

ON ULTRAPRODUCTS, THE SPECTRAL THEOREM AND RIGGED HILBERT SPACES

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ABSTRACT. We start by showing how to approximate unitary and bounded self-adjoint operators by operators in finite dimensional spaces. Using ultraproducts we give a precise meaning for the approximation. In this process we see how the spectral measure is obtained as an ultralimit of counting measures that arise naturally from the finite dimensional approximations. Then we see how generalized distributions can be interpreted in the ultraproduct. Finally we study how one can calculate kernels of operators K by calculating them in the finite dimensional approximations and how one needs to interpret Dirac deltas in the ultraproduct in order to get the kernels as propagators $\langle x_1 | K | x_0 \rangle$.

This paper continues a study of finite dimensional approximations of operators on Hilbert spaces. We started the work in [HH], motivated by a quest for mathematically justified eigenvectors in quantum mechanics. There we built eigenvectors directly as ultraproducts of eigenvectors in the small spaces. This time we look at another way of finding finite dimensional approximations, and reconstruct the spectral decomposition theorem using ultraproducts. We also find structure resembling Rigged Hilbert Spaces, thus providing generalized eigenvectors and bringing us back to our earlier motivation.

In quantum mechanics the states of a system are usually represented by unit vectors in a Hilbert space. Observables of the system are modeled by self-adjoint (often unbounded) operators whose spectrum gives the possible measurement outcomes. If the observable has a finite spectrum and a finite number of eigenvectors, this model coincides with Dirac's bra- and ket-vector representation [Di], but in general Dirac's bras and kets don't form a Hilbert space. A key ingredient in Dirac's approach, the delta-functions, were later mathematically justified by Schwartz [Sch] as distributions, functionals over a set of test functions. Gelfand and Vilenkin found a natural space for these, namely the Rigged Hilbert Spaces. These are triplets of spaces: in addition to a Hilbert space, one considers a dense subspace of it - the test functions - and the space of either linear or antilinear functions over these test functions. This last space, in which the Hilbert space embeds naturally, is the space of distributions. The linear functions correspond to bra vectors, the antilinear functions to ket vectors.

One key question about quantum systems is how they evolve over time. Depending on the approach to the system, this is described either by the kernel of the time evolution operator, or via the Feynman propagator. The kernel is a function $K(x, y)$ such that the action of the time evolution operator K on a wave function φ can be

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described by

$$K(\varphi)(y) = \int_S K(x, y) \varphi(x) d\mu(x),$$

where S is the spectrum of the observable.

We study a method of calculating the kernel via the finite dimensional approximations. Although our method builds on ultraproducts, the space studied is not the full ultraproduct, but a subspace of it. As this is not elementary, we cannot use Łos's theorem, but this is not a problem: as we are only interested in equations, it is enough to have just an embedding of the original space into the ultraproduct space. This also means that our approach does not restrict to only pseudocompact spaces (the metric analogue of pseudofinite spaces, see [GL]).

Discrete approximations of quantum systems have been used and studied extensively in physics. From the list of references of [HH] one can find examples of this. They have been studied also in mathematical physics, see e.g. [Ba]. We got inspired by the work of B. Zilber [Zi], and set out to compute concrete examples of kernels in [HH]. The question we studied there was whether it is possible to calculate the kernels of the time evolution operators by calculating them in some kind of finitely generated approximations of quantum systems which in [HH] were the free particle and the harmonic oscillator. It turned out that in these cases it is possible. One can choose the approximations to be simply finite dimensional Hilbert spaces with rather straight forward approximations of the position operator and the momentum operator. In the choice of the approximations of the time evolution operator one needed to be more creative in the case of the harmonic oscillator. To determine in which sense the approximations approximate the quantum systems was more tricky. For that we used a rather heavily modified version of the metric ultraproduct of the finite dimensional Hilbert spaces. Once we knew how the approximation works, it was rather straight forward to do the calculations with the help of number theory.

However, following Zilber, we would have liked to be able to calculate the kernels by using Dirac deltas in the propagator style $\langle x_1 | K | x_0 \rangle$. In [HH] our interpretation of Dirac deltas was as ultraproducts of eigenvectors. In hind sight that was too naive an approach, and didn't work out. The propagators calculated in the finite dimensional models did not give a correct propagator in the ultraproduct, and not even some kind of ad hoc renormalization would have made the values correct, as there were also divisibility-related discretizing effects stemming from the finite approximations. The remedy in [HH] was to calculate the kernel instead, as this could be done using the propagators in the finite dimensional models, and the method 'averaged out' the discretization effect.

In this paper we continue the work we started in [HH], but we look at a general case instead of concrete examples. Also, the method is different. The first question we look at is the following: Given a separable Hilbert space H and an operator A on it, can one find finite dimensional Hilbert spaces H_N and operators A_N on them so that the operators approximate A in the sense that H with A is isometrically isomorphic to a submodel of the metric ultraproduct of the spaces H_N with operators A_N ? The first problem in finding the pairs (H_N, A_N) is that in the case where the operator A is unbounded, the metric ultraproduct A^m of the operators A_N can not be well-defined everywhere in the metric ultraproduct H^m of the spaces H_N . In [HH] looking at the position and momentum operators, we were able to show that

A^m is well defined in a suitable part of H^m so that the isomorphism can indeed be found. In this paper we look only at bounded operators and thus this problem does not arise. Although this does restrict the direct applicability of our method, we hope the method can be extended, more on this below.

In the first two sections we consider the approximation question in the case where A is such that $AA^* = A^*A = rI$ where r is a positive real and I is the identity operator. So, e.g., A can be a unitary operator. We show how the operators A_N can be found, show how the spectral measure μ for A is obtained as an ultralimit of counting measures that arise from the approximations naturally, and how the approximations give A as a multiplication operator in the space $L_2(\sigma(A), \mu)$.

In the third section we study the same questions for bounded self-adjoint operators A and get the same results by reducing the questions to the previous case by looking at the operator e^{irA} for r a small enough real.

In the fourth section we look at generalized distributions in the context of the space $L_2(\sigma(A), \mu)$ from the previous sections. See [RS] for the classical theory of generalized distributions. Now the generalized distributions (excluding some) cannot be seen as elements of H^m . Thus we look at the space H^∞ which is obtained from the classical ultraproduct of the spaces H_N by fractioning out the equivalence relation of being infinitely close to each other in a metric that arises from the norms in H_N . This space contains infinite vectors giving us a chance. The problem here is that the ultraproduct of the inner products in the spaces H_N is not well-defined everywhere in $H^\infty \times H^\infty$. It turns out that it is well defined in a large enough subset so that one can interpret generalized distributions θ as vectors $u(\theta)$ in H^∞ so that for all continuous $f : \sigma(A) \rightarrow \mathbb{C}$, where \mathbb{C} is the field of complex numbers, $\theta(f) = \langle F^m(f) | u(\theta) \rangle$ where F^m is a natural embedding we get from the isometric embedding of $L_2(\sigma(A), \mu)$ into H^m , see above (the embedding is not into H^∞ , for details see Section 4).

In the fifth section we look at ways of calculating the kernel of an operator B on $L_2(\sigma(A), \mu)$ (or on H). We start by showing that the method from [HH] works also here assuming that we can find reasonable approximations B_N for B in the spaces H_N . In [HH] we showed how to find these for the time evolution operators B of the free particle and the harmonic oscillator (there A was the position operator). Here we show that if B has a kernel, then the reasonable approximations B_N can always be found. However, unlike in the special cases studied in [HH], here our proof is essentially existential and does not give a practical way of finding the approximations. One must keep in mind that finding the kernels is difficult and thus one can not expect to have a simple trick that gives them.

In the fifth section we also show that the kernel can be obtained as the propagator $\langle x_1 | B | x_0 \rangle$ (using the discrete approximations we can extend B to an operator that acts also on Dirac deltas) if one is very careful in choosing the interpretations of the Dirac deltas $|x_1\rangle$ and $|x_0\rangle$ and here, of course, we think of Dirac deltas as generalized distributions as is common in mathematics. The need to be very careful comes from the fact that $\langle x_1 | B | x_0 \rangle$ is very sensitive to the choice of eigenvectors, as opposed to $\langle F^m(f) | u(\theta) \rangle$. In [HH] we showed that a straightforward approach to eigenvectors can give completely wrong values. As the result, this Dirac delta method is probably not very practical if applied in a straightforward manner. From

[Ma] one can find an example of a rigorous use of Dirac deltas in the context where Dirac deltas are interpreted as generalized distributions.

Open question 1. Can one find approximations A_N for bounded normal operators A or unbounded self adjoint operators?

Although it is not immediate, we believe that if one can find finite dimensional approximations of bounded normal operators, one can combine the classical trick of looking at the operators $(A \pm iI)^{-1}$ and our technique for building unbounded operators in ultraproducts from [HH] to find approximations for unbounded self-adjoint operators. However, even in the case of bounded normal operators we have a serious problem: How to find the approximations A_N so that that they are roughly like how we chose them in Section 1 and still normal (with the assumption $AA^* = A^*A = rI$, there are no problems in guaranteeing normality, in fact, this assumption was designed to get normal approximations). The conventional trick of decomposing normal operators into two self-adjoint ones does not seem to work here, as approximating them in the spaces H_N seems to destroy commutativity, so that we cannot guarantee our approximations to be normal. But without normality, nothing that we do in this paper works.

1. THE CONSTRUCTION

This section presents the basic construction of finite-dimensional approximations that we will use and modify in later sections. Here we study a separable Hilbert space with a (scaled) unitary operator having a cyclic vector. We use the cyclic vector and polynomials to build finite-dimensional approximations such that the original Hilbert space embeds isometrically into their metric ultraproduct. We also get a vector space homomorphism from the space of polynomials (over a compact set $S \subset \mathbb{C}$) into the metric ultraproduct. In section 2 we will construct a suitable measure to make the homomorphism an isometry.

The construction will be used to find similar approximations for the case where the operator is bounded and self-adjoint in section 3. In section 4 we will study other norms for the ultraproduct space that we will be essential in studying distributions in sections 4 and 5. In section 6 we will have a short look at what one can do if the operator doesn't have a cyclic vector.

So we let H be a separable complex Hilbert space and A an operator on H of the form qU , where q is a nonzero complex number and U is unitary. Then A is normal and bounded and $A^* \circ A = A \circ A^* = r_A I$ where r_A is a positive real, I is the identity operator, and A^* is the adjoint of A . This is enough for the constructions we do in this paper (we will use the construction when A is unitary), but we use notation allowing for more general operators, as there are also other operators allowing for this sort of approximations (see the Example at the end of this section).

From a polynomial $P(X, Y) \in \mathbb{C}[X, Y]$ over the complex numbers \mathbb{C} we get an operator $P(A, A^*)$ on H the natural way, e.g. $X^2Y(A, A^*) = A \circ A \circ A^*$, and we say that $\phi \in H$ is *cyclic* if the set $\{P(A, A^*)(\phi) \mid P \in \mathbb{C}[X, Y]\}$ is dense in H . We will here study the case where there is a cyclic vector ϕ in H and we pick ϕ so that in addition its norm is 1. Notice that H can always be split into countably many complete subspaces so that they are orthogonal to each other, closed under A and

A^* and each of them has a cyclic vector, and in section 6 we will see how to combine the constructions from this decomposition.

For all $N \in \omega$, let H_N be the subspace of H generated by

$$(1) \quad \{A^i(A^*)^j(\phi) \mid i \leq N, j \leq N\}.$$

In particular, note that H_0 is the space generated by ϕ , and $H_N \subset H_{N+1}$. Notice that as finite dimensional spaces these are complete spaces. We write H_N^- for the subspace of H generated by

$$(2) \quad \{A^i(A^*)^j(\phi) \mid i < N, j \leq N\}$$

and H_N^+ for the subspace of H generated by

$$(3) \quad \{A^i(A^*)^j(\phi) \mid 0 < i \leq N, j \leq N\}.$$

As $AA^* = A^*A = r_AI$, the operators A and A^* swap these spaces: A maps H_N^- onto H_N^+ and A^* maps H_N^+ onto H_N^- . To get approximating operators on H_N we can modify A and A^* by a wraparound trick: Let W^- be the orthogonal complement of H_N^- in H_N and W^+ be the orthogonal complement of H_N^+ in H_N . Then W^+ and W^- have the same dimension and we let U be a unitary operator from W^- onto W^+ . Now we define an operator A_N on H_N as follows. If $u \in H_N^-$, then $A_N(u) = A(u)$ and if $u \in W^-$, then $A_N(u) = \sqrt{r_A}U(u)$ (by $\sqrt{r_A}$ we mean the one that is positive). We get the adjoint of A_N as follows: If $u \in H_N^+$, then we let $A_N^*(u) = A^*(u)$ and if $u \in W^+$, then $A_N^*(u) = \sqrt{r_A}U^{-1}(u)$. It is easy to check that A_N^* is indeed the adjoint of A_N and that $A_N^* \circ A_N = A_N \circ A_N^* = r_AI$ and so A_N is normal. Notice that because of how A and A^* cancel out (up to multiplication by a constant), H_N^+ is also generated by the set $\{A^i(A^*)^j(\phi) \mid i \leq N, j < N\}$. Thus H_N^- and H_N^+ are the subspaces on which A_N and A_N^* respectively behave 'correctly' and

$$(4) \quad A_N(A^i(A^*)^j(\phi)) = A(A^i(A^*)^j(\phi)) = A^{i+1}(A^*)^j(\phi) \text{ if } i < N, j \leq N$$

and

$$(5) \quad A_N^*(A^i(A^*)^j(\phi)) = A^i(A^*)^{j+1}(\phi) \text{ if } j < N, i \leq N.$$

In H_N we can also define operators $P(A_N, A_N^*)$ for polynomials $P(X, Y)$ in the natural way.

For all $N \in \omega$, we let D_N be the dimension of H_N and we choose eigenvectors $u_N(n)$, $n < D_N$, of A_N with eigenvalues $\lambda_N(n)$. Notice that

$$(6) \quad u_N(n) \text{ is also an eigenvector of } A_N^* \text{ with eigenvalue } \overline{\lambda_N(n)},$$

where $\overline{\lambda_N(n)}$ is the complex conjugate of $\lambda_N(n)$. Since A_N is normal, we can choose these so that they form an orthonormal basis of H_N . Denote $\xi_N(n) = \langle u_N(n) | \phi \rangle$, i.e.

$$(7) \quad \phi = \sum_{n=0}^{D_N-1} \xi_N(n) u_N(n),$$

and we choose the vectors $u_N(n)$ so that in addition $\xi_N(n)$ is a non-negative real number. One should note that even though the spaces H_N form an increasing sequence, the bases do not extend each other, as the operators A_N and A_N^* do not extend each other (because of the wraparound modification).

Next we want to tie a spectral decomposition of A to our finite-dimensional approximation. So we choose a natural number M so that it is strictly greater than both the operator norm of A and $\sqrt{r_A}$, and let

$$(8) \quad S = \{\lambda \in \mathbb{C} \mid -M \leq \operatorname{Re}(\lambda) \leq M, -M \leq \operatorname{Im}(\lambda) \leq M\}.$$

Notice that there is a real $\varepsilon > 0$ such that for all $N < \omega$ and $n < D_N$, $|\lambda_N(n)| < M - \varepsilon$ since the norm of A_N is at most the maximum of the norm of A and $\sqrt{r_A}$. For all $X \subseteq \mathbb{C}$, let $C(X)$ be the vector space of all bounded continuous functions from X to \mathbb{C} . Of these our main interest is in $C(S)$. Notice that since S is compact, every $f \in C(S)$ is uniformly continuous. We let $D(S)$ be the subspace of $C(S)$ that consists of all functions $f_P(\lambda) = P(\lambda, \bar{\lambda})$, where $P \in \mathbb{C}[X, Y]$ and $\bar{\lambda}$ is the complex conjugate of λ . On $D(S)$ we can again define operators A_D and A_D^* in a natural way

$$(9) \quad A_D(f_P) = f_{XP} \quad \text{and} \quad A_D^*(f_P) = f_{YP}$$

(e.g. $A_D(f_{X^i Y^j}) = f_{X^{i+1} Y^j}$). Notice that $A_D(f_P)(\lambda) = \lambda f_P(\lambda)$ and $A_D^*(f_P)(\lambda) = \bar{\lambda} f_P(\lambda)$.

Let's define some mappings that will be crucial in this paper:

Definition 1.1. For $P \in \mathbb{C}[X, Y]$, let

$$(10) \quad G(f_P) = P(A, A^*)(\phi)$$

and note that it is a homomorphism from $D(S)$ to H (as vector spaces), and that its image is dense in H (by the cyclicity assumption). Further let

$$(11) \quad F_N(f_P) = \sum_{n < D_N} \xi_N(n) f_P(\lambda_N(n)) u_N(n),$$

and note that this is a homomorphism from $D(S)$ to H_N . Notice that $F_N(f_P) = P(A_N, A_N^*)(\phi)$.

Now the inner product in H_N (inherited from H), gives us one norm $\|\cdot\|_2$ on H_N . Using this, there is a natural way of defining metric ultraproducts of the Hilbert spaces H_N . However, we want to consider also other norms in the ultraproduct, so we take as our starting point a classical ultraproduct of the structures H_N and 'dig out' various metric ultraproduct spaces from it. We look at the $\|\cdot\|_2$ -norm here, and consider the other norms in section 4.

Let \mathcal{U} be an ultrafilter on ω such that (as in [HH]) for all $m \in \omega - \{0\}$,

$$\{N \in \omega - \{0\} \mid \sqrt{N} \in \omega, m|\sqrt{N}\} \in \mathcal{U}.$$

Let H^u be the (classical) ultraproduct $\Pi_{N \in \omega} H_N / \mathcal{U}$ and \mathbb{C}^u be the ultrapower $\mathbb{C}^\omega / \mathcal{U}$ of the field of complex numbers \mathbb{C} . Then H^u is a vector space over \mathbb{C}^u and by identifying elements of \mathbb{C} with their images under the canonical embedding of \mathbb{C} into \mathbb{C}^u , also a vector space over \mathbb{C} . There is also a pairing $\langle \cdot, \cdot \rangle^u$ from H^u to \mathbb{C}^u which is obtained as the ultraproduct of the inner products $\langle \cdot, \cdot \rangle_N$ of H_N . And similarly we get a unary function $\|\cdot\|_2^u$. Notice that for all $v = (v_N)_{N \in \omega} / \mathcal{U} \in H^u$, $\|v\|_2^u = (\sqrt{\langle v_N | v_N \rangle_N})_{N \in \omega} / \mathcal{U}$ and that $((\sqrt{\langle v_N | v_N \rangle_N})_{N \in \omega} / \mathcal{U})^2 = \langle v | v \rangle^u$. Similarly for all $q = (q_N)_{N \in \omega} / \mathcal{U} \in \mathbb{C}^u$, we write $|q|^u$ for $(|q_N|)_{N \in \omega} / \mathcal{U}$. Notice that if we let X be the range of this 'absolute value', then $\mathbb{R}^u = X \cup \{-r \mid r \in X\}$ is a real closed field containing the reals. In particular, it is linearly ordered and thus we

can compare e.g. the 'norm' $\|v\|^u$ of $v \in H^u$ and a rational number. Thus for all $u, v \in H^u$, if there is $q \in \mathbb{C}$ such that q is infinitely close to $\langle u|v \rangle^u$, we write $\langle u|v \rangle$ for this q . If there is no such q , we write $\langle u|v \rangle = \infty$.

Notation 1.2. When we define an element $(v_N)_{N < \omega} / \mathcal{U}$ of H^u , it is enough to define the vectors v_N so that the definition makes sense for all N in some set that belongs to \mathcal{U} . This often simplifies notation considerably.

Now H^u contains some structures of interest to us. Let us first look at H^∞ : On H^u we define an equivalence relation \sim so that $u \sim v$ if $\|u - v\|_2^u$ is infinitesimal (i.e. smaller than $1/n$ for all natural numbers n) and let $H^\infty = H^u / \sim$. Now addition of elements of H^∞ and for all $q \in \mathbb{C}$, the multiplication by q are well-defined in H^∞ and thus H^∞ is a vector space over \mathbb{C} . We define a partial pairing $\langle \cdot | \cdot \rangle$ on H^∞ as follows: For all $u, v \in H^\infty$, if there is $q \in \mathbb{C}$ such that for all $u' \in u / \sim$ and $v' \in v / \sim$, $\langle u'|v' \rangle^u = q$, then we let $\langle u|v \rangle = q$ and otherwise $\langle u|v \rangle$ is ∞ .

The second of these structures is the metric ultraproduct H^m : We let H^m be the set of all $v / \sim \in H^\infty$ such that $\|v\|_2^u$ is finite (i.e. smaller than some natural number). Notice that H^m is still a vector space over \mathbb{C} and $\langle \cdot | \cdot \rangle$ is total on H^m and, in fact, an inner product. This also gives a norm on H^m but it is the same as $\|\cdot\|_2$ defined above. Later, when considering spaces arising from other norms, we may write H_2^m for H^m .

Remark 1.3. We could define \sim also on \mathbb{C}^u by $q \sim r$ if $|q - r|^u$ is infinitesimal and let \mathbb{C}^∞ be \mathbb{C}^u / \sim . Notice that multiplication is not well defined in \mathbb{C}^∞ if one of the elements has infinite norm. Then we could let \mathbb{C}^m be (the field of) all $q / \sim \in \mathbb{C}^\infty$ such that $|q|^u$ is finite. Notice that $\mathbb{C}^m = \{q / \sim \mid q \in \mathbb{C}\}$ and that for all $u, v \in H^m$, if $\langle u|v \rangle = q$, then $q / \sim = \langle u|v \rangle^u / \sim$.

Let A^u be the ultraproduct of the operators A_N . Since the operators A_N are uniformly bounded, A^u induces a well-defined operator A^∞ on H^∞ and the restriction A^m of A^∞ to H^m is a well-defined operator on H^m . We can do the same for the A_N^* 's and get operators A^{*u} , $A^{*\infty}$ and A^{*m} . Notice that $(A^m)^* = A^{*m}$ since being an adjoint is preserved in metric ultraproducts and that A^m and A^{*m} commute since A_N and A_N^* commute for all N and thus A^m is normal.

Now the homomorphisms F_N (from Definition 1.1) induce a homomorphism from the vector space $D(S)$ into H^m :

Definition 1.4. Let $F^m : D(S) \rightarrow H^m$ be given by

$$F^m(f_P) = (F_N(f_P))_{N < \omega} / \mathcal{U} / \sim.$$

By F we will denote the homomorphism from $D(S)$ to H^u :

$$F(f_P) = (F_N(f_P))_{N < \omega} / \mathcal{U}.$$

Notice that $F^m(A_D(f_P)) = A^m(F^m(f_P))$ and similarly for A_D^* and A^{*m} . We can also embed H into H^m :

Definition 1.5. Let $G^m : H \rightarrow H^m$ be given by

$$G^m(P(A, A^*)(\phi)) = (P(A, A^*)(\phi))_{N < \omega} / \mathcal{U} / \sim,$$

and denote $H^{Im} = \text{rng}(G^m)$.

Notice that G^m is well-defined, as

$$P(A, A^*)(\phi) \in H_N$$

for all N large enough. Notice also that for all $\psi \in H$, $G^m(A(\psi)) = A^m(G^m(\psi))$ and the same for A^* and A^{*m} . Also $\|G^m(\psi)\|_2^m = \|\psi\|_2$ and so G^m is an (isometric) embedding of the Hilbert space H into H^m and maps A to $A^m \upharpoonright H^{Im}$ and A^* to $A^{*m} \upharpoonright H^{Im}$.

So we have seen:

Lemma 1.6. *G^m is an isometric isomorphism from H onto H^{Im} , $G^m \circ A = A^m \circ G^m$, $G^m \circ A^* = A^{*m} \circ G^m$ and A^{*m} and $A^{*m} \upharpoonright H^{Im}$ are the adjoints of A^m and $A^m \upharpoonright H^{Im}$, respectively.* \square

Corollary 1.7. *For all $f \in D(S)$, $G^m(G(f)) = F^m(f)$.*

Proof. Now $f = f_P$ for some $P \in \mathbb{C}[X, Y]$. Then

$$G^m(G(f)) = G^m(P(A, A^*)(\phi)) = P(A^m, A^{*m})(G^m(\phi)) = F^m(f),$$

where the last identity follows from (4), (5) and (6) above. \square

We now return to the reason for talking about A and A^* instead of A and A^{-1} , namely there are operators that are not obtained just by multiplying a unitary operator by a constant but that still allow for the sort of approximation we developed here.

Example 1.8. Let $A = rU$ be an operator with a cyclic vector u on a Hilbert space H_1 and $B = qV$ be an operator with a cyclic vector v on a Hilbert space H_2 , where $r, q \in \mathbb{C}$, $r, q \neq 0$, $r \neq q$, and U and V are unitary. Then consider the space $H \subset H_1 \oplus H_2$ spanned by the vectors $(A \oplus B)^i (A^* \oplus B^*)^j (u \oplus v)$, for $i, j < \omega$. This gives a space (H) and an operator $(A \oplus B)$ with a cyclic vector $(u \oplus v)$ such that the operator is not a constant times a unitary operator, but one can still build spaces H_N as above (generated by vectors $(A \oplus B)^i (A^* \oplus B^*)^j (u \oplus v)$ with $i, j \leq N$).

Open question 2. What exactly is needed of the operator A for this kind of approximation technique to work?

2. SPECTRAL MEASURE

In this section we show how weighted counting measures in the finite dimensional spaces give rise to a measure on subsets of \mathbb{C} . With this measure the vector space homomorphism F from the previous section becomes an isometric embedding of $L_2(S, \mu)$ into the metric ultraproduct.

For all $N < \omega$, we define a measure μ_N for subsets X of \mathbb{C} : Recall that $\phi = \sum_{n=0}^{D_N-1} \xi_N(n) u_N(n)$ and each $\xi_N(n)$ is a non-negative real. We let $\mu_N(X)$ be the sum of all $\xi_N(n)^2$ such that $n < D_N$ and $\lambda_N(n) \in X$. Note that, as $\|\phi\| = 1$, $\mu_N(X) \in [0, 1]$ for all $N < \omega$ and $X \subset \mathbb{C}$. Then we define a naive measure $\mu^n(X)$ (not a measure) to be the ultralimit $\lim_{\mathcal{U}} \mu_N(X)$ of the measures $\mu_N(X)$, i.e., $\mu^n(X)$ is the unique real r such that for all $\varepsilon > 0$, the set

$$\{N < \omega \mid |\mu_N(X) - r| < \varepsilon\} \in \mathcal{U}.$$

Definition 2.1. For all reals $r \in [-M, M]$ we define lines $I_r = \{\lambda \in S \mid \operatorname{Re}(\lambda) = r\}$ and $J_r = \{\lambda \in S \mid \operatorname{Im}(\lambda) = r\}$. For all $\varepsilon > 0$ we let $I_r^\varepsilon = \{\lambda \in S \mid |\operatorname{Re}(\lambda) - r| < \varepsilon\}$ and $J_r^\varepsilon = \{\lambda \in S \mid |\operatorname{Im}(\lambda) - r| < \varepsilon\}$. For a point $\lambda \in S$ and $\varepsilon > 0$, we let $\lambda^\varepsilon = \{x \in S \mid |\operatorname{Re}(\lambda) - \operatorname{Re}(x)|, |\operatorname{Im}(\lambda) - \operatorname{Im}(x)| < \varepsilon\}$. We say that a real $r \in [-M, M]$ is *nice* if for all $\delta > 0$ there is $\varepsilon > 0$ such that $\mu^n(I_r^\varepsilon), \mu^n(J_r^\varepsilon) < \delta$.

Lemma 2.2. (i) μ^n satisfies all the properties of a measure when restricted to finite collections of sets e.g. if $X \subseteq Y$, then $\mu^n(X) \leq \mu^n(Y)$ and if X_i , $i < m$, are disjoint, then $\mu^n(\cup_{i < m} X_i) = \sum_{i < m} \mu^n(X_i)$.
(ii) The set of $r \in [-M, M]$ that are not nice is countable. Thus there is a countable dense subset NI of $[-M, M]$ such that every $r \in NI$ is nice and $-M, M \in NI$.

Proof. (i): The claim is clear since each μ_N is a measure and thus has these properties.

(ii): Suppose not. Since $-M$ and M are clearly nice, there must be uncountably many reals $r \in [-M, M]$ that are not nice. Let these be r_i , $i < \omega_1$. By symmetry and the pigeon hole principle we may assume that for all $i < \omega_1$, the vertical lines witness this. Then by the pigeon hole principle again, we may assume that there is $\delta > 0$ such that for all $i < \omega_1$ and $\varepsilon > 0$, $\mu^n(I_{r_i}^\varepsilon) > \delta$. Let $K < \omega$ be such that $K\delta > 1$. Now for all $i < K$ choose $\varepsilon_i > 0$ so that the sets $I_{r_i}^{\varepsilon_i}$, $i < K$, are disjoint. But now we can find $N < \omega$ such that for all $i < K$, $\mu_N(I_{r_i}^{\varepsilon_i}) > \delta$. It follows that $\mu_N(\cup_{i < K} I_{r_i}^{\varepsilon_i}) > 1$, a contradiction. \square

Definition 2.3. Let Sq^* be the set of all subsets Y such that there are $r_i \in NI$ such that

$$Y = \{\lambda \in S \mid r_0 < \operatorname{Re}(\lambda) < r_1, r_2 < \operatorname{Im}(\lambda) < r_3\}.$$

Now define $\mu^*(Y)$ for any $Y \subseteq \operatorname{Int}(S)$ (interior of S) to be the outer measure based on $\mu^n(X)$, $X \in Sq^*$, i.e.

$$\mu^*(Y) = \inf \left\{ \sum_{i=0}^{\infty} \mu^n(X_i) \mid X_i \in Sq^*, Y \subseteq \cup_{i < \omega} X_i \right\}.$$

As usual, we say that $X \subseteq \operatorname{Int}(S)$ is measurable if for all $Y \subseteq \operatorname{Int}(S)$, $\mu^*(Y) = \mu^*(Y \cap X) + \mu^*(Y - X)$. For $X \subseteq S$ (or $X \subseteq \mathbb{C}$), we let $\mu^*(X) = \mu^*(X \cap \operatorname{Int}(S))$ and X is measurable if $X \cap \operatorname{Int}(S)$ is. Since there is $\varepsilon > 0$ such that for all $N < \omega$ and $n < D_N$, $|\lambda_N(n)| < M - \varepsilon$, it is enough to look at subsets of $\operatorname{Int}(S)$ (everything close to the boundary of S has naive measure zero). We write \overline{X} for the closure of $X \subseteq \mathbb{C}$.

Lemma 2.4. Let $Y \in Sq^*$.

- (i) $\mu^*(Y) = \mu^*(\overline{Y}) = \mu^n(Y) = \mu^n(\overline{Y})$.
- (ii) Y is measurable.

Proof. (i): Clearly $\mu^*(Y) \leq \mu^*(\overline{Y})$. From the choice of Sq^* it follows easily that for all $\varepsilon > 0$, there is $X \in Sq^*$ such that $|\mu^n(X) - \mu^n(Y)| < \varepsilon$ and $\overline{Y} \subseteq X$. Thus $\mu^*(\overline{Y}) \leq \mu^n(Y)$ and $\mu^n(\overline{Y}) = \mu^n(Y)$. Now for a contradiction, suppose that $\mu^*(Y) + \varepsilon < \mu^n(Y)$ for some $\varepsilon > 0$. Then we can find $Y_i \in Sq$, $i < \omega$, such that $Y \subseteq \cup_{i < \omega} Y_i$ and $\sum_{i < \omega} \mu^n(Y_i) + \varepsilon < \mu^n(Y)$. But then we can find $X \in Sq^*$ such

that $X \subseteq Y$ and $\mu^n(Y - \overline{X}) < \varepsilon$. Since \overline{X} is compact, there is $K < \omega$ such that $\overline{X} \subseteq \cup_{i < K} Y_i$. Now

$$\mu^n(Y - \overline{X}) + \sum_{i < K} \mu^n(Y_i) < \mu^n(Y),$$

a contradiction by Lemma 2.2.

(ii): By the definition of the outer measure μ^* , and by considering coverings, it is enough to show that if $X \in Sq^*$, then $\mu^*(X - Y) + \mu^*(X \cap Y) = \mu^*(X)$. Clearly $\mu^*(X - Y) + \mu^*(X \cap Y) \geq \mu^*(X)$. On the other hand there are $n \leq 4$ and disjoint $Z_i \in Sq^*$, $i < n$, such that $\cup_{i < n} Z_i \subseteq X - Y$ and $X - Y \subseteq \cup_{i < n} \overline{Z}_i$. Notice that $X \cap Y \in Sq^*$. Now

$$\begin{aligned} \mu^*(X - Y) &\leq \mu^*(\cup_{i < n} \overline{Z}_i) \leq \sum_{i < n} \mu^*(\overline{Z}_i) = \\ &= \sum_{i < n} \mu^*(Z_i) = \sum_{i < n} \mu^n(Z_i) = (\sum_{i < n} \mu^n(Z_i)) + \mu^n(X \cap Y) - \mu^n(X \cap Y) \leq \\ &\leq \mu^n((X \cap Y) \cup \cup_{i < n} Z_i) - \mu^n(X \cap Y) \leq \mu^n(X) - \mu^n(X \cap Y) = \mu^*(X) - \mu^*(X \cap Y). \end{aligned}$$

□

Now as usual one can see that the collection of measurable sets is closed under countable unions and complements. In particular every Borel set ($\subseteq S$) is measurable. For measurable X ($\subseteq S$), we write $\mu(X) = \mu^*(X)$ and then μ is a Borel measure on S . Notice that by Lemma 2.4, $\mu(S) = \mu^*(S) = \mu^n(S) = 1$ (recall equation (7) and that ϕ has norm 1).

When working with the measure and later for constructing rigged Hilbert spaces, it will be convenient to have a sequence of refinements in Sq^* .

Definition 2.5. Let, for each $n < \omega$, $Sq_n \subseteq Sq^*$ be a finite set so that

- (a) the elements of Sq_n are disjoint,
- (b) $\cup_{X \in Sq_n} \overline{X} = S$,
- (c) for all $X \in Sq^*$ and $m < \omega$, there are $m < n < \omega$ and $Z \subseteq Sq_n$ such that $\overline{X} = \cup_{Y \in Z} \overline{Y}$,
- (d) if $n < m < \omega$, $X \in Sq_n$, then there is $Z \subseteq Sq_m$ such that $\overline{X} = \cup_{Y \in Z} \overline{Y}$, and
- (e) if $Y = \{\lambda \in S \mid r_0 < \operatorname{Re}(\lambda) < r_1, r_2 < \operatorname{Im}(\lambda) < r_3\} \in Sq_n$, then $r_0 < r_1$, $r_1 - r_0 < 2M/(n+1)$, $r_2 < r_3$ and $r_3 - r_2 < 2M/(n+1)$.

We let $Sq = \cup_{n < \omega} Sq_n$.

Remark 2.6. Now by Lemma 2.4, it is easy to see that for all $X \subseteq S$,

$$\mu^*(X) = \inf \left\{ \sum_{k=0}^{\infty} \mu^n(X_i) \mid X_i \in Sq, X \subseteq \cup_{i < \omega} \overline{X}_i \right\}.$$

We will use this frequently without mentioning it.

Now $L_2(S, \mu)$ is a Hilbert space and we write $\|\cdot\|_2$ also for the norm of this space. Elements of $L_2(S, \mu)$ are equivalence classes of the equivalence relation $\|f - g\|_2 = 0$ and we write $[f]_2$ for these equivalence classes. Then as usual we write $C(S)$ also for $\{[f]_2 \mid f \in C(S)\}$ and similarly for $D(S)$. For each equivalence class $x \in C(S)$ we fix a continuous $f_x : S \rightarrow \mathbb{C}$ so that $x = [f_x]_2$. We will see later (in Lemma 2.12 below) that this determines f_x on $\sigma(A)$ and what f_x does outside $\sigma(A)$ doesn't

matter. Now later when a function gives a value in $C(S)$ which is an equivalence class, we can think of it also as a continuous function from S to \mathbb{C} . Thus $L_2(S, \mu)$ and $L_2(\sigma(A), \mu \upharpoonright \sigma(A))$ are naturally isomorphic since every $f \in C(\sigma(A))$ extends to a (bounded uniformly) continuous function $f' \in C(S)$.

Our next step is to show that the mappings $F^m : D(S) \rightarrow H^m$ and $G : D(S) \rightarrow H$ are not only vector space homomorphisms, but also isometries under the L_2 norm, which will let us define $F^m([f]_2) = F^m(f)$ and $G([f]_2) = G(f)$ to get isometric embeddings of $L_2(S, \mu)$ into H^m and H respectively.

Lemma 2.7. *For all $P \in \mathbb{C}[X, Y]$, $\|F(f_P)\|_2 = \|f_P\|_2$.*

Proof. This proof is essentially the same as the proof that shows that continuous functions are Riemann integrable. Let $1 > \varepsilon > 0$, $m = 1 + \sup\{|f_P(\lambda)| \mid \lambda \in S\}$ and choose $n > 0$ so that if $|\lambda - \lambda'| < 4M/n$ (which happens for all $\lambda, \lambda' \in \overline{Y}$, for $Y \in Sq_n$), then $|f_P(\lambda) - f_P(\lambda')| < \varepsilon/8m$. Then we can choose $Z \in \mathcal{U}$ such that for all $N \in Z$ and $Y \in Sq_n$, both $|\mu(Y) - \mu_N(Y)|$ and $|\mu(\overline{Y}) - \mu_N(\overline{Y})|$ are $< \varepsilon/(4m^2|Sq_n|)$. Notice that if we let X be the union of all $\overline{Y} - Y$, $Y \in Sq_n$, then $\mu(X) = 0$, and

$$\left\| \sum_{\lambda_N(k) \in X} \xi_N(k) f_P(\lambda_N(k)) u_N(k) \right\|_2^2 \leq \mu_N(X) m^2 < \varepsilon/4.$$

Then

$$\begin{aligned} |(\|F_N(f_P)\|_2)^2 - (\|f_P\|_2)^2| &= \left| \sum_{k < D_N} \xi_N(k)^2 |f_P(\lambda_N(k))|^2 - \int_S |f_P(x)|^2 d\mu \right| \\ &= \left| \sum_{\lambda_N(k) \in X} \xi_N(k)^2 |f_P(\lambda_N(k))|^2 + \sum_{Y \in Sq_n} \sum_{\lambda_N(k) \in Y} \xi_N(k)^2 |f_P(\lambda_N(k))|^2 \right. \\ &\quad \left. - \sum_{Y \in Sq_n} \int_Y |f_P(x)|^2 d\mu \right| \\ &\leq \varepsilon/4 + \sum_{Y \in Sq_n} \left| \sum_{\lambda_N(k) \in Y} \xi_N(k)^2 |f_P(\lambda_N(k))|^2 - \int_Y |f_P(x)|^2 d\mu \right|, \end{aligned}$$

and for each $Y \in S_{q_n}$

$$\begin{aligned}
& \left| \sum_{\lambda_N(k) \in Y} \xi_N(k)^2 |f_P(\lambda_N(k))|^2 - \int_Y |f_P(x)|^2 d\mu \right| \\
& \leq \int_Y \left(\frac{\sum_{\lambda_N(k) \in Y} \xi_N(k)^2 ||f_P(\lambda_N(k))|^2 - |f_P(x)|^2|}{\mu(Y)} \right. \\
& \quad \left. + \left| \frac{\sum_{\lambda_N(k) \in Y} \xi_N(k)^2 |f_P(x)|^2}{\mu(Y)} - |f_P(x)|^2 \right| \right) d\mu \\
& \leq \int_Y \left(\frac{\mu_N(Y) 2m \frac{\varepsilon}{8m}}{\mu(Y)} + \left| \frac{\mu_N(Y)}{\mu(Y)} - 1 \right| m^2 \right) d\mu \\
& = \mu_N(Y) \frac{\varepsilon}{4} + |\mu_N(Y) - \mu(Y)| m^2 \\
& < \mu_N(Y) \frac{\varepsilon}{4} + \frac{\varepsilon}{4|S_{q_n}|}.
\end{aligned}$$

Putting this together (and recalling $\mu_N(S) = 1$), we get

$$|(\|F_N(f_P)\|_2)^2 - (\|f_P\|_2)^2| \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon$$

from which the claim follows. \square

Corollary 2.8. $F^m([f]_2) = F^m(f)$ and $G([f]_2) = G(f)$ are isometric embeddings of $L_2(S, \mu)$ into H^m and H respectively.

Proof. By the above, and noting that $G = (G^m)^{-1} F^m$, we see that F^m and G are well-defined on the $\|\cdot\|_2$ equivalence classes of $D(S)$. Now we notice that $\mathbb{C}[X, Y]$ is closed under the function $f \mapsto f^*$, where $f^*(\lambda) = \overline{f(\lambda)}$. And thus by Stone-Weierstrass, $D(S)$ is dense in $C(S)$ even in the uniform convergence topology (we will need this later). And since $C(S)$ is dense in $L_2(S, \mu)$, see, e.g., [Bo] Corollary 4.2.2, $D(S)$ is dense in $L_2(S, \mu)$. Thus we can extend the functions F^m and G to all of $L_2(S, \mu)$ by continuity. \square

Remark 2.9. Recall that on $D(S)$, we have $A_D(f_P)(\lambda) = \lambda f_P(\lambda)$ and $A_D^*(f_P)(\lambda) = \overline{\lambda} f_P(\lambda)$. These can also be defined on the $\|\cdot\|_2$ equivalence classes by $A_D([f]_2) = [A_D(f)]_2$ (and similarly for A_D^*), and then extend it to all of $L_2(S, \mu)$ by continuity.

Corollary 2.10. F^m is an isometric isomorphism from $L_2(S, \mu)$ to H^{Im} and $A^m \upharpoonright H^{Im} = F^m A_D (F^m)^{-1}$. Thus also $G = (G^m)^{-1} F^m$ is an isometric isomorphism from $L_2(S, \mu)$ to H and $A = G A_D G^{-1}$. \square

We have been looking at $L_2(S, \mu)$, where S was chosen to be a 'big enough' set. However, μ only puts weight on part of this set, and next we will show that that set is exactly the spectrum of A . As pointed out above, there is a rational number $0 < q < M$ such that for all $N < \omega$, the norm of A_N is $< q$ since the norm of A_N is at most the norm of A . Thus if we let $S^- = \{\lambda \in \mathbb{C} \mid -M < \operatorname{Re}(\lambda) < M, -M < \operatorname{Im}(\lambda) < M\}$ then $\mu(S^-) = \mu(S) = 1$.

Definition 2.11. Let $\sigma(A)$ be the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ does not have a bounded inverse. Notice that $\lambda \in \sigma(A)$ iff for all $\varepsilon > 0$, there is $\psi \in H$ such that $\|\psi\|_2 = 1$ and $\|A(\psi) - \lambda\psi\|_2 < \varepsilon$. We let $\sigma^*(A)$ be the set of all $\lambda \in S^-$ such that every open neighbourhood of λ that is contained in S^- has measure > 0 .

Lemma 2.12. $\sigma(A) = \sigma^*(A)$.

Proof. By Corollary 2.10, it is enough to show that $\sigma(A_D) = \sigma^*(A)$, where $\sigma(A_D)$ is defined as $\sigma(A)$ was. Also if $\lambda \notin S^-$, λ does not belong to either of the sets. So suppose that $\lambda \in S^-$.

We write $B_r(\lambda)$, r a positive real number, for the set of all $\lambda' \in \mathbb{C}$ such that $|\lambda - \lambda'| < r$. Now if for some r , $B_r(\lambda) \subseteq S^-$ and $\mu(B_r(\lambda)) = 0$, then it is easy to see that $\|A_D(f) - \lambda f\|_2 \geq r$ for all f with norm 1 and thus $\lambda \notin \sigma(A_D)$. On the other hand, if for all positive r such that $B_r(\lambda) \subseteq S^-$, $\mu(B_r(\lambda)) > 0$, then for all such r , we can find $f \in C(S)$ such that $\|f\|_2 = 1$ and $f(x) = 0$ for all $x \in S - B_r(\lambda)$. Then $\|A_D(f) - \lambda f\|_2 < r$. Thus $\lambda \in \sigma(A_D)$. \square

Corollary 2.13. $\mu(\sigma(A)) = 1$ and $\mu(S - \sigma(A)) = 0$.

Proof. $\sigma(A) = \sigma^*(A)$ is a closed set and thus measurable and thus the claims follow from Lemma 2.12 and $\mu(S) = \mu(S^-) = 1$. \square

Looking back at the constructions of this and the past section, one notes that the finite dimensional spaces H_N depend on the choice of ϕ , and thus it is likely that one gets different H^u and H^m depending on the choice of cyclic vector. We have not looked into this, as the whole of these spaces are too big for our interest. As every H_N has a part, where the operator is replaced by a wraparound-mapping, these erroneous parts will show up in the ultraproduct, and we are really interested only in parts stemming from the original operator. There are subspaces where these errors disappear, and one of them is H^{Im} , which is canonical, as we have showed it to be isometrically isomorphic to H . A natural question arises:

Open question 3. Is there a natural characterization for the subspaces H^m where the errors of the A_N 's disappear?

As for the canonicity of our measure μ we have the following:

Corollary 2.14. *The measure μ constructed is independent of the choice of cyclic vector. In particular all measures constructed in the above manner will agree on measure zero sets, and thus agree on which lines are nice.*

Proof. This follows from Corollary 2.10 and the uniqueness of the spectral measure (see, e.g., section VII.2 of [RS]). \square

However, note that in general $\sigma(A) \neq \sigma(A^m)$, as the latter can be strictly larger. Thus we are not considering elementary embeddings into the (metric) ultraproduct, but only embeddings. This is enough for our purposes, as we only use the construction for calculating equations.

3. SELF-ADJOINT OPERATORS

In this section we show that the constructions from Sections 1 and 2 can be generalized to self adjoint operators.

We will need the following facts, some of which seem hard to find in the literature (in particular item (iv) of the fact). Our version is not the best possible.

Fact 3.1. *Let H be a separable Hilbert space and B a self-adjoint, U a unitary operator on H and I the identity operator.*

- (i) $\max\{|\lambda| \mid \lambda \in \sigma(B)\} = \|B\|$, e^{iB} is unitary and $\sigma(e^{iB}) = \{e^{i\lambda} \mid \lambda \in \sigma(B)\}$.
- (ii) Let r and q be reals. $e^{i(B+rI)} = e^{iB} \circ e^{irI}$, $e^{iq} \in \sigma(e^{iB})$ iff $e^{i(q+r)} \in \sigma(e^{i(B+rI)})$ and $\lambda \in \sigma(B)$ iff $\lambda + r \in \sigma(B + rI)$.
- (iii) Suppose H_0 is a complete subspace of H , H_1 its orthogonal complement and A_i are bounded operators on H_i , $i < 2$. Then there is a unique operator A on H such that $A \upharpoonright H_i = A_i$ for both $i < 2$ and it is bounded and $\sigma(A) = \sigma(A_0) \cup \sigma(A_1)$. If in addition both operators A_i are self-adjoint, then also A is self-adjoint and if both are unitary, then also A is unitary.
- (iv) If $0 \leq r < \pi/2$ and for all $\lambda \in \sigma(U)$, $\lambda = e^{iq}$ for some $q \in [-r, r]$, then there is a unique self-adjoint operator A such that $U = e^{iA}$ and $\sigma(A) \subseteq [-r, r]$.

Proof. (i): The first claim is the spectral radius formula (for a proof see, e.g., Chapter VI of [RS]) the others are proved via polynomial approximations, see Chapter VII of [RS].

(ii): The first claim follows from the fact that B and rI commute. The rest is easy.

(iii): Obviously A must be such that if $v \in H_0$ and $u \in H_1$, then $A(v + u) = A_0(v) + A_1(u)$ and this is a bounded operator. Also clearly $\sigma(A_0) \cup \sigma(A_1) \subseteq \sigma(A)$ and since spectra are closed the other direction follows by an easy calculation. If A_0 and A_1 are self adjoint, then for all $v_0, v_1 \in H_0$ and $u_0, u_1 \in H_1$,

$$\begin{aligned} \langle A(v_0 + v_1) | u_0 + u_1 \rangle &= \langle A_0(v_0) | u_0 \rangle + \langle A_1(v_1) | u_1 \rangle = \\ &= \langle v_0 | A_0(u_0) \rangle + \langle v_1 | A_1(u_1) \rangle = \langle v_0 + v_1 | A(u_0 + u_1) \rangle \end{aligned}$$

where the first and the third identity follow from the choice of H_0 and H_1 . Finally if A_0 and A_1 are unitary, an easy calculation shows that A is norm preserving.

(iv): Let $U' = e^{i\pi}U$. Then U' is unitary, and if $\lambda \in \sigma(U')$, then $\lambda = e^{iq}$ for some $q \in [-r + \pi, r + \pi]$. Now by [St, Theorem 8.4], there is a unique self-adjoint operator A' such that $U' = e^{i2\pi A'}$ and the spectrum of A' is a subset of $[0, 1]$ and 0 is not in the point spectrum of A' . By item (i), the spectrum of $2\pi A'$ is a subset of $[\pi - r, \pi + r]$. Let $A = 2\pi A' - \pi I$. Then $e^{iA} = U$, $\sigma(A) \subseteq [-r, r]$ and the uniqueness follows from the uniqueness of A' and (ii). \square

From now on, in this section we assume that $Re(e^i) \in NI$. If this is not true, we can replace 1 by a real q close to 1 such that $Re(e^{iq}) \in NI$ and work with it.

Let B be a bounded self-adjoint operator on a separable Hilbert space H . It will turn out that it does not matter whether we look at B or rB for some positive real r , and thus we may assume that the norm of B is < 1 . Also we may assume that $B(v) \neq 0$ for all $v \in H - \{0\}$ because we can always restrict B to the orthogonal complement of its kernel (which is closed under B since B is self-adjoint). Let $A = e^{iB}$. Then A is unitary and not the identity by Fact 3.1. Also by the same fact, if $\lambda \in \sigma(A)$, then $\lambda = e^{iq}$ for some $q \in [-1, 1]$.

In the proof of the following lemma, the assumption that the norm of B is small plays an important role, stemming from Fact 3.1. We will need the uniqueness of

the self-adjoint operator in point (iv) of the fact, and for that we need to avoid a 'wraparound'. Our formulation of the fact allows for B to have norm smaller than $\pi/2$, and we have chosen 1 as it is convenient in calculations.

Lemma 3.2. *If H_0 is a complete subspace of H and closed under A and A^* , then it is closed under B .*

Proof. Suppose not. Let H_1 be the orthogonal complement of H_0 in H . Then H_1 is closed under A and A^* and there are self-adjoint operators B_0 in H_0 and B_1 in H_1 such that $e^{iB_0} = A \upharpoonright H_0$ and $e^{iB_1} = A \upharpoonright H_1$. But then by Fact 3.1 (iii) there is a self-adjoint operator B' in H such that $e^{iB'} = A$ (because $e^{iB'} \upharpoonright H_0 = A \upharpoonright H_0$ and $e^{iB'} \upharpoonright H_1 = A \upharpoonright H_1$ and A is uniquely determined by $A \upharpoonright H_0$ and $A \upharpoonright H_1$) and $B' \neq B$. This contradicts Fact 3.1 (iii). \square

By Lemma 3.2, if we may assume that H has a cyclic vector with respect to the operator A as in Section 1, we have spaces H_N , operators A_N , functions G , G^m , F^m and F_N , a spectral measure μ etc. as in the first two sections. Notice that each A_N is a unitary operator and thus for all $N < \omega$ and $n < D_N$, $\lambda_N(n) = e^{iq}$ for some $q \in]-\pi, \pi]$. It follows that $\mu(S - S_0) = 0$ when S_0 is the set $\{e^{iq} \mid q \in]-\pi, \pi]\}$.

We define operators B_N as follows: $B_N(u_N(n)) = qu_N(n)$ if $A_N(u_N(n)) = e^{iq}u_N(n)$ and $q \in]-\pi, \pi]$. We write B^m for the metric ultraproduct (with the filter \mathcal{U} as in the previous sections) of the operators B_N . Then $A_N = e^{iB_N}$ for all $N < \omega$ and since the operators B_N are uniformly bounded (by π) it is easy to see that $A^m = e^{iB^m}$. Also each B_N is self-adjoint and thus also B^m is self-adjoint. We want to show that

Theorem 3.3. $B^m \upharpoonright H^{Im} = G^m B (G^m)^{-1}$.

This follows immediately from Fact 3.1 (iv) if we know that H^{Im} is closed under B^m . So what we need to show is the following:

Lemma 3.4. H^{Im} is closed under B^m .

Proof. Suppose not. Note that from Corollaries 1.7 and 2.8 it follows that $H^{Im} := \text{rng}(G^m) = \text{rng}(F^m)$. Let H_0 be the closure of H^{Im} under B^m i.e. the complete subspace of H^m generated by the set

$$\{(B^m)^n(F^m(f_P)) \mid P \in \mathbb{C}[X, Y], n < \omega\}$$

(keep in mind that B^m is bounded). Then H^m is a complete subspace of H_0 , closed under A^m and A^{m*} , so we get a contradiction as in the proof of Lemma 3.2, unless the norm is too large to guarantee uniqueness. But this we can rule out by showing the following:

Claim 3.4.1. *The operator norm of $B^m \upharpoonright H_0$ is ≤ 1 .*

Proof. For a contradiction suppose that there is $\varepsilon > 0$, $k < \omega$ and $P \in \mathbb{C}[X, Y]$ such that $\|B^m((B^m)^k(F^m(f_P)))\| > (1 + \varepsilon)\|(B^m)^k(F^m(f_P))\|$. Since A is not the identity, $\mu(\{1\}) < 1$. So there are $\delta > 0$ and $V \subseteq \sigma(A)$ such that $\mu(V) > \delta$, for all $\lambda \in V$, $|f_P(\lambda)| > \delta$ and if $\lambda = e^{iq} \in V$, then $|q| > \delta$.

For all $N < \omega$, let H_N^0 be the subspace of H_N generated by those $u_N(n)$ for which $\lambda_N(n) = e^{iq}$ for some $q \in [-\pi, \pi]$ such that $|q| > 1$ and let H_N^1 be the subspace of H_N generated by those $u_N(n)$ for which $\lambda_N(n) = e^{iq}$ for some $q \in [-\pi, \pi]$ such

that $|q| \leq 1$. Then H_N^1 is the orthogonal complement of H_N^0 . Further let H_N^2 be the subspace of H_N generated by those $u_N(n)$ for which $\lambda_N(n) \in V$. Let $v_i(N)$ be the projection of $F_N(f_P)$ to H_N^i , $i < 3$. Let S^* be the set of all $\lambda \in S$ such that $\operatorname{Re}(\lambda) \geq \operatorname{Re}(e^i)$. Notice that since $\operatorname{Re}(e^i) \in NI$ (see the assumption immediately after the proof of Fact 3.1), $\mu(S^*)$ and $\mu(S - S^*)$ are ultralimits of $\mu_N(S^*)$ and $\mu_N(S - S^*)$. Let Z be the supremum of all $|f_P(\lambda)|$ for $\lambda \in S$.

Now

$$(a) \quad \|(B_N)^k(F_N(f_P))\| \geq \|(B_N)^k(v_2(N))\| = \\ = \|(B_N)^k \sum_{\lambda_N(n) \in V} \xi_N(n) f_P(\lambda_N(n)) u_N(n)\| \geq \delta^{k+3/2}$$

in a set of N 's that belongs to \mathcal{U} ,

$$(b) \quad \|B_N((B_N)^k(F_N(f_P)))\| = \|(B_N)^{k+1}(v_0(N) + v_1(N))\| \leq \\ \pi^{k+1} \sqrt{\mu_N(S - S^*)} Z + \|v_1(N)\|$$

in a set of N 's that belongs to \mathcal{U} ,

$$(c) \quad \mu_N(S - S^*) \leq (\varepsilon \delta^{k+3/2} / (2\pi^{k+1} Z))^2 \text{ in a set of } N\text{'s that belong to } \mathcal{U} \text{ (since } \mu(S - S^*) = 0 \text{ by the assumption } \|B\| \leq 1 \text{ and the construction of the measure for } A).$$

By putting these together, we get

$$\|B_N((B_N)^k(F_N(f_P)))\| \leq \pi^{k+1} \sqrt{\mu_N(S - S^*)} Z + \|v_1(N)\| \leq (\frac{2}{\varepsilon} + 1) \|(B_N)^k(F_N(f_P))\|$$

in a set of N 's that belongs to \mathcal{U} , which is a contradiction.

□ Claim 3.4.1

□

Now we can define an operator B_D on $L_2(S, \mu)$ the natural way: For $f \in C(S)$, we let $B_D(f)(\lambda) = K_B(\lambda)f(\lambda)$, where $K_B(\lambda) = q$ if $q \in]-\pi, \pi]$ and $\lambda = pe^{iq}$ for some real number p . Now it is easy to see that $e^{iB_D} = A_D$ and thus by Fact 3.1, $B^m \upharpoonright H^{Im} = F^m B_D(F^m)^{-1}$.

If we want for B exactly the same setup as for A we can do the following: Let $S_B =]-\pi, \pi]$ and define a Borel measure μ_B on S_B by letting $\mu_B(X) = \mu(K_B^{-1}(X))$. Then we can define an operator B'_D on $L_2(S_B, \mu_B)$ so that for all $f \in C(S_B)$, $B'_D(f)(q) = qf(q)$. Finally we notice that if we define a continuous $F_B : L_2(S, \mu) \rightarrow L_2(S_B, \mu_B)$ so that for $f \in C(S)$, $F_B(f)(q) = f(e^{iq})$ we get an isometric isomorphism and $B'_D = F_B B_D F_B^{-1}$.

4. RIGGED HILBERT SPACES

Rigged Hilbert spaces are a way of accounting for objects that don't fit into a Hilbert space, such as Dirac delta functions. A rigged Hilbert space consists of a Hilbert space H and a subspace Φ (with a finer norm topology than the one given by the norm in H). One then has

$$\Phi \subset H \subset \Phi^*,$$

where Φ^* is the dual space of Φ . The elements in Φ^* are antilinear functionals, and if Φ was suitably chosen with respect to a cyclic, essentially self adjoint operator

B , then the functionals act as generalized eigenvectors corresponding to values in the spectrum of B (see, e.g., [Ma]).

In this section we set out to find something similar to Φ^* by considering different norms in our ultraproduct space H^u , and finding extensions of H^{Im} . Here, as well as in the next section, it does not matter whether we look at a self-adjoint bounded B or e^{iB} , and thus we return to the assumptions of the first two sections, i.e., that A is bounded, $A \circ A^* = A^* \circ A = r_A I$ and ϕ is a cyclic vector. Also we let H_N , A_N etc. be as in these sections.

We will look at two norms, that end up being essentially an L_∞ -norm and an L_1 -norm. The latter of these will be called $\|\cdot\|_0$ in order to avoid confusion. Strictly speaking both of these may be only seminorms in the finite dimensional spaces H_N , but we will call them norms for simplicity. In the ultraproduct we will fraction out all vectors to which the seminorm gives an infinitesimal value, and then the seminorms become norms, and we obtain spaces H_0^m and H_∞^m somewhat similarly to how we obtained H^m for the $\|\cdot\|_2$ -norm before. Then we will show how generalized distributions embed into H_0^m such that values of the distribution can be calculated via a pairing function resembling an inner product.

Definition 4.1. Consider the following functions from H_N to \mathbb{C} : Let $X \subseteq \mathbb{C}$ be a closed set such that $X \subseteq S$. Then

$$\left\| \sum_{n=0}^{D_N-1} a_n u_N(n) \right\|_\infty^X = \sup \{ \xi_N(n)^{-1} |a_n| \mid \lambda_N(n) \in X \},$$

where by 0^{-1} we mean 0 and

$$\left\| \sum_{n=0}^{D_N-1} a_n u_N(n) \right\|_0 = \sum_{n=0}^{D_N-1} \xi_N(n) |a_n|.$$

Keep in mind that for all N and n , $\xi_N(n)$ is a non-negative real number ≤ 1 .

When constructing H^m from the $\|\cdot\|_2$ -norm we just fractioned out infinitesimals and discarded elements of infinite length. For our other norms we need to be a bit more careful. We start with $\|\cdot\|_0$. As with the Hilbert space norm, this gives first a function from H^u to \mathbb{C}^u , denoted by $\|\cdot\|_0^u$ (as before), and an equivalence relation \sim_0 .

Definition 4.2. Let $n < \omega$, $Y \subseteq Sq_n$, $R_0^Y = \cup_{R \in Y} R$, $S_0^Y = \cup_{R \in Y} \overline{R}$, $R_1^Y = \cup_{R \in Sq_n - Y} R$ and $S_1^Y = \cup_{R \in Sq_n - Y} \overline{R}$. We say that Y is *good* if the distance between $\sigma(A)$ and the closure of S_1^Y is > 0 (then $\sigma(A) \subseteq S_0^Y$). We say that

$$\left(\sum_{k=0}^{D_N-1} a_k^N u_N(k) \right)_{N < \omega} / \mathcal{U} \in H^u$$

(and $(\sum_{k=0}^{D_N-1} a_k^N u_N(k))_{N < \omega}$) is *0-good* if for all good Y

$$\left(\sum_{\lambda_N(k) \in S_1^Y} \xi_N(k) |a_k^N| \right)_{N < \omega} / \mathcal{U} \sim 0.$$

Then we let H^{m0} be the set of all 0-good $f/\mathcal{U} \in H^u$ such that $\|f/\mathcal{U}\|_0^u < n$ for some natural number n . And we let $H_0^m = H^{m0} / \sim_0$. Finally, we define the norm

$\|\cdot\|_0$ on H_0^m as before: $\|f/\sim_0\|_0 = q$ if $q \in \mathbb{C}$ is the unique element such that for all $\varepsilon > 0$,

$$\{N < \omega \mid \|f(N)\|_0 - q < \varepsilon\} \in \mathcal{U}.$$

With this norm H_0^m is a Banach space (addition and scalar multiplication are defined as before). In order to simplify the notation, for all $f \in \Pi_{N \in \omega} H_N$, we write $[f]_0 = f/\mathcal{U}/\sim_0$. Notice that if $f, g \in H^u$ and $\|f - g\|_0^u$ is infinitesimal and f is 0-good, then so is g . Notice that for all $f \in C(S)$, $F(f) \in H^{m0}$ and $\|f\|_1 = \|[F(f)]_0\|_0$: The first of the claims is immediate since f is bounded and $\mu(S)$ finite, and the second follows from the fact that f is uniformly continuous and thus Riemann integrable.

Let us then look at the norm $\|\cdot\|_\infty$. First we will need a modified version of the sup-norm on $C(S)$:

Definition 4.3. We define a seminorm $\|f\|_\infty^\sigma = \sup\{|f(\lambda)| \mid \lambda \in \sigma(A)\}$ on $C(S)$ and write $L_\infty(S, \mu)$ for $C(S)/\sim_\infty^\sigma$, where $f \sim_\infty^\sigma g$ if $\|f - g\|_\infty^\sigma = 0$.

Then $\|\cdot\|_\infty^\sigma$ is a norm on $L_\infty(S, \mu)$.

Definition 4.4. For all $\varepsilon > 0$, we let X_ε be the set of all $\lambda \in \mathbb{C}$ whose distance to $\sigma(A)$ is $\leq \varepsilon$ and we let $\|(v_N)_{N < \omega}/\mathcal{U}\|_\infty^\varepsilon$ be the unique $q \in \mathbb{C}$ such that for all $\delta > 0$

$$\{N < \omega \mid \|v_N\|_\infty^{X_\varepsilon} - q < \delta\} \in \mathcal{U}$$

if there is such a q and otherwise the value is ∞ (i.e. we do as before).

Now let $v = (\sum_{k=0}^{D_N-1} a_k^N u_N(k))_{N < \omega}/\mathcal{U} \in H^u$. Then we let $\|v\|_\infty = \lim_{\varepsilon \rightarrow 0} \|v\|_\infty^\varepsilon$ if there are $p < \omega$ and $Z \in \mathcal{U}$ for which $|a_k^N(\xi_N(k))^{-1}| < p$ for all $N \in Z$ and $k < D_N$ and otherwise we let $\|v\|_\infty = \infty$. Notice that thus having bounded $\|\cdot\|_\infty$ -norm in H^u requires the $\|\cdot\|_\infty^S$ -norm to be bounded in a \mathcal{U} -large set of H_N s although the value is determined by what happens 'close to $\sigma(A)$ '. We then let $H^{m\infty}$ be the set of all $v \in H^u$ for which $\|v\|_\infty < \infty$ and $H_\infty^m = H^{m\infty}/\sim_\infty$, where \sim_∞ is the relation $\|v - u\|_\infty = 0$. We write $[f]_\infty$ for $f/\mathcal{U}/\sim_\infty$.

Lemma 4.5. (i) For all $f \in C(S)$, $F(f) \in H^{m\infty}$ and $\|f\|_\infty^\sigma = \|F(f)\|_\infty$.
(ii) Every element of $H^{m\infty}$ has finite $\|\cdot\|_2$ -norm, and for all $v \in H^u$, $\|v\|_0^u \leq \|v\|_2$. For $f \in C(S)$, $\|F(f)\|_0 \leq \|F(f)\|_2 \leq \|F(f)\|_\infty$.

Proof. (i) The first claim is clear since f is bounded. For the second, we notice that f is uniformly continuous and thus the direction $\|f\|_\infty^\sigma \geq \|F(f)\|_\infty$ follows easily. The direction $\|f\|_\infty^\sigma \leq \|F(f)\|_\infty$ follows from the equality of $\sigma(A)$ and $\sigma^*(A)$ and the continuity of f .

(ii) The first claim is a straight forward calculation, the second follows from Hölder's inequality, as $\mu(S) = 1$. The chain of inequalities follows from the classical result. \square

Let $H_*^m = \bigcup_{k \in \{0, 2, \infty\}} H_k^m$. We define a pairing $\langle \cdot | \cdot \rangle$ on H_*^m the same way as before: For $[f]_k, [g]_r \in H_*^m$, $\langle [f]_k | [g]_r \rangle = q$ if $q \in \mathbb{C}$ is such that for all $f' \in [f]_k$ and $g' \in [g]_r$, $\langle f' | g' \rangle^u$ is infinitely close to q . If no such q exists we let $\langle [f]_k | [g]_r \rangle = \infty$ (meaning that $\langle [f]_k | [g]_r \rangle$ is not well-defined).

Lemma 4.6. If $x \in H_\infty^m$ and $y \in H_0^m$, then $\langle x | y \rangle \neq \infty$.

Proof. Let $x = [(x_N)_{N < \omega}]_\infty$, where

$$x_N = \sum_{n=0}^{D_N-1} a_n^N u_N(n)$$

and $y = [(y_N)_{N < \omega}]_0$, where

$$y_N = \sum_{n=0}^{D_N-1} b_n^N u_N(n)$$

and let $q > \|x\|_\infty$ and $r > \|y\|_0$. Let $p < \omega$, be such that for all $N < \omega$ and $n < D_N$, $|a_n^N(\xi_N(n))^{-1}| < p$. It is enough to show that

$$|\langle (x_N)_{N < \omega} | (y_N)_{N < \omega} \rangle| < qr.$$

Let $\varepsilon > 0$ be such that $\|x\|_\infty^\varepsilon < q$. Then we choose $n < \omega$ so that there is a good $Y \subseteq S_{q_n}$ such that if as before we denote $S_0^Y = \cup_{R \in Y} \overline{R}$ and $S_1^Y = \cup_{R \in S_{q_n-Y}} \overline{R}$, then $S - X_\varepsilon \subseteq S_1^Y$. Then

$$\begin{aligned} |\langle x_N | y_N \rangle| &\leq \sum_{n=0}^{D_N-1} |a_n^N| |b_n^N| < \\ &< \sum_{\lambda_N(n) \in S_0^Y} \xi_N(n) q |b_n^N| + \sum_{\lambda_N(n) \in S_1^Y} \xi_N(n) p |b_n^N| \leq \\ &\leq q \sum_{\lambda_N(n) \in S_0^Y} \xi_N(n) |b_n^N| + p \sum_{\lambda_N(n) \in S_1^Y} \xi_N(n) |b_n^N| < qr + p \sum_{\lambda_N(n) \in S_1^Y} \xi_N(n) |b_n^N| \end{aligned}$$

in a set of $N < \omega$ that belongs to \mathcal{U} . Now the claim follows from the assumption that $(y_N)_{N < \omega}/\mathcal{U}$ is 0-good. \square

Next we set out to find distributions as vectors in our space H^u .

Definition 4.7. We say that an antilinear map $f : D(S) \rightarrow \mathbb{C}$ is a *generalized distribution* if it is bounded in the sense of $\|\cdot\|_\infty^\sigma$ i.e. there is $K < \omega$ such that $|f(x)| \leq K \|x\|_\infty^\sigma$. Then f extends to $C(S)$ by continuity and thus by generalized distributions we actually mean bounded antilinear maps $f : C(S) \rightarrow \mathbb{C}$.

We now fix a generalized distribution θ for the remainder of this section.

In general a distribution emerges through its test functions, and we will construct test functions picking out rectangles in the spectrum, similarly to how the measure was built. This will let us 'copy over' the values of θ to our vector. However, we need to be careful to make sure our rectangles are well behaved.

Definition 4.8. We return to the lines I_r and J_r and to the sets I_r^ε and J_r^ε from Section 2. For $X \subseteq \mathbb{C}$, we say that $f \in C(S)$ is an *X-function* if for all $x \in S$, $|f(x)| \leq 1$ and for all $x \in S - X$, $f(x) = 0$. Notice that we do not require that $X \subseteq S$ but still $\text{dom}(f) = S$.

We say that a line I_r is *good* (for θ) if for all $\delta > 0$ there is $\varepsilon > 0$ such that for all I_r^ε -functions f , $|\theta(f)| < \delta$. J_r being good is defined similarly.

Lemma 4.9. *There are at most countably many $r \in [-M, M]$ such that I_r is not good, and the same is true for the lines J_r .*

Proof. Suppose not. Then there are $\delta > 0$ and r_i , $i < \omega_1$, such that for all $\varepsilon > 0$ and $i < \omega_1$, there is an $I_{r_i}^\varepsilon$ -function f_i^ε such that $|\theta(f_i^\varepsilon)| > \delta$. W.l.o.g. we may assume that for all $i < \omega$, $r_i < r_{i+1}$. Then for all $i < \omega$, we choose $\varepsilon_i > 0$ so that for all $i < \omega$, $r_i + \varepsilon_i < r_{i+1} - \varepsilon_{i+1}$. Also w.l.o.g. we may assume that for all $i < \omega$, the direction of $\lambda_i = \theta(f_i^{\varepsilon_i})$ is roughly the same, e.g. $\operatorname{Re}(\lambda_i) \geq \operatorname{Im}(\lambda_i) \geq 0$. Let $f = \sum_{i < \omega} (1/(i+1)) f_i^{\varepsilon_i}$. Clearly $f \in C(S)$ and $\|f\|_\infty \leq 1$ but $\theta(f) = \infty$, a contradiction. \square

Definition 4.10. We say that I_r is *very good* (for θ) if it is good and for all $\delta > 0$, there is $\varepsilon > 0$ such that $\mu^n(I_r^\varepsilon) < \delta$. J_r being very good is defined similarly. We say that $\varepsilon > 0$ is *nice* for r if both $r - \varepsilon$ and $r + \varepsilon$ are nice or $< -M$ or $> M$.

Lemma 4.11. (i) *There are at most countably many $r \in [-M, M]$ such that I_r is not very good and the same is true for lines J_r .*
(ii) *For all r there are at most countably many $\varepsilon > 0$ that are not nice for r .*
(iii) *If $\varepsilon > 0$ is nice for r , then $\mu(I_r^\varepsilon) = \mu^n(I_r^\varepsilon)$ and $\mu(J_r^\varepsilon) = \mu^n(J_r^\varepsilon)$.*

Proof. (ii): This is immediate by Lemma 2.2.

(i): This is immediate by Lemma 2.2 and Lemma 4.9.

(iii): This can be proved as Lemma 2.4 (i) was proved. \square

Definition 4.12. For all $n < \omega$, we choose $r_i^n \in [-M, M]$, $i < 2^{n+2} + 1$, so that

- (i) $r_i^n < r_{i+1}^n$ and $r_{i+1}^n - r_i^n < 2M/(n+1)$,
- (ii) $I_{r_i^n}$ and $J_{r_i^n}$ are very good for θ ,
- (iii) $r_0^0 = -M$ and $r_4^0 = M$,
- (iv) for all $i < 2^{n+2} + 1$ there is $j < 2^{n+3} + 1$ such that $r_i^n = r_j^{n+1}$.

When we talk about $I_{r_i^n}^\varepsilon$ or $J_{r_i^n}^\varepsilon$ we assume that ε is small enough so that for all $j < 2^{n+2}$, $r_j^n + \varepsilon < r_{j+1}^n - \varepsilon$.

For all $n < \omega$, $i, j < 2^{n+2}$ and $\varepsilon > 0$, we let $B_{ij}^n = \{\lambda \in S \mid r_i^n < \operatorname{Re}(\lambda) < r_{i+1}^n, r_j^n < \operatorname{Im}(\lambda) < r_{j+1}^n\}$ and $R_\varepsilon^n = \cup_{i < 2^{n+2} + 1} (I_{r_i^n}^\varepsilon \cup J_{r_i^n}^\varepsilon)$. We say that $\varepsilon > 0$ is *very nice* for $n < \omega$, if it is nice for every r_i^n , $i < 2^{n+2}$. As in Lemma 4.11, we can see that $\mu(B_{ij}^n) = \mu^n(B_{ij}^n)$, that if ε is very nice for n , then $\mu(R_\varepsilon^n) = \mu^n(R_\varepsilon^n)$, and that excluding countably many ε , every ε is very nice for n .

Lemma 4.13. *For all $n < \omega$ and $\delta > 0$, there is $\varepsilon > 0$ such that $\mu(R_\varepsilon^n) < \delta$ and for all R_ε^n -functions f , $|\theta(f)| < \delta$.*

Proof. For all R_ε^n -functions f there are $I_{r_i^n}^\varepsilon$ - and $J_{r_i^n}^\varepsilon$ -functions f_i and g_i , $i < 2^{n+2}$, such that $f = \sum_{i < 2^{n+2}} (f_i + g_i)$. The only nontrivial part here are the 'crossings' between I and J lines, and there one can define f_i to be any continuous continuation from the values it has on the upper and lower edge of the crossing and 0 on the left and right edge, and define g from the differences $f - f_i$. Then the claim follows easily. \square

For all $\lambda \in \mathbb{C}$, $n < \omega$, $i, j < 2^{n+2}$ and $\varepsilon > 0$, we choose a B_{ij}^n -function $f_{ij}^{n\varepsilon\lambda}$ so that for all $x \in \mathbb{C}$, if $r_i^n + \varepsilon < \operatorname{Re}(x) < r_{i+1}^n - \varepsilon$ and $r_j^n + \varepsilon < \operatorname{Im}(x) < r_{j+1}^n - \varepsilon$, then $f_{ij}^{n\varepsilon\lambda}(x) = \lambda$. We write $\theta(B_{ij}^n, \lambda)$ for $\lim_{\varepsilon \rightarrow 0} \theta(f_{ij}^{n\varepsilon\lambda})$. Notice that this does not depend on the choice of functions $f_{ij}^{n\varepsilon\lambda}$. Notice also that $\theta(B_{ij}^n, \lambda) = \bar{\lambda} \theta(B_{ij}^n, 1)$.

Definition 4.14. Suppose $n : \omega \rightarrow \omega$ and $\varepsilon : \omega \rightarrow \mathbb{R}_+$. We say that the pair (n, ε) is θ -good if the following holds:

- (i) For all $0 < m < \omega$, $\{N < \omega \mid n(N) > m, \varepsilon(N) < 1/m\} \in \mathcal{U}$.
- (ii) There is a $U = U(\theta, n, \varepsilon) \in \mathcal{U}$ such that for all $N \in U$ if $n = n(N) > 0$, then the following holds: Let $\varepsilon = \varepsilon(N) > 0$, $\delta' = \delta'(N) = \min\{\mu_N(B_{ij}^n) \mid i, j < 2^{n+2}, \mu(B_{ij}^n) > 0\}$ where $\mu_N(B_{ij}^n) = \sum_{\lambda_N(k) \in B_{ij}^n} \xi_N(k)^2$, and let $\delta = \delta(N) = \delta'/(2^{3(n+2)})$. Then
 - (a) for all R_ε^n -functions f , $|\theta(f)| < \delta$,
 - (b) $\sum_{\lambda_N(i) \in R_\varepsilon^n} \xi_N(i)^2 < \delta$,
 - (c) for all $i, j < 2^{n+2}$, $|\mu_N(B_{ij}^n) - \mu(B_{ij}^n)| < \delta$.
- (iii) $\varepsilon(N)$ is very nice for $n(N)$.

Notice that if (n, ε) is θ -good and for all $N < \omega$, $0 < \varepsilon'(N) < \varepsilon(N)$, and ε' is very nice for $n(N)$, then (n, ε') is θ -good. So keeping in mind that the set of ε very nice for n is dense, we can always assume e.g. that $\varepsilon(N)$ is such that for all $i < 2^{n(N)+2}$, $\varepsilon(N) < (r_{i+1}^{n(N)} - r_i^{n(N)})/3$.

Lemma 4.15. *There is a θ -good pair (n, ε) for which we can choose $U(\theta, n, \varepsilon) = \omega$.*

Proof. Simply let $n(N)$ be the largest natural number $0 < n \leq N$ for which there is $\varepsilon = \varepsilon(N)$ for which Definition 4.14 (ii) and (iii) hold, if such n and ε exist and otherwise we let $n(N) = 0$ and $\varepsilon = 1$. Since for all $\delta > 0$, $n < \omega$ and $i, j < 2^{n+2}$, the set

$$\{N < \omega \mid |\mu_N(B_{ij}^n) - \mu(B_{ij}^n)| < \delta\}$$

belongs to \mathcal{U} , it is easy to see that now also (i) holds. \square

Remark 4.16. The reader may wonder what the purpose of the sets $U(\theta, n, \varepsilon)$ is. In the context when we have infinitely many generalized distributions θ to handle, we need to diagonalize and then these sets become handy, see Remarks 4.20 and 5.5.

Now we fix a θ -good pair (n, ε) and in order to simplify the notations, we assume that $U(\theta, n, \varepsilon) = \omega$.

We are finally ready to define the vector in H^u that will correspond to the distribution θ :

Definition 4.17. We define $u(\theta) = (u_N(\theta))_{N < \omega} / \mathcal{U} \in H^u$ by letting $n = n(N)$ and defining

$$u_N(\theta) = \sum_{\mu(B_{ij}^n) \neq 0} \sum_{\lambda_N(k) \in B_{ij}^n} (\xi_N(k) \theta(B_{ij}^n, 1) / \mu_N(B_{ij}^n)) u_N(k).$$

Recall that we may assume the relevant $\mu_N(B_{ij}^n) \neq 0$, as this will happen for all N in some $Z \in \mathcal{U}$. Also notice that $u(\theta)$ is 0-good (immediate by the definition) and that for all $f \in C(S)$, one can define a generalized distribution θ_f by $\theta_f(g) = \langle g | f \rangle$ and then (if $\theta = \theta_f$) $[F(f)]_0 = [u(\theta_f)]_0$ (easy calculation).

Lemma 4.18. $\|u(\theta)\|_0 < \infty$.

Proof. Let K be such that for all $f \in C(S)$, $|\theta(f)| < K\|f\|_\infty^\sigma$. It is enough to show that for all $N < \omega$,

$$\begin{aligned} \sum_{\mu(B_{ij}^n) \neq 0} \sum_{\lambda_N(k) \in B_{ij}^n} |\xi_N(k)^2 \theta(B_{ij}^n, 1) / \mu_N(B_{ij}^n)| = \\ \sum_{\mu(B_{ij}^n) \neq 0} |\theta(B_{ij}^n, 1)| \leq 16K, \end{aligned}$$

where $n = n(N)$. Suppose not. Then one can find $X \subseteq \{B_{ij}^n \mid i, j < 2^{n+2}, \mu(B_{ij}^n) > 0\}$ such that

$$\sum_{B \in X} |\theta(B, 1)| > 2K$$

and for all $B \in X$, $\theta(B, 1)$ point roughly to the same direction, e.g. $\operatorname{Re}(\theta(B, 1)) \geq \operatorname{Im}(\theta(B, 1)) \geq 0$, see the proof of Lemma 4.9. But now for all $B \in X$, choose a B -function f_B so that $\theta(B, 1)$ is very close to $\theta(f_B)$. Let $f = \sum_{B \in X} f_B \in C(S)$. Then $\|f\|_\infty = 1$ but $|\theta(f)| > K$, a contradiction. \square

Theorem 4.19. *For all $f \in C(S)$, $\theta(f) = \langle [F(f)]_\infty [u(\theta)]_0 \rangle$.*

Proof. Let $f \in C(S)$. W.l.o.g. we may assume that $|f(x)| \leq 1$ for all $x \in S$, in particular, $\|f\|_\infty^\sigma \leq 1$. For all $0 < n < \omega$, there is $\varepsilon_1(n)$ such that if $x, y \in S$ are such that $|x - y| < 4M/(n(N) + 1)$, then $|f(x) - f(y)| < \varepsilon_1(n(N))$ and $\varepsilon_1(n)$ goes to 0 when n goes to infinity. Then for all $N < \omega$ such that $n(N) > 0$, we can find $f_N \in C(S)$ such that $\|f - f_N\|_\infty < \varepsilon_1(n(N))$, $|f_N(x)| \leq 1$ for all $x \in S$ and $f_N \upharpoonright (B_{ij}^{n(N)} - R_{\varepsilon(N)}^{n(N)})$ is constant for all $i, j < 2^{n+2}$ (here $\varepsilon(N)$ is the one from Definition 4.14, not $\varepsilon_1(n(N))$). Let c_{ij}^N be the constant value. Now we notice that $\theta(f_N)$ goes to $\theta(f)$ when $n(N)$ goes to infinity and also $\langle [F(f_N)]_\infty [u(\theta)]_0 \rangle$ goes to $\langle [F(f)]_\infty [u(\theta)]_0 \rangle$ when $n(N)$ goes to infinity by the proof of Lemma 4.6 (and Lemma 4.18). Thus it is enough to show the following: Let $\varepsilon^* > 0$ and $N < \omega$ be such that $n(N) > 0$, and $K/n(N) < \varepsilon^*$ where $K > 0$ is a natural number such that $|\theta(f)| \leq K\|f\|_\infty^\sigma$. Then

$$|\theta(f_N) - \langle F_N(f_N) | u_N(\theta) \rangle| \leq 5\varepsilon^*.$$

We write $n = n(N)$, $\varepsilon = \varepsilon(N)$, $\delta = \delta(N)$ etc.

Now for all $i, j < 2^{n+2}$, there is a B_{ij}^n -function f_{ij} such that for all $x \in B_{ij}^n - R_\varepsilon^n$, $f_{ij}(x) = f_N(x) = c_{ij}^N$. Let $f_2 = \sum_{i,j < 2^{n+2}} f_{ij}$. Then (as $\delta < \varepsilon^*$) it is enough to show that

$$|\theta(f_2) - \langle F_N(f_N) | u_N(\theta) \rangle| \leq 4\varepsilon^*.$$

For this it is enough to show that for all $i, j < 2^{n+2}$,

$$|\theta(f_{ij}) - \langle F_N(f_N \upharpoonright B_{ij}^n) | u_N(\theta) \rangle| \leq 3K/(2^{3(n+2)}),$$

where by $F_N(f_N \upharpoonright B_{ij}^n)$ we mean $\sum_{\lambda_N(k) \in B_{ij}^n} \xi_N(k) f_N(\lambda_N(k)) u_N(k)$.

Now if $\mu(B_{ij}^n) = 0$, then $B_{ij}^n \cap \sigma(A) = \emptyset$ by the definition of $\sigma^*(A)$ and Lemma 2.12 and thus $\theta(f_{ij}) = 0$ since f_{ij} is \sim_∞ -equivalent with the constant zero function. Clearly then also $\langle F_N(f_N \upharpoonright B_{ij}^n) | u_N(\theta) \rangle = 0$. Thus from now on we may assume that $\mu(B_{ij}^n) > 0$.

But then it is enough to show that

$$|\theta(f_{ij}) - \langle F_N(f_{ij}) | u_N(\theta) \rangle| \leq 2K/(2^{3(n+2)}),$$

since

(*) if $\mu(B_{ij}^n) > 0$, then

$$\left(\sum_{\lambda_N(k) \in B_{ij}^n \cap R_\varepsilon^n} |\xi_N(k)|^2 \right) / \mu_N(B_{ij}^n) \leq \delta / \delta' \leq 1 / (2^{3(n+2)}).$$

But then it is enough to show that

$$|\theta(B_{ij}^n, c_{ij}^N) - \langle F_N(f_{ij}) | u_N(\theta) \rangle| \leq K / (2^{3(n+2)}).$$

Since $\theta(B_{ij}^n, c_{ij}^N) = \overline{c_{ij}^N} \theta(B_{ij}^n, 1) = \overline{f_{ij}(x)} \theta(B_{ij}^n, 1)$ for any $x \in B_{ij}^n - R_\varepsilon^n$, using (*) above, this is easy. \square

Remark 4.20. The set of all generalized distributions form a vector space, we call it GDIS, under $(\theta + \theta')(x) = \theta(x) + \theta'(x)$ and $q\theta(x) = \theta(\overline{q}x)$. Let V be a subspace of this vector space of countable dimension and let θ_i , $i < \omega$, be a basis of this subspace. Then one can do the construction for all these generalized distributions simultaneously: One can choose the reals r_j^n so that the lines $I_{r_j^n}$ and $J_{r_j^n}$ are very good for every θ_i and then one can find a pair (n, ε) so that it is θ_i -good for every θ_i and in fact a bit more: Although we don't get one U set for all the θ_i , we can handle finitely many at a time. In Definition 4.14 (ii) (a), one can require that for all R_ε^n -functions f and $i < n$, $|\theta_i(f)| < \delta / n^2$ (see the remark immediately after the definition). Then for all $\theta \in V$, one defines $U(\theta, n, \varepsilon)$ to be the set of all $N < \omega$ such that $n(N) > 0$ and there are $a_i \in \mathbb{C}$, $i < n(N)$, such that $|a_i| \leq n(N)$ and $\theta = \sum_{i < n(N)} a_i \theta_i$. It is easy to see that by using these n and ε , $\theta \mapsto u(\theta)$ is an embedding of V to H_0^m so that Theorem 4.19 holds.

We can now compare our construction to a rigged Hilbert space. We have defined our distributions using $C(S)$ as test functions. However, our construction disregards what happens outside $\sigma(A)$, and in practice we are looking at $C(\sigma(A))$, which embeds into H_∞^m . The distributions then embed into H_0^m , so we can dig out the rigged Hilbert space in our ultraproduct. However, the construction embedding distributions into H_0^m can only handle countably many distributions at a time (the way we approximate the test functions over rectangles varies with the distributions), asking for a question to be posed:

Open question 4. Can one find an embedding of all of GDIS into H_0^m so that Theorem 4.19 and everything in Section 4 hold for it?

Another natural question that arises is how canonical the norms $\|\cdot\|_\infty$ and $\|\cdot\|_0$ are, as they were constructed from the chosen cyclic vector ϕ . However, there is no reason to believe that different cyclic vectors would generate the same ultraproduct H^u , so different norms act on different spaces. The crucial thing is that we have shown that the vectors corresponding to distributions behave in the desired way (as distributions). Thus, e.g., computing kernels (the way we consider in the next section) in spaces arising from different cyclic vectors will give the same result.

5. DIRAC DELTA FUNCTIONS AND THE FEYNMAN PROPAGATOR

We now turn to the study of the Feynman propagator. In [HH] we studied the kernel vs. Feynman propagator for two simple quantum mechanical systems. To

explain the terminology, let us consider a particle in (one-dimensional) space. Its state can be described by a wave function φ in position space, where the amplitude $\varphi(x)$ determines the particle's probability of being in position x . The evolution over a given time interval is described by the *propagator* $K(x, y)$ which expresses the probability amplitude of the particle travelling from point x to point y in the given time. (Usually the time is given as a third parameter, but with a time independent evolution operator, the question of calculating the kernel can be studied over a fixed time interval.) If the system has eigenvectors $|x\rangle$, $|y\rangle$, corresponding to the positions, one can calculate the propagator as an inner product

$$K(x, y) = \langle y | K^{\Delta t} | x \rangle$$

where $K^{\Delta t}$ is the time evolution operator for the given time interval. In a system without eigenvectors, the propagator is defined as the kernel of the integral representation of the time evolution operator

$$K^{\Delta t}(\varphi)(y) = \int_{\mathbb{R}} K(x, y) \varphi(x) dx.$$

In this section we show that one can use Dirac deltas to calculate the kernel of an operator in the style we tried to calculate it in [HH] (based on the Feynman propagator). In [HH] the first straightforward approach, to directly calculate the kernel using eigenvectors found in the ultraproduct, failed, essentially because the eigenvectors did not work properly but came in large numbers and had divisibility issues. We remedied the problem by calculating the kernel instead as an average over ever smaller areas.

Here we show how the same averaging idea as in [HH] can be used for the kind of finite dimensional approximations constructed in this paper. Assuming a kernel exists, we show it can be calculated as a limit of approximating inner products. We then show that being very careful one can actually embed the generalized distributions corresponding to Dirac deltas into our ultraproduct model and compute the kernel as a propagator using these Dirac delta vectors. However, the embedding essentially does the same averaging trick in a built in fashion, and is probably not the most convenient way of calculating. The limiting approach we present first is probably much easier to use.

So suppose B is an operator in H . Let $G^{-r} : H \rightarrow L_2(\sigma(A), \mu \upharpoonright \sigma(A))$ be the isometric isomorphism determined by the following: for all $f \in C(S)$, $G^{-r}(G(f)) = f \upharpoonright \sigma(A)$. Let G^r be the inverse of G^{-r} and $B_D = G^{-r} B G^r$. We suppose that B is such that there is a continuous function $K(x, y) : \sigma(A)^2 \rightarrow \mathbb{C}$ such that for all polynomials $P \in \mathbb{C}[X, Y]$, there is a continuous $g \in B_D(f_P)$ such that for all $y \in \sigma(A)$, $g(y) = \int_{\sigma(A)} K(x, y) (f_P \upharpoonright \sigma(A))(x) dx$. We will write $B_D(f_P)$ also for this continuous g . Notice that if we are going to calculate the kernel of B in the 'spectral basis of A ', it makes sense to assume that B has one. Also the motivation of these questions comes from physics and there all functions are continuous and thus we may assume that K is not only measurable but even continuous. Notice also that from this assumption it follows, e.g., that B is bounded and that for all $f \in D(S)$, $B_D(f) \in C(S)$.

However, we want to work in $L_2(S, \mu)$ in place of $L_2(\sigma(A), \mu \upharpoonright \sigma(A))$. For this we choose a continuation of K to S^2 and we call this continuation also K and

we call $G^{-1}BG$ also B_D and we notice that for all polynomials $P \in \mathbb{C}[X, Y]$, $g(y) = \int_S K(x, y)f_P(x)dx$ belongs to $B_D(f_P)$, since

$$\int_{\sigma(A)} K(x, y)(f_P(x) \upharpoonright \sigma(A))dx = \left(\int_S K(x, y)f_P(x)dx \right) \upharpoonright \sigma(A)$$

as one can easily see and again call also this function $B_D(f_P)$. If we want to do the calculations in the finite dimensional spaces H_N we need to find suitable operators B_N on these spaces.

Although we end up working with generalized distributions θ , the method from [HH] works also inside H^m (in fact in H^{Im} as we will see) which appears a more natural place to work and so we start by looking at the L_2 -norm. When we move to Dirac deltas we will need stronger assumptions. Our first assumption on the operators B_N is the following:

(C1) There is a natural number K_D such that for all $N < \omega$ and $v \in H_N$,

$$\|B_N(v)\| < K_D\|v\|.$$

Notice that from this (C1) it follows that the ultraproduct B^u of the operators B_N gives a well-defined bounded operator on all of H^m . We call this operator B^m .

The second assumption is the obvious requirement that if we write B^{Im} for the restriction of B^m to H^{Im} , then

(C2) $B^{Im} = G^m B (G^m)^{-1} (= F^m B_D (F^m)^{-1})$.

So how can one find the operators B_N ? First of all, in fact, they need not be linear functions as long as the ultraproduct of them is nice enough. However in practice one probably wants them to be operators. Examples of finding these can be found in [HH], although the situation there is not exactly the same as here. In the case of the free particle, B had a definition in terms of an operator C for which we already had operators C_N and we used this definition in the spaces H_N to get operators B_N . In the case of the harmonic oscillator we could have done essentially the same. However it turned out that with this definition $K(x, y)$ was very difficult to calculate. Thus we used another method that gave completely different operators B_N but whose ultraproduct was the same (upto \sim_2) in the places that mattered.

In the next lemma we show that the operators B_N can always be found. The proof is existential i.e. the method used there can not be used to find the operators in practice for an obvious reason - unless the mere existence of them is enough. And as pointed out above, the best way of choosing the operators may be such that it gives operators that are very different from those chosen in the proof of the lemma. The operators defined in the proof will be useful later.

Lemma 5.1. *There are operators B_N in the spaces H_N such that they satisfy (C1) and (C2) above.*

Proof. We define B_N as follows. Let $v = \sum_{n=0}^{D_N-1} a_n u_N(n) \in H_N$. Then we let

$$B_N(v) = \sum_{n=0}^{D_N-1} \sum_{k=0}^{D_N-1} \xi_N(k) \xi_N(n) a_k K(\lambda_N(k), \lambda_N(n)) u_N(n).$$

Clearly B_N is a linear function.

Let K_D be a natural number greater than any of the absolute values of the values of the function K . Then

$$\begin{aligned} \|B_N(v)\|_2^2 &\leq \sum_{n=0}^{D_N-1} \sum_{k=0}^{D_N-1} \xi_N(k)^2 \xi_N(n)^2 |a_k K(\lambda_N(k), \lambda_N(n))|^2 \leq \\ &\leq K_D^2 \|v\|_2^2 \sum_{n=0}^{D_N-1} \xi_N(n)^2 = K_D^2 \|v\|_2^2. \end{aligned}$$

So (C1) holds.

From the definition of the functions B_N and the uniform continuity of the functions K and f_P , it is easy to see that for all $P \in \mathbb{C}[X, Y]$, $[(B_N(F_N(f_P)))_{N < \omega}]_2 = [(F_N(B_D(f_P)))_{N < \omega}]_2$. But then (C2) follows (since the vectors $[(F_N(f_P))_{N < \omega}]_2$, $P \in \mathbb{C}[X, Y]$, are dense in H^{Im}). \square

From now on we will write \tilde{B}_N for the operators B_N from the proof of Lemma 5.1 \tilde{B}^u for their ultraproduct and \tilde{B}^m for their metric ultraproduct.

Now we want to calculate $K(\alpha, \beta)$ for $\alpha, \beta \in \sigma(A)$ from operators B_N , $N < \omega$, that satisfy (C1) and (C2). We start by looking at the method used in [HH], namely computing the kernel as a limit of kernels (or propagators, as computed in the finite dimensional spaces). Then we get the Dirac delta method as an immediate consequence. Now we choose numbers r_i^n for $n < \omega$ and $i < 2^{n+2} + 1$ (and lines $I_{r_i^n}$ and $J_{r_i^n}$) as in Section 4 so that Definition 4.12 holds when (ii) is replaced by

(ii)' for all $m < \omega$, neither α nor β is in any line $I_{r_i^m}$ or $J_{r_i^m}$, $i < 2^{m+2} + 1$, and for all $\delta > 0$ there is $\varepsilon > 0$ such that $\mu^n(R_\varepsilon^m) < \delta$, where R_ε^m is as in Section 4.

For all $p < \omega$, we let B_α^p be the B_{ij}^p containing α , and let $u_\alpha^p = (u_\alpha^p(N))_{N < \omega}$, where

$$u_\alpha^p(N) = \sum_{\lambda_N(k) \in B_\alpha^p} (\xi_N(k) / \mu_N(B_\alpha^p)) u_N(k)$$

and similarly for β in place of α . Notice that $\|u_\alpha^p\|_2^2 = 1/\mu(B_\alpha^p)$ and thus $[u_\alpha^p]_2 \in H_2^m$ whenever $\alpha \in \sigma(A)$.

Theorem 5.2. *When $\alpha, \beta \in \sigma(A)$, $K(\alpha, \beta) = \lim_{p \rightarrow \infty} \langle [u_\beta^p]_2 | B^m([u_\alpha^p]_2) \rangle$.*

Proof. We write $\chi_{B_{ij}^n}$ for the characteristic function of B_{ij}^n and $\chi'_{B_{ij}^n}$ for the function $\chi'_{B_{ij}^n}(x) = \chi_{B_{ij}^n}(x) / \mu(B_{ij}^n)$. Since we can approximate $\chi'_{B_\alpha^p}$ by polynomials in the L_2 -norm, it is easy to see that

$$[F(\chi'_{B_\alpha^p})]_2 = [u_\alpha^p]_2.$$

Notice that from this it follows that $[u_\alpha^p]_2 \in H^{Im}$. Now using this, the approximations again, (C1), (C2) and the proof of Lemma 5.1,

$$B^m([u_\alpha^p]_2) = \tilde{B}^m([u_\alpha^p]_2) = [\tilde{B}^u(u_\alpha^p)]_2.$$

We are left with an easy calculation:

$$\begin{aligned}
& \langle u_\beta^p(N) | \tilde{B}_N(u_\alpha^p(N)) \rangle = \\
& = \langle \sum_{\lambda_N(k) \in B_\beta^p} (\xi_N(k) / \mu_N(B_\beta^p)) u_N(k) | \\
& \quad \sum_{n=0}^{D_N-1} \sum_{\lambda_N(l) \in B_\alpha^p} K(\lambda_N(l), \lambda_N(n)) \xi_N(l)^2 \xi_N(n) \mu_N(B_\alpha^p)^{-1} u_N(n) \rangle = \\
& = \sum_{\lambda_N(k) \in B_\beta^p} \sum_{\lambda_N(l) \in B_\alpha^p} K(\lambda_N(l), \lambda_N(k)) \xi_N(l)^2 \xi_N(k)^2 (\mu_N(B_\alpha^p) \mu_N(B_\beta^p))^{-1}.
\end{aligned}$$

Now keeping in mind that K is a continuous function, when p is large enough, this is roughly

$$\sum_{\lambda_N(k) \in B_\beta^p} \sum_{\lambda_N(l) \in B_\alpha^p} K(\alpha, \beta) \xi_N(l)^2 \xi_N(k)^2 (\mu_N(B_\alpha^p) \mu_N(B_\beta^p))^{-1} = K(\alpha, \beta),$$

since the absolute value of the error is at most the maximum of

$$|K(\alpha, \beta) - K(\lambda_N(l), \lambda_N(k))|,$$

for $\lambda_N(k) \in B_\beta^p$ and $\lambda_N(l) \in B_\alpha^p$ as a straight forward calculation shows. \square

Now we can turn to Dirac deltas. We let θ_α be the generalized distribution such that $\theta_\alpha(f) = f(\alpha)$ for all $f \in C(S)$ and θ_β is defined similarly. Now it is easy to see that for all $n < \omega$ and $i < 2^{n+2} + 1$, $I_{r_i^n}$ and $J_{r_i^n}$ are very good for both θ_α and θ_β i.e. all the requirements of Definition 4.12 are satisfied. It follows that we can find a pair (n, ε) so that it is θ_α -good and θ_β -good (see the end of Section 4). We will need one more requirement for the pair (n, ε) . We will return to this once we have looked at the requirements for the operators B_N , $N < \omega$.

Again, recall that the pair (n, ε) determines the function u and (independent of the last requirement) notice that $\|u(\theta_\alpha)\|_0^u = 1$ and thus $[u(\theta_\alpha)]_0 \in H_0^m$ (and similarly for β). However $\|u_\alpha^p\|_\infty$ is infinite and this causes problems, the inner product in the Feynman propagator may not be well-defined. So we need to modify the assumptions (C1) and (C2).

We let H_∞^∞ be H^u / \sim_∞ . This is a natural space to consider Dirac deltas, as it is large enough to allow for 'infinite' objects, and on the other hand has a very delicate equivalence class structure, avoiding the risk of putting into the same equivalence class objects that behave differently in our calculations. Now our first assumption requires that the ultraproduct of the operators B_N gives a well-defined operator on H_∞^∞ . We will call the operator B^∞ :

(C1)' There is a natural number K_D such that for all $N < \omega$ and $v \in H_N$,

$$\|B_N(v)\|_\infty^S < K_D \|v\|_\infty^S.$$

Our second assumption ties the values of B_N to values of B_D in the sense of $\|\cdot\|_\infty$. Here we use basically the simple functions since our definition of the function u is based on them. However, notice that instead of simple functions we could use polynomials here as well as in Section 4, but this would make the definition of u much more complicated.

We write $F(\chi'_{B_{ij}^n})$ for $(F_N(\chi'_{B_{ij}^n}))_{N < \omega}$ where

$$F_N(\chi'_{B_{ij}^n}) = \sum_{n=0}^{D_N-1} \xi_N(n) \chi'_{B_{ij}^n}(\lambda_N(n)) u_N(n).$$

We also write $B_D(\chi'_{B_{ij}^n})$ for the function $g(y) = \int_S K(x, y) \chi'_{B_{ij}^n}(x) dx$. Then (C2) implies

$$B^m([F(\chi'_{B_{ij}^n})]_2) = [F(B_D(\chi'_{B_{ij}^n}))]_2.$$

And now following this we assume:

(C2)' For all B_{ij}^n ,

$$B^\infty([F(\chi'_{B_{ij}^n})]_\infty) = [F(B_D(\chi'_{B_{ij}^n}))]_\infty.$$

We recall that if $\|(v_N)_{N < \omega}/\mathcal{U}\|_\infty < \infty$ and for all $N < \omega$, $v_N = \sum_{n=0}^{D_N-1} a_n^N u_N(n)$, then there are $Z \in \mathcal{U}$ and a natural number p such that for all $N \in Z$ and $n < D_N$, $|\xi_N(n)^{-1} a_n^N| < p$. However, B^∞ images of Dirac's deltas diagonalize such functions and thus we may loose this property. So although with (C1)' and (C2)' and (*) below, $u(\theta_\alpha)$ and $u(\theta_\beta)$ do give Feynman's propagator the value $K(\alpha, \beta)$, this may be kind of accidental, they have just got lucky. The value is right for wrong reasons. So we introduce one more requirement for the operators B_N (it is a bit unnecessarily strong, but we are just making a point).

(C3)' There is a natural number p such that for all $n < \omega$, $i, j < 2^{n+2} + 1$ and $N < \omega$, $\|B_N(u_{ij}^n(N))\|_\infty^S < p$, where

$$u_{ij}^n(N) = \sum_{\lambda_N(k) \in B_{ij}^n} (\xi_N(k)/\mu_N(B_{ij}^n)) u_N(k)$$

when $\mu_N(B_{ij}^n) \neq 0$, and 0 otherwise.

Lemma 5.3. *The operators \tilde{B}_N , $N < \omega$, satisfy (C1)', (C2)' and (C3)'.*

Proof. (C1)' and (C2)' can be proved essentially as in the proof of Lemma 5.1 and (C3)' is a straight forward calculation. \square

Let us return to the choice of the pair (n, ε) and thus to the choice of u . We choose it so that in addition to what we have already said, the following holds: There are $U(\theta_\alpha, n, \varepsilon)$ and $U(\theta_\beta, n, \varepsilon)$ that witness that (n, ε) is both θ_α -good and θ_β -good and

(*) if $N \in U(\theta_\alpha, n, \varepsilon) \cap U(\theta_\beta, n, \varepsilon)$ and $p = n(N) > 0$, then

$$|\langle u_\beta^p(N) | B_N(u_\alpha^p(N)) \rangle - \langle u_\beta^p | B^u(u_\alpha^p) \rangle| < p^{-1}.$$

It is easy to see that this is possible.

Proposition 5.4. *Suppose (C1)', (C2)' and (C3)' hold, and $\alpha, \beta \in \sigma(A)$. Then*

$$K(\alpha, \beta) = \langle [u(\theta_\beta)]_0 | B^\infty([u(\theta_\alpha)]_\infty) \rangle.$$

Proof. Notice that if $p = n(N) > 0$, then $F_N(\chi_{B_\alpha^p} \mu_N (B_\alpha^p)^{-1}) = u_\alpha^p(N) = u_N(\theta_\alpha)$ and $(F_N(\chi_{B_\alpha^p} \mu_N (B_\alpha^p)^{-1}))_{N < \omega} / \mathcal{U} \in [F(\chi'_{B_\alpha^p})]_\infty$. and so from (C3)' it follows that $B^\infty[u(\theta_\alpha)]_\infty \in H_\infty^m$ and thus by Lemma 4.6,

$$\langle [u(\theta_\beta)]_0 | B^\infty([u(\theta_\alpha)]_\infty) \rangle = \overline{\langle B^\infty([u(\theta_\alpha)]_\infty) | [u(\theta_\beta)]_0 \rangle} < \infty,$$

i.e. it is well-defined. By (C2)' and Lemma 5.3, $B^\infty([u_\alpha^n]_\infty) = [\tilde{B}^u(u_\alpha^n)]_\infty$, for all $n < \omega$. But then the claim follows from (the proof of) Theorem 5.2 and (*) above. \square

Remark 5.5. Let Z be the set of all r for which there is $\delta > 0$ such that either for all $\varepsilon > 0$ $\mu^n(I_r^\varepsilon) \geq \delta$ or for all $\varepsilon > 0$, $\mu^n(J_r^\varepsilon) \geq \delta$. In Section 2 we saw that Z is countable. Let $Y = \cup_{r \in Z} (I_r \cup J_r)$. Then as in the end of Section 4, we can see that for any countable $X \subseteq \sigma(A) - Y$, we can choose a pair (n, ε) so that it is θ_α -good for all $\alpha \in X$ i.e. Theorem 4.19 holds, (*) above holds for every pair $(\alpha, \beta) \in X^2$ and in addition u gives an embedding of the vector space generated by the Dirac deltas θ_α , $\alpha \in X$, into H_0^m . So we can make everything in Sections 3 and 4 work for all θ_α , $\alpha \in X$, simultaneously. If X is dense in $\sigma(A)$, one expects that this is enough for most arguments in the contexts in which all functions are continuous.

6. THE NON-CYCLIC CASE

We finally look at the case where the operator doesn't have a cyclic vector. Then H can be decomposed into a countable direct sum $\bigoplus_{i < \omega} H_i$ of subspaces H_i , invariant under A and with a cyclic vector v_i . By the construction for the cyclic case, each H_i is isomorphic to a space $L_2(S, \mu_i)$, where every μ_i is a Borel measure with $\mu_i(S) = 1$.

Now, to combine these, let

$$\mu = \sum_i 2^{-i-1} \mu_i.$$

Then μ is a finite Borel measure, and every μ_i is absolutely continuous with respect to μ . Thus by the Radon-Nikodym theorem there are Borel functions f_i such that for all Borel sets $A \subseteq S$

$$\mu_i(A) = \int_A f_i d\mu.$$

This means that each $L_2(S, \mu_i)$ can be isometrically embedded into $L_2(S, \mu)$ by the mapping generated by

$$g(x) \mapsto \sqrt{f_i(x)} g(x).$$

Now if we wish to combine the direct sum of these into an L_2 space over a finite measure space, we can turn to the *Bochner spaces*, which are function spaces of vector valued functions (see, e.g., [HNVW]). We can recognize the space $\bigoplus_{i < \omega} L_2(S, \mu)$ as a Bochner- ℓ_2 -space with values in the space $L_2(S, \mu; \mathbb{C})$, i.e., $\ell_2(L_2(S, \mu; \mathbb{C}))$. By [HNVW, Corollary 1.2.23] $\ell_2(L_2(S, \mu; \mathbb{C})) \cong L_2(S, \mu; \ell_2(\mathbb{C}))$, where the latter is the vector valued L_2 -space over S with values in $\ell_2(\mathbb{C})$. Our sum space $\bigoplus_i H_i$ can now be isometrically embedded into this space by combining the isometric isomorphisms $H_i \cong L_2(S, \mu_i)$ obtained from our previous theory, the embeddings $L_2(S, \mu_i) \hookrightarrow L_2(S, \mu)$, and finally the isomorphism $\ell_2(L_2(S, \mu; \mathbb{C})) \cong L_2(S, \mu; \ell_2(\mathbb{C}))$.

If we denote the image under this embedding by H' , we can as before find a copy of H' within a suitable ultraproduct of finite-dimensional spaces: If we denote by

H_N^i the finite dimensional subspaces of H_i constructed as before, we can construct spaces

$$H^N = \oplus_{i \leq N} H_N^i,$$

and build the ultraproduct model from these. A canonical modification of the embedding G^m gives an embedding of $\oplus_{i < \omega} H_i$ (and thus of H') into the metric ultraproduct and allows us to use finite dimensional approximations. It is worth noting that this does not give an embedding of all of $L_2(S, \mu; \ell_2(\mathbb{C}))$ into the metric ultraproduct.

Now, although we can find a spectral decomposition of our space in an ultraproduct of finite-dimensional spaces, we don't see that this would give a general approach to kernels. If the operator B behaves differently in the different spaces H_i , one doesn't get a common complex-valued kernel that could be used for the vectors of the Bochner space. And if the H_i aren't invariant under B , one cannot treat these spaces separately, either.

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