

Łoś Ultraproduct Theorem

Anthony Wilkie

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The ultraproduct of models is itself a model.

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- ▷ This has been applied (by Micheal Makkai and more recently Jacob Lurie) to recover the syntax of a theory by using categories with an “ultraproduct” structure.

- ▷ A (single sorted, first-order) *language* \mathcal{L} is a set of symbols divided into three groups:

- ▷ *Relation symbols* R which has a specified arity n ,
- ▷ *Function symbols* f which has a specified arity n ,
- ▷ *Constant symbols* c ,

together with logical symbols $(,), \wedge, \vee, \neg, \exists, \forall, \top, \perp, \Rightarrow, =$ and an infinite collection of variables x_1, x_2, \dots

- ▷ The *terms* over \mathcal{L} are defined inductively as follows:

- ▷ A variable x is a term.
- ▷ If t_1, \dots, t_n are terms and f is a function symbol of \mathcal{L} of arity n , then $f(t_1, \dots, t_n)$ is a term.

- ▷ We write $t(\bar{x})$ to mean t is a term in which the variables $\bar{x} = x_1, \dots, x_n$ appear in t .

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- ▷ The *formulae* ϕ over \mathcal{L} with *free variables* $FV(\phi)$ are defined inductively as follows (Note: the free variables of a term $t(\bar{x})$ is $\bar{x} = x_1, \dots, x_n$):
 - ▷ If $t_1(\bar{x})$ and $t_2(\bar{x})$ are terms, then $t_1(\bar{x}) = t_2(\bar{x})$ is a formula with free variables $FV(t_1 = t_2) = \bar{x}$.
 - ▷ If $t_1(\bar{x}_1), \dots, t_n(\bar{x}_n)$ are terms and R is a relation symbol of arity n , then $R(t_1(\bar{x}_1), \dots, t_n(\bar{x}_n))$ is a formula with $FV(R) = \{\bar{x}_1, \dots, \bar{x}_n\}$.
 - ▷ If $\phi(\bar{x})$ and $\psi(\bar{x})$ are formulae, then $(\phi \vee \psi)(\bar{x})$ is a formula with $FV(\phi \vee \psi) = FV(\phi) \cup FV(\psi)$.
 - ▷ If $\phi(\bar{x})$ is a formula, then so is $\neg\phi(\bar{x})$ with $FV(\neg\phi) = FV(\phi)$.
 - ▷ If $\psi(\bar{x}, y)$ is a formula and y is a variable, then $\exists y.\psi(\bar{x}, y)$ is a formula with with free variables $FV(\psi) \setminus \{y\}$.

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- ▷ A variable x is called *bound* if it is in the scope of a quantifier, i.e. if $\exists x$ or $\forall x$.
- ▷ A formula $\sigma(\bar{x})$ is called a *sentence* if $FV(\sigma) = \emptyset$, that is, all of its variables are bound.
- ▷ We will use the common abbreviations for the following:
 - ▷ $\phi(\bar{x}) \wedge \psi(\bar{x}) \stackrel{\text{def}}{=} \neg((\neg\phi(\bar{x})) \vee (\neg\psi(\bar{x})))$.
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Example of Theory

- ▷ We define the *theory of partially ordered sets (posets)* as follows:
 - ▷ $\mathcal{L} = \{\leq\}$, where \leq is a 2-ary relation symbol.
 - ▷ The theory consists of the following axioms:
 - (i) *Reflexivity*: $\forall x.(x \leq x)$.
 - (ii) *Asymmetry*: $\forall x, y.(x \leq y \wedge y \leq x \Rightarrow x = y)$.
 - (iii) *Transitivity*: $\forall x, y, z.(x \leq y \wedge y \leq z \Rightarrow x \leq z)$.
 - ▷ We obtain the *theory of directed posets* if we add the following axiom:
 - (iv) *Directed*: $\forall x, y.\exists z.(x \leq z \wedge y \leq z)$.
- ▷ We can define the theory of groups, rings, fields, number theory, etc.
- ▷ We CANNOT define the theory of topological spaces.

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- ▷ We can define the theory of groups, rings, fields, number theory, etc.
- ▷ We CANNOT define the theory of topological spaces.

- ▷ We define a \mathcal{L} -*structure* to be a function from our language to a *set* M such that:
 - ▷ For each relation symbol R of arity n , we have an *interpretation* of R as a subset $M(R) \subseteq M^n$.
 - ▷ For each function symbol f of arity n , we have an *interpretation* of f as a function $M(f) : M^n \rightarrow M$.
 - ▷ For each constant symbol c , we have an *interpretation* of c as an element $M(c) \in M$.

- ▷ The *value* of a term $t(\bar{x})$ at $\bar{a} = a_1, \dots, a_n$ in a \mathcal{L} -structure M is defined as follows:
 - ▷ If $t = x_i$ is a variable, then $t(\bar{a}) = a_i$.
 - ▷ If $t = f(t_1, \dots, t_n)$, f a function symbol and t_i are terms, then $t(\bar{a}) = M(f)(t_1(\bar{a}), \dots, t_n(\bar{a}))$.
- ▷ Given a \mathcal{L} -structure M , a formula $\phi(\bar{x})$, and a tuple of element $\bar{a} = a_1, \dots, a_n \in M$, we define M *satisfies* ϕ with \bar{a} , denoted $M \models \phi(\bar{a})$, as follows:
 - ▷ If $\phi(\bar{x}) := t_1(\bar{x}) = t_2(\bar{x})$, then $M \models \phi(\bar{a})$ iff $t_1(\bar{a}) = t_2(\bar{a})$.
 - ▷ If $\phi(\bar{x}) := R(t_1, \dots, t_n)$, then $M \models \phi(\bar{a})$ iff $t_1(\bar{a}), \dots, t_n(\bar{a}) \in M(R)$.
 - ▷ If $\phi(\bar{x}) := (\psi_1 \vee \psi_2)(\bar{x})$, then $M \models \phi(\bar{a})$ iff $M \models \psi_1(\bar{a})$ or $M \models \psi_2(\bar{a})$.
 - ▷ If $\phi(\bar{x}) := (\neg\psi)(\bar{x})$, then $M \models \phi(\bar{a})$ iff not $M \models \psi(\bar{a})$.
 - ▷ If $\phi(\bar{x}) := \exists y. \psi(\bar{x}, y)$, then $M \models \phi(\bar{a})$ iff there exists $b \in M$ such that $M \models \psi(\bar{a}, b)$.

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Models

- ▶ If $T = \{\sigma_i\}_{i \in I}$ is a theory with axioms σ_i , then a \mathcal{L} -structure M is a *model* of T if $M \models T$, that is, $M \models \sigma_i$ for each $\sigma_i \in T$.
- ▶ A poset (P, \leq_P) is a model of the theory of posets with

$$x \leq_P y \Leftrightarrow (x, y) \in \leq_P \quad (\subseteq P \times P).$$

- ▶ Let M and N be models of a theory T . A function $f : M \rightarrow N$ is an *elementary embedding* if for every formula $\phi(\bar{x})$ and $\bar{a} \in M^n$, we have

$$M \models \phi(\bar{a}) \Leftrightarrow N \models \phi(f(\bar{a})).$$

- ▶ An elementary embedding between posets is a monotone function.

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- ▷ If $T = \{\sigma_i\}_{i \in I}$ is a theory with axioms σ_i , then a \mathcal{L} -structure M is a **model** of T if $M \models T$, that is, $M \models \sigma_i$ for each $\sigma_i \in T$.
- ▷ A poset (P, \leq_P) is a model of the theory of posets with

$$x \leq_P y \Leftrightarrow (x, y) \in \leq_P \quad (\subseteq P \times P).$$

- ▷ Let M and N be models of a theory T . A function $f : M \rightarrow N$ is an **elementary embedding** if for every formula $\phi(\bar{x})$ and $\bar{a} \in M^n$, we have

$$M \models \phi(\bar{a}) \Leftrightarrow N \models \phi(f(\bar{a})).$$

- ▷ An elementary embedding between posets is a monotone function.

- ▷ An *ultrafilter* \mathcal{U} on a set S is a subset of the powerset $\mathcal{P}(S)$ such that
 - ▷ $S \in \mathcal{U}$.
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Ultraproducts

- ▷ Given a collection of (nonempty) sets $\{X_s\}_{s \in S}$ and an ultrafilter \mathcal{U} on S , we define an equivalence relation $\sim_{\mathcal{U}}$ on $\prod_{s \in S} X_s$ by

$$(\{x_s\}_{s \in S} \sim_{\mathcal{U}} \{y_s\}_{s \in S}) \Leftrightarrow \{s \in S : x_s = y_s\} \in \mathcal{U}.$$

We let $x^{\mathcal{U}}$ denote the equivalence class of $\{x_s\}_{s \in S}$.

- ▷ Then the *ultraproduct* of $\{X_s\}_{s \in S}$ w.r.t \mathcal{U} , denoted

$$\left(\prod_{s \in S} X_s \right) / \mathcal{U} = \left\{ x^{\mathcal{U}} : \{x_s\}_{s \in S} \in \prod_{s \in S} X_s \right\},$$

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Let $\{M_s\}_{s \in S}$ be a collection of models of a theory T and let \mathcal{U} be an ultrafilter on S . Then the ultraproduct $(\prod_{s \in S} M_s) / \mathcal{U}$ is also a model of T .

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Set Functions

- ▷ We can define functions between sets, i.e. $f : X \rightarrow Y$, where $\text{dom}(f) = X$ and $\text{cod}(f) = Y$.
- ▷ For any three sets X, Y, Z and functions $f : X \rightarrow Y$, $g : Y \rightarrow Z$ between them, we are able to compose f and g to create a new function

$$g \circ f : X \rightarrow Z.$$

- ▷ This composition is *associative*, that is, for any three functions $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and $h : Z \rightarrow W$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- ▷ For any set X , we have the identity function $\text{id}_X : X \rightarrow X$.
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Category

A *category* C consists of the following data:

- ▷ a collection C_0 of *objects*: X, Y, Z, \dots
- ▷ a collection C_1 of *morphisms* (or *arrows*): f, g, h, \dots

such that

- ▷ Each $f \in C_1$ has a *domain* and a *codomain* object, with $f : X \rightarrow Y$ the notation used to denote that $\text{dom}(f) = X$ and $\text{cod}(f) = Y$.
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This data is required to satisfy the following axioms:

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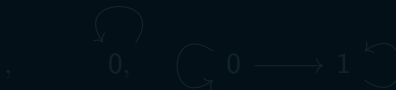
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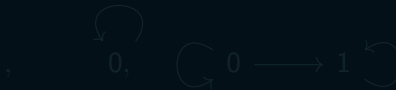
Examples

- ▷ **Set**: the category of sets and set functions.
- ▷ **Top**: the category of topological spaces and continuous functions.
- ▷ **Grp**: the category of groups and group homomorphisms.
- ▷ **Mod(T)**: the category of models of a first order theory T and elementary embeddings.
- ▷ Any set X can be regarded as a category, whose objects are the elements and the only arrows are the identity arrows. This is known as a *discrete category*.
- ▷ Any poset (P, \leq) can be regarded as a category, whose objects are the elements and there exists an arrow $f : x \rightarrow y$ if and only if $x \leq y$. In particular, we have categories $\mathbb{0}$, $\mathbb{1}$, and $\mathbb{2}$ that look as follows:



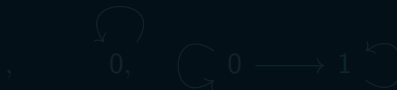
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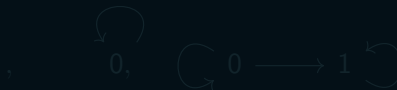
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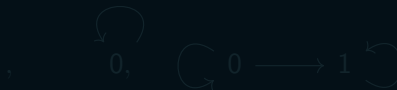
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- ▷ Any poset (P, \leq) can be regarded as a category, whose objects are the elements and there exists an arrow $f : x \rightarrow y$ if and only if $x \leq y$. In particular, we have categories $\mathbb{0}$, $\mathbb{1}$, and $\mathbb{2}$ that look as follows:



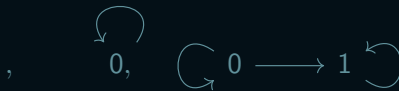
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- ▶ If C is a category, then a *subcategory* D of C consists of subcollections $D_0 \subseteq C_0$ and $D_1 \subseteq C_1$ such that:
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- ▶ Let C and D be categories. A (*covariant*) *functor* $F : C \rightarrow D$ is a morphism consisting of the following data:
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Examples of Functors

- ▶ For each category C we have the *identity functor* $\text{Id}_C : C \rightarrow C$.
- ▶ If $Y \in D_0$, we have the *constant functor* $\hat{Y} : C \rightarrow F$ with $\hat{Y}(X) = Y$ and $\hat{Y}(f) = \text{id}_Y$.
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Natural Transformations

- ▶ Given categories C and D and functors $F, G : C \Rightarrow D$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of:
 - ▶ For each object $X \in C$ an arrow $\alpha_X : F(X) \rightarrow G(X)$ in D , the collection of which define the *components* of α , so that, for any morphism $f : X \rightarrow Y$ in C , TFDC in D

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Syntactic Category

Let T be a first-order theory.

We can construct the *syntactic category of T* , $\text{Syn}(T)$, as follows:

- ▷ The objects of $\text{Syn}(T)$ will be the formulae $\phi(\bar{x})$ of the language of T (which we will denote by $[\phi(\bar{x})]$ to avoid confusion).

Intuitively, the collection of tuples \bar{x} satisfying ϕ .

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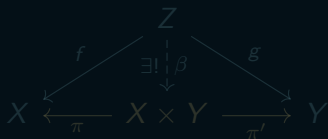
- ▶ For each category \mathcal{C} (with sufficient structure) there exists a (typed) first-order theory $T(\mathcal{C})$ such that \mathcal{C} is equivalent to $\text{Syn}(T(\mathcal{C}))$.

What structure do Set and $\text{Syn}(T)$ have?

Products

Given two sets X, Y , we can form the *Cartesian product* $X \times Y$, which comes with projection arrows $\pi : X \times Y \rightarrow X$, $\pi' : X \times Y \rightarrow Y$.

In general, given X, Y in \mathcal{C} the *product of X and Y* in \mathcal{C} is $(X \times Y, \pi, \pi')$ such that



$$\pi \circ \beta = f \text{ and } \pi' \circ \beta = g$$

In $\text{Syn}(T)$, the product of $[\phi(\bar{x})]$ and $[\psi(\bar{y})]$ (\bar{x} and \bar{y} disjoint) is $[(\phi \wedge \psi)(\bar{x}, \bar{y})]$ with arrows $[(\phi \wedge \psi \wedge (\bar{x} = \bar{x}'))]$, $[(\phi \wedge \psi \wedge (\bar{x} = \bar{x}'))]$.

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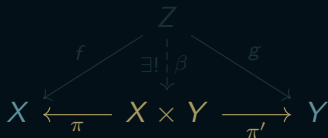
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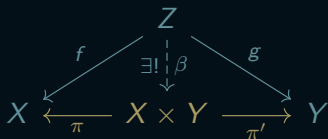
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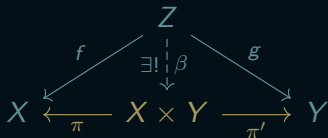
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Terminal Object

We have that any singleton set $\{*\}$ is such that for any other set X , there is a unique arrow

$$1_X : X \rightarrow \{*\} :: 1_X(x) = *.$$

In general, a *terminal object* in \mathcal{C} is an object 1 such that

$$\text{Hom}_{\mathcal{C}}(X, 1) \cong 1$$

In $\text{Syn}(T)$, the terminal object is $[T]$.

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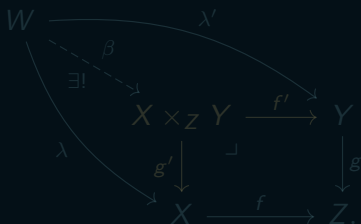
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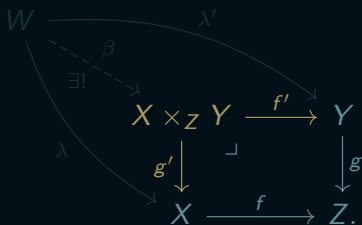
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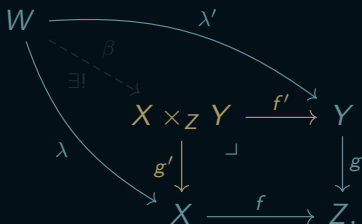
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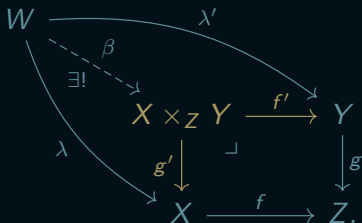
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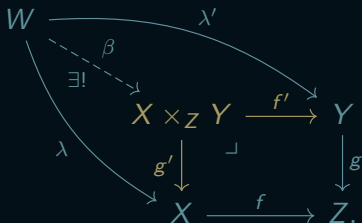
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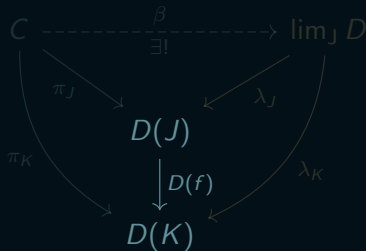
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- ▶ A *diagram* of *shape* J in a category C is a functor $D : J \rightarrow C$.
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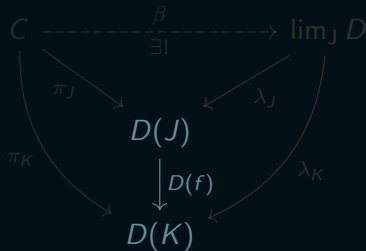
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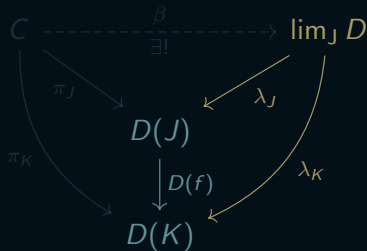
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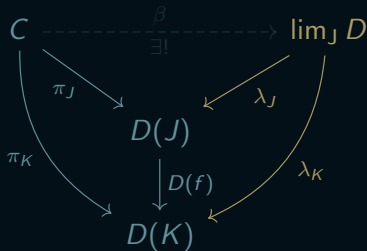
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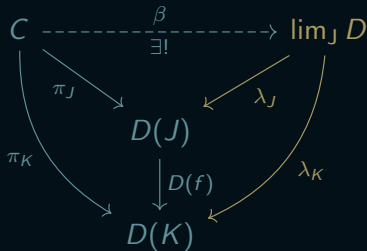
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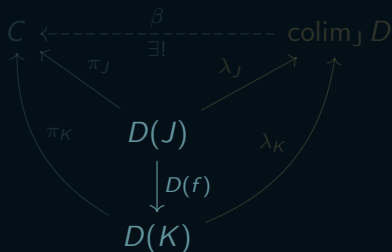
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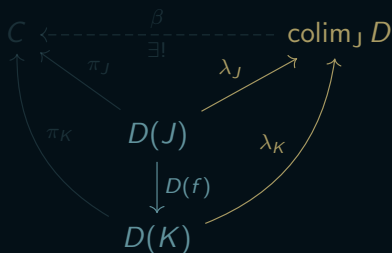
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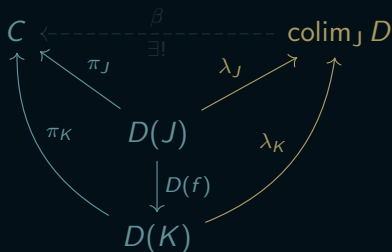
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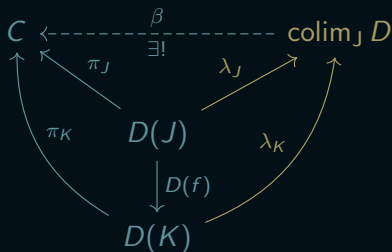
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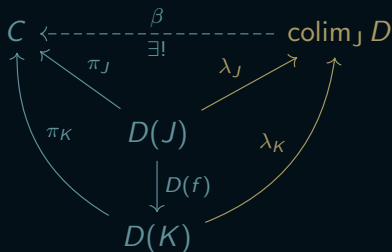
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 - ▷ For every pair of objects $X, Y \in J_0$, there exists an object $Z \in J_0$ and a pair of morphisms

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- ▷ A filtered category is to be thought of as a categorification of a *directed set*.

- ▷ Any ultrafilter \mathcal{U} on a set S is a poset (which can be regarded as a category).
- ▷ Then \mathcal{U}^{op} is a directed poset (since \mathcal{U} has is closed under intersections) and is a filtered category.
- ▷ A diagram $D : J \rightarrow C$ is called a *filtered diagram* if J is filtered.
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Let C^+ be a category and let $C \subseteq C^+$ be a full subcategory. We say that C *has ultraproducts in* C^+ if the following conditions are satisfied:

- ▶ For every collection $\{M_s\}_{s \in S}$ of objects of C indexed by S , there exists a product $\prod_{s \in S} M_s$ in the category C^+ .
- ▶ For every collection $\{M_s\}_{s \in S}$ of objects of C indexed by S and every ultrafilter \mathcal{U} on S , the diagram

$$D : \mathcal{U}^{\text{op}} \rightarrow C^+ :: (S_0 \in \mathcal{U}^{\text{op}}) \mapsto \left(\prod_{s \in S_0} M_s \right)$$

admits a colimit (in the category C^+) which belongs to the subcategory $C \subseteq C^+$.

In this case, we denote this colimit by $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$ and refer to it as the *categorical ultraproduct of* $\{M_s\}_{s \in S}$ *w.r.t.* \mathcal{U} .

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Ultraproduct Maps

- ▷ For each $S_0 \in \mathcal{U}$, we have a *quotient map* given by the colimit above:

$$q_{\mathcal{U}}^{S_0} : \prod_{s \in S_0} M_s \rightarrow \operatorname{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s.$$

- ▷ Given a collection of functions $\{f_s : M_s \rightarrow N_s\}_{s \in S}$ in \mathcal{C} , we let

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denote the unique map in \mathcal{C} such that for each $S_0 \in \mathcal{U}$, TFDC in \mathcal{C}^+ :

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Ultraproduct Functor

- ▷ We have the *ultraproduct functor*

$$\operatorname{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} (\bullet) : \mathbb{C}^S \rightarrow \mathbb{C}$$

$$\{M_s\}_{s \in S} \mapsto \operatorname{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$$

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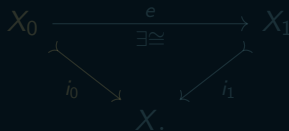
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Subobjects

- ▶ For any set X , we have a set $\mathcal{P}(X)$ of subsets of X .
- ▶ If X is an object in a category \mathcal{C} , then a *subobject* is an equivalence class of monomorphisms $i_0 : X_0 \rightarrowtail X$, where

$$i_0 \sim i_1 \Leftrightarrow i_0 = i_1 \circ e$$

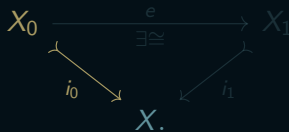


- ▶ We will usually refer to a subobject of X by its domain and assume that the monomorphism is supplied.
- ▶ We denote by $\text{Sub}(X)$ the set of subobjects of X .
- ▶ In $\text{Syn}(T)$, a subobject of $[\phi(\vec{x})]$ has the form $[\psi(\vec{x})]$ which is such that $T \models \forall \vec{x}. (\psi(\vec{x}) \Rightarrow \phi(\vec{x}))$.

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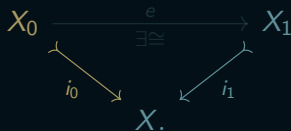


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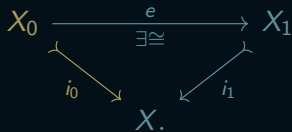


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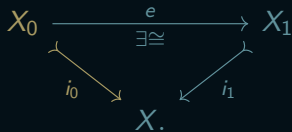


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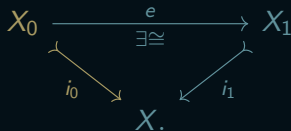


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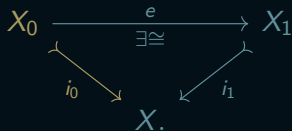


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- ▷ We can define a partial order on $\text{Sub}(X)$ by, for $X_0, X_1 \in \text{Sub}(X)$, $X_0 \leq X_1$ if and only if there is a unique morphism $e : X_0 \rightarrow X_1$ such that $i_0 = i_1 \circ e$.
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Coequalizers

- If $f, g : X \rightrightarrows Y$ are a parallel pair of arrows in \mathbf{C} , then their *coequalizer* is an object Q and arrow $q : Y \rightarrow Q$ such that

$$\begin{array}{ccccc} X & \xrightleftharpoons[g]{f} & Y & \xrightarrow{q} & Q \\ & & \searrow k & \downarrow \exists! \beta & \\ & & & Z. & \end{array}$$

$$\begin{aligned} q \circ f &= q \circ g \\ k \circ f &= k \circ g \\ k &= \beta \circ q. \end{aligned}$$

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The diagram shows a commutative square. The top row consists of object X , a double arrow from X to Y with f on top and g on the bottom, and object Y . From Y , there is a horizontal arrow q to object Q . From Y , there is a diagonal arrow k to object Z . From Q , there is a vertical arrow β to object Z . A label $\exists!$ is placed to the left of the vertical arrow β , indicating its uniqueness.

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- ▷ If $f, g : X \rightrightarrows Y$ are a parallel pair of arrows in \mathcal{C} , then their *coequalizer* is an object Q and arrow $q : Y \rightarrow Q$ such that

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y \\ & \searrow k & \downarrow \beta \\ & & Z \end{array}$$

$q : Y \rightarrow Q$

$$q \circ f = q \circ g$$

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$$k = \beta \circ q.$$

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Effective Epimorphisms

- ▶ Suppose $f : X \rightarrow Y$ is a surjection and let $R = \{(x_0, x_1) \in X \times X : f(x_0) = f(x_1)\}$ be an equivalence relation (notice that $R = X \times_Y X$).
- ▶ Then we have that $Y = X/R$, that is, Y is the coequalizer of the projection maps $\pi, \pi' : X \times_Y X \rightarrow X$.
- ▶ We say that a equivalence relation R is *effective* if it arises as the pullback $R = X \times_Y X$.
- ▶ Let \mathcal{C} be a category which admits pullbacks, and suppose that $f : X \rightarrow Y$ is a morphism in \mathcal{C} .
Then f is an *effective epimorphism* if it exhibits Y as a coequalizer of the maps $\pi, \pi' : X \times_Y X \rightrightarrows X$.
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- ▷ We say that the collection of effective epimorphisms in \mathcal{C} is *closed under pullbacks* if, in \mathcal{C} we have the following pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & \lrcorner & \downarrow f \\ Y' & \longrightarrow & Y, \end{array}$$

if f is an effective epi, then f' is also an effective epi.

- ▷ Let $f : X \rightarrow Y$ be a set function. Then we can (uniquely) *factor* f into a surjection g and an injection h :

$$\begin{array}{ccccc} & & f & & \\ & \searrow & & \nearrow & \\ X & \xrightarrow{g} & \text{im}(f) & \xrightarrow{h} & Y. \end{array}$$

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Coherent Category

- ▷ A *coherent category* \mathcal{C} is a category satisfying the axioms:
 - ▷ The category \mathcal{C} has finite limits.
 - ▷ For every object $X \in \mathcal{C}_0$, the poset $\text{Sub}(X)$ is an upper semilattice.
 - ▷ For every morphism $f : X \rightarrow Y$ in \mathcal{C} , the inverse map $f^{-1} : \text{Sub}(Y) \rightarrow \text{Sub}(X)$ is a morphism of upper semilattices.
 - ▷ Every morphism $f : X \rightarrow Z$ in \mathcal{C} can be written as a composition $X \xrightarrow{g} \text{im}(f) \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.
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Example

- ▶ Consider two theories T and T' defined as follows:
 - ▶ The language of T has no symbols and T has a single axiom $\exists!x.(x = x)$.
 - ▶ The language of T' has a single 1-ary relation R and a pair of axioms $\exists!y.R(y)$ and $\exists!z.\neg R(z)$.
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- ▶ So $\text{Mod}(T)$ is equivalent to $\text{Mod}(T')$.
- ▶ But, $\text{Syn}(T)$ is equivalent to the poset $\{0 \leq 1\} = 2$ while $\text{Syn}(T')$ is equivalent to Finset .
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Disjoint Coproducts

- ▷ Given two sets X, Y , we can form their *disjoint union* by creating isomorphic copies of X and Y :

$$X' = \{(x, 0) : x \in X\}, \quad Y' = \{(y, 1) : y \in Y\},$$

with $X' \cap Y' = \emptyset$, and taking the union: $X \sqcup Y = X' \cup Y'$.

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Pretopos Completion

- ▶ Given a small coherent category C , there exists a small pretopos C^{eq} and a coherent functor $\lambda : C \rightarrow C^{eq}$ with the property that, for any other pretopos D , composition with λ induces a functor

$$\begin{aligned}\lambda \circ : \text{Fun}^{\text{coh}}(C^{eq}, D) &\rightarrow \text{Fun}^{\text{coh}}(C, D) \\ (F : C^{eq} \rightarrow D) &\mapsto (\lambda \circ F : C \rightarrow D)\end{aligned}$$

which is an equivalence of categories.

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Let \mathcal{C} be a pretopos and let $\{M_s : \mathcal{C} \rightarrow \mathbf{Set}\}_{s \in S}$ be a collection of models of \mathcal{C} . For every ultrafilter \mathcal{U} on S , the ultraproduct $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$ (formed in the category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$) is a model of \mathcal{C} .

- ▷ The ultraproduct $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$ will be a pretopos functor from \mathcal{C} to \mathbf{Set} , which can be defined explicitly as

$$\left(\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s \right) (X) = \text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s(X).$$

Lemma

Let S be a set and let \mathcal{U} be an ultrafilter on S . Then the ultraproduct functor $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} (\bullet) : \mathbf{Set}^S \rightarrow \mathbf{Set}$ is a pretopos functor.

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Proof.

Let $\{M_s : C \rightarrow \mathbf{Set}\}_{s \in S}$ be a collection of models of C and let \mathcal{U} be an ultrafilter on the index set S . Notice that we can write the ultraproduct $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s : C \rightarrow \mathbf{Set}$ of the models of C as the following composition

$$C \xrightarrow{\{M_s\}_{s \in S}} \mathbf{Set}^S \xrightarrow{\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} (\bullet)} \mathbf{Set}.$$

Since each M_s is a model of C , that is, a pretopos functor, we have that $\{M_s\}_{s \in S}$ is a pretopos functor. Also, by the Lemma, we have the second map is a pretopos functor. Then we have that $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$ is a pretopos functor and hence a model of C . □

Classical Łoś Ultraproduct Theorem

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Let T be a first-order theory and let $\{M_s\}_{s \in S}$ be a collection of models of T . Then the ultraproduct $\text{colim}_{S_0 \in \mathcal{U}^{\text{op}}} \prod_{s \in S_0} M_s$ is also a model of T .

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Apply Łoś Ultraproduct Theorem to the pretopos completed syntactic category $\text{Syn}^{eq}(T)$. □

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- ▶ What Łoś Ultraproduct Theorem shows is that $\text{Mod}(\mathcal{C})$ has ultraproducts in $\text{Fun}(\mathcal{C}, \text{Set})$.
- ▶ Intuitively (and informally), an *ultracategory* is a category with an “ultraproduct” structure, that is, three collections of different types of ultraproduct functors satisfying some axioms.
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- ▶ For each object X in a pretopos C , define the *evaluation functor* $\text{ev}_X : \text{Mod}(C) \rightarrow \text{Set}$ by $\text{ev}_X(M) = M(X)$.
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Let C be a small pretopos. Then the evaluation map $\text{ev} : C \rightarrow \text{Fun}^{\text{Ult}}(\text{Mod}(C), \text{Set})$ is an equivalence of categories.

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Conclusion

- ▷ Pretoposes: syntax $\text{Syn}(T)$
- ▷ Ultracategories: semantics $\text{Mod}(\text{Syn}(T))$.

Thank You!

Questions?