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Quantum channels and representation theory

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In the study of d-dimensional quantum channels ($d \geqslant 2$), an assumption which includes many interesting examples, and which has a natural physical interpretation, is that the corresponding Kraus operators form a representation of a Lie algebra. Physically, this is a symmetry algebra for the interaction Hamiltonian. This paper begins a systematic study of channels defined by representations; the famous Werner-Holevo channel is one element of this infinite class. We show that the channel derived from the defining representation of $\mathfrak{su}(n)$ is a depolarizing channel for all n, but for most other representations this is not the case. Since the standard Bloch sphere only exists for the qubit representation of $\mathfrak{su}(2)$, we develop a consistent generalization of Bloch's technique. By representing the density matrix as a polynomial in Lie algebra generators, we determine a class of positive semidefinite matrices which represent quantum states for various channels defined by finite-dimensional representations of semisimple Lie algebras. We also give a general method for finding positive semidefinite matrices using Lie algebraic trace identities. This includes an analysis of channels based on the exceptional Lie algebra $\mathfrak{g}_2$ and the Clifford algebra.
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I. INTRODUCTION

A quantum channel is a model for a particular snapshot of the time evolution of a density matrix, and especially for the evolution of pure into mixed states. Let $\mathcal{H}$ be a Hilbert space of dimension n, and let $\mathfrak{gl}(\mathcal{H})$ denote the vector space of all linear transformations $\mathcal{H} \to \mathcal{H}$. A map $\mathcal{E}: \mathfrak{gl}(\mathcal{H}) \to \mathfrak{gl}(\mathcal{H})$ is said to be *completely positive* (*CP*) if it is linear and $\mathcal{E} \otimes \mathbf{1}$ is positive on $\mathcal{H} \otimes \mathcal{H}'$ for all $\mathcal{H}'$. The map is said to be *trace-preserving* if $\operatorname{Tr} \mathcal{E}(A) = \operatorname{Tr} A$ for all $A \in \mathfrak{gl}(\mathcal{H})$.

Definition 1: A CPT map *(or* stochastic map *or* channel*) is a completely positive, trace-preserving linear transformation* $\mathcal{E}: \mathfrak{gl}(\mathcal{H}) \to \mathfrak{gl}(\mathcal{H})$.

Of central importance to the current work is a famous theorem of Kraus[1] which gives a simple representation of any quantum channel.

Theorem 1 (Kraus decomposition): *For any quantum channel* $\mathcal{E}$, *there exists a finite set of operators*

$$M_0, M_1, M_2, \ldots, M_k, \quad \text{where } k \leqslant (\dim \mathcal{H})^2,$$

such that

$$\mathcal{E}(\boldsymbol{\rho}) = \sum_\mu M_\mu \boldsymbol{\rho} M_\mu^\dagger \quad \text{with} \sum_\mu M_\mu^\dagger M_\mu = \mathbf{1}. \tag{1}$$

In this situation, (1) *is called the* Kraus representation, *the* operator sum representation *or the* Stinespring form, *while* $\Sigma_\mu M_\mu^\dagger M_\mu = \mathbf{1}$ *is sometimes called the* normalization condition *and is just the statement that the map is trace preserving.*

A proof of this theorem may be found in the original article of Kraus,[1] or in the book by Nielsen and Chuang.[2] We simply note that the converse, namely that any operator of the form (1) satisfies the conditions of Definition 1, is clearly true. A stochastic map may also be obtained as the partial trace of a unitary conjugation on a larger space; see Ruskai,[3] Sec. III D for a discussion. The representation (1) is sometimes called the *Stinespring form* since its existence follows from the Stinespring dilation theorem.[4]

This is a general framework, and in order to obtain explicit results, further assumptions are necessary. A mathematically elegant assumption with important physical consequences is that the possible errors introduced in the decoherence process are not arbitrary, but that they correspond to the action of the infinitesimal generators of a Lie group G of continuous symmetries. For example, the *qubit depolarizing channel* is a model of a decohering qubit with an $\mathfrak{su}_2$ symmetry. With probability p an error occurs, which is implemented by one of the generators of the Lie algebra $\mathfrak{su}_2$ in its two-dimensional irreducible representation. These generators admit direct physical interpretations as bit-flip errors, phase-flip errors, or combinations of those. The qubit depolarizing channel admits a beautiful generalization to a channel with k possible errors based on a d-dimensional representation $\mathcal{H}$ of a k-dimensional Lie algebra $\mathfrak{g}$.

The situation just described, in which the error generators live in a matrix representation of a semisimple Lie algebra, arises naturally in the model of Markovian dynamics considered by Lidar, Chuang, and Whaley.[5] This was shown to have important consequences for the possibility of decoherence-free dynamics; see Lidar and Whaley,[6] and references therein for an up-to-date review. The present work may be considered as a further exploration of the consequences of that model.

Suppose the M_μ span the space of Hermitian operators, so that $\mu=0,\ldots,n^2-1$. Define $\lambda_0=\mathbf{1}$ and let λ_a for $a\geq 1$ denote the n-dimensional Gell-Mann matrices, which are standard generators for the Lie algebra $\mathfrak{su}_n$. Then $\{\lambda_\mu\}$ is a basis for the space of Hermitian matrices. Taking M_0 proportional to the identity, there exist constants U_{ab} such that $M_a=\Sigma_b U_{ab}\lambda_b$. If U is a unitary matrix, then we may take each M_μ proportional to λ_μ without changing the quantum channel defined by these Kraus operators.

In the latter case, one may readily calculate $\Sigma_\mu M_\mu \rho M_\mu^\dagger$, since ρ itself may be expanded in the λ_μ basis, and the Gell-Mann matrices in any dimension satisfy elegant product identities. This leads to a simple, elegant and explicit formula for the action of the $\mathfrak{su}_n$ channel; if $\rho=(1/d)\mathbf{1}+\Sigma_\mu v_\mu\lambda_\mu$ then the channel multiplies v by a scalar given in Sec. III. This is a wonderful calculational tool, and also has physical significance. A quantum channel models the interaction of a decohering system with its environment, and the identification of the M_μ as generators of a Lie algebra is related to symmetry of the interaction Hamiltonian.[5]

The generalizations of the $\mathfrak{su}_n$ calculations to other Lie algebras and to higher-dimensional representations are illuminating, and have not appeared in the literature before. These cases necessarily have the property that not all of the Hermitian matrices in that dimension are linear combinations of representation matrices, so direct generalization of the calculational method for $\mathfrak{su}_n$ will not work, and a new idea is required. This is the subject of Sec. IV. We give a detailed analysis of the spin-1 representation; however, many of the formulas we use there generalize readily to higher spin. In an interesting twist, the spin-one case turns out to be a generalization of the Werner-Holevo channel. Section V analyzes two channels, based, respectively, on the exceptional algebra $\mathfrak{g}_2$ and the Clifford algebra.

In Sec. VI we generalize some aspects of the Bloch sphere to density matrices constructed from Lie algebra representations. It is shown that for each representation, there is a class of positive semidefinite, Hermitian trace-one density operators parametrized by a closed, bounded (hence compact) submanifold of Euclidean space, which we term the Bloch manifold. Explicit bounds are given on the size of these manifolds. A general method is given for finding the Bloch manifold exactly, using trace identities.

II. QUANTUM CHANNELS FROM LIE ALGEBRA REPRESENTATIONS

This section contains our notations and conventions for the generalized depolarizing channels which will be studied in detail in later sections. The possibility of defining a quantum channel based on a representation of a compact Lie algebra was mentioned briefly, but never elaborated upon, in a paper of Gregoratti and Werner.[7] In any case, it is not necessary that the Lie algebra be compact.

A. Pure Lie algebra channels

It is a standard convention[8,9] to normalize the canonical generators for the defining representation of $\mathfrak{su}_n$ so that

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}. \tag{2}$$

This has the desirable feature that the canonical generators for $n=2$ are the Pauli matrices, and those for $n=3$ are the familiar Gell-Mann matrices, while inserting factors of 2 in certain formulas. With convention (2), these generators will be orthogonal but not orthonormal with respect to the Killing form. We return to this point below.

On a general semisimple Lie algebra, the Killing form K is defined as

$$K(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) : \mathfrak{g} \to \mathfrak{g}),$$

where the trace is taken in the adjoint representation. At the moment we focus on semisimple algebras $\mathfrak{g}$, for which the Killing form is nondegenerate. Let α be an irreducible representation of $\mathfrak{g}$, let X_i be any basis of $\mathfrak{g}$, and let X_i' denote the dual basis with respect to the Killing form. The Casimir operator

$$C_2(\alpha) = \sum_i \alpha(X_i)\alpha(X_i')$$

does not depend on the choice of basis, and by Schur's lemma is proportional to the identity, so we write $C_2(\alpha)=c_2(\alpha)\mathbf{1}$. If X_i is orthonormal with respect to K, then $C_2(\alpha)=\Sigma_i \alpha(X_i)^2$. For reducible representations, $C_2(\alpha)$ may not be proportional to the identity.

Definition 2 (Lie algebra channel: Let $\mathfrak{g}$ denote a Lie algebra of dimension k, with basis $\{X_i : i=1,\ldots,k\}$. Let α be an irreducible $\mathfrak{g}$-representation on the Hilbert space $\mathcal{H}$. The generalized depolarizing channel *or* Lie algebra channel *is defined to be the channel in which an error occurs conditionally with probability p, causing an initial state $|\psi\rangle \in \mathcal{H}$ to evolve into an ensemble of the k states $\alpha(X_i)|\psi\rangle$, all with equal likelihood.*

The Kraus operators for the channel of Definition 2 are given by

$$M_0 = \sqrt{1-p}\,\mathbf{1}, \quad M_i = \sqrt{\Lambda p}\,\alpha(X_i), \tag{3}$$

where Λ is a normalization constant which will be fixed momentarily. The operators M_μ are Hermitian if the representation is unitary and if $p \in [0,1]$, and are constrained to satisfy $\Sigma_\mu M_\mu M_\mu = \mathbf{1}$, which fixes the value of the constant Λ appearing in (3). By definition,

$$\sum_\mu M_\mu^2 = (1-p)\mathbf{1} + \Lambda p \sum_i \alpha(X_i)^2.$$

If $\Sigma_i \alpha(X_i)^2 = Z \cdot \mathbf{1}$, where Z is a constant (which in most cases we can take to be real), then

$$\Lambda = \frac{1}{Z}.$$

If X_i is orthonormal with respect to the Killing form, then $Z=c_2(\alpha)$. More generally, if the basis satisfies

$$K(X_i, X_j) = \mathfrak{n}\, \delta_{ij}, \quad \mathfrak{n} > 0,$$

then it can be rescaled to an orthonormal basis by a single constant. In this situation,

$$Z = \mathfrak{n}\, c_2(\alpha), \quad \Lambda = \frac{1}{Z}. \tag{4}$$

Defining the Killing norm by $\|x\|_K^2 = K(x,x)$, we note that if

$$\|X_i\|_K \neq \|X_j\|_K,$$

for some pair of indices i, j, then the normalization condition cannot be satisfied.

What if the representation is reducible? Suppose $\mathcal{H} = V \oplus W$ as a direct sum of irreducible $\mathfrak{g}$-modules, and X_i is orthonormal with respect to K. Then there exist independent constants Z_V and Z_W such that the operator

$$C_2(\alpha) = \sum_i \alpha(X_i)^2 = \begin{pmatrix} Z_V & 0 \\ 0 & Z_W \end{pmatrix}$$

as a block decomposition on $V \oplus W$. If $Z_V \neq Z_W$, then it is not possible for the Kraus operators (3) to give a trace-preserving map. On the other hand, if $Z_V = Z_W$ then they do define a CPT map even though the representation is reducible.

What if $p > 1$? Then $M_0 = i\sqrt{p-1}\mathbf{1}$, and we have

$$\sum_\mu M_\mu M_\mu^\dagger = (2p-1)\mathbf{1}.$$

Thus the map cannot be trace-preserving unless $p=1$, which is a contradiction. A similar argument shows that $p < 0$ does not give a trace-preserving map. Thus, if we wish to study the framework of Definition 2, then we must limit ourselves to $p \in [0,1]$.

We summarize the results of the last few paragraphs in a theorem.

Theorem 2 (normalization): *Consider the Kraus operators,*

$$M_0 = (1-p)^{1/2}\mathbf{1} \quad \textit{and } M_i = (\Lambda p)^{1/2}\alpha(X_i),$$

for $i = 1, \ldots, k$. If

(i) $p \in [0,1]$,

(ii) *The representation α of $\mathfrak{g}$ is a direct sum of irreducible representations all with the same quadratic Casimir, and*

(iii) $\exists\ \mathfrak{n} > 0$ *such that $K(X_i, X_j) = \mathfrak{n}\,\delta_{ij}$ for all i, j,*

then $\Sigma_\mu M_\mu M_\mu = \mathbf{1}$ with Λ given by Eq. (4). *Conversely, if any of (i)–(iii) is not satisfied, then (except in trivial cases) there does not exist Λ s.t. $\Sigma_\mu M_\mu M_\mu = \mathbf{1}$, and the M's do not give rise to a quantum channel.*

The coefficients of the M_μ in (3) admit a natural "probability of error" interpretation, but in Sec. II B we investigate the possibility of modifying them to complex coefficients in order to obtain a channel. We find that no new channels arise unless one is willing to promote the coefficients to operators.

Using (1), the Lie algebra channel has the explicit Kraus decomposition

$$\boldsymbol{\rho} \to \mathcal{E}(\boldsymbol{\rho}) = (1-p)\boldsymbol{\rho} + \frac{p}{Z}\sum_{i=1}^{k} \alpha(X_i)\boldsymbol{\rho}\, \alpha(X_i). \tag{5}$$

As is proven in standard textbooks[9] (see Theorem 8.9), the trace of any generator of any representation of a compact simple Lie algebra is zero, so in particular, the $\alpha(X_i)$ are traceless. Moreover, it is clear that this transformation satisfies the defining properties for a quantum channel, given here as Definition 1.

Two operator-sum representations

$$\sum_\mu M_\mu \boldsymbol{\rho} M_\mu^\dagger \quad \text{and} \; \sum_\nu N_\nu \boldsymbol{\rho} N_\nu^\dagger$$

describe the same channel if and only if there exists a unitary matrix $U_{\nu\mu}$ such that $N_\nu = U_{\nu\mu} M_\mu$. Therefore, it is immaterial which basis of the Lie algebra that we use, as long as the two bases are related by a $U(N)$ similarity transformation. As noted in Theorem 2, in order to build a channel satisfying the normalization condition, we are forced to use a basis satisfying "orthonormality," $K(X_i, X_j) = \mathfrak{n}\, \delta_{ij}$. But any two "orthonormal" bases in this sense are related by a unitary transformation, so the CPT map constructed above is independent of the basis chosen for $\mathfrak{g}$.

Given a Lie algebra $\mathfrak{g}$ and a representation α on a vector space of dimension d, the CPT map (5) is a model for decoherence through a d-level noisy quantum channel, with errors that are not completely arbitrary; rather, they transform the state in a way determined by the representation of $\mathfrak{g}$.

The channels (5) have an extremely interesting structure. For a certain subclass of possible Lie algebra representations, the channel (5) has an action which, like the qubit case, is most simply described by a Bloch parametrization with polarization vector $v \in \mathbb{R}^k$, where $k = \dim \mathfrak{g}$. In these cases, we show that (5) decreases the length of v, and so deserves the title generalized depolarizing channel. In other cases of interest, a single Bloch vector is not sufficient, but the action of the channel can be described by similar rescalings of symmetric 2-tensors or higher-rank objects.

A natural step, which we begin in the next section, is to calculate the expression (5) explicitly in certain representations of classical Lie algebras.

Remark 1: When we use the terminology "the $\mathfrak{g}$-channel," where $\mathfrak{g}$ is a semisimple Lie algebra, the fundamental representation of $\mathfrak{g}$ is implied. Examples of fundamental representations include the n-dimensional defining representation of $\mathfrak{su}_n$, and the seven-dimensional irrep of G_2.

It is easy to see that the Lie algebra channel (5) always has the property of being *doubly stochastic*, i.e., $\mathcal{E}(\mathbf{1}) = \mathbf{1}$. See, for example, Ref. 7 for further discussion.

B. A note on coefficients and extensions

As discussed prior to Theorem 2, for $p \notin [0,1]$ the transformation defined by (5) is CP but not T, and it is possible to recover a CPT map (channel) only if we consider different coefficients for the Kraus operators (3). To this end, let us first consider

$$M_0 = m_0 \mathbf{1} \quad \text{and} \; M_i = \frac{\tilde{m}}{\sqrt{Z}} \alpha(X_i), \tag{6}$$

where m_0, $\tilde{m} \in \mathbb{C}$ are some complex constants. Then to obtain a trace-preserving map, we require

$$\sum_\mu M_\mu^\dagger M_\mu = |m_0|^2 + |\tilde{m}|^2 = 1.$$

This condition is equivalent to the statement that the point $(m_0, \tilde{m}) \in \mathbb{C}^2 \cong \mathbb{R}^4$ lies in the unit 3-sphere $S^3 \subset \mathbb{R}^4$.

We can now view the coefficients of the Kraus operators (3) as the projection $S^3 \to S^1$. Introduce a parameter $q \in [-1,1]$ such that $p = q^2$, and write (3) as $M_0 = \pm\sqrt{1-q^2}\mathbf{1}$, and $M_i = (q/\sqrt{Z})\alpha(X_i)$. Then ignoring $\sqrt{Z}$, the coefficients of M_i and M_0 give a point on the unit circle. Further, m_0 and $\tilde{m}$ only enter through the square of their magnitude, so the two additional parameters associated to projecting from the 3-sphere are fictitious, and (5) is in fact the most general channel of this kind.

A nontrivial generalization is obtained by promoting m_0 and $\tilde{m}$ to *operators*. However this "generalization" is a special case of a well-known operation which extends an existing channel $\mathcal{E}_B$ using any set of operators which satisfy the normalization condition (1). Given two sets of Kraus operators $A_1, \ldots, A_r$ and $B_1, \ldots, B_s$ acting on the same vector space and satisfying

$$\sum_{i=1}^{r} A_i^\dagger A_i = \sum_{j=1}^{s} B_j^\dagger B_j = \mathbf{1},$$

we note that the set of operators

$$\{A_1, \ldots, A_{r-1}, B_1 A_r, \ldots, B_s A_r\} \tag{7}$$

also satisfies the normalization condition, because

$$\sum_{j=1}^{s} (B_j A_r)^\dagger B_j A_r = A_r^\dagger A_r.$$

This construction is *natural* with respect to the channel $\mathcal{E}_B$ defined by B_i, in the sense that if $\{B_i'\}$ is another set of Kraus operators defining the same channel, then the channel defined by (7) is also the same. Naturality does not hold for the A operators, but this will not concern us here. We call this procedure *the extension of $\mathcal{E}_B$ by the A operators, on the element A_r*.

For example, one may notice that the operators $Z^{-1/2}\alpha(X_i)$ of the preceding section satisfy the normalization condition since the sum of their squares is a Casimir element, and the normalization constant Z cancels the numerical factor. Consider this the B-channel, and extend it on every element by the same set of Kraus operators. This yields a "double $\mathfrak{g}$-channel" with Kraus operators

$$\left\{ \frac{1}{Z} X_i X_j : i, j = 1, \ldots, k \right\}. \tag{8}$$

These operators generate the image of $\mathfrak{g} \otimes \mathfrak{g}$ under the universal homomorphism expressed in the commutative diagram (18).

This underscores the fact that, aside from the basic examples of quantum channels provided by Sec. II A, many further examples may be obtained by extension, as in (7). As in the basic Lie algebra channels, computations with extended channels are facilitated by the existence of nontrivial identities which exist among the representation matrices. Channel (8) is interesting because for many representations, the matrices $\alpha(X_i)$ do not span the entire space of traceless $d \times d$ matrices, but the set of products $\alpha(X_i)\alpha(X_j)$ spans a subspace of larger dimension. Therefore the extension leading to (8) is a way of generating a channel whose Kraus operators come closer to spanning the space of all matrices in the appropriate dimension. If a density matrix were written as $\rho = \Sigma_{ij} w_{ij} X_i X_j$, and if the representation satisfies an identity for reduction of products of six generators, then one can calculate the action of (8) on ρ explicitly.

We are now in a position to interpret the channel defined by (6) with complex coefficients m_0, $\tilde{m}$ as the extension (7) of the identity channel with the unusual Kraus representation $B_1 = m_0 \mathbf{1}$, $B_2 = \tilde{m}\mathbf{1}$ by the nontrivial Lie algebra channel $A_i = Z^{-1/2}\alpha(X_i)$ on the element B_2. We may use naturality in B to rotate to the case $\tilde{m} = q \in [-1, 1]$ and $m_0 = \pm\sqrt{1-q^2}$ whence we recover (5), so there is no advantage to complexifying the coefficients. Given any channel whose set of Kraus operators do not contain $\mathbf{1}$, we can always extend it so that they do contain the identity by this method.

For the rest of this paper, we will assume that the Kraus operators take the form (3) in order to retain the beautiful probabilistic interpretation given by Definition 2. As we continue, we will keep in mind that extensions are possible, and develop methods which easily generalize.

III. THE $\mathfrak{su}_n$ CHANNEL

A. Bloch methods for $\mathfrak{su}_n$

The $\mathfrak{su}_n$ channel, our first example, is the channel built from the n-dimensional defining representation (also called standard representation) of $\mathfrak{su}_n$. It is simpler than most other channels studied in this paper, because it admits a complete solution. Its action on any arbitrary input density matrix can be calculated in closed form using the Bloch parametrization, and in all cases it is a depolarizing channel.

One reason for the beauty and simplicity of the $\mathfrak{su}_n$ channel is that any n-dimensional density matrix admits a Bloch vector parametrization in terms of $\mathfrak{su}_n$ generators. This is because $k \equiv \dim(\mathfrak{su}_n) = n^2 - 1$ is only one less than n^2, the dimension over $\mathbb{R}$ of the space of $n \times n$ Hermitian matrices.

Any $n \times n$ Hermitian matrix ρ may be represented as

$$\rho = \frac{1}{n}(\mathrm{tr}(\rho)\mathbf{1} + T), \quad T \in \mathfrak{su}_n,$$

and having chosen a basis X_a for $\mathfrak{su}_n$, it follows that

$$T = \sum_{a=1}^{k} v_a X_a \equiv v \cdot X,$$

for some coefficient vector v. In analogy with the well-known parametrization of the 2×2 density matrices as the interior of a sphere, we will refer to v as the *Bloch vector*.

For $n \geqslant 3$ it may be hard to visualize the geometry of the space of density matrices in terms of the geometry of v. This question was first considered in the $n=3$ case by MacFarlane *et al.*[8] Section VI undertakes a systematic study of the geometry of the space of v which lead to a valid density matrix in various representations. We call this space the *Bloch manifold* and give details of the geometry for a number of important examples, including all representations of $\mathfrak{su}_2$, and the n-dimensional irrep of $\mathfrak{su}_n$.

B. The standard representation

In this section, we take α to be the standard representation of $\mathfrak{su}_n$ on a vector space $\mathcal{H}$ of dimension n. For simplicity, we let X_i denote both the generator of $\mathfrak{su}_n$ and its image under this representation. One could now compute the quadratic Casimir in the standard way using roots and weights, but it will turn out that the value of this Casimir as well as all other properties we will need to obtain a complete solution to the $\mathfrak{su}_n$ channel follow from the single relation

$$X_i X_j = \beta \delta_{ij} \mathbf{1} + \sum_k Q_{ijk} X_k \tag{9}$$

for some constant β and tensor Q_{ijk}. Of course, this relation is just the decomposition of a Hermitian matrix into a trace part with trace $n\beta\delta_{ij}$, and a linear combination of the X_k, which generate the space of traceless matrices.

Elements of the standard basis of $\mathfrak{su}_n$ are called Gell-Mann matrices, and they satisfy

$$\mathrm{Tr}(X_i X_j) = 2\delta_{ij},$$

so $\beta = 2/n$. Many properties of the Q tensor already follow from the single assumption that X_i generate a Lie algebra. It is immediate that $Q_{[ij]k} = i f_{ijk}$ where $[ij]$ denotes antisymmetrization, and f_{ijk} is $1/2$ times the structural tensor of the Lie algebra. It follows that

$$Q_{ijk} = d_{ijk} + i f_{ijk}$$

for some d_{ijk} symmetric in the first two indices. Also, (9) implies

$$\{X_i, X_j\} = \frac{4}{n}\delta_{ij}\mathbf{1} + 2\sum_l d_{ijl} X_l.$$

Multiplying by X_k and taking the trace yields

$$d_{ijk} = \tfrac{1}{4}\mathrm{Tr}(\{X_i, X_j\}X_k),$$

therefore the d-tensor is *completely* symmetric, and interchange of any two indices has the effect of complex conjugating Q. Since $\Sigma_i X_i X_i$ is a multiple of the identity,

$$\sum_i d_{iik} = \frac{1}{2}\mathrm{Tr}\left(\left(\sum_i X_i X_i\right)X_k\right) = 0. \tag{10}$$

It follows from the associativity of matrix multiplication that

$$f_{ijm}f_{klm} = \frac{2}{n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + d_{ikm}d_{jlm} - d_{jkm}d_{ilm}$$

with a sum over m implied. Contracting j and k and using (10) yields

$$d_{ijm}d_{ljm} = f_{ijm}f_{jlm} + \left(2n - \frac{4}{n}\right)\delta_{il}.$$

By a general property of compact semisimple Lie algebras, the structure constants satisfy

$$f_{ijk}f_{ljk} = n\delta_{il}. \tag{11}$$

Therefore, $d_{ijm}d_{ljm} = [n-(4/n)]\delta_{il}$. Using this and (11), we obtain

$$Q_{ijm}Q_{ljm} = d_{ijm}d_{ljm} - f_{ijm}f_{ljm} = -\frac{4}{n}\delta_{il}. \tag{12}$$

C. Solution of the $\mathfrak{su}_n$ channel

For this basis of $\mathfrak{su}_n$, $Z=2k/n$, where $k=n^2-1$. The action of the channel

$$\boldsymbol{\rho} \to \mathcal{E}(\boldsymbol{\rho}) = (1-p)\boldsymbol{\rho} + \frac{pn}{2k}\sum_{i=1}^{k} X_i \boldsymbol{\rho} X_i$$

on the density matrix

$$\boldsymbol{\rho} = \frac{1}{n}(\mathrm{tr}(\boldsymbol{\rho})\mathbf{1} + v \cdot X)$$

is given by

$$\mathcal{E}(\boldsymbol{\rho}) = \frac{\mathrm{tr}(\boldsymbol{\rho})}{n}\mathbf{1} + \frac{1-p}{n}v \cdot X + \frac{p}{2k}\sum_{i,j} v_j X_i X_j X_i. \tag{13}$$

Using (9) to expand the triple product, we have

$$\sum_{j,i} v_j X_i X_j X_i = \beta v \cdot X + \beta \sum_{i,j} v_a Q_{iji}\mathbf{1} + \sum_{i,j,k,a} v_j Q_{ijk}Q_{kia}X_a.$$

Since $\mathcal{E}(\boldsymbol{\rho})$ has unit trace, it must be the case that $\Sigma_i Q_{iji}=0$. The same conclusion also follows from (10), but it is amusing to see that $\Sigma_i Q_{iji}$ must vanish because this is a CPT map. Therefore,

$$\mathcal{E}(\boldsymbol{\rho}) = \frac{\mathrm{tr}(\boldsymbol{\rho})}{n}\mathbf{1} + \frac{1-p+p/k}{n}v \cdot X + \frac{p}{2k}\sum_{i,j,k,a} v_j Q_{ijk}Q_{kia}X_a. \tag{14}$$

Using (12), we have finally

$$\mathcal{E}(\boldsymbol{\rho}_v) = \frac{1}{n}(\mathrm{tr}(\boldsymbol{\rho})\mathbf{1} + f(p,n)v \cdot X),$$

where

$$f(p,n) = 1 - p - \frac{p}{k} = \frac{(1-p)n^2 - 1}{n^2 - 1}. \tag{15}$$

In the qubit case, $f(p,2)=1-4p/3$, which is consistent with standard results.

D. Properties of the solution

The $\mathfrak{su}_n$ channel maps an initial density matrix to a linear combination of itself and the identity, i.e., it has the form

$$\Delta_\lambda(\boldsymbol{\rho}) = \lambda\boldsymbol{\rho} + \left(\frac{1-\lambda}{n}\right)\mathbf{1}. \tag{16}$$

This is the standard definition of the n-dimensional depolarizing channel. The information-carrying capacity of this channel was studied in great detail by King,[10] where notably the Amosov-Holevo-Werner[11] conjecture was established for channels which are products of a depolarizing channel with an arbitrary channel. Channels based on representations of semisimple algebras generically do not take the form (16), except possibly on special subsets of the space of density matrices; see Theorem 3.

The depolarizing channel on an n-dimensional Hilbert space satisfies complete positivity if and only if

$$\frac{1}{1-n^2} \leqslant \lambda \leqslant 1.$$

The $\mathfrak{su}_n$ channel has the form (16) for $\lambda=f(p,n)$. Note that the relation

$$\frac{-1}{n^2-1} \leqslant f(p,n) \leqslant 1$$

holds for all $n \geqslant 2$. In fact, $f(p,n)$ saturates both of these inequalities at the endpoints of the allowed range, $0 \leqslant p \leqslant 1$.

Knowing that the $\mathfrak{su}_n$ channel is depolarizing allows an easy calculation of the minimal von Neumann output entropy,

$$S_{\min} = \frac{-np}{1+n}\ln\left(\frac{np}{n^2-1}\right) - \left(1 - \frac{n\,p}{1+n}\right)\ln\left(1 - \frac{n\,p}{1+n}\right),$$

with large n behavior $\lim_{n\to\infty} S_{\min}/\ln(n)=p$.

At the special value $p=1-n^{-2}$, the $\mathfrak{su}_n$ channel is a constant map from $\mathbb{R}^{n^2-1}$ into the space of density matrices,

$$\mathcal{E}(\boldsymbol{\rho}_v) = \frac{1}{n}\mathbf{1} \quad \text{for all } v, \quad \text{at } p = p_c \equiv 1 - n^{-2}. \tag{17}$$

Physically, if the probability of error happens to be $p=p_c$, then $\mathfrak{su}_n$-decoherence evolves an arbitrary initial density matrix into a completely uniform ensemble consisting of pure states with equal probabilities. This is the "worst" value of p, in the sense that all information about the initial density matrix has been lost. This result is *stable* in the sense that if p is only approximately equal to the critical value, the initial density matrix decoheres into an approximately uniform ensemble. We will see in Sec. IV that for other Lie algebra channels, there are critical values of p which generalize (17).

IV. OTHER REPRESENTATIONS

A. General remarks

In the n-dimensional standard representation of $\mathfrak{su}_n$, the representation matrices $\alpha(X_i)$ span the space of all traceless Hermitian matrices, and thus an arbitrary initial density matrix can be expressed in terms of the $\alpha(X_i)$ and the identity. As we consider higher-dimensional representations, the representation matrices become increasingly sparse in the space of all traceless matrices, and thus only some fraction of the set of all possible density matrices can be expressed in the form $d_\alpha^{-1}\mathbf{1}+\Sigma_i v_i \alpha(X_i)$. Also for higher-dimensional irreducible representations, (9) does not hold. Therefore, ideas are needed to further extend our analysis to irreps which have higher dimension than the fundamental representation. This is the subject of Sec. IV. Let $d=d_\alpha$ denote the dimension of the representation α, and $\mathfrak{gl}_d$ as usual denotes the associative algebra of all $d\times d$ matrices.

A representation ϕ of $\mathfrak{g}$ lifts to a unique associative algebra homomorphism $\tilde{\phi}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, by the universal property most elegantly expressed in the commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\phi"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, "\tilde{\phi}"] \\ & \mathfrak{gl}_d \end{tikzcd} \tag{18}$$

The action of $\tilde{\phi}$ is simply to convert the tensor product to matrix multiplication, i.e., $\tilde{\phi}(x\otimes y)=\phi(x)\cdot\phi(y)$, etc. The interesting property about this commutative diagram, and one which gives a computational method for Lie algebra channels, is that if ϕ is an irreducible faithful representation and if $\mathfrak{g}$ is a semisimple Lie algebra, then $\tilde{\phi}$ is surjective.

This surjectivity has the consequence that for *any representation* of said Lie algebra, *every* density matrix can be represented as a linear combination of products of the representation matrices. In other words, the calculational method outlined in this section will always work. Before continuing our discussion of this, let us consider a simple but nontrivial example, the spin-1 channel, in complete detail.

B. The spin-1 channel

Consider the spin-1 representation of $\mathfrak{su}_2$. We use standard angular momentum notation, in which

$$J_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad J_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\quad J_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

Before generalizing to arbitrary density matrices, we restrict attention to the simpler example of density matrices $\boldsymbol{\rho}$ which are of the form

$$\boldsymbol{\rho}_v=\tfrac{1}{3}(\mathbf{1}+v\cdot J),\quad v\in\mathbb{R}^3. \tag{19}$$

Then

$$\mathcal{E}(\boldsymbol{\rho}_v)=\frac{1}{3}\mathbf{1}+\frac{1-p}{3}v\cdot J+\frac{p}{6}\sum_{a,b}v_bJ_aJ_bJ_a. \tag{20}$$

The relation analogous to (9) does not hold, i.e., J_aJ_b is not a linear combination of $\mathbf{1}$ and $\{J_i : i=1,\ldots,3\}$. In this special case, the triple product appearing in (20) simplifies considerably,

$$J_a J_b J_a = \delta_{ab} J_a \quad \text{(no sum)}, \tag{21}$$

which implies that

$$\mathcal{E}(\boldsymbol{\rho}_v) = \frac{1}{3}\left(\mathbf{1} + \left(1 - \frac{p}{2}\right) v \cdot J\right). \tag{22}$$

This takes the form (19) with $v \to [1-(p/2)]v$. Thus, for 3×3 density matrices admitting a Bloch parametrization, the spin-1 channel scales the Bloch vector by a number between $1/2$ and 1.

Interestingly, we can go further and find a Bloch-type picture of the spin-1 channel on a general density matrix. The six elements of the form

$$J_{(a}J_{b)} \equiv \tfrac{1}{2}(J_a J_b + J_b J_a),$$

together with J_1, J_2, and J_3, span the space of 3×3 matrices. Therefore an arbitrary 3×3 density matrix $\boldsymbol{\rho}$ can be written as

$$\boldsymbol{\rho}_{v,w} = v \cdot J + \sum_{a,b} w_{ab} J_{(a}J_{b)} \tag{23}$$

for some vector v and symmetric tensor w.

We use standard physics normalizations which entail that for the spin-s representation in $d = 2s+1$ dimensional space,

$$\sum_a J_a^2 = \lambda\,\mathbf{1} \quad \Rightarrow \quad \mathrm{tr}(J_a J_b) = \frac{d\lambda}{3}\delta_{ab},$$

where $\lambda = s(s+1)$. Then we have

$$\mathrm{tr}(\boldsymbol{\rho}_{v,w}) = \frac{d\lambda}{3}\mathrm{tr}(w).$$

It follows that in order to have a density matrix, we require $\mathrm{tr}(w) = 3(d\lambda)^{-1}$. For $s=1$, $\mathrm{tr}(w) = 1/2$.

Theorem 3 (action of the spin-1 channel): *The action of the spin-1 channel on the vector and symmetric tensor are* $v \to v'$ *and* $w \to w'$, *where*

$$\begin{aligned} v_a \to v_a{}' &= \left(1 - \frac{p}{2}\right) v_a, \\ w_{ab} \to w_{ab}{}' &= \left(1 - \frac{3p}{2}\right) w_{ab} + \frac{p}{4}\delta_{ab}. \end{aligned} \tag{24}$$

Proof: The asymmetric quadruple product identity

$$\sum_i J_i J_j J_k J_i = \delta_{jk}\mathbf{J}^2 - J_k J_j \tag{25}$$

implies the symmetrized identity

$$\sum_i J_i J_{(a}J_{b)} J_i = \delta_{ab}\mathbf{J}^2 - J_{(a}J_{b)}. \tag{26}$$

We refer to identities of the form (25) and (26) as "$4 \to 2$ identities," because they relate degree 4 polynomials in the generators to degree 2 polynomials. Using $4\to 2$ identities and (21), a straightforward calculation shows that $\mathcal{E}(\boldsymbol{\rho}_{v,w})$ is equal to

$$\left(1-\frac{p}{2}\right)v\cdot J+\left(1-\frac{3p}{2}\right)\sum_{a,b} w_{ab}J_{(a}J_{b)}+p\,\mathrm{tr}(w)\mathbf{1},$$

which implies the stated result, since for spin-1, we have $\mathrm{tr}(w)=1/2$ and $\mathbf{1}=\Sigma_{a,b}\frac{1}{2}\delta_{ab}J_{(a}J_{b)}$. □

Theorem 3 shows that $p=2/3$ is a sort of critical value; the channel at $p=2/3$ maps an arbitrary initial $\boldsymbol{\rho}$ into a density matrix with a Bloch representation; $\mathcal{E}(\boldsymbol{\rho})=d^{-1}(\mathbf{1}+v\cdot J)$. These are a three-dimensional set within the eight-dimensional space of all 3×3 density operators. This is reminiscent of the critical value $p=1-n^{-2}$ for the $\mathfrak{su}_n$ channel, but now there is no value of p for which the channel outputs pure noise.

It is possible to iterate formula (24), with interesting results. Clearly, after n applications of the channel, $v\to[1-(p/2)]^n v$. Consider a W-state, i.e., a state of the form

$$\boldsymbol{\rho}_w=\sum_{a,b} w_{ab}J_{(a}J_{b)},$$

and let $\mathcal{E}^n$ denote n applications of the spin-1 channel.

Theorem 4 (iteration formula): *The action of* $\mathcal{E}^n$ *on* w *is the following:*

$$w\underset{\mathcal{E}^n}{\longrightarrow}F^{(n)}(p)(\mathbf{1}-6w)+w,$$

where $F^{(n)}(p)$ *is a degree* n *polynomial in* p, *determined as follows.* $F^{(1)}(p)=1-3p/2$, *and the* $F^{(n)}$ *for* $n>1$ *are determined by the recursion relation*

$$F^{(n+1)}(p)=\left(1-\frac{3p}{2}\right)F^{(n)}(p)+\frac{p}{4}.$$

Interestingly, this recursion relation has the same coefficients as the transformation (24) of w itself.

C. Higher spin representations

Note that the triple product (21) and quadruple product (25) identities are simply certain elements of the ideal $\mathcal{I}=\ker(\tilde{\phi})$, where $\tilde{\phi}$ is the representation of the universal enveloping algebra, as in (18). The larger this ideal, the more product identities there will be in the representation of interest. For higher spin, we have the following $3\to 1$ identity in the spin-s representation of $\mathfrak{su}_2$:

$$\sum_{i=1}^{3} J_iJ_aJ_i=(\lambda-1)J_a,\quad \text{where } \lambda=s(s+1). \tag{27}$$

There is also a generalization of the $4\to 2$ identity valid for general spin-s,

$$\sum_{i=1}^{3} J_iJ_aJ_bJ_i=(\lambda-2)J_aJ_b+\lambda\delta_{ab}\mathbf{1}-J_bJ_a.$$

The latter has the more convenient symmetrized form,

$$\sum_{i=1}^{3} J_iJ_{(a}J_{b)}J_i=(\lambda-3)J_{(a}J_{b)}+\lambda\delta_{ab}\mathbf{1}. \tag{28}$$

Theorem 5 (higher spin channel): *Let* J_1, J_2, J_3 *be canonical generators for the spin-s representation of* $\mathfrak{su}_2$ *in dimension* $d=2s+1$, *and let* $\mathcal{E}_s$ *denote the spin-s channel. Defining* $\boldsymbol{\rho}_{v,w}=v\cdot J+\Sigma_{a,b}w_{ab}J_{(a}J_{b)}$, *we have*

$$\mathrm{tr}(\boldsymbol{\rho}_{v,w})=1\Leftrightarrow\ \mathrm{tr}(w)=\frac{3}{d\lambda},$$

where $\lambda=s(s+1)$. *The action of the spin-s channel is*

$$\mathcal{E}_s(\rho_{v,w}) = \left(1-\frac{p}{\lambda}\right)v\cdot J + \left(1-\frac{3p}{\lambda}\right)w_{ab}J_{(a}J_{b)} + p\ \mathrm{tr}(w)\mathbf{1}.$$

Proof: A straightforward application of (27) and (28). □

Thus, the action of the spin channel is more complicated than the scaling of a single vector. It is the scaling of a series of symmetric tensors, by different scale factors. This shows that the spin-s channels are never depolarizing channels.

At the critical value $p=\lambda/3$, the channel maps an arbitrary $\rho_{v,w}$ into a matrix with a Bloch representation,

$$\mathcal{E}_s(\boldsymbol{\rho}_{v,w})\big|_{p=\lambda/3} = \frac{1}{d}\mathbf{1} + \frac{2}{3}v\cdot J.$$

It follows that if $p=\lambda/3$, then the channel maps an initial density matrix of the form (23) with $v=0$ into pure noise.

For spin-1, an arbitrary density matrix may be represented as (23), and for higher spin, these are a proper submanifold of the convex cone of all density matrices. For spin-3/2, an arbitrary density matrix may be written in the form

$$\sum_{a,b} w_{ab}J_{(a}J_{b)} + \sum_{a,b,c} u_{abc}J_{(a}J_bJ_{c)},$$

where w and u are completely symmetric tensors. The U-term is traceless, and so we require the W-term to have trace one. As discussed prior to Theorem 3, this means that $\mathrm{tr}(w)=3/(\mathrm{d}\lambda)=1/5$.

D. Finding *v* and *w* from *ρ* in higher spin and pure states

In this brief section we show how to invert the relation (23) for the density operator, and find the coefficient vector v and symmetric tensor w. We do the analysis at arbitrary spin, although for spin higher than 1, not all density matrices have the form (23). The methods will generalize assuming the relevant trace identities can be found.

As in Theorem 5, let J_1, J_2, J_3 be canonical generators for the spin-s representation of $\mathfrak{su}_2$. Note that

$$\mathrm{Tr}(J_aJ_b) = \frac{1}{3}d\lambda\ \delta_{ab} \quad \text{and } \mathrm{Tr}(J_aJ_bJ_c) = i\frac{d\lambda}{6}\epsilon_{abc},$$

where $\lambda=s(s+1)$ and $d=2s+1$, and ϵ_{abc} is the Levi-Civita alternating symbol. It follows immediately from (23) that

$$v_a = \frac{3}{d\lambda}\mathrm{Tr}(\rho J_a).$$

To find w, note the trace identity

$$\mathrm{Tr}(J_{(a}J_{b)}J_{(j}J_{k)}) = f_1(s)\tfrac{1}{2}(\delta_{ak}\delta_{bj} + \delta_{bk}\delta_{aj}) + f_2(s)\delta_{ab}\delta_{jk},$$

where f_i are functions of s, given by

$$f_1 = \mathrm{tr}(J_{(1}J_{2)})^2 = \frac{\lambda d(d^2-4)}{30},$$

$$f_2 = \mathrm{tr}(J_1^2J_2^2) = \frac{\lambda d(1+2\lambda)}{30}.$$

By calculating $\mathrm{tr}(\rho J_{(j}J_{k)})$, we find

$$w_{jk} = \frac{1}{f_1}\left(\operatorname{tr}(\rho J_{(j}J_{k)}) - f_2 \operatorname{tr}(w)\delta_{jk}\right) = \frac{30}{\lambda d(d^2-4)}\operatorname{tr}(\rho J_{(j}J_{k)}) - \frac{2\lambda+1}{d^2-4}\operatorname{tr}(w)\delta_{jk}.$$

For spin-1, $f_1=1/2$, $f_2=1$, and $d\lambda=6$ so

$$v_a = \tfrac{1}{2}\operatorname{tr}(\rho J_a) \quad \text{and } w_{jk} = \operatorname{tr}(\rho J_{(j}J_{k)}) - \tfrac{1}{2}\delta_{jk}.$$

This gives another way to find pure states, if $\rho=|\psi\rangle\langle\psi|$ then

$$v_a = \tfrac{1}{2}\langle J_a\rangle_\psi \quad \text{and } w_{jk} = \langle J_{(j}J_{k)}\rangle_\psi - \tfrac{1}{2}\delta_{jk}. \tag{29}$$

In conclusion, if the pure state $\rho=|\psi\rangle\langle\psi|$ has a representation of the form (23), then we can find its Bloch vector and w-matrix easily.

E. Relation to the Werner-Holevo channel and a new conjecture

Datta[12] has shown that the spin-1 channel at $p=1$ is equivalent to the Werner-Holevo channel[13]

$$\mathcal{E}_{\mathrm{WH}}(\rho) = \frac{1}{d-1}(\operatorname{tr}(\rho)\mathbf{1} - \rho^T). \tag{30}$$

Recall that in our notation, $M_0=\sqrt{1-p}\mathbf{1}$, so taking $p=1$ eliminates the identity from the set of Kraus operators. For $p<1$ and for the spin-s representation with $s>1$, we may view the spin channel as a generalization of the WH channel.

Amosov *et al.*[11] conjectured that ν_q is multiplicative for tensor product channels,

$$\nu_q(\mathcal{E}^{\otimes m}) \equiv \sup_{\Gamma\in\mathcal{D}(\mathcal{H}^{\otimes m})} \|\mathcal{E}^{\otimes m}(\Gamma)\|_q = \nu_q(\mathcal{E})^m, \tag{31}$$

where $\nu_q(\mathcal{E})=\sup_{\gamma\in\mathcal{D}(\mathcal{H})}\|\mathcal{E}(\gamma)\|_q$ is the maximal ℓ_q-norm. Equation (31) is often called the ℓ_q *multiplicativity relation* or the *AHW conjecture*. Giovannetti *et al.*[14] have conjectured that (31) holds for the Werner-Holevo channel when $d\geqslant 2^{q-1}$.

The Werner-Holevo channel became famous as a counterexample to the AHW conjecture.[11] We infer by Datta's equivalence that the spin-1 channel at $p=1$ gives precisely the same counterexample to the AHW conjecture, stated below. Therefore, multiplicativity does not hold generically in Lie algebra channels. Once it was established that the AHW conjecture does not hold for all $q\geqslant 1$, it was natural to conjecture[15] that it holds for $1\leqslant q\leqslant 2$, and this was recently proved for the WH channel by Alicki and Fannes.[16] If this is true, one would expect additional counterexamples with values of q approaching 2. However, none have yet been reported, except for the WH channel which gives a sequence of counterexamples with q increasing from 4.79 as the dimension d increases. Ruskai, in a private communication to the author, suggested the possibility that Lie algebra channels might provide additional counterexamples with special properties.

Conjecture 1: Lie algebra channels generate counterexamples to the AHW conjecture for a sequence of values of q approaching the boundary of the region in q-space where multiplicativity begins to hold for all channels, assuming there is such a region.

A preliminary investigation in this direction seems promising, and we hope to address this more fully in a future paper.

V. EXCEPTIONAL LIE ALGEBRAS AND CLIFFORD ALGEBRAS

A. Channels based on the exceptional algebra G_2

Let e_j $(j=0,\ldots,7)$ denote the standard basis for the octonions $\mathbb{O}$, where e_0 is the unit. Our notation is compatible with that of Baez,[17] and the proofs of our statements about the octonion algebra can be found there. The Lie group G_2 is the automorphism group of $\mathbb{O}$, so the Lie algebra $\mathfrak{g}_2$ is the derivation algebra of the octonions,

$$\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O}).$$

Derivations act trivially on the identity, and the imaginary octonions $\mathrm{Im}(\mathbb{O})$ form the fundamental seven-dimensional irreducible representation of $\mathfrak{g}_2$.

It is known that if $\mathcal{A}$ is an alternative, nonassociative algebra (such as the octonions), any pair of elements $x,y \in \mathcal{A}$ define a derivation $D(x,y):\mathcal{A} \to \mathcal{A}$ by

$$D(x,y)a = [[x,y],a] - 3[x,y,a], \tag{32}$$

where $[a,b,x]$ denotes the associator $(ab)x-a(bx)$. When $\mathcal{A}$ is a normed division algebra, every derivation is a linear combination of derivations of this form. For the octonion algebra, the elements

$$D(e_1,e_i), \quad D(e_2,e_j), \quad \text{and } D(e_4,e_k)$$

for all $i>1$, $j>2$, and $k>4$, are linearly independent and there are 14 such elements, so they are a basis for $\mathfrak{g}_2$. Define the notation

$$d_{i,j} = \tfrac{1}{2}D(e_i,e_j).$$

This is one possible basis for the Lie algebra $\mathfrak{g}_2$, but we will use another more suited for our purposes. The fact[18] that the six-dimensional sphere S^6 may be viewed as a $G_2/\mathrm{SU}(3)$ coset space, implies a corresponding decomposition of the algebra:

$$\mathfrak{g}_2 = \mathfrak{m} + \mathfrak{h}, \quad \mathfrak{h} \cong \mathfrak{su}_3,$$

where $\mathfrak{m}$ is a six-dimensional subspace. We find a basis adapted to this decomposition. The basis vectors for $\mathfrak{m}$ are simply expressed as $m_i = d_{1,i+1}$, while

$$h_1 = d_{12} + 2d_{47}, \quad h_2 = d_{13} - 2d_{46}, \quad h_3 = d_{14} - 2d_{27},$$

$$h_4 = d_{15} + 2d_{26}, \quad h_5 = d_{16} - 2d_{25}, \quad h_6 = d_{17} + 2d_{24},$$

$$h_7 = \sqrt{3}d_{23}, \quad h_8 = d_{23} + 2d_{45}$$

are a basis for $\mathfrak{su}_3$. Let

$$\beta = \frac{i}{\sqrt{24}}\left(\{m_1,\ldots,m_6\} \cup \frac{1}{\sqrt{3}}\{h_1,\ldots,h_8\}\right)$$

denote a corresponding basis for $\mathfrak{g}_2$. Interestingly, β is an orthonormal basis of $\mathfrak{g}_2$ with respect to the trace form on the seven-dimensional representation space,

$$\mathrm{Tr}_{\mathrm{Im}(\mathbb{O})}(\beta_i\beta_j) = \tfrac{1}{2}\delta_{ij}, \quad \text{therefore } \sum_{i=1}^{14} \beta_i^2 = I_7.$$

The $\mathfrak{g}_2$ channel acts as

$$\mathcal{E}(\boldsymbol{\rho}) = (1-p)\boldsymbol{\rho} + p\sum_{i=1}^{14} \beta_i \boldsymbol{\rho}\, \beta_i.$$

Assume $\boldsymbol{\rho}$ has a Bloch representation with $\mathbf{v} \in \mathbb{R}^{14}$,

$$\boldsymbol{\rho} = \tfrac{1}{7}(I_7 + \mathbf{v}\cdot\boldsymbol{\beta}), \tag{33}$$

then as an intermediate step,

$$\mathcal{E}(\boldsymbol{\rho}) = \frac{1-p}{7}(I + \mathbf{v}\cdot\boldsymbol{\beta}) + \frac{p}{7}\sum_{i=1}^{14}(\beta_i^2 + v_a\beta_i\beta_a\beta_i).$$

The sum of β_i^2 gives the identity, with a factor of $p/7$ to cancel the $-p/7$, and (miraculously) the term which is cubic in β's vanishes identically. This is due to the following remarkable $3\to 0$ identity

$$\sum_i \beta_i\beta_a\beta_i = 0 \quad \text{for all } a,$$

as may be checked explicitly. Therefore, the $\mathfrak{g}_2$ channel (restricted to its Bloch manifold) is the simplest of all. It is a true depolarizing channel, shrinking its Bloch vector by a factor of $1-p$,

$$\mathcal{E}(\boldsymbol{\rho}) = \tfrac{1}{7}(I + (1-p)\mathbf{v}\cdot\boldsymbol{\beta}).$$

We emphasize, however, that the $\mathfrak{g}_2$ channel is almost certainly *not* a depolarizing channel outside the Bloch manifold, though we have not proven this.

B. Channels based on the Clifford algebra

Let $\langle,\rangle$ be a nondegenerate bilinear form on V, a d-dimensional vector space. A representation of the Clifford algebra associated to $(V,\langle,\rangle)$ is a map $\gamma: V\to\mathfrak{gl}(V)$ satisfying

$$\{\gamma(x),\gamma(y)\} = \langle x,y\rangle\mathbf{1},$$

where the left-hand side is an anticommutator. The representation is *Hermitian* if the image of γ is contained in $H(V)$, the (Hilbert) space of Hermitian operators on V.

Theorem 6 (Clifford algebra channel): *Given a Hermitian representation of the Clifford algebra, and a finite collection of nonzero vectors* $x_1,x_2,\ldots,x_n\in\mathbb{R}^d$, *then*

$$\mathcal{E}_{C\ell}(\boldsymbol{\rho}) \equiv \left(\sum_{i=1}^{n}\langle x_i,x_i\rangle\right)^{-1}\sum_{i=1}^{n}\gamma(x_i)\boldsymbol{\rho}\,\gamma(x_i) \tag{34}$$

is a CPT map.

Proof: The operator is completely positive because it is already in the form of an operator sum representation. We check that it is trace preserving. By cyclicity of the trace,

$$\mathrm{Tr}(\mathcal{E}_{C\ell}(\boldsymbol{\rho})) = \left(\sum_{i=1}^{n}\langle x_i,x_i\rangle\right)^{-1}\sum_{i=1}^{n}\mathrm{Tr}(\boldsymbol{\rho}\,\gamma(x_i)^2).$$

However, $\gamma(x_i)^2=\frac{1}{2}\{\gamma(x_i),\gamma(x_i)\}=\langle x_i,x_i\rangle\mathbf{1}$ using the Clifford algebra. The sum of such terms decouples from $\mathrm{Tr}(\boldsymbol{\rho})$ and exactly cancels the prefactor.

Although the proof of Theorem 6 is trivial, the result may not be easily obtained by inspecting any of the standard matrix representations. Taking the Weyl representation of the γ matrices in $d=4$, writing out the CPT map $\gamma(x)\boldsymbol{\rho}\gamma(x)+\gamma(y)\boldsymbol{\rho}\gamma(y)$ for general x, y, $\boldsymbol{\rho}$ as an explicit matrix takes a full page.

As we have seen in other examples, the computational methods used in this paper are most effective when an arbitrary density matrix can be written in terms of the generators of the symmetry algebra. For the Weyl representation of the Clifford algebra, there is a convenient basis consisting of antisymmetric combinations of γ matrices, which we summarize in the following:

$\mathbf{1}$	one of these
γ^μ	four of these
$\gamma^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]=\gamma^{[\mu}\gamma^{\nu]}$	six of these
$\gamma^{\mu\nu\rho}=\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}=i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$	four of these

$$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5 \qquad \text{one of these}$$

These 16 matrices form a basis for the space $\mathfrak{gl}(\mathbb{R}^4)$. One can therefore write any 4×4 density matrix as a linear combination of these matrices with coefficients that are tensors of rank 4, and use γ matrix identities to calculate the action of the CPT map (34).

VI. THE BLOCH MANIFOLD

A. General results

The Bloch manifold is a geometrical space which is naturally associated to a certain representation of a semisimple Lie algebra $\mathfrak{g}$, by asking the question, which linear combinations of the generators of $\mathfrak{g}$ in that representation are positive-definite, trace-one Hermitian matrices, i.e., physical states? For any preferred class of matrices (such as those with non-negative eigenvalues) one can define a manifold from a representation in a similarly basis-dependent way, but for the application to quantum physics, we restrict interest to density matrices.

Since the action of the $\mathfrak{su}_n$ channel is most simply expressed as a rescaling of the Bloch vector and any n-level density matrix admits a Bloch representation, all that remains for a complete mathematical description of the $\mathfrak{su}_n$ channel is to know the set of vectors $v \in \mathbb{R}^{n^2-1}$ which correspond to positive-definite matrices.

The simplest example of why this is an important question for other representations is formula (22), which gives the action of the spin-1 channel on density matrices admitting a Bloch representation as

$$\mathcal{E}(\boldsymbol{\rho}_v) = \frac{1}{3}\left(\mathbf{1} + \left(1 - \frac{p}{2}\right) v \cdot J\right).$$

Thus, the spin-1 channel also rescales the Bloch vector and it is of interest to know which Bloch vectors give rise to physical states.

In any representation which has a $3\to 1$ identity, i.e., an expression for $\Sigma_i \alpha(X_i)\alpha(X_j)\alpha(X_i)$ in terms of the generators $\alpha(X_k)$, it follows that any density matrix admitting a Bloch representation transforms very simply under the action of the Lie algebra channel.

Definition 3 (Bloch manifold): Choose a set of generators $\{X_i\}$ *of a semisimple Lie algebra* $\mathfrak{g}$, *and an irreducible representation* $\alpha:\mathfrak{g}\to\mathfrak{gl}(\mathcal{H})$ *on a* d_α*-dimensional Hilbert space* $\mathcal{H}$. *The* Bloch manifold $\mathcal{V}$ *(in the* X_i *basis) is defined to be the set of vectors* $v \in \mathbb{R}^k$ ($k = \dim \mathfrak{g}$), *such that*

$$\boldsymbol{\rho}(v) = \frac{1}{d_\alpha}\left(\mathbf{1} + \sum_i v_i \alpha(X_i)\right), \tag{35}$$

is a positive semidefinite, Hermitian matrix. A density matrix which can be written in the form (35) *is said to possess a* Bloch representation, *and the point* $v \in \mathcal{V}$ *is said to be a* valid Bloch vector.

Theorem 7 (closure property): *The Bloch manifold is a closed set in* $\mathbb{R}^k$.

Proof: The matrix $\boldsymbol{\rho}(v)$ is positive iff the lowest eigenvalue $\lambda_{\min}$ of $\mathbf{1}+\Sigma_i v_i\alpha(X_i)$ lies in the set $[0,+\infty)$. The lowest eigenvalue of a matrix is a continuous function of the matrix, so $\lambda_{\min}$ is a continuous function of v. The inverse image of the closed set $[0,+\infty)$ must be closed.

Theorem 8 (general Bloch manifold bound): *Let* α *be a d-dimensional representation of* $\mathfrak{g}$, *let* $k = \dim(\mathfrak{g})$, *and let* X_a *be an orthogonal basis of* $\mathfrak{g}$ *with respect to the Killing form. Then for some constant N*,

$$\mathrm{Tr}(\alpha(X_a)\alpha(X_b)) = Nd\,\delta_{ab}.$$

Moreover, if v is a valid Bloch vector, then

$$v^2 \leqslant \frac{d-1}{N}. \tag{36}$$

In the notation of Sec. II, $N=Z/\dim(\mathfrak{g})$.

Proof: The density matrix $\boldsymbol{\rho}=d^{-1}(\mathbf{1}+v_a\alpha(X_a))$ must satisfy $\mathrm{tr}(\boldsymbol{\rho}^2)\leqslant 1$. But

$$\mathrm{tr}(\boldsymbol{\rho}^2)=d^{-1}(1+Nv^2) \tag{37}$$

which implies the desired result. □

If $\{X_j'\}$ is a second basis of $\mathfrak{g}$, related to the original basis by a matrix A, then the Bloch manifold in the X' basis consists of A^T applied to the Bloch manifold in the X basis. If $\det(A)=1$, this yields an isometric copy of the original manifold, but otherwise the manifold has been stretched in some way. We will see examples of Lie algebra representations which are analogous to the qubit representation, in the sense that the Bloch manifold is a closed ball in some preferred basis.

B. The Bloch manifold for all $\mathfrak{su}_2$ representations

As an example, we give the Bloch manifold relevant to the spin-j representation of $\mathfrak{su}_2$. Let I_{2j+1} be the $(2j+1)$-dimensional identity matrix, and the J_i are the standard generators in the spin-j representation. Let $E_{\min}$ denote the lowest eigenvalue of a matrix. Then

$$E_{\min}\left(I_{2j+1}+\sum_{i=1}^{3}v_iJ_i\right)=1-j\|v\|.$$

The next result follows immediately.

Theorem 9: *The valid Bloch vectors for the spin-j representation of* $\mathfrak{su}_2$ *(with the standard basis) are elements of a closed ball in* $\mathbb{R}^3$ *with radius* $1/j$.

Thus, the picture of the Bloch manifold as a closed ball is not necessarily particular to the qubit system, however, it is certainly not *always* a closed ball. As we shall see below, the Bloch manifold for the defining representation of $\mathfrak{su}_n$ with $n>2$ is a proper subset of the analogous closed ball. To complete the $\mathfrak{su}_2$ case, we remark that the radius receives a multiplicative constant if we rescale the generators; however, the radius always scales as one inverse power of the dimension of the representation.

C. A Bloch submanifold from the Cartan subalgebra

We discuss a method which works for any Lie algebra representation and which always gives a nonempty set of positive semidefinite Hermitian trace-one operators which take the form $\rho=d^{-1}(\mathbf{1}+\Sigma_i v_i\alpha(X_i))$. In Bloch space this set is the interior of a convex polyhedron.

Let $H_1,\ldots,H_r$ denote a basis for the Cartan subalgebra of $\mathfrak{g}$, with $r=\mathrm{rank}(\mathfrak{g})$. As they commute, we may simultaneously diagonalize all $\alpha(H_i)$, and let $h_i^{\,j}$ denote the jth diagonal element of $\alpha(H_i)$. Each vector h^j with components $(h_1^{\,j},h_2^{\,j},\ldots,h_r^{\,j})$ is a weight vector for the given representation.

Assume that the basis $\{X_i\}$ has the Cartan generators $H_1,\ldots,H_r$ as its first r elements. Consider $v\in\mathbb{R}^k$ which are zero except for the first r components. Let $\mathcal{P}$ be the set of $v\in\mathbb{R}^r$ such that $\mathbf{1}+\Sigma_{i=1}^r v_i\alpha(H_i)$ is positive semidefinite. The positivity condition is $1+\Sigma_{i=1}^r v_ih_i^{\,j}\geqslant 0(\forall\ j=1,\ldots,d)$. The solutions of each linear inequality $v\cdot h^j\geqslant -1$ define a half-space $\mathbb{H}_j\subset\mathbb{R}^r$, and the valid Bloch vectors lie in their intersection. A bounded intersection of a finite set of half-spaces is called a *polytope*. If one of the weights h^j is the zero vector, it satisfies $v\cdot h^j\geqslant -1$ for all v.

Definition 4: Define the Bloch-Cartan polytope *as*

$$\mathcal{P}=\{v|v\cdot h^j\geqslant -1\ \ \forall\ \ j\}=\bigcap_{j=1}^{d_\alpha}\mathbb{H}_j. \tag{38}$$

For representations of non-Abelian Lie algebras, $\mathcal{P}\subsetneq\mathcal{V}$. *A priori*, an intersection of half-spaces is either a finite or semi-infinite polyhedron; however (36) implies $v^2\leqslant(d-1)/N$, so $\mathcal{P}$ is always bounded. This provides a geometric proof that a semisimple rank r algebra will not have any irreps

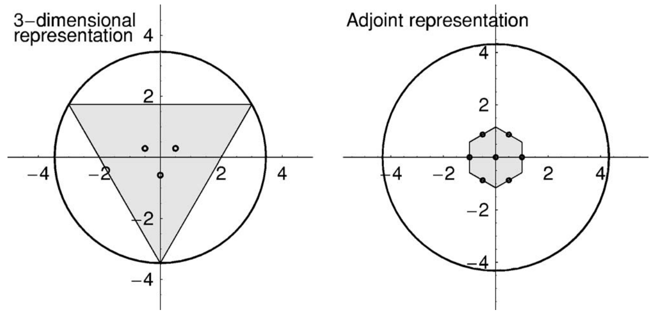

FIG. 1. The shaded region is the intersection of the linear inequalities $v \cdot h^j \geqslant -1$, where h^j are the weight vectors for the indicated representation of $\mathfrak{su}_3$, shown as small circles. Points in the shaded region represent positive semidefinite density matrices. The outer circle shows Bloch vectors which saturate the bound (36).

of dimension smaller than $r+1$, since we need $r+1$ half-spaces to define a finite polyhedron in $\mathbb{R}^r$.

Figure 1 shows two examples for $\mathfrak{su}_3$. This algebra has $r=\mathrm{rank}(\mathfrak{su}_3)=2$ Cartan generators so the Bloch-Cartan polytope can be drawn in $\mathbb{R}^2$. With the conventions of Georgi,[9] the weight vectors h^1, h^2, h^3 for the three-dimensional representation of $\mathfrak{su}_3$ are $\left(\pm\frac{1}{2}, \sqrt{3}/6\right)$ and $(0, -\sqrt{3}/3)$, and the convention $\mathrm{Tr}(T_a T_b)=\frac{1}{2}\delta_{ab}$ for the Gell-Mann matrices implies $N=1/6$. The bound (36) then gives $|v| \leqslant \sqrt{12}$. Note that $v^2=12$ is precisely the circumscribing circle of the polygon. Therefore, in this case our simple method using the Cartan subalgebra produces points up to and including the boundary of the Bloch manifold. These may or may not include pure states. For the adjoint representation $N=3/8$ and the bound (36) gives a boundary circle larger than the polygon by a factor of $\sqrt{14}$.

D. The Bloch manifold for the standard representation of $\mathfrak{su}_n$

We now discuss the structure of the Bloch manifold for the defining representation of $\mathfrak{su}_n$. First, Theorem 8 with $N=2/n$ gives a simple bound,

$$v^2 \leqslant \frac{n(n-1)}{2}. \tag{39}$$

By Descartes' rule of signs, an algebraic equation of degree D with real roots

$$\sum_{j=0}^{D} (-1)^j a_j x^{n-j} = \prod_{i=1}^{D} (x - x_i) = 0, \quad x_i \in \mathbb{R}$$

has all roots non-negative if and only if $a_i \geqslant 0$ for all $i=0,\ldots,D$. Such an equation arises as the characteristic polynomial for a Hermitian matrix. Therefore the Bloch manifold for the n-dimensional irrep of $\mathfrak{su}_n$ is given by the set of $v \in \mathbb{R}^{n^2-1}$ such that the characteristic polynomial $\mathrm{ch}_{\boldsymbol{\rho}(v)}(x)$ has only non-negative coefficients. Some related results for $\mathfrak{su}_n$ were recently reported by Kimura.[19]

The coefficients a_i can be calculated for any specific example using $a_0=1$ and Newton's formula,

$$a_j = \frac{1}{j}\sum_{q=1}^{j} (-1)^{q-1} c_q a_{j-q}, \tag{40}$$

where $c_q=\Sigma_i x_i^q=\mathrm{tr}(\boldsymbol{\rho}(v)^q)$. The largest value of j that we need to consider is $j=d$, the dimension of the Hilbert space.

Naturally, calculating $c_q=\mathrm{tr}(\boldsymbol{\rho}^q)$ reduces to calculating the traces of products of at most q generators of $\mathfrak{su}_n$. Since

$$a_1 = c_1 = 1 \quad \text{and } a_2 = \tfrac{1}{2}(1 - c_2),$$

the condition $a_2 \geqslant 0$ is equivalent to tr $\boldsymbol{\rho}^2 \leqslant 1$, which leads to (39).

Using (40), the condition $a_3 \geqslant 0$ reduces to $c_3 \geqslant 1/2(3c_2 - 1)$, but c_2 is given by (37), and a similar calculation shows that

$$c_3 \equiv \mathrm{tr}\, \boldsymbol{\rho}^3 = \frac{1}{n^3}(n + 6v^2 + 2v_a v_b v_c d_{abc}).$$

The calculations up to this point have been valid for $\mathfrak{su}_n$ for all n. To completely solve the problem for $n > 3$, we need to know $c_4, c_5, \ldots, c_n$. However, we can completely calculate the Bloch manifold for the $\mathfrak{su}_3$ channel in closed form. For $n=3$, we note that

$$\det(v \cdot \lambda) = \tfrac{2}{3} d_{ijk} v_i v_j v_k$$

so the condition $c_3 \geqslant 1/2(3c_2 - 1)$ (for $n=3$) can be expressed as $\det(v\lambda) > -1$ and $v^2 \leqslant 1 + \det(v\lambda)$. Therefore, the Bloch manifold for the **3** of $\mathfrak{su}_3$ admits the following expression, beautiful in its simplicity:

$$\mathcal{V}_{\mathfrak{su}_3} = \{v \in \mathbb{R}^8 : v^2 \leqslant \min(3, 1 + \det(v\lambda))\}. \tag{41}$$

E. Bloch manifold for the seven of G_2.

In our calculation of the $\mathfrak{g}_2$ channel, we explicitly constructed a basis β of $\mathfrak{g}_2$ using its definition as $\mathfrak{Der}(\mathbb{O})$. This basis was normalized so that

$$\sum_a \beta_a^2 = \mathbf{1}, \quad \mathrm{tr}(\beta_a \beta_b) = \frac{1}{2}\delta_{ab}.$$

Theorem 8 gives

$$|v| \leqslant 2\sqrt{21}.$$

This proves that the $\mathfrak{g}_2$ Bloch manifold is contained in a closed ball of radius about 9.2. However, the true radius is much smaller, as we will now show. By $\mathfrak{g}_2$ symmetry, the β's satisfy the identity

$$\mathrm{tr}(v\beta)^q = 0, \quad (\forall\ v \in \mathbb{R}^{14}), \quad q \text{ odd},$$

where $v\beta = \Sigma_i v_i \beta_i$. Further, for certain even values of q, $\mathrm{tr}(v\beta)^q$ may have a simple expression. For example,

$$\mathrm{tr}(v\beta)^2 = \frac{v^2}{2}, \quad \mathrm{tr}(v\beta)^4 = (\mathrm{tr}(v\beta)^2)^2 = \frac{v^4}{4}. \tag{42}$$

The $\mathfrak{g}_2$ trace identity $\mathrm{tr}(v\beta)^4 = (\mathrm{tr}(v\beta)^2)^2$ is not easy to prove. It is true because for $\mathfrak{g}_2$ and some other algebras, every fourth-order Casimir invariant is expressible in terms of the second-order invariant, as shown by Okubo.[20] Recently a simpler proof, together with other interesting trace identities, was given by Macfarlane and Pfeiffer,[21] see their Eq. (4.36).

Enforcing $c_3 \geqslant 1/2(3c_2 - 1)$ gives a refinement,

$$|v| \leqslant 2\sqrt{7} \approx 5.3.$$

Requiring $a_4 \geqslant 0$ gives $v^2 \leqslant 8(10 - \sqrt{65})$, so $|v| \leqslant 3.93$. The coefficients are such that $a_5 \geqslant 0$ for all v, and $\mathrm{tr}(v\beta)^n$ for $n \geqslant 6$ has no simple expression analogous to (42), so we have taken the simple analysis of the $\mathfrak{g}_2$ Bloch manifold as far as it will go.

F. Pure states in the Bloch manifold

Let the representation matrices be denoted by $X_a, a=1,\ldots,k$. If v is in the Bloch manifold, so that

$$\boldsymbol{\rho}_v = d^{-1}\left(\mathbf{1} + \sum_a v_a X_a\right)$$

is a density matrix, it is particularly easy to determine whether $\boldsymbol{\rho}$ is pure. If the products $X_a X_b$ are linearly independent from X_a (i.e., there is no $2\to 1$ identity) then $\boldsymbol{\rho}^2 \neq \boldsymbol{\rho}$ and the state is never pure.

On the other hand, if the representation has a $2\to 1$ identity of the type satisfied by the fundamental representation of $\mathfrak{su}_n$,

$$X_a X_b = \beta\delta_{ab}\mathbf{1} + \sum_c Q_{abc}X_c, \tag{43}$$

then there can be pure states, and we have a complete characterization of them.

Theorem 10 (pure Bloch states): *If the* $2\to 1$ *identity* (43) *holds, then a Bloch state* $\boldsymbol{\rho}_v$ *is pure if and only if*

$$1 + \beta v^2 = d \quad \textit{and} \sum_{a,b} v_a v_b Q_{abc} = \left(1 - \frac{2}{d}\right) v_c,$$

for all $c=1,\ldots,k$.

Proof: This follows from $\boldsymbol{\rho}_v{}^2=(1/d^2)(1+\beta v^2)\mathbf{1}+(1/d^2)\Sigma_{abc}(2v_c+v_a v_b Q_{abc})X_c$. □

It is interesting to see how Theorem 10 specializes to $d=2$. In that case, $Q_{abc}=0$ and also $1-2/d=0$, so the second equation is always satisfied. The first equation amounts to $v^2=1/\beta$, and $\beta=1$, so this just says that v is on the boundary of the Bloch sphere, which is the well-known characterization of pure states.

Unfortunately, $2\to 1$ identities almost never hold, excepting of course the fundamental representation of $\mathfrak{su}_n$, because products $X_a X_b$ tend to be linearly independent from the representation matrices X_a if the dimension of the vector space is large enough to allow this.

G. Summary of Bloch manifold technology

The Bloch manifold is defined to contain the positive-semidefinite density matrices and is given (in a certain basis) by the solution of a system of polynomial inequalities in the components of the Bloch vector v. These inequalities come from enforcing positivity of the density matrix, $\boldsymbol{\rho}_v \geqslant 0$. It is easy to see that the Bloch manifold is bounded within a ball, by enforcing the inequality $\mathrm{tr}(\boldsymbol{\rho}^2)\leqslant 1$. The Bloch manifold for the **3** of $\mathfrak{su}_3$ can be calculated exactly, and also in principle for $\mathfrak{g}_2$. In the latter case, it is bounded within a ball of radius <3.93. In any representation of any Lie algebra, if a $2\to 1$ identity (43) holds, then pure states lie on the surface of a sphere of squared radius $(d-1)/\beta$.

What we have defined and studied here should rightly be called the *linear Bloch manifold*, because already for the spin-1 channel, one needs to represent the density matrix as $v\cdot J + \Sigma_{a,b} w_{ab} J_{(a}J_{b}$. So the full geometry of the space of 3×3 density matrices is described by placing nontrivial conditions on both v and w, and similar remarks apply in higher dimensions.

The positivity constraint $\boldsymbol{\rho}\geqslant 0$ can always be solved by the method of Sec. VI D, but its complete solution requires knowledge of $\mathrm{tr}(\boldsymbol{\rho}^q)$ for all $q\leqslant d$ for a d-dimensional quantum system. Also, $\boldsymbol{\rho}$ itself may not be representable as a linear combination of generators, but will require products of m generators where m grows with the dimension. Thus to apply the method of Sec. VI D, one needs a trace identity for a product of md generators. The study of such identities is an active branch of research.[21]

VII. CONCLUSIONS

The main results are as follows:

(1) Definition 2 and Theorem 2 which define the Lie algebra channel and provide conditions for this channel to be trace-preserving (and hence physically realizable),
(2) identification of the $\mathfrak{su}(n)$ channel (in its standard representation) as a depolarizing channel,
(3) computation of the action of the $\mathfrak{su}(2)$ channel in its Three-dimensional representation (the spin-1 case) on a general Hermitian 3×3 matrix, and on some special pure states,
(4) definition and analysis of channels based on the exceptional Lie algebra $\mathfrak{g}_2$ and the Clifford algebra,
(5) a description of the positive semidefinite matrices which represent quantum states for various Lie algebra channels, and a general method for proving positivity from Lie algebraic trace identities.

In conclusion, the mathematical problems posed by quantum information theory and, more generally, by finite-dimensional quantum systems often involve computations which are most easily handled using techniques from representation theory of Lie algebras. In many cases, the feasibility of these calculations rests on the availability of trace identities,[20,21] of which many are known but surely many more remain to be discovered.

It is remarkable and unexpected by this author how much the theory of quantum channels and the classical theory of Lie algebras seem to dovetail. In some cases, classical results about Lie algebras receive independent proofs based on physical intuition. This is surely a math/physics bridge.

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