

Geometric Representation Theory, Spring 2021

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All ideas from these notes were shamelessly stolen from various lecture notes in the literature. I mention a few: Arkhipov, Bernstein, Gaitsgory, Ginzburg, and Gunningham. Thanks to all participants for their live comments and suggestions, often uncredited. All mistakes are mine; please notify me of corrections.

These notes are written in the order that I gave the lectures; that is, not optimally. In particular, some topics were introduced too soon (equivariant D -modules and twisted Beilinson-Bernstein), and a better plan would've had category \mathcal{O} appear a bit earlier. The reader is encouraged to proceed non-linearly, e.g. to start with category \mathcal{O} .

1 Lecture 1 (2021-02-08): Introduction

Here is a basic question in representation theory.

Question 1.1. Let G be a reductive algebraic group (resp. Lie group) over \mathbb{C} . What are the irreducible finite-dimensional rational (resp. continuous) representations of G ?

Example 1.2. Let $G = GL_2(\mathbb{C})$, and $V := \mathbb{C}^2$ denote the standard representation. The irreducible representations are in bijection with determinant twists of symmetric powers of V , i.e. $\text{Sym}^a(V) \otimes \det^b$ for $(a, b) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}$.

Irreducible $SL_2(\mathbb{C})$ -representations correspond to representations of $GL_2(\mathbb{C})$ for which the determinant twist $b = 0$. Irreducible representations of $PGL_2(\mathbb{C})$ correspond to representations of $SL_2(\mathbb{C})$ where the center acts trivially, i.e. where $a \in 2\mathbb{Z}^{\geq 0}$ is even.

The usually strategy involves linearization, i.e. passage to the *Lie algebra* $\mathfrak{g} := T_e(G)$, defined to be the tangent space at the identity. The multiplication in the group induces an additional structure on \mathfrak{g} , i.e. the *Lie bracket*. A representation of G induces a representation of \mathfrak{g} (but not necessarily conversely).

Question 1.3. Let \mathfrak{g} be a semisimple Lie algebra (if you like, take $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$). What are the irreducible finite-dimensional representations of \mathfrak{g} ?

Example 1.4. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of both SL_2 and PGL_2 , since the tangent space is unaffected by killing a dimension 0 subgroup. It has the same irreducible representations as SL_2 .

The strategy for classifying irreducible representations of \mathfrak{g} involves:

- Fixing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (or a maximal torus $T \subset G$) and writing a representation via its eigenspaces:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda.$$

One can plot the eigenspaces in \mathfrak{h}^* , and they should be a subset of a certain lattice determined by \mathfrak{g} .

- The Weyl group $W(T) = N(T)/T$ acts on the eigenspaces by conjugation. In \mathfrak{h}^* , this amounts to reflecting about certain hyperplanes. Irreducible representations are symmetric with respect to this action, so it's helpful to draw them.
- Fixing a Borel subgroup (conjugate of upper triangular matrices). This determines a notion of positivity, and thus a *dominant chamber* (or “highest chamber” in the order).
- For every point in the lattice in the dominant chamber, one has a unique irreducible finite-dimensional representation of \mathfrak{g} .

We want to know more than this: we want to have some grasp on what these irreducibles “look like.” A first basic question is: what do the eigenspaces look like in a given irreducible?

Question 1.5. For each irreducible representation of \mathfrak{g} , find a formula for the dimensions of the \mathfrak{h} -eigenspaces (equivalently, the trace of the action of any $h \in \mathfrak{h}$).

The answer to this question is given by the *Weyl character formula*. It is convenient to formulate this in terms of generating functions.

Theorem 1.6 (Weyl character formula). *Let λ be dominant, and V_λ the irreducible representation with highest weight λ . Letting e^λ denote formal variables indexed by the weight lattice and Δ^+ the set of positive roots, we have*

$$\sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda) e^\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.$$

This formula might seem mysterious, but once one is familiar with category \mathcal{O} it isn't hard to derive. Namely, it arises via a certain *BGG resolution* in category \mathcal{O} , which leads to the following formula in the Grothendieck group:

$$[L_\lambda] = \sum_{w \in W} (-1)^{\ell(w)} [M_{w(\lambda + \rho) - \rho}]$$

where L_μ are irreducible objects the M_μ are certain *standard modules* or *Verma modules*, which can roughly be thought of as “free objects in highest weight-representations.”

On the other hand, the abelian category \mathcal{O} has *finite length*, i.e. every object can be written as a finite extension of simple (irreducible) objects. Thus we can ask for the converse.

Question 1.7. Find the composition series for M_λ , with multiplicity.

That is, we want to compute matrices that relate the basis $[M_\lambda]$ and $[L_\lambda]$. But the Weyl character formula only fills in one “row” of a matrix. Thus we need an analogue of the Weyl character formula for *non-dominant highest weights*.

Question 1.8 (Kazhdan-Lusztig problem). Find a formula for $[L_\lambda]$ in terms of $[M_{w \cdot \lambda}]$ for λ not dominant (where $w \cdot \lambda = w(\lambda + \rho) - \rho$).

Conjecture 1.9 (Kazhdan-Lusztig conjecture). Kazhdan and Lusztig define some polynomials $P_{y,w}$ for $y, w \in W$ and conjectured (λ dominant):

$$[L_{w \cdot \lambda}] = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} P_{y,w}(1) [M_{y \cdot \lambda}].$$

Example 1.10. We know some things already:

- The Weyl character formula gives the answer when λ is dominant.
- By the ordering on weights, the matrix must be “upper triangular”.
- If λ is antidominant, then $L(\lambda) = M(\lambda)$.

Remark 1.11. Why is this question harder? Why is the dominant case easier? Well, we know that $L(\lambda)$ is finite-dimensional, and in particular W -symmetric. The usual argument exploits this symmetry to determine multiplicities. In general, when λ is not dominant, $L(\lambda)$ is not symmetric, and we cannot do this.

Example 1.12 ($G = SL_2$). Let us focus on the trivial block of \mathfrak{sl}_2 -representations. Then we have either $\lambda = 0, -2$, and in this order, the matrix looks like

$$\begin{pmatrix} [L_0] \\ [L_{-2}] \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [M_0] \\ [M_{-2}] \end{pmatrix}.$$

Example 1.13 ($G = SL_3$). For $G = SL_3$ we have

$$\begin{pmatrix} [L_0] \\ [L_{-\alpha_2}] \\ [L_{-\alpha_1}] \\ [L_{-2\alpha_1-\alpha_2}] \\ [L_{-\alpha_1-2\alpha_2}] \\ [L_{-2\alpha_1-2\alpha_2}] \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} [M_0] \\ [M_{-\alpha_2}] \\ [M_{-\alpha_1}] \\ [M_{-2\alpha_1-\alpha_2}] \\ [M_{-\alpha_1-2\alpha_2}] \\ [M_{-2\alpha_1-2\alpha_2}] \end{pmatrix}.$$

Remark 1.14. You might have observed a pattern; however, it does not continue for SL_4 , which is already a challenge to draw.

The answer to this seemingly combinatorial question was obtained by geometry, thanks to results by: Riemann-Hilbert, Beilinson-Bernstein (and Brylinski-Kashiwara), Beilinson-Bernstein-Deligne, and Kazhdan-Lusztig. This collection of results is usually regarded as the starting point for *geometric representation theory*.

These results will be the focus of this course. I will preview them now. As a warm-up, let us recall the Borel-Weil-Bott theorem. Recall that reductive groups G have *Borel subgroups* B (e.g. the upper triangular subgroup of GL_n) which are (1) unique up to conjugation, (2) self-normalizing, and (3) have projective homogeneous space G/B . This space G/B is called the *flag variety* and comes equipped with a special point $B \in G/B$; it has a base-point-free version \mathcal{B} called the *variety of Borels* (this uses the fact that B is self-normalizing).

Exercise 1.15. When $G = GL(V), SL(V), PGL(V)$; the flag variety G/B has a realization as the variety of *full flags* $V_0 \subset V_1 \subset \dots \subset V_n \subset V$ where $\dim(V_r) = r$.

The G -equivariant line bundles on G/B are in natural bijection with character representations of B , which are in bijection with characters of $B/[B, B] = H$, i.e. characters of a torus (sometimes called the *universal Cartan*, i.e. there are many choices of a torus inside B , but there is a canonical quotient). Let $\lambda \in X^\bullet(H)$, and $\mathcal{L}(\lambda)$ denote the corresponding line bundle on G/B .

Theorem 1.16 (Borel-Weil-Bott). *Let $w \in W$ be such that $w \cdot \lambda$ is dominant (i.e. the shifted action). Then,*

$$H^\bullet(G/B, \mathcal{L}(\lambda)) = \begin{cases} V_{w \cdot \lambda} & \bullet = \ell(w), \\ 0 & \text{else.} \end{cases}$$

Exercise 1.17. Work out the example $G = SL_2$, where $G/B = \mathbb{P}^1$.

What is the version of Borel-Weil-Bott for Lie algebra representations? We obtain an answer by differentiating the group action. Since G acts on G/B , we obtain a map $\mathfrak{g} \rightarrow \text{Vect}(G/B)$, respecting the bracket, to global vector

fields on G/B . The structure of G -equivariance on a vector bundle on G/B tells us how to identify fibers; since G acts transitively, this means a G -representation is just a representation of the stabilizer at any point. These vector fields, however, only tell us how to move fibers around *locally*. The resulting structure is that of a D -module, roughly speaking, a quasicoherent sheaf on X with an action of the tangent bundle.

Is the global sections (i.e. induction) functor

$$\Gamma(G/B, -) : \mathrm{QCoh}^G(G/B) = \mathrm{Rep}(B) \rightarrow \mathrm{QCoh}^G(\mathrm{pt}) = \mathrm{Rep}(G)$$

fully faithful? The answer is obviously no, since the $\mathrm{QCoh}^G(\mathrm{pt})$ has no Homs between simples, while $\mathrm{QCoh}^G(G/B) = \mathrm{Rep}(B)$ does. We will see, however, that the global sections on D -modules *is* fully faithful. That is, there is a global sections functor

$$\Gamma(G/B, -) : \mathcal{D}(G/B) \rightarrow \mathrm{Rep}(\mathfrak{g})$$

which is fully faithful, with a very understandable essential image (representations with trivial central character). One can further tweak the left-hand side a bit (i.e. fix a singular support, or equivalently impose U -equivariance) to restrict to category \mathcal{O} . This is the *Beilinson-Bernstein theorem*: it provides a connection between category \mathcal{O} and D -modules on the flag variety.

Exercise 1.18. Show that the B -orbits on G/B are indexed by $w \in W$.

Let $j_w : BwB/B \hookrightarrow G/B$ be the inclusion of the *Schubert cell*. We will define, for any map f , two pushforward functors $f_*, f_!$. When f is proper, they coincide; recall that in algebraic geometry, we have adjoint functors (f^*, f_*) and $(f_*, f^!)$ when f is proper. Under Beilinson-Bernstein, the standard modules correspond to $\Delta_w := j_{w,!} \mathcal{O}_{BwB/B}$, the costandard modules correspond to $\nabla_w := j_{w,*} \mathcal{O}_{BwB/B}$, and the simple objects correspond to certain *minimal extensions*. That is, much of the structure of category $\mathcal{O} \subset \mathrm{Rep}(\mathfrak{g})$ can be realized geometrically.

The question remains: why can we decompose objects in this category? The answer arises by passing through the *Riemann-Hilbert correspondence*, which provides a connection between D -modules and *perverse sheaves* (or *constructible sheaves*). Roughly, the Riemann-Hilbert correspondence says that the (abelian) “sheaf of solutions” to a D -module is constructible. Under this correspondence, the minimal extensions correspond to *intersection cohomology sheaves*, which have a topological description.

In the setting of constructible sheaves, we have the celebrated *Beilinson-Bernstein-Deligne decomposition theorem*, which was first proved using reduction to characteristic p and Frobenius weight methods (later proved using mixed Hodge modules by Saito, and again by de Cataldo-Migliorini; see this survey). The same methods, i.e. by enriching this category of sheaves to a category of *mixed sheaves* (very very roughly, an extra “grading”), give rise to a calculation of transfer matrices between these “bases” in K -theory.

2 Lecture 2 (2021-02-10): Local systems

2.1 Betti local systems

Let us begin with a general discussion of local systems.

Definition 2.1 (Betti local system). Let X be a topological space, and G a group. A G -bundle on X is a G -torsor on X .¹ A G -local system is a G -bundle \mathcal{L} over X with parallel transport, i.e. for endpoints $s, t \in X$ and a path $\gamma : s \rightarrow t$, a map of stalks $\mathcal{L}_\gamma : \mathcal{L}_s \rightarrow \mathcal{L}_t$, such that (a) if γ, γ' are homotopic, then $\mathcal{L}_\gamma = \mathcal{L}_{\gamma'}$ and (b) if $\gamma = \gamma'' \circ \gamma'$, then $\mathcal{L}_\gamma = \mathcal{L}_{\gamma''} \circ \mathcal{L}_{\gamma'}$.

Remark 2.2. If G is a topological group, then to a G -bundle one can associate the total space P , which has a projection map $P \rightarrow X$. The advantage of the sheaf-theoretic definition is one can consider bundles for groups in an arbitrary category.

For those coming from algebraic geometry, note that stalks are much more “local” in analytic settings, e.g. if X is Hausdorff.

¹It is possible in various contexts to formulate this as a sheaf of sections, which we leave as an exercise to the reader.

Example 2.3. If $G = GL_n$, then a G -bundle on X is just a rank n vector bundle. Formally, this arises via the *Borel construction*. Let V be the standard n -dimensional representation of GL_n . Then, given a (right) GL_n -bundle P , we can associate to it the vector bundle $E := P \times^{GL_n} V$, i.e. the quotient space $(P \times V)/GL_n$ where $g \in GL_n$ acts by the antidiagonal: $g \cdot (p, v) = (pg, g^{-1}v)$.

Exercise 2.4. Formulate the above in a sheaf-theoretic way.

Exercise 2.5. Give a characterization of SL_n -bundles as some kind of vector bundle. Take your favorite reductive group G , and give a characterization of G -bundles as some kind of vector bundle.

Remark 2.6. Note that the maps in a local system are invertible.

Exercise 2.7. Choose a base point $x \in X$. Show that there is a bijection between local systems on X and group homomorphisms $\pi_1(X, x) \rightarrow G$.

Remark 2.8. One can also work in a “derived” context, i.e. one can formulate what it means to be a local system with values in a *chain complex*. This turns out to be much more involved if done “homotopically correctly.” Namely, a homotopy should not determine an isomorphism of maps of chain complexes, but rather a homotopy of chain complexes, et cetera.

Finally, note that local systems sometimes have an incarnation as locally constant sheaves.

Proposition 2.9. *Assume that X is semi-locally simply connected². Then, there is a bijection between locally constant sheaves on X and local systems on X .*

Proof. Suppose that \mathcal{L} is a local system on X . We claim the underlying bundle is locally constant. Choose $x \in X$ and let U be a neighborhood with contractible loops; then the data of a local system gives a unique isomorphism of stalks between any two points, compatible with composition. Thus, claim: $\mathcal{L}|_U$ is a constant sheaf on U .

Conversely, suppose \mathcal{F} is a locally constant sheaf. We want to produce a map of stalks as specified. Take an open cover of X on which \mathcal{F} is constant. A path has compact image, so there is a finite number of opens U_1, \dots, U_r covering the path; order them sequentially. Then we have a zig-zag of invertible morphisms:

$$\mathcal{F}_s \leftarrow \mathcal{F}(U_1) \rightarrow \mathcal{F}(U_1 \cap U_2) \leftarrow \mathcal{F}(U_2) \rightarrow \dots \leftarrow \mathcal{F}(U_r) \rightarrow \mathcal{F}_s.$$

Claim: this map is independent of choice, and satisfies the desired properties. □

Exercise 2.10. Check the claims in the above argument.

Remark 2.11. In algebraic geometry, locally free \mathcal{O}_X -sheaves are vector bundles. In topology, locally constant k -sheaves are local system (which have an underlying vector bundle). While these might seem similar in form, they are very different in flavor.

2.2 de Rham local systems

Is it possible to formulate the definition of local system in algebraic geometry? Yes, but we have to take a different approach. For simplicity we will restrict to the example of vector bundles.

Definition 2.12. Let k be an algebraically closed field of characteristic zero (e.g. $k = \mathbb{C}, \overline{\mathbb{Q}}_\ell$). Let X be a smooth scheme over k , and let E be a vector bundle on X with sheaf of sections \mathcal{E} . A *connection* on E is a k -linear map of sheaves $\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ satisfying the *Leibniz rule* ($f \in \mathcal{O}_X, \sigma \in \mathcal{E}$):

$$\nabla(f\sigma) = f\nabla(\sigma) + df \otimes \sigma.$$

Remark 2.13. A similar definition can be made in the setting of differentiable manifolds or complex manifolds.

²This means for every point $x \in X$ there is a neighborhood U such that every loop in U can be contracted to a constant map via a homotopy in X (i.e. the homotopy can leave U).

The idea of a connection is that it tells you how to identify fibers of E infinitesimally. Namely, given a vector field θ , we have a *covariant derivative* $\nabla_\theta \in \mathcal{E}nd_k(\mathcal{E})$ by evaluating Ω_X^1 at θ (or equivalently, dualizing the map and treating the connection as a \mathcal{O}_X -linear map $\nabla : \mathcal{T}_X \rightarrow \mathcal{E}nd_k(\mathcal{E})$). The Leibniz rule becomes:

$$[\nabla_\theta, f] = \theta(f).$$

Remark 2.14 (\mathcal{O}_X -linearity on the left vs. right). Note that the \mathcal{O}_X -linearity in this formulation (and the tensor product over \mathcal{O}_X in the previous one) encodes “left \mathcal{O}_X -linearity”, i.e.

$$\nabla_{f\theta} = f\nabla_\theta.$$

Here, $f \in \mathcal{O}_X$ acts on the “target” \mathcal{E} , and encodes the convention that differential operators act on the left:

$$(f \frac{d}{dx})y = f(\frac{d}{dx}y).$$

More formally, $\mathcal{E}nd_k(\mathcal{E})$ is a \mathcal{O}_X - \mathcal{O}_X -bimodule, with the left action corresponding to post-composition and the right action to pre-composition. Under the right action, the rule becomes

$$\nabla_{f\theta} = \nabla_\theta f.$$

These two conventions will lead to the notions of left vs. right D -modules. Let us focus on the left version for now. We will refer to these as *left* and *right* covariant derivatives respectively.

We want to use connections to come up with an infinitesimal notion of parallel transport, but there is a problem: there’s nothing that says we have to end up with the same section if we walk around a contractible loop.

Definition 2.15. The *curvature* $\kappa_\nabla \in \Gamma(X, \Omega_X^2 \otimes_k \mathcal{E}nd_k(\mathcal{E}))$ of a connection is defined by

$$\kappa_\nabla(\theta_1, \theta_2) := \nabla_{\theta_1} \nabla_{\theta_2} - \nabla_{\theta_2} \nabla_{\theta_1} - \nabla_{[\theta_1, \theta_2]}.$$

Exercise 2.16. Let’s take $X = S^2$ to be a sphere embedded in \mathbb{R}^3 . Show that the Levi-Civita connection on the tangent bundle with nonzero curvature.

Intuitively, $[v, w]$ measures the difference between the “infinitesimal flow”: i.e. doing v then w versus w then v . What curvature measures is the difference of the “infinitesimal flow” of sections in \mathcal{E} . This leads the notion of flat connection, which we view as an infinitesimal or “de Rham” incarnation of local systems, i.e. we interpret flatness as an infinitesimal parallel transport.

Definition 2.17 (de Rham local system). We say that (\mathcal{E}, ∇) is a *flat connection* if $\kappa_\nabla = 0$, i.e. if

$$\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s).$$

2.3 Sheaf of differential operators

The above definitions can be made in an algebraic setting as well. We view the covariant derivatives as defining an “action” of \mathcal{T}_X on \mathcal{E} . Our goal is to understand this as modules for some algebra. Recall the following standard definition.

Definition 2.18. A *derivation* on X is an element $D \in \mathcal{E}nd_k(\mathcal{O}_X)$ satisfying the Leibniz rule:

$$D(fg) = D(f)g + fD(g).$$

Derivations assemble into a subsheaf $\mathcal{D}er_X$ of \mathcal{O}_X - \mathcal{O}_X -modules (by composition with multiplication).

Exercise 2.19. Let X be smooth. Show that \mathcal{T}_X can be defined in two equivalent ways: (a) the \mathcal{O}_X -linear dual of Ω_X^1 , and (b) the subsheaf of $\mathcal{E}nd_k(\mathcal{O}_X)$ consisting of derivations. Where does the choice of sided-ness come in?

Definition 2.20. Let X be a smooth scheme. The *sheaf of differential operators* $\mathcal{D}_X \subset \mathcal{E}nd_k(\mathcal{O}_X)$ is defined to be the subalgebra generated by \mathcal{O}_X (viewed as multiplication) and \mathcal{T}_X (viewed as derivations). It is a sheaf of non-commutative³ k -algebras, and a sheaf of \mathcal{O}_X - \mathcal{O}_X -bimodules.

Remark 2.21. Any “word” $f_1\theta_1 \cdots f_r\theta_r f_{r+1}$ of sections in \mathcal{O}_X and \mathcal{T}_X defines an element of \mathcal{D}_X by composition. Furthermore, we have the relation $[\theta, f] = \theta(f)$ due to the Leibniz rule. We will see soon that monomials in the θ_i form a basis for the sheaf.

Our definition doesn’t give us very much to work with in terms of highlighting the structure of the algebra. One can get a better sense using a natural filtration.

Definition 2.22. The sheaf of differential operators \mathcal{D}_X is equipped with a natural *order filtration* defined inductively as follows: $\mathcal{D}_X^{\leq 0} := \mathcal{O}_X$, and $D \in \mathcal{D}_X^{\leq k}$ if $[D, f] \in \mathcal{D}_X^{\leq k-1}$ for all $f \in \mathcal{O}_X$.

Exercise 2.23. Show (using the Jacobi identity) that the order filtration is:

- (a) “strongly” multiplicative, i.e. $\mathcal{D}_X^{\leq i} \mathcal{D}_X^{\leq j} = \mathcal{D}_X^{\leq i+j}$,
- (b) that the bracket decreases order by one, i.e. $[-, -] : \mathcal{D}_X^{\leq i} \otimes_k \mathcal{D}_X^{\leq j} \rightarrow \mathcal{D}_X^{\leq i+j-1}$,
- (c) exhaustive, i.e. $\mathcal{D}_X = \bigcup_{k \geq 0} \mathcal{D}_X^{\leq k}$,
- (d) locally free of finite rank in each stage, i.e. that $\mathcal{D}_X^{\leq k}$ is locally free of finite rank,
- (e) a quantization of the symmetric algebra of tangents, i.e. that $\text{gr}_k(\mathcal{D}_X) = \text{Sym}^k \mathcal{T}_X$.

Remark 2.24. In particular, given a (local) basis $\theta_1, \dots, \theta_n$ for \mathcal{T}_X , the monomials in the θ_i form a basis for \mathcal{D}_X .

Proposition 2.25. We have a natural isomorphism $\mathcal{D}_X \simeq \text{Sym}_{\mathcal{O}_X} \mathcal{T}_X$ as left (or right) \mathcal{O}_X -modules. In particular, \mathcal{D}_X is quasicoherent as an \mathcal{O}_X -module (on either side).

Proof. There is a natural isomorphism of \mathcal{O}_X -modules $\mathcal{D}_X \simeq \text{Sym}_{\mathcal{O}_X} \mathcal{T}_X$, e.g. viewing $\mathcal{O}_X \subset \mathcal{D}_X$ via multiplication, the short exact sequence of \mathcal{O}_X -modules splits:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{D}_X^{\leq 1} \rightarrow \mathcal{T}_X \rightarrow 0.$$

A similar argument works for the rest of the filtration. □

Even though \mathcal{D}_X is \mathcal{O}_X -quasicoherent, this only tells us how to localize \mathcal{D}_X as an \mathcal{O}_X -module. It doesn’t tell us how to localize \mathcal{D}_X as an algebra.

Example 2.26. Let $X = \mathbb{A}^1 = \text{Spec } k[x]$. Then \mathcal{T}_X is free of rank 1, generated by the derivation $\partial := d/dx$. Thus we have

$$\Gamma(X, \mathcal{D}_X) := \frac{k\langle x, \partial \rangle}{[\partial, x] = 1}.$$

Let $U := \mathbb{G}_m = \text{Spec } k[x, x^{-1}]$. We thus know, by \mathcal{O}_X -quasicoherence,

$$\mathcal{D}_X(U) = \frac{k\langle x, \partial \rangle}{[\partial, x] = 1} \otimes_{k[x]} k[x, x^{-1}].$$

However, this doesn’t tell us the algebra structure on the right-hand side. To do so, we need to know how the derivation ∂ acts on x^{-1} . Calculus tells us the answer:

$$\mathcal{D}_X(U) := \frac{k\langle x, x^{-1}, \partial \rangle}{[\partial, x] = 1, [\partial, x^{-1}] = -x^{-2}}.$$

Example 2.27 (On affine space). Let V be a vector space. Then

$$\mathcal{D}_V = \frac{k\langle V \oplus V^* \rangle}{[\theta, v] = \theta(v)}$$

where $\theta \in V^*$ and $v \in V$. More explicitly, given a basis $x_1, \dots, x_n \in V^*$ and a dual basis $\partial_1, \dots, \partial_n \in V$ (which one thinks of as the corresponding vector fields, one has the relations $[\partial_i, x_j] = \delta_{ij}$.

³The notion of a “noncommutative R -algebra” A is only well-behaved when R is central in A .

Exercise 2.28. The sheaf \mathcal{D}_X is well-defined, but not well-behaved, when X is not smooth. Show that for $X = \operatorname{Spec} k[x, y]/xy$, the module $D_X := \Gamma(X, \mathcal{D}_X)$ is not finitely generated.

Let's do one final example: \mathbb{P}^1 .

Example 2.29. Let $X = \mathbb{P}^1$. Let $U_x = \operatorname{Spec} k[x]$ and $U_y = \operatorname{Spec} k[y]$ be an open affine cover (with $xy = 1$). How do we glue \mathcal{D}_{U_x} and \mathcal{D}_{U_y} ? On the intersection, we have two derivations ∂_x and ∂_y , which are related by:

$$\frac{dx}{dy} \frac{d}{dx} = \frac{d}{dy}, \quad \text{i.e. } -x^2 \partial_x = \partial_y.$$

3 Lecture 3 (2021-02-12): D-modules

Definition 3.1. Let X be a smooth k -scheme. We define the category $\operatorname{DMod}^\ell(X)$ of *left D-modules* on X to be the (abelian) category of sheaves of \mathcal{O}_X -quasicoherent left \mathcal{D}_X -modules, and the category of *right D-modules* by $\operatorname{DMod}^r(X)$. We will omit this superscript when it is implied.

Exercise 3.2. Show that a left (resp. right) D -module structure on a locally free sheaf \mathcal{E} is equivalent to a left (resp. right) flat covariant derivative.

Remark 3.3. One can define a \mathcal{D}_X -*quasicoherent* module to be a \mathcal{D}_X -module which locally has a presentation

$$\mathcal{M}|_U = \operatorname{coker} \left(\bigoplus_{j \in J} \mathcal{D}_U \rightarrow \bigoplus_{i \in I} \mathcal{D}_U \right).$$

Since \mathcal{D}_X is quasicoherent, \mathcal{O}_X -quasicoherence implies \mathcal{D}_X -quasicoherence.

Definition 3.4. We say \mathcal{M} is \mathcal{D}_X -*coherent* if \mathcal{M} is locally finitely presented.⁴ We denote the full subcategory of \mathcal{D}_X -coherent D -modules by $\operatorname{DMod}_c(X)$.

3.1 Singular support

We are interested in compatible filtrations on D -modules. Let's work in a general context.

Definition 3.5. A *filtered algebra* A is a (possibly non-commutative) algebra equipped with an increasing, exhaustive, non-negative, multiplicative filtration. That is, there is a filtration $F_\bullet A$ (for $\bullet \geq 0$) such that $F_k A \subset F_{k+1} A$, $A = \bigcup_{n \rightarrow \infty} F_n A$, $F_k A = 0$, $1 \in F_0 A$, and $(F_i A)(F_j A) \subset F_{i+j} A$.

A *filtered left module* is an A -module M equipped with an increasing, exhaustive, left-bounded⁵ multiplicative filtration $F_\bullet M$.

There is a formal construction that is useful in working with filtered modules.

Definition 3.6. Let (M, F) be a filtered module for a filtered algebra A . Let t denote a formal symbol with weight grading 1. We define the *Rees construction*:

$$\operatorname{Rees}(M, F) := \bigoplus_{n \in \mathbb{Z}} (F_n M) t^n.$$

The associated graded is $\operatorname{gr}(M, F) = \operatorname{Rees}(M, F) \otimes_{k[t]} k$ and the underlying module can be recovered as $M = (\operatorname{Rees}(M, F) \otimes_{k[t]} k[t, t^{-1}])^{\operatorname{wt}=0}$.

Remark 3.7. If A is a filtered algebra and M a filtered module, then $\operatorname{Rees}(M)$ is a $\operatorname{Rees}(A)$ -module. However, not every $\operatorname{Rees}(A)$ -module corresponds to a filtered A -module, e.g. if there is t -torsion. What's true is that $k[t]$ -free $\operatorname{Rees}(A)$ -modules correspond to filtered A -modules.

⁴More precisely, this is only the correct definition if \mathcal{D}_X is coherent. But it is, see Proposition 1.4.9 in [HTT08].

⁵This is my lazy way of saying $F_n M = 0$ for $n \ll 0$.

Definition 3.8. We say the filtration F on M is *good* if $\text{Rees}(M, F)$ is finitely generated (i.e. coherent) over $\text{Rees}(A)$.

Exercise 3.9. The notion of a filtration being good can be formulated more explicitly (e.g. as in Section 2.1 in [HTT08]). Show that they are equivalent. Show that (M, F) is good if and only if $\text{gr}(M, F)$ is coherent over $\text{gr}(A)$.

Definition 3.10. Let \mathcal{M} be a \mathcal{D}_X -module equipped with a good filtration. In this case, the *singular support* of \mathcal{M} is defined to be

$$\text{SS}(\mathcal{M}) := \text{supp}(\text{gr}(\mathcal{M})) \subset \mathbb{T}_X^*.$$

The following comes from totally general statements in non-commutative algebra. See Theorem 2.1.3 and Appendix D in [HTT08].

Proposition 3.11. *Every \mathcal{D}_X -coherent \mathcal{D}_X -module has a good filtration. The singular support does not depend on the choice of good filtration.*

Proposition 3.12. *The singular support is a conical, closed subscheme of \mathbb{T}_X^* , whose image in X is closed and equal to the usual support.*

Proof. It is closed since support is closed. It is conical since all constructions above are manifestly \mathbb{G}_m -equivariant (i.e. respect the grading in the associated graded construction). Since it is conical and closed, its image in X is closed, and equal to its restriction to X , which is the classical support. \square

Remark 3.13. Note that $\text{gr}(\mathcal{M})$ can depend on the choice of good filtration, but the support (or more generally the characteristic cycle) does not.

Example 3.14. Let $X = \mathbb{A}^1$, and $U = \mathbb{G}_m$. Let's do some examples of D -modules. We let ξ denote the classical limit of ∂ .

- The structure sheaf \mathcal{O}_X is always a left D -module. We can write down a presentation as follows. The map $\mathcal{D}_X \rightarrow \mathcal{O}_X$ given by acting on $1 \in \mathcal{O}_X$ is evidently surjective. One can verify directly (by writing a general element $\sum f_n(x)\partial^n$) that its kernel is given by $\mathcal{D}_X\partial$, and therefore $\mathcal{O}_X \simeq \mathcal{D}_X/\mathcal{D}_X\partial$. It's sometimes convenient to think of the generator of \mathcal{O}_X as the function satisfying the differential equation $\frac{d}{dx} = 0$.

The trivial filtration is a good filtration. In the associated graded, ξ acts by zero, so $\text{gr}(\mathcal{O}_X) = \mathcal{O}_X = \mathcal{O}_{\mathbb{T}_X^*}/\xi$, i.e. the support is the zero section $X \subset \mathbb{T}_X^*$.

- There is a natural \mathcal{D}_X -module structure on $\mathcal{O}_U = k[x, x^{-1}]$. In this case, $1 \in \mathcal{O}_U$ is not a generator, but $1/x$ is. There is a surjective homomorphism $\mathcal{D}_X \rightarrow \mathcal{O}_{\mathbb{G}_m} = k[x, x^{-1}]$ sending $1 \mapsto 1/x$, and $\mathcal{D}_X\partial x \subset \ker$. To show the other inclusion, write a general element in the form $\sum f_n(x)\partial^n x + \sum c_n\partial^n$ for constants c_n (one can verify that $\{x^m\partial^n x, \partial^n \mid m \in \mathbb{Z}^{\geq 0}\}$ maps to a basis in $\text{gr}^n(\mathcal{D}_X)$).

To compute the singular support, we filter $\mathcal{O}_{\mathbb{G}_m}$ by negative power:

$$\mathcal{O}_X \subset \frac{1}{x}\mathcal{O}_X \subset \frac{1}{x^2}\mathcal{O}_X \subset \dots$$

We find that

$$\text{gr}(\mathcal{O}_{\mathbb{G}_m}) = k[x] \oplus kx^{-1} \oplus kx^{-2} \oplus \dots$$

where x acts on positive weight parts by zero, and ξ acts by multiplication by x^{-1} , i.e. $k[x] \oplus k[\xi]$, which has support given by the equation $x\xi = 0$. Note that if we start the filtration at $\frac{1}{x}\mathcal{O}_X$ instead, we get the module $k[x, \xi]/x\xi$, which has the same support.

- There is a D -module on X given by the D_X -module

$$\delta_0 := \frac{D_X}{D_X x}.$$

One can think about the generator as corresponding to the Dirac delta function. This D -module is not finitely generated as an \mathcal{O}_X -module. Its singular support is given by $x = 0$.

Exercise 3.15. Let $X = \mathbb{A}^1$. Consider the D -module given by $M_\lambda := D_X/D_X(x\partial - \lambda)$ for $\lambda \in k$. Which of these are isomorphic?

Remark 3.16. There is a notion of support for modules which are not finitely generated (but the support might not be closed). However, the need to choose a filtration means the notion of singular support does not make sense for non-coherent \mathcal{D}_X -modules (i.e. the support depends on the choice of filtration). In particular, in the absence of a notion of good filtration, one can always take the trivial filtration, where the singular support becomes the zero section.

4 Lecture 4 (2021-02-15): Functors

4.1 Left vs. right D -modules

Example 4.1. Let X be a smooth scheme. Then \mathcal{O}_X is a left $\mathcal{E}nd_k(\mathcal{O}_X)$ -module (essentially by conventions on composing functions), and thus a left D -module. It is not a right D -module.

It turns out that we can identify left and right D -modules. If A is a non-commutative algebra, then the most naive source of an identification of left and right modules is an isomorphism of algebras $A \simeq A^{op}$.

Example 4.2. Let $X = \mathbb{A}^1$. Then we can define an isomorphism $D_X \simeq D_X^{op}$ by

$$x \mapsto x, \quad \partial \mapsto -\partial.$$

For example, the left D -module $\mathcal{M} = \mathcal{D}_X/D_X(\partial x - x)$ is sent to the right D -module $\mathcal{D}_X/(x\partial + x)\mathcal{D}_X$.

However, this notion is very much coordinate-dependent.

Example 4.3. Let $X = \mathbb{G}_m$. Let x be a coordinate, and $y = 1/x$ a different coordinate. If we use the coordinate x to define the isomorphism, the isomorphism above takes (where on the right, remember that we are imposing the relation in D_X^{op}):

$$\partial_x \mapsto -\partial_x =^{op} \partial_y y^2.$$

If we use the coordinate y ,

$$\partial_x = -y^2 \partial_y \mapsto y^2 \partial_y.$$

Notation 4.4. We will use $\omega_X := \Omega_X^{\dim(X)}$ denote the top exterior power, i.e. the dualizing sheaf. In the derived context, it will come with an additional shift (which we ignore for now).

Example 4.5. The dualizing sheaf ω_X is naturally a right D -module via the *Lie derivative*. Recall the Lie derivative is defined by (ι_θ is contraction):

$$\mathcal{L}_\theta(\omega) = \iota_\theta d\omega + d\iota_\theta \omega.$$

This formula simplifies at the two extremes. On functions, $\mathcal{L}_\theta(f) = \iota_\theta(df) = \theta(f)$, i.e. satisfies $\mathcal{L}_{x\theta} = x\mathcal{L}(\theta)$ and defines a left action. On volume forms, $\mathcal{L}_\theta(\omega) = d(\iota_\theta \omega)$, and satisfies $\mathcal{L}_{x\theta}(\omega) = \mathcal{L}_\theta(x\omega)$, i.e. defines a right action:

$$\omega \cdot f := f\omega \quad (f \in \mathcal{O}_X), \quad \omega \cdot \theta := -\mathcal{L}_\theta(\omega) \quad (\theta \in \mathcal{T}_X).$$

Formulas can be found in [HTT08]. Note the presence of the “antipode” sign: it is needed since the Lie derivative still satisfies formulas like $\mathcal{L}_{[\theta_1, \theta_2]} = [\mathcal{L}_{\theta_1}, \mathcal{L}_{\theta_2}]$. It is the interaction with (function) multiples of vector fields that determines left vs. right.

Morally, the idea is that sections of ω_X give a way to integrate over compact regions, i.e. and the D -module action is given by acting on the “input” test function. Explicitly, let ω be a top form; then it defines a distribution (for f in some test function space, probably smooth compactly supported functions):

$$\langle \omega, f \rangle := \int f \omega.$$

For $D \in \mathcal{D}_X$, we have

$$\langle \omega D, f \rangle = \langle \omega, Df \rangle = \int D(f) \omega.$$

This only makes sense if D is a right action on ω . Note that this formula agrees with the Lie derivative only up to a sign:

$$\langle \omega, \theta f \rangle = - \int f L_\theta(\omega)$$

which explains the presence of the sign in the definition above.⁶

If we start with the right-module of distributions rather than functions, we can naturally identify a subalgebra of $\mathcal{E}nd_k(\omega_X)$ rather than $\mathcal{E}nd_k(\mathcal{O}_X)$. The two are related by the following twisting.

Proposition 4.6. *There is a natural isomorphism of k -algebras*

$$\begin{array}{ccc} \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} & \xrightarrow{\simeq} & \mathcal{D}_X^{op} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) \otimes_{\mathcal{O}_X} \mathcal{E}nd_k(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathrm{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) & \longrightarrow & \mathcal{E}nd_k(\omega_X). \end{array}$$

Example 4.7. Let's unwind what the above means in the affine case. In this case, the bundles ω_X and ω_X^{-1} are trivializable, and a choice of trivialization, i.e. a choice of section $\frac{d}{dx_1} \wedge \cdots \wedge \frac{d}{dx_n} \in \omega_X^{-1}$ with dual $dx_1 \cdots dx_n \in \omega_X$, determines a unit-preserving (this is why we need the sections to be dual) isomorphism $\mathcal{D}_X \simeq \mathcal{D}_X^{op}$. This isomorphism depends on our choices, as exhibited in Example 4.3. Note by contrast that the choice of generating section is unique up to scaling in Example 4.2; in particular, for $X = \mathbb{A}^n$ we there is a canonical equivalence $\mathcal{D}_X \simeq \mathcal{D}_X^{op}$.

This diagram gives rise to side-changing functors.

Definition 4.8. We define a functor $\Omega : \mathrm{DMod}^\ell(X) \rightarrow \mathrm{DMod}^r(X)$ by

$$\Omega(\mathcal{M}) := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

$$(\omega \otimes m) \cdot f := \omega f \otimes m = \omega \otimes fm, \quad (\omega \otimes m) \cdot \theta := (\omega \cdot \theta) \otimes m - \omega \otimes (\theta \cdot m).$$

It has an inverse given by

$$\Omega^{-1}(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathrm{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X)$$

$$f \cdot (m \otimes \eta) := mf \otimes \eta = m \otimes f\eta, \quad \theta \cdot (m \otimes \eta) := m \cdot \theta \otimes \eta - m \otimes (\theta \cdot \eta) = m \cdot \theta \otimes \eta(-) - m \otimes \eta(-m).$$

4.2 Functoriality

One issue in defining functors for D -modules is that the functorialities go the wrong way: one can pull back functions, but push forward tangent vectors. Or put another way, given $f : X \rightarrow Y$, there are no maps

$$\mathbb{T}_X^* \longrightarrow \mathbb{T}_Y^*, \quad \mathbb{T}_X^* \longleftarrow \mathbb{T}_Y^*.$$

However, what we do have is a correspondence:

$$\mathbb{T}_X^* \longleftarrow \mathbb{T}_Y^* \times_Y X \longrightarrow \mathbb{T}_Y^*.$$

One can think of the left arrow as capturing the pushforward of tangents (or pullback of cotangents), and the right arrow as capturing the \mathcal{O} -linear pullback and pushforward. The left map is affine (in particular, pushforward does “nothing” on objects), and the right map is smooth and base-changed from $f : X \rightarrow Y$.

⁶Thanks to Alekos Robotis for explaining much of this to me.

Philosophically, functoriality for D -modules passes through this correspondence. One can view the following definition as a “quantization” of the “middle term” $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_{\mathbb{T}^*}$:

Definition 4.9. Let $f : X \rightarrow Y$. We define the following *transfer bimodule*,⁷ i.e. a \mathcal{D}_X - $f^{-1}\mathcal{D}_Y$ -module on X :

$$\mathcal{D}_{X \rightarrow f^*Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

For this expression to make sense, we need the $f^{-1}\mathcal{O}_Y$ -linear of the middle tensor to be compatible with the left and right module structures, i.e.

$$\theta \cdot (xy \otimes D) = \theta \cdot (x \otimes yD), \quad (xy \otimes D) \cdot \theta' = (x \otimes yD) \cdot \theta'.$$

For the right tensor factor, this is obvious. For the left, it is not naively true if we just define $\theta \cdot x = \theta(x)$, since $\theta(xy) \neq \theta(x)y$. A convenient way to think about this is by defining the action to be left multiplication by θ , and “commuting” θ out of the left tensor factor (i.e. since θ does not make sense as an element of \mathcal{O}_X). As a baby example: the \mathcal{T}_X -action on $\mathcal{O}_X = \mathcal{D}_X/\mathcal{D}_X\mathcal{T}_X$ is given by commuting θ past x via the relation $\theta x = x\theta + \theta(x)$. But since $x\theta = 0$, we this is just $\theta \cdot x = \theta(x)$. In this setting, we just keep θ on the right:

$$\theta \cdot (xy \otimes D) = \theta(x)y \otimes D + x\theta y \otimes D = \theta(x)y \otimes D + x\theta \otimes yD = \theta(x)y \otimes D + x \otimes \theta yD.$$

We need to define a \mathcal{T}_X -action on $f^{-1}\mathcal{O}_Y$. The natural candidate is to use the pushforward of tangent vectors. Putting this all together, we define the left module structure by

$$\theta \cdot (x \otimes D) = \theta(x) \otimes D + x \cdot \tilde{\theta}D$$

where $\tilde{\theta}$ is the image under the pushforward of tangents $\mathcal{T}_X \rightarrow f^*\mathcal{T}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{T}_Y$.

Remark 4.10. One could try to write down a $f^{-1}\mathcal{D}_X$ - \mathcal{D}_X -module $f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \omega_X$. This has to be wrong, however. If we try to use this to define a pullback on right D -modules, we send $\omega_Y \mapsto f^*\omega_Y \otimes \omega_X$ (where we only expected ω_X). The issue is that, as above, viewing a derivation as an endomorphism of ω_X , requires us to pass through an untwist by ω^{-1} and a twist by ω_X . This is an inherent asymmetry of our set-up.

Definition 4.11. Let $f : X \rightarrow Y$. We define a functor $f^\dagger : \mathrm{DMod}^\ell(Y) \rightarrow \mathrm{DMod}^\ell(X)$ by

$$f^\dagger(-) := \mathcal{D}_{X \rightarrow f^*Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}(-) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(-).$$

That is, f^\dagger is the usual \mathcal{O} -module pullback. The \mathcal{T}_X -action is given by

$$\theta \cdot (f \otimes m) = \theta f \otimes m + f \cdot \tilde{\theta}m.$$

We define $f_\bullet(-) : \mathrm{DMod}^r(X) \rightarrow \mathrm{DMod}^r(Y)$.

$$f_\bullet(-) := f_*(- \otimes_{\mathcal{D}_X} \mathcal{D}_{X \rightarrow f^*Y}).$$

We define the functors on right (resp. left) D -modules by side-changing.

Since I prefer left D -modules, we can define the pushforward functors for left D -modules using the following transfer bimodule. See Lemma 1.3.4 in [HTT08] for details about this.

Definition 4.12. Let $f : X \rightarrow Y$. We define a $f^{-1}\mathcal{D}_Y$ - \mathcal{D}_X *transfer bimodule* on X :

$$\mathcal{D}_{Y \leftarrow f^*X} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X \rightarrow f^*Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$$

⁷The subscript $f : X \rightarrow Y$ indicates that it is a transfer bimodule for f , and that the \mathcal{D}_X -action is on the left and the \mathcal{D}_Y -action is on the right.

where \mathcal{D}_X acts via inside the “left” $\otimes_{f^{-1}\mathcal{O}_Y}$ and $f^{-1}\mathcal{D}_Y$ acts through the “right” $\otimes_{f^{-1}\mathcal{O}_Y}$ via the formulas in Definition 4.8. We have a pushforward functor on left D -modules $f_{\bullet}(-) : \mathrm{DMod}^{\ell}(X) \rightarrow \mathrm{DMod}^{\ell}(Y)$:

$$f_{\bullet}(-) := f_{*}(\mathcal{D}_{Y \leftarrow f^{-1}X} \otimes_{\mathcal{D}_X} -).$$

Remark 4.13 (Notation confusion). The notation for these functors is somewhat inconsistent in the early literature. My requirement is that the functors $f_!$, f_* , $f^!$, f^* in both the D -modules and constructible sheaves setting should agree under Riemann-Hilbert. Our functor f^{\dagger} above will later lead to the definition $f^! := Lf^{\dagger}[\dim(X) - \dim(Y)]$. Likewise, we will define f_* to be the corresponding derived (left, and right, where appropriate) functor to f_{\bullet} .

Another possible confusion is the difference between the functors for $\mathrm{QCoh}(-)$ vs. for $\mathcal{D}(-)$, which share the same notation. We will just live with this; we will only use the \mathcal{O}_X -linear functors to define the functors for D -modules.

Example 4.14. Let $j : U = \mathbb{G}_m \hookrightarrow X = \mathbb{A}^1$, and $i : \{0\} \rightarrow \mathbb{A}^1$.

- We have $i_{\bullet}k = i_{*}(k \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X) = i_{*}(k \otimes_{\mathcal{O}_{X,0}} (\mathcal{O}_{X,0} \otimes_{\mathcal{O}_X} \mathcal{D}_X)) = \mathcal{D}_X/x\mathcal{D}_X$.
- We have $j_{\bullet}\omega_U = j_{*}(\omega_U \otimes_{\mathcal{D}_U} \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{D}_U) = j_{*}\omega_U$. The right \mathcal{D}_X -action arises by restriction to U , then action. We find that $1/x \in \omega_U$ is a generator, and is killed by $x\partial$. Switching to left modules, we have that $j_{\bullet}\mathcal{O}_U \simeq \mathcal{D}_X/\mathcal{D}_X\partial x \simeq j_{*}\mathcal{O}_U$.
- We have $i^{\dagger}\mathcal{O}_X = k \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{O}_X = k$.
- We have $j^{\dagger}\mathcal{O}_X = \mathcal{O}_U$.

Example 4.15. Let $f : X \rightarrow \mathrm{pt}$. Then, $f_{\bullet}(\omega_X) = f_{*}\omega_X$. When $X = \mathbb{P}^1$, $f_{*}\omega_{\mathbb{P}^1} = 0$. So this notion seems a bit useless. We will see later that if we properly derived these functors, the pushforward will compute de Rham cohomology. We will see in the next lecture that if f is affine (e.g. a closed embedding), then the pushforward as defined above is exact.

Exercise 4.16. Show that if f, g are composable maps, then $(fg)_{\bullet} = f_{\bullet}g_{\bullet}$ and $(fg)^{\dagger} = g^{\dagger}f^{\dagger}$.

5 Lecture 5 (2021-02-17, 2021-02-19): Kashiwara’s theorem

Definition 5.1. For a singular support condition $\Lambda \subset \mathbb{T}_X^*$, we define $\mathrm{DMod}_{\Lambda}(X) \subset \mathrm{DMod}(X)$ to be the full subcategory consisting of D -modules \mathcal{M} such that $\mathrm{SS}(\mathcal{M}) \subset \Lambda$.

Somewhat confusingly, if $Z \subset X$ is a closed subscheme, we let $\mathrm{DMod}_Z(X)$ denote the full subcategory whose *classical support* is contained in Z .

Remark 5.2. This might seem horribly ambiguous, but we’ll basically never run into issues. The reason, as we will see later, is that $\dim(\Lambda) \geq \dim(X)$.

In the previous lecture I mentioned that sometimes the pushforward functor that we defined is not the correct one. It should feel a little weird, since it is not left nor right exact in general. But in the case of a closed immersion, it is the correct notion and furthermore has a right adjoint $i^!$.

Definition 5.3. Let $Z \hookrightarrow X$ be a closed embedding. We define a functor $i_! = i_{\bullet} : \mathrm{DMod}(Z) \rightarrow \mathrm{DMod}(X)$. We define the right adjoint to $i_!$:

$$i^! : \mathrm{DMod}(X) \rightarrow \mathrm{DMod}(Z), \quad i^!(\mathcal{M}) := \mathcal{H}om_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{X \leftarrow Z}, i^{-1}(\mathcal{F})).$$

Exercise 5.4. Define the D -module structure on $i^!(-)$, and show that the functors are adjoint. Show that if X is affine, then on the underlying \mathcal{O}_X -module, $i^!\mathcal{M} \simeq \mathrm{Hom}_{\mathcal{O}_X}(i_{*}\mathcal{O}_Z, \mathcal{M})$ is submodule annihilated by the ideal $\mathcal{I}_Z \subset \mathcal{O}_X$ cutting out $Z \subset X$, and define the \mathcal{D}_Z -module action on it explicitly. What happens when X is not affine? Hint: refer to 1.3.5 and 1.5.14 in [HTT08], but be aware that the notation is different from mine.

Lemma 5.5. *The functors $(i_!, i^!)$ are adjoint. The functor $i_!$ is exact, and its right adjoint $i^!$ is left exact.*

Proof. All right adjoint functors are left exact, and all left adjoint functors are right exact. So, we need to show that $i_!$ is left exact. Recall that (on right modules)

$$i_!(\mathcal{M}) := i_*(\mathcal{M} \otimes_{\mathcal{D}_Z} \mathcal{O}_Z \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X).$$

The functor i_* is automatically left exact, so we need to verify the claim for the tensor products. We can check on local rings; take generators $x_1, \dots, x_n \in \mathcal{O}_{X,x}$ such that $\mathcal{O}_{Z,x} = \mathcal{O}_{X,x}/(x_1, \dots, x_r)$ (roughly, this can be done since locally, $Z \subset X$ looks like the inclusion of a subspace of the tangent space $\mathbb{T}_{X,x}$). We have

$$\mathcal{M}_x \otimes_{\mathcal{D}_{Z,x}} \mathcal{O}_{Z,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{D}_{X,x} = \mathcal{M}_x \otimes_{\mathcal{D}_{Z,x}} \mathcal{D}_{X,x}/(x_1, \dots, x_r)\mathcal{D}_{X,x}.$$

Using the order filtration, one can show that $\mathcal{D}_{X,x}/(x_1, \dots, x_r)\mathcal{D}_{X,x}$ is free as a left $\mathcal{D}_{Z,x}$ -module, with basis given by $\partial_{r+1}, \dots, \partial_n$. \square

Exercise 5.6. Check the last claim.

Remark 5.7. We will later see that $i^! = Li^\dagger[-\text{codim}(Z/X)]$.

Exercise 5.8. Show that if j is an affine open embedding, then j_* is also exact. Why do we need j to be affine?

Let us state it the theorem we want to prove.

Theorem 5.9 (Kashiwara). *Let $Z \subset X$ be a smooth closed subscheme of X . The functor*

$$i_! : \text{DMod}(Z) \rightarrow \text{DMod}_Z(X)$$

is an equivalence of categories.

Proof. Let $i^!$ denote the restriction to $\text{DMod}_Z(X) \rightarrow \text{DMod}(Z)$. We may take local rings at points $x \in Z$, where one can pass to left D -modules via choice of coordinates. In this case, for $\mathcal{O}_Z = \mathcal{O}_X/I$, we have

$$i_!\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X I \otimes_{\mathcal{D}_Z} \mathcal{M}, \quad i^!\mathcal{M} = \text{Hom}_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{M}).$$

Furthermore, since closed smooth subschemes inside smooth subschemes are lci, we can induct on dimension (e.g. by the same Nakayama-type argument on cotangent spaces at local rings), so assume $I = (x)$. Let $\partial \in \mathcal{T}_X$ denote a derivation such that $[\partial, x] = 1$.

Consider the differential operator $D = \partial x$, with transpose⁸ $D^t = x\partial$. Let $m \in \mathcal{M}$ be an eigenvector of D with eigenvalue λ . Then:

- m has D^t -eigenvalue $\lambda - 1$,
- xm has D -eigenvalue $\lambda + 1$ (i.e. x is a “raising operator”),
- ∂m has D -eigenvalue $\lambda - 1$ (i.e. ∂ is a “lowering operator”),
- $xm = 0$ if and only if m has D -eigenvalue $\lambda = 0$ (equiv. D^t -eigenvalue $\lambda - 1 = -1$),
- $\partial m = 0$ if and only if m has D^t -eigenvalue $\lambda - 1 = 0$ (equiv. D -eigenvalue $\lambda = 1$).

That is, fixing an eigenvector $m \in \mathcal{M}$, a priori the action of x, ∂ generates a submodule which looks like:

$$\begin{array}{ccccccc}
 & \partial x = \lambda - 1 & & \partial x = \lambda & & \partial x = \lambda + 1 & \\
 \cdots & \xleftarrow{\quad} & \bullet & \xleftarrow{x} & \bullet & \xleftarrow{x} & \bullet & \xleftarrow{\quad} \cdots \\
 & \xrightarrow{\quad} & & \xrightarrow{\partial} & & \xrightarrow{\partial} & & \\
 & x\partial = \lambda - 2 & & x\partial = \lambda - 1 & & x\partial = \lambda & &
 \end{array}$$

⁸Note we could very well have swapped these roles.

The chain is bi-infinite if and only if $\lambda \notin \mathbb{Z}$. The chain is infinite in the negative direction if and only if $\lambda \in \mathbb{Z}^{\leq 0}$, and infinite in the positive direction if and only if $\lambda \in \mathbb{Z}^{\geq 1}$.

Now, suppose that \mathcal{M} is supported at Z ; then x acts by torsion, and in particular the chain associated to every eigenvector m is negatively infinite.

$$\begin{array}{ccccccc}
 & \partial x = -2 & & \partial x = -1 & & \partial x = 0 & \\
 \cdots & \xleftarrow{\quad} & \bullet & \xleftarrow{x} & \bullet & \xleftarrow{x} & \bullet \xrightarrow{\quad} 0 \\
 & \xrightarrow{\quad} & & \xrightarrow{\partial} & & \xrightarrow{\partial} & \\
 & x\partial = -3 & & x\partial = -2 & & x\partial = -1 &
 \end{array}$$

We claim that every $m \in \mathcal{M}$ is decomposable.⁹ Suppose that k is minimal such that $x^k m = 0$. Then, $x^{k-1}m$ has D -eigenvalue 0, and $\partial^{k-1}x^{k-1}m$ has eigenvalue $-k+1$. Furthermore, $m' = m - \partial^{k-1}x^{k-1}m$ satisfies $x^{k-1}m' = 0$, and one can repeat the process which must eventually terminate.

In particular, (recall that $\mathcal{M} \in \text{Coh}_Z(X)$), if \mathcal{M}_0 is the 0 D -eigenspace, then $\mathcal{M} \simeq k[\partial] \otimes_k \mathcal{M}_0$. By this description, we see that

$$i^! \mathcal{M} = \ker(x) = \mathcal{M}_0, \quad i_! \mathcal{M}_0 = \mathcal{D}_X / \mathcal{D}_X x \otimes_{\mathcal{D}_Z} \mathcal{M}_0 \xleftarrow{\simeq} k[\partial] \otimes_k \mathcal{M}_0 \simeq \mathcal{M},$$

i.e. since $\mathcal{D}_X / \mathcal{D}_X x$ is generated by the ∂^k as a right \mathcal{D}_Z -module, and ∂ must act freely as described above. \square

Exercise 5.10. Adapt the above argument to prove the following proposition.

Proposition 5.11. *A D -module on X is \mathcal{O}_X -coherent if and only if it is \mathcal{O}_X -locally free (i.e. a flat connection).*

Remark 5.12. This allows for a definition of D -module on singular schemes Z : (locally) choose an embedding into a smooth scheme $Z \subset X$, and define $\text{DMod}(Z) := \text{DMod}_Z(X)$. One has to show this definition is independent of choices. One can do this via standard arguments: embed $Z \hookrightarrow X_i$, and show that the pushforward functor induces an equivalence $\text{DMod}_Z(X_1 \times X_2) \simeq \text{DMod}_Z(X_i)$.

6 Lecture 6: Moment map and Springer resolution (Jeffrey Jiang)

See Jeffrey's notes.

7 Lecture 7 (2021-02-22): D -affinity and Barr-Beck

Definition 7.1. We say that X is D -affine if:

- Γ is exact, i.e. $R^i \Gamma(X, \mathcal{M}) = 0$ for all \mathcal{M} and $i > 0$,
- Γ is conservative, i.e. if $\Gamma(X, \mathcal{M}) = 0$, then $\mathcal{M} = 0$.

Example 7.2. Affine schemes are D -affine. We will see soon that projective spaces \mathbb{P}^n are D -affine. We will see, via Beilinson-Bernstein, that flag varieties G/B are D -affine. This can be generalized to show that partial flag varieties G/P are also D -affine.

Remark 7.3. Note that the typical definition of a conservative functor Γ is one such that if $\Gamma(\phi)$ is an isomorphism, then ϕ is an isomorphism (i.e. Γ *reflects isomorphisms*). Sometimes, in the definition of D -affine we see the second condition replaced with the condition that every D -module is *globally generated*, i.e. the map $\epsilon : \mathcal{D}_X \otimes_k \Gamma(X, \mathcal{M}) \rightarrow \mathcal{M}$ is surjective for all \mathcal{M} . These are all equivalent if Γ is left or right exact (which it is).

The following result is essentially formal (i.e. category theoretic), a consequence of Barr-Beck. The geometry takes place in the verification of D -affinity.

⁹This fails if the chain is bi-infinite; there can be “nontrivial gluings” near the “center.”

Proposition 7.4. *If X is D -affine, then the functor (on left-modules)*

$$\Gamma : \mathbf{DMod}(X) \rightarrow \mathbf{Mod}(\mathcal{D}(X))$$

is an equivalence of categories.

Definition 7.5. Consider a pair of adjoint functors

$$\mathbf{C} \xrightleftharpoons[G]{F} \mathbf{D}.$$

Let $T = GF$ denote the endofunctor of \mathbf{C} . It is a *monad*, i.e. there is a unit map of functors $\eta : \text{id}_{\mathbf{C}} \rightarrow T$ and a multiplication map of functors $\mu : T^2 = GF GF \rightarrow T = GF$ induced by the counit $FG \rightarrow \text{id}_{\mathbf{D}}$. These functors satisfy the usual identities. A *module* for T is an object $c \in \mathbf{C}$ and an action map $T(c) \rightarrow c$ satisfying the usual identities.

Theorem 7.6 (Barr-Beck). *In the set-up above, assume that \mathbf{D} has coequalizers, and that the right adjoint G is conservative and preserves coequalizers. Then, there is an equivalence:*

$$\begin{array}{ccc} \mathbf{C} & \xrightleftharpoons[G]{F} & \mathbf{D} \\ \uparrow & \nwarrow \tilde{G} & \\ \mathbf{Mod}_{\mathbf{C}}(T) & & \end{array}$$

The idea behind Barr-Beck is that the monad T gives a way to build canonical *monadic resolutions*. The content of the theorem, which is purely formal, are that under these conditions, this resolution is actually a resolution. Note that in the 1-category context, a resolution is just a presentation; there are derived generalizations of Barr-Beck (e.g. Barr-Beck-Lurie).

Example 7.7. Let $F : \mathbf{Set} \rightarrow \mathbf{Mon}$ denote the free monoid functor, and $G : \mathbf{Mon} \rightarrow \mathbf{Set}$ the forgetful functor. The monad $T = GF$ takes a set to the set X to the set of words in X . The multiplication $T^2(X) \rightarrow T(X)$ is given by interpreting “words of words” as simply words, and the unit map $X \rightarrow T(X)$ is the inclusion of single letter words. A module for T , i.e. $T(X) \rightarrow X$, is a rule for resolving words back into letters. Given a group Y , or a set X with a composition law $T(X) \rightarrow X$, there is a “free resolution” or monadic resolution of Y (resp. X) given by

$$\text{colim}(FGFG(Y) \rightrightarrows FG(Y)) = Y,$$

$$\text{colim}(GF GF(Y) \rightrightarrows GF(X)) = X.$$

These two are related by applying G to the top diagram. Unwinding this: $GF(Y)$ is the free group on the underlying set Y . We claim that Y is the coequalizer of the two maps. One map takes a word of words and formally concatenates them. The other map takes a word of words and resolves the inner grouping of words using the group law. For example:

$$(xy)(yz) \mapsto xy yz, \quad x \cdot y \cdot y \cdot z.$$

Applying the relation to words of words where the outer word has only a single letter generates the group law.

Exercise 7.8. There is a comonadic version of Barr-Beck, obtained by reserving all the arrows. Let $p : U \rightarrow X$ be an open affine cover of a topological space. Show that the comonadic resolution for the adjunction (p^{-1}, p_*) gives rise to the Čech resolution.

Proof of Barr-Beck. The functor \tilde{G} takes an object d to $G(d)$, which is naturally a T -module by the action map $GF G(d) \rightarrow G(d)$ arising via the counit $FG \rightarrow \text{id}_{\mathbf{D}}$. It has a left adjoint \tilde{F} defined to be

$$\tilde{F}(c) = \text{colim}(FT(c) = FGF(c) \xrightleftharpoons[\epsilon(F)]{F(\alpha)} F(c)).$$

Exercise 7.9. Check that (\tilde{F}, \tilde{G}) are adjoint.

Observe that if $c \in \text{Mod}_{\mathbf{C}}(T)$, the unit map $c \rightarrow T(c)$ provides a “splitting” of the diagram, exhibiting c as the colimit in $\text{Mod}_{\mathbf{C}}(T)$:

$$T^2(c) \xrightleftharpoons[\mu(T)]{T(\alpha)} T(c) \xrightarrow{\quad} c.$$

We show that the unit and counit of the adjunction (\tilde{F}, \tilde{G}) are isomorphisms. Since G preserves coequalizers, it proves that the unit is an equivalence:

$$c \rightarrow \tilde{G}\tilde{F} = G \left(\text{colim} \left(FGF(c) \xrightleftharpoons[\epsilon(F)]{F(\alpha)} F(c) \right) \right) = \text{colim} \left(GFGF(c) \xrightleftharpoons[G(\epsilon)]{GF(\alpha)} GF(c) \right).$$

For the counit, note that since G is conservative, it reflects colimits; then apply the above colimit to $c = G(d)$ to obtain:

$$\tilde{F}\tilde{G}(d) = \text{colim} \left(FGF(d) \xrightleftharpoons[\epsilon(FG)]{FG(\epsilon)} FG(d) \right) \longrightarrow d.$$

Note the unit and counit which we require to be isomorphisms are exactly the two monadic resolutions discussed in the example. \square

Proof of D -affinity. Note that right adjoints, e.g. Γ , are automatically left exact. In an abelian category, right exact implies preserves coequalizers. Apply Barr-Beck to the adjoint pair $(\mathcal{D}_X \otimes_k -, \Gamma(X, -))$, and note that the monad is given by tensoring with $\Gamma(X, \mathcal{D}_X)$, and the algebra structure on the monad is the usual algebra structure. The monadic resolution looks like:

$$\mathcal{D}_X \otimes_k \mathcal{D}(X) \otimes_k \Gamma(X, \mathcal{M}) \rightrightarrows \mathcal{D}_X \otimes_k \Gamma(X, \mathcal{M}) \longrightarrow \mathcal{M}.$$

\square

Remark 7.10. One can try to formulate the notion of D -affinity for right D -modules. Since the categories are equivalent, we expect there should be a notion. However, such a notion will necessarily take us into the setting of derived functors. For example, for $X = \mathbb{P}^1$, the functor $\Gamma(\mathbb{P}^1, -)$ cannot be an equivalence of categories for right D -modules, since $\mathcal{M} = \omega_{\mathbb{P}^1}$ has no global sections. However, by Grothendieck duality, its derived global sections are related by duality.

8 Lecture 8 (2021-02-24): Beilinson-Bernstein warm-ups

Finally, we are interested in the following application of Kashiwara, which we can view as a warm-up to Beilinson-Bernstein (or a special case for $G = SL_2$):

Theorem 8.1. *Projective spaces \mathbb{P}^n are D -affine.*

Proof. The main idea will be to factor the pushforward to a point along the sequence (where $V^\times = V - \{0\}$):

$$\mathbb{P}(V) \rightrightarrows V^\times / \mathbb{G}_m \xhookrightarrow{j} V / \mathbb{G}_m \longrightarrow B\mathbb{G}_m \longrightarrow \text{pt}.$$

All of these (sheaf, i.e. not D -module) pushforward functors are exact except for the (non-affine) open immersion j . Note that we use will implicitly stacky language, but all that’s really going on is the addition of a \mathbb{G}_m -equivariance. Recall that \mathbb{G}_m -equivariance for sheaves on affine varieties means the additional structure of a compatible \mathbb{G}_m -weight grading, and that pushing forward along $B\mathbb{G}_m \rightarrow \text{pt}$ amounts to taking \mathbb{G}_m -invariants, i.e. the weight 0 part.

We first establish exactness of $\Gamma(\mathbb{P}(V), -)$. Let $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3 \rightarrow 0$ be a sequence of D -modules on $\mathbb{P}(V)$, and let $p : V^\times \rightarrow \mathbb{P}(V)$. Recall that $p^* = p^\dagger$ on underlying \mathcal{O}_X -modules, and $j_* = j_\bullet$ for open embeddings;

this is critical since D -affinity is about exactness of the \mathcal{O} -module pushforward Γ . We have a non-exact sequence

$$0 \rightarrow j_{\bullet} p^{\dagger} \mathcal{M}_1 \rightarrow j_{\bullet} p^{\dagger} \mathcal{M}_2 \rightarrow j_{\bullet} p^{\dagger} \mathcal{M}_3 \rightarrow 0$$

whose cohomologies (as D -modules) are supported at $0 \in V$. Thus, by Kashiwara, the cohomology groups are in the essential image of $i_!$ where $i : \{0\} \hookrightarrow V$. Note that the \mathbb{G}_m -invariants functor is exact, and thus commutes with cohomology functors. Thus, we need to show that $i_! k$ has no weight 0 part, i.e. $(i_! k)^{\mathbb{G}_m} = 0$. Further note that the Euler vector field (which arises via the \mathbb{G}_m -action) $\theta := \sum x_i \partial_i$ acts on the weight n summand by multiplication by n . Morally, the fact that there are no \mathbb{G}_m -invariants corresponds to the fact that if θ acted by weight 0 in a given direction, it should locally be possible to extend a section in that direction, which is not the case for “skyscrapers.”

More precisely, note that the essential image of $i_!$ consists of sums of the module $i_! k = \mathcal{D}_V / \mathcal{D}_V I \simeq \text{Sym } \mathcal{T}_V$ where $I = \langle V^* \rangle$ is the ideal of definition. To this effect, letting δ_0 denote the generator of $i_! k$, we see that $\partial_i x_i \cdot \delta_0 = 0$, so $x_i \partial_i \cdot \delta_0 = -1$, i.e. $\theta \cdot \delta_0 = -\dim(V)$. One can verify that θ acts on $\text{Sym}^k \mathcal{T}_V \delta_0$ by multiplication by $-(k+1)\dim(V) \neq 0$.

Now, we need to establish that the functor is conservative. Suppose that $\mathcal{M} \neq 0$; then $j_{\bullet} p^{\dagger} \mathcal{M} \neq 0$. Take $m \in \mathcal{M}$ with positive \mathbb{G}_m -weight; the only way for $\mathcal{D}_V m$ to not contain a weight 0 part is if all ∂_i act by torsion, but if there is $m' \in \mathcal{M}$ killed by all ∂_i , then $\theta \cdot m' = 0$, so m' has weight 0. Thus, every positive weight $m \in \mathcal{M}$ will generate a weight 0 vector. But if \mathcal{M} was entirely in negative weights, then the x_i would act by torsion, i.e. \mathcal{M} is supported at $\{0\}$, which is impossible. \square

Exercise 8.2. Check the claim that the Euler vector field acts as claimed above by differentiating the \mathbb{G}_m -action.

Remark 8.3. Are there other projective D -affine varieties? This is an open question. It is conjectured that partial flag varieties are the only ones. See MO post and paper by Langer.

Remark 8.4. Note that D -affinity of \mathbb{P}^n does not imply D -affinity for all projective varieties.¹⁰ The reason the naive argument fails is that the D -module pushforward is not the same as the \mathcal{O} -module pushforward for closed embeddings.

We now state the Beilinson-Bernstein theorem.

Theorem 8.5 (Beilinson-Bernstein). *Let $Z\mathfrak{g} \subset U\mathfrak{g}$ denote the center of the universal enveloping algebra; it has an augmentation character k_0 . The G -action on G/B induces an action map $U\mathfrak{g} \rightarrow \mathcal{D}(G/B)$.*

1. *The action map factors through an equivalence*

$$U\mathfrak{g} \otimes_{Z\mathfrak{g}} k_0 \xrightarrow{\simeq} \mathcal{D}(G/B).$$

2. *The flag variety G/B is D -affine.*

That is, the global sections functor defines an equivalence

$$\Gamma(G/B, -) : \text{DMod}(G/B) \xrightarrow{\simeq} \text{Rep}(\mathfrak{g})_0$$

where $\text{Rep}(\mathfrak{g})_0$ is the subcategory of \mathfrak{g} -representations with trivial central character.

Remark 8.6. The equivalence restricts to an equivalence:

$$\text{DMod}(U \backslash G/B) = \text{DMod}_{\Lambda}(G/B) \simeq \mathcal{O}'_0$$

where \mathcal{O}' is a *geometric category* \mathcal{O} (where we require $Z\mathfrak{g}$ -semisimplicity rather than \mathfrak{h} -semisimplicity), $\text{DMod}(U \backslash G/B)$ denotes the category of strongly U -equivariant D -modules on G/B , and $\Lambda \subset T^*(G/B)$ is a singular support condition given by the union of conormals to U -orbits. We may discuss equivariant D -modules, or this version of the equivalence, later.

¹⁰Thanks to Balazs Elek for pointing this out.

9 Lecture 9: Beilinson-Bernstein for SL_2 (Alekos Robotis)

Recall that the flag variety has a “coordinate-free” incarnation \mathcal{B} as the variety of Borel subgroups. The choice of a Borel B fixes a presentation $\mathcal{B} \simeq G/B$, and determines a Schubert stratification of G/B via left B -orbits (equivalently, left U -orbits, where U is the unipotent radical of B). This stratification has a unique closed orbit, the point $B \in G/B$.

Example 9.1 (The map on global differential operators). We can do this totally explicitly. Recall that $[u : v] \in \mathbb{P}^1$ parameterize cosets in G/B via

$$[u : v] \mapsto \begin{pmatrix} u & * \\ v & * \end{pmatrix} B$$

where the $*$ are chosen to make the matrix invertible. Note that right multiplication by B allows the second column to take any linearly independent value. In particular, the chosen point B corresponds to $[1 : 0]$. Let x be a coordinate for the chart where $u \neq 0$, i.e. $x([1 : v]) = v$.

Let us choose paths:

$$\gamma_E(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \gamma_H(t) = \begin{pmatrix} 1+t & 0 \\ 0 & (1+t)^{-1} \end{pmatrix}, \quad \gamma_F(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Note that group elements $g \in G$ act through inversion on the coordinate x . We identify the derivations (note that derivations on \mathbb{A}^1 are determined by their value on the coordinate x by Leibniz):

$$\gamma_E(t) \cdot x([1 : v]) = x(\gamma_E(t)^{-1} \cdot [1 : v]) = x([1 - vt : v]) = \frac{v}{1 - vt}, \quad \gamma_E(t) \cdot x = \frac{x}{1 - xt}.$$

$$E \cdot x = \left. \frac{d}{dt} \frac{x}{1 - xt} \right|_{t=0} = x^2, \quad \boxed{E \mapsto x^2 \partial_x = -\partial_y.}$$

$$\gamma_H(t) \cdot x([1 : v]) = x(\gamma_H(t)^{-1} \cdot [1 : v]) = x([(1+t)^{-1} : (1+t)v]) = (1+t)^2 v, \quad \gamma_H(t) \cdot x = (1+t)^2 x$$

$$H \cdot x = \left. \frac{d}{dt} (1+t)^2 x \right|_{t=0} = 2x, \quad \boxed{H \mapsto 2x \partial_x = -2y \partial_y.}$$

$$\gamma_F(t) \cdot x([1 : v]) = x(\gamma_F(t)^{-1} \cdot [1 : v]) = x([1 : v - t]) = v - t, \quad \gamma_F(t) \cdot x = v - x.$$

$$F \cdot x = \left. \frac{d}{dt} (v - x) \right|_{t=0} = -1, \quad \boxed{F \mapsto -\partial_x = y^2 \partial_y.}$$

Here, y is the opposite coordinate where $v = 1$.

Exercise 9.2. Check that the Casimir operator

$$\Omega := \frac{1}{8} H^2 + \frac{1}{4} EF + \frac{1}{4} FE$$

vanishes in the ring of global differential operators $\mathcal{D}(\mathbb{P}^1)$.

Recall that we have already chosen a Borel subgroup; let $\mathfrak{b} := \text{Lie}(B)$, and \mathfrak{n} denote its nilpotent radical.

Definition 9.3 (Category \mathcal{O}). Fix a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We define the (algebraic) *category* \mathcal{O} to be the full subcategory of $\text{Rep}(\mathfrak{g})$ consisting of objects which are:

- (a) finitely generated,
- (b) $Z\mathfrak{g}$ and \mathfrak{n} -locally finite,
- (c) \mathfrak{h} -semisimple.

Remark 9.4 (Indecomposables in category \mathcal{O}). Recall that a *weight* is an eigenvalue of \mathfrak{h} , i.e. $\lambda \in \mathfrak{h}^*$. This vector space is equipped with a natural lattice;¹¹ we say a weight is *integral* if it lies in that lattice. There is block

¹¹I.e. the cocharacter lattice for a maximal torus in a simply-connected form of G .

decomposition

$$\mathcal{O} := \bigoplus_{\chi} \mathcal{O}_{\chi} = \bigoplus_{[\lambda] \in \mathfrak{h}^* // W} \mathcal{O}_{[\lambda]}.$$

where we sum over all central characters $\chi : Z\mathfrak{g} \rightarrow k_{\lambda}$, or by passing through Harish-Chandra, a W -orbit in \mathfrak{h}^* under the ρ -shifted¹² Weyl group action. There are five important classes of indecomposable objects in \mathcal{O} :

- The *irreducible modules* $L(\lambda)$ with highest weight λ . These are finite-dimensional if and only if λ is a dominant integral weight.
- The *standard modules* (also called *Verma modules*) $M(\lambda) := U\mathfrak{g} \otimes_{U\mathfrak{b}} k_{\lambda}$ with highest weight λ . It has $L(\lambda)$ as a unique irreducible quotient.
- The *costandard modules* (also called *dual Verma modules*)¹³ $W(\lambda) := \text{Hom}_{U\mathfrak{b}}(U\mathfrak{g}, k_{\lambda})$ with highest weight λ . It has $L(\lambda)$ as a unique irreducible submodule.
- The *projective cover* $P(\lambda)$ of $L(\lambda)$ and $M(\lambda)$. It is an extension of standard modules of larger or equal weights. If λ is dominant then $P(\lambda) = M(\lambda)$.
- The *injective hull* $I(\lambda)$ of $L(\lambda)$ and $W(\lambda)$. It is an extension of costandard modules of larger or equal weights. If λ is dominant then $I(\lambda) = W(\lambda)$.

We will not go into detail about how these come about; see [Hu08].

Example 9.5. For the trivial block of $G = SL_2$, these give us five objects.

- $L(0)$ is the trivial representation,
- $L(-2) = M(-2) = W(-2)$ is an infinite-dimensional irreducible,
- $N(0) = I(0)$ is the costandard module, which sits inside a short exact sequence

$$0 \rightarrow L(0) \rightarrow N(0) \rightarrow N(-2) \rightarrow 0.$$

- $M(0) = P(0)$ is the standard module, which sits inside a short exact sequence

$$0 \rightarrow M(-2) \rightarrow M(0) \rightarrow L(0) \rightarrow 0.$$

- $P(-2) = I(-2)$ is the big tilting module. It has a multiplicity one standard filtration and costandard filtration:

$$0 \rightarrow M(0) \rightarrow P(-2) \rightarrow M(-2) \rightarrow 0, \quad 0 \rightarrow W(-2) \rightarrow I(-2) \rightarrow W(0) \rightarrow 0.$$

Exercise 9.6. Our goal will be to see these on the D -module side. Let $X = G/B$, $i : Z := \{B/B\} \hookrightarrow G/B$ be the inclusion of the closed stratum and $j : U := (G/B - \{B/B\}) \hookrightarrow G/B$ be the inclusion of the open stratum. Show that the representations above correspond to the following D -modules. Fix an open affine U_x containing x , and let $D := \mathcal{D}(U_x)$ denote the algebra of differential operators

- the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, which restricts to $D/D\partial$ (“generated by the constant function”),
- the skyscraper D -module $i_{\bullet}k$, which restricts to D/Dx (“generated by the Dirac delta distribution”),
- the pushforward $j_{\bullet}\mathcal{O}_U$, which restricts to $D/D\partial x$ (“generated by the Heaviside step distribution $H(x)$ ”),
- the Verdier dual¹⁴ $\mathbb{D}_X(j_{\bullet}\mathcal{O}_U)$, which restricts to $D/Dx\partial$ (“generated by $1/x$ ”),

¹²I.e. $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is half the sum of positive roots, and $w \cdot \lambda = w(\lambda + \rho) - \rho$.

¹³The notation W is totally non-standard. It’s an upside-down M . Sometimes standards are denoted Δ and costandards ∇ .

¹⁴We haven’t defined this yet. For now, think of it as the D -module you get by reversing the relations.

- the big tilting sheaf \mathcal{T} , which restricts to $D/Dx\partial x$ (“generated by $H(x)/x$ ”).

Example 9.7. The D -module $D/D\partial^2$ is decomposable via the map

$$D/D\partial^2 \longrightarrow D/D\partial \oplus D/D\partial, \quad 1 \mapsto (x, 1).$$

The idea is that the general solution to the differential equation is $Ax + B$, so the generator $1 \in D/D\partial^2$ should decompose as such on the right, where the generators are given by the constant function. The inverse is given by

$$(A, B) \mapsto A\partial + B(1 - x\partial).$$

The idea here is to solve for A, B in terms of $Ax + B$. We have $(1 - x\partial)(Ax + B) = B$ while $\partial(Ax + B) = A$. The inverse map takes $1 \mapsto (x, 1)$.

Exercise 9.8. Show that the D -module $D/D\partial x\partial$ is decomposable.

Remark 9.9. There is a *Fourier transform* on $\mathrm{DMod}(\mathbb{A}^n)$ given by swapping the roles of x and ∂ (with a sign) which can be observed in the above example.

You might notice that the big tilting sheaf doesn’t quite match up. This is because the category we get out of Beilinson-Bernstein is slightly different.

Definition 9.10 (Category \mathcal{O}'). We define the (geometric) *category* \mathcal{O}' to be the full subcategory of $\mathrm{Rep}(\mathfrak{g})$ consisting of objects which are:

- (a) finitely generated,
- (b) $Z\mathfrak{g}$ and \mathfrak{n} -locally finite,
- (c) $Z\mathfrak{g}$ -semisimple.

Remark 9.11. Note that $P(-2) = Q(-2)$ is not Ω -semisimple, and the global sections of the big tilting sheaf $\Gamma(G/B, \mathcal{T})$ is not H -semisimple. Note Ω always acts on scalars by highest weight modules (i.e. modules generated by a highest weight), but $P(-2) = Q(-2)$ is not generated by a highest weight.

Roughly, the idea is that \mathcal{O}' arises via $\mathrm{DMod}(U \backslash G/B)$, while \mathcal{O} arises via $\mathrm{DMod}(B \backslash G/U)$.

Theorem 9.12 (Soergel, Webster [We11]). *The two category \mathcal{O} s are equivalent on regular blocks.*

10 Lecture 10 (2021-03-01): Global sections of the flag variety

Our goal for today will be to compute the global sections $\mathcal{D}(G/B)$.

Theorem 10.1. *The G -action on G/B induces an equivalence*

$$U\mathfrak{g} \otimes_{Z\mathfrak{g}} k_0 \xrightarrow{\cong} \mathcal{D}(G/B).$$

Recall the following from Jeffrey’s talk.

Proposition 10.2. *The associated graded of the natural map of filtered algebras $\alpha : U\mathfrak{g} \rightarrow \mathcal{D}(G/B)$ arising via the G -action on G/B is the pullback along the moment map $\mu^* : \mathcal{O}(\mathfrak{g}^*) \rightarrow \mathcal{O}(T^*(G/B))$.*

Definition 10.3. Let $\mathfrak{g}^*//G := \mathrm{Spec} k[\mathfrak{g}^*]^G$, and define the *nilpotent cone* $\mathcal{N} \subset \mathfrak{g}^*$ by

$$\mathcal{N} = \{0\} \times_{\mathfrak{g}^*//G} \mathfrak{g}^*.$$

The theorem follows from the following proposition.

Proposition 10.4. *The associated graded of the inclusion of the center $Z\mathfrak{g} \hookrightarrow U\mathfrak{g}$ is the map $k[\mathfrak{g}^*]^G \hookrightarrow k[\mathfrak{g}^*]$. Thus, the moment map $\mu : \tilde{\mathcal{N}} \rightarrow \mathfrak{g}^*$ factors through \mathcal{N} . Furthermore, $\mathcal{O}(\tilde{\mathcal{N}}) = \mathcal{O}(\mathcal{N})$.*

Let us identify the center $Z\mathfrak{g}$. The first is the Harish-Chandra isomorphism. We first define the universal Cartan.

Definition 10.5. The *universal Cartan algebra* of \mathfrak{g} is defined as follows. Choose a Borel \mathfrak{b} , and define

$$\mathfrak{h} := \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}].$$

It is “universal” (i.e. canonical, but not a subalgebra) since for any choice of Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$, we have an isomorphism $\mathfrak{t} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{h}$, and for any two choices $\mathfrak{t}, \mathfrak{t}'$, the induced automorphism of \mathfrak{h} is the identity. There is an abstract root system in \mathfrak{h} transferred over from any \mathfrak{t} , and the universal Cartan has a *universal Weyl group* \mathbb{W} is the reflection group for that root system.

We need to introduce the following shifted W -action to correctly formulate the Harish-Chandra theorem.

Definition 10.6 (Shifted W -action). Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{h}^*$ denote the Weyl vector. We define the *shifted W -action* on \mathfrak{h}^* by:

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

i.e. the reflection hyperplanes pass through the common point $-\rho$. This also defines a shifted W -action on $\text{Sym } \mathfrak{h}$ by:

$$w \cdot v = wv + w\rho(v) - \rho(v).$$

In particular, note that the action respects the filtration on $\text{Sym } \mathfrak{h}$, but not the grading (i.e. not an action on \mathfrak{h}).

Theorem 10.7 (Harish-Chandra theorem). *Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. There is an isomorphism (using the shifted W -action)*

$$Z(U\mathfrak{g}) \xrightarrow{\cong} \text{Sym}(\mathfrak{t})^W.$$

Furthermore, the associated graded of $\text{Sym}(\mathfrak{t})^W$ (under the shifted action) is naturally isomorphic to $k[\mathfrak{h}^]^{\mathbb{W}}$ (under the usual action).*

Rather than prove the theorem, let us just demonstrate it for \mathfrak{sl}_2 .

Remark 10.8 ($G = SL_2$). When $G = SL_2$, $Z\mathfrak{g}$ is generated by the Casimir element

$$\Omega := \frac{1}{8}H^2 + \frac{1}{4}(EF + FE).$$

We can write this in the given decomposition:

$$\Omega = \frac{1}{8}H^2 + \frac{1}{4}(2FE + H) = \frac{1}{8}(H^2 + 2H) + \frac{1}{2}FE.$$

If m is a highest weight, then $\Omega m = \frac{1}{8}(H^2 + 2H)m$. That is, if Ω acts by central character r , then $H^2 + 2H - 8r = 0$. The roots of this polynomial have the form $s, -s - 2$ for $s = -1 + \sqrt{1 + 8r}$. The Weyl vector here is $\rho = 1$.

We state Chevalley’s theorem without proof.

Theorem 10.9 (Chevalley restriction theorem). *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. The restriction map on functions $k[\mathfrak{h}^*] \rightarrow k[\mathfrak{g}^*]$ defines an equivalence*

$$k[\mathfrak{h}^*]^W \xrightarrow{\cong} k[\mathfrak{g}^*]^G.$$

Let us assume these two foundational theorems. Then we can prove the result. We will prove the following easy algebraic geometry lemma first.

Lemma 10.10. *Any proper birational¹⁵ map $f : X \rightarrow Y$ where Y is a normal affine variety satisfies $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof. Since f is proper, f_* takes coherent sheaves to coherent sheaves, i.e. $\mathcal{O}(X)$ is a finite $\mathcal{O}(Y)$ -module. Since f is birational, $\mathcal{O}(Y) \subset \text{Frac}(\mathcal{O}(Y))$, and in particular $\mathcal{O}(X) \subset \mathcal{O}(Y) \subset \text{Frac}(\mathcal{O}(Y))$. Since Y is normal, $\mathcal{O}(X) \subset \text{Frac}(\mathcal{O}(X))$ is integrally closed, and in particular any submodule of $\text{Frac}(\mathcal{O}(X))$ finitely generated over $\mathcal{O}(X)$ is $\mathcal{O}(X)$ itself, i.e. $\mathcal{O}(X) = \mathcal{O}(Y)$. \square

¹⁵In particular, X and Y are irreducible.

We now begin the proof of Proposition 10.4. Some claims about the basic geometry of the Springer resolution will take us a bit too far afield; we will check these claims by hand in type A afterwards.

Proof of Proposition 10.4. By Harish-Chandra and Chevalley restriction, we have that the associated graded of $Z\mathfrak{g} \hookrightarrow U\mathfrak{g}$ is given by $k[\mathfrak{g}^*]^G \hookrightarrow k[\mathfrak{g}^*]$. It remains to show that $\mathcal{O}(\tilde{\mathcal{N}}) = \mathcal{O}(\mathcal{N})$; we need to check the conditions of the above lemma.

The Springer resolution $\mu : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is evidently proper via the factorization¹⁶

$$\tilde{\mathcal{N}} = G \times^B \mathfrak{b}^\perp \hookrightarrow G \times^B \mathfrak{g}^* = G/B \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*.$$

The map μ is surjective, since every nilpotent element is contained in Borel. Since $\tilde{\mathcal{N}}$ is smooth (thus irreducible) and μ is surjective, then \mathcal{N} is irreducible of dimension $\leq \dim(\mathcal{N}) = 2(\dim(G) - \dim(B))$. On the other hand, \mathcal{N} is cut out by $\dim(\mathfrak{h})$ equations in \mathfrak{g} , i.e. has dimension $\geq \dim(G) - \dim(H)$. Then we have

$$\dim(\mathcal{N}) = 2 \dim(G) - 2 \dim(B) = 2 \dim(G) - (\dim(G) + \dim(H)) = \dim(G) - \dim(H).$$

In particular, \mathcal{N} is a complete intersection, thus Cohen-Macaulay. By Serre's criterion for normality, \mathcal{N} is normal if it is regular in codimension 2. We will take for granted that \mathcal{N} has finitely many G -orbits, but we assuming this we will prove that the orbits have even dimension following an argument by Kirillov. Since the orbits give a smooth stratification of \mathcal{N} , this establishes the desired regularity in codimension 2.

To see that the orbits have even dimension, note that the Lie bracket allows us to define symplectic structures on the coadjoint orbits $\mathcal{O} \subset \mathcal{N} \subset \mathfrak{g}^*$:

$$\omega_x(X \cdot x, Y \cdot x) = \langle x, [X, Y] \rangle$$

where $x \in \mathcal{O} \subset \mathfrak{g}^*$, we identify the tangent space $T_x(\mathcal{O}) = \mathfrak{z}(x)^\perp \subset \mathfrak{g}^*$, and $X, Y \in \mathfrak{g}$ act via differentiating the G -action on \mathfrak{g}^* . The action is transitive on the tangent space of the orbit. Note that since if $X \cdot x = x$, then $X \in \mathfrak{z}$ where \mathfrak{z} is the Lie algebra of the stabilizer of x , and in particular $x(X) = 0$; this implies the formula is well-defined as well as non-degeneracy. We leave verification that the form is closed as an exercise (see Proposition 1.1.5 in [CG97]).

Thus \mathcal{N} is normal. The map μ is birational; this follows from the fact, which we take for granted, that a regular semisimple nilpotent element is contained in a single Borel (see Proposition 3.2.10 and 3.2.14 in [CG97]), and the conclusion follows. \square

Example 10.11. The above proof relies on some heavy lifting. Let us work out what it says for $G = SL_n$:

- The map $\mathfrak{g} \rightarrow \mathfrak{g}/G$ is given by G -invariants in \mathfrak{g} , i.e. by the coefficients of the characteristic polynomial.
- The identification $\mathfrak{g}/G \simeq \mathfrak{h}/W$ given by solving the characteristic polynomial for the eigenvalues (with multiplicity), up to ordering (since $W = S_n$).
- The G -orbits in \mathcal{N} correspond to partitions of n , i.e. Jordan decompositions. The open orbit of the nilpotent cone is given by the regular nilpotent, i.e. the nilpotent consisting of a single Jordan block.
- The flag variety G/B is the variety of flags, i.e. subspaces $V_0 \subset V_1 \subset \dots \subset V_n$ where $\dim(V_i) = i$. Note that a regular nilpotent element fixes a single Borel.

11 Lecture 11 (2021-03-03): Equivariant D -modules

We will not prove exactness and conservativeness of the global sections functor (i.e. the rest of Beilinson-Bernstein) until Sections 18 and 19. We have established this in the case of SL_2 . See these notes of Bezrukavnikov for a survey of arguments, Section 1.3 of a paper of Frenkel and Gaitsgory and notes by Gannon for proofs.

See Section 25 for a possibly better treatment of equivariant D -modules.

¹⁶Note that if G acts on V , then $G \times^H V \simeq G/H \times V$ for any subgroup H , i.e. any $v \in V$ determines a trivializing section $gH \mapsto (g, g^{-1}v)$.

For this section, G will denote an affine algebraic group, and X a G -scheme. Recall the notion of a G -equivariance.

Definition 11.1. A G -equivariant sheaf on X is a sheaf \mathcal{F} on X equipped with an equivalence $\phi : a^*\mathcal{F} \simeq p^*\mathcal{F}$ where $a, p : G \times X \rightarrow X$ are the action and projection maps, respectively. The map is required to satisfy a cocycle condition. A map of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is G -equivariant if $p^*\phi = a^*\phi$.

Remark 11.2. Note that G -equivariance on a sheaf is a *structure*, while G -equivariance on a morphism is a *condition*.

Remark 11.3. Note that $(a_g^*(\mathcal{F}))_{(g,x)} = \mathcal{F}_{gx} \rightarrow p^*(\mathcal{F})_{(g,x)} = \mathcal{F}_x$.

Definition 11.4. Assume X is smooth. A *weakly G -equivariant D -module* on X is a G -equivariant sheaf \mathcal{M} on X equipped with a \mathcal{D}_X -module structure, such that the structure map $\mathcal{D}_X \otimes_k \mathcal{M} \rightarrow \mathcal{M}$ is G -equivariant.

Let G act on X , and $x \in \mathfrak{g}$ (sometimes abusively used to denote $1 \otimes x \in \mathcal{O}_X \otimes \mathfrak{g} = \mathcal{T}_G$). For an equivariant quasicoherent sheaf \mathcal{M} , we let $\alpha_\theta : \mathcal{M} \rightarrow \mathcal{M}$ denote the k -linear endomorphism obtained via differentiating the G -equivariant structure. More precisely, α_θ is the restriction of the following map on $G \times X$ to $\{e\} \times X$:

$$a^*\mathcal{F} \xrightarrow{\phi} p^*\mathcal{F} \xlongequal{\quad} \mathcal{O}_G \boxtimes \mathcal{F} \xrightarrow{x \cdot -} \mathcal{O}_G \boxtimes \mathcal{F}.$$

For a \mathcal{D}_X -module \mathcal{M} , we let $\delta_x : \mathcal{M} \rightarrow \mathcal{M}$ denote the action obtained by differentiating the G -action, i.e. via action by the differential operator obtained by pushing forward the tangent vector $(x, 0)$ along the action map.

A *strongly G -equivariant D -module* on X is a weak G -equivariant D -module \mathcal{M} such that $\alpha_x = \delta_x$ for all $x \in \mathfrak{g}$.

Exercise 11.5. Show that weakly equivariant D -modules are equivariant \mathcal{D}_X -modules \mathcal{M} where the equivariance structure map ϕ is $\mathcal{O}_G \boxtimes \mathcal{D}_X$ -linear, and strongly equivariant D -modules those which are $\mathcal{D}_G \boxtimes \mathcal{D}_X$ -linear.

Remark 11.6. There is a “homotopically correct” notion of a complex of strong equivariant D -modules: instead of insisting that $\alpha_x = \theta_x$, we ask for a (linear in \mathfrak{g}) degree -1 homotopy whose differential is $\alpha_x - \delta_x$.

Example 11.7 (\mathbb{G}_m -equivariant D -modules on $\text{Spec } k$). Let $X = \text{Spec } k = \text{pt}$. Since $\mathcal{D}_X = k$, a weakly equivariant D -module on pt is just a quasicoherent sheaf on $B\mathbb{G}_m$, i.e. a \mathbb{G}_m -representation. A strongly equivariant D -module is just a k vector space. To see this, note that vector fields act trivially, so $\delta_x = 0$, and that if V has weight n , i.e. $z \cdot v = z^n v$, then letting $x = \partial_z$, we have α_x is given by

$$v \otimes z \mapsto v \otimes z^n \mapsto v \otimes n z^{n-1}$$

i.e. $v \mapsto nv$ after restricting to $z = 1$. In particular, if $n \neq 0$, then we require $nv = 0$, i.e. $v = 0$.

Example 11.8 (\mathbb{G}_m -equivariant D -modules on \mathbb{G}_m). A \mathbb{G}_m -equivariant sheaf on \mathbb{G}_m is just given by a vector space V , or rather $V[z, z^{-1}]$ where V has weight 0 and z has weight -1. A weakly equivariant D -module has a $\mathcal{D}(\mathbb{G}_m)$ -module structure, where ∂_z has weight 1. Its action is determined by its weight 0 part, which is $\mathcal{D}(\mathbb{G}_m)^{\mathbb{G}_m} = k\langle z\partial_z, \partial_z z \rangle / [\partial_z, z] = 1$, i.e. a matrix in $\text{End}_k(V)$ given by the action of $z\partial_z$. This can be viewed as a logarithm of monodromy.

For strongly equivariant D -modules, we identify $\text{Lie}(\mathbb{G}_m)$ with left-invariant vector fields, which are spanned by $z\partial_z$. Note that $\alpha_x = 0$ on V , and δ_x is given by the action of $x = z\partial_z \in \mathcal{T}_{\mathbb{G}_m}$, i.e. the matrix above is 0. That is, a strongly equivariant D -module is just a vector space, and morphisms are homomorphisms of vector spaces.

Exercise 11.9. Write down some weakly and strongly \mathbb{G}_m -equivariant D -modules on \mathbb{A}^1 , e.g. the pushforwards.

Example 11.10. Suppose that G is discrete. Then strong equivariance is the same as weak equivariance.

Definition 11.11. We denote by $\text{DMod}(X/G) = \text{DMod}^G(X)$ the category of strongly G -equivariant D -modules on X , and $\text{DMod}(X/G) = \text{DMod}^{wG}(X)$ the category of weakly G -equivariant D -modules.

We state the following without proof (for now).

Proposition 11.12. *Suppose that G acts on X with connected centralizers. Strong G -equivariance is a property, not a structure, on the abelian category of D -modules. That is, the functor $\mathrm{DMod}(X/G) \rightarrow \mathrm{DMod}(X)$ is fully faithful.*

Remark 11.13. Note that the above will only be true for derived categories when G is not unipotent and connected, roughly because for such G all centralizers are connected and all orbits have trivial cohomology.

The following justifies the notation.

Proposition 11.14. *Suppose that P is a G -torsor over X . Then $\mathrm{DMod}(P/G) \simeq \mathrm{DMod}(X)$.*

Proof. If P is a G -torsor in the Zariski topology, one can check affine locally. Otherwise, one needs some kind of descent result. \square

Remark 11.15. Some authors define a notion of *monodromic D -module* on P to be the essential image of the functor $p^\dagger : \mathrm{DMod}(X) \rightarrow \mathrm{DMod}(P)$. Roughly, this corresponds to weakly equivariant D -modules with unipotent monodromy.

12 Lecture 12 (2020-03-05): Twisted Beilinson-Bernstein

Remark 12.1 (Conventions). If you look up statements in the literature, there are two possible sources of confusion arising from conventions.

- Shifted Weyl group actions on \mathfrak{h}^* . Consider the Harish-Chandra map $H : Z\mathfrak{g} \rightarrow \mathrm{Sym} \mathfrak{h}$. On the graded dual (denoted by \circ) it induces a map $H^\circ : \mathrm{Sym} \mathfrak{h}^* \rightarrow Z\mathfrak{g}^\circ$. This map is not W -invariant for the usual Weyl group action, but one shifted by $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, i.e.

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

i.e. the reflection hyperplanes pass through the common point $-\rho$. Some authors will build this shift into the W -action, while others shift the space \mathfrak{h}^* itself. **Our convention will be to shift the W -action.**

- Sign conventions in Borel-Weil-Bott. Let χ be a character of H inflated to B . There are two possible conventions for what it means to be positive. In one, the Borel defines a notion of positive weights, i.e. such that \mathfrak{n} consists of positive weights. In another the characters λ of $H = B/U$ which inflate and induce to ample line bundles $\mathcal{L}(\lambda)$ on G/B should be positive. These two conventions are opposite. **Our convention will be the former.** Under this convention, k_λ gives rise to a line bundle \mathcal{L}_λ . If λ is *antidominant*, then $H^0(G/B, \mathcal{L}_\lambda) = V_{w_0\lambda} = V_{-\lambda}^*$ is the irreducible representation with lowest weight λ .

Example 12.2. The second point is confusing. Let's see an example. Take $G = SL_2$, and let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathcal{B}$ be

the “base point” point in the flag variety. The cocharacter that makes the Borel positive weight is $\theta(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

The fiber of the tautological bundle $\mathcal{O}(-1)$ at B is the span of e_1 , which has weight 1 with respect to this cocharacter. Thus, the line bundle induced from k_1 is $\mathcal{O}(-1)$.

Proposition 12.3. *Let T be an algebraic torus, and let $p : \tilde{X} \rightarrow X$ denote a T -torsor. Then, a weakly equivariant \mathcal{D}_X -module on \tilde{X} is equivalent to a $(p_* \mathcal{D}_{\tilde{X}})^T$ -module on X . Furthermore, there is an identification of algebras*

$$(p_* \mathcal{D}_{\tilde{X}})^T \simeq \mathcal{D}_X \otimes_k \mathrm{Sym}_k \mathfrak{t}.$$

Proof. The functor $p_*(-)^T$ (where p_* is the \mathcal{O} -module pushforward) defines an equivalence with inverse given by p^* , i.e. the \mathcal{O} -module pullback with the natural T -equivariant structure on the fibers. The second claim is a local calculation. \square

Remark 12.4. Note that there are no twists involved in the vertical tangent vectors. That is, the T -invariant vector fields are generated by differentiating the action of T , which is globally defined.

Definition 12.5. Let $\lambda \in \mathfrak{h}^*$ denote a character of the universal Cartan subalgebra. We define the sheaf of *twisted differential operators*

$$\mathcal{D}_{G/B, \lambda} := (p_* \mathcal{D}_{G/U})^H \otimes_{\text{Sym } \mathfrak{h}} k_\lambda.$$

Remark 12.6. If λ is integral then $\mathcal{D}_{G/B, \lambda}$ can be realized as a sheaf of differential operators on $\mathcal{O}_{G/B}(-\lambda)$.

Remark 12.7. Can one assemble all the twists into one sheaf, i.e. simply consider $(p_* \mathcal{D}_{G/U})^H$ -modules? Yes; but one tends to require \mathfrak{h} to act finitely. I'm not exactly sure the reason; possibly because one tends to not consider \mathfrak{g} -representations where the center does not act locally finitely. Following this paper by Ben-Zvi and Nadler, it seems the requirement is not strictly needed.

Theorem 12.8 (Twisted Beilinson-Bernstein). *Assume that $\lambda \in \mathfrak{h}^*$ satisfies the property:*

$$\langle \lambda, \alpha^\vee \rangle \neq -1, -2, \dots, \quad \alpha \in \Delta^+.$$

Then the global sections functor $\Gamma : \text{DMod}(G/B, \lambda) \rightarrow \text{Rep}(\mathfrak{g})_{[\lambda]}$ is an equivalence of categories.

Remark 12.9 (Non-integral weights). Note that every W -orbit has a λ satisfying the above conditions, which say that λ is dominant weight, or “generic in the directions in which it is not dominant.”

Remark 12.10. For Beilinson-Bernstein for partial flag varieties, see the paper by Backelin and Kremnizer.

13 Lecture 13 (2020-03-08): Holonomic D -modules

Lemma 13.1. *Let A be a filtered algebra, and M, N filtered right and left A -modules respectively with a good filtration. Equip $M \otimes_A N$ with the tensor product of the filtrations¹⁷ Then, there is an isomorphism*

$$\text{gr}(M \otimes_A N) \simeq \text{gr}(M) \otimes_{\text{gr}(A)} \text{gr}(N).$$

Proof. Pass to the Rees construction.¹⁸ The tensor product of filtrations has the property that $\text{Rees}(M \otimes_A N) = \text{Rees}(M) \otimes_{\text{Rees}(A)} \text{Rees}(N)$. Then, use the fact that $\text{gr}(M) = \text{Rees}(M) \otimes_{k[t]} k$, and that tensor products commute. \square

Proposition 13.2. *Let $i : Z \hookrightarrow X$ be a closed embedding. Then, the singular support of (the exact functor) $i_* \mathcal{M}$ is given by the inverse image and then image along the correspondence:*

$$\mathbb{T}_Z^* \longleftarrow \mathbb{T}_X^* \times_X Z \longrightarrow \mathbb{T}_X^*.$$

Proof. This is a local calculation. Assume X is affine; we have

$$i_* \mathcal{M} = \mathcal{D}_X / \mathcal{D}_X I \otimes_{\mathcal{D}_Z} \mathcal{M}.$$

Apply the lemma; note that $\text{gr}(\mathcal{D}_X / \mathcal{D}_X I) = \mathcal{O}_{\mathbb{T}_X^* \times_X Z}$ (this can be checked in local coordinates, i.e. \mathcal{D}_X is generated by ∂_i, x_i and $I = (x_1, \dots, x_r)$). Tensor product of sheaves amounts to fiber product of supports, i.e. we have a Cartesian square

$$\begin{array}{ccc} \text{SS}(i_* \mathcal{M}) & \longrightarrow & \mathbb{T}_X^* \times_X Z \\ \downarrow & & \downarrow \\ \text{SS}(\mathcal{M}) & \longrightarrow & \mathbb{T}_Z^*. \end{array}$$

This is exactly the closed subvariety described above. \square

Proposition 13.3. *Let $p : X \rightarrow Y$ be a smooth morphism. Then, the singular support of (the exact functor) $p^\dagger \mathcal{M}$ is given by the inverse image and then image along the correspondence:*

$$\mathbb{T}_Y^* \longleftarrow \mathbb{T}_X^* \times_Y X \longrightarrow \mathbb{T}_X^*.$$

¹⁷Recall that $F_n(M \otimes_A N) = \sum_{i+j=n} \text{im}(F_i M \otimes_{A_0} F_j N)$.

¹⁸Thanks to Andres Fernandez Herrero for pointing out this argument.

Proof. Note that p^\dagger is the \mathcal{O} -module pullback on the underlying sheaf, i.e. $p^\dagger \mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{M}$. Apply the lemma again to find that $\mathrm{SS}(p^\dagger \mathcal{M}) = \mathrm{SS}(\mathcal{M}) \times_Y X$, proving the claim. \square

Remark 13.4. The obstruction to generalizing the above two results is that the functors do not always preserve D -coherent modules; see Remark 3.16.

Theorem 13.5. *Let $\mathcal{M} \in \mathrm{DMod}(X)$. Then*

$$\dim(\mathrm{SS}(\mathcal{M})) \geq \dim(X) = \frac{1}{2} \dim(\mathbb{T}_X^*).$$

Proof. Suppose that the ordinary support $\mathrm{supp}(\mathcal{M})$ has dimension $r < \dim(X)$. Then, it is supported on a closed subscheme of $i : Z \subset X$ of dimension r . If Z is not smooth, then restrict to an open subset of X where it is. Then, by Kashiwara, $\mathcal{M} = i_* \mathcal{M}_0$. By the above proposition, we pick up extra codirections with dimension $\dim(X) - \dim(Z)$, i.e. the dimension of the support is at least $\dim(Z) + (\dim(X) - \dim(Z)) = \dim(X)$. \square

Definition 13.6. A D -module is *holonomic* if $\dim(\mathrm{SS}(\mathcal{M})) = \dim(X)$ i.e. the minimum possible. We denote by $\mathrm{DMod}_h(X)$ the full subcategory of holonomic D -modules, and $D_h(\mathrm{DMod}(X))$ the full subcategory of complexes with holonomic cohomology.

Example 13.7. Every example from Lecture 9 was holonomic. The free \mathcal{D}_X -module is not holonomic.

Example 13.8. Let G be a group acting on X by finitely many orbits. Then every D -module in $\mathrm{DMod}(X/G)$ is holonomic. The reason is that the singular support of any such D -module lives in

$$\Lambda := \bigcup \mathbb{N}_{X_i/X}^*$$

where the X_i are the finitely many orbits. This gives a decomposition of Λ into irreducible components, which are all dimension $\dim(X)$.

It will be useful for us to have a slightly more refined notion of singular support.

Definition 13.9. Let \mathcal{M} be a D -module, and decompose its singular support into irreducible components $\mathrm{SS}(\mathcal{M}) = \bigcup C_i$, with generic points ξ_i . For a choice of good filtration, the sheaf $\mathrm{gr}(M)_{\xi_i}$ is a finite length over $(\mathbb{T}_X^*)_{\xi_i}$, of length m_i (via Nakayama). We define the *characteristic cycle* to be the formal sum

$$\mathrm{CC}(\mathcal{M}) := \sum m_i [C_i].$$

Note these are not elements of the Chow ring; we do not impose any sort of equivalence.

Exercise 13.10. Check that the above results on functoriality of singular supports generalizes to characteristic cycles.

Proposition 13.11. *The characteristic cycle is independent of choice of good filtration. Let d be the maximum dimension of the characteristic cycle; then the dimension d characteristic cycles are additive in short exact sequences.*

Proof. We won't prove the first claim. For the second, it is not true in general that a short exact sequence of filtered modules gives rise to a short exact sequence on associated graded.¹⁹ However, it is true if the morphisms are *strict*: we say $f : M \rightarrow N$ is strict if $f(F_i M) = f(M) \cap F_i N$. Given a filtered module $F_i M$, inclusion and quotient for any sub or quotient object (with the induced filtration) is strict. The second claim now follows since length is additive in short exact sequences and taking stalks is exact. Note that we can only pass to stalks which correspond to irreducible components of $\mathrm{SS}(M)$; these include all top dimension characteristic cycles. \square

Example 13.12. Let $X = \mathbb{A}^1$, and take the short exact sequence

$$0 \rightarrow \mathcal{D}_X \rightarrow \mathcal{D}_X \rightarrow \mathcal{D}_X / \mathcal{D}_X \partial \rightarrow 0.$$

¹⁹For example, take $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$, and give M_1 a nontrivial filtration, and M_2 and M_3 the trivial filtrations.

If we take dimension 2 characteristic cycles, we have $[\mathbb{T}_X^*] = [\mathbb{T}_X^*] + [0]$, while if we take dimension 1 characteristic cycles, we have $[0] = [0] + [X]$ which is not true.

Example 13.13. Consider filtered vector spaces, after passing to the Rees construction. The multiplication by t map $k[t] \rightarrow k[t]$ is not strict. I.e., filter $M = k$ such that the jump is at index 0, and $N = k$ such that the jump is at index -1.

Exercise 13.14. Consider a map $f : M \rightarrow N$ of filtered modules. Show that if $\text{Rees}(f)$ is surjective, then it is strict. One can factor $\text{Rees}(f) : \text{Rees}(M) \twoheadrightarrow \text{im}(\text{Rees}(f)) \hookrightarrow \text{Rees}(N)$; let i denote the inclusion and show that f is strict if and only if $i \otimes_{k[t]} k$ is injective. Prove that a strict short exact sequence of filtered modules gives rise to a short exact sequence on the associated graded. Prove that it is possible to construct strict projective resolutions.

Proposition 13.15. *The category $\text{DMod}_h(X)$ is abelian and closed under extensions. Furthermore, every holonomic D -module has finite length, and is generically a flat connection.*

Proof. The first claim follows since singular support is additive in short exact sequences, and the fact that holonomic D -modules have support of minimal dimension. For the second, we use the fact that characteristic cycles are additive, and multiplicities are positive. For the third, restrict to the smooth locus of $\text{SS}(\mathcal{M})$ (which is open), and then restrict to the zero section; the singular support on this $U \subset X$ will be supported on the zero section, and thus a flat connection. \square

Example 13.16. These assertions are not true if we do not assume \mathcal{M} is holonomic. For example, $\mathcal{M} = \mathcal{D}_X$ is evidently not generically a flat connection (it has infinite rank), and also does not have finite length since the order filtration defines a infinite chain of left ideals.

14 Lecture 14 (2020-03-17): Category \mathcal{O} (Rodrigo Horruitiner)

We'll continue from where Rodrigo left off. Rodrigo proved the following the last time:

Proposition 14.1. *The category \mathcal{O} is Noetherian and Artinian, i.e. every object has finite length.*

Thus, every object has a Jordan-Holder series.

Theorem 14.2 (Jordan-Holder). *Suppose \mathbf{C} is an abelian category which is Noetherian and Artinian. Then, every object M has a finite composition series, i.e. a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that the M_i/M_{i-1} are simple. These simples, with multiplicity, are independent of the choice of filtration (ignoring ordering), and are called the composition factors of M .

Proof. Induct on the length of M (i.e. length of minimal composition series). For length 1, the claim is obvious. Now, suppose we have a minimal length series M_i for M , and another series N_i for M (possibly not minimal length). If $N_1 = M_1$ we are done by induction, and if not we can find the first M_k such that $N_1 \cap M_k \neq 0$; since N_1 is simple this means $N_1 \subset M_k$. This means $M_k/M_{k-1} \simeq N_1$ (since both are simple) and we can take the quotient by N_1 (exercise: check this) and induct. \square

Remark 14.3. Note that D -modules do not have finite length in general; however, holonomic D -modules do.

Also, Rodrigo proved the following.

Proposition 14.4. *The simple objects in \mathcal{O} are exactly the L_λ for $\lambda \in \mathfrak{h}^*$. The standard modules Δ_λ are projective when $\lambda + \rho$ is dominant, and irreducible when $\lambda + \rho$ is antidominant.*

We also have the following more or less immediate consequence.

Definition 14.5. Let $\chi : Z\mathfrak{g} \rightarrow k$ be a central character (sometimes we write $\chi = [\lambda]$ if $\lambda \in \mathfrak{h}^*$). We define $\mathcal{O}_\chi \subset \mathcal{O}$ to be the full subcategory consisting of objects where $Z\mathfrak{g} - \chi$ acts locally nilpotently. We say that χ is *integral* if $\chi = [\lambda]$ for integral λ , and *regular* if $\chi = [\lambda]$ where λ does not lie on any (shifted) reflection hyperplane.

Proposition 14.6. *We have*

$$\mathcal{O} = \bigoplus_{\chi \in X^\bullet(Z\mathfrak{g})} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^* \text{ dom.}} \mathcal{O}_{[\lambda]}.$$

Furthermore, if χ is regular integral then \mathcal{O}_χ is indecomposable.

Proof. We need to show that $Z\mathfrak{g}$ acts locally finitely. But it does so on simples, and since \mathcal{O} is Artinian and Noetherian every object is a finite extension of simples. For the second claim, note that Δ_λ is indecomposable where λ is dominant, and contains every irreducible. \square

Exercise 14.7. See why \mathcal{O}_χ is not indecomposable if χ is not integral.

Definition 14.8. We define the $a_{\lambda\mu} \in \mathbb{N}$ and $b_{\mu\lambda}$ to be integers such that

$$[\Delta_\lambda] = \sum_{\mu} a_{\lambda\mu} [L_\mu], \quad [L_\mu] = \sum_{\mu} b_{\mu\lambda} [\Delta_\mu].$$

Remark 14.9. Some immediate observations.

- We have $a_{\lambda\mu} = b_{\mu\lambda} = 0$ unless λ and μ are W -linked (i.e. we can focus on each individual \mathcal{O}_χ).
- We have $a_{\lambda\mu} = b_{\mu\lambda} = 0$ unless $\lambda \leq \mu$ (i.e. the transition matrices are “upper triangular”).
- If λ, λ' are integral dominant, then $a_{\lambda, w \cdot \lambda} = a_{\lambda', w \cdot \lambda'}$ and likewise for b (i.e. every regular integral block looks the same).

Definition 14.10. We write

$$a_{yw} = a_{y \cdot \lambda, w \cdot \lambda}, \quad b_{yw} = a_{y \cdot \lambda, w \cdot \lambda}$$

for any λ integral dominant.

Recall from the first day that we are interested in the following problem.

Question 14.11. Compute the constants above.

We have enough now to compute the constants a_{ey} (where $e \in W$ is the identity).

Proposition 14.12. *We have $a_{ey} = (-1)^{\ell(y)}$.*

Proof. We leave the proof as an exercise. Write the character

$$\text{ch}(L_\lambda) = \sum_{y \in W} a_y \text{ch}(\Delta_{y \cdot \lambda}) = \sum_{y \in W} a_y \frac{e^{y(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.$$

Use the fact that $\text{ch}(L_\lambda)$ is W -invariant for the non-linked W -action (induct on the simple reflections). \square

Computing the rest of the a_{yw} is the content of the Kazhdan-Lusztig conjecture (which takes a lot of work). First, let's think about two cases we've ignored: non-regular and non-integral λ . We will address the non-regular case first via *translation functors*.

14.1 Standards and costandards

Recall that we have a class of objects Δ_λ which we call Verma modules or *standard objects*. We now define a dual notion.

Definition 14.13. The *Cartan involution* $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ is the unique Lie algebra morphism acting by -1 on \mathfrak{h} and exchanging \mathfrak{n} with \mathfrak{n}^- (exchanging e_α with f_α). The *contragredient* or *graded dual* of $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda \in \mathcal{O}$ is

$$(M^\vee)^\lambda := \bigoplus_{\lambda \in \mathfrak{h}^*} (M^\lambda)^*.$$

Note that M^λ are finite-dimensional, and \mathfrak{g} acts through the antipode twisted by the Cartan involution, i.e.

$$(x \cdot f)(m) = f(-x \cdot m), \quad x \in \mathfrak{g}, f \in M^\vee, m \in M.$$

Remark 14.14. The graded dual can be characterized as the maximal subspace of M^* such that \mathfrak{h} acts locally finitely. Note there is an alternative characterization of \mathcal{O} as the category of finite-dimensional, \mathfrak{b} -locally finite $\mathfrak{b}, \mathfrak{h}$ semisimple representations.

Exercise 14.15. Check that the graded dual defines a functor on \mathcal{O} (and also on each \mathcal{O}_χ). Show that $(M^\vee)^\vee = M$, and that $L_\lambda^\vee = L_\lambda$.

Definition 14.16. We define the *dual Verma module* or *costandard object* to be

$$\nabla_\lambda := \text{Hom}_{U\mathfrak{b}}(U\mathfrak{g}, k_\lambda)$$

where $U\mathfrak{b}$ acts on $U\mathfrak{g}$ by right multiplication, and $U\mathfrak{g}$ acts by multiplication on the left through the antipode-twisted Cartan involution.

Exercise 14.17. Show that $\Delta_\lambda^\vee = \nabla_\lambda$. Show that dual Verma modules have a unique maximal quotient and unique irreducible submodule L_λ .

Remark 14.18. These definitions involving duals can be a bit unwieldy; it is sometimes easier to remember that the graded dual reverses composition series. Unlike the Verma modules, the dual Vermas are not generated by \mathfrak{n}^- by a highest weight, e.g. $M^\vee(0)$ for SL_2 .

It will be useful to have the following lemma. Recall that the inflation functor $\text{Inf} : \text{Rep}(\mathfrak{h}) \rightarrow \text{Rep}(\mathfrak{b})$ is induced by the quotient $\mathfrak{b} \rightarrow \mathfrak{h}$, which has kernel $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Lemma 14.19. *We have adjunctions $(\text{Inf}, (-)^\mathfrak{n})$ and $((-)^\mathfrak{n}, \text{Inf})$, i.e. the inflation functor has left adjoint the \mathfrak{n} -coinvariants and right adjoint \mathfrak{n} -invariants. The restriction functor $\text{Res}_\mathfrak{b}^\mathfrak{g}$ functor has left adjoint compact induction and right adjoint induction. Thus,*

$$\text{Hom}_\mathcal{O}(\Delta_\lambda, M) = \text{Hom}_\mathfrak{g}(\text{cInd}(\text{Inf}(k_\lambda)), M) = \text{Hom}_\mathfrak{b}(\text{Inf}(k_\lambda), \text{Res } M) = \text{Hom}_\mathfrak{h}(k_\lambda, (\text{Res } M)^\mathfrak{n}),$$

$$\text{Hom}_\mathcal{O}(M, \nabla_\lambda) = \text{Hom}_\mathfrak{g}(M, \text{Ind}(\text{Inf}(k_\lambda))) = \text{Hom}_\mathfrak{b}(\text{Res } M, \text{Inf } k_\lambda) = \text{Hom}_\mathfrak{h}((\text{Res } M)_{\mathfrak{n}^-}, k_\lambda).$$

Note that \mathfrak{n}^- appears since we act through the Cartan involution.

Remark 14.20. The first statement just says that maps out of the standard module are given by highest weight vectors of weight λ in M . The second statement is a bit more confusing to me since it involves duals, but it's the dual statement.

Example 14.21. In some sense, the dual Verma modules are the Verma modules with the “simple pieces built up in the opposite order.” That is, if $w_0 \in W$ is the longest element, then we have for dominant λ (non-exact sequences)

$$\begin{aligned} L_{w_0 \cdot \lambda} &\hookrightarrow \Delta_\lambda \twoheadrightarrow L_\lambda, & L_\lambda &\hookrightarrow \nabla_\lambda \twoheadrightarrow L_{w_0 \cdot \lambda}, \\ \Delta_\lambda &\twoheadrightarrow L_\lambda \hookrightarrow \nabla_\lambda, & \nabla_\lambda &\twoheadrightarrow L_{w_0 \cdot \lambda} \hookrightarrow \Delta_\lambda. \end{aligned}$$

Note that, however, there are many maps $\nabla_\lambda \rightarrow \Delta_\lambda$, since $L_{w_0 \cdot \lambda}$ is not the unique quotient or sub. Using the adjunctions above and the calculation $(\nabla_\nu)^\mathfrak{n}$ is $\delta_{\lambda\mu}$ -dimensional in weight λ (or, $(\Delta_\mu)_{\mathfrak{n}^-}$ in weight λ), we see that

$$\text{Hom}(\Delta_\lambda, \nabla_\mu) \simeq \begin{cases} k & \lambda = \mu \\ 0 & \text{else.} \end{cases}$$

Let's just recap what we know.

- There are three kinds of interesting objects we know about so far: $L_\lambda, \Delta_\lambda, \nabla_\lambda$.

- The Δ_λ have L_λ as a unique quotient, but lots of subs including $L_{w_0 \cdot \lambda}$.
- The ∇_λ have L_λ as a unique sub, but lots of quotients including $L_{w_0 \cdot \lambda}$.
- There is at most one map $\Delta_\lambda \rightarrow \nabla_\mu$, but lots of maps $\nabla_\mu \rightarrow \Delta_\lambda$.

15 Lecture 15 (2021-03-19): (Co)standard filtrations, projectives

Definition 15.1. Let M be an object of \mathcal{O} . A *standard filtration* is a filtration such that the subquotients are standard objects, and likewise for a *costandard filtration*.

Let us give some examples of non-trivial objects with standard filtrations.

Proposition 15.2. Let μ_1, \dots, μ_r be the weights of a finite-dimensional V in non-increasing order. Then, $V \otimes \Delta_\lambda$ has a standard filtration with subquotients $\Delta_{\lambda+\mu_i}$ (i.e. with “highest standard” as a sub and “lowest standard” as a quotient). Likewise, if we order the weights in non-decreasing order, $V \otimes \nabla(\lambda)$ has a costandard filtration with subquotients $\nabla(\lambda + \mu_i)$ (i.e. with “highest costandard” as a quotient and “lowest costandard” as a sub).

Proof. Let us argue for standards. The idea is that the highest weight of $V \otimes \Delta_\lambda$ is $\lambda + \mu_r$; we claim that $\Delta_{\lambda+\mu_r}$ is a quotient of $V \otimes \Delta_\lambda$. Note that since V is finite-dimensional, $- \otimes V$ is exact, and its left and right adjoint are both $- \otimes V^*$. Using the adjunctions, we have

$$\mathrm{Hom}_{\mathfrak{g}}(V \otimes \Delta_\lambda, M) \simeq \mathrm{Hom}_{\mathfrak{b}}(V \otimes k_\lambda, M) \simeq \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{b}} (V \otimes k_\lambda), M).$$

Thus $V \otimes \Delta_\lambda \simeq U\mathfrak{g} \otimes_{U\mathfrak{b}} (V \otimes k_\lambda)$ by Yoneda. The \mathfrak{b} -representation V has a filtration with subquotients as described, and since $U\mathfrak{g}$ is free over $U\mathfrak{b}$, tensoring is exact, and the claim follows. For costandards, we have instead $V \otimes \nabla_\lambda \simeq \mathrm{Hom}_{U\mathfrak{b}}(U\mathfrak{g}, V \otimes k_\lambda)$. The argument for costandards follows by dualizing. \square

15.1 Projectives

Proposition 15.3. The category \mathcal{O} has enough projectives.

Proof. First, we argue that Δ_λ for λ dominant is projective. Since Δ_λ lives in the $[\lambda]$ -block, we may take summands in that block. Crucially, in this block, since λ is dominant, every λ -weight vector is killed by \mathfrak{n} . Thus we have the desired lift:

$$\begin{array}{ccc} \Delta_\lambda & & \\ \downarrow \text{dotted} & \searrow & \\ M & \twoheadrightarrow & N. \end{array}$$

Next, we argue that if L is finite-dimensional and P is projective, then $P \otimes L$ is projective. This follows since $\mathrm{Hom}_{\mathcal{O}}(P \otimes_k L, -) = \mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes -)$; $L^* \otimes -$ is exact, and $\mathrm{Hom}_{\mathcal{O}}(P, -)$ is exact.

To complete the proof, it suffices to show that every simple has a projective cover (exercise: show why this is enough). Consider L_μ , and let λ be any dominant weight such that $\lambda - \mu$ is integral dominant. By Proposition 15.2, there is a surjection

$$\Delta_\lambda \otimes L_{\lambda-\mu} \twoheadrightarrow M_\mu \twoheadrightarrow L_\mu.$$

\square

We now classify all indecomposable projectives. For a given module M , a projective cover of M is a projective $P_M \twoheadrightarrow M$ which is “minimal” in some appropriate sense.

Definition 15.4. A *projective cover* $P_M \twoheadrightarrow M$ is a projective P_M such that no proper submodule maps surjectively onto M . Note that projective covers are automatically indecomposable.

Proposition 15.5. Projective covers are indecomposable.

Proof. For any projective P and $P_1 \rightarrow M$, we have a split factorization through the any other projective $P_2 \rightarrow M$, i.e. the maps are projections onto and inclusions into various (projective) summands. The projective cover is minimal in the sense that the maps we get $P \rightarrow P_M$ must be surjective; in particular, they are indecomposable. \square

The next question is if projective covers exist, and whether they are unique up to isomorphism (i.e. $P_M \simeq P'_M$ commuting with the projections). We simply cite the following (though only the first claim is “involved”):

Proposition 15.6. *Projective covers exist in any Artinian category, and in any category they are unique given they exist. In this setting, projective covers are the only indecomposable projectives.*

Definition 15.7. We denote the projective cover of L_λ by P_λ . There is a dual notion of *injective envelope* which we denote by I_λ .

We note the following.

Proposition 15.8. *Projective covers have standard filtrations.*

Proof. This follows from the proof of existence; the projective cover of $L(\lambda)$ was obtained as a quotient of $M(\lambda) \otimes V$ for a finite-dimensional V , with the quotient arising via a standard filtration. \square

Finally, the following follows from Morita theory.

Proposition 15.9. *Let $\lambda \in \mathfrak{h}^*$, and let $P = \bigoplus_{\mu \in W \cdot \lambda} P_\mu$. Then P is a projective generator, and there is an equivalence of categories:*

$$\text{Hom}(P, -) : \mathcal{O}_{[\lambda]} \rightarrow \text{Mod}^r(\text{End}(P)).$$

Remark 15.10. Note that we could have used injective objects instead of projectives, giving rise to a “dual” module theory. Later we will introduce the notion of tilting objects which are self-dual and in a sense “in-between”.

We also have the following characterization of homomorphisms out of projective covers.

Proposition 15.11. *Let $M \in \mathcal{O}$. There is a canonical equivalence*

$$\text{Hom}(P_\lambda, M) \simeq M^{\text{wt}=\lambda}.$$

Proof. One can verify the above for simples $M = \Lambda_\mu$. Next, we induct on length of Jordan-Holder decomposition, i.e. on short exact sequences, using the lifting property of projectives. \square

16 Lecture 16 (2021-03-22): Category \mathcal{O} for \mathfrak{sl}_2

Recall our list of indecomposable objects in \mathcal{O}_0 from Section 9. Let us justify this list.

Remark 16.1 (Isomorphism classes). A priori, we have objects: $L_0, L_{-2}, \Delta_0, \Delta_{-2}, \nabla_0, \nabla_{-2}, P_0, P_{-2}, I_0, I_{-2}$.

- For anti-dominant weight, Δ_{-2} (and thus ∇_{-2}) is irreducible, i.e. $L_{-2} \simeq \Delta_{-2} \simeq \nabla_{-2}$.
- For dominant weight, standard is projective (and thus costandard is injective), i.e. $\Delta_0 \simeq P_0$ and $\nabla_0 \simeq I_0$.
- For the anti-dominant weight, $P_{-2} \simeq I_{-2}$.

Thus, we have five objects: $L_0, L_{-2} \simeq \Delta_{-2} \simeq \nabla_{-2}, \Delta_0 \simeq P_0, \nabla_0 \simeq I_0, P_{-2} \simeq I_{-2}$.

Let $P = P_{-2} \oplus P_0$. We can understand $\text{End}(P)$ via a quiver. Recall that $\text{Hom}(P_\lambda, L_\mu) = \delta_{\lambda\mu}$; we can compute dimensions of Hom spaces by lifting maps in the Jordan-Holder series. We have the identifies in the Grothendieck group:

$$[P_0] = [L_{-2}] + [L_0], \quad [P_{-2}] = 2[L_{-2}] + [L_0].$$

We find:

$$\dim(\text{Hom}(P_{-2}, P_{-2})) = 2, \dim(\text{Hom}(P_{-2}, P_0)) = 1, \dim(\text{Hom}(P_0, P_{-2})) = 1, \dim(\text{Hom}(P_0, P_0)) = 1.$$

Proposition 16.2. *The algebra $\text{End}(P)$ is equivalent to the path algebra for the quiver with relations:*

$$\bullet_{-2} \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{q} \end{array} \bullet_0 \quad pq = 0.$$

We define $m_{-2} = qp$ and $m_0 = pq$, i.e. we have $m_{-2}^2 = 0$ and $m_0 = 0$. Thus the functor $\mathbb{P} := \text{Hom}_{\mathcal{O}_0}(P, -)$ defines an equivalence to modules for the quiver with relations:

$$\bullet_{-2} \begin{array}{c} \xleftarrow{p^*} \\ \xrightarrow{q^*} \end{array} \bullet_0 \quad q^*p^* = 0.$$

Proof. Let us establish the relations. The only endomorphism of P_0 is the identity. Recall that we have a surjection $P_{-2} \twoheadrightarrow \Delta_{-2}$; by BGG reciprocity, the kernel must be Δ_0 , and q is this inclusion. The map p is given by the above surjection composed with the inclusion $\Delta_{-2} \hookrightarrow \Delta_0$. Thus we see that qp is not the identity, and that $pq = 0$. Furthermore, we have (up to renormalization), $(qp)^2 = qp$. \square

We can now see where our five objects go (we let $k = \mathbb{C}$, though we can take any algebraically closed field of characteristic 0), e.g. using Proposition 15.11:

$$\begin{aligned} \mathbb{P}(L_{-2}) &= \mathbb{C} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} 0 \\ \mathbb{P}(L_0) &= 0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathbb{C} \\ \mathbb{P}(\Delta_0) &= \mathbb{C} \begin{array}{c} \xleftarrow{0} \\ \xrightarrow{1} \end{array} \mathbb{C} \\ \mathbb{P}(\nabla_0) &= \mathbb{C} \begin{array}{c} \xleftarrow{1} \\ \xrightarrow{0} \end{array} \mathbb{C} \\ \mathbb{P}(P_{-2}) &= \mathbb{C}^2 \begin{array}{c} \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \\ \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \end{array} \mathbb{C} \end{aligned}$$

Exercise 16.3. Classify indecomposable representations of the above quiver. Thus our five objects are a complete list of indecomposables.

Remark 16.4. One can do the same in higher rank, though the quiver obviously gets more complicated. I am not sure if the resulting quiver admits a clean description of indecomposables, or whether they are given by the analogous list.

Finally, note that non-integral blocks all split into two summands isomorphic to \mathbf{Vect}_k , and that the singular block is isomorphic to \mathbf{Vect}_k since $\Delta_{-1} \simeq \nabla_{-1} \simeq L_{-1} \simeq P_{-1} \simeq I_{-1}$.

17 Lecture 17 (2021-03-24): BGG reciprocity, translation functors

17.1 BGG reciprocity

Proposition 17.1. *We have $\text{Ext}^i(\Delta_\lambda, \nabla_\mu) = 0$ for all i . In particular, the $i = 1$ case means we have a short exact sequence*

$$0 \rightarrow \nabla_\mu \rightarrow M \rightarrow \Delta_\lambda \rightarrow 0$$

then it must split.

Proof. We first consider the $i = 1$ case. Suppose that we have

$$0 \rightarrow \nabla_\mu \rightarrow M \rightarrow \Delta_\lambda \rightarrow 0.$$

Take $v_\lambda \in \Delta_\lambda$ the highest weight vector. If it lifts to a non-zero vector in M killed by \mathfrak{n} , then by the adjunctions we have a splitting $\Delta_\lambda \rightarrow M$. If it is not killed by \mathfrak{n} , its image under the action of \mathfrak{n} must land somewhere in ∇_μ , so in particular $\mu > \lambda$. Apply the graded dual, so we now have a sequence

$$0 \rightarrow \nabla_\lambda \rightarrow M \rightarrow \Delta_\mu \rightarrow 0.$$

Repeat the argument; we either get a splitting, or we find that $\lambda > \mu$, which cannot be.

For $i > 1$, we induct on i . Consider the short exact sequence

$$0 \rightarrow K \rightarrow P_\lambda \rightarrow \Delta_\lambda \rightarrow 0.$$

Note that K has a standard filtration. The long exact sequence gives us an exact sequence

$$0 = \text{Ext}^i(P_\lambda, \nabla_\mu) \rightarrow \text{Ext}^i(\Delta_\lambda, \nabla_\mu) \rightarrow \text{Ext}^{i-1}(K, \nabla_\mu) \rightarrow \text{Ext}^{i-1}(P_\lambda, \nabla_\mu) = 0.$$

We claim that $\text{Ext}^{i-1}(K, \nabla_\mu) = 0$ by inducting on the length of a standard filtration of K . The base case is simply the case where K is standard, which we know by induction. Now, suppose we have

$$0 \rightarrow K' \rightarrow K \rightarrow \Delta_\nu \rightarrow 0.$$

By induction, $\text{Ext}^{i-1}(-, \nabla_\mu) = 0$ on the ends, so it is zero in the middle. Thus, we are done. \square

We leave the proof of the following as an exercise.

Corollary 17.2. *Suppose that M has a standard filtration and N has a costandard filtration. Then $\text{Ext}^i(M, N) = 0$ for $i > 0$.*

Definition 17.3. An object in \mathcal{O} is *tilting* if it has both a standard and a costandard filtration.

Remark 17.4. If M, N are tilting objects, then $\text{Ext}^i(M, N) = 0$ for $i > 0$. Thus, if T is a (sum of) tilting objects generating the category \mathcal{O}_χ (in the derived sense), then there is an equivalence via derived Morita theory $D(\mathcal{O}_\chi) \simeq D(\text{End}(T))$. There is a general tilting theory containing statements like this, see for example Keller's "Derived categories and tilting" or Rickard's Morita theory for derived categories.

Theorem 17.5 (BGG reciprocity). *Let $(\Delta_\mu : P_\lambda)$ denote the multiplicity of M_μ in a standard filtration of P_λ (and likewise for costandards), and $[L_\lambda : M]$ denote the multiplicity of the simple L_λ in a Jordan-Holder series for M . We have*

$$(\Delta_\mu : P_\lambda) = [L_\lambda : \nabla_\mu] = [L_\lambda : \Delta_\mu] = (\nabla_\mu : I_\lambda).$$

Proof. First, note that the above multiplicity is well-defined; this can be observed via characters. Next, we compute the standard multiplicity, which we claim is equal to $\dim(\text{Hom}(P_\lambda, \nabla_\mu))$. This follows by induction on the standard filtration, i.e. given a short exact sequence $0 \rightarrow M_i \rightarrow M_{i+1} \rightarrow \Delta_{i+1} \rightarrow 0$, we have a short exact sequence

$$0 \rightarrow \text{Hom}(\Delta_{i+1}, \nabla_\mu) \rightarrow \text{Hom}(M_i, \nabla_\mu) \rightarrow \text{Hom}(M_{i+1}, \nabla_\mu) \rightarrow \text{Ext}^1(\Delta_{i+1}, \nabla_\mu) = 0$$

and note that $\dim(\text{Hom}(\Delta_{i+1}, \nabla_\mu))$ is 1-dimensional if $\Delta_{i+1} = \Delta_\mu$ and 0 otherwise.

Next, we claim that $\dim(\text{Hom}(P_\lambda, M)) = [L_\lambda : M]$; taking $M = \nabla_\mu$ gives the result. To see this, we induct on the composition series for M , i.e. given a short exact sequence

$$0 \rightarrow M_{i-1} \hookrightarrow M_i \twoheadrightarrow L_i \rightarrow 0$$

the dimension of maps out of P_λ is additive (i.e. since $\text{Hom}(P_\lambda, -)$ is exact). The claim follows since $\dim \text{Hom}(P_\lambda, L_\mu) = \delta_{\lambda\mu}$. \square

17.2 Translation functors

Definition 17.6. Let $\lambda_2 - \lambda_1$ be integral, and $\mu := w(\lambda_2 - \lambda_1)$ be the dominant weight in the (unshifted) W -orbit. We define *translation functors* $T_{\lambda_2\lambda_1} : \mathcal{O}_{[\lambda_1]} \rightarrow \mathcal{O}_{[\lambda_2]}$ as the composition

$$\mathcal{O}_{[\lambda_1]} \hookrightarrow \mathcal{O} \xrightarrow{-\otimes L_\mu} \mathcal{O} \xrightarrow{\text{pr}} \mathcal{O}_{[\lambda_2]}.$$

Proposition 17.7. *Translation functors are exact; in particular they preserve projective and injective objects.*

Proposition 17.8. *Suppose that λ is dominant and integral. Then $P_{w_0\cdot\lambda} \simeq I_{w_0\cdot\lambda}$. In particular, this object is tilting.*

Proof. At the most singular weight $-\rho$, the anti-dominant weight is the same as the dominant weight, thus $P_\lambda = I_\lambda$. Now, translate out of $-\rho$ to $w_0 \cdot \lambda$. We now show that $\Xi := T_{w_0\cdot\lambda, -\rho} P_{-\rho} \simeq P_{w_0\cdot\lambda}$. We first show that there is a nonzero map $\Xi \rightarrow L_{w_0\cdot\lambda}$, which means that the projective cover $P_{w_0\cdot\lambda}$ is a summand of Ξ . At the same time, we will show that the standard filtration of Ξ has multiplicity 1 for all standards, and thus $P_{w_0\cdot\lambda} \simeq \Xi$.

We defined Ξ to be the λ -linked summand of $\Delta_{-\rho} \otimes L_{\lambda+\rho}$. The quotient standard has weight $-\rho + w_0(\lambda + \rho) = w_0 \cdot (\lambda)$ as desired. Furthermore, by the same formula,

$$w \cdot \lambda = -\rho + w(\lambda + \rho)$$

i.e. the W -linkage class for λ corresponds to the unshifted W -orbit of the extremal weight in $L_{\lambda+\rho}$, which has multiplicity 1. \square

Remark 17.9. By 7.16 in [Hu08], the above is true for non-integral weights as well, but takes more work.

Remark 17.10. The object above is sometimes called the *big tilting object* and denoted Ξ . We will see why later: it is in fact the unique indecomposable tilting object which contains every simple as a subquotient.

Let's do an example.

Example 17.11 (\mathfrak{sl}_2). The “wall-crossing” functor $T_{0,-1} \circ T_{-1,0} : \mathcal{O}_0 \rightarrow \mathcal{O}_0$ sends

$$L_0 \mapsto 0 \mapsto 0, \quad L_{-2} \mapsto L_{-1} \mapsto \Xi.$$

By exactness, we can deduce:

$$\Delta_0 \mapsto L_{-1} \mapsto \Xi, \quad \nabla_0 \mapsto L_{-1} \mapsto \Xi, \quad \Xi \mapsto L_{-1} \oplus L_{-1} \mapsto \Xi \oplus \Xi.$$

Proposition 17.12. *Assume $\lambda - \mu$ is integral. The functors $T_{\lambda,\mu}$ and $T_{\mu,\lambda}$ are biadjoint.*

Proof. The projection and inclusion functors of summands of any category are biadjoint. Letting $\nu = w \cdot (\lambda - \mu)$ denote the dominant representative in the linkage class, $-\otimes L_\nu$ is biadjoint with $-\otimes L_\nu^*$. Thus, it suffices to show that $L_\nu^* \simeq L_{-w_0\cdot\nu}$ (note that $w_0 \cdot (-\nu)$ is dominant), but this is standard. \square

18 Lecture 18 (2021-03-29): Exactness of Beilinson-Bernstein

We never proved Beilinson-Bernstein. We will do so now. Recall Section 12.

Theorem 18.1 (Twisted Beilinson-Bernstein). *Assume that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}^{\leq -1}$ for all positive simple coroots α^\vee . Then the global sections functor factors through an equivalence from the Serre quotient $\Gamma' : \mathrm{DMod}(G/B, \lambda) / \ker(\Gamma) \rightarrow \mathrm{Rep}(\mathfrak{g})_{[\lambda]}$. If in addition, $\lambda \notin \mathbb{Z}^{\leq 0}$, then $\ker(\Gamma) = 0$.*

Proof. As discussed earlier, the statement is completely formal except for: (1) a calculation of global sections of $\mathcal{D}_{G/B, \lambda}$ (which is analogous to our earlier argument and we will omit), (2) exactness, (3) conservativeness. We will argue (2) and (3). \square

Example 18.2 (Monodromic D -modules for $G = SL_2$). Let $G = SL_2$. Then, $G/U = \mathbb{A}^2 - \{0\}$, and

$$\mathcal{D}(G/U) = \frac{k\langle s, \partial_s, t, \partial_t \rangle}{[\partial_s, s] = 1, [\partial_t, t] = 1}.$$

The Euler vector field is given by $s\partial_s + t\partial_t$, and we set it equal to the monodromy λ . Let $x = t/s$ and $y = s/t$. Then we have $\partial_x = s\partial_t$ and $\partial_y = t\partial_s$ (and thus $x\partial_x = t\partial_t$ and $y\partial_y = s\partial_s$). The Lie algebra \mathfrak{sl}_2 maps

$$e \mapsto -t\partial_s = -\partial_y, \quad h \mapsto t\partial_t - s\partial_s = 2x\partial_x - \lambda = \lambda - 2y\partial_y, \quad f \mapsto -s\partial_t = -\partial_x.$$

Take H -invariants (i.e. the weight 0 part), so we have the global sections

$$\begin{aligned} \mathcal{D}(G/U)_\lambda^H &= \frac{k\langle \partial_x, \partial_y, t\partial_t, s\partial_s \rangle}{[\partial_x, \partial_y] = s\partial_s - t\partial_t, [s\partial_s, \partial_x] = \partial_x, [t\partial_t, \partial_y] = -\partial_x, [s\partial_s, \partial_y] = -\partial_y, [t\partial_t, \partial_y] = \partial_y, s\partial_s + t\partial_t = \lambda} \\ &= \frac{k\langle \partial_x, \partial_y, x\partial_x, y\partial_y \rangle}{[\partial_x, \partial_y] = y\partial_y - x\partial_x, x\partial_x + y\partial_y = \lambda} \\ &= \frac{k\langle e, f, h \rangle}{[e, f] = h, [h, e] = 2e, [h, f] = -2f, H^2 + 2EF + 2FE = \lambda^2 + 2\lambda} = U\mathfrak{sl}_2 \otimes_{Z\mathfrak{sl}_2} k_{[\lambda]} \end{aligned}$$

where $[\lambda]$ is the central character corresponding to the W -orbit of λ . Note that this ring depends only on the (shifted) W -orbit of $\lambda \in \mathfrak{h}^*$.

Exercise 18.3 (Irreducibles for various λ). Let us restrict our attention to the singular support condition given by $x\xi_x = 0$ and $\xi_y = 0$. We know what to expect for $U\mathfrak{sl}_2 \otimes_{Z\mathfrak{sl}_2} k_{[\lambda]}$ -modules:

- When λ is regular integral, there are five indecomposables (and two simples).
- When λ is non-integral, then there are two simples (and no extensions).
- When λ is singular, there is one simple.

Show that:

- When $\lambda \geq 0$ is integral, there are five indecomposables, with simples given by a skyscraper and a line bundle, and the global sections functor is an equivalence.
- When $\lambda = -1$ the global sections functor kills the line bundle. However, the functor is still exact.
- When $\lambda \leq -2$ is integral, the global sections functor kills the line bundle. However, this means it cannot be exact.
- When λ is non-integral, there is one skyscraper simple (but two characters for the central character $[\lambda]$). This can be explained by the mismatch between $\lambda \in \mathfrak{h}^*$ and $[\lambda] \in \mathfrak{h}^*/W$. This mismatch arises above as well; it's just that we ignore the anti-dominant weights since the functor is not well-behaved.

There will be some useful set-up. Note that G is affine, thus D -affine.

Proposition 18.4. *We have maps of Lie algebras (i.e. $[a_\ell(x), a_\ell(y)] = a_\ell([x, y])$ and $[a_r(x), a_r(y)] = -a_r([x, y])$):*

$$\alpha_\ell : \mathfrak{g} \longrightarrow \Gamma(G, \mathcal{T}_G) \longleftarrow \mathfrak{g} : \alpha_r$$

such that their images commute (i.e. $[\alpha_\ell(x), \alpha_r(y)] = 0$). In particular, we have induced G -equivariant maps (where G acts on $U\mathfrak{g}$ via the adjoint action):

$$a_\ell : U\mathfrak{g} \otimes \mathcal{O}_G \longrightarrow \mathcal{D}(G) \longleftarrow \mathcal{O}_G \otimes U\mathfrak{g} : a_r.$$

Furthermore, both of these maps are isomorphisms.

Proof. The map α_ℓ identifies \mathfrak{g} with right-invariant²⁰ vector fields (likewise α_r with left-invariant vector fields). That the maps are isomorphisms is immediate to verify by passing to the associated graded. \square

Remark 18.5. The sign difference above explains the opposite sign conventions in Remark 12.1.

Proposition 18.6. *Assume that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{\leq -1}$ for all positive simple coroots α^\vee . Then the global sections functor $\Gamma : \mathrm{DMod}(G/B, \lambda) \rightarrow \mathrm{Mod}(U\mathfrak{g})_\lambda$ is exact.*

Proof. We sketch the proof (due to Frenkel and Gaitsgory). Let $q : G \rightarrow G/B$. We factor the global sections functor:

$$\mathrm{DMod}(G/B, \lambda) \xrightarrow{q^*} \mathrm{DMod}^{wB, \lambda}(G) \xrightarrow{\Gamma} \mathrm{Rep}^{B, \lambda}(\mathfrak{g}) \xrightarrow{\mathrm{Lie}} \mathrm{Mod}(U\mathfrak{g} \otimes_k U\mathfrak{b}^{op}) \xrightarrow{(-)^{\mathfrak{b}, \lambda} = \mathrm{Hom}_{\mathfrak{b}}(k_\lambda, -)} \mathrm{Rep}(\mathfrak{g}).$$

Let us explain what these categories and functors are:

- The first functor takes a $(p_* \mathcal{D}_{G/U})^H$ -module on G/B and pulls it back to G . The resulting sheaf is automatically (right) B -equivariant (i.e. a weakly B -equivariant D -module), and is also a \mathcal{D}_G -module (i.e. since its pullback to G/U is a $\mathcal{D}_{G/U}$ -module). It is exact since q is smooth.

The critical observation is the above proposition: the \mathcal{D}_G -module structure allows us to recover the left $U\mathfrak{g}$ -action, and the right $U\mathfrak{g}$ -action, thus a right $U\mathfrak{b}$ -action. On the other hand, differentiating the right B -action leads to a right $U\mathfrak{b}$ -action. These two right $U\mathfrak{b}$ -actions are compatible after twisting by the character k_λ .

- The second functor is the \mathcal{O} -module pushforward along an affine map, which is also exact, i.e. we forget the \mathcal{D}_G -module structure but retain the left $U\mathfrak{g}$ -module structure. Note that these are in fact equivalent data.
- The third functor arises by using the left $U\mathfrak{g}$ -action to recover the right $U\mathfrak{g}$ -action, and then forget down to a $U\mathfrak{b}$ -action, and subsequently forgetting the B -equivariance. This is exact. Morally, what we are doing is replacing right B -equivariance with a twist by k_λ of its differentiated $U\mathfrak{b}$ -action (which commutes with the left $U\mathfrak{g}$ -action).
- The fourth map (combined with the third) realizes the B -invariants functor. Per the discussion above, the B -invariants functor is realized by $\mathrm{Hom}(k_\lambda, -)$ rather than $\mathrm{Hom}(k, -)$. Note this does not involve the left $U\mathfrak{g}$ -action at all, which is retained.

We examine the fourth map. Note that there are two \mathfrak{g} -actions on the global sections of a D -module on G : one by left and one by right multiplication. Considering the right multiplication, we have the usual tensor-hom adjunction (viewing $U\mathfrak{g}$ as a $U\mathfrak{b}$ - $U\mathfrak{g}$ -bimodule):

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta_{-\lambda}, -) = \mathrm{Hom}_{\mathfrak{g}}(k_\lambda \otimes_{U\mathfrak{b}} U\mathfrak{g}, -) \simeq \mathrm{Hom}_{\mathfrak{b}}(k_\lambda, \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, -)) = \mathrm{Hom}_{\mathfrak{b}}(k_\lambda, -).$$

If M is a right B -equivariant D -module on G , then in particular it is right U -equivariant, i.e. \mathfrak{n} has to act locally finitely through the right \mathfrak{g} -action, as well as T -equivariant, i.e. \mathfrak{t} has to act semisimply. Thus, the category takes values²¹ in category \mathcal{O} , where Δ_0 is projective, so this functor is exact. \square

²⁰E.g. because $\frac{d}{dt}\gamma(t)h|_{t=0}$ is not left-invariant, but is right-invariant.

²¹Note this is the “right category \mathcal{O} ”, while the \mathfrak{g} -representation structure we are left with is the left one, which of course does not have to take values in category \mathcal{O} .

19 Lecture 19 (2020-03-31): Conservativeness of Beilinson-Bernstein

We aim to prove the following.

Proposition 19.1. *Assume that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{\leq 0}$ for all positive simple coroots α^\vee . Then the global sections functor $\Gamma : \mathrm{DMod}(G/B, \lambda) \rightarrow \mathrm{Mod}(U\mathfrak{g})_\lambda$ is conservative.*

Proof. Suppose that λ is antidominant, and let $p : G/B \rightarrow \mathrm{pt.}$ Borel-Weil-Bott gives us an equivalence $L_{w_0\lambda} \rightarrow p_*\mathcal{L}_\lambda$, which by adjunction gives us a map $p^*L_{w_0\lambda} \rightarrow \mathcal{L}_\lambda$. We define

$$\mathcal{V}_\lambda := p^*p_*\mathcal{L}_\lambda \simeq p^*L_{w_0\lambda} \in \mathrm{QCoh}^G(G/B).$$

Note that $\mathrm{QCoh}^G(G/B) \simeq \mathrm{Rep}(B)$, and the \mathcal{V}_λ corresponds to the B -representation $L_{w_0\lambda}$. By the usual arguments, this representation has a filtration by characters, with submodule at weight $w_0\lambda$ and quotient at weight λ (exhibited by the natural map above).

Now, let \mathcal{M} be a $\mathcal{D}_{G/B} := (\pi_*\mathcal{D}_{G/B})^H$ -coherent D -module (where $\pi : G/U \rightarrow G/B$). It has a set of generators, and thus some twist $\mathcal{M} \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_\mu$ for μ very antidominant will be globally generated over $(p_*\mathcal{D}_{G/B})^H$. Choose such a μ , and consider the map of $\mathcal{D}_{G/B}$ -modules:

$$\mathcal{M} \otimes_{\mathcal{O}_{G/B}} \mathcal{V}_\mu \twoheadrightarrow \mathcal{M} \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_\mu$$

which is a surjection since $\mathcal{V}_\mu \twoheadrightarrow \mathcal{L}_\mu$ is. We claim it is split; if so, then by the projection formula, the global sections of the left-hand side is

$$p_*(\mathcal{M} \otimes \mathcal{V}_\mu) = p_*(\mathcal{M} \otimes p^*L_{w_0\mu}) \simeq \Gamma(G/B, \mathcal{M}) \otimes L_{w_0\mu}.$$

It contains $\Gamma(G/B, \mathcal{M} \otimes \mathcal{L}_\mu)$ as a summand, which is non-zero since it is globally generated (over $(p_*\mathcal{D}_{G/B})^H$).

We now prove the claim. Note that $\mathcal{M} \otimes \mathcal{V}_\mu$ has a filtration with graded pieces $\mathcal{M} \otimes \mathcal{L}_\nu$ for various weights $w_0\nu$ in $L_{w_0\mu}$. Since $Z\mathfrak{g}$ acts on global sections, it acts on the sheaves itself, and by assumption \mathcal{M} is acted on through central character $[\lambda]$, thus $\mathcal{M} \otimes \mathcal{L}_\nu$ is acted on through central character $[\lambda + \nu]$. The claim now follows from the following fact: if $w_0\lambda + \rho$ is regular dominant and ν is dominant, then each extremal weight in $L_{w_0\lambda} \otimes k_\nu$ is unique in its linkage class. \square

19.1 Translation functors and Beilinson-Bernstein

On the D -modules side, we define certain translation functors.

Definition 19.2. Let $p : \tilde{X} \rightarrow X$ be a H -torsor, for some torus H . Let $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu - \lambda = \chi \in X^\bullet(H)$ is integral, i.e. a character corresponding to a one-dimensional representation V_χ of H . Let \mathcal{L}_χ be the sheaf of sections of the line bundle $\tilde{X} \times^{\mathbb{G}_m} V_\chi$. Then, we have an equivalence

$$\Theta_{\mu, \lambda} := \mathcal{L}_\chi \otimes_{\mathcal{O}_X} - : \mathrm{DMod}^{wH, \lambda}(\tilde{X}) \rightarrow \mathrm{DMod}^{wH, \mu}(\tilde{X}).$$

Proof. We need to explain how the sheaf of twisted differential operators acts. Let \mathcal{M} be a $\mathcal{D}_{G/B, \lambda} = (p_*\mathcal{D}_{G/U})^H \otimes_{k[\mathfrak{h}^*]} k_\lambda$ module. We can view it instead as a $(p_*\mathcal{D}_{G/U})^H$ -module via the forgetful functor, with an additional $k[\mathfrak{h}^*]$ -module structure. Now, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_\chi$ still has a $(p_*\mathcal{D}_{G/U})^H$ -module structure, but $k[\mathfrak{h}^*]$ acts through $\lambda - \chi = \mu$. \square

Our goal is to prove the following. Note that $U_\lambda\mathfrak{g} = U\mathfrak{g} \otimes_{Z\mathfrak{g}} k_{[\lambda]}$, i.e. the central character acts on $\mathrm{Mod}(U_\lambda\mathfrak{g})$ by strict central character (not generalized).

Proposition 19.3. *Suppose that $\lambda + \rho, \mu + \rho$ are dominant, and that $\lambda - \mu$ is integral and that $W^\lambda \subset W^\mu$ (where $W^\lambda \subset W$ is the stabilizer, i.e. μ is allowed to be “deeper in walls”). Then there is a commuting diagram*

$$\begin{array}{ccc} \mathrm{DMod}(G/B, \lambda) & \xrightarrow{\Gamma} & \mathrm{Mod}(U_\lambda\mathfrak{g}) \\ \Theta_{\mu, \lambda} \downarrow & & \downarrow T_{\mu, \lambda} \\ \mathrm{DMod}(G/B, \mu) & \xrightarrow{\Gamma} & \mathrm{Mod}(U_\mu\mathfrak{g}). \end{array}$$

Proof. Let $\nu = \lambda - \mu$, which is automatically dominant, and let $p : G/B \rightarrow \text{pt.}$ Consider

$$T_{\mu\lambda}(\Gamma(G/B, M)) = \text{pr}_\mu(L_\nu \otimes p_* M) \simeq \text{pr}_\mu(p_*(p^* L_\nu \otimes M))$$

by the projection formula. We leave it to the reader to verify that we have a decomposition and projection functors $\text{pr}_\lambda : \text{DMod}_f(G/U/H) \rightarrow \text{DMod}_f(G/B, \lambda)$, and that the projections commute under Γ , as well as its left adjoint localization Loc . It suffices to prove that the left adjoints commute. Going around counterclockwise, we have

$$\text{Loc}(T_{\lambda\mu} M) = \text{Loc}(\text{pr}_\lambda(M \otimes L_\nu^*)) = \text{pr}_\lambda(\text{Loc}(M \otimes L_\nu^*)) \simeq \text{pr}_\lambda(p^*(U\mathfrak{g} \otimes M) \otimes \mathcal{V}_{-w_0\nu}).$$

The localization $p^*(U\mathfrak{g} \otimes M)$ is in the μ -block. As above, $\mathcal{V}_{-w_0\nu}$ has a filtration whose graded pieces are $\mathcal{L}_{\nu'}$; in particular, $-\nu$ is an extremal weight and is unique in its linkage class. Since μ is dominant, the projection to the λ block is given by $-\otimes \mathcal{L}_{-\nu}$. \square

20 Lecture 20: Constructible sheaves (Balasz Elek)

See Balasz's notes.

21 Lecture 21 (2021-04-07): Derived functors, constructibility of holonomic D -modules

It's been awhile since we talked about D -modules. Let us recap. Let all schemes be smooth.

- We have abelian categories of left and right D -modules, $\text{DMod}^\ell(X)$ and $\text{DMod}^r(X)$ with side-changing functors (equivalences) that allow us to pass between left and right D -modules (see Definition 4.8).
- For $f : X \rightarrow Y$, we have functors $f^\dagger : \text{DMod}^\ell(Y) \rightarrow \text{DMod}^\ell(X)$ and $f_\bullet : \text{DMod}^r(X) \rightarrow \text{DMod}^r(Y)$ (see Definitions 4.9 and 4.12).
- There is an abelian category of coherent bounded D -modules $\text{DMod}_c(X)$ and holonomic D -modules $\text{DMod}_h(X)$. Holonomic D -modules have singular support of minimal dimension.

Our goal will be to prove a certain “constructibility” theorem for holonomic D -modules, develop the “six functors” formalism for them, and give a description of simple holonomic D -modules.

21.1 Derived categories and functors

To do so we need to define some derived categories and derived functors.

Definition 21.1. We will denote the derived categories by

$$D(\text{DMod}(X)), D^b(\text{DMod}(X)), D^+(\text{DMod}(X)), D^-(\text{DMod}(X)).$$

Strictly speaking, we will take $D(\text{DMod}(X))$ to be the full subcategory of the derived category of sheaves of \mathcal{D}_X -modules on X with quasicoherent cohomology. We denote by $D_c(\text{DMod}(X))$ (resp. $D_h(\text{DMod}(X))$) to be the full subcategory with \mathcal{D}_X -coherent (resp. bounded holonomic) cohomology.

Definition 21.2. Let $f : X \rightarrow Y$ be a map of smooth schemes. We use the notation $\dim(f) := \dim(X) - \dim(Y)$ to denote the (expected) relative dimension of f . We define

$$f^! : D \text{DMod}(Y) \rightarrow D \text{DMod}(X), \quad f^! = Lf^\dagger[\dim(f)]$$

$$f_* : D \text{DMod}^r(X) \rightarrow D \text{Mod}^r(Y), \quad f_*(-) = Rf_*(- \otimes_{\mathcal{D}_X}^L \mathcal{D}_{X \rightarrow Y}).$$

We state the following without proof.

Proposition 21.3. *The natural functors $D \operatorname{DMod}_c(X) \rightarrow D_c \operatorname{DMod}(X)$ and $D \operatorname{DMod}_h(X) \rightarrow D_h \operatorname{DMod}(X)$ are equivalences.*

Remark 21.4. One can motivate the shift in $f^!$ as follows. Morally, if f is proper, e.g. a closed embedding, then we should have an adjunction $(f_*, f^!)$. One can show that these two are in fact adjoint given the above shift. For example, take $f : \{0\} \hookrightarrow \mathbb{A}^1$. Then, $\operatorname{Hom}(f_! k, \mathcal{M}) = \operatorname{Hom}(D/Dx, \mathcal{M}) = \ker(x : \mathcal{M} \rightarrow \mathcal{M})$. On the other hand, $Lf^! \mathcal{M}$ is the complex $x : \mathcal{M} \rightarrow \mathcal{M}$ in degrees $[-1, 0]$; to get the kernel in degree 0 we need to shift up by 1, and $\dim(f) = -1$.

Example 21.5. If $f : X \rightarrow \operatorname{pt}$, the pushforward is computed by the *Spencer resolution*. The Spencer resolution $\operatorname{dR}_X^\bullet(\mathcal{D}_X)$ is a locally free resolution of the right D -module ω_X :

$$0 \rightarrow \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \cdots \rightarrow \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \omega_X \rightarrow 0$$

with differential in local coordinates:

$$\omega \otimes P \mapsto d\omega \otimes P + \sum_i dx_i \wedge \omega \otimes \frac{\partial}{\partial x_i} P.$$

That is, it is a certain deformation of $\operatorname{dR}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We leave the proof that this is a resolution as an exercise; roughly, the idea is to filter the complex take the associated graded, and show that the pieces are acyclic (i.e. they are the pieces of a Koszul resolution).

Now, the functor $f_\bullet : \operatorname{DMod}^\ell(X) \rightarrow \operatorname{DMod}^\ell(\operatorname{pt})$ is given by (see Definition 4.12):

$$f_\bullet(-) := Rf_* \left(\mathcal{D}_{Y^*f-X} \otimes_{\mathcal{D}_X}^L - \right).$$

The “piece” involving the derived tensor product is roughly $\omega_X \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{M}$; thus we can resolve ω_X via the Spencer resolution. Unwinding everything, we define the *de Rham complex* for M by:

$$\operatorname{dR}_X^\bullet(\mathcal{M}) := \operatorname{dR}_X^\bullet(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M}.$$

Then, we have that

$$f_* \mathcal{M} = R\Gamma(X, \operatorname{dR}_X^\bullet(\mathcal{M})).$$

In particular, if $\mathcal{M} = \mathcal{O}_X$, then we obtain the (shifted) de Rham cohomology of X :

$$f_* \mathcal{O}_X \simeq H_{\operatorname{dR}}^\bullet(X; k)[\dim_{\mathbb{C}} X].$$

Remark 21.6. The pushforward for a projection $f : X \times Y \rightarrow X$ is similar. In general, there are formulas for the pushforward but they are quite complicated. See Gauss-Manin connection.

21.2 Statements of theorems and duality

Our goal is to prove the following two theorems.

Theorem 21.7 (Preservation). *The functors f_* and $f^!$ preserve holonomicity.*

Theorem 21.8 (Constructibility). *A complex \mathcal{M}^\bullet is holonomic if and only if there is a stratification $i_\alpha : S_\alpha \hookrightarrow X$ such that the cohomology sheaves of $i_{\alpha*} \mathcal{M}^\bullet$ are flat connections. Furthermore, in this case the stratification is obtained via the irreducible components of the singular support, restricted to the zero section.*

In the following, for a left D -module \mathcal{M} we let $\mathcal{M}^\vee := R\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ (temporarily) denote the derived dual module, which is a complex of right D -modules. This allows us to define sheafy $\operatorname{Ext}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$. We state the following theorem; we will postpone its proof (and maybe skip it for time reasons). The proof involves a lot of homological algebra and algebra of filtrations.

Theorem 21.9 (Roos). *Suppose that \mathcal{M} is coherent. Then $\dim(\mathrm{SS}(H^k(\mathcal{M}^\vee))) \leq \dim(\mathbb{T}_X^*) - k$, and \mathcal{M}^\vee has cohomological amplitude $[\mathrm{codim}(\mathrm{SS}(\mathcal{M})), n]$.*

The first consequence is the following definition. Note that if \mathcal{M} is holonomic, then \mathcal{M}^\vee is concentrated in degree n . Thus we shift to get a t -exact functor, i.e. on abelian categories.

Definition 21.10. We define the *duality functor* $\mathbb{D}_X : D^-(\mathrm{DMod}(X)) \rightarrow D^+(\mathrm{DMod}(X))^{\mathrm{op}}$ by

$$\mathbb{D}_X(\mathcal{M}) := \Omega^{-1}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))[\dim(X)].$$

Often we restrict \mathbb{D}_X to $D_h(\mathrm{DMod}(X))$.

Example 21.11. Let's just check what happens on $X = \mathbb{A}^1$. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X w$ where $w \in \mathcal{D}_X$. We replace \mathcal{M} with the complex $\mathcal{D}_X \xrightarrow{w} \mathcal{D}_X$ in degrees $[-1, 0]$, where the map is right multiplication by w . Then, we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = \mathcal{D}_X \xrightarrow{w} \mathcal{D}_X$$

i.e. a complex in degrees $[0, 1]$ where the map is left multiplication by w . Shifting by $\dim(X)$, we obtain in degree 0 the right D -module $\mathcal{D}_X/w\mathcal{D}_X$. The side changing functor takes this to $\mathcal{D}_X/w^t\mathcal{D}_X$, where w^t is the transpose of w .

By Roos' theorem, duality does not make sense on the abelian category of coherent D -modules. However, in the derived category we have the following.

Proposition 21.12. *The functor $\mathbb{D}_X : D_c(\mathrm{DMod}(X)) \rightarrow D_c(\mathrm{DMod}(X))^{\mathrm{op}}$ is an equivalence, and $\mathbb{D}_X^2 \simeq \mathrm{id}_{D_c(\mathrm{DMod}(X))}$.*

Proof. The first claim essentially follows from homological algebra: that a coherent D -module has a finite resolution by finite rank projectives. We leave the second claim as an exercise. \square

However, duality is well-behaved on holonomic D -modules. In a sense this explains the shift. We have the following consequences; we leave the proofs as exercises.

- If \mathcal{M} is in the heart, then $\mathbb{D}(\mathcal{M})$ has cohomological amplitude $[-\dim(X), 0]$.
- The global dimension of $\mathrm{DMod}(X)$ is n .
- A module $\mathcal{M} \in \mathrm{DMod}(X)$ is holonomic if and only if $\mathbb{D}(X) \in (D\mathrm{DMod}(X))^\heartsuit = \mathrm{DMod}(X)$.
- The duality functor \mathbb{D} restricts to an autoequivalence on the abelian category $\mathrm{DMod}_h(X)$.

22 Lecture 22 (2020-04-09): Proof of lemma

Let's give some examples of functors not preserving coherence.

Example 22.1. Let $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the open immersion. Then $j_*\mathcal{D}_{\mathbb{G}_m}$ is not coherent. However, $j_*\mathcal{O}_{\mathbb{G}_m}$ is coherent. Now let $i : \{0\} \hookrightarrow \mathbb{A}^1$ be the closed immersion. Then $i^!\mathcal{D}_{\mathbb{A}^1} = k[\partial]$ is not finite-dimensional (i.e. not coherent).

Remark 22.2. This is what makes the fact that $f_*, f^!$ preserve holonomicity somewhat unexpected. Note that this is why we cannot define functors $f_! = \mathbb{D}f_*\mathbb{D}$ and $f^* = \mathbb{D}f^!\mathbb{D}$ in general; the functor \mathbb{D} is only defined on coherent complexes.

Lemma 22.3. *Let $i : S \hookrightarrow X$ be a locally closed embedding. Then, i_* restricts to a functor $i_* : D_h\mathrm{DMod}(S) \rightarrow D_h\mathrm{DMod}(X)$. If i is affine then it restricts to a functor $i_* : \mathrm{DMod}_h(S) \rightarrow \mathrm{DMod}_h(X)$.*

Proof. Suppose that i is a closed immersion. Recall we had a formula for computing the singular support of pushforward along a closed immersion, i.e. as the pull-push along:

$$\mathbb{T}_S^* \leftarrow \mathbb{T}_X^* \times_X S \hookrightarrow \mathbb{T}_X^*.$$

The left map is a quotient bundle with kernel the normal bundle $\mathbb{N}_{S/X}^*$, and in particular taking inverse image increases dimension by $\dim(X) - \dim(S)$. The right map is a closed immersion so it does not affect dimension. The first claim follows for closed immersions; the second claim follows since for closed immersions, i_* is exact (see Lemma 5.5).

Now, suppose that i is open. We can assume the complex is in the heart. Note that the underlying \mathcal{O} -module is given by the \mathcal{O} -module pushforward in this case; in particular, if i is affine then i_* is exact, establishing the second claim for open immersions.

We may assume X is affine. We make the following reduction; cover S by affine opens and, compute i_* using the Čech resolution instead. Thus we've reduced to the case where S is affine, i.e. it is a distinguished open of affine X (and i_* is exact).

Thus, we are in the setting where $U = \operatorname{Spec} \mathcal{O}(X)[1/f] \subset X = \operatorname{Spec} \mathcal{O}(X)$. We may also assume that $M = \Gamma(U, \mathcal{M}) = \Gamma(X, i_* \mathcal{M})$ is generated over $\mathcal{D}(U)$ by a single generator $m_0 \in M$. Let us first show that M is $\mathcal{D}(X)$ -coherent. That is, we need to show that

$$\frac{1}{f^n} m_0 \in \mathcal{D}(X) m_0$$

for every $n \geq 0$.²² This follows from the following lemma, by taking n to be very negative and possibly replacing m_0 with $f^{-k} m_0$ to avoid the zeroes of $b(t)$.

Lemma 22.4 (Lemma on b -functions). *Let t be a formal polynomial variable. There is a polynomial operator $D(t) \in \mathcal{D}(X)[t]$ and a nonzero polynomial $b(t) \in k[t]$ such that*

$$b(n) f^n \cdot m_0 = D(n) f^{n+1} \cdot m_0 \in M.$$

Proof of Lemma on b -functions. Let $K = k(t)$ be the field of rational functions on a formal variable t . We take the base change $U_K \hookrightarrow X_K$, and define a D -module M_K on U_K by

$$M_K := f^t K \otimes_k M$$

where f^t is a formal symbol that tells us how tangents act, i.e. its underlying \mathcal{O} -module is the base change and tangents act by:

$$\theta \cdot f^t m = \frac{t\theta(f)}{f} f^t h + f^t \theta m$$

i.e. via the formula $[\theta, f^t] = t f^{t-1} \theta(t)$. We use the *extension lemma* (proof later) to claim that $i_* M_K$ has a holonomic D -submodule $H \subset M_K$ such that $H|_{U_K} = M_K$, and therefore M_K/H is supported at the complement $X - U$. Our generator $f^t m_0 \in M_K$ is annihilated by some power of f in the quotient M_K/H , say $f^r (f^t m_0) = 0$, i.e. $f^r (f^t m_0) \in H$. Replace m_0 with $f^r m_0$ (i.e. replace $t + r$ with t , i.e. a change of coordinates in K), so now we may assume that the generator $f^t m_0 \in H$.

Since N is holonomic, it has finite length, so the chain

$$\mathcal{D}(X_K)(f^t m_0) \supset \mathcal{D}(X_K) f(f^t m_0) \supset \cdots$$

eventually terminates, i.e.

$$\mathcal{D}(X_K) f^t m_0 \supset \mathcal{D}(X_K) f^{t+1} m_0 \supset \cdots \supset \mathcal{D}(X) f^{t+r} m_0 = \mathcal{D}(X) f^{t+r+1} m_0.$$

²²Think about the example $U = \mathbb{G}_m \subset X = \mathbb{A}^1$.

Replace m_0 with $f^r m_0$ again.²³ Then, there is a $D_0 \in \mathcal{D}(X_K)$ such that

$$D_0 \cdot f^{t+1} m_0 = f^t m_0.$$

Now write $D_0 = D/b$ where $D \in \mathcal{D}(X)[t]$ and $b \in k[t]$. Note that by replacing m_0 with $f^{-r} m_0$ for large r , we can guarantee that the zeroes of b are positive, and in particular, $H = M_K$. \square

We now prove the extension lemma.

Lemma 22.5 (Extension). *Let $\mathcal{M} \in \text{DMod}(X)$, and $U \subset X$ an open subscheme. If $\mathcal{M}|_U$ is holonomic, then there is a holonomic extension of $\mathcal{M}|_U$ inside \mathcal{M} as a submodule.*

Proof. First, we can find a coherent extension of $\mathcal{M}|_U$ inside \mathcal{M} ; we can write \mathcal{M} as a sum of all coherent subsheaves, and take the extension to be a finite sum of these coherent subsheaves. Thus, we can replace \mathcal{M} with this extension, so we assume it is coherent.

Take $\mathbb{D}_X(\mathcal{M})$; since this complex lives in degrees $[-\dim(X), 0]$, there is a map $\mathbb{D}_X(\mathcal{M}) \rightarrow H^0(\mathbb{D}_X(\mathcal{M}))$ (roughly, it's a quotient). We have $\dim(\text{SS}(H^0(\mathbb{D}_X(\mathcal{M})))) = \dim(X)$ by Roos' theorem, i.e. it is holonomic. Taking duals, we get an injective (verify as an exercise) map $\mathbb{D}_X(H^0(\mathbb{D}_X(\mathcal{M}))) \rightarrow \mathcal{M}$. Restrict this map to U . Using that duality and cohomology commutes with restriction to opens, and since $\mathcal{M}|_U$ is holonomic (thus duality is t -exact), we obtain

$$\mathbb{D}_X(H^0(\mathbb{D}_X(\mathcal{M})))|_U = \mathbb{D}_U(H^0(\mathbb{D}_U(\mathcal{M}|_U))) = \mathcal{M}|_U \rightarrow \mathcal{M}|_U.$$

Since $\mathcal{M}|_U$ is holomorphic it has finite length, so the map, a priori just an embedding, is an isomorphism. \square

We return to our main proof. Let us retain the notation from the proof of the b -lemma. The lemma showed that $M_K = H$ is in fact holonomic. Recall that $M_K = f^t K \otimes_k M$. The idea will be to specialize t . That is, we need to lift our “data” defined over K to $k[t][1/p]$ for p some product of “bad” roots, and then specialize at a “good point” $t = n$.

Let's say what this “data” is, other than X_K, U_K, M_K . That M_K involved f^t in its definition means we should specialize at an integer if we want to recover M as a D -module (it still makes sense to specialize at any non-integral point, but we get a D -module non-isomorphic to the original M). In addition, choose a good filtration of M_K such that m_0 is in the lowest piece of the filtration. Let $D_1, \dots, D_r \in \mathcal{D}(X_K)$ generate the annihilator of $f^t m_0 \in M_K$; in the associated graded they cut out the singular support, which must be a dimension $\dim_K(X_K)$ subvariety of $\mathbb{T}_{X_K/K}^*$. We also need to avoid the roots in the denominators which appear in these D_i .

Subject to the above restrictions, we do our lift and specialization (which evidently can be done). Then, the $D_i(n)$ generate the annihilator of $f^n m_0$, which is also a generator of M (as we've shown above), and their images in the associated graded cut out a dimension $\dim(X)$ subvariety in \mathbb{T}_X^* . Thus, M is holonomic. \square

23 Lecture 23 (2021-04-14): Inducting on dimension

We skipped some basic techniques for devissage-type arguments, which we will fill in now. Let $Z \subset X$ a closed subscheme with complement U ; then there is a localization sequence of derived categories:

$$D\text{Coh}_Z(X) \longrightarrow D\text{Coh}(X) \longrightarrow D\text{Coh}(U).$$

This leads to an exact triangle

$$R\Gamma_Z(\mathcal{F}^\bullet) \rightarrow \mathcal{F}^\bullet \rightarrow j_* \mathcal{F}^\bullet|_U.$$

Suppose we have some property we want to prove for a sheaf on X , and suppose we know it is true generically. Then we can choose an open U for which it is true; it remains to show that it is true for $R\Gamma_Z(X, \mathcal{F}^\bullet)$. Generally, $R\Gamma_Z(\mathcal{F}^\bullet)$ is not the same as a sheaf on the scheme Z – for example, take $X = \mathbb{A}^1$ and $Z = \{0\}$, then $R\Gamma_Z(\mathcal{O}_X) =$

²³Note that it is this step that fails if we do not pass to the fraction field. Namely, this step would otherwise only tell us that there is a generator m_0 such that $\frac{1}{f} m_0 \in \mathcal{D}(X)m_0$. We would need to iterate the argument, but we cannot, because if the inclusions in the chain are strict then we do not have $\frac{1}{f^2} m_0 \in \mathcal{D}(X)m_0$.

$k[x, x^{-1}]/k[x][{-1}]$ is locally x -torsion but of arbitrarily high order. Some techniques are required to overcome this difference (the “devissage step”), but if it can be done, then one can reduce to a smaller-dimensional subvariety and induct.

In the setting of D -modules, the devissage step is trivial by Kashiwara. However, there is the additional technicality that we can only induct on smooth schemes. Let us first define the local cohomology functor.

Definition 23.1. Let $\mathcal{M} \in \mathrm{DMod}(X)$, and $Z \subset X$ a closed subscheme. We define $\Gamma_Z(\mathcal{M}) \in \mathrm{DMod}(X)$ by

$$\Gamma_Z(\mathcal{M})(U) := \{s \in \mathcal{M}(U) \mid \mathrm{supp}(s) \subset Z\} \subset \mathcal{M}(U).$$

This subsheaf is a D -module.²⁴ The functor is left-exact and we can define its right derived functor $R\Gamma_Z$.

Proposition 23.2 (Exact triangle). *Let $i : Z \hookrightarrow X$ be a closed subscheme, and $j : U \hookrightarrow X$ the inclusion of its complement. There is a short triangle of functors*

$$\Gamma_Z(-) \longrightarrow \mathrm{id}_{\mathrm{DMod}(X)} \longrightarrow j_* j^!.$$

Furthermore, if Z is smooth there is a natural identification $\Gamma_Z \simeq i_* i^!$.

Proof. For the first claim, note that for an open embedding j , the functor on D -modules $j^!$ is the \mathcal{O} -module pullback, which is just the sheaf restriction, i.e. this follows from the same claim in the setting of arbitrary sheaves on any topological space:

$$R\Gamma_Z(\mathcal{M}^\bullet) \longrightarrow \mathcal{M}^\bullet \longrightarrow Rj_* \mathcal{M}^\bullet|_U.$$

For this claim it suffices to check exactness when \mathcal{M}^\bullet is injective. We can check on stalks. We claim that if $z \in Z$ then $R\Gamma_Z(\mathcal{M})_z = \mathcal{M}_z$ and $(Rj_* \mathcal{M}|_U)_z = 0$. We will use the fact that injective sheaves are *flasque*, i.e. the restriction maps are all surjective. For the first claim, it suffices to show that $\Gamma_Z(\mathcal{M})_z = \mathcal{M}_z$ for any $z \in Z$ and \mathcal{M} injective. Clearly $\Gamma_Z(\mathcal{M})_z \subset \mathcal{M}_z$; for the opposite inclusion, note that every $s \in \mathcal{M}_z$ can be extended to any open neighborhood of z ; lift the section to some $\tilde{s} \in \mathcal{M}(U')$. Since $\mathrm{supp}(\tilde{s})$ is closed, we take $U'' = U' - (\mathrm{supp}(\tilde{s}) - Z)$; the image of \tilde{s} in $\mathcal{M}(U'')$ has support contained in Z , so $s \in \mathcal{M}_z$. The second claim is left as an exercise.

For the second claim, note that $R\Gamma_Z(\mathcal{M}^\bullet)$ is supported at Z . Then by Kashiwara, the counit map $i_* i^! R\Gamma_Z(\mathcal{M}^\bullet) \rightarrow R\Gamma_Z(\mathcal{M}^\bullet)$ is an equivalence. We wish to show that $i^! R\Gamma_Z(\mathcal{M}^\bullet) \simeq i^! \mathcal{M}^\bullet$. Applying $i^!$ to the exact triangle, it suffices to show that $i^! Rj_* \mathcal{M}^\bullet|_U = 0$. This is a calculation:

$$i^! Rj_* \mathcal{M}^\bullet|_U = \mathcal{O}_Z \otimes_{i^{-1}\mathcal{O}_X} i^{-1} Rj_* \mathcal{M}^\bullet|_U \simeq Rj_*(j^* \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet|_*) \simeq 0$$

with the last equivalence via the projection formula.²⁵ □

Remark 23.3. Later we will define functors $i_!$ and j^* . When i is a closed embedding, $i_! = i_*$ and when j is an open embedding, $j^! = j^*$. Thus, we can write the above

$$i_! i^! \longrightarrow \mathrm{id}_{\mathrm{DMod}(X)} \longrightarrow j_* j^*.$$

Proposition 23.4 (Base change). *Suppose we have a Cartesian square:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then, there is a natural isomorphism of derived functors $f'_ g'^! \simeq g^! f_*$.*

²⁴I.e. for $\theta \in \mathcal{D}_X(U)$, we have $\mathrm{supp}(\theta \cdot s) \subset \mathrm{supp}(s)$. Recall that $x \in \mathrm{supp}(s)$ if the image $s_x \in \mathcal{O}_{X,x}$ is nonzero, i.e. we want to show that if $s_x = 0$ then $(\theta \cdot s)_x = \theta_x \cdot s_x = 0$, but this is obvious. Morally, the idea is if a section is zero, it is zero in some neighborhood, and differentiating gives zero again.

²⁵Note that it's not true that $\mathcal{O}_Z \otimes_{\mathcal{O}_X} j_* \mathcal{M} = 0$; for example, take $X = \mathbb{A}^2$, $Z = \{(0,0)\}$; then $j_* \mathcal{O}_U = \mathcal{O}_X$. However, when we take the derived functor, we resolve via Čech complex of distinguished affine opens; each of these is obtained by inverting a function which is in the ideal defining Z , thus tensoring with \mathcal{O}_Z kills all terms.

Proof. Every map $g : X \rightarrow Y$ may be factored as a closed immersion followed by a projection:

$$Y' \xrightarrow{\Gamma_g} Y' \times Y \xrightarrow{p} Y.$$

We leave the case when g is a projection as an exercise. Now assume that i is a closed embedding, and take $Z = Y'$. Let $j : U \hookrightarrow Y$ be its open complement:

$$\begin{array}{ccccc} X_Z & \xrightarrow{i_X} & X & \xleftarrow{j_X} & X_U \\ \downarrow f_Z & & \downarrow f & & \downarrow f_U \\ Z & \xrightarrow{i} & Y & \xleftarrow{j} & U \end{array}$$

Then we have exact triangles:

$$i_* i^! f_* \mathcal{M} \rightarrow f_* \mathcal{M} \rightarrow j_* j^! f_* \mathcal{M}, \quad f_* i_{X*} i_X^! \mathcal{M} \rightarrow f_* \mathcal{M} \rightarrow f_* j_{X*} j_X^! \mathcal{M}.$$

Base change is satisfied by open embeddings, so we obtain an identification of the right terms compatibly with the map. By the two-out-of-three property, we have an identification of the left terms. By Kashiwara, we may identify $\mathrm{DMod}_Z(Y)$ and $\mathrm{DMod}_{Z_X}(X)$, removing the functors i_* and i_{X*} from the formulas. \square

We now prove constructibility.

Theorem 23.5 (Constructibility). *Let $\mathcal{M}^\bullet \in D_c \mathrm{DMod}(X)$ be a coherent complex. The following are equivalent.*

1. *The complex \mathcal{M}^\bullet is holonomic.*
2. *There is a smooth stratification $i_\alpha : S_\alpha \hookrightarrow X$ such that $i_\alpha^! \mathcal{M}^\bullet$ is a flat connection²⁶ for all α .*
3. *For every closed point $i_x : \{x\} \hookrightarrow X$ we have $i_x^! \mathcal{M}^\bullet$ is finite-dimensional.*

Proof. We have (1) implies (2) by inducting on Proposition 13.15, i.e. every holonomic D -module is generically a flat connection. Let $j : U \hookrightarrow X$ be an open on which this is true with complement $i : Z \hookrightarrow X$. By assumption $j^! \mathcal{M}$ is a flat connection. By the lemma, $j_* j^! \mathcal{M}$ is also holonomic, and thus by the exact triangle, $i_* i^! \mathcal{M}$ is holonomic. In particular, we add U to the stratification.

We can almost induct on dimension, except that Z might not be smooth. To remedy this, let $j' : U' \subset X$ be an open such that $U' \cap Z \subset Z$ is dense and smooth, and let Z' be its complement in Z (which is closed in Z thus X). It is clear that $\Gamma_{Z'} \Gamma_Z = \Gamma_{Z'}$, thus we have an exact triangle

$$\Gamma_{Z'}(\mathcal{M}^\bullet) \rightarrow \Gamma_Z(\mathcal{M}^\bullet) \rightarrow j'_* \Gamma_Z(\mathcal{M}^\bullet)|_{U'} = j'_* i'_* i'^! j'^! \mathcal{M}^\bullet$$

where the right identification commutes the restriction to U and Γ_Z via a (trivial case of) base change. Further, restriction to an open clearly preserves holonomicity, and so the right term is holonomic. We add $Z \cap U'$ to the stratification and continue.

We have (2) claim implies (3) by commutativity of pullback functors, and since $i_x^! \mathcal{M}^\bullet$ is finite-dimensional for any \mathcal{M}^\bullet with finite rank \mathcal{O}_X -locally free cohomology sheaves by a standard spectral sequence argument.

Thus we only need to show that (3) implies (1). To this end, we induct on the dimension of the classical support of \mathcal{M}^\bullet . Choose a dense open $U \subset X$ such that the cohomology of $\mathcal{M}^\bullet|_U$ is locally \mathcal{O}_X -free. Then, since for $u \in U$ we have that $i_u^! \mathcal{M}^\bullet$ (which is isomorphic to the \mathcal{O} -module pullback up to a shift we can ignore) is finite-dimensional, and thus we know that in fact $\mathcal{M}^\bullet|_U$ is finite rank, thus a flat connection, thus holonomic. We now induct in a similar way as the first part of this proof. That is, we have an exact triangle

$$\Gamma_Z(\mathcal{M}^\bullet) \rightarrow \mathcal{M} \rightarrow j_* j^* \mathcal{M}.$$

It suffices to show that $\Gamma_Z(\mathcal{M}^\bullet)$ is holonomic; note that it has property (3) via base change. We run into the same issue that Z is not smooth, and resolve it in an analogous way as above. \square

²⁶I.e. its cohomology sheaves are flat connections.

24 Lecture 24 (2020-04-16): Functors on holonomic D -modules

Theorem 24.1 (Preservation of holonomicity). *The functors $f^!$, f_* preserve holonomic D -modules.*

Proof. Let us consider $f^!$ first. Suppose that f is smooth; then the singular support is given by pull-push along:

$$\mathbb{T}_Y^* \leftarrow \mathbb{T}_Y^* \times_Y X \hookrightarrow \mathbb{T}_X^*$$

i.e. if $\text{SS}(\mathcal{M})$ has dimension $\dim(Y)$, then its pullback has dimension $\dim(Y) + \dim(X) - \dim(Y) = \dim(X)$ as desired.

Now suppose that i is a closed embedding with complement j . We have a the exact triangle

$$i_* i^! \mathcal{M} \rightarrow \mathcal{M} \rightarrow j_* j^! \mathcal{M}.$$

By the lemma and the above, we have the $j_* j^! \mathcal{M}$ is holonomic, thus $i_* i^! \mathcal{M}$ is. But i_* preserves the codimension of the singular support, so $i^! \mathcal{M}$ must be holonomic.

We now consider f_* . We already know the result when f is a closed embedding, so assume $f : X = Y \times S \rightarrow Y$ is a projection; since we know the result for locally closed pushforward, we may also assume that S is proper. We use the criterion and base change: $i_y^! f_* \mathcal{M}^\bullet$ is the de Rham cohomology of the pullback to the fiber. Since pullbacks preserve holonomicity, and since proper pushforward preserves coherent D -modules (check as an exercise for projective morphisms; hint: embed), $i_y^! f_* \mathcal{M}^\bullet$ is finite-dimensional, thus holonomic. We now apply the criterion. \square

This allows us to define the following functors. Recall Example 22.1.

Definition 24.2. Let $f : X \rightarrow Y$. We define functors

$$f^* := \mathbb{D}_X \circ f^! \circ \mathbb{D}_Y : D_h \text{DMod}(Y) \rightarrow D_h \text{DMod}(X), \quad f_! := \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X : D_h \text{DMod}(X) \rightarrow D_h \text{DMod}(Y).$$

We state the following theorem without proof.

Theorem 24.3. *If f is proper, then $f_* = f_!$ on $D_c \text{DMod}(-)$.*

Exercise 24.4. Prove the case when f is a closed immersion (this is the only case we will use).

Theorem 24.5. *The functors (f^*, f_*) are adjoint. The functors $(f_!, f^!)$ are adjoint on $D_h \text{DMod}(-)$.*

Exercise 24.6. Prove the above theorem. Hint: use

$$R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \simeq (\omega_X \otimes_{\mathcal{O}_X}^L \mathbb{D}_X \mathcal{M})[-\dim(X)] \otimes_{\mathcal{D}_X}^L \mathcal{N}.$$

Our goal is to now classify simple holonomic D -modules.

Definition 24.7. Let $f : S \hookrightarrow X$ be a locally closed embedding. There is a natural transformation of functors $f_! \rightarrow f_*$ defined as follows. Let us factor

$$S \xrightarrow[i]{\text{cl}} U \xrightarrow[j]{\text{op}} X.$$

We have that $i_! = i_*$, so it suffices to define the natural transformation for j . In this case, $j^* = j^!$, and the unit and counit maps $\mathcal{M}^\bullet \rightarrow j^! j_! \mathcal{M}^\bullet$ and $j^* j_* \mathcal{M}^\bullet \rightarrow \mathcal{M}^\bullet$ are isomorphisms. The natural transformation $j_! \mathcal{M}^\bullet \rightarrow j_* \mathcal{M}^\bullet$ is defined to be the adjoint to the identity $\mathcal{M} \rightarrow j^! j_* \mathcal{M}^\bullet = \mathcal{M}^\bullet$, or equivalently the adjoint to the identity $j^* j_! \mathcal{M}^\bullet \rightarrow \mathcal{M}^\bullet$. Assembling everything, we have a natural transformation $f_! \rightarrow f_*$.

We define the *middle extension* or *minimal extension* to be the object in the abelian category of holonomic D -modules:

$$f_{!*} \mathcal{E} := \text{im}(H^0(f_! \mathcal{E}) \rightarrow H^0(f_* \mathcal{E})) \in \text{DMod}_h(X).$$

Exercise 24.8. Prove that the two definitions of the map $f_* \rightarrow f_!$ are equivalent (hint: use duality).

Theorem 24.9. *Let $f : S \hookrightarrow X$ be an affine embedding with S simple and \mathcal{E} a simple flat connection on S . Then, $f_! \mathcal{E}$ is simple is the unique simple submodule of $f_* \mathcal{E}$, the unique simple quotient of $f_! \mathcal{E}$, and the unique simple composition factor of either which has nonzero restriction to S . Furthermore, every simple object is of the above form (for some S and some \mathcal{E}).*

Proof. First, we reduce to the case where f is an open embedding. We may factor any locally closed f as an open embedding followed by a closed embedding, and the claims are easy to check in the case of a closed embedding.

We show that $f_! \mathcal{E}$ has a unique simple quotient (by duality, this shows that $f_* \mathcal{E}$ has a unique simple submodule). Since $(f_! \mathcal{E})|_U = \mathcal{E}$ is simple, there is only one component factor which is nonzero on U . Take any simple quotient $f_! \mathcal{E} \twoheadrightarrow \mathcal{M}$; since restriction to U is exact, then either $\mathcal{M}|_U = 0$ or $(f_! \mathcal{E})|_U = \mathcal{E} \twoheadrightarrow \mathcal{M}|_U$ is an iso; thus this unique simple subquotient (nonzero on U) must be a quotient. Furthermore, the composition

$$f_! \mathcal{E} \twoheadrightarrow f_{!*} \mathcal{E} \hookrightarrow f_* \mathcal{E}$$

is the identity when restricted to U ; thus $f_! \mathcal{E} \twoheadrightarrow f_{!*} \mathcal{E}$ is nonzero, so $f_{!*} \mathcal{E}$ is this unique simple quotient.

Next, suppose that \mathcal{M} is simple. Choose an affine open U such that $\mathcal{E} := \mathcal{M}|_U$ is a local system on its support $Z \subset U$ – we can make this choice such that Z is smooth. Letting $i : U \hookrightarrow X$, and using the unit and counit adjunctions, we have maps

$$i_! \mathcal{E} \twoheadrightarrow \mathcal{M} \hookrightarrow i_* \mathcal{E}$$

where we have surjectivity and injectivity by virtue of the maps being nonzero (i.e. they restrict to nonzero maps on U) and \mathcal{M} being simple. If \mathcal{E} were simple, we would be done. To see it is simple, note that the kernel of $i_! \mathcal{E} \twoheadrightarrow \mathcal{M}$ is supported at $Z = X - U$. For any subobject $\mathcal{E}' \subset \mathcal{E}$, the image of $i_! \mathcal{E}' \subset i_! \mathcal{E}$ must also surject onto \mathcal{M} (since it cannot be in the kernel for support reasons), but this means that $\mathcal{E}' = i^! i_! \mathcal{E}' = i^! i_! \mathcal{E} = \mathcal{E}$. \square

Remark 24.10. The requirement that i is affine is not strictly needed above, but must then take the 0th cohomology. We do not obtain any new irreducibles that way.

Example 24.11. If i is a closed embedding, then $i_! \mathcal{E} = i_* \mathcal{E} = i_{!*} \mathcal{E}$.

Example 24.12. Take $X = \mathbb{A}^1$, with $i : Z = \{0\} \hookrightarrow X$ and $j : U = \mathbb{G}_m \hookrightarrow X$. We have that $i_{*!} k$ is the skyscraper D -module. There are nontrivial local systems on U (where the latter isomorphism is as \mathcal{O} -modules)

$$\mathcal{E}_\lambda := \mathcal{D}_U / \mathcal{D}_U(x\partial - \lambda) \simeq \mathcal{O}_U.$$

One can think of λ as the logarithm of monodromy $\exp(2\pi i \lambda)$; note that multiplying by $x^n \in \mathcal{O}(\mathbb{G}_m)$ for $n \in \mathbb{Z}$ isomorphisms between various $\mathcal{E}_\lambda \simeq \mathcal{E}_\mu$ for $\lambda - \mu \in \mathbb{Z}$. We have

$$\mathcal{M}_{*\lambda} := \mathcal{D}_X / \mathcal{D}_X(x\partial - \lambda), \quad \mathcal{M}_{!\lambda} := \mathcal{D}_X / \mathcal{D}_X(\partial x - \lambda).$$

Consider the following table.

\cdots	\mathcal{M}_{*1}	\mathcal{M}_{*0}	$\mathcal{M}_{!,-1}$	$\mathcal{M}_{*,-2}$	\cdots
\cdots	$D/x\partial - 1$	$D/x\partial$	$D/t\partial + 1$	$D/x\partial + 2$	\cdots
\cdots	$D/\partial x - 2$	$D/\partial x - 1$	$D/\partial x$	$D/\partial x + 1$	\cdots
\cdots	$\mathcal{M}_{!2}$	$\mathcal{M}_{!1}$	$\mathcal{M}_{!0}$	$\mathcal{M}_{!,-1}$	\cdots

We claim that everything to the left of the vertical line are isomorphic, and everything to the right is isomorphic. It's easy to see that the modules in a given column are isomorphic just by using the commuting relation. For the horizontal equivalences, verify that the following chain of maps are well-defined

$$D/t\partial + 1 \xleftarrow{\partial} D/x\partial + 2 \xleftarrow{\frac{1}{2}\partial} D/x\partial + 3$$

and have inverses given by

$$D/t\partial + 1 \xrightarrow{x} D/x\partial + 2 \xrightarrow{x} D/x\partial + 3$$

so that the right side of the line are all isomorphic. Now, note that we have short exact sequences

$$0 \rightarrow D/x \rightarrow \mathcal{M}_{!0} = D/\partial x \rightarrow D/\partial \rightarrow 0,$$

$$0 \rightarrow D/\partial \rightarrow \mathcal{M}_{*0} = D/x\partial \rightarrow D/x \rightarrow 0.$$

Thus, the middle extension is

$$\mathcal{M}_{!*,n} = D/x = \mathcal{O}_X, \quad n \in \mathbb{Z}.$$

If $\lambda \notin \mathbb{Z}$, then show that $\mathcal{M}_{*,\lambda} \simeq \mathcal{M}_{*,\lambda+1}$ for every λ (i.e. the “vertical bar” above disappears). Thus,

$$\mathcal{M}_{!*,\lambda} \simeq \mathcal{M}_{!,\lambda} \simeq \mathcal{M}_{*,\lambda}, \quad \lambda \notin \mathbb{Z}.$$

Exercise 24.13. Can you classify all irreducible holonomic D -modules on \mathbb{A}^1 ?

Example 24.14. Take $j : U := \mathbb{A}^2 - \{0\} \hookrightarrow X = \mathbb{A}^2$. The D -module pushforward is just the \mathcal{O} -module pushforward, which has higher cohomology. But $H^0(j_*\mathcal{O}_U) = \mathcal{O}_X$.

Definition 24.15. Let $i : S \subset X$ be a locally closed subscheme, and \mathcal{E} an irreducible local system on S . We define the simple holonomic D -module

$$L(S, \mathcal{E}) := H^0(i_{!*}\mathcal{E}) \in \text{DMod}_h(X).$$

The $L(S, \mathcal{E})$ form a possibly redundant set of simples. We leave the following as an exercise.

Proposition 24.16. We have $L(S, \mathcal{E}) = L(S', \mathcal{E}')$ if and only if $\overline{S} = \overline{S'}$ and if there is an $U \subset S \cap S'$ dense and open in both, such that $\mathcal{E}|_U \simeq \mathcal{E}'|_U$.

Minimal extensions in general can be difficult to compute.

Remark 24.17. For the middle extensions off nilpotent orbits, see Section 3 of a paper by Levasseur (result due to Hotta and Kashiwara) for formulas.

Remark 24.18. In summary we have the following very favorable properties of holonomic D -modules.

- The abelian category is finite length.
- Every complex of holonomic D -modules is “constructible” with respect to a stratification which can be read off of its singular support.
- The derived functors $f_*, f^!$ preserve holonomicity, allowing us to define dual functors $f_!, f^*$.
- These functors satisfy adjunctions: (f^*, f_*) and $(f_!, f^!)$. When f is proper, $f_* = f_!$ and when f is smooth, $f^! = f^*[2 \dim(f)]$.
- There are internal Hom and tensor products, i.e. a “six functors” formalism.
- The simple objects $i_{!*}\mathcal{E}$ in the abelian category are exactly the minimal extensions off of affine embeddings, realized as the unique simple quotient of certain standard objects $i_!\mathcal{E}$ and unique simple submodule of certain costandard objects $i_*\mathcal{E}$.

25 Lecture 25 (2021-04-21): Equivariant Beilinson-Bernstein

Recall Section 11. We’ll now discuss how G -actions on X give rise to two kinds of infinitesimal actions on weakly equivariant D -modules. The first is a “flow” map: if G acts on X , we expect a map $\mathfrak{g} \rightarrow \mathcal{T}_X(X)$.

Example 25.1. How does one compute this in practice? Let's consider the adjoint action of G on $X = \mathfrak{g}$. Let $x \in \mathfrak{g}$; what is the map

$$\mathfrak{g} \rightarrow T_x(\mathfrak{g}) = \mathfrak{g}, \quad y \mapsto \theta_y?$$

This should be given by the formula:

$$(e + \epsilon y) \cdot x = x + \theta_y \epsilon, \quad (e + \epsilon y) \in G^{(1)}, x \in X = \mathfrak{g}, y \in T_e(G) = \mathfrak{g},$$

$$\theta_y = \frac{(e + y\epsilon) \cdot x - x}{\epsilon}.$$

We can compute

$$\epsilon \theta_y = (e + y\epsilon)x(e + y\epsilon)^{-1} - x = (e + y\epsilon)x(e - y\epsilon) - x = (yx - xy)\epsilon = [y, x]\epsilon.$$

Thus $\theta_y = [y, x]$.

Remark 25.2. We now generalize the above formula, and write it in terms of functions. Suppose G acts on X , and let f be a (local) function on X . Let $y \in \mathfrak{g}$ and take an “infinitesimal element” $e + y\epsilon \in G$. Then, the tangent vector θ_y defined by (for $x \in X$):

$$\theta_y(x) := \frac{(e + y\epsilon) \cdot x - x}{\epsilon}$$

gives rise to the derivation (a familiar formula from calculus):

$$\delta(f)(x) := \frac{f((e + y\epsilon) \cdot x) - f(x)}{\epsilon}$$

$$(e + y\epsilon)^{-1} \cdot f = f + \delta(f)\epsilon.$$

This motivates the following definition.

Definition 25.3. We define $\theta : \mathfrak{g} \rightarrow \mathcal{T}(X) = \Gamma(X, \mathcal{T}_X)$ as follows, and denote $\theta_y := \theta(y) \in \Gamma(X, \mathcal{T}_X)$. Let $G^{(1)} = \text{Spec}(k \oplus \mathfrak{g}^* \epsilon) \subset G$ denote the first-order neighborhood of G at the identity. The coaction map on functions locally gives

$$\mathcal{O}_U \rightarrow \mathcal{O}_U \otimes \mathcal{O}(G^{(1)}) = \mathcal{O}_U \otimes_k (k \oplus \mathfrak{g}^* \epsilon).$$

The unit condition tells us that this map has the form $f \mapsto (f, \delta(f))$, and that it is a map of algebras means it is a \mathfrak{g}^* -valued derivation. Evaluating at $y \in \mathfrak{g}$ gives rise to a derivation of \mathcal{O}_U , which glue to a derivation \mathcal{O}_X , i.e. an element of $\theta_y \in \Gamma(X, \mathcal{T}_X)$.

Exercise 25.4. Show that associativity of the group action shows that the resulting map is a map of Lie algebras, i.e.

$$\theta_{[y, y']} = [\theta_y, \theta_{y'}].$$

In our example above, this is the Jacobi identity.

We now define the second kind of infinitesimal action, this time using the G -equivariant structure on the sheaf. Let us discuss, morally, how this should work.

Remark 25.5. Let $f \in \mathcal{O}(U)$, i.e. a function on X defined only on $U \subset X$. The group G acts on sections by $g \cdot f(-) := f(g^{-1} \cdot -) \in \mathcal{O}(gU)$. Recall that the equivariant structure map is given by $\phi : a^* \mathcal{O}_X \rightarrow p^* \mathcal{O}_X$. Taking sections over $\{g\} \times U \subset G \times X$ gives a map $\mathcal{O}(g \cdot U) \rightarrow \mathcal{O}(U)$, so the action map above is the fiber over $g^{-1} \in G$. The action map on functions is usually written twisted by an inverse.

Definition 25.6. We define $\alpha : \mathfrak{g} \rightarrow \text{End}_k(\mathcal{M})$ at $y \in \mathfrak{g}$ (we denote $\alpha_y \in \text{End}_k(\mathcal{M})$) as follows. Restricting to $G^{(1)} \subset G$ and working affine locally over $U \subset X$, and letting $\phi : a^* \mathcal{O}_X \rightarrow p^* \mathcal{O}_X$ denote the equivariant structure map, we have

$$\mathcal{M}(U) \longrightarrow a_* a^* \mathcal{M}(U) = a^* \mathcal{M}(U \times G^{(1)}) \xrightarrow{\phi} \mathcal{M}(U) \otimes \mathcal{O}(G^{(1)}) = \mathcal{M}(U) \oplus \mathcal{M}(U) \otimes \mathfrak{g}^* \epsilon \xrightarrow{\text{pr}_\epsilon^* \circ \text{ev}_y} \mathcal{M}(U).$$

The associativity ensures that it is a map of Lie algebras (exercise). The \mathcal{O} -linearity of equivariance implies the following compatibility of α with \mathcal{O} -linear multiplication:

$$[\alpha_y, f] \cdot m = \langle \theta_y, df \rangle m.$$

Remark 25.7. Morally, $\alpha_y(m)$ is given by the formula

$$\alpha_y(m) = \frac{(e + y\epsilon) \cdot m - m}{\epsilon}.$$

Working this way allows us to prove the compatibility somewhat easily using the usual “product rule trick” from calculus:

$$\alpha_y(fm) = \frac{(e + y\epsilon) \cdot (fm) - fm}{\epsilon} = \frac{((e + y\epsilon) \cdot f)((e + y\epsilon) \cdot m) - fm}{\epsilon} = (y \cdot f)m + f(y \cdot m)$$

i.e. $\alpha_y(fm) - f\alpha_y(m) = \langle \theta_y, df \rangle m$.

Exercise 25.8. The above is not truly rigorous. Make it rigorous using the “product rule trick”.

Exercise 25.9. Show that $[\alpha_y, P](m) = [\theta_x, P] \cdot m$, where $P \in \mathcal{D}_X$ and $m \in \mathcal{M}$ is a local section of a D -module.

Example 25.10. Take $\mathcal{M} = \mathcal{O}$. What is the canonical equivariant structure? We can check fiber-wise: $(a^*\mathcal{O}_X)_{(g,x)} = \mathcal{O}_{X,g \cdot x} \rightarrow \mathcal{O}_{X,x}$ takes a function $f(-) \mapsto f(g \cdot -) = g^{-1} \cdot f(-)$. This is exactly the coaction map, so we find that $\alpha_y = \theta_y$.

Now consider $\mathcal{M} = \mathcal{T}_X = \mathcal{D}er_X(\mathcal{O}_X)$; again taking the fiber at (g, x) we have:

$$\begin{array}{ccc} \mathcal{O}_{g \cdot x} & \longrightarrow & \mathcal{O}_{g \cdot x} \\ \downarrow & & \downarrow \\ \mathcal{O}_x & \longrightarrow & \mathcal{O}_x \end{array}$$

where the vertical arrows take $f \mapsto g^{-1} \cdot f$; thus the map $(a^*\mathcal{T})_{(g,x)} \rightarrow \mathcal{T}_x$ “moves the tangent via $g \in G$, then differentiates” or equivalently “moves the function by $g \in G$, differentiates, then moves it back by g^{-1} ”, i.e.

$$\theta(\bullet) \mapsto g^{-1} \cdot \theta(g \cdot \bullet).$$

Thus we find that $\alpha(y)(\theta) = [\theta_y, \theta]$. Finally, putting $\mathcal{M} = \mathcal{D}_X$, we have $\alpha(y)(P) = [\theta_y, P]$.

Definition 25.11. We regard θ now as a map $\theta : \mathfrak{g} \rightarrow \Gamma(X, \mathcal{T}_X) \hookrightarrow \Gamma(X, \mathcal{D}_X) \hookrightarrow \text{End}_k(\mathcal{M})$. We define the difference

$$\eta = \theta - \alpha : \mathfrak{g} \rightarrow \text{End}_{\mathcal{D}_X}(\mathcal{M}) \subset \text{End}_k(\mathcal{M}).$$

Like before, we denote $\eta_x \in \text{End}_{\mathcal{D}_X}(\mathcal{M})$.

Remark 25.12. Unlike the above two, the resulting maps are \mathcal{D}_X -linear. That is,

$$\begin{aligned} \eta_x(Pm) &= \theta_x(Pm) - \alpha_x(Pm) = \theta_x(Pm) - (P\alpha_x(m) + [\theta_x, P]m) = \theta_x(Pm) - P\alpha_x(m) - \theta_x Pm + P\theta_x m \\ &= P\theta_x(m) - P\alpha_x(m) = P\eta_x(m). \end{aligned}$$

Example 25.13. On $\mathcal{M} = \mathcal{O}_X$, we have $\gamma = 0$. On $\mathcal{M} = \mathcal{T}_X$, we have $\gamma(y)(\theta) = -\theta\alpha(y)$. On $\mathcal{M} = \mathcal{D}_X$, we have $\gamma(y)(P) = -P\alpha(y)$, which is in particular \mathcal{D}_X -linear for the left action.

We now prove the U -equivariant variant of Beilinson-Bernstein. I forgot to talk about the following.

Definition 25.14. The *Beilinson-Bernstein localization functor* $\text{Loc} : \text{Mod}(U_0\mathfrak{g}) \rightarrow \text{DMod}(G/B)$ is the inverse functor to the global sections functor Γ . Explicitly (i.e. without using a theorem), it is the left adjoint to Γ , i.e.

$$\text{Loc}(M) = \mathcal{D}_X \otimes_{\mathcal{D}(X)} M.$$

That is, for $U \subset X$ open, the restriction map $\mathcal{D}(X) \rightarrow \mathcal{D}(U)$ is a map of algebras, making $\mathcal{D}(U)$ a $\mathcal{D}(X)$ -bimodule, and we define $\text{Loc}(M)(U) = \mathcal{D}(U) \otimes_{\mathcal{D}(X)} M$.

Theorem 25.15 (Strict Beilinson-Bernstein to category \mathcal{O}'). *Assume that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}^{\leq -1}$ for all positive simple coroots α^\vee . Then the global sections*

$$\Gamma : \text{DMod}_c^U(G/B, \lambda) / \ker(\Gamma) \rightarrow \mathcal{O}'_{[\lambda]}$$

is exact and essentially surjective, and if in addition $\lambda \notin \mathbb{Z}^{\leq 0}$, then it is an equivalence.

Proof. I will only prove this for $\lambda = 0$. We need to check that two conditions that define category \mathcal{O}' match: finite generation, and local \mathfrak{n} -finiteness (semisimplicity of the $Z\mathfrak{g}$ -action is implied since we restrict to the trivial block). For finite generation, we say an object $M \in \mathbf{A}$ in an abelian category is *compact* if $\text{Hom}(M, -)$ commutes with (infinite) direct sums. We leave it to the reader to verify that the coherent D -modules and the finitely generated $U\mathfrak{g}$ -modules are exactly the compact objects in their respective categories.

Now, let $\mathcal{M} \in \text{DMod}(G/B, \lambda)$ be a strongly U -equivariant D -module, in particular a U -equivariant sheaf. Then, $\Gamma(G/B, \mathcal{M})$ has the structure of an U -representation, which is automatically locally finite (since it must be a rational representation). The exponential map defines an isomorphism $\exp : \mathfrak{n} \simeq U$ (with group multiplication defined via Baker-Campbell-Hausdorff, whose expression terminates since \mathfrak{n} is nilpotent), and strong equivariance means their actions on $\Gamma(G/B, \mathcal{M})$ agree, and in particular, are locally finite.

We now prove the converse. A locally finite \mathfrak{n} -action on M gives rise to an action of U on M via the exponential map as above. Thus, we have a U -equivariant structure on $\mathcal{D}_X \otimes M$, where \mathfrak{n} acts through the comultiplication on $U\mathfrak{n}$, i.e. for $n \in \mathfrak{n}$,

$$\exp(n) \cdot (P \otimes m) = \exp(n)P \otimes m + P \otimes \exp(n)m.$$

To show that we have a U -equivariant structure on the quotient $\text{Loc}(M) = \mathcal{D}_X \otimes_{U\mathfrak{g}} M$, we need to show that for $u \in U$ and $Q \in \mathfrak{g}$ (we extend $\theta : \mathfrak{g} \rightarrow \Gamma(X, \mathcal{T}_X)$ to $\theta : U\mathfrak{g} \rightarrow \mathcal{D}_X$ multiplicatively, i.e. let $\theta_{xy} := \theta_x \theta_y$), we have

$$u \cdot (P\theta_Q \otimes m) = u \cdot (P \otimes Qm).$$

By exponentiating, it suffices to show that (for $y \in \mathfrak{n}$):

$$\alpha_y(P\theta_Q \otimes m) = \alpha_y(P \otimes Qm).$$

We leave this as an exercise: use that $\alpha_y(P) = [\theta_y, P]$, and that $[\theta_Q, \theta_y] = \theta_{[Q, y]}$. Thus, $\text{Loc}(M)$ naturally has the structure of a weakly U -equivariant D -module.

Next, we verify strict equivariance:

$$\eta_y(P \otimes m) = \theta_y(P \otimes m) - \alpha_y(P \otimes m) = \theta_y P \otimes m - [\theta_y, P] \otimes m - P \otimes y \cdot m = P\theta_y \otimes m - P \otimes y \cdot m = 0.$$

□

We also sketch a proof of the “other” Beilinson-Bernstein. I am not sure if it is the easiest proof; it will use much of the machinery we have developed regarding holonomic D -modules.

Theorem 25.16. *The global sections functor Γ defines an equivalence*

$$\text{DMod}_c^B(G/U) \xrightarrow{\simeq} \mathcal{O}_0.$$

Proof. We first consider the global sections functor as defining a functor $\text{DMod}^B(G/U) \rightarrow \text{Mod}(U\mathfrak{g})$. Finite generation follows as above. By left strong U -equivariance, we also have locally \mathfrak{n} -finiteness. By left strong T -equivariance, we have that the \mathfrak{t} -action on global sections are induced by the T -representation structure, which must be semisimple, thus we have \mathfrak{t} -semisimplicity. To see that the center acts nilpotently, note that there are finitely many B -orbits on G/U , and that they are fibers under the H -torsor $G/U \rightarrow G/B$. Thus, the simple objects, which are middle extensions off of orbits, have trivial H -monodromy, and thus the center acts nilpotently.

Thus, we have a functor $\Gamma : \mathrm{DMod}_c^B(G/U) \rightarrow \mathcal{O}_0$. We need to show that it is an equivalence. It is fully faithful by Beilinson-Bernstein, so we just need to show it is essentially surjective. This can be done by counting simples. \square

26 Lecture 26: Intersection cohomology (Isaac Goldberg)

27 Lecture 27 (2021-04-29, 2021-05-03): Matching objects

We will match (co)standard objects in by the following characterization in \mathcal{O}' . Recall that in the trivial block, -2ρ is the lowest weight in the W linkage class. For shorthand, we let $\nabla_w := \nabla_{w \cdot (-2\rho)}$ and likewise for L_w . In this convention, as w gets longer, ∇_w gets bigger.

Lemma 27.1. *Suppose that $M \in \mathcal{O}'_0$ and ∇_w have the same composition factors, and that $\mathrm{Hom}(L_{w'}, M) = 0$ for $\ell(w') < \ell(w)$. Then, $M \simeq \nabla_w$.*

Dually, and equivalently, we have the following.

Lemma 27.2. *Suppose that $M \in \mathcal{O}'_0$ and Δ_w have the same composition factors, and that $\mathrm{Hom}(M, L_{w'}) = 0$ for $\ell(w') < \ell(w)$. Then, $M \simeq \Delta_w$.*

Proof. Since the two have the same character, they have the same Jordan-Holder series, consisting of $L_{w'}$ for $\ell(w') \leq \ell(w)$ (i.e. “more antidominant”). Since M has a maximal weight $w \cdot (-2\rho)$, there is a non-zero map $\phi : \Delta_w \rightarrow M$.

Consider the exact sequence:

$$0 \rightarrow \ker(\phi) \rightarrow \Delta_w \rightarrow M \rightarrow \mathrm{coker}(\phi) \rightarrow 0.$$

Note that since Δ_w and M have the same composition series, if one of $\ker(\phi)$ or $\mathrm{coker}(\phi)$ are zero, then ϕ is an isomorphism. Suppose that $\mathrm{coker}(\phi) \neq 0$. Then we have a surjection $M \rightarrow \mathrm{coker}(\phi) \twoheadrightarrow L_{w'}$ for some w' with $\ell(w') \leq \ell(w)$. This violates the assumption unless $w = w'$. But if $[L_w]$ is a composition factor of $\mathrm{coker}(\phi)$, then $[L_w]$ must also be a composition factor of $\ker(\phi)$ (since it is multiplicity 1 in both Δ_w and M by assumption). But since L_w is the unique quotient of Δ_w , it cannot be in $\ker(\phi)$. We have a contradiction, which completes the proof of the claim. \square

If two objects have the same composition series, it is clear they have the same formal character (i.e. dimension of eigenspaces). The converse will be useful. Let $\mathrm{ch} : K_0(\mathcal{O}'_0) \rightarrow \mathbb{Z}[X^\bullet(H)]$ denote the character function, i.e.

$$\mathrm{ch}(M) = \sum_{\lambda \in X^\bullet(H)} \dim(M_\lambda) e^\lambda$$

where e^λ is a formal symbol. It is clear that ch is multiplicative²⁷ and additive.

Proposition 27.3. *The map ch is injective. Thus if two modules have the same character, they have the same composition series.*

Proof. Left as an exercise; take either basis $[M_w]$ or $[L_w]$ and use the fact that there is an ordering on the highest weights. \square

Recall that $X = G/B$ has a stratification by B -orbits called the *Schubert stratification*. We let $X_w = BwB/B \subset G/B$. In this labeling, X_e is a point, and X_{w_0} is the open orbit. We let $i_w : X_w \hookrightarrow X$ denote the inclusion.

Theorem 27.4. *We have*

$$\Gamma(G/B, i_{w,*} \mathcal{O}_{X_w}) \simeq \nabla_w, \quad \Gamma(G/B, i_{w,!} \mathcal{O}_{X_w}) \simeq \Delta_w.$$

²⁷Multiplication is by convolution, i.e. $e^\lambda e^\mu = e^{\lambda+\mu}$ (rather than 0, which is the pointwise multiplication). Strictly speaking, we need to restrict to a smaller ring of “weight bounded above” functions to multiply such functions. We will suppress this detail.

Proof. Let us argue the theorem, assuming the calculation of characters that buys us the conditions in the above characterization of (co)standard objects. Matching characters aside, what the characterization says is that the standard objects filter category \mathcal{O}'_0 compatibly with the length ordering on W . On the D -modules side, we have a corresponding filtration by support.

Let us formalize the above; we induct on length. When $w = e$, costandard, standard, and irreducible all coincide in all cases, since i_e is a closed immersion (so $i_{e,*} = i_{e,!}$). Thus the claim follows by a calculation of characters. We induct on the length of $w \in W$. We need to show that $\text{Hom}(L_{w'}, \Gamma(G/B, i_{w,*}\mathcal{O}_{X_w})) = 0$ for $\ell(w') < \ell(w)$. Now, $\text{Loc}(L_{w'}) \subset i_{w',*}\mathcal{O}_{X_{w'}}$ is a submodule, so its support is contained in the closure $\overline{X_{w'}}$. Since $\ell(w') < \ell(w)$, this closure is disjoint from X_w . By Beilinson-Bernstein and adjunction,

$$\text{Hom}_{\mathcal{O}'_0}(L_{w'}, \Gamma(G/B, i_{w,*}\mathcal{O}_{X_w})) \simeq \text{Hom}_{\mathcal{D}_{G/B}}(\text{Loc}(L_{w'}), i_{w,*}\mathcal{O}_{X_w}) \simeq \text{Hom}_{\mathcal{D}_{X_w}}(i_w^*L_{w'}, \mathcal{O}_{X_w}) = \text{Hom}_{\mathcal{D}_{X_w}}(0, \mathcal{O}_{X_w}) = 0$$

as desired. The analogous argument follows for standards; note that the embeddings i_w are all affine (we will see this soon if you're unfamiliar).

It remains to compute the characters, and show they agree. We discuss some basic geometry of Schubert varieties. Namely, let $U \subset B$ denote the unipotent radical, and let $U^- \subset B^-$ denote the unipotent radical of an opposite Borel such that $B \cap B^- = T$. Further, note that for any root α , there is a unipotent subgroup $U_\alpha \subset G$ isomorphic to \mathbb{G}_a , obtained by exponentiating the corresponding nilpotent element. For any subset of roots S we denote by $U_S = \prod_{\alpha \in S} U_\alpha$ the corresponding (non-direct) product; note that U_S is unipotent if and only if it does not contain opposite roots, and that $U = U_{\Delta^+}$ while $U^- = U_{\Delta^-}$. Some observations:

- The Bruhat orbits BwB/B can be generated by acting by U , i.e. $X_w = UwB/B$.
- The T -fixed points of G/B are given by wB/B , and each X_w contains exactly one.
- The Bruhat orbit X_w is acted on transitively by a unipotent group $U_w \subset U$, thus is isomorphic to an affine space. To see this, let us compute the stabilizer of wB/B . Note that $U_\alpha w = w(w^{-1}U_\alpha w) = wU_{w^{-1}\alpha}$, thus $U_\alpha \subset U$ fixes wB/B if and only if $w^{-1}\alpha \in \Delta^+$, i.e. $\alpha \in w(\Delta^+)$. Thus, the stabilizer of is $U_{\Delta^+ \cap w(\Delta^+)}$, and any complimentary subgroup inside U will act transitively, e.g. $U_{\Delta^+ \cap w(\Delta^-)} \subset U$.
- The subvariety $Y_w := wU^-B/B$ is automatically affine (i.e. isomorphic to \mathbb{A}^d for some d , i.e. it is isomorphic to the orbit for a unipotent group), and contains wB/B . It also contains $X_w = UwB/B = w(w^{-1}Uw)B/B$, i.e. since the positive roots in $w^{-1}Uw$ are eaten up by the right B -action.
- We will show that there is a group $U_w^- \subset U^-$ which acts transitively on Y_w and such $U_w^- \cdot X_w = Y_w$. In particular, this implies that $X_w \subset Y_w$ is closed. We choose $U_w^- \subset U^-$ to be the subgroup which acts on X_w in normal directions. We restrict our attention to the point wB/B . Again, $U_\alpha w = wU_{w^{-1}\alpha}$, so $U_\alpha \subset U^-$ fixes wB/B if and only if $\alpha \in \Delta^- \cap w(\Delta^+)$, i.e. define $U_w^- = U_{\Delta^- \cap w(\Delta^+)} \subset U^-$, which we know acts on wB/B transitively.

Now, note that the subsets defining U_w and U_w^- are disjoint, and their union is $w(\Delta^-)$. We know they must generate a subvariety of dimension equal to G/B , and by a dimension count we know their orbits are normal.

- For example, when $w = e$, we have $U_e = \{1\}$ (i.e. the orbit is just a point) and $U_e^- = U^-$, while for the longest element $U_{w_0} = U$ and $U_{w_0}^- = \{1\}$.

In summary, we have an closed immersion into an affine followed by a open immersion:

$$X_w = U_w wB/B \hookrightarrow Y_w := U_w^- U_w wB/B = U_w^- U_w B/B \hookrightarrow G/B$$

for some subgroups $U_w \subset U$ and $U_w^- \subset U^-$, where U_w and $U_w^- U_w$ act transitively on X_w and Y_w respectively. For an open embedding, the \mathcal{D} -module pushforward agrees with the \mathcal{O} -module $*$ -pushforward (here, the argument only applies to costandards), so we restrict our attention to the closed. Let $i : X_w \hookrightarrow Y_w$ be the closed embedding. There is a filtration on $i_*\mathcal{O}_{X_w} = i_!\mathcal{O}_{X_w}$ such that the associated graded is

$$\text{Sym}_{\mathcal{O}_{X_w}}(\mathcal{N}_{X_w/Y_w}) \otimes_{\mathcal{O}_{X_w}} \det(\mathcal{N}_{X_w/Y_w})$$

where the twisting by determinant arises due to side changing, i.e. twisting by $\omega_{Y_w}^\vee \otimes \omega_{X_w} \simeq \det(\mathcal{N}_{X_w/Y_w})$. By the above, \mathcal{N}_{X_w/Y_w} is constant with fiber \mathfrak{n}_w^- , and since $X_w \simeq \mathfrak{u}_w$, we have $\mathcal{O}(X_w) \simeq \mathrm{Sym}_k \mathfrak{u}_w^*$. Thus the global sections of the left tensor factor is $\mathrm{Sym}_k(\mathfrak{u}_w^* \oplus \mathfrak{u}_w^-) = \mathrm{Sym}_k \mathfrak{u}^-$, whose character is $\mathrm{ch}([\Delta_0])$. On the other hand, $\det(\mathcal{N}_{X_w/Y_w})$ contributes a shift in weight equal to the sum of the roots in $\Lambda_w^- = \Delta^- \cap w(\Delta^-)$, which is the same as $w(-\rho) - \rho = w \cdot (-2\rho)$ (i.e. if we take half the sum over Λ_w^- we get the sum, and if we take half the sum over Λ_w^+ we get zero).

For standards, we note that duality does not change the composition factors, thus the character, and the result follows. \square

The following is clear, by the characterization of both as unique irreducible submodules or quotients.

Corollary 27.5. *We have an isomorphism*

$$\Gamma(G/B, L(X_w, \mathcal{O}_{X_w})) \simeq L_w.$$

28 Lecture 28 (2020-05-03, 2020-05-10): de Rham functor, Riemann-Hilbert

From now on we will take $k = \mathbb{C}$ (if we haven't been doing so already), since we will need analytic methods. Recall we defined the de Rham functor in Example 21.5 to compute the derived global sections of a (left) D -module. Let us recall it. There is a similar version for complexes but we omit it.

Definition 28.1. Let $\mathcal{M} \in \mathrm{DMod}(X)$. We define²⁸

$$\mathrm{dR}_X^\bullet(\mathcal{M}) := \omega_X \otimes_{\mathcal{D}_X}^L \mathcal{M} = \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}[\dim_{\mathbb{C}} X]$$

with differential given in local coordinates x_1, \dots, x_n and $\partial_1, \dots, \partial_n$ by:

$$\omega \otimes m \mapsto d\omega \otimes m + \sum \omega dx_i \otimes \partial_i m.$$

Remark 28.2. To compute the above we can also resolve \mathcal{M} over \mathcal{D}_X . This is often easier to do.

Remark 28.3. The differential given above is not \mathcal{O}_X -linear; it is only k -linear. Earlier, when we took global sections, we ended up with a k -linear chain complex so this didn't matter in some sense. Now, we ask: what category does $\mathrm{dR}_X(\mathcal{M})$ live in?

Example 28.4. Since the differential is not \mathcal{O}_X -linear, even on affine varieties the global sections functor is not exact. Take $X = \mathbb{A}^1$, and let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(\partial - 1)$. As a \mathcal{O}_X -module, we have $M \simeq \mathcal{O}_X$, “generated” by the function $f(x) = e^x$ (i.e. the solution to the differential equation $\partial f = f$). Its de Rham complex is:

$$\mathcal{O}_X \xrightarrow{f \mapsto (f+f') dx} \mathcal{O}_X dx$$

where the map morally can be thought of as taking

$$fe^x \mapsto d(fe^x) = (f'e^x + fe^x) dx.$$

One can check that this map is acyclic on global sections. However, if one restricts to $U = \mathbb{G}_m$, then $x^{-1}e^x$ is a nonzero element of the cokernel.

We will do something different: we will pass to the analytification.

²⁸Recall that $\mathrm{dR}_X^\bullet(\mathcal{D}_X)$ is a free resolution of ω_X ; this is where the shift comes from.

Definition 28.5. Let X be a scheme over \mathbb{C} . We denote by $X^{\text{an}} := X(\mathbb{C})$ equipped with the complex analytic topology. There is a canonical sheaf of complex analytic functions $\mathcal{O}_{X^{\text{an}}}$. There is a map of topological spaces $n : X^{\text{an}} \rightarrow X$, realizing $a^{-1}\mathcal{O}_X$ as a sheaf of $\mathcal{O}_{X^{\text{an}}}$ -algebras on X^{an} . We define

$$\mathrm{dR}_X^{\text{an}}(\mathcal{M}) := \mathrm{dR}_X(\mathcal{M})^{\text{an}} = \Omega_X^{\bullet, \text{an}} \otimes_{\mathcal{O}_X^{\text{an}}} a^{-1}\mathcal{M}.$$

This defines an exact functor (and thus on derived categories):

$$\mathrm{dR}_X^{\text{an}} : D\mathrm{DMod}(X) \rightarrow D\mathrm{Sh}(X^{\text{an}}; \mathbb{C})$$

to the derived category of sheaves of \mathbb{C} -vector spaces on X^{an} .

Example 28.6. We examine the example above again. The complex

$$\mathcal{O}_{X^{\text{an}}} \xrightarrow{f \mapsto (f+f') dx} \mathcal{O}_{X^{\text{an}}} dx$$

is now surjective as sheaf, since the differential equation $f'(x) + f(x) = g(x)$ has a solution for f , for any g , locally. But now, the differential equation $f'(x) + f(x) = 0$ also has a non-zero solution, namely $f(x) = e^{-x}$, so the complex is quasi-isomorphic to the constant sheaf \mathbb{C}_X in degree 0.

Exercise 28.7. Show that for $\mathcal{M} = \mathcal{O}_X$, we have $\mathrm{dR}_X^{\text{an}}(\mathcal{O}_X) \simeq \mathbb{C}_X[\dim(X)]$. In particular the de Rham functor is not fully faithful. Note the shift: this means that if X is proper, then $\mathrm{dR}_X^{\text{an}}(\mathcal{O}_X)$ is symmetric about 0 due to Poincare duality.

Example 28.8. Let $X = \mathbb{A}^1$. Let $i : \{0\} \hookrightarrow \mathbb{A}^1$ and let us take $\mathcal{M} = i_*\mathbb{C}$. We compute $\mathrm{dR}_X^{\text{an}}(\mathcal{M}) = k[\partial] \longrightarrow k[\partial]$ where the differential is

$$\partial^n \mapsto 0 + dx \otimes \partial^{n+1}.$$

Thus we see that $\mathrm{dR}_X^{\text{an}}(\mathcal{M}) = \mathbb{C}_0$ is the skyscraper sheaf at 0.

Example 28.9. Check that $\mathrm{dR}_X^{\text{an}}(\mathcal{D}_X) \simeq \Omega_{X^{\text{an}}}^n[-n]$ (where $n = \dim(X)$). This sheaf is not constructible (it has infinite-dimensional stalks).

Definition 28.10. Let X be the analytification of a complex algebraic variety.²⁹ A subset of X is *constructible* can be obtained as a finite intersection, union, or complement of Zariski closed subsets. Recall from Balasz's talk that a sheaf is *constructible* if there is a stratification of X by constructible subsets, such that the sheaf is locally constant of finite dimension on each stratum.

We let $D_c\mathrm{Sh}(X; \mathbb{C})$ denote the full subcategory of complexes with constructible cohomology. If S is a specific stratification, we let $D_S(\mathrm{Sh}(X; \mathbb{C}))$ denote the full subcategory of sheaves constructible with respect to that stratification.

We now state Riemann-Hilbert. We will take for given that there is a six-functors package on the constructible side, just to state the theorem. We also note there is a notion of “regular” D -module, which will correct the failure of the de Rham functor to be fully faithful (as seen above).

Theorem 28.11 (Riemann-Hilbert correspondence). *The analytic de Rham functor defines an equivalence:*

$$\mathrm{dR}_X^{\text{an}} : D_{rh}\mathrm{DMod}(X) \rightarrow D_c(\mathrm{Sh}(X; \mathbb{C})).$$

Furthermore, this functor intertwines duality functors, as well as the (derived) functors $f^, f_*, f^!, f_!$, tensor products and internal Hom (and middle extensions as well). Furthermore, there is a certain category of perverse sheaves on the right-hand side, defined by a certain perverse t -structure. This functor is t -exact for the usual t -structure on D -modules, and the perverse t -structure on constructible sheaves.*

Remark 28.12. One can ask if there is a perverse t -structure on D -modules; the answer is yes, due to Kashiwara.

²⁹As Isaac discussed in his talk, this can be made more general, but we won't do it since we don't need it.

Remark 28.13. One might be tempted to claim that $H^{-\dim(X)} \circ \mathrm{dR}_X^{\mathrm{an}}$ defines an equivalence between flat connections on X and locally constant sheaves on X . However, this is not true without regularity assumptions either. Take $X = \mathbb{A}^1$. There is only one local system on \mathbb{A}^1 . However there are many D -modules isomorphic to \mathcal{O}_X (as \mathcal{O} -modules): for example, $\mathcal{D}_X/\mathcal{D}_X(\partial - \lambda)$ for any λ .

Example 28.14 (Regularity). Except for the statement that $\mathrm{dR}_X^{\mathrm{an}}(\mathcal{M})$ is constructible, regularity is required for other statements of Riemann-Hilbert to hold. Let's see these in examples. Let $f : X = \mathbb{A}^1 \rightarrow Y = \mathrm{pt}$. We have that $Rf_*(\mathcal{O}_X e^x) = 0$, since its global sections are acyclic as we have checked. However, $\mathrm{dR}_X^{\mathrm{an}}(\mathcal{O}_X e^x) = \mathbb{C}_X$ is the constant sheaf, which has nonzero global sections.

Now, let $f : X = \{0\} \hookrightarrow Y = \mathbb{A}^1$ and take $\mathcal{M} = D/D(x^2\partial + 1)$. Morally, these are the solutions to the differential equation $x^2 \frac{dy}{dx} = y$, i.e. $y = e^{1/x}$, which has an essential singularity at 0. The de Rham complex is given by

$$\mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{x^2\partial + 1} \mathcal{O}_{X^{\mathrm{an}}} dx.$$

For shorthand, let $P = x^2\partial + 1$ be the differential operator. We wish to compute the $*$ -stalk at 0. For the cokernel, looking at power series at $x = a$, we see that $P(1) = 1$, but

$$\begin{aligned} P((x-a)^k) &= kx^2(x-a)^{k-1} + (x-a)^k = (a^2 + 2a(x-a) + (x-a)^2)(x-a)^{k-1} + (x-a)^k \\ &= a^2(x-a)^{k-1} + (2a+1)(x-a)^k + (x-a)^{k+1} \end{aligned}$$

for $k \geq 1$. So $x-a$ is nonzero in the cokernel at every point, including 0. For the kernel, there is a unique solution away from $x=0$ (as above) but no solution near $x=0$. Thus we have that $f^* \mathrm{dR}_Y^{\mathrm{an}}(\mathcal{M}) = \mathbb{C}$ (recall the shift).

On the D -modules side, we use Verdier duality to compute the $*$ -pullback. The Verdier dual of \mathcal{M} is $\mathbb{D}_Y(\mathcal{M}) = \mathcal{D}_X/\mathcal{D}_X(\partial x^2 + 1)$. We want to compute

$$i^! \mathbb{D}_Y(\mathcal{M}) = k \otimes_{k[x]} \mathcal{D}(X)/\mathcal{D}(X)(\partial x^2 + 1).$$

To do this, we can resolve the right with the usual D_X -free resolution, which is $\mathcal{O}(X)$ -flat. We identify $k \otimes_{k[x]} \mathcal{D}_X \simeq k[\partial]$, and we have

$$i^! \mathbb{D}_Y(\mathcal{M}) = k[\partial] \xrightarrow{\partial x^2 + 1} k[\partial]$$

where the map is right multiplication, and the game we need to play is to use the commuting relations to move the x symbols to the left where they are killed. Exercise: check that

$$[\partial^n, x] = n\partial^{n-1}, \quad [\partial^m, x^2] = 2 \sum_{i+j=m-1} \partial^i x \partial^j = 2x\partial^{m-1} + 2(m-1)\partial^{m-2}.$$

Thus, the map sends $\partial^k \mapsto 2(k-1)\partial^{k-1} + \partial^k$ for $k \geq 2$, and $1 \mapsto 1$ and $\partial \mapsto \partial + 2$. We see that it is an isomorphism, thus the complex is acyclic, and $i^! \mathbb{D}_Y(\mathcal{M}) = 0$.

Remark 28.15. There are some examples where the functors above always commute with the de Rham functor. For example, by Proposition 4.7.5 in [HTT08], $f_! = f_*$ commutes when f is proper, and by Corollary 4.3.3 in [HTT08], $f^! = f^*[2 \dim(f)]$ do when f is smooth.

Remark 28.16 (Riemann-Hilbert problem). Let X be a curve, and choose points $x_1, \dots, x_n \in X$. Fix an n -dimensional representation $\rho : \pi_1(X - \{x\}) \rightarrow GL_n(\mathbb{C})$. Now, consider a system of differential equation:

$$\frac{dy}{dx} = A(x)y$$

where y is vector valued and $A(x)$ is a matrix with values in rational functions. We say such a linear system is *Fuchsian* if the poles have at most order 1. Note that this is a *global* condition; it is a theorem that every regular differential equation (i.e. whose solutions locally have polynomial growth) can *locally* be written as a Fuchsian system.

The Riemann-Hilbert problem asks: when does a given monodromy representation come from a Fuchsian system of differential equations? Fuchsian implies regular, but not conversely. This problem is still open; I am not sure what its precise status is.

29 Lecture 29 (2020-05-12): Regular singularities

We begin with a digression into differential equations on algebraic curves or Riemann surfaces X . For simplicity we often let $X = \mathbb{P}^1$ when writing formulas. We are interested in either the setting of a homogeneous linear meromorphic ODE:

$$y^{(n)}(x) + c_1(x)y^{(n-1)}(x) + \cdots = c_n(x)y(x) = 0$$

or more generally, a homogeneous system of first order meromorphic ODEs:

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x)$$

where the $c_i(x)$ and matrix entries $a_{ij}(x)$ are *meromorphic functions* on X . Recall the notion of a *fundamental system of solutions*, essentially a matrix $Y(x)$ with a basis of solutions as columns. Recall that the first setting is a special case of the second, where the matrix $A(x)$ is in rational canonical form (with one block).

Regularity of the system is defined as a bound on the growth of the solutions.

Definition 29.1. In the setting above, we say the system is *regular* if, for any critical point $x_0 \in X$, its solutions have moderate growth in any sector of x_0 . More concretely, locally at x_0 , in any sector $\arg(x - x_0) \in (a, b)$ with angle $b - a < \pi$, any solution has $|\mathbf{y}(t)| \leq |x - x_0|^{-d}$ for some d .

On the other hand, we are interested in the following condition on the ODEs themselves.

Definition 29.2. A *gauge transformation* on a system of linear first order ODEs with fundamental matrix $Y(x)$, by a matrix $T(x)$, is the corresponding system with solutions $T(x)Y(x)$. This transformation acts on the coefficient matrix $A(x)$ by:

$$A(x) \mapsto T(x)A(x)T(x)^{-1} - T^{-1}(x)\frac{d}{dx}T(x).$$

We say a gauge transformation is *holomorphic* (resp. *meromorphic*) if $T(x)$ and $T(x)^{-1}$ have holomorphic (resp. meromorphic) coefficients.

A system of differential equations $\mathbf{y}'(x) = A(x)\mathbf{y}$ is *Fuchsian* if the matrix entries of $A(x)$ have poles of order ≤ 1 , up to meromorphic gauge equivalence (i.e. conjugation by a matrix $T(x)$ with meromorphic terms).³⁰

We won't prove this, but this is Theorem 5.1.4 in [HTT08]. It deduces the equivalence between (a) Fuchsian systems with (c) regular solutions, as well as showing that every Fuchsian system has a convenient form (b) which is easier to solve.

Theorem 29.3. Let \mathcal{O} denote the ring of regular functions on a small disk containing a puncture, and let \mathcal{K} denote the ring of functions on the punctured disk. The following conditions on a system of ODEs are equivalent:

- (a) the system is meromorphically gauge equivalent to a system $\mathbf{y}'(x) = \frac{1}{x}A(x)\mathbf{y}(x)$ where $a_{ij} \in \mathcal{O}$, i.e. the system is Fuchsian,
- (b) the system is meromorphically gauge equivalent to a system $\mathbf{y}'(x) = \frac{1}{x}A\mathbf{y}(x)$ with A a matrix with constant coefficients,
- (c) the system is regular.

Exercise 29.4. Find that the fundamental matrix to the system $\mathbf{y}'(x) = \frac{1}{x}A\mathbf{y}(x)$ is given by $Y(t) = x^A = \exp(A \ln(x))$. Thus the monodromy is $M = \exp(2\pi i A)$.

Remark 29.5. We consider meromorphic gauge equivalence because it preserves the one property we care about: regularity (i.e. multiplying by a meromorphic matrix does not change moderate growth). However, meromorphic gauge equivalence can change monodromy, et cetera, while holomorphic gauge equivalence preserves these properties.

³⁰Some authors use terms such as Fuchsian vs. truly Fuchsian, et cetera to distinguish between a system where $A(x)$ has such poles versus a system equivalent to such a system.

Remark 29.6 (Fuchsian vs. regular). The theorem above says that a system of differential equations, when considered locally, is Fuchsian if and only if the solutions are regular. However, this might not be true globally. That is, a system is globally Fuchsian implies that it is regular, but not conversely.

Let us consider an example.

Example 29.7 (Euler system). Consider the ODE:

$$y^{(n)}(x) + \frac{c_1}{x}y^{(n-1)}(x) + \cdots + \frac{c_n}{x^n}y = 0.$$

Check that all solutions have the form x^λ for some λ . This ODE is equivalent to the linear system:

$$\mathbf{y}'(x) = \begin{pmatrix} -c_1/x & -c_2/x^2 & \cdots & -c_n/x^n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{y}(x).$$

We have meromorphic gauge equivalence

$$\begin{aligned} & \text{diag}(x^n, x^{n-1}, \dots, 1) \begin{pmatrix} -c_1/x & -c_2/x^2 & \cdots & -c_n/x^n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{diag}(x^{-n}, x^{-n+1}, \dots, 1) + \text{diag}(-n/x, \dots, -1/x) \\ &= \begin{pmatrix} -c_1/x - n/x & -c_2/x & \cdots & -c_{n-1}/x & -c_n/x \\ 1/x & -(n-1)/x & \cdots & 0 & 0 \\ 0 & 1/x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2/x & 0 \\ 0 & 0 & \cdots & 1/x & -1/x \end{pmatrix}. \end{aligned}$$

Example 29.8. Consider

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$$

It has linearly independent solutions: $y(x) = 1/x$ and $y(x) = x$.

Definition 29.9. The *Poincaré rank* of a system is the minimum integer r such that the system is meromorphically gauge equivalent to:

$$\mathbf{y}'(x) = \frac{1}{x^{r+1}}A(x)\mathbf{y}(x)$$

for $a_{ij}(x) \in \mathcal{O}$. In particular, Fuchsian just means Poincaré rank 0. Note that this notion is preserved by holomorphic gauge transformations, but not meromorphic gauge transformations, as we've seen above (which also do not preserve monodromy of solutions; however, they do not affect whether the solutions are meromorphic or not).

Remark 29.10. There is a classification of Fuchsian systems up to holomorphic gauge, called the *Poincaré-Dulac-Levelt normal form*. See [IY08], Chapter 15.

The following is useful. A flat connection gives rise to a system of differential equations. By the theorem, in local coordinates around a puncture, it can be written:

$$\mathbf{y}'(x) = \frac{1}{x}A(x)\mathbf{y}(x).$$

That is, there is a \mathcal{K} -basis such that $x\partial$ acts by multiplication by the \mathcal{O} -matrix $A(x)$. Thus, the *lattice*, i.e. \mathcal{O} -submodule generated by this basis, is stable under the monodromy action $x\partial$. Conversely, given such a lattice, one recovers such a system. This leads to our algebraic definition.

Definition 29.11. Let X be a smooth scheme of dimension 1; it has a unique compactification \overline{X} . Let $x_0 \in \partial X$ and $X' = X \cup \{x_0\} \subset \overline{X}$. Denote $i : X \hookrightarrow X'$, and let t be a local coordinate at x_0 . We say a flat connection \mathcal{E} on X is *regular* if $i_*\mathcal{E}$ the action of $t\partial_t$ is locally \mathcal{O} -finite (equivalently, if it possesses a lattice which is $t\partial_t$ -stable). We say a holonomic D -module on X , which is a flat connection on $U \subset X$, is regular if it is regular on U .

Now, suppose X has positive dimension. We say that a holonomic D -module \mathcal{M} is *regular* if its $!$ -restriction to any curve $C \subset X$ is regular.

The main result is the following, which we won't prove.

Theorem 29.12. *The category of regular holonomic D -modules is closed under extensions, and preserved by all six functors. Furthermore, the irreducible regular holonomic D -modules are exactly the middle extensions of regular flat connections.*

Remark 29.13. If G acts on X with finitely many orbits, then all holonomic D -modules are regular. Roughly, this follows from the classification of irreducibles: any local system on an orbit must be constant.

References

- [BGS98] Alexander Beilinson, Victor Ginzburg and Wolfgang Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.
- [CG97] Neil Chriss, Victor Ginzburg, *Representation Theory and Complex Geometry*, Birkhauser, 1997.
- [HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, Toshiyuki Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Birkhauser, 2008.
- [Hu08] James Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, 94. American Mathematical Society, Providence, RI, 2008.
- [IY08] Yulij Ilyashenko and Sergei Yakovenko, *Lectures on Analytic Differential Equations*, Graduate Studies in Mathematics, 86. American Mathematical Society, Providence, RI, 2008.
- [We11] Ben Webster, *Singular blocks of parabolic category \mathcal{O} and finite W -algebras*, arXiv:0909.1860v4, 2011.