vdash_{\vec{x}} \psi)$ is a sequent over Σ interpretable in C, we say σ is *satisfied in* M, denoted by $M \vDash \sigma$ if $[\![\phi(\vec{x})]\!]_M \leq [\![\psi(\vec{x})]\!]_M$ in $\mathrm{Sub}(M(A_1, ..., A_n))$.

The functor Σ -Str (F) preserves the satisfaction of sequents.

Theorem 4.0.12. *Let $F : \mathsf{C} \to \mathsf{D}$ be a regular (coherent, Hatting, geometric) functor between categories of the appropriate kind. Let M be a Σ-structure in C and let σ be a sequent over Σ interpretable in C. If $M \vDash \sigma$ in C, then Σ-$\mathsf{Str}(F)(M) \vDash \sigma$ in D.*

Proof. By induction. $\square$

Definition 4.0.13. Let M be a Σ-Structure in a category C. If T is a theory over Σ interpretable in C, we say that M is a *model* of T, denoted $M \vDash T$ if all of the axioms of T are satisfied in M.

It follows that we have a full subcategory of Σ -Str (C), which we denote by $\mathsf{Mod}(T, \mathsf{C})$, whose objects are models and whose morphisms are Σ-structure homomorphisms. We denote by $\mathsf{Mod}(T, \mathsf{C})_e$ the non-full subcategory of Σ -Str (C) whose objects are models and whose morphisms are elementary morphisms. If T is a regular (coherent, first-order, geometric) theory over Σ, then for any regular (coherent, first-order, geometric) functor $F : \mathsf{C} \to \mathsf{D}$, the functor Σ-$\mathsf{Str}(F)$ restricts to a functor $\mathsf{Mod}(T, F) : \mathsf{Mod}(T, \mathsf{C}) \to \mathsf{Mod}(T, \mathsf{D})$.

5 Syntactic Categories

In this section, we will consider the category that bridges the connection between Category Theory and Logic. This category is known as the *syntactic category*, will be constructed using the formulae of the signature of a theory T. As we will see, this syntactic category enjoys much of the nice structure discussed in Section 3, that is, every syntactic category will be a coherent category.

5.1 The Syntactic Category

Definition 5.1.1. Let T be a first-order theory. We can construct the *syntactic category of T*, $\mathsf{Syn}(T)$, as follows:

- The objects of $\mathsf{Syn}(T)$ will be the formulae $\phi(\overline{x})$ of the language of T (which we will denote by $[\phi(\overline{x})]$ to avoid confusion).
- Let $[\phi(\overline{x})]$ and $[\psi(\overline{y})]$ be objects of $\mathsf{Syn}(T)$. Then a morphism from $[\phi(\overline{x})]$ to $[\psi(\overline{y})]$ will be an equivalence class of formulae $\theta(\overline{x},\overline{y})$ satisfying

$$T \vDash (\forall \overline{x},\overline{y}.[\theta(\overline{x},\overline{y}) \Rightarrow \phi(\overline{x}) \wedge \psi(\overline{y})] \wedge \forall \overline{x}.[\phi(\overline{x}) \Rightarrow \exists! \overline{y}.\theta(\overline{x},\overline{y})]),$$

  where two formulae $\theta(\overline{x},\overline{y}), \theta'(\overline{x},\overline{y})$ equivalent if

$$T \vDash \forall \overline{x},\overline{y}.[\theta(\overline{x},\overline{y}) \Leftrightarrow \theta'(\overline{x},\overline{y})].$$

This definition may seem very technical, but we are able to think of $\mathsf{Syn}(T)$ in the following intuitive way:

- The objects of $\mathsf{Syn}(T)$ are the collection of tuples $\overline{x}$ satisfying ϕ.
- The formulae $\theta(\overline{x},\overline{y})$ defining arrows can be thought as T implies that θ defines the graph from $\overline{x}$ satisfying ϕ to $\overline{y}$ satisfying ψ.

Let $[\phi(\overline{x})]$ be an object of $\mathsf{Syn}(T)$ and $\bar{x'}$ be variables not appearing in $\phi(\overline{x})$. Then the morphism

$$[\phi(\overline{x})] \xrightarrow{[\phi(\overline{x}) \wedge (\overline{x} = \bar{x'})]} [\phi(\bar{x'})]$$

is the identity morphism for $[\phi(\overline{x})]$. Let $[\theta(\overline{x},\overline{y})] : [\phi(\overline{x})] \to [\psi(\overline{y})]$ and $[\theta'(\overline{y},\overline{z})] : [\psi(\overline{y})] \to [\gamma(\overline{z})]$ be morphisms in $\mathsf{Syn}(T)$. Then their composition is $[\rho(\overline{x},\overline{z})] = [\theta'(\overline{y},\overline{z})] \circ [\theta(\overline{x},\overline{y})]$, where

$$\rho(\overline{x},\overline{z}) := \exists \overline{y}.[\theta(\overline{x},\overline{y}) \wedge \theta'(\overline{y},\overline{z})].$$

We have that $\mathsf{Syn}(T)$ is a category for every first order theory T, with the unitality and associativity axioms of definition Definition 2.1.1 being easily satisfied. We will wish to state some examples of syntactic categories to have a better understanding of these categories.

Example 5.1.2. 1. The *category of definable sets* $\mathsf{Def}(T)$ has, for each model $M \vDash T$:

- For each formula $\phi(\overline{x})$ of T we have a *definable set* $M[\phi] = \{\overline{a} \in M^n : M \vDash \phi(\overline{a})\}$.
- Each arrow $f : M[\phi] \to M[\psi]$ is *definable* by a formula $\theta(\overline{x},\overline{y})$ with

$$M[\theta] = \left\{(\overline{c},\overline{d}) \in M^{n+m} : M \vDash \phi(\overline{c}) \text{ and } f(\overline{c}) = \overline{d}\right\}.$$

2. Suppose T is a propositional theory (that is, a typed first order theory with no types). Recall that every formula of T is a sentence and that for each pair of sentences ϕ and ψ, we have

$$\mathrm{Hom}_{\mathsf{Syn}(T)}([\phi],[\psi]) = \begin{cases} * & \text{if } T \vDash (\phi \Rightarrow \psi), \\ \emptyset & \text{otherwise} \end{cases}.$$

Then we have that $\mathsf{Syn}(T)$ is equivalent to a poset. Since we have the first-order operations $\vee$ and $\neg$, this poset becomes a Boolean algebra.

3. For each Boolean coherent category C (that is, a coherent category where for each object X, $\mathrm{Sub}(X)$ is a Boolean algebra) there exists a (typed) first-order theory $T(\mathsf{C})$ such that C is equivalent to $\mathsf{Syn}(T(\mathsf{C}))$. The signature $\Sigma(\mathsf{C})$ of the theory $T(\mathsf{C})$ is defined as follow:

   - The types $\hat{X}$ of $T(\mathsf{C})$ are the objects X of C.
   - For any arrow $f : X \to Y$ in C, there is a function symbol $\hat{f} : \hat{X} \to \hat{Y}$.
   - For each subobject $R \rightarrowtail X$, there is a relation symbol $\hat{R} \rightarrowtail \hat{X}$.

   A Σ-structure M (in Set) will be a function that assigns each objects $X \in \mathsf{C}_0$ to a set $M(X)$, each arrow $f : X \to Y$ to a set function $M(\hat{f}) : M(X) \to M(Y)$, and each subobject $R : X_0 \rightarrowtail X$ to a relation $M(\hat{R}) \subseteq M(X)$. We add the following axioms to $T(\mathsf{C})$, which are given by the definition of a category:

   - For every $f : X \to Y$, we have the axiom $\forall x.\exists! y.\hat{f}(x) = y$.
   - If $\mathrm{id}_X : X \to X$ is the identity morphism, then we have the axiom $\forall x.\widehat{\mathrm{id}_X}(x, x)$.
   - If $f : X \to Y$ and $g : Y \to Z$ are a composable pair of arrows, then we have the axiom $\forall x, y, z.\hat{f}(x, y) \wedge \hat{g}(y, z) \Rightarrow \widehat{g \circ f}(x, z)$.

   The next set of axioms, as we will see, ensure that the syntactic category $\mathsf{Syn}(T(\mathsf{C}))$ of $T(\mathsf{C})$ has the structure of a coherent category (in fact, all of the syntactic categories we will be working with will be coherent):

   - If 1 is the terminal object of C and $e : 1$ (e is of type 1), then we have the axiom $\exists! e.e = e$.
   - For every pullback square

     $$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

     in C, we have a axiom $\forall x, y, y'.(\hat{f}(x, y) \wedge \hat{g}(y', y) \Rightarrow \exists! x'.(\widehat{f'}(x', x) \wedge \widehat{g'}(x', y'))$.
   - If $f : X \to Y$ is an effective epimorphism in C, then we have the axiom $\forall y \exists x.\hat{f}(x, y)$.
   - If $X \in \mathsf{C}_0$ and $i_0 : X_0 \rightarrowtail X$ is a monomorphism which exhibits X_0 as the smallest element of $\mathrm{Sub}(X)$, then we have the axiom $\neg\exists x_0.x_0 = x_0$.
   - If X is an object of C which is given as the join of subobjects $f : Y \rightarrowtail X$ and $g : Z \rightarrowtail X$, then we have an axiom $\forall x.(\exists y.\hat{f}(x, y) \wedge \exists z.\hat{g}(x, z))$.

   We can then define a functor $\lambda : \mathsf{C} \to \mathsf{Syn}(T(\mathsf{C}))$ as follows:

   - For each object $X \in \mathsf{C}_0$, we set $\lambda(X) = [x = x]$, where $x : X$ is a variable.

- For each arrow $f : X \to Y$ in C, we let $\lambda(f) : \lambda(X) \to \lambda(Y)$ denote the arrow in $\mathsf{Syn}(T(\mathsf{C}))$ defined by the formula $\hat{f}(x, y)$.

We then have the following bijection:

$$\{\text{Models of } T(\mathsf{C})\} \cong \{\text{Models of } \mathsf{C}\} .$$

It turns out that $\lambda : \mathsf{C} \to \mathsf{Syn}(T(\mathsf{C}))$ is the equivalence of categories as stated at the beginning of this example.

We are able to perform many of the same limits in $\mathsf{Syn}(T)$ that we were able to do in Set. For example, the product of $[\phi(\overline{x})]$ and $[\psi(\overline{y})]$ ($\overline{x}$ and $\overline{y}$ disjoint) is $[(\phi \wedge \psi)(\overline{x}, \overline{y})]$ with arrows $[(\phi \wedge \psi \wedge (\overline{x} = \bar{x'})]$, $[(\phi \wedge \psi \wedge (\overline{x} = \bar{x'})]$. The terminal object in $\mathsf{Syn}(T)$ is the formula that is always true, $[\top]$. The pullback of $[\theta(\overline{x}, \overline{z})]$ and $[\theta'(\overline{y}, \overline{z})]$ is $[\rho(\overline{x}, \overline{y})] := \exists \overline{z}.(\theta(\overline{x}, \overline{z}) \wedge \theta'(\overline{y}, \overline{z}))$. By Theorem 2.2.12, then we can see that $\mathsf{Syn}(T)$ has finite limits. $\mathsf{Syn}(T)$ coherent

5.2 Pretopos Completion

Let us take a look at some examples of theories and their their corresponding syntactic categories.

Example 5.2.1. Consider two theories T and T' defined as follows:

- The language of T has no symbols and T has a single axiom $\exists! x.(x = x)$.
- The language of T' has a single 1-ary relation R and a pair of axioms $\exists! y.R(y)$ and $\exists! z.\neg R(z)$.

We have that (up to isomorphism) T and T' have a single model, $M = \{x\}$ and $M' = \{y, z\}$. So $\mathsf{Mod}(T)$ is equivalent to $\mathsf{Mod}(T')$. But, $\mathsf{Syn}(T)$ is equivalent to the poset $\{0 \leq 1\} = \mathbb{2}$ while $\mathsf{Syn}(T')$ is equivalent to Finset. We will wish to use Makkai-Reyes Conceptual Completeness Theorem to show that if the category of models of two theories are equivalent, then their syntactic categories will be equivalent as well. However, the example above shows that this is not always the case. The reason for the syntactic categories are not equivalent is because $\mathbb{2}$ is a coherent category while Finset is a pretopos. It turns out that the structure of a coherent category is not enough.

We will want our syntactic categories that we will use to have the larger structure of a pretopos. If T is a first-order theory, then we say that T *eliminates imaginaries* if the syntactic category $\mathsf{Syn}(T)$ of T is a pretopos. This is not a coincidence. First, let us recall the construction of an equivalent theory T^{eq} of a theory T which eliminates imaginaries. Let T be a first-order theory and let $X = [\phi(x)]$ be a formula of the signature Σ of T such $FV(\phi) = \{x\}$, where $x : t$. Suppose $R \rightarrowtail X \times X$ is an equivalence relation defined by some formula $\psi(x, x')$. We will enlarge Σ by adding a new type $s = X/R$ and a new binary relation $R' \rightarrowtail t \times s$, where R' is the graph of $X \to X/R$, and we add the following axioms:

$$\forall x, y.R'(x, y) \geq \phi(x)$$

$$\forall x.(\phi(x) \Rightarrow \exists! y.R'(x, y))$$

$$\forall x, x', y, y'.(R'(x, y) \wedge R'(x', y') \Rightarrow (y = y' \Leftrightarrow \psi(x, x')))$$

$$\forall y \exists x.R'(x, y).$$

The result of this construction is a typed first-order theory T^{eq} which eliminates imaginaries and has the same models as T, that is $\mathsf{Mod}(T)$ is equivalent to $\mathsf{Mod}(T^{eq})$ In fact, the elimination of imaginaries is a special case of the more general pretopos completion below:

Theorem 5.2.2. *Given a small coherent category* C*, there exists a small pretopos* C^{eq} *and a coherent functor* $\lambda : \mathsf{C} \to \mathsf{C}^{eq}$ *with the property that, for any other pretopos* D*, composition with* λ *induces a functor*

$$\begin{aligned} \lambda\circ : \mathrm{Fun}^{\mathrm{coh}}(\mathsf{C}^{eq}, \mathsf{D}) &\to \mathrm{Fun}^{\mathrm{coh}}(\mathsf{C}, \mathsf{D}) \\ (F : \mathsf{C}^{eq} \to \mathsf{D}) &\mapsto (\lambda \circ F : \mathsf{C} \to \mathsf{D}) \end{aligned}$$

which is an equivalence of categories. If $\mathsf{D} = \mathsf{Set}$*, then* $\lambda\circ : \mathsf{Mod}(\mathsf{C}^{eq}) \to \mathsf{Mod}(\mathsf{C})$ *is an equivalence of categories.*

In the case that $\mathsf{C} = \mathsf{Syn}(T)$ is a syntactic category for a first-order theory T, then we are able to construct a pretopos, which we denote $\mathsf{Syn}^{eq}(T)$, from $\mathsf{Syn}(T)$ for which $\mathsf{Syn}^{eq}(T)$ eliminates imaginaries and is such that $\mathsf{Mod}(\mathsf{Syn}(T))$ is equivalent to $\mathsf{Mod}(\mathsf{Syn}^{eq}(T))$. Looking back at Example 5.2.1, we can see that $\mathsf{Syn}(T')$ (which is equivalent to Finset) while $\mathsf{Syn}(T)$ (which is equivalent to $\mathbb{2}$) does not. One reason that $\mathbb{2}$ is not a pretopos is that it does not have disjoint coproducts, while Finset does. We may intuitively think of Theorem 5.2.2 as freely adjoining disjoint coproducts and coequalizers of equivalence relations. We omit the proof of Theorem 5.2.2 as we will need the theory of Grothendieck topologies and sheaves (see Lectures 8 - 13 of [Lur18] for an introduction to these concepts and the proof). From this point forward, we will assume our syntactic category eliminates imaginaries, possibly through pretopos completion.

6 Ultraproducts

6.1 Ultrafilters

Definition 6.1.1. A *Boolean algebra* is a partially ordered set $(B, \leq)$ with the following properties:

- Every finite subset of B has a least upper bound. Equivalently, B contains a least element 0 and every pair of elements $x, y \in B$ have a least upper bound $x \vee y$.
- Every finite subset of B has a greatest lower bound. Equivalently, B contains a largest element 1 and every pair of elements $x, y \in B$ have a greatest lower bound $x \wedge y$.
- The distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds.
- Every element $x \in B$ has a *complement* $\overline{x}$, characterized by the identities

$$x \wedge \overline{x} = 0 \quad x \vee \overline{x} = 1.$$

If B and B' are Boolean algebras, a *homomorphism of Boolean algebras* from B to B' is a function $\mu : B \to B'$ satisfying the identities:

$$\begin{aligned} \mu(0) &= 0 & \qquad \mu(x \vee y) &= \mu(x) \vee \mu(y) \\ \mu(1) &= 1 & \qquad \mu(x \wedge y) &= \mu(x) \wedge \mu(y). \end{aligned}$$

Definition 6.1.2. Let S be a set and let $P(S)$ denote the Boolean algebra of all subsets of S. An *ultrafilter on* S is a Boolean algebra homomorphism $\mu : P(S) \to \{0, 1\}$. For each ultrafilter μ on S, we have a subset $\mathcal{U}_\mu \subseteq P(S)$ defined by $\mathcal{U}_\mu = \{S_0 \subseteq S : \mu(S_0) = 1\}$. We can see that $\mathcal{U}_\mu$ is closed under finite intersections and, for each $S_0 \subseteq S$, exactly one of S_0 or $S \setminus S_0$ are in $\mathcal{U}_\mu$.

Example 6.1.3. An important and commonly used ultrafilter is the principal ultrafilter, defined as follows: Let S be a set and let $s \in S$. Then there is an ultrafilter, called the *principal ultrafilter associated to* s, δ_s on S given by the formula

$$\delta_s : P(S) \to \{0,1\} :: \delta_s(S_0) = \begin{cases} 1 & \text{if } s \in S_0 \\ 0 & \text{if } s \notin S_0. \end{cases}$$

We say that an ultrafilter μ on S is *principal* if it is of the form δ_s for some $s \in S$.

Construction 6.1.4. [Composition of Ultrafilters] Let S and T be sets, and let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by the set S. For each ultrafilter μ on S, we let $\int_S \nu_s d\mu$ denote the ultrafilter on T given by the formula

$$\left(\int_S \nu_s d\mu\right)(T_0) = \mu(\{s \in S : \nu_s(T_0) = 1\}).$$

Example 6.1.5. Let S and T be sets and let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by S. Let $s_0 \in S$ and let δ_{s_0} denote the corresponding principal ultrafilter. Then the composite ultrafilter $\int_S \nu_s d\delta_{s_0}$ is equal to ν_{s_0}, since, for all $T_0 \subseteq T$,

$$\begin{aligned} \left(\int_S \nu_s d\delta_{s_0}\right)(T_0) &= \delta_{s_0}(\{s \in S : \nu_s(T_0) = 1\}) \\ &= \begin{cases} 1 & \text{if } s_0 \in \{s \in S : \nu_s(T_0) = 1\} \\ 0 & \text{if } s_0 \notin \{s \in S : \nu_s(T_0) = 1\} \end{cases} \\ &= \begin{cases} 1 & \text{if } \nu_{s_0}(T_0) = 1 \\ 0 & \text{if } \nu_{s_0}(T_0) = 0 \end{cases}. \end{aligned}$$

Construction 1 (Pushforward of Ultrafilter)**.** Let $f : S \to T$ be a map of sets and let μ be an ultrafilter on S. We define an ultrafilter $f_*\mu$ of T by the formula $(f_*\mu)(T_0) = \mu(f^{-1}(T_0))$. We call $f_*\mu$ the *pushforward of* μ *along* f.

We also have that $f_*\mu = \int_S \delta_{f(s)} d\mu$, which follows from

$$\begin{aligned} \left(\int_S \delta_{f(s)} d\mu\right)(T_0) &= \mu(\{s \in S : \delta_{f(s)}(T_0) = 1\}) \\ &= \mu(\{s \in S : f(s) = T_0\}) \\ &= \mu(f^{-1}(T_0)) \\ &= (f_*\mu)(T_0). \end{aligned}$$

If $f : S \to T$ is an injection, then so is f_*:

$$\{\text{Ultrafilters on } S\} \hookrightarrow \{\text{Ultrafilters on } T\},$$

whose image consists of those ultrafilters μ on T satisfying $\mu(f(S)) = 1$.

The composition of ultrafilters yields a nice associativity law: Let T be a set, let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T, let $\{\mu_r\}_{r\in R}$ be a collection of ultrafilters on S, and let λ be an ultrafilter on R. Then

$$\int_R \left(\int_S \nu_s d\mu_r\right) d\lambda = \int_S \nu_s d\left(\int_R \mu_r d\lambda\right).$$

Both sides coincide with the ultrafilter ρ on T given by the formula

$$\rho(T_0) = \lambda(\{r \in R : \mu_r(\{s \in S : \nu_s(T_0) = 1\}) = 1\}).$$

For all $T_0 \subseteq T$,

$$\begin{aligned}\left(\int_R \left(\int_S \nu_s d\mu_r\right) d\lambda\right)(T_0) &= \lambda\left(\left\{r \in R : \left(\int_S \nu_s d\mu_r\right)(T_0) = 1\right\}\right) \\ &= \lambda\left(\{r \in R : \mu_r\left(\{s \in S : \nu_s(T_0) = 1\}\right) = 1\}\right) \\ &= \left(\int_R \mu_r d\lambda\right)(\{s \in S : \nu_s(T_0) = 1\}) \\ &= \left(\int_S \nu_s d\left(\int_R \mu_r d\lambda\right)\right)(T_0).\end{aligned}$$

Proposition 6.1.6. *Let S be a set and let $\mathcal{U}$ be a collection of subsets of S which is closed under finite intersections. If $\emptyset \notin \mathcal{U}$, then there exists an ultrafilter μ on S such that $\mu(S_0) = 1$ for each $S_0 \in \mathcal{U}$.*

The proof of this fact can be found in any introductory book on Model Theory, such as [CK90].

6.2 Ultraproducts

Recall that if μ is an ultrafilter on S, the set $\mathcal{U}_\mu = \{S_0 \subseteq S : \mu(S_0) = 1\}$ is a partially ordered set with respect to inclusion and closed under finite intersections. Then the opposite partially ordered set $\mathcal{U}^{\mathrm{op}}$ is directed, that is, for all $S_0, S_1 \in \mathcal{U}^{\mathrm{op}}$ we have $S_0 \cup S_1 \in \mathcal{U}^{\mathrm{op}}$.

Construction 6.2.1. Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a full subcategory. We say that M *has ultraproducts in* M^+ if the following conditions are satisfied:

- For every collection $\{M_s\}_{s\in S}$ of objects of M indexed by S, there exists a product $\prod_{s\in S} M_s$ in the category M^+.
- For every collection $\{M_s\}_{s\in S}$ of objects of M indexed by S and every ultrafilter μ on S, the diagram

  $$(S_0 \in \mathcal{U}_\mu^{\mathrm{op}}) \mapsto \left(\prod_{s\in S_0} M_s\right)$$

  admits a colimit (in the category M^+) which belongs to the subcategory $\mathsf{M} \subseteq \mathsf{M}^+$. In this case, we denote this colimit by $\int_S M_s d\mu$ and refer to it as the *categorical ultraproduct of* $\{M_s\}_{s\in S}$ *indexed by* μ.

Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a full subcategory which has ultraproducts in M^+. Fix a set S and an ultrafilter μ on S. Then for every collection of objects $\{M_s\}_{s\in S}$ of M, the categorical ultraproduct $\int_S M_s d\mu$ comes equipped with a family of maps

$$q_\mu^{S_0} : \prod_{s\in S_0} M_s \to \int_S M_s d\mu,$$

indexed by those subsets $S_0 \subseteq S$ such that $\mu(S_0) = 1$. These are the maps from the colimit cone from our construction. If we have that $S_0 = S$, the we denote $q_\mu^{S_0}$ by q_μ.

Suppose that we are given a collection $\{f_s : M_s \to N_s\}_{s\in S}$ of morphism of M indexed by S. We will let $\int_S f_s d\mu$ denote the unique morphism from $\int_S M_s d\mu$ to $\int_S N_s d\mu$ in the category M with the property that, for every subset $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$, the diagram

$$\begin{tikzcd}
\prod_{s\in S} M_s \arrow[r, "\prod_{s\in S_0} f_s"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} N_s \arrow[d, "q_\mu^{S_0}"] \\
\int_S M_s d\mu \arrow[r, "\int_S f_s d\mu"] & \int_S N_s d\mu
\end{tikzcd}$$

commutes (in M^+).

We define the *categorical ultraproduct functor* $\int_s (\bullet) d\mu : \mathsf{M}^S \to \mathsf{M}$ given by

$$\{M_s\}_{s\in S} \mapsto \int_S M_s d\mu \quad \{f_s\}_{s\in S} \mapsto \int_S f_s d\mu.$$

This is indeed a functor. For all collections $\{f_s : M_s \to N_s\}_{s\in S}$, $\{g_s : N_s \to P_s\}_{s\in S}$, we have for every $S_0 \subseteq S$ such that $\mu(S_0) = 1$, the two inner squares commute of the left diagram and the right diagram

$$\begin{tikzcd}
\prod_{s\in S} M_s \arrow[r, "\prod_{s\in S_0} f_s"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} N_s \arrow[r, "\prod_{s\in S_0} g_s"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} P_s \arrow[d, "q_\mu^{S_0}"] \\
\int_S M_s d\mu \arrow[r, "\int_S f_s d\mu"] & \int_S N_s d\mu \arrow[r, "\int_S g_s d\mu"] & \int_S P_s d\mu
\end{tikzcd}, \quad
\begin{tikzcd}
\prod_{s\in S} M_s \arrow[r, "\prod_{s\in S_0} (g_s\circ f_s)"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} P_s \arrow[d, "q_\mu^{S_0}"] \\
\int_S M_s d\mu \arrow[r, "\int_S (g_s\circ f_s) d\mu"] & \int_S P_s d\mu
\end{tikzcd}$$

both commute. We have that $\prod_{s\in S_0} g_s \circ \prod_{s\in S_0} f_s = \prod_{s\in S_0}(g_s \circ f_s)$. So the right diagram corresponds to the outer rectangle of the left diagram. By uniqueness, we must have that $\int_S g_s d\mu \circ \int_S f_s d\mu = \int_S (g_s \circ f_s) d\mu$. Now given a collection $\{M_s\}_{s\in S}$ and for every $S_0 \subseteq S$ such that $\mu(S_0) = 1$, the following diagram commutes:

$$\begin{tikzcd}
\prod_{s\in S} M_s \arrow[r, "\prod_{s\in S_0} \mathrm{id}_{M_s}"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} M_s \arrow[d, "q_\mu^{S_0}"] \\
\int_S M_s d\mu \arrow[r, "\int_S \mathrm{id}_{M_s} d\mu"] & \int_S M_s d\mu
\end{tikzcd}$$

Since $\int_S \mathrm{id}_{M_s} d\mu$ is unique, we must have that $\int_S \mathrm{id}_{M_s} d\mu = \mathrm{id}_{\int_S M_s d\mu}$.

Now, for each $S_0 \subseteq S$ such that $\mu(S_0) = 1$, we have that

$$\{M_s\}_{s\in S} \mapsto \left(q_\mu^{S_0} : \prod_{s\in S_0} M_s \to \int_S M_s d\mu \right)$$

is a natural transformation, which follows from above.

Warning 6.2.2. Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a subcategory which has ultraproducts in M^+. For any ultrafilter μ on a set S, the ultraproduct functor $\int_S (\bullet) d\mu : \mathsf{M}^S \to \mathsf{M}$ is given by the formula $\int_S M_s d\mu = \operatorname{colim}_{\mu(S_0)=1} \prod_{s\in S_0} M_s$. These products are formed in M^+ and need not be in M. In particular, the ultraproduct functors $\int_S (\bullet) d\mu : \mathsf{M}^S \to \mathsf{M}$ depend on the structure of the larger category M^+.

Example 6.2.3. Let M be a category which admits small products and small filtered colimits. Then M has ultraproducts in itself.

Example 6.2.4. Let $\{M_s\}_{s\in S}$ be a collection of sets indexed by a set S, let μ be an ultrafilter on S, and let $\int_S M_s d\mu$ be the categorical ultraproduct. Let $q_\mu : \prod_{s\in S} M_s \to \int_S M_s d\mu$ be as defined above. Then we can identify q_μ with a filtered direct limit of projection maps

$$\pi_{S_0} : \prod_{s\in S} M_s \to \prod_{s\in S_0} M_s.$$

If each of the sets M_s is nonempty, then each of the maps π_{S_0} is surjective and therefore q_μ is also surjective. In this case, we can identify $\int_S M_s d\mu$ with the quotient of the Cartesian product $\prod_{s\in S} M_s$ by an equivalence relation $=_\mu$, given by

$$\left(\{x_s\}_{s\in S} =_\mu \{y_s\}_{s\in S}\right) \Leftrightarrow \left(\mu\left(\{s \in S : x_s = y_s\}\right) = 1\right).$$

If one of the M_s's are empty, then q_μ need not be surjective and then $\prod_{s\in S} M_s = \emptyset$, but $\int_S M_s d\mu$ need not be.

Example 6.2.5. Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a full subcategory which has ultraproducts in M^+. Let S be a set containing an element $s_0 \in S$, and let δ_{s_0} be the principal ultrafilter associated to s_0. Then the partially ordered set $\mathcal{U}_{\delta_{s_0}} = \{S_0 \subseteq S : s_0 \in S_0\}$ has a least element $\{s_0\}$. It follows that, for any collection of objects $\{M_s\}_{s\in S}$, we have a canonical isomorphism

$$\epsilon_{S,s_0} : \int_S M_s d\delta_{s_0} \cong \prod_{s\in\{s_0\}} M_s = M_{s_0}.$$

Let $\mathrm{ev}_{s_0} : \mathsf{M}^s \to \mathsf{M}$ denote the evaluation functor $\{M_s\}_{s\in S} \mapsto M_{s_0}$. Then the construction $\{M_s\}_{s\in S} \mapsto \epsilon_{S,s_0}$ determines a natural isomorphism $\epsilon_{S,s_0} : \int_S(\bullet)d\delta_{s_0} \to \mathrm{ev}_{s_0}$ of functors from M^S to M, i.e. For every collection $\{f_s : M_s \to N_s\}_{s\in S}$ of morphisms in M, the following diagram commutes:

$$\begin{array}{ccc}
\int_S M_s d\delta_{s_0} & \xrightarrow{\cong} & \mathrm{ev}_{s_0}\left(\{M_s\}_{s\in S}\right) = M_{s_0} \\
\Big\downarrow{\scriptstyle \int_S f_s d\delta_{s_0}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_{s_0}\left(\{f_s\}_{s\in S}\right) = f_{s_0}} \\
\int_S N_s d\delta_{s_0} & \xrightarrow{\cong} & \mathrm{ev}_{s_0}\left(\{N_s\}_{s\in S}\right) = N_{s_0}
\end{array}$$

Proposition 6.2.6. *Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a full subcategory which has ultraproducts in M^+. Let $\{M_t\}_{t\in T}$ be a collection of objects of M indexed by a set T, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S. Let μ be an ultrafilter on S and let $\int_S \nu_s d\mu$ denote the composite ultrafilter defined above. Then there is a unique morphism*

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d\left(\int_S \nu_s d\mu\right) \to \int_S \left(\int_T M_t d\nu_s\right) d\mu$$

in the category M with the following property:

(*) Let $S_0 \subseteq S$ and $T_0 \subseteq T$ be subsets such that $\mu(S_0) = 1$ and $\nu_s(T_0) = 1$ for each $s \in S_0$ (so that we also have $\left(\int_S \nu_s d\mu\right)(T_0) = 1$). Then the diagram

$$\begin{array}{ccc}
\prod_{t\in T_0} M_t & \xrightarrow{\left\{q_{\nu_s}^{T_0}\right\}_{s\in S_0}} & \prod_{s\in S_0}\left(\int_T M_t d\nu_s\right) \\
\Big\downarrow{\scriptstyle q^{T_0}_{\int_S \nu_s d\mu}} & & \Big\downarrow{\scriptstyle q_\mu^{S_0}} \\
\int_T M_t d\left(\int_S \nu_s d\mu\right) & \xrightarrow{\Delta_{\mu,\nu_\bullet}} & \int_S \left(\int_T M_t d\nu_s\right) d\mu
\end{array}$$

commutes (in the category M^+).

We refer to the map $\Delta_{\mu,\nu_\bullet}$ as the *categorical Fubini transformation*. This depends functorially on $\{M_t\}_{t\in T}$, that is, we can regard $\Delta_{\mu,\nu_\bullet}$ as a natural transformation of functors from M^T to M, fitting into a diagram

$$\begin{tikzcd}
 & \mathsf{M}^S \arrow[dr, "\int_S(\bullet)d\mu"] & \\
\mathsf{M}^T \arrow[ur, "\left\{\int_T(\bullet)d\nu_s\right\}_{s\in S}"] \arrow[rr, "\int_T(\bullet)d\left(\int_S \nu_s d\mu\right)"'] \arrow[rr, phantom, bend left=30, "\Uparrow \Delta_{\mu,\nu_\bullet}"] & & \mathsf{M}
\end{tikzcd}$$

To show that $\Delta_{\mu,\nu_\bullet}$ is a natural transformation, let $\{f_t : M_t \to N_t\}_{t\in T}$ be a collection of morphisms in M indexed by a set T, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, and let μ be an ultrafilter on S. We must show that the following diagram commutes:

$$\begin{tikzcd}
\int_T M_t d\left(\int_S \nu_s d\mu\right) \arrow[r, "\int_T f_t d\left(\int_S \nu_s d\mu\right)"] \arrow[d, "\Delta_{\mu,\nu_\bullet}"] & \int_T N_t d\left(\int_S \nu_s d\mu\right) \arrow[d, "\Delta_{\mu,\nu_\bullet}"] \\
\int_S\left(\int_T M_t d\nu_s\right)d\mu \arrow[r, "\int_S\left(\int_T f_t d\nu_s\right)d\mu"] & \int_S\left(\int_T N_t d\nu_s\right)d\mu
\end{tikzcd}$$

We can extend the diagram above as follows: Let $S_0 \subseteq S$ and $T_0 \subseteq T$ be subsets such that $\mu(S_0) = 1$ and $\nu_s(T_0) = 1$ for each $s \in S_0$. Then by definition of $\int_T f_t d\left(\int_S \nu_s d\mu\right)$ and $\int_S\left(\int_T f_t d\nu_s\right)d\mu$, we have the commuting squares on the top and bottom of the cubical diagram below. Also, by the definition of $\Delta_{\mu,\nu_\bullet}$, we have the commuting squares on the left and right sides of the cube below. The back face commutes by properties of products.

$$\begin{tikzcd}
\prod_{t\in T_0} M_t \arrow[rr, "\prod_{t\in T_0} f_t"] \arrow[dr, "q^{T_0}_{\int_S \nu_s d\mu}"] \arrow[dd, "\left\{q^{T_0}_{\nu_s}\right\}_{s\in S_0}"'] & & \prod_{t\in T_0} N_t \arrow[dr, "q^{T_0}_{\int_S \nu_s d\mu}"] \arrow[dd, "\left\{q^{T_0}_{\nu_s}\right\}_{s\in S_0}"] & \\
 & \int_T M_t d\left(\int_S \nu_s d\mu\right) \arrow[rr, "\int_T f_t d\left(\int_S \nu_s d\mu\right)"] \arrow[dd, "\Delta_{\mu,\nu_\bullet}"] & & \int_T N_t d\left(\int_S \nu_s d\mu\right) \arrow[dd, "\Delta_{\mu,\nu_\bullet}"] \\
\prod_{s\in S_0}\left(\int_T M_t d\nu_s\right) \arrow[rr, "\prod_{s\in S_0}\left(\int_T f_t d\nu_s\right)"] \arrow[dr, "q^{S_0}_\mu"'] & & \prod_{s\in S_0}\left(\int_T N_t d\nu_s\right) \arrow[dr, "q^{S_0}_\mu"] & \\
 & \int_S\left(\int_T M_t d\nu_s\right)d\mu \arrow[rr, "\int_S\left(\int_T f_t d\nu_s\right)d\mu"'] & & \int_S\left(\int_T N_t d\nu_s\right)d\mu
\end{tikzcd}$$

We have that by commutativity

$$\Delta_{\mu,\nu_\bullet} \circ \int_T f_t d\left(\int_S \nu_s d\mu\right) \circ q^{T_0}_{\int_S \nu_s d\mu} = \int_S\left(\int_T f_t d\nu_s\right)d\mu \circ \Delta_{\mu,\nu_\bullet} \circ q^{T_0}_{\int_S \nu d\mu}.$$

Since the maps $\left\{q^{T_0}_{\int_S \nu_s d\mu} : \prod_{t\in T_0} M_t \to \int_T M_t d\left(\int_S \nu_s d\mu\right)\right\}$ exhibit $\int_T M_t d\left(\int_S \nu_s d\mu\right)$ as a colimit of the diagram $\left\{\prod_{t\in T_0} M_t\right\}_{\left(\int_S \nu_s d\mu\right)(T_0)=1}$, by the universality of colimits, we have that

$$\Delta_{\mu,\nu_\bullet} \circ \int_T f_t d\left(\int_S \nu_s d\mu\right) = \int_S\left(\int_T f_t d\nu_s\right)d\mu \circ \Delta_{\mu,\nu_\bullet}$$

and hence $\Delta_{\mu,\nu_\bullet}$ is a natural transformation.

7 Łos Ultraproduct Theorem

We now come to the main theorem of this paper, the Łos Ultraproduct theorem. We will state the Łos Ultraproduct Theorem in the categorical setting and then show how we can recover the theorem in first order logic. First we will need the following lemma, which says that the ultraproduct functor is a pretopos functor.

Lemma 7.0.1. *Let S be a set and let μ be an ultrafilter on S. Then the ultraproduct functor $\int_S(\bullet)d\mu : \mathsf{Set}^S \to \mathsf{Set}$ is a pretopos functor.*

A proof of Lemma 7.0.1 can be found in [Mak85] and [Lur19].

Łos Ultraproduct Theorem. *Let C be a pretopos and let $\{M_s\}_{s\in S}$ be a collection of models of C. For every ultrafilter μ on S, the ultraproduct $\int_S M_s d\mu$ (formed in the category* $\mathrm{Fun}(\mathsf{C}, \mathsf{Set})$*) is a model of* C.

Notice that we can describe the ultraproduct stated in the Łos Ultraproduct Theorem, being a functor from C to Set, as

$$\left(\int_S M_s d\mu\right)(C) = \int_S (M_s(C))d\mu = \underset{\mu(S_0)=1}{\operatorname{colim}} \prod_{s\in S_0} M_s(C).$$

Proof. Let $\{M_s : \mathsf{C} \to \mathsf{Set}\}_{s\in S}$ be a collection of models of C and let μ be an ultrafilter on the index set S. Notice that we can write the ultraproduct $\int_S M_s d\mu : \mathsf{C} \to \mathsf{Set}$ (which is formed in $\mathrm{Fun}(\mathsf{C}, \mathsf{Set})$) of the models of C can be thought of as the following composition

$$\mathsf{C} \xrightarrow{\{M_s\}_{s\in S}} \mathsf{Set}^S \xrightarrow{\int_S(\bullet)d\mu} \mathsf{Set}\ .$$

Since each M_s is a model of C, that is, a pretopos functor, we have that $\{M_s\}_s$ is a pretopos functor. Also, by Lemma 7.0.1, we have the second map is a pretopos functor. Then we have that $\int_S M_s d\mu$ is a pretopos map and hence a model of C. □

Now we can recover the classical first-order logic version of the Łos Ultraproduct Theorem.

Classical Łoś Ultraproduct Theorem. *Let T be a first-order theory. Let $\{M_s\}_{s\in S}$ be a collection of models of T and let μ be an ultrafilter on S. Then the ultraproduct $\int_S M_s d\mu$ is a model of T.*

Proof. Apply Łoś Ultraproduct Theorem to the pretopos completed syntactic category $\mathsf{Syn}^{eq}(T)$. □

8 Ultracategories

8.1 Ultracategories

Definition 8.1.1. Let M be a category. An *ultrastructure* of M consists of the following data:

(1) For every set S and every ultrafilter μ on S, a functor

$$\int_S (\bullet)d\mu : \mathsf{M}^S \to \mathsf{M}.$$

We will denote the value of this functor on an object $\{M_s\}_{s\in S} \in \mathsf{M}^S$ by $\int_S M_s d\mu$, and refer to it as the *ultraproduct of $\{M_s\}_{s\in S}$ with respect to μ.*

(2) For every family of objects $\{M_s\}_{s\in S}$ and every element $s_0 \in S$, an isomorphism

$$\epsilon_{S,s_0} : \int_S M_s d\delta_{s_0} \cong M_{s_0};$$

here δ_{s_0} denotes the principal ultrafilter associated to s_0. We require that, for fixed S and s_0, these isomorphisms depend functorially on $\{M_s\}_{s\in S}$: that is, they determine a natural isomorphism $\epsilon_{S,s_0} : \int_S(\bullet)d\delta_{s_0} \cong \mathrm{ev}_{s_0}$, where $\mathrm{ev}_{s_0} : \mathsf{M}^S \to \mathsf{M}$ denotes the evaluation function $\{M_s\}_{s\in S} \mapsto M_{s_0}$.

(3) For every family of objects $\{M_s\}_{t\in T}$ indexed by a set T, every family $\nu_\bullet = \{\nu_s\}_{s\in S}$ of ultrafilters on T indexed by a set S, and every ultrafilter μ on S, a morphism

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d\left(\int_S \nu_s d\mu\right) \to \int_S \left(\int_T M_t d\nu_s\right) d\mu$$

which we call the *Fubini transformation*.

For fixed S, T, μ, and $\nu_\bullet$, we require that these morphisms depend functorially on the family $\{M_t\}_{t\in T}$. That is, they determine a natural transformation of functors

$$\Delta_{\mu,\nu_\bullet} : \int_T (\bullet) d\left(\int_S \nu_s d\mu\right) \to \int_T \left(\int_S (\bullet) d\nu_s\right) d\mu$$

of functors from M^T to M, fitting into a diagram

$$\begin{tikzcd}
 & \mathsf{M}^S \arrow[dr, "\int_S(\bullet)d\mu"] & \\
\mathsf{M}^T \arrow[ur, "\left\{\int_T(\bullet)d\nu_s\right\}_{s\in S}"] \arrow[rr, "\int_T(\bullet)d\left(\int_S \nu_s d\mu\right)"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow \Delta_{\mu,\nu_\bullet}"] & & \mathsf{M}
\end{tikzcd}$$

These data are required to satisfy the following axioms:

(A) Let $\{M_t\}_{t\in T}$ be a collection of objects of M indexed by a set T, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, and let δ_{s_0} be the principal ultrafilter on S associated to an element $s_0 \in S$. Then the Fubini transformation

$$\Delta_{\delta_{s_0},\nu_\bullet} : \int_T M_t d\left(\int_S \nu_s d\delta_{s_0}\right) \to \int_S \left(\int_T M_t d\nu_s\right) d\delta_{s_0}$$

is the inverse of the isomorphism

$$\textstyle\int_S \left(\int_T M_t d\nu_s\right) d\delta_{s_0} \xrightarrow{\epsilon_{S,s_0}} \int_T M_t d\nu_{s_0} = \int_T M_t d\left(\int_S \nu_s d\delta_{s_0}\right).$$

(B) Let $\{M_t\}_{t\in T}$ be a collection of objects of M indexed by a set T, let $f : S \to T$ be a monomorphism of sets, and let μ be an ultrafilter on S, so that the pushforward ultrafilter $f_*(\mu)$ is given by $\int_S \delta_{f(s)} d\mu$. Then the Fubini transformation

$$\Delta_{\mu,\delta_{f(s)}} : \int_T M_t d(f_*\mu) \to \int_S \left(\int_T M_t d\delta_{f(s)}\right) d\mu$$

is an isomorphism.

(C) Let $\{M_t\}_{t\in T}$ be a collection of objects of M indexed by a set T, let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, let $\{\mu_r\}_{r\in R}$ be a collection of ultrafilters on S indexed by a set R, and let λ be an ultrafilter on R. Let ρ denote the ultrafilter on T given by $\rho = \int_R \left(\int_S \nu_s d\mu_r\right) d\lambda = \int_S \nu_s d\left(\int_R \mu_r d\lambda\right)$. Then the diagram of Fubini transformations

$$\begin{array}{ccc} \int_T M_t d\rho & \xrightarrow{\Delta_{\lambda, \int_S \nu_s d\mu_\bullet}} & \int_R \left(\int_T M_t d\left(\int_S \nu_s d\mu_r\right)\right) d\lambda \\ \Big\downarrow{\scriptstyle \Delta_{\int_R \mu_r d\lambda, \nu_\bullet}} & & \Big\downarrow{\scriptstyle \int_R \Delta_{\mu_r,\nu_\bullet} d\lambda} \\ \int_S \left(\int_T M_t d\nu_s\right) d\left(\int_R \mu_r d\lambda\right) & \xrightarrow{\Delta_{\lambda,\mu_\bullet}} & \int_R \left(\int_S \left(\int_T M_t d\nu_s\right) d\mu_r\right) d\lambda \end{array}$$

commutes in the category M.

An *ultracategory* is a category M together with an ultrastructure on M.

Definition 8.1.2. Let M be an ultracategory. Suppose we are given a collection of objects $\{M_t\}_{t\in T}$ of M, a map of sets $f : S \to T$, and an ultrafilter μ on S. We let $\Delta_{\mu,f} : \int_T M_t d\left(f_*\mu\right) \to \int_S M_{f(s)} d\mu$ denote the composite map

$$\int_T M_t d\left(f_*\mu\right) = \int_T M_t d\left(\int_S \delta_{f(s)} d\mu\right) \xrightarrow{\Delta_{\mu,\delta_{f(\bullet)}}} \int_S \left(\int_T M_t d\delta_{f(s)}\right) d\mu \xrightarrow{\int_S \epsilon_{S,s_0} d\mu} \int_S M_{f(s)} d\mu.$$

We refer to $\Delta_{\mu,f}$ as the *ultraproduct diagonal map*.

Since the composition of isomorphisms is also an isomorphism, we have that axiom (B) of the ultrastructure definition is equivalent to that requirement that if f is injective then $\Delta_{\mu,f}$ is an isomorphism in M.

Example 8.1.3. Let M be an ultracategory and let μ be an ultrafilter on a set S. For each object $M \in \mathsf{M}$, we let $M^\mu = \int_S M d\mu$ denote the object of M obtained by applying the ultraproduct functor $\int_S (\bullet) d\mu$ to the constant map $S \to \mathsf{M}$ taking the value M. We will refer to M^μ as the *ultrapower of M bu μ.*

Example 8.1.4. Let X be a set and recall we can regard X as a category whose objects are the elements of X and whose only arrows are the identity arrows. We can define an ultrastructure on X as follows:

1. For every map of sets $S \to X$ and every ultrafilter μ on S, we have an element $\int_S f(x) d\mu \in X$.

This is required to satisfy the following:

2. For every map of sets $f : S \to X$ and every element $s_0 \in S$, we have $\int_S f(s) d\delta_{s_0} = f(s_0)$.

3. For every map of sets $f : T \to S$, every family $\nu_\bullet = \{\nu_s\}_{s\in S}$ of ultrafilters on T, and every ultrafilter μ on S, we have an identity

$$\int_T f(t) d\left(\int_S \nu_s d\mu\right) = \int_S \left(\int_T f(t) d\nu_s\right) d\mu.$$

We say X is an *ultraset* if it has an ultrastructure as defined above.

In [Lur19], Lurie showed that there is a bijection

$$\{\text{Ultrastructures on } X\} \cong \{\text{Compact Hausdorff topologies on } X\}.$$

Proposition 8.1.5. *Let M^+ be a category and let $\mathsf{M} \subseteq \mathsf{M}^+$ be a full subcategory which has ultraproducts in M^+. Then the functors $\int_S (\bullet) d\mu : \mathsf{M}^S \to \mathsf{M}$ of Construction 6.2.1, together with the isomorphisms ϵ_{S,s_0} of Example 6.2.5 and the categorical Fubini transformation $\Delta_{\mu,\nu_\bullet}$ of Proposition 6.2.6, determines an ultrastructure on M.*

The proof of Proposition 8.1.5 can be found in [Lur18].

Example 8.1.6. If M is a category with small products and small filtered colimits, then by Proposition 8.1.5, we obtain an ultrastructure on M and M is an ultracategory. We call this ultrastructure the *categorical ultrastructure on* M.

8.2 Ultrafunctors

In this section, we will introduce the notion of an ultrastructure on a functor between ultracategories. We formulate this definition through the notion of a left ultrastructure on the functor. This functor between ultracategories will be called an ultrafunctor.

Definition 8.2.1. Let M and N be ultracategories and let $F : \mathsf{M} \to \mathsf{N}$ be a functor. A *left ultrastructure on F* consists of the following data:

- For every collection of objects $\{M_s\}_{s \in S}$ of M and every ultrafilter μ on S, there is a morphism $\sigma_\mu : F\left(\int_S M_s d\mu\right) \to \int_S F(M_s) d\mu$ in N.

The morphisms $\{\sigma_\mu\}$ are required to satisfy the following axioms:

1. We can regard σ_μ as a natural transformation between $F \circ \int_S (\bullet) d\mu, \int_S (\bullet) d\mu \circ F : \mathsf{M}^S \to \mathsf{N}$:
$$\sigma_\mu : F \circ \int_S (\bullet) d\mu \to \int_S (\bullet) d\mu \circ F.$$
Concretely, for every collection of morphisms $\{f_s : M_s \to M'_s\}_{s \in S}$ in M and every ultrafilter μ on S, we the following diagram commutes in N:
$$\begin{tikzcd} F\left(\int_S M_s d\mu\right) \arrow[r, "\sigma_\mu"] \arrow[d, "F\left(\int_S f_s d\mu\right)"] & \int_S F(M_s) d\mu \arrow[d, "\int_S F(f_s) d\mu"] \\ F\left(\int_S M'_s d\mu\right) \arrow[r, "\sigma_\mu"] & \int_S F(M'_s) d\mu. \end{tikzcd}$$

2. For every collection of objects $\{M_s\}_{s \in S}$ of M and every element of $s_0 \in S$, the following diagram commutes in N:
$$\begin{tikzcd} F\left(\int_S M_s d\delta_{s_0}\right) \arrow[rr, "\sigma_{\delta_{s_0}}"] \arrow[dr, "F(\epsilon_{S,s_0})"'] & & \int_S F(M_s) d\delta_{s_0} \arrow[dl, "\epsilon_{S,s_0}"] \\ & F(M_{s_0}). & \end{tikzcd}$$

3. For every collection of objects $\{M_t\}_{t \in T}$ of M indexed by a set T, every collection of ultrafilters $\nu_\bullet = \{\nu_s\}_{s \in S}$ on T indexed by a set S, and every ultrafilter μ on S, the following diagram commutes in N:
$$\begin{tikzcd} F\left(\int_T M_t d\left(\int_S \nu_s d\mu\right)\right) \arrow[rr, "\sigma_{\int_S \nu_s d\mu}"] \arrow[d, "F(\Delta_{\mu,\nu_\bullet})"'] & & \int_T F(M_t) d\left(\int_S \nu_s d\mu\right) \arrow[d, "\Delta_{\mu,\nu_\bullet}"] \\ F\left(\int_S \left(\int_T M_t d\nu_s\right) d\mu\right) \arrow[r, "\sigma_\mu"] & \int_S F\left(\int_T M_t d\nu_s\right) d\mu \arrow[r, "\int_S \sigma_{\nu_s} d\mu"] & \int_S \left(\int_T F(M_t) d\nu_s\right) d\mu. \end{tikzcd}$$

An *ultrastructure* on F is a left ultrastructure $\{\sigma_\mu\}$ for which each of the maps σ_μ is an isomorphism.

A *left ultrafunctor* form M to N is a pair $(F, \{\sigma_\mu\})$, where $F : \mathsf{M} \to \mathsf{N}$ is a functor and $\{\sigma_\mu\}$ is a left ultrastructure on F. We say $(F, \{\sigma_\mu\})$ is an *ultrafunctor* from M to N if each σ_μ is an isomorphism.

In most cases, we will not need the looser notion of a left ultrafunctor outside of some of the proofs below. There is also a notion of a *right ultrastructure* in which each of the σ_μ above goes in the other direction.

We can define the composition of (left) ultrafunctors as follows: Let M, M', and M'' be ultracategories and let $F : \mathsf{M} \to \mathsf{M}'$ and $F' : \mathsf{M}' \to \mathsf{M}''$ be left ultrafunctors with left ultrastructures $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$ respectively. Then we can define a left ultrastructure on the composite $F' \circ F$ where for each collection of objects $\{M_s\}_{s\in S}$ in M and each ultrafilter μ on S we have the map

$$(F' \circ F)\left(\int_S M_s d\mu\right) \xrightarrow{F'(\sigma_\mu)} F'\left(\int_S F(M_s) d\mu\right) \xrightarrow{\sigma'_\mu} \int_S (F' \circ F)(M_s) d\mu.$$

If we have that $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$ are ultrastructures, then we obtain an ultrastructure on $F' \circ F$.

Definition 8.2.2. Let M and N be ultracategories and let $F, F' : \mathsf{M} \to \mathsf{N}$ be left ultrafunctors with ultrastructures $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$, respectively. We will say that a natural transformation $\alpha : F \Rightarrow F'$ is a *natural transformation of left ultrafunctors* if, for every collection of objects $\{M_s\}_{s\in S}$ of M and every ultrafilter μ on S, the following diagram commutes in N:

$$\begin{tikzcd} F\left(\int_S M_s d\mu\right) \arrow[r, "\sigma_\mu"] \arrow[d, "\alpha\left(\int_S M_s d\mu\right)"] & \int_S F(M_s) d\mu \arrow[d, "\int_S \alpha(M_s) d\mu"] \\ F'\left(\int_S M_s d\mu\right) \arrow[r, "\sigma_\mu"] & \int_S F'(M_s) d\mu. \end{tikzcd}$$

We denote by $\mathrm{Fun}^{\mathrm{LUlt}}(\mathsf{M}, \mathsf{N})$ the category of left ultrafunctors from M to N and natural transformations between left ultrafunctors. We denote by $\mathrm{Fun}^{\mathrm{Ult}}(\mathsf{M}, \mathsf{N})$ the full subcategory of $\mathrm{Fun}^{\mathrm{LUlt}}(\mathsf{M}, \mathsf{N})$ whose objects are the ultrafunctors between M and N.

Proposition 8.2.3. *Let M^+ and N^+ be categories and let $\mathsf{M} \subseteq \mathsf{M}^+$ and $\mathsf{N} \subseteq \mathsf{N}^+$ be categories with ultraproducts in M^+ and N^+, respectively. Let $F^+ : \mathsf{M}^+ \to \mathsf{N}^+$ be a functor which carries M into N which satisfies the following:*

- *For every collection of objects $\{Ms\}_{s\in S}$ of M and every ultrafilter μ on S, the maps*

$$F^+\left(q_\mu^{S_0}\right) : F^+\left(\prod_{s\in S_0} M_s\right) \to F^+\left(\int_S M_s d\mu\right)$$

  exhibit $F^+\left(\int_S M_s d\mu\right)$ as a colimit of the diagram $\left\{F^+\left(\prod_{s\in S_0} M_s\right)\right\}_{\mu(S_0)=1}$ in the category N^+.

Let $F = F^+ \restriction_{\mathsf{M}} : \mathsf{M} \to \mathsf{N}$ and regard M and N as ultracategories with the ultrastructure from Proposition 8.1.5. Then:

1. *For every collection of objects $\{M_s\}_{s\in S}$ of M and every ultrafilter μ on S, there is a unique map $\sigma_\mu : F\left(\int_S M_s d\mu\right) \to \int_S F(M_s) d\mu$ having the property that, for each subset $S_0 \times S$ with*

$\mu(S_0) = 1$, *the following diagram commutes in* N*:*

$$\begin{tikzcd} F^+\left(\prod_{s\in S_0} M_s\right) \arrow[r] \arrow[d, "F^+(q_\mu^{S_0})"'] & \prod_{s\in S_0} F(M_s) \arrow[d, "q_\mu^{S_0}"] \\ F\left(\int_S M_s d\mu\right) \arrow[r, "\sigma_\mu"] & \int_S F(M_s) d\mu. \end{tikzcd}$$

2. *The morphisms* $\{\sigma_\mu\}$ *of (1) determine a left ultrastructure of the functor* F.

3. *If* F^+ *preserves small products, then* $\{\sigma_\mu\}$ *is an ultrastructure on* F.

The proof of Proposition 8.2.3 can be found in [Lur18].

Example 8.2.4. Let M and N be categories with small products and small filtered colimits, and equip M and N with the categorical ultrastructure of Example 8.1.6. Let $F : \mathsf{M} \to \mathsf{N}$ be a functor which preserves small filtered colimits. Then by Proposition 8.2.3, F is a left ultrafunctor. If F also preserves small products, then F is an ultrafunctor.

9 Applications of Ultracategories

In this section, we will discuss some applications of ultracategories to Mathematical Logic. One of these, known as the Makkai Strong Conceptual Completeness Theorem, will allow us to recover the syntax of a first order theory T from using the category of models of T.

Let C be a pretopos. Then by Łoś Ultraproduct Theorem, $\mathsf{Mod}(\mathsf{M})$ has ultraproducts in $\mathrm{Fun}(\mathsf{C}, \mathsf{Set})$ in the sense of Construction 6.2.1. Applying Proposition 8.1.5, we obtain an ultrastructure on $\mathsf{Mod}(\mathsf{C})$. What this implies is that ultracategories are the best category to encode the semantics of a theory T.

Definition 9.0.1. Let C be a pretopos. Then for each object $X \in \mathsf{C}_0$, we have a functor $\mathrm{ev}_X : \mathsf{Mod}(\mathsf{C}) \to \mathsf{Set}$ defined by $\mathrm{ev}_X(M) = M(X)$, that is, evaluation of the model $M : \mathsf{C} \to \mathsf{Set}$ at X. We have ev_X is the restriction to $\mathsf{Mod}(\mathsf{C})$ of a functor $\mathrm{Fun}(\mathsf{C}, \mathsf{Set}) \to \mathsf{Set}$ which preserves small limits and colimits.

By Proposition 8.2.3, the functor ev_X is an ultrafunctor. We can define the *evaluation map* $\mathrm{ev} : \mathsf{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Set})$ by $\mathrm{ev}(X) = \mathrm{ev}_X$.

Makkai Strong Conceptual Completeness Theorem. *Let* C *be a small pretopos. Then the evaluation map* $ev : \mathsf{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathsf{C}, \mathsf{Mod}(\mathsf{C}), \mathsf{Set})$ *is an equivalence of categories.*

If T is a first-order theory, then the Makkai Strong Conceptual Completeness Theorem says that the (pretopos completed) syntactic category $\mathsf{Syn}(T)$ of T is equivalent to the category of ultrafunctors from the category of models $\mathsf{Mod}(T)$ of T to Set. Using this, we are able to recover the syntax of T if all we know is the category of models of T.

Another consequence of Makkai Strong Conceptual Completeness Theorem is that for any pretopos C, $\mathrm{Fun}^{\mathrm{Ult}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Set})$ is a pretopos. We actually have that for any ultracategory M, $\mathrm{Fun}^{\mathrm{Ult}}(\mathsf{M}, \mathsf{Set})$ is a pretopos (see Corollary 29X.6 of [Lur18]).

Let $\lambda : \mathsf{C} \to \mathsf{D}$ be a pretopos functor between pretoposes C and D. Then we have that *precomposition* with λ induces a functor

$$\circ\lambda : \mathsf{Mod}(\mathsf{D}) \to \mathsf{Mod}(\mathsf{D}) :: (F : \mathsf{D} \to \mathsf{Set}) \mapsto (F \circ \lambda : \mathsf{C} \to \mathsf{Set}).$$

Applying Proposition 8.2.3, we have that $\circ\lambda$ is an ultrafunctor between $\mathsf{Mod}(\mathsf{D})$ and $\mathsf{Mod}(\mathsf{C})$. Denote by $\mathrm{Fun}^{\mathrm{Pretop}}(\mathsf{C}, \mathsf{D})$ the category of pretopos functors from C to D. Then we have for every $\lambda \in \mathrm{Fun}^{\mathrm{Pretop}}(\mathsf{C}, \mathsf{D})$, there is the precomposition map $\circ\lambda \in \mathrm{Fun}^{\mathrm{Ult}}(\mathsf{Mod}(\mathsf{D}), \mathsf{Mod}(\mathsf{C}))$.

Makkai Duality. *Let C and D be pretoposes. If C is small, then the construction $\lambda \mapsto \circ\lambda$ induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{Pretop}}(\mathsf{C}, \mathsf{D}) \cong \mathrm{Fun}^{\mathrm{Ult}}(\mathsf{Mod}(\mathsf{D}), \mathsf{Mod}(\mathsf{C})).$$

We can apply the Makkai Duality to obtain the following theorem.

Makkai-Reyes Conceptual Completeness Theorem. *Let C and D be small pretoposes and let $\lambda : \mathsf{C} \to \mathsf{D}$ be a pretopos functor. Then if $\circ\lambda : \mathsf{Mod}(\mathsf{D}) \to \mathsf{Mod}(\mathsf{C})$ is an equivalence of categories, then λ is an equivalence of categories.*

In the context of first-order logic, if we have two theories T and T', then the Makkai-Reyes Conceptual Completeness Theorem says that $\mathsf{Mod}(T)$ and $\mathsf{Mod}(T')$ are equivalent, then $\mathsf{Syn}^{eq}(T)$ and $\mathsf{Syn}^{eq}(T')$ are equivalent as well.

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