CATEGORICAL REPRESENTATION THEORY AND THE COARSE QUOTIENT

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ABSTRACT. This is essentially the content of the author's PhD thesis, and has been superseded by [Gan22a] and [Gan22b], which contain more general results.

We show that a localized version of the 2-category of all categories with an action of a reductive group is equivalent to the 2-category of categories with an action of sheaves on a space defined only using the data of the Weyl group action on a maximal torus. As an application of our methods, we upgrade the equivalence of [Gin18] and [Lon18], which identifies the category of bi-Whittaker $\mathcal{D}$-modules on a reductive group with the category of W^{aff}-equivariant sheaves on a maximal Cartan subalgebra which satisfy Coxter descent, to a monoidal equivalence (which equips the bi-Whittaker category with a symmetric monoidal structure), and compute a restriction on the essential image of parabolic restriction of very central adjoint equivariant sheaves, providing evidence for a conjecture of [BZG17] on the essential image of enhanced parabolic restriction. Along the way, we develop a 'pointwise' criterion for a W-equivariant sheaf on a maximal Cartan of a semisimple Lie algebra to descend to a sheaf on the coarse quotient, and use this to reprove a result of Lonergan which states that such a sheaf descends to the coarse quotient of a maximal Cartan by the Weyl group if and only if for any simple reflection the sheaf descends to the coarse quotient of a maximal Cartan by the order two subgroup generated by that reflection.

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1. Introduction

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1.1. **Main Results.** Much of modern geometric representation theory can be interpreted as the study of groups acting on *categories* and the natural symmetries that the various invariants obtain; we will see specific examples of this in Section 1.2. Therefore it is of natural interest to study the class of all categories with an action of a split reductive group G. Our main object of study will be a localized class of G-categories known as *nondegenerate categories*.

Definition 1.1. Assume G is simply connected. A *nondegenerate* G-category is a G-category $\mathcal{C}$ such that for every rank one parabolic P_α, the invariants $\mathcal{C}^{[P_\alpha, P_\alpha]}$ vanish.

The general definition of a nondegenerate G-category is given in Definition 5.12. Furthermore, any G-category $\mathcal{C}$ admits a functor $\mathcal{C} \to \mathcal{C}_{\mathrm{nondeg}}$ that, informally speaking, has the same properties as the map $j^! : \mathcal{D}(X) \to \mathcal{D}(U)$ for an open subset $j : U \hookrightarrow X$. This is made precise in Definition 2.52 and Example 2.53.

Our main result states that the 2-category of nondegenerate G-categories admits a coherent description as modules over sheaves on an ind-scheme $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ defined only in terms of the action of the extended affine Weyl group $\tilde{W}^{\mathrm{aff}} := X^\bullet(T) \rtimes W$ on $\mathfrak{t}^*$. Specifically, we have $\Gamma_{\tilde{W}^{\mathrm{aff}}} \simeq \pi_1(G^\vee) \times \Gamma_{W^{\mathrm{aff}}}$, where $G^\vee$ denotes the Langlands dual group and $\Gamma_{W^{\mathrm{aff}}}$ denotes the union of graphs in $\mathfrak{t}^* \times \mathfrak{t}^*$ given by the W^{aff}-action on $\mathfrak{t}^*$. As we will see below in Proposition 3.14, one can identify $\Gamma_{\tilde{W}^{\mathrm{aff}}} \simeq \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$ for a certain prestack $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ defined below and referred to as the *coarse quotient*. This implies that one can use the convolution formalism of [GR17a, Section 5.5] to equip $\mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$ with a monoidal structure. Our main result can be summarized as follows:

Theorem 1.2. There is an equivalence of 2-categories

$$G\text{-mod}_{\mathrm{nondeg}} \simeq \mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})\text{-mod}$$

where the left hand side denotes the 2-category of all DG categories with a strong G action.

As we will review in Section 1.4, this result can reinterpreted as an equivalence of monoidal categories $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\text{nondeg}} \simeq \text{IndCoh}(\Gamma_{\tilde{W}^{\text{aff}}})$; see Theorem 1.13 for the full statement.

A key difference in the theory of groups G acting on categories $\mathcal{C}$ from the theory of groups acting on vector spaces is the existence of nontrivial maps between invariants $\mathcal{C}^{H_1}$ and $\mathcal{C}^{H_2}$, where each H_i is a closed subgroup of G (and, in particular, we do not require $H_1 \subseteq H_2$ or vice versa). These relations will prove a key technical tool in the proof of our theorem, and are summarized in Section 1.3. In particular, we will see in Theorem 1.8 that, for any G-category, the category $\mathcal{C}^N_{\text{nondeg}}$ admits a W-action, and that there is a fully faithful functor $\mathcal{C}^{N^-,\psi} \hookrightarrow \mathcal{C}^{N,W}_{\text{nondeg}}$. When $\mathcal{C}$ itself is given by Whittaker $\mathcal{D}$-modules on G, we will see in Corollary 5.4 that $\mathcal{C}^N \simeq \mathcal{C}^N_{\text{nondeg}}$, and, using this special case of Theorem 1.8, in Theorem 1.4 we derive a monoidal equivalence between the category of bi-Whittaker $\mathcal{D}$-modules on G and $\tilde{W}^{\text{aff}}$-equivariant sheaves on $\mathfrak{t}^*$ which satisfy Coxeter descent, providing a monoidal upgrade of [Gin18] and [Lon18].

1.2. **Motivation and Survey of Known Results.** Assume we are given a finite dimensional vector space V over an algebraically closed field k, equipped with an endomorphism $T : V \to V$. A familiar paradigm in representation theory and algebraic geometry is to regard V as a module over the ring $k[x]$, where x acts by the transformation T, and to write V as a direct sum of its generalized eigenspaces V_α. Furthermore, the vector space V can be recovered from the various V_α. We may equivalently view V as a sheaf over the line, and then each V_α can be identified as the subsheaf which lives over α. This particular example gives the well known Jordan normal form of a matrix, but there are analogues of this process for any k-algebra A and any module $M \in \text{QCoh}(\text{Spec}(A))$.

We can also apply this idea to other representation theoretic contexts. For example, let $\mathfrak{g}$ be a semisimple Lie algebra, and let M be a representation of the Lie algebra. Then it is known (see [Hum08]) that $Z\mathfrak{g}$ is a polynomial algebra, and furthermore that we may identify $\text{Spec}(Z\mathfrak{g}) \simeq \mathfrak{t}^* /\!\!/ W$. Therefore we may spectrally decompose a given $\mathfrak{g}$-representation by viewing it as a sheaf on the space $\mathfrak{t}^* /\!\!/ W$.

We will discuss analogues for these results one categorical level higher. Specifically, our notion of vector space will be replaced with that of a category. The analogue of an algebraic group acting on a vector space is a *group acting on a category*. For example, if G acts on a variety X, then the category $\mathcal{D}(X)$, the category of $\mathcal{D}$-modules on X, obtains a canonical G-action. Similarly, we can obtain a G-action on the category $\mathfrak{g}$-mod.

Analogous to the case of a group acting on a vector space, we can define the invariants of a group acting on a category. For example, one can understand representations of the Lie algebra $\mathfrak{g}$ of a semisimple algebraic group G via the invariants $\text{Rep}(G) \simeq \mathfrak{g}\text{-mod}^G$, or the associated zero block of the BGG category $\mathcal{O}$, which can be identified (via the Beilinson-Bernstein localization theorem) with objects of $\mathcal{D}(G/B)^N$. Similarly, one can also study the other blocks of category $\mathcal{O}$ via the notion of *twisted invariants* $\mathcal{D}(G/_\lambda B)^N$ for $\lambda \in \mathfrak{t}^*$.

Certain twisted invariants play a special role in geometric representation theory. Specifically, for a reductive group G acting on a category $\mathcal{C}$, one can take the *Whittaker invariants* $\mathcal{C}^{N^-,\psi}$. This category can be interpreted as the generically twisted N^- invariants of $\mathcal{C}$–the specific definition is given below. Often, Whittaker subcategories can be easier to understand than the usual N-invariants. For example, we have seen above that one may identify $\mathcal{D}(G/B)^N$ contains all of the information of the BGG category $\mathcal{O}_0$, whereas one can use the results of [BBM04] discussed below to show that $\mathcal{D}(G/B)^{N^-,\psi} \simeq \text{Vect}$.

The Whittaker invariants of a category have often been used to 'bootstrap' information about the original category, see, for example, [AB09] or [BY13]. In fact, our results below can be viewed as an attempt to generalize the work done by [BY13] at generalized central character 0 to the setting of varying central character. One can formally argue that the category of *bi*-Whittaker invariants of $\mathcal{D}(G)$, denoted $\mathcal{H}_\psi := \mathcal{D}(N^-_\psi\backslash G/_{-\psi}N^-)$, acts on the Whittaker invariants of any category with a G-action. It is therefore of interest to determine an explicit description for $\mathcal{H}_\psi$. This was identified in terms of sheaves on $\mathfrak{t}^*$ which are equivariant with respect to the (partially extended) affine Weyl group $\tilde{W}^{\text{aff}} := X^\bullet(T) \rtimes W$ for the Langlands dual group, where $X^\bullet(T)$ is the character lattice $\text{Hom}_{\text{AlgGp}}(T, \mathbb{G}_m)$.

Theorem 1.3. ([Lon18], [Gin18]) There is an equivalence identifying the abelian category $\mathcal{H}^\heartsuit_\psi$ with the full subcategory $\text{QCoh}(\mathfrak{t}^*)^{\tilde{W}^{\text{aff}},\heartsuit}$ of objects which *satisfy Coxeter descent* (see Definition 4.22).

Ginzburg [Gin18] and Ben-Zvi–Gunningham [BZG17, Section 1.2] also recorded the expectation that a derived, monoidal variant of Theorem 1.3 should hold (see Section 2 for our exact categorical conventions).

Ben-Zvi and Gunningham further noted [BZG17, Section 2.7.3] that this result admits an interpretation with respect to the *coarse quotient* $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, the main object of study in Section 3.

To state this precisely, we first recall the notion of the *Mellin transform*, a symmetric monoidal, W-equivariant equivalence $\mathrm{FMuk} : \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \xrightarrow{\sim} \mathcal{D}(T)$, where the notation follows [Lur18]. Here, we use ind-coherent sheaves rather than quasi-coherent sheaves since our $\mathcal{D}$-modules are right $\mathcal{D}$-modules in the sense of [GR], although since T is smooth, there is also a similar equivalence for left $\mathcal{D}$-modules $\mathrm{QCoh}(\mathfrak{t}^*/X^\bullet(T)) \xrightarrow{\sim} \mathcal{D}^\ell(T)$. With this, we can now state the derived, monoidal version of Theorem 1.3:

Theorem 1.4. There is a monoidal, t-exact, fully faithful functor $\widetilde{\mathrm{Av}}_* : \mathcal{H}_\psi \hookrightarrow \mathcal{D}(T)^W$. Under the Mellin transform, this functor induces monoidal equivalence F' such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) \arrow[r, "\pi^!"] \arrow[d, "F'"'] & \mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}}) \arrow[d, "\mathrm{FMuk}"] \\
\mathcal{H}_\psi \arrow[r, "\widetilde{\mathrm{Av}}_*"] & \mathcal{D}(T)^W
\end{tikzcd}$$

and such that $F'[\dim(\mathfrak{t}^*)]$ is t-exact.

Remark 1.5. The heart of any t-exact functor of DG categories (or, more generally, triangulated categories) equipped with t-structures is an exact functor of abelian categories, see [BBD82, Proposition 1.3.17]. Therefore, taking the heart of the equivalence in Theorem 1.4, our methods show that there is an *exact* equivalence of the abelian categories in Theorem 1.3.

Remark 1.6. The t-exactness of $\widetilde{\mathrm{Av}}_*$ is essentially due to Ginzburg, which we review in Theorem 2.43. Moreover, as we will see below, the composite functor

$$\mathcal{H}_\psi \xrightarrow{\widetilde{\mathrm{Av}}_*} \mathcal{D}(T)^W \xrightarrow{\mathrm{oblv}^W} \mathcal{D}(T)$$

can be identified, up to cohomological shift, with an averaging functor Av_*^N, where oblv^W denotes the forgetful functor.

1.3. **Generalization to Nondegenerate G-categories.** Using categorical representation theory, we provide an alternate proof of Theorem 1.4 as a special case of Theorem 1.8, which is different than the proofs of [Gin18] or [Lon18]. For example, the ideas in [Lon18] pass through the geometric Satake equivalence, whereas we do not. We view our proof as closer in spirit to the proof of [Gin18]; for example, both use versions of the Gelfand-Graev action (see Section 5.3). However, we also use the idea of [Lon18] and [BZG17] of *defining* the coarse quotient as the colimit of the groupoid determined by $\Gamma_{\tilde{W}^{\mathrm{aff}}}$, see Section 3.2. Our new input in the proof of Theorem 1.4 is showing that the map $\Gamma_{\tilde{W}^{\mathrm{aff}}} \to \mathfrak{t}^*$ is ind-finite flat. This implies that we may largely regard the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ as an object of classical algebraic geometry (see Proposition 3.17), while simultaneously using the benefits of the sheaf theory provided to us in derived algebraic geometry (for example, the full strength of the base change theorem).

Furthermore, much of our proof is phrased in the language of categorical representation theory. This can provide another conceptual explanation for Theorem 1.4 which makes certain aspects of this equivalence follow from general, categorical principles. For example, in [Gin18], Ginzburg noted that a derived version of Theorem 1.3 would show that $\mathcal{H}_\psi$ is symmetric monoidal, which was unknown at the time the first edition of [Gin18] was written. However, because we will see that the functor $\widetilde{\mathrm{Av}}_*$ is monoidal in Proposition 7.1, the fully faithfulness of $\widetilde{\mathrm{Av}}_*$ immediately implies the symmetric monoidality of $\mathcal{H}_\psi$, since we can identify it as a monoidal subcategory of a symmetric monoidal category. This differs from the original approach of [BZG17] of showing that $\mathcal{H}_\psi$ is symmetric monoidal, and in particular avoids the useage of bialgebroids entirely.

The principles of categorical representation theory will also allow us to prove the following generalization of Theorem 1.4 to all nondegenerate G-categories (discussed above in Section 1.1), see Theorem 1.8. Specifically, note that we may interpret the symmetric monoidality of Theorem 1.4 as a statement regarding spectrally decomposing categories with a G-action. For example, Theorem 1.4 says that if $\mathcal{C}$ is a category with a G-action, then for each $[\lambda] \in \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, we may consider the eigencategories of its Whittaker invariants

$(\mathcal{C}^{N^-,\psi})_{[\lambda]}$. Unfortunately, some categories do not admit Whittaker invariants. For example, one can show that $\mathrm{Vect}^{N^-,\psi} \simeq 0$, see Example 2.39.

On the other hand, work of [BZGO20] (which we will summarize below in Theorem 1.12) shows that if $\mathcal{C}$ is any G-category, the subcategory $\mathcal{C}^N$ (with its natural symmetries) determines $\mathcal{C}$. It is therefore of interest to relate the N-invariants of a category to the Whittaker invariants. To do this, we recall the following well known result, which we provide a proof for in Section 2.3.2:

Proposition 1.7. The restriction functor provides a canonical equivalence of categories

$$\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(N^-B/N)^{N^-,\psi} \simeq \mathcal{D}(T).$$

Therefore, we may reinterpret the statement of Theorem 1.4 in the language of groups acting on categories. Specifically, Theorem 1.4 in fact says that for the left G-category $\mathcal{C} := \mathcal{D}(G)^{N^-,-\psi}$, the averaging functor $\mathrm{Av}_*^N : \mathcal{C}^{N^-,\psi} \to \mathcal{C}^N$, after applying cohomological shift, lifts to a fully faithful, t-exact functor:

$$\mathcal{C}^{N^-,\psi} \xhookrightarrow{\widetilde{\mathrm{Av}}_*} \mathcal{C}^{N,W}. \tag{1}$$

Now let $\mathcal{C}$ be any G-category. Since $\mathcal{C}^N$ determines $\mathcal{C}$, one may ask whether a similar technique can be applied. Unfortunately, for example, in the universal case $\mathcal{C} = \mathcal{D}(G)$, the category $\mathcal{C}^N$ is not expected to admit a natural W-action. However, as we will see in Corollary 5.37, W *does* act on any nondegenerate G-category $\mathcal{C}$, and moreover for such $\mathcal{C}$ the analogue of Theorem 1.4 holds:

Theorem 1.8. For any nondegenerate G-category $\mathcal{C}$, the category $\mathcal{C}^N$ acquires a canonical W-action and there is an induced, fully faithful functor $\widetilde{\mathrm{Av}}_* : \mathcal{C}^{N^-,\psi} \hookrightarrow \mathcal{C}^{N,W}$.

Example 1.9. Let $\mathcal{C} = \mathcal{D}(G)^{N^-,-\psi}$ with its canonical left G-action. Then $\mathcal{C}$ is nondegenerate, as we will see below in Proposition 2.40. Furthermore, by Proposition 1.7 we have $C^N \simeq \mathcal{D}(T)$ and a result of Ginzburg's we review in Proposition 5.23 states that this isomorphism is W-equivariant. Thus a special case of Theorem 1.8 gives the fully faithfulness statement in Theorem 1.4.

1.4. **A Universal Nondegenerate G-category.** Recall that if H is any algebraic group acting on a category $\mathcal{C}$, we may also define its *weak invariants*. This is defined by forgetting the action of the category $\mathcal{D}(H)$ down to an action of $\mathrm{IndCoh}(H)$ and taking invariants of $\mathcal{C}$ as an $\mathrm{IndCoh}(H)$-module category. The notion of weak invaraints is specific to groups acting on categories (as opposed to vector spaces). Moreover, for any discrete group F the data of a weak action is equivalent to a strong action since the forgetful functor $\mathrm{oblv} : \mathcal{D}(F) \to \mathrm{IndCoh}(F)$ is an equivalence.

Example 1.10. The category $\mathcal{D}(G)^{G,w} \simeq \mathfrak{g}\text{-mod}$, while $\mathcal{D}(G)^G \simeq \mathrm{Vect}$. We also note that the category $\mathcal{D}(G)$ obtains two commuting G-actions (one from the left action of G on itself and one from the right). Therefore, we may define the category $\mathcal{D}(G)^{G\times G,w}$, and this category identifies with the *Harish-Chandra category* HC_G the category of $U\mathfrak{g}$ bimodules with an integrable diagonal action. Note we also see from this example a natural way to interpret the G-action on $\mathfrak{g}$-mod.

Example 1.11. The category $\mathfrak{g}$-mod acquires a G-action, and so, in particular, the category $\mathfrak{g}\text{-mod}^N$ acquires a $T \cong B/N$ action. We can identify the category $\mathfrak{g}\text{-mod}^{N,(T,w)}$ with the *universal category* $\mathcal{O}$, see [KS]. We will review other connections to the BGG category $\mathcal{O}$ in Section 6. In particular, we will see that the left adjoint to the functor $\widetilde{\mathrm{Av}}_*$ at a fixed central character can be identified with an enhanced version of Soergel's functor $\mathbb{V}$. This admits one interpretation of the left adjoint of $\widetilde{\mathrm{Av}}_*$ in the universal case–it is an analogue of Soergel's $\mathbb{V}$ which does not require a fixed character.

The following theorem then states that a category $\mathcal{C}$ with a G-action can be recovered from $\mathcal{C}^{N,(T,w)}$ with its natural symmetries.

Theorem 1.12. [BZGO20, Theorem 1.2] The monoidal categories $\mathcal{D}(G)$, $\mathcal{D}(N\backslash G/N)$, and $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ are all Morita equivalent.

Therefore, to understand results on G-categories, it suffices to understand the monoidal category $\mathcal{D}(N\backslash G/N)^{T\times T,w}$. In particular, via application of Theorem 1.3 of [BZGO20], we may similarly understand nondegenerate G-categories via understanding the localized monoidal category $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\mathrm{nondeg}}$.

We are now in a position to recast Theorem 1.2 as an equivalence of monoidal categories:

Theorem 1.13. There is a canonical monoidal equivalence of categories
$$\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\mathrm{nondeg}} \simeq \mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$$
which is t-exact up to cohomological shift.

In particular, the formalism of [BZG17, Theorem 1.1] applies[1] and we obtain an $\mathbb{E}_2$ functor $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) \to \mathcal{Z}(\mathcal{D}(G)_{\mathrm{nondeg}}) \simeq \mathcal{D}(G)^G_{\mathrm{nondeg}}$, where given a monoidal category $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$ denotes its *center* $\mathcal{Z}(\mathcal{A}) := \underline{\mathrm{End}}_{\mathcal{A}\times\mathcal{A}}(\mathcal{A})$. Using this, we may consider the eigencategories for any nondegenerate G-category over the category $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$, see [BZG17, Section 2.8.2].

Remark 1.14. The statement of Theorem 1.13 can be interpreted at the level of abelian categories as follows, which we state for G adjoint type for the ease of exposition. Let $\Gamma_{W^{\mathrm{aff}}}$ be the union of the graphs of the affine Weyl group W^{aff}. Then $\Gamma_{W^{\mathrm{aff}}}$ is an ind-scheme, and so, in particular, every compact object in $\mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})$ can be realized as the pushforward $i^{\mathrm{IndCoh}}_{S,*}(\mathcal{F}_S)$ for $i_S : \Gamma_S \to \Gamma_{W^{\mathrm{aff}}}$ the closed embedding of the union of some finite collection graphs of the affine Weyl group, and $\mathcal{F}_S$ an object of the abelian category of coherent sheaves on Γ_S (see chapter 3, section 1 of [GR17b]). Therefore, every object of $\mathcal{D}(N\backslash G/N)^{T\times T,w,\heartsuit}$ admits a quotient which can be viewed as a filtered colimit of such sheaves.

Remark 1.15. We can also interpret nondegeneracy as a localization of 2-categories
$$\mathcal{D}(G)\text{-mod} \to \mathcal{D}(G)\text{-mod}_{\mathrm{nondeg}}.$$
This perspective may prove useful in the local geometric Langlands correspondence, which studies twisted representations of the *loop group.* Our localization can be interpreted as an upgraded version of the functor $\mathcal{C} \mapsto \mathrm{Whit}(\mathcal{C})$. The functor Whit is of importance to the local geometric Langlands program, see [Ras18]. However, one may not need knowledge of this program for the results below.

Remark 1.16. This result, along with Theorem 1.2, admits an interpretation in the theory of 2 ind-coherent sheaves, in upcoming work of Arinkin-Gaitsgory and di Fiore-Stefanich [DFS]. In this vein, an informal interprtation of Theorem 1.2 is that we can identify a generic part of G-categories as free of rank one over $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$, and furthermore we have complete understanding of the singular support behavior which can occur.

1.5. **Parabolic Restriction.** Consider the category $\mathcal{Z}(\mathcal{D}(G)) := \underline{\mathrm{End}}_{G\times G}(\mathcal{D}(G)) \simeq \mathcal{D}(G)^G$. Here, as with all invariants in this subsection, G is acting via the adjoint action. This category is canonically the center of all categories with a G-action. Associated to it is a functor known as *parabolic restriction* $\mathrm{Res} : \mathcal{D}(G)^G \to \mathcal{D}(T)^T$. For an excellent survey on parabolic restriction in many of its guises in representation theory, see [KS]. We will define parabolic restriction in terms of a related functor, known as the *horocycle functor* hc, which is defined as the composite:
$$\mathcal{D}(G)^G \xrightarrow{\mathrm{oblv}^G_B} \mathcal{D}(G)^B \xrightarrow{\mathrm{Av}^{N\times N}_*} \mathcal{D}(N\backslash G/N)^T.$$
Let $i : N\backslash B/N \hookrightarrow N\backslash G/N$ denote the closed embedding.

Definition 1.17. The *parabolic restriction* functor is the composite
$$\mathcal{D}(G)^G \xrightarrow{\mathrm{hc}} \mathcal{D}(N\backslash G/N)^T \xrightarrow{i^!} \mathcal{D}(T)^T.$$

Definition 1.18. We say a sheaf $\mathcal{F} \in \mathcal{D}(G)^{G,\heartsuit}$ is *very central* if $\mathrm{oblv}^T \circ \mathrm{hc}(\mathcal{F}) \in \mathcal{D}(N\backslash G/N)$ is supported on $N\backslash B/N$.

It was proved that parabolic restriction is t-exact in [BYD]. Therefore, we see that the category $\mathcal{V}$ of very central $\mathcal{D}$-modules in $\mathcal{D}(G)^{G,\heartsuit}$ is an abelian category. In [BZG17], the authors conjecture that parabolic restriction of objects in $\mathcal{V}$ can be viewed as sheaves on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$. More precisely, the authors argue that ideas from the generalized Springer correspondence in the Lie algebra case [Gun17], [Gun18] will allow one to construct an enhanced parabolic restriction functor ${}^W\mathrm{Res} : \mathcal{D}(G)^G \to \mathcal{D}(T)^W$, see [Gun]. With motivation coming from Ngô's proof of the Fundamental Lemma [Ngô10], Ben-Zvi and Gunningham made the following conjecture:

[1]For adjoint G, this particular example for the category $\mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$ is, in fact, given in [BZG17, Section 2.7.3]. The new input is here is providing a description of $\mathrm{IndCoh}(\Gamma_{\tilde{W}^{\mathrm{aff}}})$ in terms of $\mathcal{D}$-modules on G, see Theorem 1.13.

Conjecture 1.19. [BZG17, Conjecture 2.7] The restricted functor of abelian categories ${}^{W}\mathrm{Res} : \mathcal{V} \to \mathcal{D}(T)^{W,\heartsuit}$ has essential image given by those sheaves satisfying Coxeter descent.

Remark 1.20. We note that enhanced parabolic induction preserves the heart of the t-structure since parabolic restriction is t-exact (again by [BYD]), and the functor which forgets W-invariants is t-exact and conservative.

Since [Gun] is not yet available, we are able to work around defining ${}^{W}\mathrm{Res}$ via the W-action on nondegenerate categories. In Section 10 provide some evidence for Conjecture 1.19. Specifically, we show:

Theorem 1.21. The horocycle functor on the nondegenerate category lifts to a functor

$$\tilde{\mathrm{hc}} : \mathcal{D}(G)^{G}_{\mathrm{nondeg}} \to \mathcal{D}(N\backslash G/N)^{T\rtimes W}_{\mathrm{nondeg}}$$

such that if $\mathcal{F} \in \mathcal{V}$, then $\tilde{hc}(\mathcal{F})$ satisfies a Coxeter descent property.

We make this Coxeter descent property precise in Theorem 10.1(4).

1.6. **Coxeter Descent for Finite Coxeter Groups.** In the course of defining and better understanding the space $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, we will require better understanding of the essential image of the pullback functor induced by the map $\phi : \mathfrak{t}^*/W \to \mathfrak{t}^* /\!\!/ W$. After showing this pullback is fully faithful, we provide a 'pointwise' criterion for a given sheaf to lie in the essential image of this map. In fact, we state this criterion more generally for a reflection group H acting by reflections on some vector space $V^\vee$ in Proposition 4.7. Using this, we provide many other characterizations of the essential image of the pullback functor in Proposition 4.16, and in particular we provide an alternate proof of the main result of [Lon17]:

Theorem 1.22. [Lon17, Theorem 1.1] An object $M \in \mathrm{QCoh}(V^\vee)^{H,\heartsuit}$ lies in the essential image of $\phi_H^*|_{\mathrm{QCoh}(V^\vee /\!\!/ H)^\heartsuit}$ if and only if the object $\mathrm{oblv}^{H}_{\langle r\rangle}(M) \in \mathrm{QCoh}(V^\vee)^{\langle r\rangle,\heartsuit}$ lies in the essential image of $\phi_r^*|_{\mathrm{QCoh}(V^\vee /\!\!/ \langle r\rangle)^\heartsuit}$ for every reflection $r \in H$, where $\phi_r : V^\vee/\langle r\rangle \to V^\vee /\!\!/ \langle r\rangle$ is the quotient map.

Our proof of Theorem 1.22 is more geometric than that of [Lon17]. Specifically, using the fact that the pullback by ϕ is *derived* fully faithful (Proposition 4.5), we obtain that the essential image of this pullback is closed under extensions and thus obtain our pointwise criterion for lying in the essential image. Using this, we therefore are able to avoid using the nil-Hecke algebra in any essential way.

1.7. **Outline of Paper.** In Section 2, we survey some of the basic results and definitions of groups acting on DG categories. In Section 3, we construct the space $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ and prove some basic properties. We then study the notion of *Coxeter descent* in Section 4. In particular, in this section we show that sheaves on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ may be identified as those $\tilde{W}^{\mathrm{aff}}$-equivariant sheaves on $\mathfrak{t}^*$ which satisfy Coxeter descent in Proposition 4.18, provide the 'pointwise' description for a W-equivariant sheaf on $\mathfrak{t}^*$ to satisfy Coxeter descent, and use this to prove Theorem 1.22.

We first define the notion of a nondegenerate G-category in Section 5. In Section 5.3, we show that the N-invariants of any category $\mathcal{C}$ with a strong G-action admits a W-action. This is the main construction of the paper. In Section 6, we provide an in depth computation of the nondegenerate variant of (weak) blocks of the BGG category $\mathcal{O}_\lambda$ indexed by regular weights. In Section 7, we prove Theorem 1.4 by utilizing these computations. After reviewing a few preliminaries in Section 8, in Section 9, we prove Theorem 1.13. In Section 10, we state Theorem 1.21 more precisely in Theorem 10.1 and prove it.

There are two appendices. In the first, Appendix A, we make computations for the classical scheme given by the union of graphs of a finite, closed subset of elements of the affine Weyl group. The main theorem of Appendix A is Corollary A.21, where we show that the union of graphs of any finite closed subset of W^{aff} is finite-flat, which may be of independent interest. In Appendix B, we show that two a priori distinct notions of the natural higher categorical variant of the BGG category $\mathcal{O}_\lambda$ agree. This technical result may be skipped at first pass.

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2. Categorical Preliminaries

2.1. **Conventions.** Derived algebraic geometry provides a convenient framework to define the space $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ and an associated category of sheaves $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$. Unless otherwise stated, all of our derived algebraic geometry conventions will be following [GR17a]. We highlight some of the conventions here.

2.1.1. *Categorical Conventions.* We work over a field k of characteristic zero and categories will, by default, be DG categories in the sense of Chapter 1, section 10 of [GR17a]. In other words, in the notation of [GR17a], we are by default working in the category $\mathrm{DGCat}_{\mathrm{cont}}$. By definition, the objects of this category are DG categories, which are presentable stable ∞-categories $\mathcal{C}$ which are equipped with the data exhibiting $\mathcal{C}$ as a module category for the monoidal stable ∞-category Vect of k-vector spaces, and the functors between them are by definition *continuous*, i.e. they preserve all colimits, and in particular *exact*, by assumption, meaning that they preserve cofiber sequences.[2] Occasionally, to emphasize our ground field, we will write $\mathrm{DGCat}_{\mathrm{cont}}^k$ for $\mathrm{DGCat}_{\mathrm{cont}}$.

We always highlight when our categories involved are not DG categories. One prominent example will occur when, in the notation of [GR17a], when we are working in $\mathrm{DGCat}^{\text{non-cocmpl}}$ we will say that we are working with *not necessarily cocomplete* DG categories. Furthermore, when we work with a DG category $\mathcal{C}$ equipped with a t-structure, we regard the categories $\mathcal{C}^{\leq m}$ and $\mathcal{C}^{\geq m}$ for each integer m as ordinary ∞-categories, and the *eventually coconnective subcategory* is $\mathcal{C}^+ := \cup_{n\in\mathbb{N}}\mathcal{C}^{\geq -n}$ as a not necessarily cocomplete DG category.

In particular, we use cohomological indexing for our t-structure. We will say a functor $F : \mathcal{C} \to \mathcal{D}$ of stable ∞-categories *reflects* the t-structure if $F(X) \in \mathcal{D}^{\leq 0}$ if and only if $X \in \mathcal{C}^{\leq 0}$ and $F(X) \in \mathcal{D}^{\geq 0}$ if and only if $X \in \mathcal{C}^{\geq 0}$. In particular, any conservative t-exact functor reflects the t-structure.

Furthermore, for any $\mathcal{F}, \mathcal{G}$ in a DG category $\mathcal{C}$, we let the notation $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ denote the internal mapping object in Vect, see Chapter 1, Section 10.3.7 in [GR17a], and similarly for $\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}) := \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F})$, and reserve the notation $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$, for the underlying space of maps. When the underlying category is clear, we also use the notation $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ or $\underline{\mathrm{End}}(\mathcal{F})$. Finally, given two DG categories $\mathcal{C}, \mathcal{D}$, we let $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ denote the DG category of maps between $\mathcal{C}, \mathcal{D}$ [GR17a, Chapter 1, Section 10.3.6].

2.1.2. *Other DAG Conventions.* Since our field k has characteristic 0, our analogue of affine schemes $\mathrm{Sch}^{\mathrm{aff}}$ is defined so that $\mathrm{Sch}^{\mathrm{aff,op}} := \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$, where the right hand side denotes the category of commutative algebra objects in the category of connective vector spaces, see [GR17a, Chapter 2, Section 1]. We will define objects such as $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ as objects in the category of *prestacks*, i.e. the $(\infty, 1)$ category of functors $\mathrm{Fun}(\mathrm{Sch}^{\mathrm{aff,op}}, \mathrm{Spc})$, where Spc denotes the $(\infty, 1)$-category of spaces. By field, we mean a classical field, which is in particular an object in $\mathrm{ComAlg}(\mathrm{Vect}^{\heartsuit})$.

We also follow the convention of [GR17a] of writing $\mathcal{D}(X) := \mathrm{IndCoh}(X_{dR})$ for the DG category of right $\mathcal{D}$-modules on a locally almost of finite type prestack X. A *sheaf* on some prestack X will mean an object of $\mathrm{QCoh}(X)$ or $\mathrm{IndCoh}(X)$ (when the latter is defined), and context will always dictate which category we are referring to. In particular, a given sheaf need not lie in the heart of the t-structure.

2.1.3. *Representation Theoretic Notation.* We fix once and for all a split reductive algebraic group G with Lie algebra $\mathfrak{g}$, and we additionally fix a choice of Borel B, maximal torus T with Weyl group $W := N_G(T)/T$, and associated character lattice $X^\bullet(T)$. Set $N := [B, B]$ and $N^- := [B^-, B^-]$, and let Φ be the induced root system with root lattice $\mathbb{Z}\Phi$. We set $\Lambda := \{\mu \in X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q} : \langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. We also view Λ and $X^\bullet(T)$ as ind-closed subschemes of $\mathfrak{t}^*$.

We also fix a pinning for G, which we record as a group homomorphism $\psi : N^- \to \mathbb{G}_a$, which factors through $N^-/[N^-, N^-]$ and is *nondegerate* (by definition of pinning) in the sense that its restriction to any one dimensional subscheme of $\mathrm{Lie}(N^-/[N^-, N^-])$ corresponding to a root vector for a simple root is nontrivial. By *reflection* of a vector space, we mean a diagonalizable linear endomorphism of the vector

[2] On the other hand, functors need not be t-exact, since not all DG categories even come equipped with a t-structure.

space fixing some hyperplane and having -1 as an eigenvalue, so that the square of any reflection is the identity.

2.2. **Groups Acting on Categories.** In this subsection, we will review some basic properties of groups acting on categories. For further surveys, see, for example, [Ras] or [CGYD19, Section 1.1].

2.2.1. *Definitions.* Let H be any affine algebraic group, and let $\mathcal{D}(H)$ denote the category of (right) $\mathcal{D}$-modules on H. Note that $\mathcal{D}(H)$ obtains a co-monoidal structure via pullback by the multiplication map. Similarly, let $\tilde{H}$ denote any group ind-scheme.

Definition 2.1. If $\mathcal{C}$ is equipped with the structure of a co-module for the co-monoidal category $\mathcal{D}(\tilde{H})$ (respectively $\mathrm{IndCoh}(\tilde{H})$), we say that the category $\mathcal{C}$ has a *strong* (respectively *weak*) *action* of $\tilde{H}$. If $\mathcal{C}$ is a category with a strong action of $\tilde{H}$, we will also say that $\mathcal{C}$ is an $\tilde{H}$-category.

Remark 2.2. In fact, the notion of an action of any group object of the category of prestacks can be defined using the notion of *sheaves of categories*, see [Gai].

Remark 2.3. One of the major interests in studying groups acting on categories came from the strong action of loop groups on categories, see [Ber13, Remark 1.3.1]. In particular, a strong action of H on a category is often referred to as an *action* of H on the category, and we will continue this practice below.

Remark 2.4. We may equivalently define an action of H on $\mathcal{C}$ as data realizing $\mathcal{C}$ as a module for the monoidal category $\mathcal{D}(H)$ under convolution, see [CGYD19, Section 1.1.1]. We will occasionally use this perspective below as well.

Remark 2.5. Assume we have an action of $\tilde{H}$ on some prestack $\mathcal{X}$ such that $\tilde{H}(K)$ and $\mathcal{X}(K)$ are discrete spaces (i.e. sets). (For example, this occurs for the action of the partially extended affine Weyl group $\tilde{W}^{\mathrm{aff}}$ on $\mathfrak{t}^*$). Then the notion of *orbit* and *stabilizer* with respect to K are defined in the classical sense.

2.2.2. *Invariants.* If $\mathcal{C}, \mathcal{D}$ are categories with an action of $\tilde{H}$, we let $\underline{\mathrm{Hom}}_{\mathcal{D}(\tilde{H})}(\mathcal{C}, \mathcal{D})$ denote the DG category of $\mathcal{D}(\tilde{H})$-linear maps between them.

Definition 2.6. Given a category $\mathcal{C}$ with an $\tilde{H}$-action, we define its *invariants* as the category $\mathcal{C}^{\tilde{H}} := \underline{\mathrm{Hom}}_{\mathcal{D}(\tilde{H})}(\mathrm{Vect}, \mathcal{C})$, where Vect acquires a trivial $\tilde{H}$-structure.

Proposition 2.7. Let X be any ind-scheme, and assume $\mathscr{G}$ is an algebraic group (in particular, a classical scheme) or is a group ind-scheme whose underlying prestack is a discrete set of points. Then

(1) The category $\mathrm{IndCoh}(X)$ canonically acquires a weak action of $\mathscr{G}$ such that canonical functor $\mathrm{IndCoh}(X/\mathscr{G}) \xrightarrow{q} \mathrm{IndCoh}(X)$ induced by the quotient $X \xrightarrow{q} X/\mathscr{G}$ lifts to an equivalence $\mathrm{IndCoh}(X/\mathscr{G}) \xrightarrow{\sim} \mathrm{IndCoh}(X)^{\mathscr{G}}$.
(2) The category $\mathcal{D}(X)$ canonically acquires an action of $\mathscr{G}$ such that canonical functor $\mathcal{D}(X/\mathscr{G}) \to \mathcal{D}(X)$ induced by the quotient map lifts to an equivalence $\mathcal{D}(X/\mathscr{G}) \xrightarrow{\sim} \mathcal{D}(X)^{\mathscr{G}}$.

Proof. We may identify category $\mathrm{IndCoh}(X/\mathscr{G})$ with the totalization of the cosimplicial object given by $\mathrm{IndCoh}(X^{\bullet}/(X/\mathscr{G})) \simeq \mathrm{IndCoh}(\mathscr{G}^{\bullet} \times X)$ by the fact that IndCoh satisfies fppf descent by [Gai13, Corollary 10.4.5] (when $\mathscr{G}$ is an algebraic group) or ind-proper descent by [GR17b, Chapter 3, Proposition 3.3.3] (when $\mathscr{G}$ is a discrete set of points). Similarly, by definition, the category $\mathrm{IndCoh}(X)^{\mathscr{G}}$ may be identified with the totalization of the cosimplicial object given by $\mathrm{IndCoh}(H^{\bullet}) \otimes \mathrm{IndCoh}(X)$. Furthermore, we have a map of cosimplicial objects $\boxtimes : \mathrm{IndCoh}(\mathscr{G}^{\bullet}) \otimes \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathscr{G}^{\bullet} \times X)$, and, for any fixed i, the associated functor $\boxtimes : \mathrm{IndCoh}(\mathscr{G}^{i}) \otimes \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathscr{G}^{i} \times X)$ is an equivalence if $\mathrm{IndCoh}(\mathscr{G}^{i})$ is dualizable, see [Gai13, Corollary 10.3.6].

However, if $\mathcal{X}$ denote any ind-inf-scheme, such as $\mathscr{G}^{i}$ or $(\mathscr{G}^{i})_{dR} \simeq (\mathscr{G}_{dR})^{i}$, then the category $\mathrm{IndCoh}(\mathcal{X})$ is in fact self-dual, see [GR17b, Chapter 3, Section 6.2.3]. Therefore, this equivalence holds for IndCoh, and, furthermore, the analogous proof shows for the associated categories of $\mathcal{D}$-modules since $\mathrm{IndCoh}(X_{dR}) \simeq \mathcal{D}(X)$ and the functor $(-)_{dR}$ commutes with limits and colimits, see [GR, Lemma 1.1.4]. $\square$

Definition 2.8. If $\mathcal{C}$ is a category with an H-action, we define the category of *monodromic objects* $\mathcal{C}^{H\text{-}\mathrm{Mon}}$ as the full subcategory of $\mathcal{C}$ generated by the essential image of the forgetful functor $\mathcal{C}^{H} \xrightarrow{\mathrm{oblv}^{H}} \mathcal{C}$.

Remark 2.9. In general, the functor oblv^H need not be fully faithful unless H is unipotent, see Lemma 2.36.

Using Remark 2.4, we can similarly define the *coinvariants* of H acting on $\mathcal{C}$ as $\mathcal{C}_H := \mathrm{Vect} \otimes_{\mathcal{D}(H)} \mathcal{C}$.

Theorem 2.10. We have the following:

(1) ([Gai20, Appendix B], [Gai20, Corollary 3.1.5])) If $\mathcal{C}$ is a category with an H-action, the forgetful functor $\mathrm{oblv} : \mathcal{C}^H \to \mathcal{C}$ admits a continuous right adjoint, denoted Av_*^H. This functor induces an equivalence $\mathcal{C}_H \xrightarrow{\sim} \mathcal{C}^H$.

(2) (Gaitsgory-Lurie, [Ber13, Theorem 2.5.7]) If $\mathcal{C}$ is a category with a weak H-action, the forgetful functor $\mathrm{oblv}^{H,w} : \mathcal{C}^{H,w} \to \mathcal{C}$ admits a continuous right adjoint, denoted $\mathrm{Av}_*^{H,w}$. This functor induces an equivalence $\mathcal{C}_{H,w} \xrightarrow{\sim} \mathcal{C}^{H,w}$.

Remark 2.11. This theorem has the following trivial but conceptually important consequence. Let M be any closed subgroup of an affine algebraic group H, and assume $\mathcal{C}$ is some category with an H-action. Then we have a canonical equivalence:

$$\underline{\mathrm{Hom}}_H(\mathcal{D}(H/M), \mathcal{C}) \simeq \mathcal{C}^M$$

for which an $F \in \underline{\mathrm{Hom}}_H(\mathcal{D}(H/M), \mathcal{C})$ is canonically isomorphic to the functor $F(\delta_{1M}) \star^H -$, where $F(\delta_{1M}) \in C^H$. This is because:

$$\underline{\mathrm{Hom}}_H(\mathcal{D}(H/M), \mathcal{C}) \simeq \underline{\mathrm{Hom}}_H(\mathcal{D}(H)^M, \mathcal{C}) \simeq \underline{\mathrm{Hom}}_H(\mathcal{D}(H)_M, \mathcal{C}) \simeq \mathcal{C}^M$$

where the last step uses the explicit description of coinvariants as a colimit, as well as the dualizability of $\mathcal{D}(H)$. In particular, suppose $F(\delta_{1M})$ is contained in some H subcategory of $\mathcal{C}$, say $\mathcal{C}'$. Then the entire essential image of F is also contained in $\mathcal{C}'$. Of course, this entire discussion also applies to weak invariants and weak actions.

2.2.3. *Character Sheaves and Twisted Invariants.* We briefly review the ideas of *twisted invariants* of groups acting on categories in this section and Section 2.3, see, for example, [CD21, Section 2] for a more thorough treatment. Let $\mathcal{L}$ be some character sheaf on H. Then we may define the **twisted invariants** $\mathcal{C}^{H,\mathcal{L}}$ associated to $\mathcal{L}$.

Remark 2.12. The natural analogue to Theorem 2.10, i.e. $\mathcal{C}_{H,\mathcal{L}} \xrightarrow{\sim} \mathcal{C}^{H,\mathcal{L}}$, holds for twisted invariants of any H acting on a category, see [Ber17, Section 2.4].

Now fix any field L and let $[\lambda]$ denote some L-point of $\mathfrak{t}^*/X^\bullet(T)$. This gives rise to a monoidal functor $\mathcal{D}(T) \simeq \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \to \mathrm{Vect}_L$ via the Mellin transform. By Remark 2.4, this in turn gives rise to a co-monoidal functor $\mathrm{Vect}_L \to \mathcal{D}(T)$, which sends the one dimensional vector space to a character sheaf $\mathcal{L}_{[\lambda]}$. Using this character sheaf, one can similarly define the twisted invariants $\mathcal{C}^{T,\mathcal{L}_{[\lambda]}}$ of any category $\mathcal{C}$ with a T-action or $\mathcal{C}^{B,\mathcal{L}_{[\lambda]}} := (\mathcal{C}^N)^{T,\mathcal{L}_{[\lambda]}}$ for any category with a B-action.

This category admits an alternative description which we now record:

Proposition 2.13. For any category $\mathcal{C}$ with an action of T, choose some lift $\lambda \in \mathfrak{t}^*(L)$ of $[\lambda]$. Then the canonical functor $\mathcal{C}^{T,w} \otimes_{\mathrm{IndCoh}(\mathfrak{t}^*)} \mathrm{Vect}_L \to \mathcal{C}^{T,\mathcal{L}_{[\lambda]}}$ is an equivalence.

Proof. This follows from the isomorphism

$$\mathcal{C}^{T,w} \otimes_{\mathrm{IndCoh}(\mathfrak{t}^*)} \mathrm{Vect}_L \simeq \mathcal{C} \otimes_{\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))} \mathrm{IndCoh}(\mathfrak{t}^*) \otimes_{\mathrm{IndCoh}(\mathfrak{t}^*)} \mathrm{Vect}_L \simeq \mathcal{C} \otimes_{\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))} \mathrm{Vect}_L$$

since the $\mathcal{L}_{[\lambda]}$-invariants and coinvariants agree. ☐

Remark 2.14. We recall that to any character $H \xrightarrow{\chi} \mathbb{G}_a$, one can pull back the exponential $\mathcal{D}$-module to create a character sheaf $\mathcal{L}_\chi$ on H, see section 2.4 of [Ber17]. We shift this character sheaf so that it has unique nonzero cohomological degree $-\dim(H)$, and denote this category by $\mathcal{C}^{H,\chi}$.

2.2.4. *Recovering G-Categories From Their Invariants.* In this section, we first recall the main result of [BZGO20], which says that all G-categories are highest weight, and then provide an extension in Proposition 2.16.

To do this, let $\mathcal{H}_N$ denote the monoidal category $\mathcal{D}(N\backslash G/N)$ and let $\mathcal{H}_{N,(T,w)}$ denote the monoidal category $\mathcal{D}(N\backslash G/N)^{(T\times T),w}$. Then the bimodule $\mathcal{D}(G/N)$ defines a functor G-mod(DGCat) $\to \mathcal{H}_N$-mod(DGCat), and similarly the bimodule $\mathcal{D}(G/N)^{T,w}$ defines a functor G-mod(DGCat) $\to \mathcal{H}_{N,(T,w)}$-mod. These functors are equivalently given by the functors $\mathcal{C} \mapsto \mathcal{C}^N$ and $\mathcal{C} \mapsto \mathcal{C}^{N,(T,w)}$ respectively, by Theorem 2.10. The main result of [BZGO20] states that these functors are Morita equivalences:

Theorem 2.15. Let $\mathcal{C}$ be a category with a G-action.

(1) [BZGO20] The functors $\mathcal{C} \mapsto \mathcal{C}^N$, $\mathcal{C} \mapsto \mathcal{C}^{N,(T,w)}$ are conservative.
(2) (Gaitsgory-Lurie, [Ber13, Theorem 2.5.7]) The functor $\mathcal{C} \mapsto \mathcal{C}^{G,w}$ is conservative.

Equivalently, the canonical functors $\mathcal{D}(G) \otimes_{\mathcal{H}_N} \mathcal{C}^N \to \mathcal{C}$, $\mathcal{D}(G) \otimes_{\mathcal{H}_{N,(T,w)}} \mathcal{C}^{N,(T,w)} \to \mathcal{C}^{N,(T,w)}$, $\mathcal{D}(G)^{G,w} \otimes_{HC_G} \mathcal{C} \to \mathcal{C}$ are equivalences, where HC_G is as in Example 1.10.

We now record the following proposition, which can be viewed as an extension of the results of Theorem 2.15.

Proposition 2.16. Let $F : \mathcal{C} \to \mathcal{D}$ denote some functor which admits a continuous adjoint. We have the following:

(1) If F additionally has the structure of a map of $\operatorname{IndCoh}(\mathfrak{t}^*)$-categories, then F is fully faithful (respectively, an equivalence) if and only if the induced functor
$$F \otimes_{\operatorname{IndCoh}(\mathfrak{t}^*)} \operatorname{id}_{\operatorname{Vect}_L} : \mathcal{C} \otimes_{\operatorname{IndCoh}(\mathfrak{t}^*)} \operatorname{Vect}_L \to \mathcal{D} \otimes_{\operatorname{IndCoh}(\mathfrak{t}^*)} \operatorname{Vect}_L$$
is fully faithful (respectively, an equivalence) for all field-valued points $\lambda \in \mathfrak{t}^*(L)$.
(2) If F is a functor of G-categories, then F is a fully faithful (respectively, an equivalence) if and only if the induced functor $F : \mathcal{C}^{B,\mathcal{L}_{[\lambda]}} \to \mathcal{D}^{B,\mathcal{L}_{[\lambda]}}$ is fully faithful (respectively, an equivalence) for all field-valued points $\lambda \in \mathfrak{t}^*(L)$.

Remark 2.17. We note in passing that there is a symmetric monoidal equivalence $\Upsilon_{\mathfrak{t}^*} : \operatorname{QCoh}(\mathfrak{t}^*) \xrightarrow{\sim} \operatorname{IndCoh}(\mathfrak{t}^*)$ [GR17a, Chater 6] since $\mathfrak{t}^*$ is smooth. In particular, we may replace $\operatorname{IndCoh}(\mathfrak{t}^*)$ with $\operatorname{QCoh}(\mathfrak{t}^*)$ in the statement (1) of Proposition 2.16.

We will prove Proposition 2.16 after showing the following lemma:

Lemma 2.18. Let A be a classical Noetherian commutative k-algebra. An object $\mathcal{F}$ of an A-module category $\mathcal{C}$ is zero if and only if $\operatorname{Av}_*^{\lambda}(\mathcal{F}) \simeq 0$ for all field-valued points $A \xrightarrow{\lambda} L$, where we denote the composite
$$\mathcal{C} \simeq \mathcal{C} \otimes_{\operatorname{IndCoh}(\operatorname{Spec}(A))} \operatorname{IndCoh}(\operatorname{Spec}(A)) \xrightarrow{\operatorname{id}\otimes\lambda^!} \mathcal{C} \otimes_{\operatorname{IndCoh}(\operatorname{Spec}(A)))} \operatorname{Vect}_L$$
by $\operatorname{Av}_*^{\lambda}$.

Proof. Apply a Cousin filtration to $\omega \in \operatorname{IndCoh}(\operatorname{Spec}(A))$ to see that we may write any $\mathcal{F} \in \mathcal{C}$ as a successive extension of objects of the form $\operatorname{oblv}^{\lambda}\operatorname{Av}_*^{\lambda}(\mathcal{F})$, where $\operatorname{oblv}^{\lambda}$ is the right adjoint to $\operatorname{Av}_!^{\lambda}$ and λ is a field-valued point. This is because the field-valued points generate the category $\operatorname{IndCoh}(\operatorname{Spec}(A))$, which is a direct consequence of [Gai13, Proposition 8.1.2]. Since the functor $\operatorname{oblv}^{\lambda}$ is conservative, the claim follows. $\square$

Proof of Proposition 2.16. The statements about equivalences follow from their respective fully faithful statements since if a functor F is a functor of $\mathcal{D}(G)$-categories (respectively, $\operatorname{IndCoh}(\mathfrak{t}^*)$-categories), then any adjoint is automatically a functor of $\mathcal{D}(G)$-categories (respectively, $\operatorname{IndCoh}(\mathfrak{t}^*)$-categories) by Theorem 2.19 (respectively, the fact that $\operatorname{IndCoh}(\mathfrak{t}^*) \simeq \operatorname{Sym}(\mathfrak{t})$-Mod is rigid monoidal, see [GR17a, Chapter 1, Section 9]) and the fact that a functor with an adjoint is an equivalence if and only if the functor and its adjoint are fully faithful. Furthermore, by Theorem 2.15, statement (2) of the proposition follows from statement (1), using Proposition 2.13.

Now assume $F : \mathcal{C} \to \mathcal{D}$ is a functor of $\operatorname{QCoh}(\mathfrak{t}^*)$-module categories which admits an adjoint. We assume that F admits a continuous right adjoint $R : \mathcal{D} \to \mathcal{C}$, a dual proof is valid if F admits a (necessarily continuous) left adjoint. To show that F is fully faithful, it suffices to show that the unit map $\operatorname{id} \to RF$ is an equivalence. This is a natural transformation of functors, and therefore it is a natural isomorphism if and only if the induced map $C \to RF(C)$ is an equivalence for every $C \in \mathcal{C}$.

The map $C \to RF(C)$ is an equivalence if and only if its cofiber K vanishes. By Lemma 2.18, we may check that this cofiber vanishes at all points. However, since R is necessarily $\operatorname{Sym}(\mathfrak{t})$-linear by the rigid monoidality of $\operatorname{Sym}(\mathfrak{t})$-mod, and so one can check that for any field-valued $\lambda \in \mathfrak{t}^*(L)$, $K \otimes L \in \mathcal{C} \otimes_{\operatorname{IndCoh}(\mathfrak{t}^*)} \operatorname{Vect}_L$ vanishes if and only if the associated unit map $\operatorname{id}_{\mathcal{C}} \otimes \operatorname{id}(C \otimes L) \to RF \otimes \operatorname{id}(C \otimes L)$ is an equivalence, which is true since $RF \otimes \operatorname{id} \simeq (R \otimes \operatorname{id})(F \otimes \operatorname{id})$ and $(F \otimes \operatorname{id})$ is fully faithful by assumption. $\square$

2.2.5. *Rigidity and Semi-Rigidity of Categories Related to $\mathcal{D}(G)$.* We recall that, as defined in [BZN15], a monoidal category is *semi-rigid* if it admits a set of compact generators which are dualizable on both the right and left, and, that a compactly generated category is *rigid* in the sense of [GR17a, Chapter 1, Section 9] if and only if it is semi-rigid and the monoidal unit is compact, see [BZN15, Proposition 3.3]. In this section, we survey and prove results regarding rigid and semi-rigid monodality of categories related to $\mathcal{D}(G)$, such as those in Theorem 2.15. We first record the following more general result of Gaitsgory:

Theorem 2.19. ([Gai20, Lemma D.4.4]) Let $\mathscr{G}$ denote any placid group ind-scheme. Then any datum of a lax or oplax equivariance on a functor of $\mathcal{D}(\mathscr{G})$-module categories is automatically strict. In particular, any adjoint of a $\mathcal{D}(\mathscr{G})$-equivariant functor acquires a canonical datum of $\mathcal{D}(\mathscr{G})$-equivariance.

Furthermore, we have the following theorem of Beraldo:

Theorem 2.20. ([Ber17, Proposition 2.3.11]) For any affine algebraic group H of finite type, the Harish-Chandra category $\mathcal{D}(H)^{H\times H,w}$ is rigid monoidal.

The 'rigid' version of the following observation was used in [Ber17] to prove Theorem 2.20; we record it and the semi-rigid variant for future use:

Lemma 2.21. Assume that $\mathcal{A}$ is a compactly generated, monoidal category, $\mathcal{B}$ is a monoidal category, and $F : \mathcal{A} \to \mathcal{B}$ is a monoidal functor of monoidal categories which admits a continuous, conservative right adjoint. Then if $\mathcal{A}$ is semi-rigid (respectively, rigid), then so too is $\mathcal{B}$.

Proof. Assume $\mathcal{A}$ is semi-rigid. Then, by definition of semi-rigidity, $\mathcal{A}$ admits a set of compact generators which are both left and right dualizable, say by the collection of $A_\alpha \in \mathcal{A}$. Because F is a left adjoint with continuous right adjoint, the collection of $F(A_\alpha)$ are compact in $\mathcal{B}$, and because the right adjoint is conservative, this collection generates. In particular, $\mathcal{B}$ is compactly generated.

Since functors with continuous right adjoints preserve compact objects and monoidal functors preserve the monoidal unit $\mathbf{1}_\mathcal{A}$, if the monoidal unit of $\mathcal{A}$ is compact, then so too is the monoidal unit $\mathbf{1}_\mathcal{B} \simeq F(\mathbf{1}_\mathcal{A})$ of $\mathcal{B}$. Since a compactly generated monoidal category is rigid if and only if it is semi-rigid and has a compact monoidal unit ([BZN15, Proposition 3.3]), it remains to show the semi-rigid version of this claim.

We have seen that $F(A_\alpha)$ are compact generators of $\mathcal{B}$. Let A_α^L denote the left dual to A_α. We claim $F(A_\alpha^L)$ is the left dual to $F(A_\alpha)$. To see this, note that, by definition of dualizability, there exists a coevaluation map $\mathbf{1}_\mathcal{A} \xrightarrow{u} A_\alpha^L \star_\mathcal{A} A$ and a counit map $A_\alpha^L \star_\mathcal{A} A \to \mathbf{1}_\mathcal{A}$ such that the compositions are equivalent to their respective identity maps (see [BZN15, Definition 2.4]). Using the monoidality of F, one can check, for example, that the map

$$\mathbf{1}_\mathcal{B} \simeq F(\mathbf{1}_\mathcal{A}) \xrightarrow{F(u)} F(A_\alpha^L \star_\mathcal{A} A) \simeq F(A_\alpha^L) \star_\mathcal{B} F(A)$$

where the last equivalence uses the monoidality of F, gives the coevaluation map of the duality datum. An identical proof (with the roles of left and right reversed) shows that $F(A_\alpha)$ is also right dualizable, and so therefore $F(A_\alpha)$ is a set of compact generators of $\mathcal{B}$ which are both left and right dualizable, and so $\mathcal{B}$ is rigid monoidal. ☐

We now prove a related result:

Proposition 2.22. The category $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is rigid monoidal and compactly generated.

We prove this after first recalling the following result of Ben-Zvi and Nadler:

Theorem 2.23. ([BZN15, Theorem 6.2]) The category $\text{IndCoh}(B\backslash G/B)$ is semi-rigid.

Remark 2.24. In [BZN15], Theorem 2.23 is originally stated for the category of (right) $\mathcal{D}$-modules $\mathcal{D}(B\backslash G/B)$. However, the entire proof of this theorem goes through for the category $\text{IndCoh}(B\backslash G/B) \simeq \text{IndCoh}(BB \times_{BG} BB)$, with the standard modifications that the Verdier duality functor $\mathbb{D}$ is replaced with the Serre duality functor on $\text{IndCoh}(B\backslash G/B)$, defined in [DG13, Section 4.4], and the constant sheaf for $\mathcal{D}$-modules is replaced with the constant sheaf for IndCoh.

In particular, note that the same proper map $p : BB \to BG$ and diagonal map $BB \to BB \times BB$ are used in both the $\mathcal{D}$-module and IndCoh cases. The analogue of the semi-rigid monoidality of $\mathcal{D}(X) = \mathcal{D}(BB)$ in [BZN15, Section 6] is replaced with the *rigid* monoidality of the compactly generated category $\text{IndCoh}(BB) \simeq \text{Rep}(B)$.

Proof of Proposition 2.22. Note that the adjoint pair given by the standard adjunction of right $\mathcal{D}$-modules (see [GR])

$$\mathrm{ind}_{dR} : \mathrm{IndCoh}(N\backslash G/N) \leftrightarrow \mathcal{D}(N\backslash G/N) : \mathrm{oblv}^r$$

Let $H := N_{dR}T$, so that $H_{dR} = B_{dR}$. We therefore see that the functor q_*^{IndCoh} induced by $q : B\backslash G/B \to H\backslash G_{dR}/H$ admits a continuous right adjoint.

We also note that q_*^{IndCoh} is monoidal, because, for example, because it is equivalently the functor given by pushforward by the Cartesian product of the quotient maps $BB \times_{BG} BB \to BH \times_{BG_{dR}} BH$. Moreover the associated right adjoint $q^!$ is conservative–for example, pulling back by the cover $s : G/B \to B\backslash G/B$, we can identify $s^!q^!$ with the composite of two forgetful functors, one which forgets the $\mathcal{D}$-module structure to the IndCoh module structure and one which forgets the $N, (T, w)$-equivariance. Therefore, q_*^{IndCoh} falls into the setup of Lemma 2.21, and furthermore we have seen that $\mathrm{IndCoh}(B\backslash G/B)$ is semi-rigid by Theorem 2.23. Therefore, using Lemma 2.21, we see $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is semi-rigid and thus in particular compactly generated.

Because $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is semi-rigid, it is in particular compactly generated. A compactly generated is rigid if and only if it is semi-rigid and the monoidal unit is compact ([BZN15, Proposition 3.3]), so it remains to verify the monoidal unit is compact. Note that we have a functor $\mathcal{D}(T)^{T\times T,w} \xhookrightarrow{i_*^{\mathrm{IndCoh}}} \mathcal{D}(N\backslash G/N)^{T\times T,w}$ induced by the composite $T \cong N\backslash B/N \overset{i}{\hookrightarrow} N\backslash G/N$ which is monoidal. Furthermore, it admits a continuous right adjoint–the fact that $i^! : \mathcal{D}(N\backslash G/N) \to \mathcal{D}(T)$ is $T\times T$-linear follows from Theorem 2.19. Therefore, the monoidal unit of $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is mapped to by a compact object in $\mathcal{D}(T)^{T\times T,w}$ (namely, the monoidal unit, which is compact by Theorem 2.20) via a functor admitting a continuous right adjoint, and thus is compact. $\square$

Finally, we prove a result analogous to Theorem 2.19:

Corollary 2.25. Assume $F : \mathcal{C} \to \mathcal{D}$ is a map of G-categories and denote by $F^N : \mathcal{C}^N \to \mathcal{D}^N$ the associated functor on the N-invariant subcategory. Assume F^N admits a continuous right adjoint (respectively, left adjoint). Then F itself admits a continuous, G-equivariant right adjoint $R : \mathcal{D} \to \mathcal{C}$ (respectively, G-equivariant left-adjoint). In particular, the adjoint of F^N is $\mathcal{H}_N$-linear.

Proof. The induced map $F^N : \mathcal{C}^N \to \mathcal{D}^N$ is necessarily T-equivariant, and so by Theorem 2.19, the adjoint functor is also T-equivariant. In particular, applying $(-)^{T,w}$ to this adjoint pair, we see that the functor $F^{N,(T,w)} : \mathcal{C}^{N,(T,w)} \to \mathcal{D}^{N,(T,w)}$ admits a continuous right adjoint, and therefore so too does the functor $\mathrm{id} \otimes F : \mathcal{D}(G/N)^{T,w} \otimes_{\mathrm{Vect}} \mathcal{C}^{N,(T,w)} \to \mathcal{D}(G/N)^{T,w} \otimes_{\mathrm{Vect}} \mathcal{C}^{N,(T,w)}$. Furthermore, by Theorem 2.15, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(G/N)^{T,w} \otimes_{\mathrm{Vect}} \mathcal{C}^{N,(T,w)} \arrow[r, "\mathrm{id}\otimes F^{N,(T,w)}"] \arrow[d, "\mathrm{act}"] & \mathcal{D}(G/N)^{T,w} \otimes_{\mathrm{Vect}} \mathcal{D}^{N,(T,w)} \arrow[d, "\mathrm{act}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

where the vertical arrows are given by the action maps. The category $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ is rigid monoidal (Proposition 2.22) and so the action map here admits a continuous right adjoint, see [GR17a, Chapter 1, Lemma 9.3.3]. However, the right adjoint to the action map is also conservative. Therefore, we may check whether the right adjoint to F commutes with colimits after applying the right adjoint to the action map on $\mathcal{C}$. However, the diagram given by the right adjoints of the above diagram commutes, and the right adjoints given by the upper and rightmost arrows are both continuous. Therefore, we see that the right adjoint R to F commutes with colimits. Since F is G-equivariant, its adjoint is by Theorem 2.19. Therefore, R^N will be $\mathcal{D}(N\backslash G/N)$-linear, as desired. Now assume F^N admits a left adjoint. Recall the functor hc of Section 1.5. Since hc is defined as a composite of the forgetful functor and averaging functor, we see that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(G) \otimes_G \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{hc}"] & \mathcal{D}(G) \otimes_G \mathcal{D} \arrow[d, "\mathrm{hc}"] \\
\mathcal{D}(G/N) \otimes_T \mathcal{C}^N \arrow[r, "\mathrm{id} \otimes F^N"] & \mathcal{D}(G/N) \otimes_T \mathcal{D}^N
\end{tikzcd}$$

It is a standard result that the functor hc admits a left adjoint ch–see, for example, [BZG17, Section 2.6.1] for an alternate description of this functor and its left adjoint. Moreover, following [BZG17, Section 2.6.1] and [Ras20a, Section 4.6], the functor hc is conservative, since, for example, the composite $\mathrm{hc}^L \circ \mathrm{hc} \simeq \mathrm{ch} \circ \mathrm{hc}$ can be identified with the functor which convolves with the Springer sheaf, a sheaf which has the sheaf $\delta_{1G} \in \mathcal{D}(G/G)$ as a direct summand. Therefore, to see that F commutes with limits it suffices to show that $\mathrm{hc}F$ commutes with limits, and by the commutativity of the diagram, this follows since hc and $\mathrm{id} \otimes F^N$ are both right adjoints. □

2.2.6. *Completeness of t-Structures.* We now recall the following definition for later use:

Definition 2.26. Let $\mathcal{C}$ be a category equipped with a t-structure. We say an object $\mathcal{F} \in \mathcal{C}$ is its *right-completion* (respectively, is its *left-completion*) with respect to the t-structure if the natural map $\mathrm{colim}_n \tau^{\leq n}\mathcal{F} \to \mathcal{F}$ (respectively, the natural map $\mathcal{F} \to \lim_m \tau^{\geq m}\mathcal{F}$) is an equivalence. A t-structure on a stable or DG category $\mathcal{C}$ is said to be *right-complete* (respectively, *left-complete*) if all of its objects are their right-completions (respectively, their left-completions).

Lemma 2.27. Let $\mathcal{C}$ and $\mathcal{D}$ be DG categories equipped with t-structures, and let $F : \mathcal{C} \to \mathcal{D}$ be a (continuous) functor between them which is t-exact and conservative.

(1) If $\mathcal{D}$ is right-complete with respect to its t-structure, then $\mathcal{C}$ is also right-complete with respect to its t-structure.
(2) If $\mathcal{D}$ is left-complete with respect to its t-structure, then if F commutes with (small) limits, $\mathcal{C}$ is also left-complete with respect to its t-structure.

Proof. Assume $C \in \mathcal{C}$, and consider the map $C \xleftarrow{\phi} \mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(C)$ in $\mathcal{C}$, where $\iota^{\leq n}$ denotes the inclusion. By the conservativity of F, it suffices to show that $F(\phi)$ is an isomorphism. However, we have:

$$F(\mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(C)) \simeq \mathrm{colim}_n F(\iota^{\leq n}\tau^{\leq n}(C)) \simeq \mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(F(\iota^{\leq n}\tau^{\leq n}(C))) \simeq \mathrm{colim}_n \iota^{\leq n}\tau^{\leq n}(F(C))$$

where the first equivalence uses the continuity of F, the second step uses the right t-exactness of F, and the third step uses the left t-exactness of F. By the right-completeness of $\mathcal{D}$, this composite map is an equivalence. However, we can identify this composite map with $F(\phi)$, so we see that $F(\phi)$ is an isomorphism, as desired. An argument dual to the above gives (2). Specifically, if $C \in \mathcal{C}$, then we have equivalences:

$$F(\lim_m \iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m F(\iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m \iota^{\geq m}\tau^{\geq m} F(\iota^{\geq m}\tau^{\geq m}(C)) \simeq \lim_m \iota^{\geq m}\tau^{\geq m} F(C)$$

where the first step uses the assumption that F commutes with small limits, the second step uses the fact that F is left t-exact, and the third uses the right t-exactness of F. Therefore, as above, the conservativity of F gives the t-structure on $\mathcal{C}$ is left-complete provided the t-structure on $\mathcal{D}$ is. □

2.2.7. *t-Structures on H-Categories.* For later use, we will also recall the t-structure on categories with a group action. We first recall a result of [Gai16]:

Lemma 2.28. ([Gai16, Lemma 4.1.3]) Assume $\mathcal{C}, \mathcal{D}_1$, and $\mathcal{D}_2$ are DG categories equipped with t-structures and we are given a (continuous) functor $F : \mathcal{D}_1 \to \mathcal{D}_2$. Then the categories $\mathcal{C} \otimes \mathcal{D}_i$ each inherit a t-structure where the objects of $(\mathcal{C} \otimes \mathcal{D}_i)^{\leq 0}$ are generated under colimits by objects of the form $c \otimes d$ where $c \in \mathcal{C}^{\leq 0}$ and $d \in \mathcal{D}^{\leq 0}$. Furthermore:

(1) If $F : \mathcal{D}_1 \to \mathcal{D}_2$ is right t-exact, then so too is the functor $\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$.
(2) If $F : \mathcal{D}_1 \to \mathcal{D}_2$ is left t-exact, then so too is the functor $\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$, if the t-structure on $\mathcal{C}$ is compactly generated, i.e. $\mathcal{C}^{\leq 0}$ is generated under colimits by the objects of $\mathcal{C}^{\leq 0}$ which are compact in $\mathcal{C}$.

Now, as above, we let H be any affine algebraic group. We note that in particular $\mathrm{IndCoh}(H)$ and $\mathcal{D}(H)$ satsifies the hypotheses of Lemma 2.28 since, for example, each category is the derived category of its heart since H is smooth–see [GR, Proposition 4.7.3] for the $\mathcal{D}(H)$ case. We make the following defininition:

Definition 2.29. ([Ras18] Appendix B.4) We say that a t-structure on a H-category $\mathcal{C}$ is *compatible* with the G action if the coaction map $\mathrm{coact}[-\dim(H)] : \mathcal{C} \to \mathcal{D}(H) \otimes \mathcal{C}$ is t-exact.

Example 2.30. Assume H acts on some scheme X. Then the action map $a : H \times X \to X$ is smooth (since there is an isomorphism $H \times X \cong H \times X$ which takes the action map to the projection map, which is smooth), and so in particular the coaction map $a^![-\dim(H)]$ is t-exact.

Let $\mathcal{C}$ denote a DG category with an action of an algebraic group, and let $\chi : H \to \mathbb{G}_a$ be any character. Recall the construction $\mathcal{C}^{H,\chi}$ of Remark 2.14. If $\mathcal{C}$ is equipped with a t-structure compatible with the action of H, then, following [Ras18, Appendix B1-B4], we can equip $\mathcal{C}^{H,\chi}$ with a unique t-structure such that the forgetful functor is t-exact.

Proposition 2.31. If $F : \mathcal{C} \to \mathcal{D}$ is a t-exact functor of G-categories and $H \leq G$ is some closed subgroup of G, then the induced functor $\tilde{F} : \mathcal{C}^{H,\chi} \to \mathcal{D}^{H,\chi}$ is t-exact.

Proof. By assumption, the following diagram canonically commutes:

$$\begin{tikzcd} \mathcal{C}^{H,\chi} \arrow[r, "\tilde{F}"] \arrow[d, "\mathrm{oblv}"] & \mathcal{D}^{H,\chi} \arrow[d, "\mathrm{oblv}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

where, oblv denotes the forgetful functor. Assume $\mathcal{F} \in \mathcal{C}^{H,\chi,\geq 0}$. Then we have $\mathrm{oblv}(\mathcal{F}) \in \mathcal{C}^{\geq 0}$ and so by the commutativity of the diagram above, we see that $\mathrm{oblv}(\tilde{F}(\mathcal{F})) \in \mathcal{D}^{\geq 0}$. In particular, we see that $\mathrm{oblv}(\tau^{<0}(\mathcal{F})) \simeq 0$ by the t-exactness of oblv. Since oblv is conservative, we see that $\tau^{<0}(\mathcal{F}) \simeq 0$, and therefore $\tilde{F}(\mathcal{F}) \in \mathcal{D}^{H,\chi,\geq 0}$. Repeating this same argument replacing the coconnective categories with the respective connective categories (and replacing $\tau^{<0}$ with $\tau^{>0}$), an identical argument shows $\tilde{F}$ preserves connective objects and thus $\tilde{F}$ is t-exact. □

Definition 2.32. Let $\mathcal{C}$ be a category equipped with a t-structure.

(1) We say an object $C \in \mathcal{C}$ is *cohomologically bounded* if there exists $m, n \in \mathbb{Z}$ such that $\tau^{>n}C$ and $\tau^{<m}C$ vanish, so that $C \simeq \tau^{\geq n}\tau^{\leq m}C \simeq C$.
(2) We say $\mathcal{C}$ has *bounded cohomological dimension* if there exists some $n > 0$ such that if $X \in \mathcal{C}^{\geq n}$ and $Y \in \mathcal{C}^{\leq 0}$, then the space of maps $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is connected.
(3) Let $\mathcal{D}$ be a category equipped with a t-structure, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between them. We say that F has *cohomological amplitude in* $[m, n]$ for some $m, n \in \mathbb{Z}$ if for all $C \in \mathcal{C}^{\heartsuit}$, the cohomology of $F(C)$ given by the t-structure is concentrated in degrees $[m, n]$, and furthermore we say it has *bounded cohomological amplitude* if it has cohomological amplitude in $[m, n]$ for some integers m and n.

Proposition 2.33. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of DG categories (or any stable ∞-categories) equipped with t-structures.

(1) Assume further that the t-structure on $\mathcal{C}$ is right-complete as in Definition 2.26 and the t-structure on $\mathcal{D}$ is compatible with filtered colimits. Then if F maps $\mathcal{C}^{\heartsuit}$ to $\mathcal{D}^{\geq m}$ (which in particular occurs for some m if F has bounded cohomological amplitude), then F maps $\mathcal{C}^{\geq 0}$ to $\mathcal{D}^{\geq m}$.
(2) Further assume that F commutes with cofiltered limits. Then if the t-structure on $\mathcal{C}$ is left-complete and the t-structure on $\mathcal{D}$ is compatible with cofiltered limits, then if F maps $\mathcal{C}^{\heartsuit}$ to $\mathcal{D}^{\leq n}$, then F maps $\mathcal{C}^{\leq 0}$ to $\mathcal{D}^{\leq n}$.

Proof. We first claim that F sends $\mathcal{C}^{[0,q]} := \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq q}$ to $\mathcal{C}^{\geq m}$. To see this, we induct on q, noting that the base case follows from definition of having cohomological amplitude in $[m, n]$. By the inductive step, note that any $\mathcal{F} \in \mathcal{C}^{[0,q]}$ for $q > 0$ admits a cofiber sequence $\tau^{\leq q-1}\mathcal{F} \to \mathcal{F} \to H^q(\mathcal{F})[-q]$, where we omit the inclusion functors given by the t-structure. The inductive hypothesis implies that $F(\tau^{\leq q-1}\mathcal{F}) \in \mathcal{D}^{\geq m}$ and the fact that F is exact implies that $F(H^q(\mathcal{F})[-q]) \simeq F(H^q(\mathcal{F}))[-q]$ lies in $\mathcal{D}^{[m+q,n+q]} \subseteq \mathcal{D}^{\geq m}$. Now, for a general $\mathcal{F} \in \mathcal{C}^{\geq 0}$, by right-completeness we may write $\mathcal{F} \xleftarrow{\sim} \mathrm{colim}_n \tau^{\geq n}(\mathcal{F})$. Apply F, which commutes with colimits, to see that $F(\mathcal{F}) \xleftarrow{\sim} \mathrm{colim}_n F(\tau^{\geq n}(\mathcal{F}))$. Therefore, by above, we see that $F(\mathcal{F})$ is a filtered colimit of objects of $\mathcal{D}^{\geq m}$, and therefore, since the t-structure on $\mathcal{D}$ is compatible with filtered colimits, we see that $F(\mathcal{F}) \in \mathcal{D}^{\geq m}$.

The dual proof holds to prove claim (2), where one similarly first inducts on q to show that $\mathcal{C}^{[-q,0]} \in \mathcal{C}^{\leq n}$ and uses left-completeness of $\mathcal{C}$, the fact that F commutes with cofiltered limits, and the fact that $\mathcal{D}^{\leq 0}$ is closed under cofiltered limits to show the general claim. $\square$

We now state two results in Appendix B in [Ras18]:

Proposition 2.34. Assume we are given a category $\mathcal{C}$ which additionally is equipped with an action of H compatible with the t-structure, and let $\chi : H \to \mathbb{G}_a$ be any character.

1. ([Ras18, Lemma B.4.1]) The functor $\mathrm{Av}_*^H : \mathcal{C} \to \mathcal{C}^{H,\chi}$ has cohomological amplitude in $[0, \dim(H)]$.
2. ([Ras18, Lemma B.6.1]) Assume $\mathcal{C}$ has a compactly generated t-structure. Then the partially defined left adjoint to the forgetful functor, denoted $\mathrm{Av}_!^{H,\chi} : \mathcal{C} \to \mathcal{C}^{H,\chi}$, has cohomological amplitude $[-\dim(H), 0]$.

Proof. We show the methods of [Ras18, Appendix B] adapt to show the more general character case as in (1). Consider the underlying sheaf of our character sheaf, $\mathcal{L}_\chi \in \mathcal{D}(H)$, associated to χ. By our conventions above, $\mathcal{L}_\chi$ is in cohomological degree $-\dim(H) = -1 + (-\dim(H) + 1)$. Since we may identify the averaging functor with the functor $m_{*,\mathrm{dR}}(- \boxtimes \mathcal{L}_\chi[2\dim(H)])$, the fact that m is affine implies that the pushforward $m_{*,dR}$ is right t-exact, and so in particular this functor will have cohomology only in degrees bounded by $\dim(H)$. Since $\mathrm{Av}_*^{H,\chi}$ is a right adjoint to a t-exact functor, it is in particular left t-exact and thus, combining these results, we obtain (1) when $\mathcal{C} = \mathcal{D}(H)$.

To show demonstrate the argument for general $\mathcal{C}$, note that, by assumption, the functor $\mathrm{coact}[-\dim(H)] : \mathcal{C} \to \mathcal{D}(H) \otimes \mathcal{C}$ is t-exact and therefore in particular by Proposition 2.31 we obtain that the induced functor on invariants $\mathcal{C}^{H,\chi} \to \mathcal{D}(H)^{H,\chi} \otimes \mathcal{C}$ is t-exact. We also have that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "\mathrm{Av}_*^{H,\chi}"] & \mathcal{D}(H) \otimes \mathcal{C} \arrow[d, "\mathrm{Av}_*^{H,\chi} \otimes \mathrm{id}"] \\
\mathcal{C}^{H,\chi} \arrow[r] & \mathcal{D}(H)^{H,\chi} \otimes \mathcal{C}
\end{tikzcd}$$

where the horizontal arrows are the shifted coaction maps as above. Since the coaction maps are conservative (for example, one can pull back by the identity morphism $* \to H$ to obtain the identity map) and t-exact, we obtain our claim by Lemma 2.28. $\square$

2.3. **Whittaker Invariants.** We now recall the notion of the Whittaker invariants of a category, discussed above in Section 1.2.

2.3.1. *Whittaker Invariants and Unipotent Groups.* Recall we have fixed a pinned reductive group in Section 2.1.3; in particular, we have fixed an additive character $\psi : N^- \to \mathbb{G}_a$ which is nondegenerate.

Definition 2.35. Given a category $\mathcal{C}$ with a G-action, we define its *Whittaker invariants* as the category $\underline{\mathrm{Hom}}_{\mathcal{D}(N^-)}(\mathrm{Vect}_\psi, \mathcal{C})$.

We record one result on the twisted invariance of *unipotent groups*:

Lemma 2.36. ([Ber13] 3.3.5) The forgetful functors $\mathcal{C}^{N^-,\psi} \xrightarrow{\mathrm{oblv}} \mathcal{C}$ and $\mathcal{C}^N \xrightarrow{\mathrm{oblv}} \mathcal{C}$ are fully faithful.

Remark 2.37. Strictly speaking, [Ber13] above does not discuss case $\mathcal{C}^{N^-,\psi} \xrightarrow{\mathrm{oblv}} \mathcal{C}$. However, after replacing the constant sheaf on N^- with the sheaf ψ gives identical results, largely due to the unipotence of N^-.

Because of these results, we will view C^N and $C^{N^-,\psi}$ as full subcategories of $\mathcal{C}$.

2.3.2. *A Universal Example of a Whittaker Category.* By Theorem 2.15, the category $\mathcal{D}(G/N)^{N^-,\psi}$ provides a universal example of the Whittaker invariants of a G-category. We compute this explicitly by showing $\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(T)$, i.e. by proving Proposition 1.7. We prove Proposition 1.7 after proving the following lemma:

Lemma 2.38. For any category $\mathcal{C}$ with an action of $\mathbb{G}_a$, if $\psi : \mathbb{G}_a \to \mathbb{G}_a$ is a nontrivial character, then $\mathcal{C}^{\mathbb{G}_a,\psi\text{-mon}} \cap \mathcal{C}^{\mathbb{G}_a\text{-mon}} \simeq 0$.

Proof. If $\mathcal{F} \in \mathcal{C}^{\mathbb{G}_a\text{-mon}}$, there exists some isomorphism $\mathrm{act}^!(\mathcal{F}) \cong \mathcal{F} \boxtimes \omega_{\mathbb{G}_a}$. In this case, if $t : \mathbb{G}_a \to *$ is the terminal map, then we see that

$$\mathcal{F} \simeq \mathcal{F} \otimes k \simeq (\mathrm{id}_{\mathcal{C}} \otimes t_{*,dR})(\mathcal{F} \boxtimes \omega_{\mathbb{G}_a})[-2] \cong (\mathrm{id}_{\mathcal{C}} \otimes t_{*,dR})(\mathrm{act}^!(\mathcal{F}))[-2]$$

Similarly, if $\mathcal{F} \in \mathcal{C}^{\mathbb{G}_a,\chi\text{-mon}}$, there is an isomorphism $(\mathrm{act}^!(\mathcal{F})) \cong \mathcal{F} \boxtimes \mathcal{L}_\psi$, where $\mathcal{L}_\psi$ is the shifted exponential $\mathcal{D}$-module on $\mathbb{G}_a \simeq \mathbb{A}^1$. Thus, continuing the chain of isomorphisms above, we see that if $\mathcal{F}$ is also ψ-monodromic, then $\mathcal{F}$ is equivalent to

$$(\mathrm{id}_{\mathcal{C}} \otimes t_{*,dR})(\mathcal{F} \boxtimes \mathcal{L}_\psi[-2]) \simeq \mathcal{F} \otimes t_{*,dR}(\mathcal{L}_\psi)[-2] \simeq 0$$

since de Rham pushforward to a point is given by de Rham cohomology (up to shift), which vanishes on the exponential $\mathcal{D}$-module on $\mathbb{A}^1$. □

Example 2.39. Consider the category Vect with the trivial G-action where G is not its maximal torus. Then $\mathrm{Vect}^{N^-,\psi} \simeq 0$ since every object is N^--monodromic.

Proof of Proposition 1.7. Fix some $\mathcal{F} \in \mathcal{D}(G/N)^{N^-,\psi}$, and let $j : N^-B/N \hookrightarrow G/N$ denote the open embedding. We wish to show $\mathcal{F} \xrightarrow{\sim} j_{*,dR}j^!(\mathcal{F})$. To do this, it suffices to show that the restriction to the complementary closed subset vanishes. In turn, by backwards induction on the length of the Weyl group element, it suffices to show that the restriction of $\mathcal{F}$ to each Schubert cell $N^-\dot{w}B/N$ vanishes, where $\dot{w} \in N_G(T)$ is some arbitrarily chosen lift of a given element $w \in W$ where $w \neq 1$.

We claim that, furthermore, we have $\mathcal{D}(N^-\dot{w}B/N)^{N^-,\psi} \simeq 0$ if $w \neq 1$. To see this, note that because $\psi : N^- \to \mathbb{G}_a$ induces a map of group schemes $\tilde{\psi} : N^-/[N^-,N^-] \to \mathbb{G}_a$, we may equivalently show the identity

$$\mathcal{D}([N^-,N^-]\backslash N^-\dot{w}B/N)^{N^-/[N^-,N^-],\tilde{\psi}} \simeq 0.$$

We may write $N^-/[N^-,N^-] \simeq \prod_\alpha \mathbb{G}_a^{-\alpha}$ as a product of copies of the additive group $\mathbb{G}_a$, where α varies over the simple roots. Since $w \neq 1$, there exists some simple root α such that $\dot{w}^{-1}\mathbb{G}_a^{-\alpha}\dot{w} \in N$. We therefore see that this action of $\mathbb{G}_a^{-\alpha}$ is trivial, since if m is some R-point of $[N^-,N^-]\backslash N$ and b is some R-point of B/N, then if $x \in \mathbb{G}_a^{-\alpha}(R)$ then

$$x(m\dot{w}b) = mx\dot{w}b = m\dot{w}(\dot{w}^{-1}x\dot{w})b = m\dot{w}b$$

in G/N, where the last step follows because $(\dot{w}^{-1}x\dot{w}) \in N$ and uses the fact that the action of N on B/N is trivial. We therefore see that the $\mathbb{G}_a^{-\alpha}$ action on $\mathcal{D}([N^-,N^-]\backslash N^-\dot{w}B/N)$ is trivial. In particular, all objects are monodromic with respect to this $\mathbb{G}_a^{-\alpha}$, and so no objects are monodromic with respect to some nontrivial character by Lemma 2.38. □

In fact, Proposition 1.7 can be extended more generally, which we record for later use:

Proposition 2.40. Let $\mathcal{C}$ be some category with an action of a reductive group G, and let α be the negative of a simple root with associated parabolic subgroup P_α. Then if $\mathcal{F} \in \mathcal{C}^{N^-,\psi}$ is $Q_\alpha := [P_\alpha, P_\alpha]$ monodromic, $\mathcal{F} \simeq 0$.

Proof. Let $s \in W$ denote the simple reflection associated to α, and consider the Levi decomposition $P_\alpha = U_{w_0s} \rtimes L_s$. Then we see that Q_α contains a connected simple closed algebraic subgroup M_s of rank one. Note that the averaging functor $\mathcal{C} \xrightarrow{\mathrm{Av}_*^{Q_\alpha}} \mathcal{C}^{Q_\alpha}$ is conservative on the subcategory of Q_α monodromic objects. In particular, the averaging with respect to the M_s action is conservative on this subcategory.

We consider the restriction of the averaging functor $\mathrm{Av}_*^{M_s}$ to the Whittaker subcategory, which is in particular given by the composite:

$$\mathcal{C}^{N^-,\psi} \xhookrightarrow{\mathrm{oblv}} \mathcal{C}^{\mathbb{G}_a^\alpha,\omega_{\exp}} \xhookrightarrow{\mathrm{oblv}^\alpha} \mathcal{C} \xrightarrow{\mathrm{Av}_*^{M_s}} \mathcal{C}^{M_s}$$

where the two leftmost functors are fully faithful by Lemma 2.36. However, by Remark 2.11 we have that the composite functor $\mathrm{Av}_*^{M_s}\mathrm{oblv}^\alpha$ is given by convolution with an object in $\mathcal{D}(M_s/M_s)^{\mathbb{G}_a^\alpha,\omega_{\exp}} \simeq 0$. Therefore, if an object of $\mathcal{C}$ lies in the subcategory $\mathcal{C}^{N^-,\psi}$ and the subcategory of Q_α-monodromic objects of $\mathcal{C}$, it must be the zero object. $\square$

2.3.3. *Averaging Functors.* Let $\mathcal{C}$ be a category with a G-action, and consider the partially defined left adjoint [DG14, Appendix A] to the forgetful functor $\mathrm{oblv} : \mathcal{C}^{N^-,\psi} \to \mathcal{C}$, which we denote, $\mathrm{Av}_!^\psi$. We will refer to this functor as the *Whittaker averaging functor* and, in the case of a category with two commuting G-actions (such as $\mathcal{D}(G)$), we will use the term *left Whittaker averaging* and *right Whittaker averaging* to emphasize which action we are averaging with respect to.

Let d denote the dimension of N^-. Then, following [Ber13, Corollary 3.3.8], we obtain a canonical map $\mathrm{Av}_!^\psi \to \mathrm{Av}_*^\psi[2d]$. With this, we can now review one of the main results of [BBM04]; see [Ras18, Section 2.7.1], the proof of [Ras18, Corollary 7.3.1], and [Ras16, p14] for further discussion:

Theorem 2.41. The canonical map $\mathrm{Av}_!^\psi \to \mathrm{Av}_*^\psi[2d]$ is an equivalence when restricted to the full subcategory $\mathcal{C}^N$. In particular, the functor $\mathrm{Av}_*^N : \mathcal{C} \to \mathcal{C}^N$ admits a left adjoint when restricted to $\mathcal{C}^{N^-,\psi}$.

Corollary 2.42. In the notation of Theorem 2.41, if $\mathcal{C}$ has a G-action compatible with the t-structure, then the functor $\mathrm{Av}_!^\psi[-\dim(N)] \simeq \mathrm{Av}_*^\psi[\dim(N)]$ is t-exact.

Proof. We have that $\mathrm{Av}_!^\psi[-\dim(N)] \simeq \mathrm{Av}_*^\psi[\dim(N)]$ simultaneously has cohomological amplitude $[0, \dim(N)]$ and $[-\dim(N), 0]$ by Proposition 2.34 (2) and (3) respectively. $\square$

As a consequence of Corollary 2.42, we see that the right adjoint functor $\mathrm{Av}_*^N[\dim(N)]$ is left t-exact. We also recall the following result of Ginzburg which also gives right t-exactness:

Theorem 2.43. [Gin18, Theorem 1.5.4] The functor $\mathrm{Av}_*^N[\dim(N)] : \mathcal{D}(G)^{N^-,\psi} \to \mathcal{D}(G)^N$ is t-exact.

Remark 2.44. To compare the above with the notation of [Gin18], we have $\Omega_N \simeq \omega_N[-\dim(N)] \simeq \underline{k}_N[\dim(N)]$, where $\underline{k}_N$ denotes the constant sheaf. (The notation Ω_N will not be used outside this remark.)

Corollary 2.45. If $\mathcal{C}$ is a G-category equipped with a compactly generated t-structure compatible with the G-action, the functor
$$\mathcal{C}^{N^-,\psi} \hookrightarrow \mathcal{C} \xrightarrow{\mathrm{Av}_*^N[\dim(N)]} \mathcal{C}^N$$
is t-exact.

Proof. When $\mathcal{C} = \mathcal{D}(G)$, this result follows from Theorem 2.43. For general $\mathcal{C}$ equipped with a compactly generated t-structure compatible with the G-action, this follows since the diagram
$$\begin{tikzcd}
\mathcal{C}^{N^-,\psi} \arrow[r, "\mathrm{oblv}"] \arrow[d] & \mathcal{C} \arrow[r, "\mathrm{Av}_*^N[\dim(N)]"] \arrow[d] & \mathcal{C}^N \arrow[d] \\
\mathcal{D}(G)^{N^-,\psi} \otimes \mathcal{C} \arrow[r, "\mathrm{oblv}\otimes\mathrm{id}_\mathcal{C}"] & \mathcal{D}(G) \otimes \mathcal{C} \arrow[r, "\mathrm{Av}_*^N[\dim(N)]\otimes\mathrm{id}_\mathcal{C}"] & \mathcal{D}(G)^N \otimes \mathcal{C}
\end{tikzcd}$$
commutes, where the vertical arrows are the coaction maps, which are conservative and t-exact (up to shift), as in the proof of Proposition 2.34. $\square$

2.4. **Other Categorical Preliminaries.**

2.4.1. *A Base Change Lemma.* As in [GR17a], we let $1\text{-}\mathrm{Cat}_{\mathrm{cont}}^{\mathrm{St,cocmpl}}$ denote the category of presentable stable ∞-categories, and whose functors commute with all colimits. Assume $\mathbf{A}$ is a commutative algebra object in $1\text{-}\mathrm{Cat}_{\mathrm{cont}}^{\mathrm{St,cocmpl}}$. Assume $\mathcal{A}$ is an associative algebra object of $\mathbf{A}$ and $\mathcal{L}$ is a commutative algebra object of $\mathbf{A}$. We have a symmetric monoidal functor $\mathrm{ind}_\mathcal{L} : \mathbf{A} \to \mathcal{L}\text{-mod}(\mathbf{A})$ given by the left adjoint to the forgetful functor [Lur17, Corollary 4.2.4.8], where the symmetric monoidal structure on $\mathcal{L}\text{-mod}(\mathbf{A})$ is given by [Lur17, Theorem 4.5.2.1], and the symmetric monoidality of the functor is given by [Lur17, Theorem 4.5.3.1]. In particular, $\mathcal{A}' := \mathrm{ind}_{\mathcal{L}}(\mathcal{A})$ is an associative algebra object of $\mathcal{L}$-mod($\mathbf{A}$), and we obtain an induced functor $\widetilde{\mathrm{ind}}_{\mathcal{L}} : \mathcal{A}\text{-mod}(\mathbf{A}) \to \mathcal{A}'\text{-mod}(\mathcal{L}\text{-mod}(\mathbf{A}))$.

Proposition 2.46. There is an equivalence of categories making the following diagram canonically commute:

$$\begin{tikzcd}
\mathcal{A}\text{-mod}(\mathbf{A}) \arrow[d, "\widetilde{\mathrm{ind}}_{\mathcal{L}}"'] \arrow[dr, "\mathcal{A}'\otimes_{\mathcal{A}}(-)"] & \\
\mathcal{A}'\text{-mod}(\mathcal{L}\text{-mod}(\mathbf{A})) & \mathcal{A}'\text{-mod}(\mathbf{A}) \arrow[l, "\sim"']
\end{tikzcd}$$

where the bottom arrow is the left adjoint to the forgetful functor induced by $\mathcal{L}\text{-mod}(\mathbf{A}) \xrightarrow{\mathrm{oblv}} \mathbf{A}$.

Proof. The arrows are all the left adjoints to the respective forgetful functors, so the diagram commutes. The fact that the bottom functor is an equivalence is a direct consequence of [GR17a, Proposition 8.5.4], where, in their notation, $\mathcal{A}_1, \mathcal{A}_2, \mathbf{M}_1, \mathbf{M}_2$, and $\mathbf{A}$ correspond to our $\mathcal{A}, \mathcal{L}$-mod, $\mathbf{A}, \mathcal{L}$-mod, and $\mathbf{A}$, respectively. ☐

Let L/k be an extension of (classical) fields. Then we can set $\mathbf{A} := \mathrm{DGCat}^k_{\mathrm{cont}}$ and $\mathcal{L} := L$-mod, so that $\mathcal{L}\text{-mod}(\mathrm{DGCat}^k_{\mathrm{cont}})$ can be identified with the associated category of DG categories over L, $\mathrm{DGCat}^L_{\mathrm{cont}}$. Let $\mathrm{ind}^L_k : \mathrm{DGCat}^k_{\mathrm{cont}} \to \mathrm{DGCat}^L_{\mathrm{cont}}$ denote the functor $\mathrm{ind}_{\mathcal{L}}$ as above.

Lemma 2.47. Let X be any k-scheme, and let $X_L := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ denote the base change.

1. There are canonical equivalences $\mathrm{ind}^L_k(\mathrm{IndCoh}(X)) \simeq \mathrm{IndCoh}(X_L)$ and $\mathrm{ind}^L_k(\mathcal{D}(X)) \simeq \mathcal{D}(X_L)$.
2. If X is equipped with an action of an affine algebraic group H and $\chi : H \to \mathbb{G}_a$ is any character, then $\mathrm{ind}^L_k(\mathcal{D}(X)^{H,w}) \simeq \mathcal{D}(X_L)^{H_L,w}$ and $\mathrm{ind}^L_k(\mathcal{D}(X)^{H,\chi}) \simeq \mathcal{D}(X_L)^{H_L,\chi_L}$, where $\chi_L := \chi \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$.

Proof. Both claims of (1) are general properties of IndCoh. For example, we can show this via the fact that IndCoh on any ind-inf-scheme (such as X or X_{dR}) is dualizable (in fact, it is self dual by [GR17b, Chapter 3, Section 6.2.3]) and therefore the tensor product of the category of ind-coherent sheaves on ind-inf-schemes is IndCoh of the product of the ind-inf-schemes [Gai13, Corollary 10.3.6].

We show the first claim of (2), the second follows by an identical argument. We have that $\mathcal{D}(X)^{H,w} \xleftarrow{\sim} \mathcal{D}(X)_{H,w}$ by Theorem 2.10. Write $\mathcal{D}(X)_{H,w}$ as a colimit of maps of objects of the form $\mathrm{IndCoh}(X_{dR}) \otimes \mathrm{IndCoh}(H)^{\otimes i}$. Since ind^L_k is a left adjoint, and therefore commutes with colimits, so we see that $\mathrm{ind}^L_k(\mathcal{D}(X)^{H,w})$ is a colimit of objects of the form $\mathrm{ind}^L_k(\mathrm{IndCoh}(X_{dR}) \otimes \mathrm{IndCoh}(H)^{\otimes i})$. Using the symmetric monoidality of ind^L_k above and Lemma 2.47(1), we see that this colimit diagram is precisely the diagram whose colimit is $\mathcal{D}(X_L)_{H_L,w}$. Applying Theorem 2.10 once again, we obtain our desired claim. ☐

Corollary 2.48. For any map $\mathrm{Sym}_k(\mathfrak{t}) \to L$, there is a canonical equivalence

$$\mathcal{D}(X)^{T,w} \otimes_{\mathrm{Sym}_k(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L \simeq \mathcal{D}(X_L)^{T_L,w} \otimes_{\mathrm{Sym}_L(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L$$

of objects in $\mathrm{DGCat}^L_{\mathrm{cont}}$.

Proof. Note that

$$\mathcal{D}(X)^{T,w} \otimes_{\mathrm{Sym}_k(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L \simeq \mathrm{ind}^{\mathrm{Sym}_L(\mathfrak{t})}_{\mathrm{Sym}_k(\mathfrak{t})}(\mathcal{D}(X)^{T,w}) \otimes_{\mathrm{Sym}_L(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L$$

and so by applying Proposition 2.46 and then applying Lemma 2.47 we see that the above category is equivalent to

$$\mathrm{ind}^L_k(\mathcal{D}(X)^{T,w}) \otimes_{\mathrm{Sym}_L(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L \simeq \mathcal{D}(X_L)^{T_L,w} \otimes_{\mathrm{Sym}_L(\mathfrak{t})\text{-mod}} \mathrm{Vect}_L$$

of objects of $\mathrm{DGCat}^L_{\mathrm{cont}}$. ☐

2.4.2. *Categorical Lemmas from Adjunctions.* The following lemma will later be used to determine the essential image of a fully faithful functor.

Lemma 2.49. Assume we have an adjoint pair (L, R) where $L : \mathcal{C} \to \mathcal{D}$. Let u (respectively c) denote the unit (respectively counit) of this adjunction such that R is fully faithful. Then we have:

1. For any $X \in \mathcal{C}$, $L(u(X))$ is an isomorphism.

(2) The essential image of R is precisely those objects $C \in \mathcal{C}$ for which the unit map $C \to RL(C)$ is an equivalence.

Proof. General properties of adjunction give that the composite

$$L(X) \xrightarrow{L(u(X))} LRL(X) \xrightarrow{c(R(X))} L(X)$$

can be identified with the identity map. However, since R is fully faithful, the counit is an equivalence, and therefore so is $L(u(X))$.

We now show (2). If the unit map is an equivalence, then $C \xrightarrow{\sim} R(L(C))$. Conversely, assume that $C \in \mathcal{C}$ is in the essential image. Then C is isomorphic to $R(D)$ for some $D \in \mathcal{D}$. We then obtain the composite:

$$R(D) \xrightarrow{u(R(D))} RLR(D) \xrightarrow{R(c(D))} R(D)$$

can be identified with the identity, and the rightmost arrow is an isomorphism since R is fully faithful. Therefore, $u(R(D))$ is an isomorphism, and thus so too is the map $u(C)$. ☐

2.4.3. *A Lemma on Compact Objects.* We will repeatedly use the following lemma on compact objects of a subcategory of a compactly generated category, so we recall it here:

Lemma 2.50. [Nee92, Lemma 2.2] Assume $\mathcal{D}$ is a compactly generated DG category and R is a subset of compact objects of $\mathcal{D}^c$ closed under cohomological shifts (i.e. suspensions). Let $\mathcal{R}$ denote the full subcategory of $\mathcal{D}$ generated under colimits by R. Then the inclusion functor $\mathcal{R} \hookrightarrow \mathcal{D}$ preserves compact objects and in particular admits a continuous right adjoint.

2.4.4. *A Lemma on AB5 Categories.* Recall that an abelian category $\mathcal{A}$ is said to be *AB5* if it is closed under colimits and filtered colimits of short exact sequences exist and are exact. We now present a lemma which is likely well known, but we were unable to locate a reference for. We thank Rok Gregorič for explaining its proof to us:

Lemma 2.51. Fix an AB5 abelian category $\mathcal{A}$ and some filtered colimit $\mathcal{F} := \text{colim}_i\mathcal{F}_i$ in $\mathcal{A}$. Let $\phi_i : \mathcal{F}_i \to \mathcal{F}$ denote the structure maps, and let $\tilde{\mathcal{F}}_i$ denote the image of ϕ_i for each i. Then the canonical filtered colimit $\text{colim}_i\tilde{\mathcal{F}}_i$ is equivalent to $\mathcal{F}$, i.e. $\mathcal{F}$ may be written as an increasing union of the images of the ϕ_i.

Proof. We may realize the image $\tilde{\mathcal{F}}_i$ as the limit of the two arrows $\mathcal{F} \rightrightarrows \mathcal{F}\coprod_{\mathcal{F}_i}\mathcal{F}$ which are given by the universal property of the coproduct. We then see that

$$\text{colim}_i\tilde{\mathcal{F}}_i \cong \text{colim}_i\text{lim}(\mathcal{F} \rightrightarrows \mathcal{F}\coprod_{\mathcal{F}_i}\mathcal{F}) \cong \text{lim}(\text{colim}_i\mathcal{F} \rightrightarrows \text{colim}_i(\mathcal{F}\coprod_{\mathcal{F}_i}\mathcal{F})) \cong \text{lim}(\mathcal{F} \rightrightarrows \mathcal{F}\coprod_{\text{colim}_i(\mathcal{F}_i)}\mathcal{F})$$

and since $\text{colim}_i\mathcal{F}_i \cong \mathcal{F}$ by assumption, we see that this expression is equivalently given by the image of the identity, i.e. $\mathcal{F}$. Here, the commutation of limits and colimits is given by the axiom of AB5 categories because, by assumption, the colimit functor is left exact, and left exact functors preserve all finite limits. ☐

2.4.5. *t-Structures on Quotient Categories.* We set the following notation for this subsection: Assume $I_* : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a fully faithful embedding in $\text{DGCat}^k_{\text{cont}}$ which admits a (continuous) right adjoint $I^!$. Furthermore, assume $\mathcal{C}$ is equipped with an accessible t-structure (i.e. the ∞-category $\mathcal{C}^{\geq 0}$ is accessible) which is compatible with filtered colimits, i.e. $\mathcal{C}^{\geq 0}$ is closed under filtered colimits. Furthermore assume the essential image of I_* is closed under truncation functors (and so, in particular, $\mathcal{C}_0$ admits a unique t-structure so that I_* is t-exact) and the essential image of $I_*^{\heartsuit} : \mathcal{C}_0^{\heartsuit} \hookrightarrow \mathcal{C}^{\heartsuit}$ is closed under subobjects. Let $\mathring{\mathcal{C}} := \ker(\mathcal{C} \xrightarrow{I^!} \mathcal{C}_0)$. Note that the inclusion functor $J_* : \mathring{\mathcal{C}} \hookrightarrow \mathcal{C}$ admits a (necessarily continuous) left adjoint $J^!$ because $I^!$ preserves small limits.

By definition, the functor $J^!$ is a localization:

Definition 2.52. A functor with a fully faithful right adjoint is called a *localization.*

When clear from context, we will refer to $J^!$ as *the quotient functor*.

Example 2.53. If $j : U \hookrightarrow X$ is an open embedding of algebraic varieties, the functor $j^! : \mathcal{D}(X) \to \mathcal{D}(U)$ is a localization.

We will now recall the following general result of [Ras20a].

Proposition 2.54. [Ras20a, Section 10.2] There is a unique t-structure on $\mathring{\mathcal{C}}$ such that $J^!$ is t-exact, uniquely determined by setting $\mathring{\mathcal{C}}^{\geq 0}$ as the full subcategory of objects $\mathcal{F} \in \mathring{\mathcal{C}}$ for which $J_*(\mathcal{F}) \in \mathcal{C}^{\geq 0}$.

Note that, in particular, this t-structure is accessible, since the category $\mathring{\mathcal{C}}^{\geq 0}$ is the fiber product of accessible categories $\mathring{\mathcal{C}} \times_{\mathcal{C}} \mathcal{C}^{\geq 0}$ and therefore is accessible by [Lur09, Section 5.4.6].

Corollary 2.55. The inclusion functor J_* identifies $\mathring{\mathcal{C}}^+$ with the kernel of $I^!|_{\mathcal{C}^+}$, and the inclusion functor I_* identifies $\mathcal{C}_0^+$ with the kernel of $J^!|_{\mathcal{C}^+}$.

Proof. Because I_* is t-exact, its right adjoint $I^!$ is left t-exact and thus preserves the eventually coconnective subcategory. In particular, if $\mathcal{F} \in \mathring{\mathcal{C}}^+$ then $J_*(\mathcal{F}) \in \mathring{\mathcal{C}}^+$, and $I^!(J_*(\mathcal{F})) \simeq 0$. Conversely if $X \in \mathcal{C}^+$ lies in the kernel of $I^!$ then $J^!(X) \in \mathring{\mathcal{C}}^+$ by the t-exactness of $J^!$. Similarly if $Z \in \mathcal{C}_0^+$ then $I_*(Z) \in \mathcal{C}^+$ by the t-exactness of I_* and if $X \in \mathcal{C}^+$ lies in the kernel of $J^!$ then $I^!(X) \in \mathcal{C}_0^+$ since $I^!$ is left t-exact, being the right adjoint to a t-exact functor I_*. □

Lemma 2.56. The t-structure on $\mathring{\mathcal{C}}$ is compatible with filtered colimits.

Proof. Assume we are given a filtered diagram I in $\mathring{\mathcal{C}}^{\geq 0}$. Let $X := \text{colim}_{i \in I} X_i$ denote the colimit of this diagram. Then we see that the continuity of J_* gives $J_*(X) \simeq \text{colim}_i(J_*(X_i))$, and therefore since the t-structure on $\mathcal{C}$ is compatible with filtered colimits by assumption, we see $\text{colim}_i(J_*(X_i)) \in \mathcal{C}^{\geq 0}$ and so by Proposition 2.54 we have $X \in \mathring{\mathcal{C}}^{\geq 0}$, and so the t-structure compatible with filtered colimits by definition. □

Corollary 2.57. If the t-structure on $\mathcal{C}$ is right-complete, then the t-structure on $\mathring{\mathcal{C}}$ is right-complete.

Proof. Let $\mathcal{F} \in \mathring{\mathcal{C}}$, and let $\phi^{\mathcal{F}} : \mathcal{F} \to \text{colim}(\tau^{\leq n}\mathcal{F})$ denote the canonical map. Using the fact that the t-structure on $\mathring{\mathcal{C}}$ is compatible with filtered colimits, we see that for all $n \in \mathbb{Z}$ since $\tau^{\leq n}(\phi^{\mathcal{F}})$ is an equivalence. Therefore, if we denote by $\mathcal{K}$ the fiber of $\phi^{\mathcal{F}}$, we see $\tau^{\leq n}(\mathcal{K}) \simeq 0$ for all n. Thus $J_*(\mathcal{K}) \in \cap_n C^{\geq n} \simeq 0$, so $\phi^{\mathcal{F}}$ is an equivalence, as required. □

Although J_* is not right t-exact in general, we do have the following statement:

Lemma 2.58. If $X \in \mathring{\mathcal{C}}^{\leq 0}$, then there is a canonical equivalence $X \simeq J^!(\tau^{\leq 0}J_*(X))$.

Proof. For such an X, we obtain a cofiber sequence $\tau^{\leq 0}J_*X \to J_*X \to \tau^{>0}X$. Applying $J^!$ to this cofiber sequence (which is continuous and thus exact by construction), we obtain the cofiber sequence

$$J^!(\tau^{\leq 0}J_*X) \to J^!J_*X \to J^!(\tau^{>0}X) \tag{2}$$

and furthermore we see that the unit map $X \xrightarrow{\sim} J_*J^!(X)$ identifies the middle term of (2) with X. The left and middle terms of (2) thus both lie in $\mathring{\mathcal{C}}^{\leq 0}$ while the rightmost term lies in $\mathring{\mathcal{C}}^{>0}$, so the rightmost term vanishes. Thus the rightmost term in (2) vanishes and so the composite

$$J^!(\tau^{\leq 0}J_*X) \xrightarrow{\sim} J^!J_*X \xleftarrow{\sim} X$$

gives our desired equivalence. □

Finally, we record the interaction with the above discussion when our underlying category DGCat is replaced with G-categories:

Proposition 2.59. Assume G acts on $\mathcal{C}$ compatibly with the t-structure and the subcategory $\mathcal{C}_0$ is closed under the G-action. Then $\mathring{\mathcal{C}}$ acquires a G-action and moreover G acts compatibly with the t-structure.

Proof. First note that the fact that I_* is a functor of G-categories gives that $I^!$ is a functor of G-categories by Theorem 2.19. Therefore we see that $\mathring{\mathcal{C}} := \ker(I^!)$ acquires a G-action. We have that J_* is G-equivariant and so $J^!$ is as well by Theorem 2.19. Let c, c_0 (respectively) denote the coaction map of G on $\mathcal{C}, \mathring{\mathcal{C}}$ (respectively) shifted by $[-\dim(G)]$ so that c is by assumption t-exact.

We now wish to show c_0 is t-exact if c is. To see this, first note if $X \in \mathring{\mathcal{C}}^{\geq 0}$ then $(\text{id}_G \otimes J_*)c_0(X) \simeq c(J_*(X))$ by the G-equivariance of J_* and so $c_0(X) \in (\mathcal{D}(G) \otimes \mathring{\mathcal{C}})^{\geq 0}$ by Lemma 2.28. Now assume $X \in \mathring{\mathcal{C}}^{\leq 0}$. By Lemma 2.58, we may write $X \simeq J^!(\tau^{\leq 0}J_*(X))$. Since c is t-exact and G-equivariant, we can use this description of X to obtain equivalences

$$c(X) \simeq (\text{id}_{\mathcal{D}(G)} \otimes J^!)c_0\tau^{\leq 0}J_*(X) \simeq (\text{id}_{\mathcal{D}(G)} \otimes J^!)\tau^{\leq 0}c_0J_*(X) \simeq (\text{id}_{\mathcal{D}(G)} \otimes J^!)\tau^{\leq 0}(\text{id}_{\mathcal{D}(G)} \otimes J_*)(c_0(X))$$

so that in particular $c(X) \simeq (\mathrm{id}_{\mathcal{D}(G)} \otimes J^!)(Y)$ for some $Y \in (\mathcal{D}(G) \otimes \mathcal{C})^{\leq 0}$. Therefore, $c_0(X) \in (\mathcal{D}(G) \otimes \mathring{\mathcal{C}})^{\leq 0}$ by the t-exactness of $\mathrm{id}_{\mathcal{D}(G)} \otimes J^!$ given by Lemma 2.28. □

2.4.6. *t-Structures on Quotient Categories Given by the Eventually Coconnective Kernel.* Assume $\mathcal{C}$ is a DG category with a t-structure compatible with filtered colimits, and let $L : \mathcal{C} \to \mathcal{D}$ be a t-exact functor of stable ∞-categories equipped with t-structures. Set $\mathcal{C}_0$ to denote the full subcategory generated under filtered colimits by eventually coconnective objects in the kernel of L, i.e. the full subcategory generated by objects $X \in \mathcal{C}^+$ for which $L(X) \simeq 0$. By t-exactness of L, $\mathcal{C}_0$ is closed under the truncation functors $\tau^{\leq 0}$ and $\tau^{\geq 0}$, where the latter also uses that the t-structure on $\mathcal{C}$ is compatible with filtered colimits. The t-exactness of L also gives essential image of $\mathcal{C}_0^{\heartsuit} \hookrightarrow \mathcal{C}^{\heartsuit}$ is closed under subobjects. We equip the full subcategory $\mathring{\mathcal{C}} \xhookrightarrow{J_*} \mathcal{C}$ with the t-structure so that the quotient functor $J^!$ is t-exact as in Proposition 2.54. Let $\tilde{L} := L \circ J_*$.

Proposition 2.60. The functor $\tilde{L}$ is t-exact.

Proof. By definition, $\tilde{L}$ is the composite of two left t-exact functors, and therefore is left t-exact. Now assume that $X \in \mathring{\mathcal{C}}^{<0}$. Then, because $X \xrightarrow{\sim} J^! J_*(X)$, we have that $\tau^{\geq 0}(J_*(X))$ lies in the kernel of $J^!$, since $J^!$ is t-exact, and thus $\tau^{\geq 0}(J_*(X)) \in \mathcal{C}_0$. By definition of $\mathcal{C}_0$, we see that $L(\tau^{\geq 0}(J_*(X))) \simeq 0$. Therefore we see that $\tau^{\geq 0}\tilde{L}(X) \simeq \tau^{\geq 0} L J_*(X) \simeq L(\tau^{\geq 0} J_*(X)) \simeq 0$, so $\tilde{L}$ is also right t-exact. □

Now, with the above notation, further assume that $\mathcal{D}$ is a DG category and that L is a functor in $\mathrm{DGCat}^k_{\mathrm{cont}}$ which admits a (continuous) right adjoint $R : \mathcal{D} \to \mathcal{C}$. Because $I^! R$ is a right adjoint, one can easily verify that $I^! R \simeq 0$, and so the canonical map $R \to J_* J^! R$ is an equivalence. Let $\tilde{R}$ denote the functor $J^! R$.

Proposition 2.61. In the above notation, $\tilde{R}$ is left adjoint to $\tilde{L}$.

Proof. We have equivalences, for all $X \in \mathring{\mathcal{C}}$ and $Y \in \mathcal{D}$:

$$\mathrm{Hom}_{\mathcal{D}}(\tilde{L}(X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\tilde{L} J^! J_*(X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(L J_*(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(J_*(X), R(Y))$$

via the fully faithfulness of J_*, the definition of $\tilde{L}$, and the adjoint property respectively. Since, $R \xrightarrow{\sim} J_* J^! R$, so the above expression is equivalent to:

$$\mathrm{Hom}_{\mathcal{C}}(J_*(X), J_* J^! R(Y)) \simeq \mathrm{Hom}_{\mathring{\mathcal{C}}}(X, J^! R(Y)) \simeq \mathrm{Hom}_{\mathring{\mathcal{C}}}(X, \tilde{R}(Y))$$

by the fully faithfulness of J_* and the definition of $\tilde{R}$, respectively, which completes our proof. □

3. The Coarse Quotient

In this section, we define the coarse quotient $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ and determine some of its basic properties. After reviewing basic properties in derived algebraic geometry which follow from the fact $\tilde{W}^{\mathrm{aff}}$ is discrete in Section 3.1, we define this quotient in Section 3.2, and show that sheaves on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ are sheaves on $\mathfrak{t}^*$ which satisfy Coxeter descent in Section 4.5.

In order to compute the category of sheaves on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, we will use the fact that the map $\mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ is an ind-finite flat cover. Preliminary computations to show this are done in Appendix A, and then this is proved in Corollary 3.16.

3.1. **Discrete Sets of Points in Derived Algebraic Geometry.** The primary result of this subsection is Proposition 3.5, which says that quotienting by $X^{\bullet}(T)$ does not affect formal completions. The reader who is willing to take this informal statement on faith may skip this subsection.

3.1.1. *Maps into* Λ. In this subsection, let Λ denote any discrete set of points over our field. If A is a classical ring, then $\mathrm{Spec}(A)$ is well known to be quasicompact as a topological space. Therefore, the identity $\mathrm{Hom}_{{}^{\mathrm{cl}}\mathrm{Sch}}(\mathrm{Spec}(A), \Lambda) \simeq \mathrm{colim}_S \mathrm{Hom}_{{}^{\mathrm{cl}}\mathrm{Sch}}(\mathrm{Spec}(A), S)$ holds, where S varies over the finite subsets of Λ. In this subsection, we show this identity holds in derived algebraic geometry:

Proposition 3.1. If A is any affine scheme and Λ is a discrete set of k-points, then $\mathrm{Hom}_{\mathrm{PreStk}}(\mathrm{Spec}(A), \Lambda) \simeq \mathrm{colim}_S \mathrm{Hom}_{\mathrm{PreStk}}(\mathrm{Spec}(A), S)$ where S ranges over the finite subsets of Λ.

Proof. The prestack Λ is a classical scheme. Therefore, it is in particular a 0-truncated prestack, meaning that, in the notation of [GR17a], Λ is an object of the fully faithful embedding ${}^{\leq 0}\mathrm{PreStk} \hookrightarrow \mathrm{PreStk}$. This embedding admits a left adjoint $\tau^{\geq 0}$ which sends the prestack $\mathrm{Spec}(A)$ to $\mathrm{Spec}(H^0(A))$, see Chapter 2, 1.3.2 of [GR17a]. Therefore, we see that for any affine test scheme $\mathrm{Spec}(A)$, we have:

$$\mathrm{Hom}_{\mathrm{PreStk}}(\mathrm{Spec}(A), \Lambda) \simeq \mathrm{Hom}_{\geq 0\mathrm{PreStk}}(\mathrm{Spec}(H^0(A)), \Lambda) \simeq \mathrm{colim}_S \mathrm{Hom}_{\geq 0\mathrm{PreStk}}(\mathrm{Spec}(H^0(A)), S)$$

where the second equivalence follows from the claim for classical schemes. Therefore, applying the adjunction to the terms in the colimit above, we obtain our claim. ☐

Corollary 3.2. If Λ is any discrete set of points, the canonical map $\Lambda \to \Lambda_{dR}$ is an equivalence.

Proof. The claim is clear when Λ is a finite set since Λ is an affine scheme in this case and the result can be computed explicitly. If Λ is infinite, we have that for any test scheme T

$$\mathrm{Hom}(T, \Lambda_{dR}) \simeq \mathrm{Hom}(T^{red}, \Lambda) \simeq \mathrm{colim}_S \mathrm{Hom}(T^{red}, S) \simeq \mathrm{colim}_S \mathrm{Hom}(T, S) \simeq \mathrm{Hom}(T, \Lambda)$$

using the adjunction property of the de Rham prestack, Proposition 3.1, the result of Corollary 3.2 for a finite set of points, and Proposition 3.1, respectively. ☐

Remark 3.3. Alternatively, one can show Corollary 3.2 by using the fact that the de Rham functor commutes with colimits, as it is a left adjoint.

Using Corollary 3.2, we immediately obtain a monoidal equivalence of categories $\mathcal{D}(\Lambda) := \mathrm{IndCoh}(\Lambda_{dR}) \simeq \mathrm{IndCoh}(\Lambda)$. We use this to show the following:

Corollary 3.4. Assume $\tilde{\Lambda}$ is a group ind-scheme whose underlying set of points is discrete, and that $\mathcal{C}$ is a category with an action of $\tilde{\Lambda}$. Then the canonical map $\mathcal{C}_\Lambda \to \mathcal{C}^\Lambda$ is an equivalence.

Proof. By Corollary 3.2, we need only show this for the weak action of Λ. However, since Λ is proper, the associated action maps are ind-proper, and therefore have left adjoints given by IndCoh pushforward (see chapter 3 of [GR17b]). Therefore, our claim follows by [GR17a, Chapter 1, Corollary 2.5.7]. ☐

3.1.2. *Formal Completions of* $\mathfrak{t}^*$ *and* $\mathfrak{t}^*/X^\bullet(T)$ *at Field-Valued Points.* We now record the following corollary of Corollary 3.2:

Proposition 3.5. Pick some field-valued $\lambda \in \mathfrak{t}^*(L)$ and let $[\lambda] \in \mathfrak{t}^*/X^\bullet(T)$ denote its projection to the quotient. Then the canonical map of formal completions $\mathfrak{t}^{*,\wedge}_\lambda \to (\mathfrak{t}^*/X^\bullet(T))^\wedge_{[\lambda]}$ is an equivalence.

Proof. We have that $(\mathfrak{t}^*/X^\bullet(T))^\wedge_{[\lambda]}$ is equivalent to

$$(\mathfrak{t}^*/X^\bullet(T)) \times_{(\mathfrak{t}^*/X^\bullet(T))_{dR}} \{[\lambda]\}_{dR} \simeq \mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^*_{dR}/X^\bullet(T)_{dR}} \{[\lambda]\}_{dR} \simeq \mathfrak{t}^*/X^\bullet(T) \times_{\mathfrak{t}^*_{dR}/X^\bullet(T)} (\lambda + X^\bullet(T))/X^\bullet(T)$$

where the first equivalence follows because the de Rham prestack commutes with colimits [GR17b, Chapter 4, Section 1.1.2], and the second equivalence follows from applications of Corollary 3.2 and the definition of $[\lambda]$. Therefore, since geometric realizations in PreStk commute with fiber products, we see this expression is equivalent to

$$(\mathfrak{t}^* \times_{\mathfrak{t}^*_{dR}} (\lambda + X^\bullet(T)))/X^\bullet(T) \simeq (\coprod_{\lambda' \in \lambda + X^\bullet(T)} \mathfrak{t}^{*,\wedge}_{\lambda'})/X^\bullet(T) \simeq (X^\bullet(T) \times \mathfrak{t}^{*,\wedge}_\lambda)/X^\bullet(T) \simeq \mathfrak{t}^{*,\wedge}_\lambda$$

where the first equivalence follows from the fact that we can write $X^\bullet(T)$ as a filtered colimit of its finite subsets and the fact that filtered colimits commute with finite limits, and the final two steps follow from a $X^\bullet(T)$-equivariant equivalence $X^\bullet(T) \times \mathfrak{t}^{*,\wedge}_\lambda \simeq \coprod_{\lambda' \in \lambda + X^\bullet(T)} \mathfrak{t}^{*,\wedge}_{\lambda'}$. ☐

3.2. **GIT Quotients and Groupoid Objects.** In this section, after recalling some basics on the finite coarse quotient $\mathfrak{t}^* /\!\!/ W$, we will define the prestacks $\mathfrak{t}^* /\!\!/ W^{\mathrm{aff}}$ and $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$, where $\tilde{W}^{\mathrm{aff}} := X^\bullet(T) \rtimes W$ is the extended affine Weyl group for G and $W^{\mathrm{aff}} \subseteq \tilde{W}^{\mathrm{aff}}$ is the affine Weyl group for the adjoint quotient, which is a Coxeter group. For a thorough treatment of groupoid objects in the (1,1) and $(\infty, 1)$ setting, see [Lur09, Section 6.1.2]. We will be terse here, because we are largely interested in the category of *sheaves* on spaces like $\mathfrak{t}^* /\!\!/ W^{\mathrm{aff}}$ and $\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ W^{\mathrm{aff}}} \mathfrak{t}^*$, which are often more explicit. In particular, the reader who is willing to black box the claim that sheaves on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ are $\tilde{W}^{\mathrm{aff}}$-equivariant sheaves satisfying Coxeter descent (Proposition 4.18) and that, for G adjoint, the (derived) product $\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$ can be realized as the (classical) union of graphs of $\tilde{W}^{\mathrm{aff}}$ in $\mathfrak{t}^* \times \mathfrak{t}^*$ (and so we can apply the toolkit of derived algebraic geometry to this classical algebro-geometric object) will lose no information skipping this section (Section 3.2).

3.2.1. *GIT Quotients.* Recall the notion of a *GIT quotient* as in [MFK94]; we will only use the affine version so we content ourselves to define, for any classical affine scheme $X = \mathrm{Spec}(A)$ with an action of a reductive group H, the *GIT* or *coarse quotient* $X /\!\!/ H := \mathrm{Spec}(A^H)$ [MFK94, Theorem 1.1]. Direct computation shows that:

Proposition 3.6. If H is a finite group acting on an affine scheme $X = \mathrm{Spec}(A)$ then the canonical map

$$X /\!\!/ H \simeq \mathrm{colim}(X \times_{X /\!\!/ H} X \rightrightarrows X)$$

is an equivalence, where the colimit is taken in the (1,1)-category of classical affine schemes.

3.2.2. *Classical Groupoid Objects.* We would like to extend the ideas in Section 3.2.1 to replace W above with $\tilde{W}^{\mathrm{aff}}$. Let Γ^{fin} denote the union of graphs of the various $w \in W$, cut out by the intersection of ideals (which is, in general, distinct from the product of the ideals, see Remark A.4). Then Γ^{fin} is a *classical groupoid object* over $\mathfrak{t}^*$ of the (1,1)-category of schemes. This means that Γ^{fin} is equipped with the data of maps relating Γ^{fin} and $\mathfrak{t}^*$ satisfying certain conditions. In particular, Γ^{fin} is equipped with the data of two maps $\Gamma^{\mathrm{fin}} \rightrightarrows \mathfrak{t}^*$, which in this case are the two projection maps. Then we have:

Proposition 3.7. [Lon18, p22] We have an equivalence $\Gamma^{\mathrm{fin}} \xrightarrow{\sim} \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ W} \mathfrak{t}^*$.

Remark 3.8. The equivalence Proposition 3.7 holds in either the category of classical affine schemes or derived affine schemes since the map $\mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ W$ is finite flat; in turn, this is a direct consequence of the Chevalley-Shephard-Todd theorem [Che55], [ST54], which gives the analogous statement for any vector space V and finite subgroup of $\mathrm{GL}(V)$ generated by reflections. In particular, the equivalence of Proposition 3.7 also holds where the product is taken in the category of prestacks.

Furthermore, we recall for later use that the Chevalley-Shephard-Todd theorem states that $\mathfrak{t}^* /\!\!/ W$ is smooth–in fact, it is a polynomial algebra.

3.2.3. *Groupoid Objects in Higher Algebra.* In derived algebraic geometry, the notion of a classical groupoid in Section 3.2.2 is replaced with the notion of a *groupoid* or an ∞-groupoid of an $(\infty, 1)$-category. We use an equivalent formulation of the notion of a groupoid (see [Lur17, Section 6.1.2.6]):

Definition 3.9. A *groupoid object* of an $(\infty, 1)$ category $\mathcal{C}$ is a simplicial object U of $\mathcal{C}$ such that for every $n \geq 0$ and every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element s, the canonical map $U([n]) \to U(S) \times_{U(\{s\})} U(S')$ is an equivalence (and, in particular, the latter term is defined).

We now recall the basic results about groupoid objects in the $(\infty, 1)$ category of spaces Spc, which immediately implies the analogous fact for the category of prestacks since limits and colimits in functor categories are computed termwise:

Proposition 3.10. [Lur09, Corollary 6.1.3.20] Every groupoid object of Spc is *effective*. In particular, if $U_\bullet$ is a groupoid object of Spc, then a geometric realization U_{-1} of it exists and the canonical map $U_1 \to U_0 \times_{U_{-1}} U_0$ is an equivalence.

3.3. **The Coarse Quotient.** We now wish to apply the general framework above to our specific case of interest. Let $\Gamma_{W^{\mathrm{aff}}}$ denote the union of graphs of each $w \in W^{\mathrm{aff}}$, where by graph we mean the closed subschemes $\mathfrak{t}^* \xhookrightarrow{(w,\mathrm{id})} \mathfrak{t}^* \times \mathfrak{t}^*$, and by union we mean the union given by the intersection of ideals, see Remark A.4. In particular, we can realize $\Gamma_{W^{\mathrm{aff}}}$ as a classical ind-scheme, and in particular an ind-closed subscheme of $\mathfrak{t}^* \times \mathfrak{t}^*$.

3.3.1. *Definition of the Coarse Quotient.* Let $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ denote the balanced product $\tilde{W}^{\mathrm{aff}} \overset{W^{\mathrm{aff}}}{\times} \Gamma_{W^{\mathrm{aff}}}$. Note that $\Gamma_{\tilde{W}^{\mathrm{aff}}}$ admits canonical maps to $s, t : \Gamma_{\tilde{W}^{\mathrm{aff}}} \to \mathfrak{t}^*$ given by the maps $s(\sigma, (w\lambda, \lambda)) = \lambda$ and $t(\sigma, (w\lambda, \lambda)) = \sigma w\lambda$. We now record the following general fact which remains valid if the subgroup $W^{\mathrm{aff}} \leq \tilde{W}^{\mathrm{aff}}$ is replaced with any closed subgroup $H \leq \tilde{H}$ and $\Gamma_{W^{\mathrm{aff}}}$ is replaced with any Γ with an H-action:

Proposition 3.11. We have an isomorphism $\Gamma_{\tilde{W}^{\mathrm{aff}}} \xrightarrow{\sim} \tilde{W}^{\mathrm{aff}}/W^{\mathrm{aff}} \times \Gamma_{W^{\mathrm{aff}}}$ in such a way that the following diagram commutes:

$$\begin{tikzcd} \Gamma_{\tilde{W}^{\mathrm{aff}}} := \tilde{W}^{\mathrm{aff}} \overset{W^{\mathrm{aff}}}{\times} \Gamma_{W^{\mathrm{aff}}} \arrow[r, "\sim"] \arrow[dr, "t"'] & \tilde{W}^{\mathrm{aff}}/W^{\mathrm{aff}} \times \Gamma_{W^{\mathrm{aff}}} \arrow[d, "\mathrm{proj}"'] \\ & \mathfrak{t}^* \end{tikzcd}$$

Proof. The isomorphism is induced by the map $(\tilde{w}, g) \mapsto (\tilde{w}, \tilde{w}g)$, and the inverse map is induced by $(\tilde{w}, g') \mapsto (\tilde{w}, \tilde{w}^{-1}g')$. $\square$

We now may construct a groupoid object $\Gamma_\bullet$ over $\mathfrak{t}^*$ such that $\Gamma_1 \simeq \Gamma_{\tilde{W}^{\mathrm{aff}}}$–specifically, set $\Gamma_n := \Gamma_{\tilde{W}^{\mathrm{aff}}} \times_{\mathfrak{t}^*} ... \times_{\mathfrak{t}^*} \Gamma_{\tilde{W}^{\mathrm{aff}}}$. However, working with this object in general would be a technical nusance, a priori: all of our fiber products are inherently derived. However, the following proposition allows us to argue that the Γ_n systematically remain in the classical (1,1)-categorical setting.

Proposition 3.12. The map $s : \Gamma_{\tilde{W}^{\mathrm{aff}}} \to \mathfrak{t}^*$ is ind-finite flat.

Proof. By Proposition 3.11, it suffices to show that the map $\Gamma_{W^{\mathrm{aff}}} \to \mathfrak{t}^*$ is flat. In this case, W^{aff} is a Coxeter group and has a length function ℓ. For each positive integer m set $S_m = \{w \in W^{\mathrm{aff}} : \ell(w) \leq m\}$, and let G_m denote the union of graphs of those $w \in S_m$. Then we clearly have $\Gamma_{W^{\mathrm{aff}}} = \cup_m G_m$, and so it suffices to show that $s_m : G_m \to \mathfrak{t}$ is finite flat. However, this follows from Corollary A.21. $\square$

Corollary 3.13. The $\Gamma_\bullet$ above forms a groupoid object in prestacks.

Proof. To show that $\Gamma_\bullet$ is a groupoid object in the category of prestacks, we must check that for each S and S' such that $|S \cap S'| = 1$ the diagram:

$$\begin{tikzcd} \Gamma_{[n]} \arrow[r] \arrow[d] & \Gamma_S \arrow[d] \\ \Gamma_{S'} \arrow[r] & \Gamma_{S \cap S'} \simeq \mathfrak{t}^* \end{tikzcd}$$

is Cartesian. However, our flatness result above implies that each $\Gamma_S \to \mathfrak{t}^*$ is flat, and so, in particular, for any $S, S' \in \Delta$ fiber product $\Gamma_S \times_{\mathfrak{t}^*} \Gamma_{S'}$ is a colimit of classical schemes of the form

$$G_1 \times_{\mathfrak{t}^*} G_2 \times_{\mathfrak{t}^*} \times ... \times G_\ell$$

where each $G_i \to \mathfrak{t}^*$ is a flat map of classical schemes. In particular, this fiber product itself is a classical scheme because each map is flat. Because filtered colimits commute with finite limits, we see that the product in the category of prestacks agrees with the product in the category of classical ind-schemes. Since $\Gamma_\bullet$ is a groupoid object of classical ind-schemes, we see that $\Gamma_\bullet$ is also a groupoid object in prestacks in the sense of Definition 3.9. $\square$

Since the category of prestacks admits all geometric realizations, we define $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ as the geometric realization of $\Gamma_\bullet$. Then we see that Proposition 3.10 immediately implies:

Proposition 3.14. We have a canonical equivalence $\Gamma_{\tilde{W}^{\mathrm{aff}}} \xrightarrow{\sim} \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$.

Remark 3.15. One may also wish to sheafify $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ for some topology. This parallels the case of the finite coarse quotient. Because, as $\mathfrak{t}^* /\!\!/ W$ is a classical scheme, a theorem of Grothendieck implies it is a sheaf in the fpqc topology. In particular, since $\mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ W$ is fpqc (see Remark 3.8) we obtain the canonical map

$$\mathrm{colim}(\ ... \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \Gamma^{\mathrm{fin}} \underset{\mathrm{s}}{\overset{\mathrm{t}}{\rightrightarrows}} \mathfrak{t}^*\ ) \to \mathfrak{t}^* /\!\!/ W$$

is an equivalence, where we take the colimit in the category of fpqc stacks and use Proposition 3.7.

Sheafification will not affect our analysis below since the functor IndCoh factors through sheafification, and our critical Proposition 3.14 is not affected since sheafification commutes with finite limits: see [GR17a, Section 2.3.6] for the results of [Lur09] specific to our setup.

We once and for all let $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ denote the quotient map.

Corollary 3.16. The map $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ is ind-finite flat.

Proof. We may check whether a map is ind-finite after base change by $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$. However, by Proposition 3.10, our based changed map is canonically $t : \Gamma \to \mathfrak{t}^*$, which is ind-finite flat by Proposition 3.12. $\square$

We now proceed to discuss sheaves on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ and other associated prestacks. We would like to use the critical result that IndCoh is defined on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ and satisfies ind-proper descent, since we now have that the map $\overline{s} : \mathfrak{t} \to \mathfrak{t} /\!\!/ \tilde{W}^{\mathrm{aff}}$ is ind-schematic and ind-proper. It remains to verify the following technical condition which may be skipped at first pass:

Proposition 3.17. Any prestack which is a countable discrete set of points is 0-coconnective locally finite type (lft). Furthermore, the respective quotient prestacks $\mathfrak{t}^*/X^\bullet(T)$ and $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ are 0-coconnective lft prestacks.

Proof. The n-coconnective prestacks (respectively, the n-coconnective lft prestacks) are those prestacks which are in the essential image of a certain left adjoint–namely, the left Kan extension of the inclusion of n-coconnective affine schemes (respectively, the left Kan extension of the inclusion of n-coconnective finite type affine schemes in the sense of [GR17a, Chapter 2, Section 1.5]). Therefore, the condition of being n-coconnective and the condition of being n-coconnective lft are conditions closed under colimits. Classical finite type affine schemes are 0-coconnective lft. Therefore since this condition is closed under colimits, any discrete set of points is also 0-coconnective lft. In turn, since $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ is a certain colimit of 0-coconnective lft prestacks, we see that $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ is a 0-coconnective lft prestack. $\square$

Thus by Proposition 3.17, we obtain that $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ is defined. In particular, Corollary 3.16 implies that $\overline{s}$ is in particular ind-schematic and ind-proper surjection between laft prestacks. We now record two immediate consequences of this fact.

Corollary 3.18. [GR17b, Chapter 3, Section 0.1.2] The functor $\overline{s}^!$ admits a left adjoint satisfying base change against !-pullbacks.

We denote this left adjoint by s_*^{IndCoh}.

Corollary 3.19. [GR17b, Chapter 3, Section 0.4.3] If $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ denotes the quotient map, the pullback functor $\overline{s}^! : \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) \to \mathrm{IndCoh}(\mathfrak{t}^*)$ induces an equivalence

$$\overline{s}^! : \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}) \xrightarrow{\sim} \mathrm{Tot}(\mathrm{IndCoh}(\mathfrak{t}^{*\bullet}))$$

where $\mathfrak{t}^{*\bullet}$ is the cosimplicial prestack given by the Čech nerve of $\overline{s}$.

3.3.2. *t-Structure on* $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathit{aff}})$ *and* $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathit{aff}})$. We use the following proposition define a t-structure on the category $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ and determine some of its basic properties in Proposition 3.21.

Proposition 3.20. The maps $s^! : \mathrm{IndCoh}(\mathfrak{t}^*) \to \mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})$ and s_*^{IndCoh} are t-exact.

Proof. Because the t-structure on $\mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})$ is by definition compatible with filtered colimits [GR17b, Chapter 3, Section 1.2.1] it suffices to show this claim when $\tilde{W}^{\mathrm{aff}} = W^{\mathrm{aff}}$. Note that the map s_*^{IndCoh} is t-exact because it is ind-affine [GR17b, Chapter 3, Lemma 1.4.9], and therefore since s is ind-proper, we have that $s^!$ is the right adjoint to the t-exact functor s_*^{IndCoh} (see Corollary 3.18) and therefore is left t-exact. We now show that $s^!$ is right t-exact.

We first claim that, to show $s^!$ is right t-exact, it suffices to show that $s^!(\mathcal{O}_{\mathfrak{t}^*}) \in \mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})^{\leq 0}$. This follows since $\mathrm{IndCoh}(\mathfrak{t}^*)^{\leq 0}$ is equivalently smallest ∞-category of $\mathrm{IndCoh}(\mathfrak{t}^*)$ containing $\mathcal{O}_{\mathfrak{t}^*}$ and closed under colimits (which can be seen, for example, by the t-exact equivalence $\Psi_{\mathfrak{t}^*} : \mathrm{IndCoh}(\mathfrak{t}^*)^{\leq 0} \xrightarrow{\sim} \mathrm{QCoh}(\mathfrak{t}^*)^{\leq 0}$ given by the fact $\mathfrak{t}^*$ is smooth and classical). Since $s^!$ commutes with colimits and the subcategory $\mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})^{\leq 0}$

is closed under colimits, we see that it remains to show that $s^!(\mathcal{O}_{\mathfrak{t}^*}) \in \mathrm{IndCoh}(\Gamma_{W^{\mathrm{aff}}})^{\leq 0}$. In turn, to show this, we first note that

$$s^!(\mathcal{O}_{\mathfrak{t}^*}) \simeq s^!(\omega_{\mathfrak{t}^*}[-d]) \simeq \omega_\Gamma[-d] \simeq \mathrm{colim}_m i^{\mathrm{IndCoh}}_{m,*}(\omega_{\Gamma_m})[-d]$$

where the first equivalence follows from the fact that $\mathfrak{t}^*$ is smooth, the second follows from by the definition of the dualizing complex and the functoriality of !-pullback, and the third follows since we have an equivalence $\mathrm{IndCoh}(\Gamma) \xleftarrow{\sim} \mathrm{colim}_m \mathrm{IndCoh}(\Gamma_m)$. We claim that each dualizing complex ω_{Γ_m} is concentrated in a single cohomological degree, i.e. Γ_m is Cohen-Macaulay. This follows from the fact that the map $\Gamma_m \to \mathfrak{t}^*$ is a finite flat map (Corollary A.21) to affine space, and therefore Γ_m is Cohen-Macaulay. Thus each object of the above colimit is contained entirely in cohomological degree zero [GR17b, Chapter 4, Lemma 1.2.5] and therefore so too is $s^!(\mathcal{O}_{\mathfrak{t}^*})$ since the t-structure is compatible with filtered colimits. □

Recall the canonical quotient map $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$. Define a t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ by declaring $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{\leq 0}$ to be the full ordinary ∞-subcategory closed under colimits and containing $\overline{s}^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{t}^*})$. Similarly, we define a t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ by declaring $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ to be the full ordinary ∞-subcategory closed under colimits and containing $q^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{t}^*})$. Note that these do indeed define t-structures since the inclusion functor preserves colimits, and therefore admits a right adjoint. We now record further properties of these t-structures:

Proposition 3.21. With the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ defined as above, we have the following:

(1) The right adjoint to the inclusion $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{\leq 0} \to \mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ is continuous.
(2) The functor $\overline{s}^!\overline{s}^{\mathrm{IndCoh}}_*$ is t-exact.
(3) The map $\overline{s}^!$ is t-exact and reflects the t-structure.
(4) The map $\overline{s}^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathfrak{t}^*) \to \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ is t-exact.
(5) The t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ is compatible with filtered colimits.

Proof. The object $\overline{s}^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{t}^*})$ is compact since the image of a compact object under a functor with a continuous right adjoint (Corollary 3.18) is compact. Therefore, by Lemma 2.50, the right adjoint to this inclusion functor is continuous.

The second claim follows by base change (Corollary 3.18) of the Cartesian diagram in Proposition 3.14, since we may identify this functor with the composite of $s^!$, t-exact by Proposition 3.20, with the functor s^{IndCoh}_*, which is t-exact since s is ind-affine [GR17b, Chapter 3, Lemma 1.4.9].

Next, we show that $\overline{s}^!$ is right t-exact. If $\mathcal{G} \in \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{\leq 0}$ we may write $\mathcal{G}$ as some colimit $\mathrm{colim}(\overline{s}^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{t}^*}))$. Since $\overline{s}^!$ is continuous, we see that by (2) that $\overline{s}^!(\mathcal{G})$ is a colimit of objects in the heart of the t-structure, and thus lies in $\mathrm{IndCoh}(\mathfrak{t}^*)^{\leq 0}$.

To see the left t-exactness of $\overline{s}^!$, let $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})^{>0}$. We wish to show that $\overline{s}^!(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*)^{>0}$, and to show this it suffices to show that $\mathrm{Hom}_{\mathrm{IndCoh}(\mathfrak{t}^*)}(\mathcal{O}_{\mathfrak{t}^*}, \overline{s}^!(\mathcal{F}))$ vanishes. However, since $\overline{s}$ is ind-proper, we see that by adjunction (Corollary 3.18) it suffices to show $\mathrm{Hom}_{\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})}(\overline{s}^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{t}^*}), \mathcal{F})$ vanishes, which follows by the definition of the t-structure. Thus the functor $\overline{s}^!$ is t-exact, and this along with its conservativity gives (3).

Now, to show (4), note that (3) gives that $\overline{s}^!$ reflects the t-structure, so it suffices to show that $\overline{s}^!\overline{s}^{\mathrm{IndCoh}}_*$ is t-exact, which is precisely (2). Finally, (5) follows from the fact that $\overline{s}^!$ is continuous and reflects the t-structure, along with the fact that t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*)$ is compatible with filtered colimits. □

Proposition 3.22. The analogous claims of Proposition 3.21 for the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ obtained by replacing $\overline{s}$ by q all hold.

Proof. This entire proof of Proposition 3.21 also holds for $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ after substituting $\overline{s}$ with q. This is because q^{IndCoh}_* also admits a continuous right adjoint since q is ind-proper, which follows since we have a Cartesian diagram

$$\begin{tikzcd}
\tilde{W}^{\mathrm{aff}} \times \mathfrak{t}^* \arrow[r, "\mathrm{act}"] \arrow[d, "\mathrm{proj}"] & \mathfrak{t}^* \arrow[d, "q"] \\
\mathfrak{t}^* \arrow[r, "q"] & \mathfrak{t}^*/\tilde{W}^{\mathrm{aff}}
\end{tikzcd}$$

for which base change analogously gives (1). Since the pullback by the projection map is t-exact (which can be seen, for example, since the t-structure on the ind-scheme $\mathrm{IndCoh}(\tilde{W}^{\mathrm{aff}} \times \mathfrak{t}^*)$ is defined so as to be compatible with filtered colimits, see [GR17b, Chapter 3, Section 1.2.1]. Moreover, the t-exactness of $\mathrm{act}_*^{\mathrm{IndCoh}}$ follows since the action map is ind-affine. The remaining claims follow mutatis mutandis to the proof of Proposition 3.21. ☐

Corollary 3.23. The t-structures on $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ and $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}})$ are both left-complete and right-complete.

Proof. By Proposition 3.22 and Proposition 3.21, both categories admit conservative, t-exact functors to $\mathrm{IndCoh}(\mathfrak{t}^*)$ which commute with limits (since they are right adjoints). Any category which admits a conservative, t-exact functor which commutes with limits to a left-complete category is left-complete, therefore the left-completeness holds in this case, where $\mathrm{IndCoh}(\mathfrak{t}^*)$ admits a t-exact equivalence to $\mathrm{QCoh}(\mathfrak{t}^*)$ (since $\mathfrak{t}^*$ is a smooth classical scheme) and therefore is left-complete. Similarly, each functor to $\mathrm{IndCoh}(\mathfrak{t}^*)$ is continuous and so the right-completeness follows from the fact that $\mathrm{IndCoh}(\mathfrak{t}^*)$ is also right-complete. ☐

4. Coxeter Descent for Finite and Affine Quotients

In this section, we show that sheaves on the coarse quotient can be viewed as certain equivariant sheaves satisfying a condition which we will refer to, following [Lon17], as *satisfying Coxeter descent*. We first study the notion of Coxeter descent in the local case in Section 4.1. We then define Coxeter descent for a finite group acting by reflections on a vector space and use our local criterion to determine a 'pointwise criterion' for a given equivariant sheaf to satisfy Coxeter descent for a finite group in Section 4.3. In Section 4.4 we use this to show other possible notions of satisfying Coxeter descent are equivalent, and, in particular, prove Theorem 1.22. Finally, we upgrade this to the affine case and show that sheaves on $\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ satisfy Coxeter descent in Section 4.5.

4.1. **Local Coxeter Descent.** Assume H is some finite Weyl group acting on an affine space $V^\vee$ such that there exists a k-point of $V^\vee$ where H acts by reflections about this k-point. Choose this k-point as the origin to view $V^\vee$ as a vector space. Let C denote the coinvariant algebra for the action of H on V. Denote by $\alpha : \mathrm{Spec}(C) \to \mathrm{Spec}(k)$ the terminal map, and let $i : \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(C)$ denote the inclusion of the k-point.

Recall the functor $\Xi_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$ defined for a classical (or, more generally, an eventually coconnective) scheme X. In particular, recall this functor is fully faithful and the left adjoint to $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$, see [GR17a, Chapter 4, Lemma 1.1.7]. Recall also the symmetric monoidal functor $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$ whose underlying functor sends $\mathcal{F} \to \mathcal{F} \otimes \omega_X$, see [GR17a, Chapter 4, Proposition 1.2.10] and [GR17a, Chapter 6].

Proposition 4.1. With the above notation, we have:

(1) The functor $\Upsilon_{\mathrm{Spec}(C)}$ and $\Xi_{\mathrm{Spec}(C)}$ are canonically isomorphic as functors of DG categories.
(2) We have a canonical isomorphism of functors $\Xi_{\mathrm{Spec}(C)}\alpha^* \simeq \alpha^!\Xi_{\mathrm{Spec}(k)}$.
(3) We have a canonical isomorphism of functors $i^!\Xi_{\mathrm{Spec}(C)} \simeq i^*\Xi_{\mathrm{Spec}(k)}$. Moreover, the functors $\Xi_{\mathrm{Spec}(C)}$ and $\Xi_{\mathrm{Spec}(k)}$ are canonically H-equivariant and the following diagram canonically commutes

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Spec}(C))^H \arrow[r, "i^!"] & \mathrm{IndCoh}(\mathrm{Spec}(k)) \\
\mathrm{QCoh}(\mathrm{Spec}(C))^H \arrow[r, "i^*"] \arrow[u, "\Xi_{\mathrm{Spec}(C)}"] & \mathrm{QCoh}(\mathrm{Spec}(k)) \arrow[u, "\Xi_{\mathrm{Spec}(k)}"']
\end{tikzcd}$$

where the left vertical arrow is induced by the H-equivariance. Furthermore, $\Xi_{\mathrm{Spec}(k)}$ is an equivalence.

Proof. Both functors in (1) are continuous and are entirely determined by where they send $\mathcal{O}_{\mathrm{Spec}(C)}$. Note that, by definition, since $\mathcal{O}_{\mathrm{Spec}(C)}$ is a perfect complex, $\Xi(\mathcal{O}_{\mathrm{Spec}(C)}) \simeq \mathcal{O}_{\mathrm{Spec}(C)}$ and $\Upsilon_{\mathrm{Spec}(C)}(\mathcal{O}_{\mathrm{Spec}(C)}) \simeq \omega_{\mathrm{Spec}(C)}$. However, since C is a Gorenstein local ring (for example, it is a finite dimensional k-algebra satisfying Poincaré duality by Borel's theorem), $\omega_{\mathrm{Spec}(C)} \simeq \mathcal{O}_{\mathrm{Spec}(C)}$ and so (1) follows, and an identical argument shows $\Upsilon_{\mathrm{Spec}(k)} \simeq \Xi_{\mathrm{Spec}(k)}$. Using these facts and the fact that Υ intertwines ! and $*$-pullbacks of schemes, we obtain (2).

Finally, (3) follows since the analogue of (3) holds if the Ξ are replaced with Υ and (1) (applied to the rings C and k). The H-equivariance of Ξ and the fact $\Xi_{\mathrm{Spec}(k)}$ is an equivalence follow by construction of Ξ via the ind-extension of the inclusion map of perfect complexes into coherent complexes. By this construction, we see that $\Xi_{\mathrm{Spec}(C)}$ is manifestly H-equivariant, and furthermore we see that $\Xi_{\mathrm{Spec}(k)}$ an equivalence since the inclusion $\mathrm{Perf}(\mathrm{Spec}(k)) \hookrightarrow \mathrm{Coh}(\mathrm{Spec}(k))$ is an equivalence. □

Proposition 4.2. Assume $M \in C\text{-mod}^H$ has the property that the (derived) fiber $i^*(M) \simeq k \otimes_C M \in \mathrm{Rep}_k(H)$ lies in the full subcategory generated by the trivial H-representation. Then the unit map $M \to k \otimes_C M$ induces an equivalence $M^H \xrightarrow{\sim} (k \otimes_C M)^H$, where we view M and $k \otimes_C M \simeq i_*i^*(M)$ as objects of $\mathrm{Rep}(H)$ via the composite $C\text{-mod}^H \xrightarrow{\alpha_*} \mathrm{Vect}^H \simeq \mathrm{Rep}(H)$.

Proof. Consider the cofiber sequence
$$C^+ \to C \to k$$
induced by the short exact sequence of classical C-modules equipped with H-equivariance. Tensoring with our M, we obtain a cofiber sequence
$$C^+ \otimes_C M \to M \to k \otimes_C M \tag{3}$$
of objects of $C\text{-mod}^H$. Therefore, it suffices to show that if M has the property that the derived fiber $i^*(M)$ is a complex of trivial H-representations, then $(C^+ \otimes_C M)^H \simeq 0$.

Note that C admits a filtration $C_0 \subseteq C_1 \subseteq ... \subseteq C_\ell$ induced by the degree of $\mathrm{Sym}(V)$, and the H-action preserves this filtration. In particular, we may filter $C^+ \cong C/C_0$ (where here we take the classical quotient) by objects whose C-module structure factors through the ring map $C \to C/C^+$ and such that the associated graded of this filtration are H-representations. Furthermore, since C itself is isomorphic as a representation of H to the regular H-representation [Kan01, Theorem 24-1], we see that no representation in this filtration of C^+ is trivial. Therefore, for each of these subquotients S in the filtration of C^+, we see that
$$(S \otimes_C M)^H \simeq ((S \otimes_k C/C^+) \otimes_C M)^H \simeq (S \otimes_k (k \otimes_C M))^H$$
is the tensor product of some nontrivial H-representation S over k with an entirely trivial representation (by assumption on M). Therefore we see that $(S \otimes_C M)^H \simeq 0$, and so $(C^+ \otimes_C M)^H \simeq 0$, as required. □

Lemma 4.3. The H-equivariance of α induces fully faithful functors $\dot{\alpha}^*$ and $\dot{\alpha}^!$ fitting into the following commutative diagram

$$\begin{tikzcd} \mathrm{IndCoh}(\mathrm{Spec}(k)) \arrow[r, "\dot{\alpha}^!"] & \mathrm{IndCoh}(\mathrm{Spec}(C))^H \\ \mathrm{QCoh}(\mathrm{Spec}(k)) \arrow[u, "\sim"] \arrow[r, "\dot{\alpha}^*"] & \mathrm{QCoh}(\mathrm{Spec}(C))^H \arrow[u, "\Upsilon^H_{\mathrm{Spec}(C)}"'] \end{tikzcd} \tag{4}$$

where the right arrow is obtained by the H-equivariance of Proposition 4.1(3). For a given $\mathcal{F} \in \mathrm{IndCoh}(\mathrm{Spec}(C))$, $\mathcal{F}$ lies in the essential image of $\dot{\alpha}^!$ only if $\mathcal{F} \simeq \Upsilon_{\mathrm{Spec}(C)}(\mathcal{F}')$ for some $\mathcal{F}' \in \mathrm{QCoh}(\mathrm{Spec}(C))^H$, and this $\mathcal{F}'$ is unique up to isomorphism. Moreover, *if such an $\mathcal{F}'$ exists*, the following conditions are equivalent:

1. The essential image of $\dot{\alpha}^!$ contains $\mathcal{F}$.
2. The object $i^*(\mathcal{F}') \in \mathrm{QCoh}(\mathrm{Spec}(k))^H \simeq \mathrm{Rep}(H)$ yields a trivial H-representation.
3. The object $i^!(\mathcal{F}) \in \mathrm{IndCoh}(\mathrm{Spec}(k))^H \simeq \mathrm{Rep}(H)$ yields a trivial H-representation.

Proof. The fully faithfulness of both functors follow since a functor $\mathrm{Vect} \to \mathcal{C}$ is fully faithful if and only if it sends the object $k \in \mathrm{Vect}$ to an object $C \in \mathcal{C}$ such that $\underline{\mathrm{End}}_{\mathcal{C}}(C) \simeq k$. This holds for both $\dot{\alpha}^!$ and $\dot{\alpha}^*$ since

$$\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathrm{Spec}(C))^H}(\Upsilon^H_{\mathrm{Spec}(C)}(C)) \simeq \underline{\mathrm{End}}_{C\text{-mod}^H}(C)) \simeq \underline{\mathrm{End}}_{C\text{-mod}}(C)^H \simeq C^H \simeq k$$

identifying $\mathrm{QCoh}(\mathrm{Spec}(C)) \simeq C$-mod and using the commutativity and fully faithfulness of $\Upsilon^H_{\mathrm{Spec}(C)}$ of Proposition 4.1(1). Furthermore, the commutativity of (4) gives the 'only if' statement on the essential image, also using the fact that $\Upsilon_{\mathrm{Spec}(k)}$ is an equivalence, and the fully faithfulness gives the condition that $\mathcal{F}'$ is unique up to isomorphism.

Note that the functor $\dot{\alpha}^!$ admits a left adjoint given by the composite:

$$\mathrm{IndCoh}(\mathrm{Spec}(C))^H \xrightarrow{\alpha_*^{\mathrm{IndCoh},H}} \mathrm{IndCoh}(\mathrm{Spec}(k))^H \simeq \mathrm{Rep}(H) \xrightarrow{(-)^H} \mathrm{Vect}$$

using the fact that k has characteristic zero, so that $(-)_H \xrightarrow{\sim} (-)^H$. To show the equivalence of (1) and (2), it therefore suffices to show that this adjoint functor is conservative on nonzero objects $\mathcal{F}$ for which $\mathcal{F} \simeq \Upsilon_{\mathrm{Spec}(C)}(\mathcal{F}')$ for some $\mathcal{F}' \in \mathrm{QCoh}(\mathrm{Spec}(C))^H$ and which satisfy the condition in (2). However, given such an $\mathcal{F}' \in \mathrm{QCoh}(\mathrm{Spec}(C))^H$ we have that the fiber $i^*(\mathcal{F}') \simeq k \otimes_C \mathcal{F}'$ does not vanish. In fact, $\Xi_{\mathrm{Spec}(k)} i^*(\mathcal{F}') \simeq i^!(\Xi_{\mathrm{Spec}(C)}\mathcal{F}')$ does not vanish by the fully faithfulness of $\Xi_{\mathrm{Spec}(C)}$ and the corresponding fact for $i^!$, see [GR17a, Chapter 4, Proposition 2.2.2].

Thus we see that $k \otimes_C M$ is nonzero and, by assumption, acquires a trivial H-representation structure. Therefore we see by Proposition 4.2 that $M^H \xrightarrow{\sim} (k \otimes_C M)^H \simeq k \otimes_C M$ does not vanish, thus obtaining the desired conservativity. Finally, the equivalence of (2) and (3) follows from taking H-invariants as in Proposition 4.1(3) so that the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Spec}(C))^H \arrow[r, "i^!"'] & \mathrm{IndCoh}(\mathrm{Spec}(k))^H \simeq \mathrm{Rep}(H) \\
\mathrm{IndCoh}(\mathrm{Spec}(C))^H \arrow[u, "\Xi^H_{\mathrm{Spec}(C)}"] \arrow[r, "i^*"'] & \mathrm{QCoh}(\mathrm{Spec}(k))^H \simeq \mathrm{Rep}(H) \arrow[u, "\Xi^H_{\mathrm{Spec}(k)}"']
\end{tikzcd}$$

commutes, noting that the right vertical arrow is an equivalence. $\square$

Definition 4.4. If $\mathcal{F} \in \mathrm{IndCoh}(\mathrm{Spec}(C)/H)$ lies in the essential image of $\dot{\alpha}^!$, we say $\mathcal{F}$ *satisfies local Coxeter descent* for the coinvariant algebra C. When C is clear from context, we will also say that $\mathcal{F}$ *satisfies local Coxeter descent.*

With this terminology, Lemma 4.3 gives necessary and sufficient conditions for a given sheaf to satisfy Coxeter descent.

4.2. **Fully Faithfulness for Finite Weyl Group.** Continuing the notation of Section 4.1, we globalize the fully faithfulness of Lemma 4.3.

Proposition 4.5. The map $\phi^! : \mathrm{IndCoh}(V^\vee /\!\!/ H) \to \mathrm{IndCoh}(V^\vee)^H$ is fully faithful.

Proof. Since $V^\vee /\!\!/ H$ is known to be smooth (Remark 3.8), the category $\mathrm{IndCoh}(V^\vee /\!\!/ H)$ has a compact generator, namely $\omega_{V^\vee /\!\!/ H}$. Since $\phi^!$ is continuous, it suffices to show that the induced map given by

$$\phi^! : \underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee /\!\!/ H)}(\omega_{V^\vee /\!\!/ H}) \to \underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee/H)}(\phi^!\omega_{V^\vee /\!\!/ H}) \simeq \underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee/H)}(\omega_{V^\vee/H})$$

is an equivalence. However, this directly follows since

$$\underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee/H)}(\omega_{V^\vee/H}) \simeq \underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee)}(\mathrm{oblv}^H(\omega_{V^\vee/H}))^H \simeq \underline{\mathrm{End}}_{\mathrm{IndCoh}(V^\vee)}(\omega_{V^\vee})^H \simeq \mathrm{Sym}(V)^H$$

since oblv^H is equivalently given as !-pullback by the quotient map $V^\vee \to V^\vee/H$. $\square$

Definition 4.6. If a given $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ lies in the essential image of $\phi^!$, we will say $\mathcal{F}$ *satisfies Coxeter descent* for the action of H. If the action of H on $V^\vee$ is clear from context, we will simply state that $\mathcal{F}$ *satisfies Coxeter descent.*

4.3. **Pointwise Criterion for (Global) Coxeter Descent.** Fix H and V as in Section 4.1. For a field-valued point $x \in V^\vee(K)$, we will fix the following notation. Let $\overline{x} \in V^\vee/H(K)$ denote its image. Since the quotient map $\phi : V^\vee/H \to V^\vee /\!\!/ H$ induces a bijection on K-points, we may also equivalently regard $\overline{x}$ as a K-point of $V^\vee /\!\!/ H$. Let $\overline{x}' : \mathrm{Spec}(C_x)/W_x \to V^\vee/W$ induced by the map of (classical) rings $\mathrm{Sym}_k(V) \to C_x$, where C_x is the coinvariant algebra for the action of the stabilizer group H_x on $V^\vee$, which is in particular an K-algebra. Later, we will also use the notation $\overline{x}$ for the image under the quotient map $q : \mathfrak{t}^* \to \mathfrak{t}^*/\tilde{W}^{\mathrm{aff}}$, but context will always dictate which quotient we are taking with respect to.

Proposition 4.7. Let $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$, and let $\mathcal{F}' \in \mathrm{QCoh}(V^\vee/H)$ denote the object for which $\Upsilon^H_{V^\vee}(\mathcal{F}') \simeq \mathcal{F}$. The sheaf $\mathcal{F}$ satisfies Coxeter descent if and only if the following equivalent conditions hold for all every field-valued point $x : \mathrm{Spec}(K) \to V^\vee$:

(1) The pullback $\overline{x}'^!(\mathcal{F})$ satisfies local Coxeter descent (in the sense of Definition 4.4) for the K-algebra of coinvariants C_x.
(2) The pullback $x^!(\mathrm{oblv}^H_{W_x}(\mathcal{F}))$, which canonically acquires a W_x-representation in Vect_K, is the trivial W_x representation.
(3) The canonical W_x-representation (over K) on $x^*(\mathrm{oblv}^H_{W_x}(\mathcal{F}'))$ is trivial.
(4) The sheaf $x^*\Psi_{V^\vee}(\mathrm{oblv}^H_{W_x}(\mathcal{F}))$, which canonically acquires a W_x-representation, is the trivial W_x representation.

Lemma 4.8. The following diagrams are Cartesian:

$$\begin{tikzcd}
H \overset{H_x}{\times} \mathrm{Spec}(C_x) \arrow[r] \arrow[d] & \mathrm{Spec}(C_x)/H_x \arrow[r, "\dot{\alpha}"] \arrow[d, "\overline{x}'"] & \mathrm{Spec}(K) \arrow[d, "\overline{x}"] \\
V^\vee \arrow[r] & V^\vee/H \arrow[r, "\phi"] & V^\vee /\!\!/ H
\end{tikzcd}$$

where the unlabeled horizontal maps are the quotient maps, and the unlabeled vertical arrow is induced by the map $\mathrm{Spec}(C_x) \to V^\vee$ and the action map of H.

Proof. Direct computation and the fact that the quotient map $V^\vee \to V^\vee /\!\!/ H$ is flat (Remark 3.8) gives an isomorphism

$$V^\vee \times_{V^\vee /\!\!/ H} \mathrm{Spec}(L) \cong H \overset{H_x}{\times} \mathrm{Spec}(C_x)$$

which is H-equivariant. Therefore we may take the quotient by H on both sides to obtain $(V^\vee \times_{V^\vee /\!\!/ H} \mathrm{Spec}(L))/H \cong \mathrm{Spec}(C_x)/H_x$. Using the fact that sifted colimits commute with products, we obtain our claim. □

Lemma 4.9. In the setup and notation of Lemma 4.8, the functors $\phi^!$ and $\dot{\alpha}^!$ admit left adjoints.

Proof. By the adjoint functor theorem, it suffices to show $\phi^!$ commutes with limits. Since pullback by the covering map $V^\vee \to V^\vee/H$ is conservative, it suffices to show that $(V^\vee \to V^\vee /\!\!/ H)^!$ commutes with limits. However, the map $V^\vee \to V^\vee /\!\!/ H$ is finite (Remark 3.8), and therefore in particular proper, and therefore $(V^\vee \to V^\vee /\!\!/ H)^!$ commutes with limits. An identical argument, using the fact that $\dot{\alpha} : \mathrm{Spec}(C_\lambda) \to \mathrm{Spec}(L)$ is proper, gives the latter claim. □

We denote these left adjoints by ϕ_*^{IndCoh} and $\dot{\alpha}_*^{\mathrm{IndCoh}}$ respectively. This light abuse of notation is justified by the following:

Lemma 4.10. The Cartesian diagrams in Lemma 4.8 all satisfy base change. In particular, for the rightmost Cartesian diagram, the canonical morphism $\dot{\alpha}_*^{\mathrm{IndCoh}}\overline{x}'^! \to \overline{x}\phi_*^{\mathrm{IndCoh}}$ is an isomorphism.

Proof. The outer diagram and the left square in the diagram of Lemma 4.8 are diagrams of schemes and therefore satisfy base change by [GR17a, Chapter 5, Corollary 3.1.4]. Since the quotient map $V^\vee \to V^\vee/H$ is in particular proper (since H is finite), we see that the pushforward of the structure sheaf is a compact generator of $\mathrm{IndCoh}(V^\vee/H)$. To show that the rightmost square satisfies base change, it suffices to show the canonical base change morphism is in fact an isomorphism on this compact generator. However, this follows by first base changing by the outer Cartesian square and then base changing by the left Cartesian square. □

Finally, we record a corollary of Proposition 4.1:

Corollary 4.11. For any $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ and any field-valued point x, the pullback $\overline{x}'^!(\mathcal{F})$ necessarily factors through the full subcategory given by the essential image of $\Xi_{\mathrm{Spec}(C_x)}$.

Proof. Note that the fact that Υ intertwines !-pullback and $*$-pullback of schemes [GR17a, Chapter 6, Section 0.3] and the fact that $\Upsilon_{\mathrm{Spec}(C_x)} \simeq \Xi_{\mathrm{Spec}(C_x)}$ (Proposition 4.1) implies that the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(V^\vee)^H \arrow[r, "\overline{x}'^!"] & \mathrm{IndCoh}(\mathrm{Spec}(C_x))^{H_x} \\
\mathrm{QCoh}(V^\vee)^H \arrow[r, "\overline{x}'^*"] \arrow[u, "\Upsilon^H_{V^\vee}"] & \mathrm{QCoh}(\mathrm{Spec}(C_x))^{H_x} \arrow[u, "\Xi"']
\end{tikzcd}$$

canonically commutes. Therefore, since $\Upsilon_{V^\vee}$ can be identified with the functor $[-d]$ by the smoothness of $V^\vee$, we obtain our claim. □

Proof of Proposition 4.7. Assume $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ satisfies Coxeter descent, and fix some field-valued $x : \mathrm{Spec}(K) \to V^\vee$. The fact the diagram in Lemma 4.8 commutes and the functoriality of !-pullback implies that $\overline{x}'^!(\mathcal{F})$ satisfies local Coxeter descent. Since $\phi^!$ factors through the full subcategory of those $\mathcal{G} \in \mathrm{IndCoh}(V^\vee)^H$ satisfying local Coxeter descent at all field-valued points and moreover admits a left adjoint ϕ_*^{IndCoh} by Lemma 4.9, it remains to show that the left adjoint is conservative on such $\mathcal{G}$.

Fix such a nonzero $\mathcal{G} \in \mathrm{IndCoh}(V^\vee)^H$. There exists some field-valued point $x : \mathrm{Spec}(K) \to V^\vee$ for which $x^!(\mathcal{G})$ does not vanish (for example, this is a direct consequence of [Gai13, Proposition 8.1.2]). By the fact that $\mathrm{Spec}(K) \xrightarrow{i_x} \mathrm{Spec}(C_x)$ is surjective on geometric points, we have that $i_x^!$ is conservative and therefore we see that $\overline{x}'^!(\mathcal{G})$ does not vanish. By assumption, $\overline{x}'^!(\mathcal{G})$ satisfies local Coxeter descent, and so since it is nonzero we see that the adjoint functor $\dot{\alpha}_*^{\mathrm{IndCoh}}$ does not vanish. Applying base change (Lemma 4.10), we therefore see that $\phi_*^{\mathrm{IndCoh}}(\mathcal{F})$ also does not vanish. Therefore, we see that $\mathcal{F}$ satisfies Coxeter descent if and only if every pullback $\overline{x}'^!(\mathcal{F})$ satisfies local Coxeter descent.

To show the remaining equivalences, recall that $\overline{x}^!(\mathcal{F})$ necessarily factors through the full subcategory given by the essential image of $\Upsilon_{\mathrm{Spec}(C_x)}$ by Corollary 4.11. Therefore we see that for each fixed x we obtain the equivalence of (1) and (2) in Proposition 4.7 from the analogous equivalence of (1) and (2) in Lemma 4.3. The equivalence of (2) and (3) follows from the fact that we have a canonical identification of the following diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(V^\vee/H) \arrow[r, "x^!"] & \mathrm{IndCoh}(\mathrm{Spec}(L)/H_x) \\
\mathrm{QCoh}(V^\vee/H) \arrow[r, "x^*"] \arrow[u, "\Upsilon^H_{V^\vee}"] & \mathrm{QCoh}(\mathrm{Spec}(L)/H_x) \arrow[u, "\Upsilon^{H_x}_{\mathrm{Spec}(L)}"']
\end{tikzcd}$$

using again the fact that Υ intertwines the ! and $*$-pullback for finite type schemes. Finally, we note that we may identify $\Upsilon_{V^\vee}$ with the shift functor $[-d]$. Since $V^\vee$ is smooth, $\Psi_{V^\vee}$ is an equivalence with inverse $\Xi_{V^\vee} \simeq \Upsilon_{V^\vee}[-d]$, thus establishing the equivalence of (3) and (4). □

Corollary 4.12. A given $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ satisfies Coxeter descent if and only if for every simple reflection $r \in H$, the sheaf $\mathrm{oblv}^H_{\langle r\rangle}(\mathcal{F}) \in \mathrm{IndCoh}(V^\vee/\langle r\rangle)$ satisfies Coxeter descent.

Proof. First, fix some K-point $x \in V^\vee(K)$, and let $r \in H_x$ denote a reflection in the stabilizer of x. We have the following diagram of Deligne-Mumford stacks commutes

$$\begin{tikzcd} \mathrm{Spec}(K)/H_x \arrow[d, "x"] & \mathrm{Spec}(K)/\langle r\rangle \arrow[l] \arrow[d, "x"] \\ V^\vee/H_x & V^\vee/\langle r\rangle \arrow[l] \end{tikzcd}$$

and so the functor obtained by their pullbacks

$$\begin{tikzcd} \mathrm{Rep}_K(H_x) \arrow[r, "\mathrm{oblv}^{H_x}_{\langle r\rangle}"] & \mathrm{Rep}_K(\langle r\rangle) \\ \mathrm{IndCoh}(V^\vee)^{H_x} \arrow[u, "x^!"] \arrow[r, "\mathrm{oblv}^{H_x}_{\langle r\rangle}"] & \mathrm{IndCoh}(V^\vee)^{\langle r\rangle} \arrow[u, "x^!"] \end{tikzcd} \tag{5}$$

also commutes.

Now, assume $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ satisfies Coxeter descent, and fix a reflection r. By Proposition 4.7, to show that $\mathcal{F}$ satisfies Coxeter descent for $\langle r\rangle$, may check that for all $x \in V^\vee$ that the pullback $x^!(\mathcal{F})$ is the trivial $\langle r\rangle_x$ representation. The only nontrivial case to check is the case when $\langle r\rangle_x = \langle r\rangle$, i.e. $rx = x$ and so $r \in H_x$. By assumption that $\mathcal{F}$ satisfies Coxeter descent for H, we see by Proposition 4.7 that for every $x \in V^\vee(K)$, we have that $x^!(\mathcal{F})$ is the trivial H_x-representation. Therefore, since diagram (5) commutes, we have that for every reflection $r \in H_x$, $x^!(\mathrm{oblv}^H_{\langle r\rangle}(\mathcal{F}))$ is the trivial $\langle r\rangle$-representation, and thus satisfies Coxeter descent for $\langle r\rangle$ by Proposition 4.7.

Conversely, if $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ has the property that $\mathrm{oblv}^H_{\langle r\rangle}(\mathcal{F})$ satisfies Coxeter descent for all reflections r, and fix some $x \in V^\vee(K)$. We wish to show the associated H_x-representation of $x^!(\mathcal{F})$ is trivial. However, the stabilizer H_x is itself a finite Weyl group since H is. Furthermore, we may demonstrate that object of $\mathrm{Rep}_K(H_x)$ lies in the trivial by subcategory by showing the representation of of all of its cohomology grousp are trivial, and therefore we may show that the associated H_x-representation is trivial on $x^!(\mathcal{F})$ by showing that $\mathrm{oblv}^{H_x}_{\langle r\rangle}(x^!(\mathcal{F}))$ is trivial for all reflections $r \in H_x$. By the commutativity of diagram (5), we see that this is equivalent to showing that $x^!(\mathrm{oblv}^{H_x}_{\langle r\rangle}(\mathcal{F}))$ is the trivial $\langle r\rangle$-representation, which follows by Proposition 4.7 since, by assumption, $\mathrm{oblv}^{H_x}_{\langle r\rangle}(\mathcal{F})$ satisfies Coxeter descent. □

Remark 4.13. We will now demonstrate that, in condition (3) of Proposition 4.7, that the condition '$x^*(\mathcal{F}')$ is a trivial W_x-representation.' cannot be replaced with 'the W_x-representation on the classical tensor product $H^0(x^*\mathcal{F}') \cong H^0(\mathcal{F}' \otimes_C k)$ is trivial.'

To see this, set $x = 0$ and $H \cong \mathbb{Z}/2\mathbb{Z}$ acting on $V^\vee$ by a reflection about the origin, and let $\mathcal{F}' := k \in C\text{-mod}^\heartsuit$. Then, direct computation shows the H-representation on the classical tensor product $H^0(k \otimes_C k)$ is trivial, while the cofiber sequence (3) shows the (derived) tensor product has a nontrivial H-action. This $\mathcal{F}'$, of course, does not lie in the essential image, since if it were, the object $V \in \mathrm{Vect}$ would satisfy that $\alpha^*(V) \cong k$ which, by conservativity, implies $V \in \mathrm{Vect}^\heartsuit$. However, we have that for such V that $\alpha^*(V) \cong C \otimes_k V$ has even k-dimension.

4.4. **Equivalent Conditions for Global Coxeter Descent.** With H and V as in Section 4.1, recall our quotient map $\phi : V^\vee/H \to V^\vee /\!\!/ H$. We have seen that $\phi^!$ is fully faithful in Proposition 4.5, and we now wish to classify the essential image. Following [Lon17], we make the following definition:

Definition 4.14. We say that $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ *satisfies Coxeter descent* for the action of H on $V^\vee$ if $\mathcal{F}$ lies in the essential image of $\phi^!$. When the action of H on $V^\vee$ is clear from context, we say that $\mathcal{F}$ *satisfies Coxeter descent.*

We first classify the sheaves satisfying Coxeter descent in the case where H is generated by a single reflection. For a fixed reflection $r \in H$, let $\phi_r : V^\vee/\langle r\rangle \to V^\vee /\!\!/ \langle r\rangle$ denote the quotient map for the action of the order two Weyl group $\langle r\rangle$ acting on $V^\vee$, and let $i_r : Z_r := V^{\vee,\langle r\rangle} \hookrightarrow V$ denote the inclusion of the classical closed subscheme of fixed points. Note that $\mathrm{IndCoh}(Z_r/\langle r\rangle) \simeq \mathrm{IndCoh}(Z_r) \otimes \mathrm{Rep}(\langle r\rangle)$ since the action of r is trivial on the fixed point locus.

Proposition 4.15. Let r denote a reflection. An object $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/\langle r\rangle)$ satisfies Coxeter descent for the action of the order two group $\langle r\rangle$ on $V^\vee$ if and only if the pullback $i_r^!(\mathcal{F}) \in \mathrm{IndCoh}(Z_r) \otimes \mathrm{Rep}(\langle r\rangle)$ lies entirely in the summand indexed by the trivial $\langle r\rangle$ representation.

Proof. The closed subscheme $Z_r \xhookrightarrow{i_r} V$ and complementary open subscheme $U_r := V \setminus Z_r \xhookrightarrow{j_r} V$ are both affine and induce two Cartesian squares as follows

$$\begin{tikzcd} Z_r/\langle r\rangle \arrow[r, "i_r"] \arrow[d, "\phi|_{Z_r}"'] & V^\vee/\langle r\rangle \arrow[d, "\phi"'] & U_r/\langle r\rangle \arrow[l, "j_s"'] \arrow[d, "\phi|_{U_r}"'] \\ Z_r /\!\!/ \langle r\rangle \arrow[r, "i_r"] & V^\vee /\!\!/ \langle r\rangle & U_r /\!\!/ \langle r\rangle \arrow[l, "j_s"'] \end{tikzcd}$$

where each vertical arrow is obtained from the map ϕ. Since $\phi^!$ is fully faithful (Proposition 4.5), its essential image is closed under extensions. In particular, an object $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ lies in the essential image if and only if $i_r^!(\mathcal{F})$ and $j_r^!(\mathcal{F})$ lie in the essential image of the respective pullbacks. However, since the action of $\langle r\rangle$ is free, the rightmost vertical arrow is an equivalence, so $j_r^!(\mathcal{F})$ is always in the essential image of $\phi^!$. Therefore, $\mathcal{F}$ lies in the essential image if and only if $i_r^!(\mathcal{F})$ does.

Note that the action of r on Z_r is trivial, and therefore we see that $Z_r /\!\!/ \langle r\rangle \cong Z_r$. Furthermore, we may identify the pullback $\phi|_{Z_r}^!$ with the functor

$$\mathrm{IndCoh}(Z_r) \xrightarrow{\mathrm{id}\otimes\mathrm{triv}} \mathrm{IndCoh}(Z_r) \otimes \mathrm{Rep}(\langle r\rangle)$$

and so we see that an object of the form $i_r^!(\mathcal{F})$ is in the essential image of the pullback $\phi^!$ if and only if the restriction lies entirely in the trivial summand. Combining this with the assertion that $\mathcal{F}$ lies in the essential image if and only if $i_r^!(\mathcal{F})$ does, we obtain our desired characterization of the essential image. $\square$

We now give various equivalent conditions for a given $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ to satisfy Coxeter descent, which can be regarded as the derived analogue of the main theorem of [Lon17]:

Proposition 4.16. An object $\mathcal{F} \in \mathrm{IndCoh}(V^\vee/H)$ satisfies Coxeter descent in the sense of Definition 4.14 if and only if one of the following equivalent conditions hold:

(1) For each reflection $r \in H$, $\mathrm{oblv}^H_{\langle r\rangle}(\mathcal{F})$ satisfies Coxeter descent for $\langle r\rangle$.
(2) For each simple reflection $s \in H$, $\mathrm{oblv}^H_{\langle s\rangle}(\mathcal{F})$ satisfies Coxeter descent for $\langle s\rangle$.
(3) Each cohomology group $H^i(\mathcal{F}) \in \mathrm{IndCoh}(V^\vee)^{H,\heartsuit} \simeq \mathrm{IndCoh}(V^\vee)^{\heartsuit,H}$ lies in the essential image of $\phi^!|_{\mathrm{IndCoh}(V^\vee /\!\!/ H)^\heartsuit}$.
(4) For every reflection $r \in H$, the cohomology group $\mathrm{oblv}^H_{\langle r\rangle}(H^i(\mathcal{F})) \in \mathrm{IndCoh}(V^\vee)^{\langle r\rangle,\heartsuit}$ lies in the essential image of the pullback $\phi_r^!$ restricted to $\mathrm{IndCoh}(V^\vee /\!\!/ \langle r\rangle)^\heartsuit$ for all $i \in \mathbb{Z}$.
(5) For every simple reflection $s \in H$, the cohomology group $\mathrm{oblv}^H_{\langle s\rangle}(H^i(\mathcal{F})) \in \mathrm{IndCoh}(V^\vee)^{\langle s\rangle,\heartsuit}$ lies in the essential image of the pullback $\phi_s^!$ restricted to $\mathrm{IndCoh}(V^\vee /\!\!/ \langle s\rangle)^\heartsuit$ for all $i \in \mathbb{Z}$.
(6) For each simple reflection $s \in H$, the sheaf $i_s^!(\mathrm{oblv}^W_{\langle s\rangle}(\mathcal{F})) \in \mathrm{IndCoh}(V^{\langle s\rangle}) \otimes \mathrm{Rep}(\langle s\rangle)$ lies entirely in the summand indexed by the trivial representation.
(7) For every reflection $r \in H$, the sheaf $i_r^!(\mathrm{oblv}^W_{\langle r\rangle}(\mathcal{F})) \in \mathrm{IndCoh}(Z_r) \otimes \mathrm{Rep}(\langle r\rangle)$ lies entirely in the summand indexed by the trivial representation.

Remark 4.17. We recall that, in addition to the criterion above, we have established a pointwise condition for a given $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ to satisfy Coxeter descent in Proposition 4.7.

Proof of Proposition 4.16. The fact that the $\mathcal{F} \in \mathrm{IndCoh}(V^\vee)^H$ satisfying Coxeter descent are precisely the sheaves satisfying (1) follows from Corollary 4.12.

We now show (2) $\Rightarrow$ (1). Fix some reflection $r \in H$, and choose some $w \in H$ for which $w^{-1}rw$ is a simple reflection s. Then the following diagram commutes

$$\begin{tikzcd} V^\vee /\!\!/ \langle r \rangle \arrow[d, "w"'] & V^\vee/\langle r \rangle \arrow[l, "\phi_r"'] \arrow[d, "w"'] \arrow[dr] & \\ V^\vee /\!\!/ \langle s \rangle & V^\vee/\langle s \rangle \arrow[l, "\phi_s"'] \arrow[r] & V^\vee/W \end{tikzcd}$$

where the vertical arrows are the maps induced by the action of $w \in H$ and the unlabeled arrows are the quotient maps. We then see if $\mathrm{oblv}^W_{\langle s \rangle}(\mathcal{F}) \simeq \phi_s^!(\mathcal{F}')$ for some $\mathcal{F}'$ then

$$\mathrm{oblv}^W_{\langle r \rangle}(\mathcal{F}) \simeq w^! \mathrm{oblv}^W_{\langle s \rangle}(\mathcal{F}) \simeq w^!(\phi_s^!(\mathcal{F}')) \simeq \phi_r^!(w^!(\mathcal{F}'))$$

showing that (1) holds. Conversely, the implication (1) $\Rightarrow$ (2) follows since simple reflections are reflections.

The equivalence of a given $\mathcal{F}$ satisfying Coxeter descent and the given $\mathcal{F}$ satisfying (3) follows from the t-exactness and fully faithfulness of $\phi^!$, where the t-exactness follows from the fact that oblv^W reflects the t-structure and the fact that the quotient map $(V^\vee \to V^\vee /\!\!/ H)$ is finite-flat, see Remark 3.8. Replacing the map $\phi^!$ with $\phi_r^!$ and $\phi_s^!$, this argument also gives the equivalences (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (5). Finally, the equivalences (1) $\Leftrightarrow$ (6) and (2) $\Leftrightarrow$ (7) follow directly from Proposition 4.15. $\square$

Proof of Theorem 1.22. We first claim that the following diagram commutes

$$\begin{tikzcd} \mathrm{IndCoh}(V^\vee/H) \arrow[r, "\Psi^H_{V^\vee}"] & \mathrm{QCoh}(V^\vee/H) \\ \mathrm{IndCoh}(V^\vee /\!\!/ H) \arrow[u, "\phi^!"] \arrow[r, "\Psi_{V^\vee /\!\!/ H}"] & \mathrm{QCoh}(V^\vee /\!\!/ H) \arrow[u, "\phi^*"'] \end{tikzcd} \tag{6}$$

and the horizontal maps are equivalences. The fact that this diagram commutes can be seen, for example, by the fact that both composites send the compact generator $\mathcal{O}_{V^\vee /\!\!/ W}$ to the object $\mathcal{O}_{V^\vee/W} \in \mathrm{QCoh}(V^\vee/W)^\heartsuit$. Under the equivalence $\mathrm{QCoh}(V^\vee/W)^\heartsuit \simeq \mathrm{QCoh}(V^\vee)^{\heartsuit, W}$, we may identify this object as the equivariant (classical) quasicoherent sheaf on $V^\vee$ given by $\mathcal{O}_{V^\vee} \in \mathrm{QCoh}(V^\vee)^\heartsuit$ (or, equivalently, $\omega_{V^\vee}[-\dim(V^\vee)] \in \mathrm{QCoh}(V^\vee)^\heartsuit$) equipped with its canonical equivariance. The fact that the horizontal arrows are equivalences follows directly from the fact that $V^\vee$ and $V^\vee /\!\!/ W$ are smooth, so that $\Psi_{V^\vee /\!\!/ W}$ and $\Psi_{V^\vee}$ are equivalences, and the fact that the functor $\Psi_{V^\vee}$ is W-equivariant follows because it is the ind-extension of the identity functor on $\mathrm{Perf}(V^\vee) \xrightarrow{\sim} \mathrm{Coh}(V^\vee)$. Furthermore, the functors of diagram (6) are all t-exact; the t-exactness of the horizontal arrows follow from the fact that oblv^W reflects the t-structure and the fact that Ψ_X is t-exact [GR17a, Chapter 4, Proposition 1.2.2] and the t-exactness of the horizontal arrows follow from the fact that ϕ is finite-flat, see Remark 3.8.

We therefore may identify $\mathrm{QCoh}(V^\vee)^{H,\heartsuit}$ with the full subcategory of objects of $\mathrm{IndCoh}(V^\vee)^H$ concentrated in cohomological degree zero. With this, our claim follows from the equivalence of (3) and (4) in Proposition 4.16. $\square$

4.5. **Coxeter Descent for Affine Weyl Groups.** The quotient map $\overline{s} : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$ induces a canonical map of prestacks $\tilde{\phi} : \mathfrak{t}^*/\tilde{W}^{\mathrm{aff}} \to \mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}$. We now study the pullback functor $\tilde{\phi}^!$ and show that this functor behaves similarly to the case where $\tilde{W}^{\mathrm{aff}}$ is replaced with a finite Weyl group. For example, we show that the functor $\tilde{\phi}^!$ is fully faithful in Proposition 4.18. We define those sheaves in $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ in the essential image of $\tilde{\phi}^!$ as those sheaves *satisfying Coxeter descent* for $\tilde{W}^{\mathrm{aff}}$, and provide descriptions of those sheaves satisfying Coxeter descent for $\tilde{W}^{\mathrm{aff}}$ in Section 4.5.2 which parallel the description for the finite Weyl group case in Proposition 4.16.

4.5.1. *Fully Faithfulness of Affine Pullback.*

Proposition 4.18. The functor $\tilde{\phi}^!$ is fully faithful.

This subsection will be dedicated to the proof of Proposition 4.18. For a given $x \in \mathfrak{t}^*(K)$, let C_x denote the coinvariant algebra of Remark A.20 for the action of W_x^{aff} on $\mathfrak{t}^*$, which is in particular a K-algebra. Furthermore, let $[x]$ denote the image of x under the quotient map $\mathfrak{t}^* \to \mathfrak{t}^*/X^\bullet(T)$, and let $\overline{x}$ denote the image of x under the quotient map $q : \mathfrak{t}^* \to \mathfrak{t}^*/\tilde{W}^{\text{aff}}$. Since the map $\tilde{\phi}$ induces a bijection on K-points, so we abuse notation in also regarding $\overline{x}$ as a K-point of $\mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}$.

Proposition 4.19. Fix some $x \in \mathfrak{t}^*(K)$. There is a $\tilde{W}^{\text{aff}}$-equivariant isomorphism

$$\Gamma_{\tilde{W}^{\text{aff}}} \times_{\mathfrak{t}^*} \operatorname{Spec}(K) \simeq \tilde{W}^{\text{aff}} \overset{W^{\text{aff}}}{\times} \coprod_{x' \in \text{orbit}_{W^{\text{aff}}}(x)} \operatorname{Spec}(C_{x'})$$

and, moreover, the rectangles of the following diagram are (derived) Cartesian:

$$\begin{tikzcd}
\tilde{W}^{\text{aff}} \overset{W^{\text{aff}}}{\times} \coprod_{x' \in \text{orbit}_{W^{\text{aff}}}(x)} \operatorname{Spec}(C_{x'}) \arrow[r, "s"] \arrow[d, "t"] & \operatorname{Spec}(C_x)/W_x^{\text{aff}} \arrow[r, "\dot{\alpha}"] \arrow[d, "q|"] & \operatorname{Spec}(K) \arrow[d, "\overline{x}"] \\
\mathfrak{t}^* \arrow[r, "q"] & \mathfrak{t}^*/\tilde{W}^{\text{aff}} \arrow[r, "\tilde{\phi}"] & \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}
\end{tikzcd}$$

Proof. We first claim the outer rectangle is Cartesian. Applying Proposition 3.11, we may prove this first claim when $\tilde{W}^{\text{aff}} = W^{\text{aff}}$. Write $\Gamma_{W^{\text{aff}}}$ as a union of Γ_S where $S \subseteq W^{\text{aff}}$ varies over the finite subsets. Because this set is filtered, colimits over it commute with all finite limits (and, in particular, Cartesian products), and so we obtain

$$\Gamma_{W^{\text{aff}}} \times_{\mathfrak{t}^*} \operatorname{Spec}(L) \simeq \coprod_S (\Gamma_S \times_{\mathfrak{t}^*} \operatorname{Spec}(L)) \simeq \coprod_S \coprod_{x' \in \text{orbit}(x)} (\Gamma_{S \cap \text{stab}(x')} \times_{\mathfrak{t}^*} \operatorname{Spec}(L))$$

where the second equivalence follows from Lemma A.22. Therefore we see that the outer rectangle in Proposition 4.19 is Cartesian.

The fact that the left box is Cartesian follows from the fact that the stack quotient $\mathfrak{t}^*/\tilde{W}^{\text{aff}}$ is defined as the colimit over a groupoid U such that $U_1 \simeq \tilde{W}^{\text{aff}} \times \mathfrak{t}^*$, and so in particular $\mathfrak{t}^* \times_{\mathfrak{t}^*/\tilde{W}^{\text{aff}}} \mathfrak{t}^* \simeq \tilde{W}^{\text{aff}} \times \mathfrak{t}^*$ by Proposition 3.10. Now, because the outer rectangle is Cartesian and all of the maps are $\tilde{W}^{\text{aff}}$-equiariant, we may take the quotient by $\tilde{W}^{\text{aff}}$. This is a sifted colimit because the opposite category of the simplex category is sifted, and in particular, taking the quotient by $\tilde{W}^{\text{aff}}$ preserves the Cartesian product and shows the rightmost rectangle is Cartesian. $\square$

Lemma 4.20. In the setup and notation of Proposition 4.19, the functor $\tilde{\phi}$ admits a (continuous) left adjoint.

Proof. To show that $\tilde{\phi}^!$ admits a left adjoint, it suffices to show that $\tilde{\phi}^!$ commutes with (small) limits, by the adjoint functor theorem (see [Lur09, Chapter 5]). To see that $\tilde{\phi}^!$ commutes with small limits, consider the following commutative diagram:

(7)
$$\begin{tikzcd}
\mathfrak{t}^* \arrow[r, "\text{id}"] \arrow[d, "q"] & \mathfrak{t}^* \arrow[d, "\overline{s}"] \\
\mathfrak{t}^*/\tilde{W}^{\text{aff}} \arrow[r, "\tilde{\phi}"] & \mathfrak{t}^* /\!\!/ \tilde{W}^{\text{aff}}
\end{tikzcd}$$

Since q is ind-proper and surjective on geometric points, we have that $q^!$ is conservative. Therefore we may check that a map in $\operatorname{IndCoh}(\mathfrak{t}^*/W^{\text{aff}})$ is an isomorphism after applying $q^!$. However, since $q^!\tilde{\phi}^! \simeq \overline{s}^!$, we see that $q^!\tilde{\phi}^!$ commutes with small limits (since $\overline{s}^!$ is also a left adjoint since $\overline{s}$ is ind-proper by Corollary 3.16) and so $\tilde{\phi}^!$ commutes with small limits as well, and thus admits a left adjoint by the adjoint functor theorem [Lur09]. $\square$

Denote the left adjoint to $\tilde{\phi}^!$ by $\tilde{\phi}_*^{\mathrm{IndCoh}}$. This light abuse of notation is justified by the following:

Corollary 4.21. The three Cartesian diagrams of Proposition 4.19 satisfy base change. In particular, the canonical map $\dot{\alpha}_*^{\mathrm{IndCoh}} q|^! \to \overline{x}^! \tilde{\phi}_*^{\mathrm{IndCoh}}$ is an isomorphism.

Proof. The left and the 'large' Cartesian diagrams satisfy base change since the maps q and $\overline{s}$ are ind-schematic and so satisfy base change by [GR17b, Chapter 3, Theorem 5.4.3], see also Corollary 3.18. To show base change for the other Cartesian diagram, note that we may check that the map is an isomorphism on a compact generator of $\mathfrak{t}^*/W^{\mathrm{aff}}$. We choose the generator $\mathcal{G} := q_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$. The uniqueness of left adjoints then gives that $\tilde{\phi}_*^{\mathrm{IndCoh}}(\mathcal{G}) \simeq \overline{s}_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$. Base change by the outer Cartesian diagram of Proposition 4.19 then gives the desired claim. $\square$

Proof of Proposition 4.18. To show that $\tilde{\phi}^!$ is fully faithful, it suffices to show that the counit map $\tilde{\phi}_*^{\mathrm{IndCoh}}\tilde{\phi}^! \to \mathrm{id}$ is an equivalence. Because $\overline{s}$ admits a left adjoint (Corollary 3.18), we have that $\mathcal{G} := \overline{s}_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$ is a compact generator of $\mathrm{IndCoh}(\mathfrak{t}^* /\!\!/ W^{\mathrm{aff}})$. Therefore to show that the counit is an equivalence, by continuity it suffices to show that the map $c(\mathcal{G}) : \tilde{\phi}_*^{\mathrm{IndCoh}}\tilde{\phi}^!(\mathcal{G}) \to \mathcal{G}$ is an equivalence. Since $\overline{s}$ is an ind-proper cover (see Corollary 3.16), $\overline{s}^!$ is conservative, and so it suffices to show $\overline{s}^!\tilde{\phi}^!\tilde{\phi}_L(\mathcal{G}) \to \overline{s}^!(\mathcal{G})$ is an equivalence.

However, our map $\overline{s}^!\tilde{\phi}_L\tilde{\phi}^!(\mathcal{G}) \to \overline{s}^!(\mathcal{G})$ is a map in $\mathrm{IndCoh}(\mathfrak{t}^*)$, which is generated by the skyscraper sheaves associated to all field-valued points, a direct consequence of [Gai13, Proposition 8.1.2]. Therefore, we may show this map is an isomorphism when restricted to each field-valued point $x \in \mathfrak{t}^*$. By Corollary 4.21, we see that $\overline{x}^! c(\mathcal{G}) \simeq c_{\dot{\alpha}}(\overline{x}^!(\mathcal{G}))$, where $c_{\dot{\alpha}}$ denotes the counit of the adjunction $(\dot{\alpha}_*^{\mathrm{IndCoh}}, \dot{\alpha}^!)$ of Lemma 4.9. However, we have that $\dot{\alpha}^!$ is fully faithful by Lemma 4.3. Therefore, since this holds for every field-valued point x, $\tilde{\phi}^!$ is also fully faithful. $\square$

4.5.2. *Characterization of Coxeter Descent.* We have seen in Proposition 4.18 that the functor $\tilde{\phi}^!$ is fully faithful. In analogy with the finite dimensional case of Definition 4.14, we make the following definition:

Definition 4.22. We say that a sheaf $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))^W$ *satisfies Coxeter descent* for $\tilde{W}^{\mathrm{aff}}$ if it lies in the essential image of $\tilde{\phi}^!$. We will say a sheaf $\mathcal{G} \in \mathcal{D}(T)^W$ *satisfies Coxeter descent* for $\tilde{W}^{\mathrm{aff}}$ if its image under the Mellin transform satisfies Coxeter descent in the sense above. When the $\tilde{W}^{\mathrm{aff}}$-action is clear from context, we will simply say the given sheaf *satisfies Coxeter descent*.

We now provide many alternative characterizations of a sheaf satisfying Coxeter descent, noting that many of the following conditions involve the usual affine Weyl group W^{aff} as opposed to the extended affine Weyl group $\tilde{W}^{\mathrm{aff}}$.

Proposition 4.23. A sheaf $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\tilde{W}^{\mathrm{aff}}}$ satsifies Coxeter descent if and only if it satisfies one of the following equivalent conditions:

(1) For every field-valued point $x \in \mathfrak{t}^*(K)$, the canonical W_x^{aff}-representation on $\overline{x}^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ is trivial.
(2) For every finite parabolic subgroup W' of W^{aff}, the object $\mathrm{oblv}_{W'}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*/W')$ satisfies Coxeter descent for W'.
(3) The object $\mathrm{oblv}_{\langle r\rangle}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*/\langle r\rangle)$ satisfies Coxeter descent for every reflection $r \in W^{\mathrm{aff}}$.
(4) The object $\mathrm{oblv}_{\langle s\rangle}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*/\langle s\rangle)$ satisfies Coxeter descent for every simple reflection $s \in W$.
(5) The object $\mathrm{oblv}_{W}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^*/W)$ satisfies Coxeter descent for W.
(6) For each n, each cohomology group $\tau^{\geq n}\tau^{\leq n}(\mathcal{F})$ given by the t-structure in Section 3.3.2 satisfies Coxeter descent for $\tilde{W}^{\mathrm{aff}}$.
(7) For every simple coroot γ with associated simple reflection s of the (finite) Weyl group W and associated closed subgroup scheme $\mathbb{G}_m^\gamma \hookrightarrow T$, the functor
$$\mathcal{D}(T)^W \xrightarrow{\mathrm{oblv}} \mathcal{D}(T)^{\langle s\rangle} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\gamma}} \mathcal{D}(T/\mathbb{G}_m^\gamma)^{\langle s\rangle} \simeq \mathcal{D}(T/\mathbb{G}_m^\gamma) \otimes \mathrm{Rep}(\langle s\rangle)$$
maps $M(\mathcal{F})$ into the subcategory $\mathcal{D}(T/\mathbb{G}_m^\gamma) \simeq \mathcal{D}(T/\mathbb{G}_m^\gamma) \otimes \mathrm{Vect}_{\mathrm{triv}} \hookrightarrow \mathcal{D}(T/\mathbb{G}_m^\gamma) \otimes \mathrm{Rep}(\langle s\rangle)$, where $\mathbb{G}_m^\gamma$ is the rank 1 subgroup scheme of T associated to γ, and $\mathrm{Vect}_{\mathrm{triv}}$ is the full subcategory generated by the trivial representation of the order two group $\langle s\rangle$.

Proof. We first show that $\mathcal{F}$ satisfies Coxeter descent if and only if $\mathcal{F}$ satisfies (1). For a given $x \in \mathfrak{t}^*(K)$, note that the fact that the right box in Proposition 4.19 commutes implies that any object in the essential image of $\tilde{\phi}^!$ has the property that the pullback to $\mathrm{IndCoh}(\mathrm{Spec}(K))^{W_x^{\mathrm{aff}}} \simeq \mathrm{Rep}_K(W_x^{\mathrm{aff}})$ is trivial. Therefore, it remains to show that the left adjoint $\tilde{\phi}_*^{\mathrm{IndCoh}}$ of Lemma 4.20 is conservative on this subcategory, since a functor with an adjoint is an equivalence if and only if it is fully faithful and its adjoint is conservative.

Assume we are given some nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\tilde{W}^{\mathrm{aff}}}$ has the property that, for every field-valued point x of $\mathfrak{t}^*$, the pullback to $\mathrm{IndCoh}(\mathrm{Spec}(K))^{W_x^{\mathrm{aff}}} \simeq \mathrm{Rep}_K(W_x^{\mathrm{aff}})$ is trivial. Since $\mathcal{F}$ is nonzero, its pullback $q^!(\mathcal{F})$ is nonzero, and so in particular there exists a field-valued point for which $x^!q^!(\mathcal{F}) \simeq \overline{x}^!(\mathcal{F})$ is nonzero. Furthermore, by Corollary 4.11, we see that $\overline{x}^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ lies in the full subcategory determined by the fully faithful functor $\Xi_{\mathrm{Spec}(C)}^H$. Therefore, by Lemma 4.3, we see that the assumption that $\overline{x}^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ is the trivial representation implies that, in the notation of Proposition 4.19, $q|^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ lies in the essential image of $\dot{\alpha}^!$. Moreover, this sheaf is nonzero since $\overline{x}^!(\mathcal{F})$ is nonzero and, since $i_x : \mathrm{Spec}(K) \to \mathrm{Spec}(C_x)$ is surjective on geometric points, it is conservative [GR17a, Chapter 4, Proposition 6.2.2], and therefore the pullback functor $i_x^!$ is conservative. Thus we in particular see that $\dot{\alpha}_*^{\mathrm{IndCoh}}(q|^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F})))$ is nonzero. Applying base change (Corollary 4.21), we therefore see that $\overline{x}^!\tilde{\phi}_*^{\mathrm{IndCoh}}(\mathcal{F})$ does not vanish, and therefore neither does $\tilde{\phi}_*^{\mathrm{IndCoh}}(\mathcal{F})$, as required.

Now, to show that (1) $\Rightarrow$ (2), let $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\tilde{W}^{\mathrm{aff}}}$ be some sheaf satisfying (1) and assume W' is some parabolic subgroup of W^{aff}. We wish to show that $\mathcal{G} := \mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F})$ satisfies Coxeter descent for W_x^{aff}. By Proposition 4.7(2), it suffices to show that the canonical W'_x-representation on $x^!(\mathcal{G})$ is trivial. However, note that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Rep}(W_x^{\mathrm{aff}}) \arrow[r, "\mathrm{oblv}_{W'_x}^{W_x^{\mathrm{aff}}}"] & \mathrm{Rep}(W'_x) \\
\mathrm{IndCoh}(\mathfrak{t}^*)^{W_x^{\mathrm{aff}}} \arrow[u, "x^!"] \arrow[r, "\mathrm{oblv}_{W'_x}^{W_x^{\mathrm{aff}}}"] & \mathrm{IndCoh}(\mathfrak{t}^*)^{W'_x} \arrow[u, "x^!"']
\end{tikzcd}$$

and so the fact that the associated W_x^{aff}-representation structure on $x^!(\mathcal{G})$ is trivial implies that the associated W'_x-representation is trivial, as desired.

Conversely, if we are given some $\mathcal{F}$ satisfying (2) and some field-valued $x \in \mathfrak{t}^*(K)$, it is standard (see, for example, [Lon18, Proposition 5.3]) that the subgroup W_x^{aff} is a finite parabolic subgroup. Therefore we see that, by assumption, $\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F})$ satisfies Coxeter descent for W_x^{aff}, and so that by Proposition 4.7(2), the canonical $(W_x^{\mathrm{aff}})_x = W_x^{\mathrm{aff}}$-representation on $x^!(\mathrm{oblv}_{W_x^{\mathrm{aff}}}^{\tilde{W}^{\mathrm{aff}}}(\mathcal{F}))$ is trivial, as required.

The equivalence (2) $\Leftrightarrow$ (3) follows directly from the fact that one can check if a given $\mathcal{G} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{W'}$ satisfies Coxeter descent for W' if and only if $\mathrm{oblv}_{\langle r\rangle}^{W'}(\mathcal{G}) \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\langle r\rangle}$ satisfies Coxeter descent for $\langle r\rangle$ for all reflections $r \in W'$, see Proposition 4.16. Similarly, the equivalence (4) $\Leftrightarrow$ (5) by varying r over all *simple* reflections of W, see Proposition 4.16(2).

The proof of the equivalence (3) $\Leftrightarrow$ (4) follows identically to the proof of the claim '(2) $\Leftrightarrow$ (1)' of Proposition 4.16. The relevant addition is the standard fact (see, for example, [BM13, Lemma 2.1.1]) that any reflection of W^{aff} is conjugate in $\tilde{W}^{\mathrm{aff}}$ to some simple reflection of W.

Because $\tilde{\phi}^!$ is fully faithful (Proposition 4.18) and t-exact (Proposition 3.21) we have that the essential image is closed under truncations, thus showing that if $\mathcal{F}$ satisfies Coxeter descent, then so too does its cohomology groups. Since the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ is left-complete and right-complete (Corollary 3.23), a given $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})$ has a canonical isomorphism

$$\mathcal{F} \simeq \lim_m \mathrm{colim}_n \tau^{\geq m}\tau^{\leq n}(\mathcal{F})$$

where $-m, n \in \mathbb{Z}^{\geq 0}$. Therefore, since $\tilde{\phi}^!$ is a continuous right adjoint, its essential image is closed under both limits and colimits. Thus since the essential image of $\phi^!$ is also closed under extensions (by fully faithfulness) we see that if all cohomology groups of $\mathcal{F}$ satisfy Coxeter descent for $\tilde{W}^{\mathrm{aff}}$, so too does $\mathcal{F}$.

Finally, we now show that the full subcategories given by (4) and (7) of Proposition 4.23 are equivalent. To this end, fix a simple reflection s, and note that if F denotes the composite functor

$$\mathcal{D}(T)^{\langle s\rangle} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\gamma}} \mathcal{D}(T/\mathbb{G}_m^\gamma)^{\langle s\rangle} \simeq \mathcal{D}(T/\mathbb{G}_m^\gamma) \otimes \mathrm{Rep}(\langle s\rangle)$$

then for any $\mathcal{F} \in \mathcal{D}(T)^W$, $F(\mathcal{F})$ lies in the subcategory $\mathcal{D}(T/\mathbb{G}_m^\gamma) \simeq \mathcal{D}(T/\mathbb{G}_m^\gamma) \otimes \mathrm{Vect}_{\mathrm{triv}}$ if and only if the object $\mathrm{Av}_*^{T/\mathbb{G}_m^\gamma,w}F(\mathcal{F})$ lies in the full subcategory $\mathcal{D}(T/\mathbb{G}_m^\gamma)^{T/\mathbb{G}_m^\gamma,w} \simeq \mathcal{D}(T/\mathbb{G}_m^\gamma)^{T/\mathbb{G}_m^\gamma,w} \otimes \mathrm{Vect}_{\mathrm{triv}}$, by the conservativity of weak averaging, see [Gai15]. Furthermore, if we assume s has associated coroot γ such that s reflects across the hyperplane $\mathfrak{t}^*_{\gamma=0} := \{\gamma = 0\} \hookrightarrow \mathfrak{t}^*$, the Mellin transform allows us to identify the functor $\mathrm{Av}_*^{T/\mathbb{G}_m^\gamma,w}F$ with the composite:

$$\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))^{\langle s\rangle} \to \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}}/X')^{\langle s\rangle} \simeq \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}}/X')\otimes\mathrm{Rep}(\langle s\rangle) \xrightarrow{\mathrm{oblv}^{X'}\otimes\,\mathrm{id}} \mathrm{IndCoh}(\mathfrak{t}^*_{\gamma\in\mathbb{Z}})\otimes\mathrm{Rep}(\langle s\rangle)$$

where X' is the lattice of weights generated by the fundamental weights distinct from the fundamental weight associated to γ. Therefore, any sheaf satisfying (4) immediately satisfies (7), and any sheaf satisfying (7) also satisfies (4). $\square$

5. Nondegenerate Categories and the Weyl Group Action

In this section, we introduce the notion of a nondegenerate G-category and show it admits a Weyl group action.

5.1. **Degenerate G-Categories.**

5.1.1. *Definition of The Universal Degenerate Category.* Define the category $\mathcal{D}(N\backslash G)_{\deg}$ to denote the full subcategory of $\mathcal{D}(N\backslash G)$ generated by eventually coconnective objects in the kernel of $\mathrm{Av}_!^\psi : \mathcal{D}(N\backslash G) \to \mathcal{D}(N_\psi^-\backslash G)$. Let $\mathcal{D}(N\backslash G/N)_{\deg}$ denote the full subcategory of $\mathcal{D}(N\backslash G/N)$ generated by eventually coconnective objects in the kernel of the right N-invariants of the left Whittaker averaging functor. We can similarly define the category $(\mathcal{D}(N\backslash G)^{N,\alpha})_{\deg}$ where $\alpha : N \to \mathbb{G}_a$ is any character, or the right N is replaced with N^-. These categories are related as follows:

Lemma 5.1. If $\alpha : N \to \mathbb{G}_a$ is any character then the inclusion $(\mathcal{D}(N\backslash G)^{N,\alpha})_{\deg} \hookrightarrow (\mathcal{D}(N\backslash G)_{\deg})^{N,\alpha}$ is an equivalence, and an identical result holds if N is replaced with any unipotent subgroup.

Proof. We show this functor is essentially surjective. Assume $\mathcal{F} \in \mathcal{D}(N\backslash G)^{N,\alpha}$ has the property that $\mathrm{oblv}^{N,\alpha}(\mathcal{F}) \simeq \mathrm{colim}_\alpha(C_\alpha)$ where $C_\alpha \in \mathcal{D}(N\backslash G)^+$ and are in the kernel of the left Whittaker averaging functor $\mathrm{Av}_!^\psi$. We then see that

$$\mathcal{F} \xrightarrow{\sim} \mathrm{Av}_*^{N,\alpha}\mathrm{oblv}^{N,\alpha}(\mathcal{F}) \simeq \mathrm{colim}_\alpha(\mathrm{Av}_*^{N,\alpha}(C_\alpha))$$

where the first step uses Lemma 2.36 and the second step uses the fact that $\mathrm{Av}_*^{N,\alpha}$ is continuous, see Theorem 2.10. Since $\mathrm{oblv}^{N,\alpha}$ is t-exact, its right adjoint is left exact. Therefore, each $\mathrm{Av}_*^{N,\alpha}(C_\alpha)$ is eventually coconnective. Furthermore, since $\mathrm{Av}_*^{N,\alpha}(C_\alpha)$ is the *right* averaging, the left Whittaker averaging vanishes. Thus $\mathcal{F}$ is a colimit of eventually coconnective objects of $D(N\backslash G)^{N,\alpha}$ which are in the kernel of the left Whittaker averaging functor, as desired. $\square$

We now justify the use of the subscript 'deg' in the notation '$\mathcal{D}(N\backslash G/N)_{\deg}$' as opposed to a subscript such as 'left-deg'.

Proposition 5.2. The following categories are equivalent:

(1) The full subcategory of $\mathcal{D}(N\backslash G/N)$ generated by eventually coconnective objects in the kernel of the left Whittaker averaging functor $\mathrm{Av}_!^\psi : \mathcal{D}(N\backslash G/N) \to \mathcal{D}(N_\psi^-\backslash G/N)$.
(2) The full subcategory of $\mathcal{D}(N\backslash G/N)$ generated by eventually coconnective objects in the kernel of the right Whittaker averaging functor $\mathrm{Av}_!^{-\psi} : \mathcal{D}(N\backslash G/N) \to \mathcal{D}(N\backslash G/_{-\psi}N^-)$.
(3) The full subcategory of $\mathcal{D}(N\backslash G/N)$ generated by eventually coconnective objects in the kernel of the bi-Whittaker averaging functor $\mathrm{Av}_!^{\psi\times-\psi} : \mathcal{D}(N\backslash G/N) \to \mathcal{H}_\psi$.

We will prove this after showing the following lemma:

Lemma 5.3. The functor $\mathcal{D}(T) \xrightarrow{\sim} \mathcal{D}(N\backslash G/_{-\psi}N^-) \xrightarrow{\mathrm{Av}_!^{\psi}} \mathcal{D}(N^-_{\psi}\backslash G/_{-\psi}N^-)$, where the first arrow is given by Proposition 1.7, is conservative.

Proof. We may equivalently show $\mathrm{Av}_!^{\psi}[-\dim(N)]$ is conservative. Because both functors $\mathrm{Av}_*^N[\dim(N)]$ below average with respect to the right N action and both $\mathrm{Av}_!^{\psi}[-\dim(N)]$ below average with respect to the left action, the following diagram commutes

$$\begin{tikzcd}
\mathcal{D}(N^-_{\psi}\backslash G/N) & \mathcal{D}(N\backslash G/N) \arrow[l, "\mathrm{Av}_!^{-\psi}[-\dim(N)]"'] \\
\mathcal{H}_{\psi} \arrow[u, "\mathrm{Av}_*^N[\dim(N)]"] & \mathcal{D}(N\backslash G/_{-\psi}N^-) \arrow[l, "\mathrm{Av}_!^{\psi}[-\dim(N)]"] \arrow[u, "\mathrm{Av}_*^N[\dim(N)]"']
\end{tikzcd}$$

and so to verify conservativity of $\mathrm{Av}_!^{\psi}[-\dim(N)]$ we may verify the composite given by $\mathrm{Av}_!^{\psi}[-\dim(N)]\mathrm{Av}_*^N[\dim(N)]$ as in the diagram is conservative. Both functors in the diagram are t-exact by Corollary 2.42 and Theorem 2.43. Let $i : T \cong N\backslash B \hookrightarrow N\backslash G$ denote the closed embedding. Since the category $\mathcal{D}(T)$ is the derived category of its heart [GR, Section 4.7]), we may equivalently show that the composite $\mathrm{Av}_!^{\psi}[-\dim(N)]\mathrm{Av}_*^N[\dim(N)]$ on objects of the form $m_{*,dR}(\mathcal{F} \boxtimes -\psi)$, where $m : N\backslash B \times N^- \hookrightarrow G$ is the open embedding induced by multiplication.

By t-exactness of $\mathrm{Av}_!^{\psi}[-\dim(N)]$ Corollary 2.42, we may verify conservativity for objects $\mathcal{F} \in \mathcal{D}(T)^{\heartsuit}$. If $\mathcal{F} \in \mathcal{D}(T)^{\heartsuit}$ then $\mathrm{Av}_*^N[\dim(N)](m_{*,dR}(\mathcal{F} \boxtimes -\psi))$ contains $i_{*,dR}(\mathcal{F}) \in \mathcal{D}(T)^{\heartsuit}$ as a subobject, since the composite $\mathcal{D}(T) \simeq \mathcal{D}(N\backslash G/_{-\psi}N^-) \xrightarrow{\mathrm{Av}_*^N} \mathcal{D}(N\backslash G/N) \xrightarrow{i^!} \mathcal{D}(T)$ is the identity. We similarly see that $\mathrm{Av}_!^{\psi}[-\dim(N)]$ is the identity when restricted to $\mathcal{D}(T) \xrightarrow{i_{*,dR}} \mathcal{D}(N\backslash G/N)$ by the uniqueness of left adjoints. Therefore, we see that $\mathrm{Av}_!^{\psi}[-\dim(N)]\mathrm{Av}_*^N[\dim(N)](\mathcal{F})$ contains $i^!(\mathcal{F}) \simeq \mathcal{F}$ as a subobject, and therefore if $\mathcal{F}$ is nonzero, so too is $\mathrm{Av}_*^N[\dim(N)]\mathrm{Av}_!^{\psi}[-\dim(N)](\mathcal{F})$ and therefore $\mathrm{Av}_!^{\psi}[-\dim(N)](\mathcal{F})$. $\square$

Proof of Proposition 5.2. We show the categories generated as in (1) and (3) are equivalent; a symmetric argument gives that the categories in (2) and (3) are equivalent. Since bi-Whittaker averaging can be realized as a composite of right and left Whittaker averaging, we see that the subcategory as in (1) is contained in the subcategory of (3). This factorization, along with the conservativity of the averaging functor $\mathrm{Av}_!^{-\psi} : \mathcal{D}(N^-_{\psi}\backslash G/N) \to \mathcal{H}_{\psi}$ in Lemma 5.3, gives that the category in (3) is contained in the category in (1). $\square$

We now record some corollaries of the above results.

Corollary 5.4. We have an equivalence of categories $(\mathcal{D}(N\backslash G)_{\mathrm{deg}})^{N^-,\psi} \simeq 0$.

Proof. Any object of $(\mathcal{D}(N\backslash G)_{\mathrm{deg}})^{N^-,\psi}$ can be written as a colimit of eventually coconnective objects of $\mathcal{D}(N\backslash G)^{N^-,\psi}$ which are in the kernel of the left Whittaker averaging functor by Lemma 5.1. By Lemma 5.3, the only such object is zero. $\square$

Corollary 5.5. The category $\mathcal{D}(N\backslash G/N)_{\mathrm{deg}}$ is closed under the left action of $\mathcal{D}(N\backslash G/N)$.

Proof. The category $\mathcal{D}(N\backslash G/N)$ is compactly generated, and therefore it suffices to show the action of the compact objects $\mathcal{F} \in \mathcal{D}(N\backslash G/N)$ preserve the category $\mathcal{D}(N\backslash G/N)_{\mathrm{deg}}$. Since the t-structure on $\mathcal{D}(N\backslash G/N)$ is right-complete (see Definition 2.26), $\mathcal{F}$ is eventually connective. Moreover, the action of convolution of $\mathcal{D}(N\backslash G/N)$ on itself preserves the eventually connective subcategory, since the pullback by a smooth map is t-exact up to shift and the pushforward map has finite cohomological amplitude, see [HTT08, Proposition 1.5.29].

Let $\mathcal{G}$ be a generator of $\mathcal{D}(N\backslash G/N)_{\mathrm{deg}}$, i.e. an eventually coconnective object in the kernel of left Whittaker averaging. Since each averaging functor is continuous, by Proposition 5.2 we see that $\mathcal{G}$ is also in the kernel of the *right* averaging functor $\mathrm{Av}_!^{-\psi} : \mathcal{D}(N\backslash G/N) \to \mathcal{D}(N\backslash G/_{-\psi}N^-)$ as well. Therefore since the right averaging functor is left $\mathcal{D}(N\backslash G/N)$-equivariant, we see that $\mathcal{F} \star^N \mathcal{G}$ is an eventually coconnective object in the kernel of the right averaging functor, and therefore in $\mathcal{D}(N\backslash G/N)_{\mathrm{deg}}$. $\square$

Corollary 5.6. Let $\mathcal{D}(N\backslash G)_{\text{deg}}$ denote the full subcategory generated by eventually coconnective objects in the kernel the left Whittaker averaging functor $\mathrm{Av}_!^{\psi} : \mathcal{D}(N\backslash G) \to \mathcal{D}(N^-_{\psi}\backslash G)$. Then $\mathcal{D}(N\backslash G)_{\text{deg}}$ is closed under the left action of $\mathcal{D}(N\backslash G/N)_{\text{deg}}$.

Proof. Let $\mathcal{D}$ denote the full (left) $\mathcal{D}(N\backslash G/N)_{\text{deg}}$-subcategory of $\mathcal{D}(N\backslash G)$ generated by eventually coconnective objects in the kernel of left Whittaker averaging. This naturally acquires a right G-action, and we have a natural inclusion functor of right G-categories $\mathcal{D}(N\backslash G)_{\text{deg}} \hookrightarrow \mathcal{D}$. Moreover, by Theorem 2.15, we may check that this functor is an equivalence after applying the right invariants $(-)^N$. Therefore this result follows directly from Corollary 5.5. ☐

5.1.2. *Definition of Degenerate G-Categories in General.*

Definition 5.7. Let $\mathcal{C}$ denote a G-category. We say $\mathcal{C}$ is *degenerate* if the canonical map $\mathcal{D}(N\backslash G)_{\text{deg}} \otimes_G \mathcal{C} \to \mathcal{C}^N$ is an equivalence.

Proposition 5.8. A G-category $\mathcal{C}$ is degenerate if and only if for any Borel B' with $N' := [B', B']$, the canonical map $\mathcal{C}^{N'} \leftarrow \mathcal{D}(N'\backslash G)_{\text{deg}} \otimes_G \mathcal{C}$ is an equivalence.

Proof. The 'if' direction follows from taking $B' := B$. For the other direction, fix some B', N' as in Proposition 5.8. There exists some $g \in G(k)$ such that $gBg^{-1} \cong B'$ so that $gNg^{-1} \cong N'$. Furthermore, g determines a character ψ', defined as the composite $N'^{-}/[N'^{-}, N'^{-}] \xrightarrow{\mathrm{Ad}_{g^{-1}}} N^-/[N^-, N^-] \to \mathbb{G}_a$, which is nondegenerate. Therefore, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(N^-_{\psi}\backslash G) \arrow[d, "\ell_g"] & \mathcal{D}(N\backslash G) \arrow[l, "\mathrm{Av}_!^{\psi}"'] \arrow[d, "\ell_g"] \\
\mathcal{D}(N'^{-}_{\psi'}\backslash G) & \mathcal{D}(N'\backslash G) \arrow[l, "\mathrm{Av}_!^{-\psi'}"']
\end{tikzcd}$$

where the right vertical arrows are given by left multiplication by g, which is manifestly right G-equivariant. The fact that the action of G on itself yields an action on $\mathcal{D}(G)$ compatible with the t-structure implies that Ad_g induces an isomorphism $\mathcal{D}(N\backslash G)_{\text{deg}} \xrightarrow{\sim} \mathcal{D}(N'\backslash G)_{\text{deg}}$. Furthermore, the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}^N \arrow[d, ":="] & \mathcal{D}(N\backslash G) \otimes_G \mathcal{C} \arrow[l, "\sim"'] \arrow[d, "\mathrm{Ad}_g \otimes \mathrm{id}"] & \mathcal{D}(N\backslash G)_{\text{deg}} \otimes_G \mathcal{C} \arrow[l, "I_* \otimes \mathrm{id}"'] \arrow[d, "\mathrm{Ad}_g \otimes \mathrm{id}"] \\
\mathcal{C}^{N'} & \mathcal{D}(N'\backslash G) \otimes_G \mathcal{C} \arrow[l, "\sim"'] & \mathcal{D}(N'\backslash G)_{\text{deg}} \otimes_G \mathcal{C} \arrow[l, "I'_* \otimes \mathrm{id}"']
\end{tikzcd}$$

where the maps I_* and I'_* are the inclusions and the leftmost horizontal arrows are the canonical maps, which are equivalences by Theorem 2.10. We have that all three vertical maps are equivalences, and, assuming that $\mathcal{C}$ is degenerate, all maps in the top row of the diagram are equivalences. Therefore, all maps in the bottom row of the diagram are equivalences. ☐

5.2. **Nondegenerate G-Categories.**

5.2.1. *Definition and Basic Properties.* Let $I_* : \mathcal{D}(G/N)_{\text{deg}} \hookrightarrow \mathcal{D}(G/N)$ denote the inclusion functor. We first note:

Proposition 5.9. The essential image of I_* is closed under truncation functors.

Proof. Assume $\mathcal{F}$ lies in the essential image of I_*, i.e. $\mathcal{F}$ is a colimit of eventually coconnective objects in the kernel of $\mathrm{Av}_!^{\psi}$. By definition of t-structures, we obtain a cofiber sequence

$$\tau^{\leq 0}\mathcal{F} \to \mathcal{F} \to \tau^{>0}\mathcal{F}$$

where we omit the inclusion functors. Since $\mathrm{Av}_!^{\psi}$ is t-exact up to shift by Corollary 2.42, we see that $\tau^{>0}\mathcal{F}$ is an eventually coconnective object in the kernel of $\mathrm{Av}_!^{\psi}$, and thus lies in the essential image of I_*. Rotation gives a cofiber sequence

$$\mathcal{F} \to \tau^{>0}\mathcal{F} \to \tau^{\leq 0}(\mathcal{F})[1]$$

and so $\tau^{\leq 0}(\mathcal{F})[1]$ is a colimit of objects in the essential image of I_* and thus lies in the essential image of I_*. Finally, since the shift functor $[-1]$ is an equivalence of categories, it commutes with all colimits and so the essential image of I_* is closed under all cohomological shifts. Therefore, we obtain that $\tau^{\leq 0}(\mathcal{F})$ lies in the essential image of I_*, as required. ☐

By Proposition 5.9, we see that the inclusion I_* falls into the setup of Section 2.4.5. We defer the proof of the following statement to Section 5.2.2.

Proposition 5.10. The functor I_* admits a continuous, $\mathcal{D}(N\backslash G/N)$-equivariant right adjoint $I^! : \mathcal{D}(G/N) \to \mathcal{D}(G/N)_{\mathrm{deg}}$, and the category $\mathcal{D}(G/N)_{\mathrm{deg}}$ is compactly generated.

We set $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ to denote the right orthogonal category to $\mathcal{D}(G/N)_{\mathrm{deg}}$. Following Section 2.4.5, we let J_* denote the inclusion of $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ into $\mathcal{D}(G/N)$. From Proposition 5.10, we therefore obtain:

Proposition 5.11. The inclusion functor of the nondegenerate category $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ admits a $\mathcal{D}(N\backslash G/N)$-equivariant left adjoint $J^! : \mathcal{D}(G/N) \to \mathcal{D}(G/N)_{\mathrm{nondeg}}$, and there is a unique t-structure on $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ for which this quotient functor is t-exact.

Note that the $\mathcal{D}(N\backslash G/N)$-equivariance follows from Corollary 2.25. We now make the following definition:

Definition 5.12. We say a G-category $\mathcal{C}$ is *nondegenerate* if $\mathcal{D}(N\backslash G)_{\mathrm{nondeg}} \otimes_G \mathcal{C} \simeq \mathcal{C}^N$ (or, equivalently, $\mathcal{D}(N\backslash G)_{\mathrm{deg}} \otimes_G \mathcal{C} \simeq 0$).

We now record some first properties of our category $\mathcal{D}(N\backslash G)_{\mathrm{nondeg}}$.

Proposition 5.13. The canonical map $\mathcal{D}(N\backslash G)^{N^-,\psi} \to (\mathcal{D}(N\backslash G)_{\mathrm{nondeg}})^{(N^-,\psi)}$ is an equivalence.

Proof. The map $I^! : \mathcal{D}(N\backslash G) \to \mathcal{D}(N\backslash G)_{\mathrm{deg}}$ is manifestly right G-equivariant. Therefore we can identify $(\mathcal{D}(N\backslash G)_{\mathrm{nondeg}})^{(N^-,\psi)}$ with the kernel of the induced functor on Whittaker invariants $I^{!,N^-,\psi} : \mathcal{D}(N\backslash G)^{N^-,\psi} \to \mathcal{D}(N\backslash G)_{\mathrm{deg}}^{N^-,\psi}$. By Corollary 5.4, the codomain of this functor vanishes, and so all objects are nondegenerate. ☐

Proposition 5.14. The functor $\mathrm{Av}_*^N : \mathcal{D}(N_{\psi}^-\backslash G) \to \mathcal{D}(N\backslash G)$ canonically factors through the nondegenerate subcategory $D(N\backslash G)_{\mathrm{nondeg}}$, i.e. we have an equivalence $\mathrm{Av}_*^N \xrightarrow{\sim} J_*J^!\mathrm{Av}_*^N$. The functor $J^!\mathrm{Av}_*^N[\dim(N)]$ is t-exact.

Proof. This from the fact that, for any $\mathcal{G} \in \mathcal{D}(N\backslash G)^+_{\mathrm{deg}}$, we have by adjunction that $\underline{\mathrm{Hom}}_{\mathcal{D}(N\backslash G)}(\mathcal{G}, \mathrm{Av}_*^N(\mathcal{F})) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}(N_{\psi}^-\backslash G)}(\mathrm{Av}_!^{\psi}(\mathcal{G}), \mathcal{F}) \simeq 0$, and such $\mathcal{G}$ generate $\mathcal{D}(N\backslash G)_{\mathrm{deg}}$ by Corollary 5.6. The t-exactness follows from Theorem 2.43 and the fact the quotient functor is t-exact, see Section 2.4.5. ☐

For a rank one parabolic P_α associated to a simple root α, let $Q_\alpha := [P_\alpha, P_\alpha]$.

Proposition 5.15. The left Whittaker averaging functor $\mathrm{Av}_!^{\psi}$ sends objects of $\mathcal{D}(N\backslash G)$ which are Q_α-monodromic for some simple root α (i.e. objects which are equivalent to objects of the form $q^!(\mathcal{F})$ for $\mathcal{F} \in \mathcal{D}(Q_\alpha\backslash G)$ and $q : N\backslash G \to Q_\alpha\backslash G$ the quotient map) to zero. Furthermore, $\mathrm{Av}_!^{\psi}$ vanishes on the full G-subcategory of $\mathcal{D}(N\backslash G)$ generated under shifts and colimits by the various Q_α-monodromic objects.

Proof. Let α denote a simple root. Since the composite of the G-functors $\mathrm{Av}_*^{Q_\alpha^r}\mathrm{Av}_*^N$ has integral kernel in the right Q_α-monodromic objects of $\mathcal{D}(N_{\psi}^-\backslash G/N)$, we see that its integral kernel is zero by Proposition 1.7. Therefore, its left adjoint $\mathrm{Av}_!^{\psi}\mathrm{oblv}^{Q_\alpha}$ vanishes. The latter claim follows from the continuity and exactness of the left adjoint $\mathrm{Av}_!^{\psi}$. ☐

5.2.2. *Proof of Proposition 5.10.* In this section, we prove Proposition 5.10 after showing the following proposition:

Proposition 5.16. The following categories are equivalent:

(1) The full subcategory of $\mathcal{D}(G/N)$ generated under colimits by the compact objects in the kernel of $\mathrm{Av}_!^{\psi}$.
(2) The full subcategory of $\mathcal{D}(G/N)$ generated under colimits by cohomologically bounded objects in the kernel of $\mathrm{Av}_!^{\psi}$.
(3) The full subcategory of $\mathcal{D}(G/N)$ generated under colimits by eventually coconnective objects in the kernel of $\mathrm{Av}_!^{\psi}$.

Proof. The compact objects in $\mathcal{D}(G/N)$ are in particular cohomologically bounded [DG13, Section 5.1.17], and cohomologically bounded objects are in particular eventually coconnective. Now, assume we are given an eventually coconnective $\mathcal{F}$. Since the t-structure on $\mathcal{D}(G/N)$ is right-complete (see Definition 2.26), we may write $\mathcal{F} \simeq \mathrm{colim}\tau^{\leq n}\mathcal{F}$ as the colimit of objects which have nonzero cohomology in only finitely many degrees, and therefore we assume $\mathcal{F}$ has nonzero cohomology in only finitely many degrees. Furthermore, we have that $\mathrm{Av}_!^{\psi}$ is t-exact, so $\mathrm{Av}_!^{\psi}(\mathcal{F})$ vanishes if and only if $\mathrm{Av}_!^{\psi}$ takes each cohomology group to zero. Therefore, we may assume $\mathcal{F}$ is concentrated in a single cohomological degree, and, since $\mathrm{Av}_!^{\psi}$ commutes with shifts, we may assume $\mathcal{F} \in \mathcal{D}(G/N)^{\heartsuit}$. However, $\mathcal{D}(G/N)^{\heartsuit}$ is a Grothendieck abelian category, and furthermore it is compactly generated, see [GR, Corollary 3.3.3]. Furthermore, the compact objects are closed under subquotients (see [GR, Section 4.7.7]) and so by Lemma 2.51 we see that every object is a union of compact subobjects. Since $\mathrm{Av}_!^{\psi}$ is t-exact, we therefore see that this object $\mathcal{F}$ is a colimit of compact objects in the kernel of $\mathrm{Av}_!^{\psi}$. ☐

Proof of Proposition 5.10. The compact generation follows by Proposition 5.16. By Lemma 2.50 with R taken to be the compact objects concetrated in a single cohomological degree, we see further that the inclusion functor of the degenerate subcategory $\mathcal{D}(G/N)_{\mathrm{deg}}$ preserves compact objects and therefore admits a continuous right adjoint. Finally, the right adjoint $I^!$ to the inclusion functor $I_* : \mathcal{D}(G/N)_{\mathrm{deg}} \hookrightarrow \mathcal{D}(G/N)$ is necessarily $\mathcal{D}(N\backslash G/N)$-linear by Corollary 2.25. ☐

5.2.3. *t-Structures on Universal Nondegenerate Categories.* We equip $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ and $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}$ with t-structures as in Section 2.4.5.

Remark 5.17. A priori, we may equip $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}} \simeq (\mathcal{D}(G/N)_{\mathrm{nondeg}})^N$ with a t-structure determined by the condition that the forgetful functor $(\mathcal{D}(G/N)_{\mathrm{nondeg}})^N \xhookrightarrow{\mathrm{oblv}^N} \mathcal{D}(G/N)_{\mathrm{nondeg}}$ is t-exact. However, we claim these two t-structures are equivalent. To see this, assume we equip $(\mathcal{D}(G/N)_{\mathrm{nondeg}})^N$ with the other t-structure. Since the map $J_* : \mathcal{D}(G/N)_{\mathrm{nondeg}} \hookrightarrow \mathcal{D}(G/N)$ is G-equivariant, we see that the following diagram canonically commutes

$$\begin{tikzcd}
\mathcal{D}(G/N)_{\mathrm{nondeg}} \arrow[r, "J_*"] & \mathcal{D}(G/N) \\
\mathcal{D}(G/N)^N_{\mathrm{nondeg}} \arrow[u, "\mathrm{oblv}"] \arrow[r, "J_*"] & \mathcal{D}(G/N)^N \arrow[u, "\mathrm{oblv}"']
\end{tikzcd}$$

and, by assumption on the t-structure on $\mathcal{D}(G/N)^N_{\mathrm{nondeg}}$ and $\mathcal{D}(G/N)^N$, the forgetful functors reflect the t-structure. Therefore, we see that the t-structure on $\mathcal{D}(G/N)^N_{\mathrm{nondeg}}$ determined by the requirement that oblv^N is t-exact also has the property that $X \in \mathcal{D}(G/N)^{N,\geq 0}_{\mathrm{nondeg}}$ if and only if $J_*(X) \in \mathcal{D}(G/N)^{N,\geq 0}$, and thus these two t-structures agree.

Since $T \times T$ acts on $\mathcal{D}(N\backslash G/N)_{\mathrm{nondeg}}$ compatibly with the t-structure by Proposition 2.59, we may equip $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\mathrm{nondeg}}$ with a t-structure such that the forgetful functor is t-exact.

Proposition 5.18. The t-structures on the categories $\mathcal{D}(G/N)_{\mathrm{nondeg}}$, $(\mathcal{D}(G/N)_{\mathrm{nondeg}})^N$ and $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\mathrm{nondeg}}$ are right-complete.

Proof. The t-structures on $\mathcal{D}(G/N)_{\text{nondeg}}$ and $(\mathcal{D}(G/N)_{\text{nondeg}})^N$ are right-complete by construction and Corollary 2.57, since the t-structures on $\mathcal{D}(G/N)$ and $\mathcal{D}(N\backslash G/N)$ are right-complete. Here, the t-structure on $\mathcal{D}(N\backslash G/N)$ is right-complete because it admits a t-exact conservative functor $\text{oblv}^N : \mathcal{D}(N\backslash G/N) \hookrightarrow \mathcal{D}(G/N)$ to a category for which the t-structure is right-complete. Since the forgetful functor is t-exact and conservative, it reflects the t-structure. Therefore, we also obtain the right-completeness of the t-structure on $\mathcal{D}(N\backslash G/N)^{T\times T,w}_{\text{nondeg}}$. ☐

5.3. **The Weyl Group Action on Nondegenerate Categories.**

5.3.1. *Algebraic Construction of Gelfand-Graev Action.* We will soon argue that for nondegenerate categories $\mathcal{C}$ (defined in Definition 5.12), the T-action on the invariants $\mathcal{C}^N$ lifts to a $T\rtimes W$ action. We will first construct this $T \rtimes W$ action for the category $D(G/N)_{\text{nondeg}}$. To do this, we recall the following theorem of [GK]:

Theorem 5.19. [GK, Theorem 1.2.2] If G is simply connected, we have a natural algebra isomorphism:

$$\mathrm{H}^0\,\Gamma(\mathcal{D}_{G/N}) \cong (\Gamma(\mathcal{D}_T) \otimes_{Z\mathfrak{g}} \Gamma(\mathcal{D}_{G/_\psi N^-}))^{Z\mathfrak{g}} \tag{8}$$

where $\mathcal{D}_{G/_\psi N^-}$ denotes the Whittaker differential operators on G/N^- with respect to the character $-\psi$.

Additionally, this isomorphism respects the G, T action of both sides, and the W-action on the right hand side the gives rise to the W-action on the ring $\mathrm{H}^0\,\Gamma(\mathcal{D}_{G/N})$ given by the Gelfand-Graev action. The Harish-Chandra datum (see [Ras20b, Section 1.2]) is also explicitly computed, and so in particular we may rephrase this result in the language of groups acting on categories:

Corollary 5.20. The group $G, T \rtimes W$ acts strongly on $\mathrm{H}^0\,\Gamma(\mathcal{D}_{G/N})$-Mod, where the W action is given by the Gelfand-Graev action.

We note in passing that the results of Theorem 5.19 and Corollary 5.20 also hold when G is the product of a simply connected group and a torus, and we will use this in what follows. To show that $\mathcal{D}(G/N)_{\text{nondeg}}$ acquires a $G,(T \rtimes W)$-action, we will relate this category to the category $\mathrm{H}^0\,\Gamma(\mathcal{D}_{G/N})$-Mod. We do so in the following lemma:

Lemma 5.21. We have the following:

(1) The global sections functor induces an equivalence $\mathcal{D}(G/N) \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/N})$-mod.

(2) The forgetful functor $\Gamma(\mathcal{D}_{G/N})$-mod $\xrightarrow{\text{oblv}}$ $\mathrm{H}^0\Gamma(\mathcal{D}_{G/N})$-mod induced by the ring map $\mathrm{H}^0\Gamma(\mathcal{D}_{G/N}) \to \Gamma(\mathcal{D}_{G/N})$ is fully faithful.

Remark 5.22. It is known that G/N is a locally closed subscheme of $\mathbb{A}^n_k$ for some $n \in \mathbb{Z}^{\geq 0}$, see, for example, [RW21, Section 29.2]. In particular, G/N is quasi-affine, and so we may realize G/N as an open subset of its affine closure $\overline{G/N}$. Furthermore, since $\overline{G/N}$ is known to be normal (which follows directly from the smoothness of G and [Gro97, Theorem 18.4], for example), we have an equivalence $\mathrm{H}^0(\mathcal{D}_{\overline{G/N}}) \cong \mathrm{H}^0\Gamma(\mathcal{D}_{G/N})$. However, this affine closure is not smooth if G is semisimple and not a product of copies of SL_2. In particular, the category $\mathcal{D}(\overline{G/N}) := \mathrm{IndCoh}(\overline{G/N}_{dR})$ need not be equivalent to the category of right modules for the ring $\mathrm{H}^0\Gamma(\mathcal{D}_{G/N})$, and we do not know if there is such an equivalence.

Proof of Lemma 5.21. To show (1), we note that since G/N is quasi-affine (see Remark 5.22), the global sections functor $\mathrm{Hom}_{\mathcal{D}(G/N)}(\mathcal{D}_{G/N}, -)$ is conservative. Therefore, since the global sections functor also commutes with colimits, by Barr-Beck we have a canonical equivalence $\mathcal{D}(G/N) \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/N})$-mod.

Now, to show (2), consider the map of rings $\mathrm{H}^0\Gamma(\mathcal{D}_{G/N}) \to \Gamma(\mathcal{D}_{G/N})$. We obtain a standard adjunction

$$\Gamma(\mathcal{D}_{G/N}) \otimes_{\mathrm{H}^0\Gamma(\mathcal{D}_{G/N})} (-) : \mathrm{H}^0\Gamma(\mathcal{D}_{G/N})\text{-Mod} \rightleftarrows \Gamma(\mathcal{D}_{G/N})\text{-Mod} : \text{oblv}$$

and therefore it suffices to show that the counit of the adjunction is an equivalence on compact generators, i.e. the natural map

$$\Gamma(\mathcal{D}_{G/N}) \otimes_{\mathrm{H}^0\Gamma(\mathcal{D}_{G/N})} \Gamma(\mathcal{D}_{G/N}) \to \Gamma(\mathcal{D}_{G/N})$$

is an equivalence. Note this map is filtered, and so in particular to show this map is an isomorphism we may show the induced map on associated graded is an isomorphism. However, the associated graded of $\Gamma(\mathcal{D}_{G/N})$ is the (not necessarily classical) ring A of global functions on the cotangent bundle $T^*(G/N)$, and the associated graded of $\mathrm{H}^0\Gamma(\mathcal{D}_{G/N})$ is similarly the ring $\mathrm{H}^0(A)$. In particular, we wish to show that the natural map:

$$A \otimes_{\mathrm{H}^0 A} A \to A$$

is an equivalence.

First, note the fact that G/N is quasi-affine (Remark 5.22) also implies that $T^*(G/N)$ is quasi-affine, since the map $T^*(G/N) \to G/N$ is affine (as $T^*(G/N)$ is a vector bundle as G/N is smooth), affine morphisms are quasiaffine [Sta22, Lemma 29.13.3], and so the composite morphism $T^*(G/N) \to G/N \to *$ is quasi-affine since the composite of quasi-affine morphisms is quasi-affine [Sta22, Lemma 29.13.4]. In particular, $T^*(G/N)$ is an affine open subset of $\mathrm{Spec}(\mathrm{H}^0(A))$ [Sta22, Lemma 29.13.3]. Let j denote this open embedding. Since the terminal map $T^*(G/N) \to *$ is quasi-affine, by [GR17a, Chapter 3, Proposition 3.3.3] we obtain an equivalence $\mathrm{QCoh}(T^*(G/N)) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Spec}(A)) := A\text{-Mod}$. Then, in particular, our natural map may be identified with the counit of the adjunction

$$j^*j_*(A) \to A$$

applied to the compact generator $A \in A$-Mod. However, j_* is fully faithful (see, for example, the proof of [GR17a, Chapter 3, Proposition 3.3.3]) and so this map is an equivalence. □

Recall that $\mathcal{D}(G/N)^{N^-,\psi}$ is a full subcategory of $\mathcal{D}(G/N)$, see Lemma 2.36. We record one result of interest with respect to the Gelfand-Graev action, see [Gin18, Proposition 5.5.2]:

Proposition 5.23. The Gelfand-Graev functors on $\mathcal{D}(G/N)^{N^-,\psi}$ are compatible with the $(W,\cdot)$-action on $\mathcal{D}(T)$. In particular, this category is a full W-subcategory of $H^0(\Gamma_{\mathcal{D}_{G/N}})$-mod.

Remark 5.24. Since the category $\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(T)$ is the derived category of its heart, the result of Proposition 5.23 in [Gin18, Proposition 5.5.2], stated for the classical derived category $\mathrm{ho}(\mathcal{D}(T))$, immediately gives W-equivariance of the equivalence of DG categories $\mathcal{D}(G/N)^{N^-,\psi} \xrightarrow{\sim} \mathcal{D}(T)$ as well.

5.3.2. *Reduction to Simply Connected Derived Subgroup Case.* Let $\tilde{G} := G_{sc} \times Z(G)^\circ$, where G_{sc} denotes the simply connected cover of the semisimple group $[G,G]$ and $Z(G)^\circ$ denotes the connected component of the identity. The following result is well known; however, we were unable to locate a reference for the exact formulation we will use, so we provide a proof here:

Lemma 5.25. The canonical map $\tilde{G} \to G$ has kernel Z for Z a finite central subscheme, and thus induces an isomorphism $\tilde{G}/Z \xrightarrow{\sim} G$. Furthermore, $Z(G)^\circ$ is a split torus, so any reductive group in particular admits a central isogeny from a group which is the product of a semisimple simply connected group and a split torus.

Proof. By [Mil17, Remark 19.30], we may write G as the quotient $(G_{sc}\times Z(G))/\tilde{F}$, where $\tilde{F}$ is the finite group scheme given by the scheme theoretic image of the map $Z(G') \to G_{sc} \times Z(G)$ defined by $\xi \mapsto (\xi, \xi^{-1})$. Now let F denote the scheme theoretic image of the closed subscheme $Z(G_{sc}) \times_{Z(G)} Z(G)^\circ$ under the above map to $G' \times Z(G)$. Then the map $G_{sc} \times Z(G)^\circ \to (G' \times Z(G))/\tilde{F}$ induces an isomorphism $(G_{sc} \times Z(G)^\circ)/F \xrightarrow{\sim} (G_{sc} \times Z(G))/\tilde{F}$ (see [Mil17, Theorem 5.82] with $H := G_{sc} \times Z(G)^\circ$ and $N := \tilde{F}$).

Let $\mathscr{T} := Z(G)^\circ$. Then $\mathscr{T}$ is smooth, since k has characteristic zero, and connected by assumption. Furthermore, $\mathscr{T}$ is diagonalizable since $Z(G)$ is [Mil17, Proposition 21.8], and subgroups are diagonalizable groups are diagonalizable [Mil17, Theorem 12.9(c)]. Therefore, we see that $\mathscr{T}$ is a smooth connected diagonalizable group. We therefore see that $\mathscr{T}$ is a split torus by [Mil17, Chapter 12, Section e]. □

We let $\tilde{G}$ and let $\phi : \tilde{G} \to G$ denote the central isogeny of Lemma 5.25. Let $\tilde{B} := \phi^{-1}(B)$, $\tilde{T} := \phi^{-1}(T)$, Furthermore, let $\tilde{N} := [\phi^{-1}(B), \phi^{-1}(B)]$ denotes the unipotent radical of $\tilde{B}$ (which is not necessarily the same group as $\phi^{-1}([B,B])$), and let $\tilde{N}^- := [\phi^{-1}(B^-), \phi^{-1}(B^-)]$. Given any parabolic subgroup P of G, we set $\tilde{P} := \phi^{-1}(G)$ to be the corresponding parabolic subgroup of $\tilde{G}$, and, given a P, we set $\tilde{Q} := [\tilde{P}, \tilde{P}]$. If P is the rank one parabolic subgroup P_α, we will also denote this subgroup by $\tilde{P}_\alpha$ and $\tilde{Q}_\alpha := [\tilde{P}_\alpha, \tilde{P}_\alpha]$. This notation is justified by the following:

Proposition 5.26. With the above notation, we have the following:

(1) [Bor91, Theorem 22.6] The group $\tilde{B}$ is a Borel subgroup of $\tilde{G}$ containing a maximal torus $\tilde{T}$, and $\tilde{B}$ (respectively $\tilde{T}, \tilde{P}$) is the unique Borel (respectively, torus, parabolic subgroup) for which the image under ϕ is B (respectively, T, P). Furthermore, ϕ induces an isomorphism of Weyl groups.
(2) The group $\tilde{N}$ is the unipotent radical of $\tilde{B}$, and ϕ induces an isomorphism $\phi|_{\tilde{N}} : \tilde{N} \xrightarrow{\sim} N$.
(3) The closed subgroup scheme $\tilde{N}$ factors through the subgroup scheme $G_{sc} \times 1$.
(4) Let $\alpha : N \to \mathbb{G}_a$ be any character, and let $\tilde{\alpha}$ denote the additive character $\alpha\phi|_{\tilde{N}}$. Then the pullback functor $\mathcal{D}(G/_{\alpha}N) \to \mathcal{D}(\tilde{G}/_{\tilde{\alpha}}\tilde{N})$ induces an equivalence of categories $\mathcal{D}(G/_{\alpha}N) \xrightarrow{\sim} \mathcal{D}(\tilde{G}/_{\tilde{\alpha}}\tilde{N})^Z$.

The analogous claims to these statements also hold when $\tilde{N}$ is replaced with $\tilde{N}^-$ and $\tilde{B}$ with $\tilde{B}^- := \phi^{-1}(B^-)$.

Proof. As noted above, the first claim is precisely [Bor91, Theorem 22.6], using the fact that central isogenies are in particular surjective. The first part of claim (2) follows by definition, and the second part follows from the fact the kernel of the map $\phi|_{\tilde{N}}$ in particular lies in the kernel of ϕ, which contains only semisimple elements. Thus any element in the kernel of $\phi|_{\tilde{N}}$ both semisimple and unipotent, and thus trivial. This also shows the third claim, since there are no nontrivial semisimple elements of a torus.

Let Z denote the kernel of ϕ. Then, using the fact that ϕ induces an isomorphism $\tilde{G}/Z \xrightarrow{\sim} G$ and Theorem 2.10, we obtain (4) via:

$$\mathcal{D}(G/_{\alpha}N) \simeq \mathcal{D}(G)^{N,\alpha} \simeq \mathcal{D}((G_{sc} \times Z(G)^\circ)/Z)^{\tilde{N},\tilde{\alpha}} \simeq D(G_{sc} \times Z(G)^\circ)^{(\tilde{N}\times Z,\tilde{N},\tilde{\alpha}\times 1)} \simeq D(G_{sc}/_{\tilde{\alpha}}\tilde{N} \times Z(G)^\circ)^Z$$

where the final step uses the fact that any element of $\tilde{N}$ lies in $\tilde{G}$. $\square$

Remark 5.27. The result of Proposition 5.26 in particular implies that we have a canonical induced nondegenerate character given by the composite $\tilde{N}^- \xrightarrow{\phi|_{\tilde{N}^-}} N \xrightarrow{\psi} \mathbb{G}_a$. We will lightly abuse notation and also denote this character by ψ.

5.3.3. *Symplectic Fourier Transform Construction and Reminders.* In this section, we recall the construction of the symplectic Fourier transformations associated to each simple reflection s and recall some basic properties of the various Fourier transformations which will be used later. Let P_s denote the associated rank one parabolic subgroup, and let $\alpha : T \to \mathbb{G}_m$ denote the associated root. Following [KL88] and [Pol01], we can construct a rank two symplectic vector bundle on $G/[P_s, P_s]$ *in the case* $[G, G]$ *is simply connected* as follows.

Let $N_w := N \cap \dot{w}^{-1}\dot{w}_0 N \dot{w}_0 \dot{w}$, where, for each $w \in W$, $\dot{w}$ is some arbitrarily chosen lift of w. Choose a Levi decomposition of the parabolic $P_s \cong L_s \ltimes N_{w_0 s}$ and set $Q_s := [P_s, P_s]$ and $M_s := [L_s, L_s]$. Our choice of ψ in particular determines an isomorphism $\mathbb{G}_a \xrightarrow{\sim} N_s$, so we can identify $M_s \cong \mathrm{SL}_{2,\alpha}$. With this choice, set $V_s := G/N_{w_0 s} \times^{\mathrm{SL}_{2,\alpha}} \mathbb{A}^2$. Then the projection map $V_s \to G/[P_s, P_s]$ can be upgraded to the structure of a rank two symplectic vector bundle, and the vector bundle V_s has complement to the zero section $G/Q_\alpha \xhookrightarrow{z} V_s$ given by the universal flag variety $G/N \xhookrightarrow{j} V_s$ [KL88]. Furthermore, the composite πj may be identified with the quotient map $q : G/N \to G/Q_\alpha$. Let $F_s := j^!\mathbf{F}j_{*,dR}$, where $\mathbb{F}$ is the usual Fourier transform and $\mathbf{F} := \omega^!\mathbb{F}$ is the symplectic Fourier transform for the symplectic form $\omega : V_s \xrightarrow{\sim} V_s^\vee$. To emphasize which vector bundle $\mathcal{V}$ we are taking the Fourier transform with respect to, we write $\mathbb{F}_{\mathcal{V}} : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{V}^\vee)$ for the usual Fourier transform.

We can also define a line bundle $\mathcal{L}_s$ over G/Q_s and a map of G/Q_s-schemes $\ell_s : \mathcal{L}_s \to V_s$ whose fiber at every k-point of G/Q_s yields the inclusion of a one dimensional vector space into a two dimensional vector space. To do this, let $G^{\langle s\rangle}$ denote the locally closed subscheme of G given by those Bruhat cells labelled by those $w \in W$ which are minimal in their right $\langle s\rangle$ cosets. Note that our above notations give an identification $N_{w_0 s}\backslash(B \cap Q_s) \cong B_\alpha$, a Borel subgroup of $\mathrm{SL}_{2,s}$. Set $\mathcal{L}_s := G^{\langle s\rangle}/N_{w_0 s} \times^{B_\alpha} \mathbb{A}^1$. Then we similarly see that we have a map $\mathcal{L}_s \to G/[P_s, P_s]$ and the locally closed embedding $G^{\langle s\rangle} \to G$ induces our desired map $\ell_s : \mathcal{L}_s \to V_s$ over $G/[P_s, P_s]$.

Remark 5.28. Occasionally, to emphasize the root α associated to a simple reflection s, we will denote the objects P_s, Q_s, M_s and V_s as $P_\alpha, Q_\alpha, M_\alpha$ and V_α.

We record two observations in the following remark.

Remark 5.29. By light abuse of notation, let ω denote the isomorphism $V_s \xrightarrow{\sim} V_s^\vee$. Duality gives a canonical map $\ell_s^\vee : V_s^\vee \to \mathcal{L}_s^\vee$. Furthermore, note that the composite $p\omega\ell_s$ vanishes. Therefore there exists an isomorphism of schemes a over G/Q_s making the following diagram commute:

$$\begin{tikzcd} V_s \arrow[r, "\omega"] \arrow[d, "p"] & V_s^\vee \arrow[d, "\ell_s^\vee"] \\ V_s/\mathbb{G}_a \arrow[r, "a"] & \mathcal{L}_s^\vee \end{tikzcd}$$

where a is an isomorphism since V_s has rank two. Dually, there is an isomorphism of schemes $\omega|_{\mathcal{L}_s}$ over G/Q_s making the following diagram commute:

$$\begin{tikzcd} \mathcal{L}_s \arrow[r, "\omega|_{\mathcal{L}_s}"] \arrow[d, "\ell_s"] & (V_s/\mathbb{G}_a)^\vee \arrow[d, "p^\vee"] \\ V_s \arrow[r, "\omega"] & V_s^\vee \end{tikzcd} \tag{9}$$

We now a result on the usual Fourier transform, which can be proven by taking Verdier dual versions of all the functors in the proof of [Lau87, Théorèm 1.2.2.4]:

Proposition 5.30. Assume $f : E \to V$ is a morphism of vector bundles, where E has rank e and V has rank r. Then there is an isomorphism of functors $\mathbb{F}_E(f_{*,dR}(-)) \simeq f^{\vee,!}\mathbb{F}_V(-)[e-r]$, where $f^\vee : V^\vee \to E^\vee$ is the induced morphism.

5.3.4. *The Symplectic Fourier Transformations Preserve the Universal Nondegenerate Category.* Using the notation of Section 5.3.3 and Section 5.3.2, we show that the symplectic Fourier transformation preserves the nondegenerate category:

Lemma 5.31. Assume that $\mathcal{F} \in \mathcal{D}(\tilde{G}/\tilde{N})_{\text{nondeg}}$. Then the canonical map $\mathbf{F}j_{*,dR}\mathcal{F} \to j_{*,dR}j^!\mathbf{F}j_{*,dR}\mathcal{F}$ is an isomorphism.

Proof. Let $\mathcal{G} \in \mathcal{D}(\tilde{G}/\tilde{Q}_\alpha)$. We may equivalently show that $\underline{\text{Hom}}_{\mathcal{D}(V_s)}(z_{*,dR}(\mathcal{G}), \mathcal{F})) \simeq 0$. The category $\mathcal{D}(\tilde{G}/\tilde{Q}_\alpha)$ is compactly generated [GR, Corollary 3.3.3], and therefore it suffices to show this vanishing when $\mathcal{G}$ is compact. We also have

$$\underline{\text{Hom}}_{\mathcal{D}(V_s)}(z_{*,dR}(\mathcal{G}), \mathbf{F}j_{*,dR}(\mathcal{F})) \simeq \underline{\text{Hom}}_{\mathcal{D}(V_s)}(\mathbf{F}z_{*,dR}(\mathcal{G}), j_{*,dR}(\mathcal{F})) \simeq \underline{\text{Hom}}_{\mathcal{D}(V_s)}(\pi^!(\mathcal{G})[-2], j_{*,dR}(\mathcal{F}))$$

where $\pi : V_s \to \tilde{G}/\tilde{Q}_\alpha$ is the projection map, and the second equivalence follows applying Proposition 5.30 to the inclusion map of the zero section. Therefore, if $q : \tilde{G}/\tilde{N} \to \tilde{G}/\tilde{Q}_\alpha$ is the quotient map, we see that this expression above is equivalent to $\underline{\text{Hom}}_{\mathcal{D}(\tilde{G}/\tilde{N})}(q^!(\mathcal{G})[-2], \mathcal{F})$. However, the compact objects of $\mathcal{D}(\tilde{G}/\tilde{Q}_\alpha)$ are cohomologically bounded (this holds if $\tilde{G}/\tilde{Q}_\alpha$ is replaced with any smooth, classical scheme X and is an immediate consequence of the t-exactness of the composite oblv : $\mathcal{D}(X) \to \text{IndCoh}(X) \xrightarrow{\sim} \text{QCoh}(X)$ of conservative functors, [GR, Proposition 4.2.11], and so $q^!(\mathcal{G})$ is in particular eventually coconnective since $q^!$ is t-exact up to shift. Furthermore, $q^!(\mathcal{G})[-2]$ is in the kernel of $\text{Av}_!^\psi$ by Proposition 5.15, so $q^!(\mathcal{G})[-2]$ is a generator of $\mathcal{D}(\tilde{G}/\tilde{N})_{\text{deg}}$. Therefore, $\underline{\text{Hom}}_{\mathcal{D}(\tilde{G}/\tilde{N})}(q^!(\mathcal{G})[-2], \mathcal{F}) \simeq 0$. $\square$

First, note that the symplectic Fourier transform is t-exact [KL88, Theorem 1.2.1(iii)], so the action of $\langle s\rangle$ on $\mathcal{D}(V_s)$ reduces to the action of $\langle s\rangle$ on the abelian subcategory $\mathcal{D}(V_s)^\heartsuit$, since $\mathcal{D}(V_s)$ is the derived category of its heart. This action is computed in [Pol01, Section 4]. Statement (1) in the following proposition may be viewed as a derived, D-module variant [Pol01, Lemma 6.1.1]:

Proposition 5.32. We temporarily use the notation $\mathcal{D}(\tilde{G}/\tilde{N})_{\tilde{Q}_\alpha\text{-nondeg}}$ to denote the full subcategory of objects $\mathcal{F} \in \mathcal{D}(\tilde{G}/\tilde{N})$ for which $q_{*,dR}(\mathcal{F})$ vanishes, which we may equivalently view as a quotient category by Section 2.4.5.

(1) The category $\mathcal{D}(\tilde{G}/\tilde{N})_{\tilde{Q}_\alpha\text{-nondeg}} \xhookrightarrow{j_{*,dR}} \mathcal{D}(V_s)$ is closed under the action of the order two group $\langle s\rangle$.

(2) The functor Av_*^N lifts to a functor $\mathrm{Av}_*^N : \mathcal{D}(\tilde{G}/_{-\psi}\tilde{N}^-) \to \mathcal{D}(\tilde{G}/\tilde{N})^{\langle s \rangle}_{\tilde{Q}_\alpha\text{-nondeg}}$.

(3) The functor $\mathrm{Av}_!^\psi : \mathcal{D}(\tilde{G}/\tilde{N}) \to \mathcal{D}(\tilde{G}/_{-\psi}\tilde{N}^-)$ has the property that, for all $\mathcal{F} \in \mathcal{D}(\tilde{G}/\tilde{N})$, there is a canonical isomorphism $\mathrm{Av}_!^\psi(\mathcal{F}) \simeq \mathrm{Av}_!^\psi(F_s(\mathcal{F}))$.

Proof. Consider the full subcategory of $\mathcal{D}(V_s)$ generated by elements in the essential image of $z_{*,dR}$ and $\pi^!$. Since $\pi^![-2] \simeq \mathbf{F}z_{*,dR}$ (Proposition 5.30), this subcategory is closed under the symplectic Fourier transform. The right orthogonal complement can be identified with those objects $\mathcal{F} \in \mathcal{D}(V_s)$ for which $z^!(\mathcal{F}) \simeq 0$ and $\pi_{*,dR}(\mathcal{F}) \simeq 0$. The first condition is equivalent to the condition that the canonical map $\mathcal{F} \to j_{*,dR}(j^!(\mathcal{F}))$ is an equivalence, and therefore this subcategory may be identified with the full subcategory of objects of those $\mathcal{F}' \in \mathcal{D}(G/N)$ such that $\pi_{*,dR}(j_{*,dR}\mathcal{F}') \simeq q_{*,dR}(\mathcal{F}')$ vanishes, i.e. $\mathcal{D}(\tilde{G}/\tilde{N})_{\tilde{Q}_\alpha\text{-nondeg}}$.

For claim (2), note that $\mathcal{D}(\tilde{G}/\tilde{N})_{\tilde{Q}_\alpha\text{-nondeg}}$ is a full G-subcategory by construction, and therefore it is a full $G \times W$ subcategory. Therefore the functor Av_*^N is given by an integral kernel in $\mathcal{D}(\tilde{N}^-_\psi\backslash\tilde{G}/\tilde{N}) \simeq \mathcal{D}(\tilde{T})$. Direct computation (or Proposition 7.1 below) shows that the kernel is given by $\delta_1 \in \mathcal{D}(\tilde{T})$ up to cohomological shift. Since the isomorphism $\mathcal{D}(\tilde{N}^-_\psi\backslash\tilde{G}/\tilde{N}) \simeq \mathcal{D}(\tilde{T})$ is compatible with the Gelfand-Graev action (see Proposition 5.23), we see that we may lift this kernel to an object of $\mathcal{D}(\tilde{N}^-_\psi\backslash\tilde{G}/\tilde{N})^{\langle s \rangle}$, and therefore we have obtained our lift.

Claim (3) follows since the left adjoint $\mathrm{Av}_!^\psi : \mathcal{D}(\tilde{G}/\tilde{N}) \to \mathcal{D}(\tilde{G}/_{-\psi}\tilde{N}^-)$ factors through $\mathcal{D}(\tilde{G}/\tilde{N})_{\tilde{Q}_\alpha\text{-nondeg}}$ by Proposition 5.15 and the adjoint is necessarily $\tilde{G} \times W$ linear by Theorem 2.19 and (2), so that in particular $\mathrm{Av}_!^\psi$ factors through the coinvariant category. Equivalently, by Theorem 2.10, we have that $\mathrm{Av}_!^\psi$ factors through the category of invariants $\mathcal{D}(\tilde{G}/\tilde{N})^{\langle s \rangle}_{\tilde{Q}_\alpha\text{-nondeg}}$. □

Corollary 5.33. The functor F_s preserves the category $\mathcal{D}(G/N)_{\mathrm{deg}}$.

Proof. First, let Z denote the kernel of the map $\tilde{G} \to G$ as in Section 5.3.2. Then the forgetful functor $\mathrm{oblv}^Z : \mathcal{D}(G/N) \to \mathcal{D}(\tilde{G}/\tilde{N})$ (see Proposition 5.26) is t-exact because Z acts on G compatibly with the t-structure, and is conservative. Therefore, the forgetful functor oblv^Z reflects the property of being an eventually coconnective object in the kernel of the associated $\mathrm{Av}_!^\psi$ functor, and thus we may prove this claim for those groups with $[G, G]$ simply connected, so that F_s acts by a symplectic Fourier transform.

In this case, note that F_s preserves the eventually coconnective subcategory because $j_{*,dR}$ has finite cohomological amplitude [HTT08, Proposition 1.5.29], the symplectic Fourier transform is t-exact [KL88, Theorem 1.2.1(iii)], and, if j is *any* open embedding of smooth schemes, $j^!$ is t-exact. Therefore F_s preserves the generating set of $\mathcal{D}(G/N)_{\mathrm{deg}}$ by claim (3) of Proposition 5.32. □

Corollary 5.34. Assume $\mathcal{F} \in \mathcal{D}(\tilde{G}/\tilde{N})_{\mathrm{nondeg}}$ and write $\mathbf{F}j_{*,dR}\mathcal{F} \simeq j_{*,dR}(\mathcal{F}')$ for some $\mathcal{F}' \in \mathcal{D}(\tilde{G}/\tilde{N})$ via Lemma 5.31. Then $\mathcal{F}' \in \mathcal{D}(\tilde{G}/\tilde{N})_{\mathrm{nondeg}}$.

Proof. Now assume $\mathcal{G} \in \mathcal{D}(\tilde{G}/\tilde{N})_{\mathrm{deg}}$. Then

$$\underline{\mathrm{Hom}}_{\mathcal{D}(\tilde{G}/\tilde{N})}(\mathcal{G}, \mathcal{F}') \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{D}(V_s)}(j_{*,dR}(\mathcal{G}), \mathbf{F}j_{*,dR}(\mathcal{F})) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}(V_s)}(\mathbf{F}j_{*,dR}(\mathcal{G}), j_{*,dR}(\mathcal{F}))$$

where the first equivalence follows since $j_{*,dR}$ is an open embedding and therefore fully faithful, the second follows since $\mathbf{F}$ is canonically its own adjoint. Therefore, we see that this space of maps is equivalent to $\underline{\mathrm{Hom}}_{\mathcal{D}(\tilde{G}/\tilde{N})}(F_s(\mathcal{G}), \mathcal{F})$, since $j^!$ is the left adjoint to $j_{*,dR}$, and this vanishes by Corollary 5.33. □

Since each F_w for a given w may be written as a composite of symplectic Fourier transformations preserves the degenerate and nondegenerate subcategories, we obtain the following theorem for the case $G = \tilde{G}$:

Corollary 5.35. The full subcategory $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ of $H^0(\Gamma(\mathcal{D}_{G/N}))$-mod is preserved by the action of T and the Gelfand-Graev action. In particular, the category $\mathcal{D}(G/N)_{\mathrm{nondeg}}$ obtains a $G, T \rtimes W$-action, where the W-action is given by the Gelfand-Graev action, and in particular each simple reflection acts by F_s.

Proof. We have shown that W acts on $\mathcal{D}(\tilde{G}/\tilde{N})$ in a manner which, in the notation of Proposition 5.26, commutes with the Z-action. Taking the Z-invariants and again using Proposition 5.26, we obtain our claim for general G. □

Remark 5.36. See also [BK99], where an analytic version of this W-action on the Schwartz space of the universal flag variety is constructed.

5.3.5. *Weyl Group Action on Nondegerate Categories.* Using Corollary 5.35, by the definition of nondegenerate G-categories, we obtain:

Corollary 5.37. For any nondegenerate G-category $\mathcal{C}$, the category $\mathcal{C}^N$ admits a canonical $T \rtimes W$ action.

We also have the following:

Corollary 5.38. For any nondegenerate G-category $\mathcal{C}$, the category $\mathcal{C}^{N,(T,w)}$ is a module for the monoidal category $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$, given a monoidal structure by the convolution formalism. In particular, $\mathcal{C}^{N,(T,w)}$ is acted on by the group $\tilde{W}^{\text{aff}}$.

Proof. We claim, in fact, for any category $\mathcal{D}$ with a $T \rtimes W$ action, the category $\mathcal{D}^{T,w}$ acquires an action of $\tilde{W}^{\text{aff}}$. This follows since $\mathcal{D}^{T,w} \simeq \mathcal{D}(T \rtimes W)^{T_\ell,w} \otimes_{\mathcal{D}(T\rtimes W)} \mathcal{D}$ and because of an equivalence of categories

$$\underline{\text{End}}_{T\rtimes W}(\mathcal{D}(T \rtimes W)^{T,w}) \xrightarrow{\sim} \underline{\text{End}}_{\mathcal{D}(T)^W}(\mathcal{D}(T)^{T,w}) \simeq \underline{\text{End}}_{\text{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\text{aff}})}(\text{IndCoh}(\mathfrak{t}^*))$$

where the first functor is given by $(-)^W : \mathcal{D}(T \rtimes W)\text{-mod}(\text{DGCat}^k_{\text{cont}}) \to \mathcal{D}(T \rtimes W)^{W\times W}\text{-mod}(\text{DGCat}^k_{\text{cont}})$, which is an equivalence by Theorem 2.15, and the second equivalence is given by the Mellin transform. The convolution formalism [GR17a, Chapter 5, Section 5] therefore yields a monoidal functor from $\text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$ to this endomorphism category. The second claim follows since we have a monoidal functor $\text{IndCoh}(\tilde{W}^{\text{aff}}) \to \text{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\tilde{W}^{\text{aff}}} \mathfrak{t}^*)$, which yields a weak $\tilde{W}^{\text{aff}}$ action, which automatically upgrades to a strong action by Corollary 3.2. □

5.3.6. *Symplectic Fourier Transformations and Stalks.* For the remainder of this section, fix a simple reflection $s \in W$ and write $W^s := \{w \in W : \ell(w) \leq \ell(ws)\}$ so that $W = W^s \sqcup W^s s$. For any subset R of the Weyl group, set $X_R \xhookrightarrow{l_R} G/N$ to denote the union of the cells NwB/N for every $w \in R$. In particular, this variety need not be closed. Let $\mathbb{G}_m^\alpha$ denote the image of the closed subscheme given by $\alpha^\vee : \mathbb{G}_m \to G$.

Proposition 5.39. Fix an $\mathcal{F} \in \mathcal{D}(G/N)$.

1. Assume that the average with respect to the right $\mathbb{G}_m^\alpha$-action vanishes on $\mathcal{F}$. Then if $l^!_{W^s s}(\mathcal{F}) = 0$, we have $l^!_{W^s}(F_s(\mathcal{F})) = 0$.
2. Assume that $\mathcal{F}$ is left N-equivariant (which, by Lemma 2.36, is a property and not additional structure). If $l^!_{W^s}(\mathcal{F}) \simeq 0$, then $l^!_{W^s}(F_s(\mathcal{F}))$ is nonzero.

We prove this after proving the following lemma, which will be used only in the proof of (2):

Lemma 5.40. Assume $\mathcal{F} \in \mathcal{D}(V_s)^N$ and let $\ell_s : \mathcal{L}_s \hookrightarrow V_s$ denote the map and line bundle in Section 5.3.3. Then if $\ell^!_s(\mathcal{F}) \simeq 0$, $\mathcal{F}$ is $\mathbb{G}_a$-equivariant.

Proof. Consider the decomposition

$$V_s \cong \bigsqcup_{w\in W^s} (N_w \dot{w} P_s/N_{w_0 s} \overset{\text{SL}_{2,\alpha}}{\times} \mathbb{A}^2)$$

induced by the parabolic Bruhat decomposition, where $N_w \leq N$ is some closed subgroup depending on w. The condition of being $\mathbb{G}_a$-equivariant is closed under colimits, since the forgetful functor commutes with colimits. Therefore we may check that the !-restriction to each locally closed subscheme $C^s(w)$ of this decomposition. Furthermore, by assumption that $\ell^!_s(\mathcal{F}) \simeq 0$, we may check this upon further !-restriction to the open subset of $C^s_0(w)$. However, on this subset, the right $\mathbb{G}_a$-action agrees with left multiplication by some dimension one subgroup of N. Therefore, the property of N invariance immediately gives the property of right $\mathbb{G}_a$-invariance, where again by Lemma 2.36 this is a property by the fact that $\mathbb{G}_a$ is unipotent. □

Proof of Proposition 5.39. We use the notation of Section 5.3.3. Assume that $\mathcal{F}$ is such that $\text{Av}_*^{\mathbb{G}_m^\alpha}(\mathcal{F}) \simeq 0$ and that $l_{s,*,dR} l^!_s(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$. Let $\mathcal{F}' := j_{*,dR}(\mathcal{F})$ where $j : G/N \to V_s$ is the open embedding. We now claim that $\mathrm{Av}_*^{\mathbb{G}_a}(\mathcal{F}') \simeq 0$ as well. To see this, note the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\mathring{j}"] \arrow[dr] & \mathcal{L}_s \arrow[d, "p|_{\mathcal{L}_s}"] \\ & G/Q_s \end{tikzcd} \tag{10}$$

where $U := \mathcal{L}_s\backslash(G/Q_s)$ is the complement of the zero section in $\mathcal{L}_s$ and both downward pointing arrows are the structural maps. Pushing forward by the left arrow gives the $\mathbb{G}_m$-averaging and the pushforward by the right arrow gives the $\mathbb{G}_a$-averaging. Therefore we see that

$$\ell_s^!(\mathbf{F}(j_{*,dR}(\mathcal{F}))) \simeq (\omega\ell_s)^!\mathbb{F}(\mathcal{F}') \simeq \omega|^!_{\mathcal{L}_s}p^{\vee,!}\mathbb{F}(\mathcal{F}')[1] \simeq \omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(\mathrm{Av}_*^{\mathbb{G}_a}\mathcal{F}')[1] \simeq 0$$

where the first step uses the definition of the symplectic Fourier transform and of $\mathcal{F}'$, the second step uses the functoriality of !-pullback and the commutivity of Eq. (9), and the third step uses Proposition 5.30 and, in particular, $\mathbb{F}_{\mathcal{L}_s} : \mathcal{D}(V_s/\mathbb{G}_a) \xrightarrow{\sim} \mathcal{D}((V_s/\mathbb{G}_a)^\vee)$ is the Fourier transform. Because $l^!_{W^s}$ is given by the composite $\mathring{j}^!\ell_s^!$, we obtain our first claim.

We now show the second claim. We first note that $j_{*,dR}(\mathcal{F})$ is left N equivariant since $j : G/N \hookrightarrow V_s$ is left G-equivariant, and the assumption that $l^!_{W^s}(\mathcal{F}) \simeq 0$ vanishes implies that $\ell_s^!\mathcal{F}' \simeq 0$ as well. Therefore, by Lemma 5.40, we have that $\mathcal{F}' \simeq p^!(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{D}(V_s/\mathbb{G}_a)$. In particular, we see that

$$\ell_s^!\mathbf{F}(\mathcal{F}') \simeq (\omega\ell_s)^!\mathbb{F}(\mathcal{F}') \simeq \omega|^!_{\mathcal{L}_s}p^{\vee,!}\mathbb{F}(\mathcal{F}') \simeq \omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(p_{*,dR}(\mathcal{F}')) \simeq \omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(\mathcal{G})$$

where the first step is the definition of $\mathbb{F}$, the second step follows by the functoriality of !-pullback and the commutivity of Eq. (9), and the third follows from the above diagram, and the fourth follows from Proposition 5.30, and the fifth follows from Lemma 2.36. Note also that, if $z : G/Q_s \hookrightarrow V_s$ denotes the zero section of V_s and $\tilde{z} : G/Q_s \hookrightarrow V_s/\mathbb{G}_a$ denotes the zero section of $V_s/\mathbb{G}_a$, then

$$z^!\omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(\mathcal{G}) \simeq \tilde{z}^!(\mathbb{F}_{\mathcal{L}_s}(\mathcal{G})) \simeq \tilde{t}_{*,dR}(\mathcal{G}) \simeq t_{*,dR}(\mathcal{F})$$

where $\tilde{t}$ and t are the respective maps to the base G/Q_s. Here, the first step uses the fact that the relevant map of zero sections commutes, the second equivalence follows from Proposition 5.30, and the third equivalence is by the fact that $\mathcal{G} \simeq p_{*,dR}(\mathcal{F}')$. In particular, if $\omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(\mathcal{G}) \xleftarrow{\sim} z_{*,dR}z^!(\omega|^!_{\mathcal{L}_s}\mathbb{F}_{\mathcal{L}_s}(\mathcal{G}))$, then the sheaf $\mathcal{G}$ would be $\mathbb{G}_a$-monodromic since $\mathbb{F}_{\mathcal{L}_s}z_{*,dR} \simeq t_1^!$ by Proposition 5.30, where $t_1 : \mathcal{L}_s^\vee \to G/Q_s$ is the structural map. In particular, $\mathcal{F}' \simeq p^!(\mathcal{G})$ would be monodromic with respect to the $\mathbb{G}_a \times \mathbb{G}_a$ action. This violates the assumption that $l^!_{W^s}(\mathcal{F}) \simeq 0$ if $\mathcal{G}$ is nonzero, so $\mathcal{G} \simeq 0$. ☐

Induction on length in the Weyl group gives the following result, which will be used below.

Corollary 5.41. The category $\mathcal{D}(N\backslash G/N)_{\text{nondeg}}$ is generated as a W category by the essential image of the functor $\mathcal{D}(B/N)^N \to \mathcal{D}(N\backslash G/N)_{\text{nondeg}}$ given by composing the pushforward of the inclusion map $N\backslash B/N \xhookrightarrow{i} N\backslash G/N$ with the quotient functor.

Corollary 5.42. The category $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}}$ is generated by the objects in the set $\{w\delta : \tilde{W}^{\text{aff}}\}$, where δ is the monoidal unit of $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\text{nondeg}}$.

Proof. This follows from the fact that the set $\{\lambda\delta' : \lambda \in X^\bullet(T)\}$ is a set of compact generators in the Harish-Chandra category $\mathcal{D}(T)^{(T\times T,\text{ w})}$, where δ' denotes the monoidal unit, and Corollary 5.41. ☐

5.3.7. *Triviality of Gelfand-Graev Action on Equivariant Categories.* In this section, we make precise and prove in Corollary 5.45 that the Gelfand-Graev action is trivial on equivariant categories associated to $\mathcal{D}(G/N)_{\text{nondeg}}$. Recall the map of vector bundles $\mathcal{L}_s \xrightarrow{\ell_s} V_s$ as Section 5.3.3, a morphism of vector bundles over the base $S := G/[P_s, P_s]$. We now prove a variation of [Lau87, Proposition 1.2.3.1] in our context.

Corollary 5.43. Let $t : \mathcal{L}_s \to S$ be the terminal map of S-schemes. Then we have a canonical $B \times T$-equivariant isomorphism of functors $\omega^!\mathbb{F}_{V_s}\ell_{s,*,dR}t^! \simeq \ell_{s,*,dR}t^!$.

Proof. Since our vector bundle is a rank two symplectic vector bundle, we obtain maps such that each square of the following diagram is Cartesian:

$$\begin{tikzcd} \mathcal{L}_s \arrow[r, "\ell_s"] \arrow[d, "t"] & V_s \arrow[r, "\omega"] \arrow[d, "p"] & V_s^{\vee} \arrow[d, "\ell_s^{\vee}"] \\ S \arrow[r, "z"] & V_s/\mathbb{G}_a \arrow[r, "\sim"] & \mathcal{L}_s^{\vee} \end{tikzcd}$$

where we use the notation of Remark 5.29. Therefore, setting $\tilde{z}$ to denote the composite of the two lower horizontal arrows, we obtain isomorphisms via various base change morphisms

$$\omega^! \mathbb{F}_{V_s} \ell_{s,*,dR} t^! \simeq \omega^! \ell_s^{\vee,!} \mathbb{F}_{\mathcal{L}_s} t^! [1] \simeq \omega^! \ell_s^{\vee,!} \tilde{z}_{*,dR} \simeq \ell_{s,*,dR} t^!$$

where, specifically, the first two isomorphisms are given by the base changes required in constructing the functor of Proposition 5.30 and the third is given by base changing along our Cartesian diagram above. □

Proposition 5.44. Fix a simple reflection $s \in W$, and, as above, let $F_s : \mathcal{D}(G/N)_{\text{nondeg}} \to \mathcal{D}(G/N)_{\text{nondeg}}$ denote the endofunctor given by the W-action above indexed by some simple root α (which, for $[G,G]$ simply connected, is the sympectic Fourier transformation given by s). Then the induced map on right $\mathbb{G}_m^{\alpha}$-equivariant objects $F_s : \mathcal{D}(G/N)_{\text{nondeg}}^{\mathbb{G}_m^{\alpha}} \to \mathcal{D}(G/N)_{\text{nondeg}}^{\mathbb{G}_m^{\alpha}}$ can be identified with the identity functor.

Proof. Since the W-action commutes with the central action, it suffices to prove this in the case where the canonical map $\tilde{G} \to G$ of Section 5.3.2 is an isomorphism. Let $H := \mathbb{G}_m^{\alpha} N$. By Remark 2.11, it suffices to provide an isomorphism $F_s(J^!(\delta_{1H})) \cong J^!(\delta_{1H})$ in $\mathcal{D}(G/H)^H_{\text{nondeg}}$. Since the forgetful functor is fully faithful at the level of abelian categories (see, for example [Ras20a, Section 10.3]), we may equivalently show that $F_s(J^!(i_{*,dR}\omega_{H/N})) \cong J^!(i_{*,dR}\omega_{H/N})$, where by Kashiwara's lemma we identify $\omega_{H/N} \in \mathcal{D}(G/N)$. Note that we may identify $J^!(i_{*,dR}\omega_{H/N})$ with $J^!(\tilde{i}_{*,dR}(\omega_{\overline{H/N}}))$, where $\tilde{i} : \overline{H/N} \hookrightarrow V_s$ is the closure of H/N in V_s. Therefore, we see:

$$F_s(J^!(i_{*,dR}\omega_{H/N})) \xleftarrow{\sim} F_s(J^!(\tilde{i}_{*,dR}(\omega_{\overline{H/N}}))) \simeq J^!(\omega^! \mathbf{F}(\tilde{i}_{*,dR}(\omega_{\overline{H/N}}))) \simeq J^!(\tilde{i}_{*,dR}(\omega_{\overline{H/N}})) \simeq J^!(i_{*,dR}\omega_{H/N})$$

where the second equivalence follows from Corollary 5.35 and the definition of the symplectic Fourier transform, the second to last equivalence is given by Corollary 5.43, since $\overline{H/N}$ is naturally a closed subscheme of $\mathcal{L}_s$. □

Corollary 5.45. The $\langle s_\alpha \rangle$-action on $\mathcal{D}(G/N)_{\text{nondeg}}^{\mathbb{G}_m^{\alpha}}$ is the trivial action.

As above, let $H := N\mathbb{G}_m^{\alpha}$. We prove this after first proving the following lemma:

Lemma 5.46. For δ_{1H} the monoidal unit of $\mathcal{D}(H\backslash G/H)$, then $H^0\underline{\text{End}}_{\mathcal{D}(H\backslash G/H)_{\text{nondeg}}}(J^!(\delta_{1H})) \cong k$.

Proof. We may equivalently show that the only maps $\delta_{1H} \to J_*J^!(\delta_{1H})$ are the scalar multiples of unit of the adjunction $(J_*, J^!)$. Since the functor which forgets H-equivariance is fully faithful at the level of abelian categories (see, for example [Ras20a, Section 10.3]) we may equivalently show the the space of maps $\text{Hom}_{\mathcal{D}(N\backslash G/H)^{\heartsuit}}(\delta_{1H}, Y)$ is one dimensional, where $Y := \text{oblv}^H \tau^{\leq 0} J_*J^!(\delta_{1H})$.

Let $\delta_{1H} \to Y$ denote any map. We note that δ_{1H} is simple (for example, it is the pushforward by a simple holonomic $\mathcal{D}$-module at a point). Furthermore, note that δ_{1H} is not degenerate, since it does not lie in the kernel of $\text{Av}_!^{\psi}$. By the long exact sequence associated to the cofiber sequence $I_*I^!\delta_{1H} \to \delta_{1H} \to J_*J^!\delta_{1H}$ see that the quotient $J_*J^!(L)/\delta_{1H}$ lies in the kernel of $J^!$. However, if the composite $\delta_{1H} \to Y \to Y/\delta_{1H}$ were nonzero, by the simplicity of δ_{1H} we would see that this map is an injection. This would, however, imply that δ_{1H} lies in $\mathcal{D}(G/H)_{\text{deg}}$, since this category is closed under subobjects–a contradiction. □

Proof of Corollary 5.45. Since the W-action commutes with the central action, it suffices to prove this in the case where the canonical map $\tilde{G} \to G$ of Section 5.3.2 is an isomorphism. The action of $\langle s_\alpha \rangle$ on $\mathcal{D}(G/H)_{\text{nondeg}}$ is entirely determined by the map of pointed spaces $A : B\langle s_\alpha \rangle \to G\text{-Cat}^{\sim}$, where the basepoint goes to the category $\mathcal{D}(G/H)_{\text{nondeg}}$. We identify $B\langle s_\alpha \rangle \simeq \mathbb{R}P^{\infty}$ with its standard cell structure. By Proposition 5.44, we see that the one cell of $B\langle s_\alpha \rangle$ is sent to an object $I \in \underline{\text{End}}_G(\mathcal{D}(G/H))$ equivalent to the identity. Since the identity of $\underline{\mathrm{End}}_G(\mathcal{D}(G/H)_{\mathrm{nondeg}})$ is sent to the sheaf $J^!(\delta_{1H}) \in \mathcal{D}(H\backslash G/H)_{\mathrm{nondeg}}$ under the equivalence $\underline{\mathrm{End}}_G(\mathcal{D}(G/H)_{\mathrm{nondeg}}) \simeq \mathcal{D}(H\backslash G/H)_{\mathrm{nondeg}}$ of Theorem 2.10, we see that the endomorphisms of the identity functor are discrete. Therefore, our map A is entirely determined by the object I and the image of the two cell, say $E \in \mathrm{Hom}(\mathrm{id}_{\mathcal{D}(G/H)_{\mathrm{nondeg}}}, I^2)$. Moreover, any two such pairs (I_1, E_1) and (I_2, E_2) are equivalent if there exists some natural transformation $i : I_1 \xrightarrow{\sim} I_2$ such that $E_1 i^2 \simeq E_2$. In particular, since we have shown that F_s acts by the identity functor, it suffices to show that the canonical functor

$$\mathrm{id}_{\mathcal{D}(G/H)_{\mathrm{nondeg}}} \xrightarrow{E_s} F_sF_s \xrightarrow{F_sE_s^{-1}} F_s \xrightarrow{E_s^{-1}} \mathrm{id}_{\mathcal{D}(G/H)} \tag{11}$$

is equal to the identity natural transformation of G-categories, where E_s is the equivalence of Proposition 5.44. By Lemma 5.46, we see that this endomorphism is given by scaling by some nonzero $x \in k$. However, the category $\mathcal{D}(G/H)_{\mathrm{nondeg}}$ contains the full $\langle s_\alpha\rangle$-subcategory $\mathcal{D}(N_\psi^-\backslash G/N)^{\mathbb{G}_m^\alpha}$ by Proposition 1.7 and Proposition 5.23. In particular, for objects of this category, the natural transformation of (11) scales by 1. Therefore, we see that $x = 1$ and so the natural transformation of (11) is the identity, and thus the $\langle s_\alpha\rangle$-action is trivial. □

6. Nondegenerate Category $\mathcal{O}$

Fix some $\lambda \in \mathfrak{t}^*(k)$. In this section, we study the category $\mathcal{D}(G/_\lambda B)^N$. A priori, this category is defined as a limit of DG categories. However, it turns out that this category is the derived category of its heart, which we show in Appendix B. The category $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ identifies with the ind-completion of the category $\mathcal{O}_\lambda$, the abelian subcategory of the BGG category $\mathcal{O}$ of objects[3] whose central character is given by χ_λ, where χ denotes the Harish-Chandra map, [Hum08, Chapter 1.9].

By the Beilinson-Bernstein localization theorem, we then see that the category $\mathcal{D}(G/_\lambda B)^N$ is the (unbounded) derived category of the ind-completion of the BGG category $\mathcal{O}_\lambda$. Using this, we can provide an explicit description of the nondegenerate and degenerate subcategories associated to $\mathcal{D}(G/_\lambda B)^N$. We do so after reviewing a few results about the BGG category $\mathcal{O}$.

Remark 6.1. Note that, in particular, $\mathcal{O}_\lambda$ is not a block if λ is not integral. However, the fact that $\mathcal{O}_\lambda$ is generated by projective modules implies that one has the block decomposition

$$\mathcal{D}(G/_\lambda B)^N \simeq \oplus_{\lambda' \in S}\mathcal{D}(G/_\lambda B)^{(T,\mathcal{L}_{[\lambda']})\text{-mon}}$$

where S denotes the set of elements in $\mathfrak{t}^*/\Lambda(k)$ in the $W(k) = W$ orbit of λ (for a direct explanation of this in terms of sheaves on $\mathcal{D}(G/_\lambda B)$, see [LY20, Lemma 2.10]). In particular, this set is finite.

Remark 6.2. Often, the results and definitions for the BGG (abelian) category $\mathcal{O}$ are only stated for semisimple Lie algebras. However, once a central character $\lambda : Z(\mathfrak{g}) \to k$, is fixed, we note that the category $\mathcal{O}_\lambda^{\mathfrak{g}}$, defined with the same axioms in [Hum08, Section 1.1] but for our reductive Lie algebra $\mathfrak{g}$, is equivalent to the category of representations at a given central character for the associated semisimple Lie algebra $\mathcal{O}_\lambda^{\mathfrak{g}'}$, where $\mathfrak{g}' := \mathrm{Lie}([G, G])$.

This follows since $U\mathfrak{g} \simeq U\mathfrak{g}' \times U(\mathrm{Lie}(Z(G)^\circ)$ by Lemma 5.25, and so the generalized central character requirement, as well as the axiom that the maximal torus acts by a character on any object of $\mathcal{O}_\lambda^{\mathfrak{g}}$ implies that the action of $U(\mathrm{Lie}(Z(G)^\circ)$ is entirely determined by the central character. Therefore $U(\mathfrak{g})$-representation is entirely determined by its restriction to the $U\mathfrak{g}'$-factor, as desired.

In particular, while some references below only refer to the BGG category $\mathcal{O}$ for representations of semisimple Lie algebras at a given central character, the results all hold mutatis mutandis for reductive Lie algebras.

6.1. Soergel's Classification of $\mathcal{O}_\lambda$.

We now briefly recall Soergel's classification of $\mathcal{O}_\lambda$ for some field-valued point λ of $\mathfrak{t}^*$. Fix some antidominant λ' of $\mathfrak{t}^*$ in the $W, \cdot$ orbit of λ, and let $I_{\lambda'}$ be the indecomposable injective hull of the simple indexed by λ'.

Theorem 6.3. [Soe90] For λ as above, we have the following:

[3] Note that $\mathcal{O}_\lambda$ is not cocomplete–its objects are, by definition, finitely generated $U\mathfrak{g}$-modules. It seems that these notations minimize conflicts with existing literature. Note in particular that what in the introduction we called the universal category $\mathcal{O}$, which we denoted $\mathfrak{g}\text{-Mod}^{N,(T,w)}$, is defined to be a DG category.

(1) (Endomorphismensatz) The canonical map $Z\mathfrak{g} \to \underline{\mathrm{End}}_{\mathcal{D}(N\backslash G/_\lambda B)}(I_{\lambda'}) \in \mathrm{Vect}^{\heartsuit}$ surjects and has the same kernel as the surjective composite $Z\mathfrak{g} \xrightarrow{\chi} \mathrm{Sym}(\mathfrak{t}) \to C_{\lambda'}$, where $C_{\lambda'} := \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})_+^{W_{[\lambda]}}$ is the coinvariant algebra associated to $W_{[\lambda]}$, and therefore induces an isomorphism of classical vector spaces $\underline{\mathrm{End}}_{\mathcal{D}(N\backslash G/_\lambda B)}(I_{\lambda'}) \cong C_{\lambda'}$.
(2) (Struktursatz) Let $\underline{I} := \oplus_\mu I_\mu$ denote the direct sum of the indecomposable injective hulls of all simple objects indexed by antidominant μ of $\mathfrak{t}^*$ in the $W, \cdot$ orbit of λ, and let $\tilde{\mathbb{V}}_I : \mathcal{D}(N\backslash G/_\lambda B) \to \tilde{C}_\lambda\text{-mod}$ denote the contravariant functor $\tilde{\mathbb{V}}_I(-) = \underline{\mathrm{Hom}}(-, \underline{I})$, where $\tilde{C}_\lambda := \underline{\mathrm{End}}_{\mathcal{O}_\lambda}(\underline{I})$. Then the functor $\tilde{\mathbb{V}}$ is fully faithful on injective objects in $\mathcal{O}_\lambda$.

Remark 6.4. Soergel's Struktursatz is often phrased for a block indexed by antidominant λ' of $\mathfrak{t}^*$ as the claim that the functor $\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_{\lambda'}}(P_{\lambda'}, -)$, where $P_{\lambda'}$ is the indecomposable projective cover associated to λ', is fully faithful on projective objects in the block of $\mathcal{O}$ containing $P_{\lambda'}$. However, this is equivalent to our formulation above, as we explain now.

Recall that any indecomposable projective cover of some L_λ labelled by an antidominant λ is also its injective hull [Hum08, Theorem 4.11]. Therefore, the duality functor $\mathbb{D}$ gives an isomorphism $\tilde{\mathbb{V}}_I \simeq \tilde{\mathbb{V}}_P \circ \mathbb{D}$, where $\tilde{\mathbb{V}}_P := \underline{\mathrm{Hom}}(\underline{P}_\lambda, -)$ for $\underline{P}_\lambda$ is the direct sum of the indecomposable projective cover associated to antidominant k-points of $\mathfrak{t}^*$ whose images agree under the Harish-Chandra map.

Furthermore, each block of $\mathcal{O}_\lambda$ contains a unique antidominant projective [Hum08, Chapter 4.9]. We therefore have that, in the notation of the Struktursatz in Theorem 6.3, we have that $\tilde{C}_\lambda \cong \oplus_\mu \underline{\mathrm{End}}_{\mathcal{O}_\lambda}(I_\mu) \cong \oplus_\mu \underline{\mathrm{End}}_{\mathcal{O}_\lambda}(P_\mu)$, and so, in particular, $\tilde{\mathbb{V}}_P \cong \oplus_\mu \underline{\mathrm{Hom}}(P_\mu, -)$.

Soergel's results will allow us to realize the category $\mathcal{D}(G/_\lambda B)^N$ as the category R_λ-mod for some classical ring R_λ as follows. Specifically, let $\mathcal{G} \in \mathcal{O}_\lambda$ denote the direct sum of all $|W|$-many indecomposable projectives in $\mathcal{O}_\lambda$. Then $\mathcal{G}$ is a compact generator for $\mathcal{D}(G/_\lambda B)^N \simeq \mathrm{Ind}(\mathcal{D}^b(\mathcal{O}_\lambda))$, and in particular setting $R_\lambda := \underline{\mathrm{End}}_{\mathcal{D}(G/_\lambda B)^N}(\mathcal{G}) \simeq \underline{\mathrm{End}}_{\mathcal{O}_\lambda}(\mathcal{G})$, we see that R_λ is concentrated in degree zero by the projectivity of $\mathcal{G}$ and so $\mathcal{D}(G/_\lambda B)^N \simeq R_\lambda$-mod. The projectivity of our generators gives that this equivalence t-exact, and so we recover the equivalence of abelian categories $\mathcal{O}_\lambda \simeq R_\lambda\text{-mod}^{\heartsuit, c}$. Through the struktursatz, one can provide an alternate description of the ring R_λ, which we will not use here (for an excellent recent survey of this in the case where λ is integral, see [EMTW20, Chapter 15]).

6.2. **The Functor $\mathbf{Av}_!^\psi$ on $\mathcal{O}_\lambda$.** Fix some $\lambda \in \mathfrak{t}^*(k)$.

Proposition 6.5. The left action of $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ on the category $\mathcal{D}(N\backslash G/_\lambda B)$ factors through the action of C_λ-mod via the monoidal functor

$$\mathrm{QCoh}(\mathrm{Spec}(C_\lambda)) \xrightarrow{\Upsilon_{\mathrm{Spec}(C_\lambda)}} \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda)) \xrightarrow{f^!} \mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$$

where C_λ is as in Theorem 6.3 and f denotes the composite $\mathrm{Spec}(C_\lambda) \hookrightarrow \mathfrak{t}^* \to \mathfrak{t}^*/X^\bullet(T)$.

Proof. The block decomposition $\mathcal{D}(G/_\lambda B)^N \simeq \oplus_{[\lambda'] \in S} \mathcal{D}(G/_\lambda B)^{(T, \mathcal{L}_{[\lambda']})\text{-mon}}$ discussed above implies that the action of $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$ factors through the action of $\mathrm{IndCoh}((\mathfrak{t}^*/X^\bullet(T))^\wedge_S) \simeq \mathrm{IndCoh}(\coprod_{\lambda'}(\mathfrak{t}^*/X^\bullet(T))^\wedge_{[\lambda']}) \simeq \mathrm{IndCoh}(\coprod_{\lambda'} \mathfrak{t}^{*;\wedge}_{\lambda'})$, where λ' are arbitrarily chosen antidominant lifts of $[\lambda']$ and the second equivalence is given by Proposition 3.5. We forget this action to an action of $\Upsilon_{\mathfrak{t}^*} : \mathrm{QCoh}(\mathfrak{t}^*) \xrightarrow{\sim} \mathrm{IndCoh}(\mathfrak{t}^*)$ by the pullback map induced by $\coprod_{\lambda'} \mathfrak{t}^{*;\wedge}_{\lambda'} \to \mathfrak{t}^*$.

Let $\mathcal{G}$ be a direct sum of all indecomposable projective objects of $\mathcal{O}_\lambda$. Then $\mathcal{G}$ is a projective generator of $\mathcal{D}(G/_\lambda B)^N$, the action of $\mathrm{QCoh}(\mathfrak{t}^*) \simeq \mathrm{Sym}(\mathfrak{t})$-mod is determined by the map $\mathrm{Sym}(\mathfrak{t}) \to \underline{\mathrm{End}}_{\mathcal{D}(G/_\lambda B)^N}(\mathcal{G})$. The projectivity of $\mathcal{G}$ implies that this endomorphism ring is concentrated in degree zero. The functor of $\tilde{\mathbb{V}}$ can be canonically equipped with a $\mathrm{Sym}(\mathfrak{t})$-linear structure, and so in particular the following diagram commutes

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{t}) \arrow[r] \arrow[d, "\exists"] & \underline{\mathrm{End}}_{\mathcal{D}(N\backslash G/_\lambda B)}(\mathcal{G}) \arrow[d, "\sim"] \\
C_\lambda \arrow[r] & \underline{\mathrm{End}}_{C_\lambda\text{-mod}}(\tilde{\mathbb{V}}(\mathcal{G}))
\end{tikzcd}$$

where the right equivalence is given by the struktursatz. Therefore, the action of $\mathrm{QCoh}(\coprod_{\lambda'} \mathfrak{t}^{*,\wedge}_{\lambda'})$ factors through an action of C_λ-mod, and so the claim follows since Υ intertwines pullbacks for QCoh and IndCoh for laft prestacks [GR17b, Chapter 6, Section 3.3.5]. □

Proposition 6.6. Using the notation of Theorem 6.3, we have the following:

1. The induced functor $\mathrm{Vect} \simeq \mathcal{D}(G/_\lambda B)^{N^-,\psi} \xrightarrow{\mathrm{Av}_*^N[\dim(N)]} \mathcal{D}(G/_\lambda B)^N$ sends $k \in \mathrm{Vect}$ to an object isomorphic to $\underline{I}_\lambda$.
2. We may identify the left adjoint to this composite functor (which by abuse of notation we also denote $\mathrm{Av}_!^\psi[-\dim(N)]$) with the functor $\mathbb{D}_{\mathrm{Vect}}\tilde{\mathbb{V}}\mathbb{D}$, where $\mathbb{D}$ denotes the Verdier duality functor on $\mathcal{D}(G/_\lambda B)^N$.

Proof. The composite is t-exact by Theorem 2.43, and so if we let A denote the object $\mathrm{Av}_*^N[\dim(N)](k)$, $A \in \mathcal{D}(G/_\lambda B)^{N,\heartsuit}$. Because the adjoint to $\mathrm{Av}_*^N[\dim(N)]$ is also t-exact by Corollary 2.42, A is an injective object. Every injective object of $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ is a direct sum of indecomposable injective objects and therefore it suffices to compute $\underline{\mathrm{Hom}}(L, A) \simeq \underline{\mathrm{Hom}}(\mathrm{Av}_!^\psi[-\dim(N)](L), k) \cong \mathrm{Av}_!^\psi[-\dim(N)](L)$ for each simple object L.

Direct computation shows that $\mathrm{Av}_!^\psi[-\dim(N)] : \mathcal{D}(G/N)^N \to \mathcal{D}(N^-_\psi\backslash G/N) \simeq \mathcal{D}(T)$ is the identity upon restriction to the torus. Therefore, we see that in our case, $\mathrm{Av}_!^\psi[-\dim(N)]$ sends the Verma Δ_λ to the one dimensional vector space of $\mathrm{Vect} \simeq \mathcal{D}(G/_{-\psi}N^-)^{B_{[\lambda]}}$. For any other Verma module Δ, there exists some bi-equivariant sheaf $\tilde{\Delta}$ for which $- \star \tilde{\Delta}$ is an equivalence and that the sheaf $\Delta_\lambda \star \tilde{\Delta}$ is concentrated in degree zero and isomorphic to Δ [LY20, Lemma 3.4, Lemma 3.5]. The residual equivariance from the fact that $\mathrm{Av}_!^\psi[-\dim(N)] : \mathcal{D}(N^-_\psi\backslash G) \to \mathcal{D}(N\backslash G)$ is G-equivariant therefore implies that the images of *all* Verma modules are isomorphic to the unique one dimensional vector space concentrated in degree zero. Furthermore, we have that $\mathrm{Av}_!^\psi[-\dim(N)]$ preserves projective objects since it is t-exact Corollary 2.42 and all objects in $\mathrm{Vect}^\heartsuit$ are projective.

Now, assume we are given a simple $L(\mu) \in \mathcal{O}$ for which μ is not antidominant. Then $L(\mu)$ can be written as a subobject of a quotient of a Verma module by another Verma module by Verma's theorem [Hum08, Theorem 4.6]. Therefore, by t-exactness of $\mathrm{Av}_!^\psi[-\dim(N)]$ and the fact that this injective map of Verma modules is sent to an isomorphism, $\mathrm{Av}_!^\psi(L(\mu))$ vanishes. Furthermore, the simples $L(\mu)$ corresponding to antidominant μ are isomorphic to their corresponding Verma modules, and therefore, by the above, $\mathrm{Av}_!^\psi$ sends them to the one dimensional vector space. Thus we see that A is an injective object whose vector space of maps from antidominant simples is one dimensional, and whose vector space of maps from any other simple is zero, implying claim (1).

The above argument also implies that $\mathbb{D}_{\mathrm{Vect}}\mathrm{Av}_!^\psi[-\dim(N)] \simeq \underline{\mathrm{Hom}}(-, \underline{I}_\lambda)$. Projectives associated to such antidominant objects of $\mathfrak{t}^*(k)$ are self dual [Hum08, Chapter 7.16], and the Beilinson-Bernstein localization intertwines the duality functor on $\mathcal{O}$ with Verdier duality. Therefore, we see that $\mathbb{D}_{\mathrm{Vect}}\mathrm{Av}_!^\psi[-\dim(N)]\mathbb{D} \simeq \underline{\mathrm{Hom}}(\underline{P}_\lambda, -)$. Since Verdier duality is an equivalence, we obtain (2). □

6.3. **Nondegenerate Category $\mathcal{O}$.** We now proceed to explicitly classify the category $(\mathcal{D}(G/_\lambda B)_{\mathrm{nondeg}})^N$, which we informally think of as the nondegenerate BGG category $\mathcal{O}_\lambda$ or, more accurately, the derived category of its ind-completion. We provide an alternate description of this category in Section 6.3.1 and use this to provide an explicit description of nondegenerate category $\mathcal{O}_\lambda$.

6.3.1. *Equivalence of Alternate Definition of Nondegenerate Category $\mathcal{O}_\lambda$.* In this section, we prove the following technical result which may be skipped at first pass. Let $\mathcal{D}(B_\lambda\backslash G/N)_{\mathrm{cecko}}$ denote the full subcategory of $\mathcal{D}(B_\lambda\backslash G/N)$ generated under colimits by eventually coconnective objects in the kernel of the right Whittaker averaging functor $\mathrm{Av}_!^\psi$. This category admits a fully faithful functor $\mathcal{D}(B_\lambda\backslash G/N)_{\mathrm{cecko}} \xhookrightarrow{I_{*,\mathrm{cecko}}} \mathcal{D}(B_\lambda\backslash G/N)_{\mathrm{deg}}$, for any object $\mathcal{F} \in \mathcal{D}(B_\lambda\backslash G/N)$ which is a colimit of eventually coconnective objects in the kernel of $\mathrm{Av}_!^\psi : \mathcal{D}(B_\lambda\backslash G/N) \to \mathcal{D}(B\backslash_\lambda G/_{-\psi}N^-)$ has the property that $\mathrm{oblv}^{B_\lambda}(\mathcal{F})$ is a colimit of eventually coconnective objects in the kernel of $\mathrm{Av}_!^\psi : \mathcal{D}(G/N) \to \mathcal{D}(G/_{-\psi}N^-)$ since this forgetful functor is t-exact and continuous. A priori, this map need not be an equivalence. However, we have the following result:

Proposition 6.7. The inclusion functor $\mathcal{D}(B_\lambda\backslash G/N)_{\mathrm{cecko}} \hookrightarrow (\mathcal{D}(G/N)_{\mathrm{deg}})^{B_\lambda}$ is an equivalence.

We will now reduce the proof of Proposition 6.7 to Proposition 6.8 after setting some notation. By right-completeness of the category $\mathcal{D}(B_\lambda\backslash G/N)_{\text{cecko}}$ with respect to its t-structure and Lemma 2.51, we can identify $\mathcal{D}(B_\lambda\backslash G/N)_{\text{cecko}}$ as the full subcategory generated by *compact* objects in the kernel of the right Whittaker averaging functor, using the fact that $\text{Av}_!^{\psi}$ is t-exact up to cohomological shift i.e. Corollary 2.42. Therefore, Lemma 2.50 shows that the inclusion functor also preserves compacts, and thus the inclusion functor $I_{*,\text{cecko}}$ falls into the setup of Section 2.4.5. Let $\mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}}$ denote the resulting quotient category. Since we can identify $\mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}}$ as the kernel of the right adjoint to the inclusion functor $I^!_{\text{cecko}}$, the proof of Proposition 6.7 immediately reduces to the following result:

Proposition 6.8. The canonical quotient functor $\mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}} \to (\mathcal{D}(G/N)_{\text{nondeg}})^{B_\lambda}$ is an equivalence.

Proof. We first note that the forgetful functor $\mathcal{D}(B_\lambda\backslash G/N) \xrightarrow{\text{oblv}} \mathcal{D}(G/N)$ reflects the property of being an eventually coconnective object in the kernel of right Whittaker averaging $\text{Av}_!^{-\psi}$. Therefore, we obtain a canonical equivalence $\mathcal{D}(B_\lambda\backslash G/N)^+_{\text{cecko}} \xrightarrow{\sim} (\mathcal{D}(G/N)_{\text{deg}})^{B_\lambda,+}$. By Corollary 2.55, we see that the quotient functor of Proposition 6.8 is an equivalence on the eventually coconnective subcategories.

We claim that both categories of Proposition 6.8 have a compact generator which is eventually coconnective. For the category $\mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}}$, this is a direct consequence of the fact that $J^!$ kills any non-antidominant simple because non-antidominant simples are eventually coconnective objects of $\mathcal{D}(B_\lambda\backslash G/N)$ in the kernel of the right Whittaker averaging functor, and so $J^!(\oplus_a L_a)$ is a compact generator where a varies over all antidominant weights in the $(W,\cdot)$-orbit of λ. An identical argument says the image under the quotient functor $J^!_{\text{non-cecko}}: \mathcal{D}(B_\lambda\backslash G/N) \to \mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}}$ of the non-antidominant simples form a set of compact generators for $\mathcal{D}(B_\lambda\backslash G/N)_{\text{non-cecko}}$. However, the image of any object in the heart is in particular eventually coconnective, we see that the quotient functor of Proposition 6.8 is fully faithful and maps a compact generator to a compact generator, and thus is an equivalence. $\square$

6.3.2. *Nondegeneracy at a Fixed Character - The Eventually Coconnective Case.* We now study the eventually coconnective objects of the nondegenerate and degenerate category $\mathcal{O}_\lambda$:

Lemma 6.9. The following not necessarily complete DG categories are equivalent:

(1) The subcategory of $\mathcal{D}(G/N)^{B_\lambda,+}$ given by those objects $\mathcal{F}$ for which $\text{oblv}^{B_\lambda}(\mathcal{F})$, which lies in $\mathcal{D}(G/N)^+$ since the t-structure on $\mathcal{D}(G/N)^{B_\lambda}$ is such that the forgetful functor is t-exact, lies in $\mathcal{D}(G/N)^+_{\text{deg}}$.
(2) The subcategory of $\mathcal{D}(G/N)^{B_\lambda,+}$ given by those objects $\mathcal{F}$ for which $\text{Av}_!^{\psi}(\mathcal{F}) \simeq 0$ in $\mathcal{D}(G/_{-\psi}N^-)^{B_\lambda,+}$.
(3) The full subcategory of $\mathcal{D}(G/N)^{B_\lambda,+}$ generated under filtered colimits and extensions[4] by the simples L_μ such that μ lies in the $(W,\cdot)$-orbit of λ and μ is not antidominant.

Proof. The forgetful functor oblv^{B_λ} is t-exact and conservative by assumption, and furthermore we have $\text{oblv}^{B_\lambda}\text{Av}_!^{\psi} \simeq \text{Av}_!^{\psi}\text{oblv}^{B_\lambda}$ since the functor $\text{Av}_!^{\psi}$ averages with respect to the right action. Therefore, the functor oblv^{B_λ} reflects the property of being an eventually coconnective object of the kernel of the respective right Whittaker averaging $\text{Av}_!^{\psi}$. Therefore, by the definition of $\mathcal{D}(G/N)_{\text{deg}}$ (see Corollary 5.6), we see the equivalences of the not necessarily cocomplete DG categories of (1) and (2).

It remains to show the equivalence of the not necessarily cocomplete DG categories of (2) and (3). By the identification $\text{Av}_!^{\psi}: \mathcal{D}(G/N)^{B_\lambda} \to \mathcal{D}(G/_{-\psi}N^-)^{B_\lambda}$ with dual of the functor $\tilde{\mathbb{V}}$ (Proposition 6.6), all of the simples L_μ as in (3) lie in the kernel of this functor. Thus, by exactness and continuity of $\text{Av}_!^{\psi}$, we see that the category of (3) lies in the subcategory of (2).

Conversely, assume we are given some eventually coconnective object $\mathcal{F} \in \mathcal{D}(G/N)^{B_\lambda,+}$. This category is the derived category of its heart by Proposition B.2 and the t-structure on $\mathcal{D}(G/N)^{B_\lambda}$ is right-complete by Lemma 2.27, since it admits a t-exact, conservative functor oblv^{B_λ} to a category $\mathcal{D}(G/N)$ with a right-complete t-structure. In particular, we can write $\mathcal{F}$ as a filtered colimit of objects with finitely many nonzero cohomology groups. In turn, this implies that we can write $\mathcal{F}$ as a filtered colimit of objects obtained by successive extensions (i.e. cofiber sequences) of objects in the heart. Because $\text{Av}_!^{\psi}$ also commutes with cohomological shifts, it suffices to show that if $M \in \mathcal{D}(G/N)^{B_\lambda,\heartsuit}$ lies in the kernel of $\text{Av}_!^{\psi}$, then M is a successive extension of objects L_μ as in (3).

[4]In other words, if any two of the three elements in a cofiber sequence lie in our category, so too does the third.

The category $\mathcal{D}(G/N)^{B_\lambda,\heartsuit}$ can be identified with the ind-completion of the BGG category $\mathcal{O}_\lambda$ Appendix B, see Remark B.7. In particular, by Lemma 2.51, we may write M as a union of its compact subobjects. Since $\mathcal{O}_\lambda$ is closed under direct summands, this implies that we may write M as a union of objects of $\mathcal{O}_\lambda$. In particular, either M lies in the abelian category generated by extensions of the simples as in (3) or there exists some compact subobject which has some L_ν such that ν is antidominant and in the $(W,\cdot)$-orbit of λ. In the latter case, though, we see that by Proposition 6.6 that $\text{Av}_!^\psi(L_\nu)$ does not vanish. Therefore this latter case does not occur, so M can be realized as a filtered colimit of simple objects labeled by non-antidominant weights, as desired. □

Lemma 6.10. The induced functor $\underline{\text{Hom}}(\underline{P}_\lambda, -) : \mathcal{D}(G/_\lambda B)^{N,+}_{\text{nondeg}} \to \text{Vect}^+$ is monadic, and induces an equivalence

$$\mathcal{D}(G/_\lambda B)^{N,+}_{\text{nondeg}} \xrightarrow{\tilde{\mathbb{V}}} \oplus_w C_{w\lambda}\text{-mod}^+$$

where w varies over the $w \in W$ such that $w\lambda$ is antidominant.

Proof. We check the conditions of Barr-Beck. We have that all indecomposable projective covers of simples are in the abelian category $\mathcal{O}$ itself, and therefore the direct sum of all indecomposable projective covers is compact in $\underline{\text{Hom}}(\underline{P}_\lambda, -)$ by Appendix B, and in particular Lemma B.8. Therefore this functor commutes with geometric realizations, and thus it remains to verify its conservativity. Assume $\mathcal{F} \in \mathcal{D}(G/_\lambda B)^{N,+}_{\text{nondeg}}$ is nonzero. By the exactness of $\underline{\text{Hom}}(\underline{P}_\lambda, -)$ (i.e. because this functor commutes with shifts), it suffices to assume that $\mathcal{F} \in \mathcal{D}(G/_\lambda B)^{N,\geq 0}_{\text{nondeg}}$ and that $H^0(\mathcal{F}) \cong \tau^{\leq 0}(\mathcal{F})$ is nonzero. Note that this uses the fact that $\mathcal{F}$ is bounded by below–in any category $\mathcal{D}$ with a t-structure and some $\mathcal{G} \in \mathcal{D}^+$, we may choose a maximal m such that $\mathcal{F} \in \mathcal{D}^{\geq m}$, so that in particular $\tau^{\geq m}\mathcal{G} \simeq H^0(\mathcal{G})$ does not vanish.

Therefore, we similarly see that $H^0(J_*(\mathcal{F}))$ is nonzero, since $J^!$ is t-exact and so $J^!(H^0(J_*(\mathcal{F}))) \cong H^0(J^!J_*(\mathcal{F})) \simeq H^0(\mathcal{F})$, where the last step follows by the fully faithfulness of J_*. Then, since we have a right adjoint J_* to a quotient functor $J^!$ we have seen is t-exact in Proposition 2.54, we have that $J_*(\mathcal{F})$ lies in the subcategory $\mathcal{D}(G/_\lambda B)^{N,\geq 0}$. and $H^0(J_*(\mathcal{F}))$ is nonzero.

By the t-exactness of $\underline{\text{Hom}}(\underline{P}_\lambda, -)$, we have that $\underline{\text{Hom}}(\underline{P}_\lambda, H^0(J_*(\mathcal{F}))) \simeq H^0\underline{\text{Hom}}(\underline{P}_\lambda, J_*(\mathcal{F}))$, and so in particular it suffices to show that $\underline{\text{Hom}}(\underline{P}_\lambda, H^0(J_*(\mathcal{F})))$ is nonzero. However, $H^0(J_*(\mathcal{F}))$ is a nonzero object in the ind-completion of $\mathcal{O}_\lambda$. In particular, by Lemma 2.51, it can be written as an increasing union of objects of $\mathcal{O}_\lambda$. It cannot be the case that all objects in this increasing union have composition factors L_μ for μ not antidominant, for this would imply that $J^!(H^0(\mathcal{F}))$ vanishes, which we have seen above cannot happen. Therefore, there exists some object $M \in \mathcal{O}_\lambda$ which is a subobject of $H^0(J_*(\mathcal{F}))$ and such that there exists some antidominant μ for which L_μ is a subquotient of M. By the t-exactness of $\underline{\text{Hom}}(\underline{P}_\lambda, -)$, we see that $\underline{\text{Hom}}(\underline{P}_\lambda, M)$ contains $\underline{\text{Hom}}(\underline{P}_\lambda, L_\mu)$ as a subquotient, and therefore $\underline{\text{Hom}}(\underline{P}_\lambda, M)$ cannot vanish. Therefore, again using t-exactness of $\underline{\text{Hom}}(\underline{P}_\lambda, -)$, we see that $\underline{\text{Hom}}(\underline{P}_\lambda, H^0(J_*(\mathcal{F}))) \simeq H^0\underline{\text{Hom}}(\underline{P}_\lambda, J_*(\mathcal{F}))$ cannot vanish either, thus verifying the conservativity condition of Barr-Beck. Since the conditions of Barr-Beck apply and that the left adjoint to $\underline{\text{Hom}}(\underline{P}_\lambda, -)$ is $\underline{P}_\lambda \otimes_k -$, we see that the Endomorphismensatz of Theorem 6.3 gives our desired equivalence. □

6.3.3. *Nondegenerate Category $\mathcal{O}_\lambda$ - The General Case.* We now use the above to give a coherent description of $\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}$:

Proposition 6.11. There is an equivalence of categories $\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}} \simeq \text{IndCoh}(W \overset{W_{[\lambda]}}{\times} \text{Spec}(C_\lambda))$.

We prove this after proving the following lemma:

Lemma 6.12. Let C be a local ring with unique maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^N = 0$ for $N \gg 0$. Then any classical C-module (i.e. an element of $C\text{-mod}^\heartsuit$) contains a submodule isomorphic to $C/\mathfrak{m}$.

Proof. Let M be any nonzero C-module. Then since M is not zero, there exists a nonzero map $C \to M$. Let I denote its kernel, so that we have an induced injection $C/I \hookrightarrow M$. Let $\mathfrak{m} := C^+$; we wish to exhibit a nonzero map $C/\mathfrak{m} \to C/I$. There exists a unique nonnegative n such that $\mathfrak{m}^{n+1} \subseteq I \subsetneq \mathfrak{m}^n$, since $\mathfrak{m}^N = 0$ for $N \gg 0$. Let $x \in \mathfrak{m}^n \setminus I$, and consider the map of C-modules $C \to C/I$ sending 1 to x. Then the kernel of this map is precisely $\mathfrak{m}$ since the kernel contains $\mathfrak{m}$ and does not contain 1. Therefore, we obtain an injective map $C/\mathfrak{m} \hookrightarrow C/I$. Composing with our injection $C/I \hookrightarrow M$, we obtain our claim. □

Proof of Proposition 6.11. By construction, we have a quotient functor $J^! : \mathcal{D}(G/_\lambda B)^N \to \mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}$ admitting a continuous right adjoint. In particular, $J^!$ preserves the set of compact generators $\oplus_w L_{w\cdot\lambda}$ where w varies over W. However, we see that for any w such that $w \cdot \lambda$ is not antidominant, $\mathrm{Av}_!^\psi(L_{w\cdot\lambda})$ vanishes by Proposition 6.6 and the fact that all indecomposable injective hulls of simples of $\mathcal{O}_\lambda$ are distinct. Therefore, we see that $\underline{L}_a^\lambda := \oplus_w J^!(L_{w\cdot\lambda})$ is a compact generator, where w varies only over those $w \in W$ such that $w \cdot \lambda$ is also antidominant. Furthermore, since each block of the abelian category $\mathcal{O}_\lambda$ contains precisely one simple labeled by an antidominant weight, we see that the category $\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}$ is equivalent to modules over the ring

$$A_\lambda := \underline{\mathrm{End}}_{\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}}(\underline{L}_a^\lambda) \simeq \oplus_w \underline{\mathrm{End}}_{\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}}(J^!(L_{w\cdot\lambda}))$$

where we note that there are precisely $[W : W_\lambda]$ many (nonzero) objects in the right direct sum. Furthermore, we may compute the right hand side explicitly as follows. Let L_a be any simple indexed by an antidominant weight. Then, by Lemma 6.10, we see that these endomorphisms are equivalently given by the C_λ-module structure on $\tilde{\mathbb{V}}(L_a) \simeq k$. However, since C_λ is an Artinian ring with a unique maximal ideal, we see that, by Lemma 6.12, there is precisely one module structure which we may place on $\tilde{\mathbb{V}}(L_a)$–namely, the trivial one. Therefore we see that the category $\mathcal{D}(G/_\lambda B)^N_{\text{nondeg}}$ is a direct sum of $[W : W_{[\lambda]}]$ many copies of E_λ-mod for $E_\lambda := \underline{\mathrm{End}}_{C_\lambda\text{-mod}}(k)$.

An identical description holds for $\mathrm{IndCoh}(W \overset{W_{[\lambda]}}{\times} \mathrm{Spec}(C_\lambda))$. Specifically, we note that this category is equivalent to a direct sum of $[W : W_{[\lambda]}]$ many copies of $\mathrm{IndCoh}(\mathrm{Spec}(C_\lambda))$ and that moreover if $i : \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(C_\lambda)$ denotes the closed embedding of the unique closed point, then $i_*^{\mathrm{IndCoh}}(k)$ is a compact generator of $\mathrm{IndCoh}(\mathrm{Spec}(C_\lambda))$, which, for example, directly follows from [GR17a, Chapter 4, Proposition 6.2.2]. Furthermore, since $i_*^{\mathrm{IndCoh}}(k) \in \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda))^\heartsuit$ and the quotient functor

$$\Psi_{\mathrm{Spec}(C_\lambda)} : \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda)) \to \mathrm{QCoh}(\mathrm{Spec}(C_\lambda)) \simeq C_\lambda\text{-mod}$$

is t-exact and induces an equivalence on the heart [GR17a, Chapter 4, Proposition 1.2.2] and is compatible with pushforwards [GR17a, Chapter 4, Proposition 2.1.2], we see that $\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathrm{Spec}(C_\lambda))}(i_*^{\mathrm{IndCoh}}(k)) \simeq \underline{\mathrm{End}}_{C_\lambda\text{-mod}}(k) =: E_\lambda$. Thus $\mathrm{IndCoh}(W \overset{W_{[\lambda]}}{\times} \mathrm{Spec}(C_\lambda))$ is a direct sum of $[W : W_{[\lambda]}]$ many copies of E_λ-mod, as desired. $\square$

6.4. **Classification of Degenerate Subcategory of Universal Flag Variety.** In this section, we provide an alternate description of $\mathcal{D}(G/N)_{\text{deg}}$, see Theorem 6.13. Using the notation of Section 5.3.2, we are able to set the following notation. Let $\mathcal{D}(\tilde{G}/\tilde{N})_{\text{mon}}$ denote those objects which are $\tilde{Q}_\alpha$-monodromic for some simple root α, and let $\mathcal{D}(\tilde{G}/\tilde{N})_{\overline{\text{mon}}}$ denote the full right $\mathcal{D}(\tilde{N}\backslash\tilde{G}/\tilde{N})$-subcategory of $\mathcal{D}(\tilde{G}/\tilde{N})$ generated by $\mathcal{D}(\tilde{G}/\tilde{N})_{\text{mon}}$. (This notation will not be used outside this section.) The main result of this section is the following theorem:

Theorem 6.13. There is an equivalence of right $\mathcal{D}(\tilde{N}\backslash\tilde{G}/\tilde{N})$-categories

$$\mathcal{D}(\tilde{G}/\tilde{N})_{\overline{\text{mon}}} \xhookrightarrow{\sim} \mathcal{D}(\tilde{G}/\tilde{N})_{\text{deg}}$$

induced by the inclusion $\mathcal{D}(\tilde{G}/\tilde{N})_{\text{mon}} \hookrightarrow \mathcal{D}(\tilde{G}/\tilde{N})_{\text{deg}}$ given by Proposition 5.15.

We prove this theorem below after proving some preliminary statements.

Lemma 6.14. Fix a simple root α, and let $j : \tilde{G}/\tilde{N} \hookrightarrow V_s$ be the open embedding as in Section 5.3.3. Assume $\mathcal{F} \in \mathcal{D}(\tilde{G}/\tilde{N})$ is right $(\mathbb{G}_m^\alpha, \chi)$-monodromic for some non-integral $\chi \in (\mathrm{Lie}(\mathbb{G}_m^\alpha)/\mathbb{Z})(k)$. Then we have a canonical isomorphism $j_!(\mathcal{F}) \to j_{*,dR}(\mathcal{F})$ (and, in particular, $j_!(\mathcal{F})$ is defined).

Proof. We equip V_s with a $\mathbb{G}_m$-action by scalar multiplication. This makes the embedding $\tilde{G}/\tilde{N} \overset{j}{\hookrightarrow} V_s$ $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts by $\mathbb{G}_m^\alpha$ on $\tilde{G}/\tilde{N}$. In particular, the object $j_{*,dR}(\mathcal{F})$ is $(\mathbb{G}_m^\alpha, \chi)$-monodromic. Similarly, let $z : \tilde{G}/Q_s \hookrightarrow V_s$ denote the embedding of the complementary closed subscheme, which is the zero section of V_s. Then we see that any object of the form $z_{*,dR}(\mathcal{G})$ has the property that the essential image is $\mathbb{G}_m^\alpha$-monodromic. In particular, because χ is not integral, we see $\underline{\mathrm{Hom}}(j_{*,dR}(\mathcal{F}), z_{*,dR}(\mathcal{G})) \simeq 0$. Therefore, if $\mathcal{F}' \in \mathcal{D}(V_s)$ we see that the composite

$$\underline{\mathrm{Hom}}_{\mathcal{D}(V_s)}(j_{*,dR}(\mathcal{F}), \mathcal{F}') \to \underline{\mathrm{Hom}}_{\mathcal{D}(V_s)}(j_{*,dR}(\mathcal{F}), j_{*,dR}j^!(\mathcal{F}')) \simeq \underline{\mathrm{Hom}}_{\mathcal{D}(\tilde{G}/\tilde{N})}(\mathcal{F}, j^!(\mathcal{F}'))$$

is an isomorphism, where the second equivalence follows from the fully faithfulness of $j_{*,dR}$ and the first arrow can be seen to be an isomorphism by applying $\underline{\mathrm{Hom}}(j_{*,dR}(\mathcal{F}), -)$ to the cofiber sequence $i_{*,dR}i^!(\mathcal{F}') \to \mathcal{F}' \to j_{*,dR}j^!(\mathcal{F}')$. $\square$

Lemma 6.15. Let $\mu \in \mathfrak{t}^*(k)$ be such that $\langle \alpha^\vee, \mu \rangle \notin \mathbb{Z}$. Then $F_{s_\alpha}(L_\mu) \cong L_{s_\alpha \mu}$.

Proof. Since $\mathcal{D}(G/N)^{B_\lambda} \xrightarrow{\sim} \mathcal{D}(\tilde{G}/\tilde{N})^{\tilde{B}_{\tilde{\lambda}}}$, we may assume that G is semisimple and simply connected. Recall the block decomposition of [LY20], $\mathcal{D}(B_\lambda\backslash G/N) \simeq \oplus_{\lambda' \in \mathrm{orbit}_W(\lambda)} \mathcal{D}(B_\lambda \backslash G)^{B_{\lambda'}\text{-mon}}$. We have that L_μ is $(T, \mathcal{L}_{[\mu]})$-monodromic, so, in particular, with respect to the $\mathbb{G}_m^\alpha$ action, we have that L_μ is $(\mathbb{G}_m^\alpha, \mathcal{L}_{[\overline{\mu}]})$-monodromic, where $\overline{\mu}$ is the image of μ under the projection map $\mathfrak{t}^* \to \mathrm{Lie}(\mathbb{G}_m^\alpha)^*$. Note that $\langle \alpha^\vee, \overline{\mu} \rangle = \langle \alpha^\vee, \mu \rangle$ is not an integer. In particular, we see that $F_{s_\alpha,*}(L_\mu) \in \mathcal{D}(B_\lambda\backslash G/N)^\heartsuit$ since the canonical map $F_{s_\alpha,!}(L_\mu) \to F_{s_\alpha,*}(L_\mu)$ is an isomorphism by Lemma 6.14 and the functor $F_{s_\alpha,!} \simeq j^!\mathbf{F}j_!$ is left t-exact and the latter functor $F_{s_\alpha,*} \simeq j^!\mathbf{F}j_{*,dR}$ is right t-exact.

Now, any object of $\mathcal{D}(B_\lambda\backslash G/N)^\heartsuit$ is a union of its compact subobjects by Lemma 2.51. Let L_ν be a simple subobject of $F_{s_\alpha}(L_\mu)$. Then, since F_{s_α} preserves those sheaves for which the $\mathbb{G}_m^\alpha$-averagings vanish (because the pushforward of the sheaf $\omega_{\mathbb{G}_m^\alpha}$ to the torus is canonically $\langle s_\alpha \rangle$-equivariant), we see that ν has the property that $\langle \alpha^\vee, \nu \rangle \notin \mathbb{Z}$. In particular, we may apply an identical argument to see that $F_{s_\alpha}(L_\nu) \simeq H^0 F_{s_\alpha}(L_\nu)$ is a subobject of $F_{s_\alpha}F_{s_\alpha}(L_\mu)$. However, the canonical map $F_{s_\alpha}F_{s_\alpha}(L_\mu) \to L_\mu$ is an isomorphism because the kernel is $\mathrm{SL}_{2,\alpha}$-monodromic (in other words, the fiber of the map $F^2_{s_\alpha}(\delta_1) \to \delta_1$ is the constant sheaf up to shift, and therefore the fiber of this map is given by $\mathrm{SL}_{2,\alpha}$-averaging up to shift, which vanishes on our L_μ). Therefore, we see that $F_{s_\alpha}(L_\nu)$ is a subobject of L_μ, and therefore $F_{s_\alpha}(L_\nu) \simeq L_\mu$ since $F^2_{s_\alpha}(L_\nu) \simeq L_\nu$ is nonzero.

We now show $\nu = s_\alpha \cdot \mu$. Write $\mu = w \cdot \lambda$ for some $w \in W$, which is unique because, by assumption, λ is regular with respect to the $(W, \cdot)$-action. We first assume that w is such that $s_\alpha w > w$ in the Bruhat ordering. Consider the locally closed embedding given by the union of Schubert cells of elements in the subset $\overline{\{w\}} \cup \overline{\{s_\alpha w\}}$. We note this subset contains an element of maximal length, namely $s_\alpha w$. Furthermore, by (1) of Proposition 5.39 we have that the restriction to the Schubert cell labeled by w vanishes. Therefore, the restriction to the Schubert cell labeled by $s_\alpha w$ does not vanish, since the symplectic Fourier transform is a transformation over G/Q_{s_α}. Thus, $F_{s_\alpha}(L_\mu)$ is a simple object in the heart whose restriction to the cell labeled by $s_\alpha w$ is nonzero and whose restriction to any other cell for u with $\ell(u) \geq \ell(w)$ vanishes. Therefore, we see that this simple object is the simple $L_{s_\alpha w \cdot \lambda} = L_{s_\alpha \cdot \mu}$.

If instead we had $s_\alpha w < w$ in the Bruhat ordering, then we may repeat the above argument swapping the roles of ν and μ to see that $F_{s_\alpha}(L_{s_\alpha \mu}) = L_\mu$, so the claim follows by applying F_{s_α} to this equality since $F^2_{s_\alpha}(L_{s_\alpha\mu}) \xleftarrow{\sim} L_{s_\alpha \mu}$. $\square$

Corollary 6.16. Fix some non-antidominant weight μ. Then there exists some expression $s_1, ..., s_r$ such that, letting $F_{s_i} : \mathcal{D}(G/N)^{B_\lambda} \to \mathcal{D}(G/N)^{B_\lambda}$ denote the induced functor on B_λ-invariant categories, we have $F_{s_1}...F_{s_r}(L_\mu) \cong L_\nu$ for some $\nu \in \mathfrak{t}^*(k)$ such that $\langle \alpha^\vee, \nu \rangle \in \mathbb{Z}^{\geq 0}$ for some simple root α. In particular, $L_\nu \in \mathcal{D}(B_\lambda\backslash G)^{Q_\alpha\text{-mon}}$.

Proof. Since $\mathcal{D}(G/N)^{B_\lambda} \xrightarrow{\sim} \mathcal{D}(\tilde{G}/\tilde{N})^{\tilde{B}_{\tilde{\lambda}}}$, we may assume that G is semisimple and simply connected. For any given $\mu \in \mathfrak{t}^*(k)$, let $\Xi^{\geq 0}_\mu$ denote the set of positive coroots γ such that $\langle \gamma, \mu \rangle \in \mathbb{Z}^{\geq 0}$. Fix an antidominant weight μ. By definition of antidominance, μ has the property that $\Xi^{\geq 0}_\mu$ is nonempty. We induct on the minimal height of an element in $\Xi^{\geq 0}_\mu$. For the base case, we note that if there is an element of $\Xi^{\geq 0}_\mu$ of height one, then it contains a simple coroot $\alpha^\vee$ and so setting $\nu = \alpha^\vee$ gives the claim.

Now assume $\gamma \in \Xi^{\geq 0}_\mu$ is some minimal height coroot of height larger than one. In particular, γ is not simple and so there exists some simple coroot $\alpha^\vee$ such that $\langle \mu, \alpha^\vee \rangle > 0$, so that the height of $s_{\alpha^\vee}$ is smaller than the height of γ. Note that this in particular implies that $\gamma - \alpha^\vee$ is a positive coroot since otherwise we would have that $\alpha^\vee = \gamma + (\alpha^\vee - \gamma)$, contradicting the simplicity of $\alpha^\vee$.

We claim for such $\alpha^\vee$ that $\langle \mu, \alpha^\vee \rangle$ is not an integer. indeed, if $\langle \mu, \alpha^\vee \rangle$ were a negative integer then $\langle \mu, \gamma - \alpha^\vee \rangle = \langle \mu, \gamma \rangle - \langle \mu, \alpha^\vee \rangle$ would be a positive integer, and so $\gamma - \alpha^\vee \in \Xi^{\geq 0}_\mu$, and if $\langle \mu, \alpha^\vee \rangle$ were a nonnegative integer then $\alpha^\vee \in \Xi^{\geq 0}_\mu$, both of which violate minimality of γ.

Therefore, by Lemma 6.15, we see that $F_{s_\alpha}(L_\mu) \simeq L_{s_\alpha \mu}$. By induction, since we have the inclusion $s_{\alpha^\vee}(\gamma) \in \Xi^{\geq 0}_{s_\alpha \mu}$ and $s_{\alpha^\vee}(\gamma)$ has smaller height than γ, we may write $F_{s_1}...F_{s_{r-1}}(L_{s_\alpha\mu}) \cong L_\nu$. Setting $s_r = s_{\alpha^\vee}$, we obtain our desired isomorphism.

The final sentence follows from [Hum08, Theorem 9.4]. Specifically, this theorem states that our simple $L_\nu \in (\mathfrak{g}\text{-mod})^{N,\heartsuit}_{\chi_\lambda}$ is in particular locally finite for the action of $\mathrm{Lie}(Q_\alpha)$, where we recall that Q_α is the unipotent radical of the associated parabolic P_α. Any $\mathrm{Lie}(Q_\alpha)$-module in the heart of the associated category which is $\mathrm{Lie}(Q_\alpha)$-locally finite can, essentially by definition, be written as a filtered colimit of finite dimensional $\mathrm{Lie}(Q_\alpha)$-representations. The assumption that G is simply connected gives that Q_α is simply connected (we have seen that in this case Q_α may be non-canonically written as a semidirect product of simply connected groups $U_{w_0s} \rtimes \mathrm{SL}_2$) so finite dimensional $\mathrm{Lie}(Q_\alpha)$-representations are precisely those which lift to representations of Q_α. Therefore we see that $L_\nu \in (\mathfrak{g}\text{-mod})^{Q_\alpha\text{-mon},\heartsuit}_{\chi_\lambda} \simeq \mathcal{D}(B_\lambda\backslash G)^{Q_\alpha\text{-mon}}$ which establishes our claim. □

Proof of Theorem 6.13. We show the functor is essentially surjective. Let $\mathcal{F} \in \mathcal{D}(G/N)^{N,+}_{\mathrm{deg}}$. Since the essential image is closed under colimits, we may assume that $\mathcal{F}$ is eventually coconnective, and, since the inclusion functor is a map of left $\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T))$-categories, we may check that the essential image of the associated B_λ-invariants agree for all field-valued λ by Proposition 2.16. Using Proposition 2.46, we may assume that $\lambda \in \mathfrak{t}^*(k)$. Furthermore, any eventually coconnective object can be written as a colimit of its cohomology groups since the t-structure on $\mathcal{D}(B_\lambda\backslash G/N)$ is right-complete by Lemma 2.27.

Note further that by Proposition 6.6 the functor $\mathrm{Av}_!^\psi$ has the same kernel as the contravariant functor $\tilde{\mathbb{V}}_I$. An object $M \in \mathcal{D}(B_\lambda\backslash G/N)^\heartsuit$ is in the kernel of $\tilde{\mathbb{V}}_I$ if and only if it does not admit a subquotient of some antidominant simple. Therefore by Lemma 6.9, it suffices to prove that each L_μ for which μ lies in the $(W,\cdot)$-orbit of λ and μ is not antidominant, L_μ lies in the right $\mathcal{D}(N\backslash G/N)$-orbit of some object of $\mathcal{D}(B_\lambda\backslash G)^{Q_\alpha\text{-mon}}$ for some simple root α. However, this is precisely the content of Corollary 6.16. □

In particular, Theorem 6.13 gives that if G is simply connected that a nondegenerate G-category in the sense of Definition 5.12, it is equivalently nondegenerate in the sense of Definition 1.1.

Remark 6.17. If $G = \mathrm{PGL}_2$, it is not the case that $\mathcal{D}(G/N)_{\mathrm{deg}}$ is not generated by right G-monodromic objects. For example, an immediate consequence of Proposition 6.6 is the fact that the two dimensional representation k^2 of $\mathfrak{pgl}_2$, is an eventually coconnective object in the kernel of $\mathrm{Av}_!^\psi$; however, it is not G-monodromic since, for example, if we let μ denote the central character of the two dimensional representation, then we have that $k^2 \in \mathcal{D}(B_\mu\backslash G/N)^\heartsuit$ and

$$\mathcal{D}(B_\lambda\backslash G/N)^{G\text{-mon}} \simeq \mathfrak{g}\text{-mod}^G_{\chi_\mu} \simeq \mathrm{Rep}(\mathrm{PGL}_2)_{\chi_\mu}$$

is zero.

7. Proofs of Theorem 1.4 and Theorem 1.8

In this section, we prove Theorem 1.4 and Theorem 1.8.

7.1. Monoidality of Averaging Functor.

In this section, we prove the following proposition:

Proposition 7.1. The composite functor $\mathcal{H}_\psi \xrightarrow{\tilde{\mathrm{Av}}_*^N} \mathcal{D}(N\backslash G/_{-\psi}N^-)^W \simeq \mathcal{D}(T)^W$ is monoidal.

Proof. The functoriality of Av_*^N gives that the following diagram canonically commutes:

$$\begin{tikzcd}
\underline{\mathrm{End}}_G(\mathcal{D}(G/_{-\psi}N^-)) \arrow[r, "(-)^N"] \arrow[d, "\mathrm{ev}_{\delta_{N^-,\psi}}"] & \underline{\mathrm{End}}_{T\rtimes W}(\mathcal{D}(N\backslash G/_{-\psi}N^-)) \arrow[d, "\mathrm{ev}_{\mathrm{Av}_*^N(\delta_{N^-,\psi})}"] \\
\mathcal{H}_\psi \arrow[r, "\mathrm{Av}_*^N\mathrm{oblv}^{N^-,\psi}"] & \mathcal{D}(N\backslash G/_{-\psi}N^-)^W
\end{tikzcd} \tag{12}$$

where the vertical arrows are the evaluation maps and the top vertical arrow is the functor induced by Proposition 5.23. Similarly, note if I denotes the isomorphism of Proposition 1.7 (which we recall is W-equivariant by Proposition 5.23) then we have a canonical identification of the following diagram

$$\begin{tikzcd} \underline{\mathrm{End}}_{T \rtimes W}(\mathcal{D}(N\backslash G/_{-\psi}N^-)) \arrow[r, "I\circ - \circ I^{-1}"] \arrow[d, "\mathrm{ev}_{\mathrm{Av}_*^N(\delta_{N^-,\psi})}"] & \underline{\mathrm{End}}_{T \rtimes W}(\mathcal{D}(T)) \arrow[d, "\mathrm{ev}_{\delta_1}"] \\ \mathcal{D}(N\backslash G/_{-\psi}N^-)^W \arrow[r, "I"] & \mathcal{D}(T)^W \end{tikzcd} \tag{13}$$

since $I(\mathrm{Av}_*^N(\delta_{N^-,\psi})) \simeq \delta_1 \in \mathcal{D}(T)$. Now, note that the functors given by the top horizontal arrows of (12) and (13) are monoidal, the left vertical arrow of (12) is a monoidal equivalence, and the right vertical arrow of (13) is a monoidal equivalence. Thus, since these diagrams commute, we see that $I\mathrm{Av}_*^N$ is a composite of monoidal functors. $\square$

7.2. **Proof of Theorem 1.8.** In this subsection, we prove Theorem 1.8. Let $\mathcal{C}$ be a G-category. Then, using the fact that invariants and coinvariants agree (see Theorem 2.10) and that tensor products commute with colimits, we may identify Av_*^N with the functor

$$\mathcal{C}^{N^-,\psi} \simeq \mathcal{C}_{N^-,\psi} \simeq \mathcal{D}(G)^{N^-,\psi} \otimes_G \mathcal{C} \xrightarrow{\mathrm{Av}_*^N \otimes_G \mathrm{id}_{\mathcal{C}}} \mathcal{D}(G)^{N,W}_{\mathrm{nondeg}} \otimes_G \mathcal{C} \simeq \mathcal{D}(G)_{N,W,\mathrm{nondeg}} \otimes_G \mathcal{C} \simeq \mathcal{C}_{N,W,\mathrm{nondeg}} \simeq \mathcal{C}^{N,W}_{\mathrm{nondeg}}$$

and similarly for the adjoint $\mathrm{Av}_!^\psi$. We therefore obtain that the fully faithfulness Av_*^N follows by proving the general universal case:

Theorem 7.2. The functor of G-categories $\mathrm{Av}_*^N : \mathcal{D}(G)^{N^-,\psi} \to \mathcal{D}(G)^N$ lifts to a fully faithful functor of G-categories $\mathrm{Av}_*^N : \mathcal{D}(G)^{N^-,\psi} \hookrightarrow \mathcal{D}(G)^{N,W}_{\mathrm{nondeg}}$. Moreover, the functor $\mathrm{Av}_*^N[\dim(N)]$ lifts to a fully faithful t-exact functor of G-categories $\widetilde{\mathrm{Av}}_* : \mathcal{D}(G)^{N^-,\psi} \hookrightarrow \mathcal{D}(G)^{N,W}_{\mathrm{nondeg}}$.

Proof. Note that all functors of DG categories (or any stable ∞-categories) are by definition exact, so they commute with cohomological shifts. Therefore to construct the lift of $\mathrm{Av}_*^N[\dim(N)]$ it suffices to construct the lift of Av_*^N, where the t-exactness of $\widetilde{\mathrm{Av}}_*$ follows since the forgetful functor oblv^W reflects the t-structure.

To construct the lift of Av_*^N, note that the G-functor Av_*^N is given by an integral kernel in $\mathcal{D}(N\backslash G)^{N^-,-\psi}_{\mathrm{nondeg}}$, and, under the equivalences $\mathcal{D}(N\backslash G)^{N^-,-\psi}_{\mathrm{nondeg}} \simeq \mathcal{D}(N\backslash G)^{N^-,-\psi} \simeq \mathcal{D}(T)$ given by Proposition 5.14 and Proposition 1.7 respectively, this kernel is given by $\delta_1[-\dim(N)] \in \mathcal{D}(T)$ by Proposition 7.1. Furthermore, these equivalences are canonically W-equivaraint by Proposition 5.23, and therefore the integral kernel can be canonically equipped with W-equivariant structure since $* \hookrightarrow T$ is W-equivariant.

We wish to show this lift is fully faithful. By Proposition 2.16, it suffices to show that the resulting functor on invariants $\mathrm{Vect} \simeq \mathcal{D}(G/_\lambda B)^{N^-,\psi} \to \mathcal{D}(G/_\lambda B)^{N,W}_{\mathrm{nondeg}}$ is fully faithful. By Lemma 2.47, it suffices to assume that λ is a k-point. By continuity of Av_*^N, we may show the counit of the adjunction is an isomorphism on the one dimensional vector space $k \in \mathrm{Vect}$. By Proposition 6.6(1), this object is a cohomological shift of the direct sum of the indecomposable antidominant injectives $\underline{I}$, or, equivalently by self duality, the direct sum of the antidominant projectives $\underline{P}$. Using the results of Theorem 6.3, we have $W \overset{W_{[\lambda]}}{\times} C_\lambda \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^N_{\mathrm{nondeg}}}(\underline{P})$. Therefore, since $T \rtimes W$ acts on $\mathcal{D}(G/_\lambda B)^N_{\mathrm{nondeg}}$ we have that by Proposition 6.5 that this equivalence of classical algebras is W-equivariant. Therefore we see

$$\underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^{N,W}_{\mathrm{nondeg}}}(\underline{P}_\lambda) \simeq \underline{\mathrm{End}}_{\mathcal{D}(G/B_\lambda)^N_{\mathrm{nondeg}}}(\underline{P}_\lambda)^W \simeq (W \overset{W_{[\lambda]}}{\times} C_\lambda)^W \simeq k$$

where the second to last equivalence follows since $\underline{P}_\lambda \simeq \mathrm{Av}_*^N(k)$ lies in the nondegenerate subcategory, and the last equivalence follows since C_λ^W can be identified with the regular W-representation by the endomorphismensatz, see Theorem 6.3. Therefore, we see our functor is fully faithful. $\square$

7.3. **Proofs of Theorem 1.4 and Theorem 1.3 from Theorem 1.8.** In this subsection, we verify the essential image of our shifted and lifted functor $\widetilde{\mathrm{Av}}_*$ and complete the proofs of Theorem 1.4 and Theorem 1.3. We first make the following computation on the essential image:

Proposition 7.3. Fix some simple root α. The composite given by

$$\mathcal{D}(G)^{N^-,\psi} \xrightarrow{\mathrm{Av}_*^N} \mathcal{D}(G/N)^{\langle s_\alpha\rangle}_{\mathrm{nondeg}} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\alpha}} \mathcal{D}(G/N)^{\mathbb{G}_m^\alpha \rtimes \langle s_\alpha\rangle}_{\mathrm{nondeg}} \simeq \mathcal{D}(G/N)^{\mathbb{G}_m^\alpha}_{\mathrm{nondeg}} \otimes \mathrm{Rep}(\langle s_\alpha\rangle)$$

where the final equivalence is given by Corollary 5.45, factors through the subcategory labelled by the trivial representation.

Proof. Let A denote the composite functor. By Remark 2.11, it suffices to show that

$$A(\delta_{N^-,\psi}) \in \mathcal{D}(G/N)^{(N^-,\psi),\mathbb{G}_m^\alpha}_{\mathrm{nondeg}} \otimes \mathrm{Rep}(\langle s_\alpha\rangle)$$

lies in the full (G-)subcategory labelled by the trivial representation. However, by direct computation or Proposition 7.1 below, we have that the sheaf $\mathrm{Av}_*^N(\delta_{N^-,\psi})$ can be identified with $\delta_1 \in \mathcal{D}(T)$ by Proposition 1.7. Furthermore, by Proposition 5.23 we see that the given W-equivariance on $\mathrm{Av}_*^N(\delta_{N^-,\psi})$ can be identified with the W-equivariance on δ_1 given by the W-equivariant closed embedding $* \hookrightarrow T$. However, for the equivariant sheaf $\delta_1 \in \mathcal{D}(T)^W$, we see that $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}(\delta_1)$ acquires a trivial $\langle s_\alpha\rangle$-representation. Therefore, the $\langle s_\alpha\rangle$ action is trivial and so the same holds for $A(\delta_{N^-,\psi}) \simeq \mathrm{Av}_*^{\mathbb{G}_m^\alpha}\mathrm{Av}_*^N(\delta_{N^-,\psi})$. ☐

Corollary 7.4. The functor $\mathrm{Av}_*^N : \mathcal{H}_\psi \to \mathcal{D}(T)^W$ factors through the full subcategory of objects of $\mathcal{D}(T)^W$ satisfying Coxeter descent in the sense of Definition 4.22.

Proof. By the final point of Proposition 4.23, it suffices to show that if $\mathcal{F} \in \mathcal{H}_\psi$ then the canonical $\langle s_\alpha\rangle$-representation on $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\mathrm{Av}_*^N(\mathcal{F})$ is trivial. However, this directly follows from taking (N^-,ψ)-invariants of the composite functor of Proposition 7.3. ☐

Lemma 7.5. Fix some field-valued point λ. We have a canonical isomorphism of functors:

$$\begin{tikzcd}
\mathcal{D}(B_\lambda\backslash G/N)^{T,w}_{\mathrm{nondeg}} \arrow[r, "\mathrm{Av}_!^{-\psi}\mathrm{oblv}^{N,(T,w)}"] & \mathcal{D}(B_\lambda\backslash G/_{-\psi}N^-) \simeq \mathrm{Vect} \\
\mathcal{D}(N^-_\psi\backslash G/N)^{T,w} \arrow[u, "\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N^-,\psi}"] \arrow[r, "\mathrm{Av}_!^{-\psi}\mathrm{oblv}^{N,(T,w)}"] & \mathcal{H}_\psi \arrow[u, "\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N^-,-\psi}"']
\end{tikzcd}$$

Proof. In the above diagram, the horizontal arrows are averaging with respect to the right action, and the vertical arrows are averaging with respect to the left action. Therefore since all four functors are maps of G-categories, the diagram canonically commutes. ☐

In the diagram of Lemma 7.5, we claim that the associated right adjoints to the horizontal arrows are functors of $\tilde{W}^{\mathrm{aff}}$-categories, where we take the $\tilde{W}^{\mathrm{aff}}$-action to be trivial on the categories of the right side of the diagram. To see this, note that via the Mellin transform, we may identify the right adjoint to the bottom functor as the composite of a W-equivariant functor and, via the Mellin transform, the forgetful functor $\mathrm{oblv}^{X^\bullet(T)}$. In particular, the fact that $\tilde{W}^{\mathrm{aff}}$ is placid allows us to apply Theorem 2.19 to show that the adjoint is also $\tilde{W}^{\mathrm{aff}}$-equivariant and induces a functor on coinvariants. By Corollary 3.4, we therefore see:

Lemma 7.6. Fix some field-valued $\lambda \in \mathfrak{t}^*$. Then the following diagram canonically commutes:

$$\begin{tikzcd}
\mathcal{D}(B_\lambda\backslash G/N)^{W}_{\mathrm{nondeg}} \arrow[r, "\mathrm{Av}_!^{-\psi}"] & \mathcal{D}(B_\lambda\backslash G/_{-\psi}N^-) \simeq \mathrm{Vect} \\
\mathcal{D}(N^-_\psi\backslash G/N)^{W} \arrow[u, "\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N^-,\psi}"] \arrow[r, "\mathrm{Av}_!^{-\psi}"] & \mathcal{H}_\psi \arrow[u, "\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N^-,\psi}"]
\end{tikzcd}$$

Proof of Theorem 1.4. We have seen in Theorem 7.2 that the functor $\mathrm{Av}_*^N : \mathcal{H}_\psi \to \mathcal{D}(G/N)^{N^-,\psi,W} \xrightarrow{\sim} \mathcal{D}(T)^W$ is fully faithful, and it is t-exact up to cohomological shift in Theorem 2.43. Furthermore, we have shown that this functor factors through the full subcategory of objects satisfying Coxeter descent in Corollary 7.4. Therefore it remains to show that the adjoint $\mathrm{Av}_!^\psi$ is conservative on this the subcategory of objects satisfying Coxeter descent. Let $\mathcal{F} \in \mathcal{D}(T)^W$ denote a nonzero object satisfying Coxeter descent. One can directly check on the compact generators labelled by simples/delta sheaves that the following diagram commutes

$$\begin{tikzcd}
\mathcal{D}(B_\lambda\backslash G/N)^W_{\mathrm{nondeg}} \arrow[r, "\tilde{\mathbb{V}}"] \arrow[d, "\mathrm{Av}_!^{N_\ell^-,\psi}\mathrm{oblv}^{B_\lambda}[-\dim(N)]"'] & \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda) \overset{W_\lambda^{\mathrm{aff}}}{\times} W)^W \arrow[drr, "\zeta_*^{\mathrm{IndCoh}}"] & & \\
\mathcal{D}(N_\psi^-\backslash G/N)^W \arrow[rr, "j^!"] & & \mathcal{D}(T)^W \arrow[r, "M"] & \mathrm{IndCoh}(\mathfrak{t}^*/\tilde{W}^{\mathrm{aff}})
\end{tikzcd}$$

where $\zeta : \mathrm{Spec}(C_\lambda) \overset{W_\lambda^{\mathrm{aff}}}{\times} W \to \mathfrak{t}^*/X^\bullet(T)$ is the canonical map, the top arrow is the equivalence of Proposition 6.11, $j^!$ is the equivalence of Proposition 1.7, and we recall that M is the Mellin transform. Passing to the associated right adjoints we see that there exists some field-valued point λ such that $\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N^-,\psi}(\mathcal{F})$ does not vanish. By applying the categorical extension of scalars of Section 2.4.1, it suffices to assume λ is a k-point.

Now note that the following diagram commtues

$$\begin{tikzcd}
\mathcal{D}(B_\lambda\backslash G)^{N^-,-\psi} \arrow[r, "\sim"] & \mathrm{IndCoh}(*) \\
\mathcal{D}(B_\lambda\backslash G/N)^W_{\mathrm{nondeg}} \arrow[u, "{\mathrm{Av}_!^{-\psi}[-\dim(N)]}"] \arrow[r, "\tilde{\mathbb{V}}"] & \mathrm{IndCoh}(\mathrm{Spec}(C_\lambda) \overset{W_\lambda^{\mathrm{aff}}}{\times} W)^W \arrow[u, "(\alpha_*^{\mathrm{IndCoh}}(-))^W"']
\end{tikzcd}$$

since again we may identify the images the left adjoints via the image of $k \in \mathrm{Vect}$. Thus since by assumption $\mathcal{F} \in \mathcal{D}(T)^W$ satisfies Coxeter descent, we see that, by Proposition 4.7(1) and the fact that satisfying Coxeter descent for $\tilde{W}^{\mathrm{aff}}$ is equivalent to satisfying Coxeter descent for all finite subgroups of W^{aff} generated by reflections i.e. Proposition 4.23, that the sheaf $\mathrm{Av}_!^{-\psi}\mathrm{Av}_*^{B_\lambda}\mathrm{oblv}^{N_\ell^-,\psi}(\mathcal{F})$ does not vanish. Thus by Lemma 7.6, we see that $\mathrm{Av}_!^\psi(\mathcal{F})$ does not vanish. $\square$

Finally, note that to derive the exact equivalence of abelian categories in Theorem 1.3 from Theorem 1.4, as in Remark 1.5 it suffices to show that each functor in Theorem 1.4 is t-exact. We have seen in Theorem 2.43 that $\mathrm{Av}_*^N[\dim(N)]$ is t-exact, and $\widetilde{\mathrm{Av}}_*$ is t-exact since oblv^W reflects the t-structure. Furthermore, we have seen in Proposition 3.21 that $\phi^!$ is t-exact. Therefore it remains to show the following, completing the proof of an exact equivalence of abelian categories as in Theorem 1.3:

Proposition 7.7. The shifted Mellin transform $\mathrm{FMuk}[d]$ is t-exact, where $d := \dim(\mathfrak{t}^*)$.

Proof. It is standard that the functors

$$\mathrm{IndCoh}(T_{dR}) \xrightarrow{\phi^!} \mathrm{IndCoh}(T) \xrightarrow{\Gamma^{\mathrm{IndCoh}}} \mathrm{Vect}$$

correspond, under the associated Fourier-Mukai transformations, to the functors

$$\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \xrightarrow{\Gamma^{\mathrm{IndCoh}}[-d]/X^\bullet(T)} \mathrm{IndCoh}(*/X^\bullet(T)) \xrightarrow{c^!} \mathrm{Vect}$$

where $c : * \to */X^\bullet(T)$ is the quotient map, see the proof of [Lau96, Théorème 6.3.3(ii)], whose proof also applies to IndCoh in the DG categorical context.

Let F denote the composite $\Gamma^{\mathrm{IndCoh}}\phi^!$ and let G denote the composite $\Gamma^{\mathrm{IndCoh}}[-d]/X^\bullet(T) \circ c^!$. Then by this observation we see that there is a canonical identification exhibiting that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathfrak{t}^*/X^\bullet(T)) \arrow[r, "G"] \arrow[d, "\mathrm{FMuk}"] & \mathrm{Vect} \arrow[d, "\mathrm{id}"] \\
\mathcal{D}(T) := \mathrm{IndCoh}(T_{dR}) \arrow[r, "F"] & \mathrm{Vect}
\end{tikzcd}$$

since the Fourier-Mukai transform for the trivial group is the identity. Because the functors F, $G[-d]$, and id are t-exact, we see that $\mathrm{FMuk}[d]$ is t-exact as well. $\square$

8. Preliminary Results for Theorem 1.13

8.1. **Barr-Beck-Lurie.** We will use the Barr-Beck-Lurie theorem. We will recall the result here for the reader's convenience–this is summarized in much more depth and proved in [Lur17, Theorem 4.7.3.5]. Given a functor of DG categories $L : \mathcal{C} \to \mathcal{D}$ which admits a right adjoint $R : \mathcal{D} \to \mathcal{C}$, we can obtain a comonad in $\mathcal{D}$ which we denote LR. The functor L canonically lifts to a functor $L^{\mathrm{enh}} : \mathcal{C} \to LR\text{-comod}(\mathcal{D})$.

Remark 8.1. If $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is any adjoint pair of functors, we will always reserve the superscript 'enh' for the corresponding lift $L^{\mathrm{enh}} : \mathcal{C} \to LR\text{-comod}(\mathcal{D})$. Furthermore, if $\mathcal{D} \simeq A\text{-mod}$ for some algebra A, then $LR(A)$ is a coalgebra, and $LR\text{-comod}(\mathcal{D}) \simeq LR(A)\text{-comod}$. We will abuse notation and also denote the composite functor $\mathcal{C} \to LR(A)\text{-comod}$ by L^{enh}.

Definition 8.2. We say that L is *comonadic* if L^{enh} is an equivalence.

Theorem 8.3. (Barr-Beck-Lurie Theorem for Comonads) The following are equivalent:

- The functor L is comonadic.
- For all $C \in \mathcal{C}$, the canonical map $C \to \mathrm{Tot}(RL)^{\bullet+1}(C)$ is an equivalence.
- The functor L is conservative and the canonical map $L(\mathrm{Tot}((RL)^{\bullet+1}(C))) \to \mathrm{Tot}(L(RL)^{\bullet+1}(C))$ is an equivalence.

8.2. **A Comonadicity Condition.** In this section, we state and prove a condition for the comonadicity of functors, see Corollary 8.6. The results of this subsection are modifications of ideas contained in the proof of [Ras20b, Proposition 3.7.1].

Proposition 8.4. If $L : \mathcal{C} \to \mathcal{D}$ is any functor of DG categories equipped with t-structures such that:

(1) The t-structures on $\mathcal{C}$ and $\mathcal{D}$ are right-complete in the sense of Definition 2.26
(2) L is t-exact and
(3) L is conservative on $\mathcal{C}^\heartsuit$

then the induced functor $L : \mathcal{C}^{\geq 0} \to \mathcal{D}^{\geq 0}$ commutes with arbitrary totalizations.

We first begin with a standard lemma on cosimplicial sets, whose proof can be found, for example, in the third paragraph of the proof of [Ras20b, Proposition 3.7.1]:

Lemma 8.5. For any DG category $\mathcal{C}$ equipped with a right-complete t-structure and any cosimplicial object $\mathcal{F}^\bullet$ of $\mathcal{C}$ such that $\mathcal{F}^i \in \mathcal{C}^{\geq 0}$ for all i, the totalization $\mathrm{Tot}(\mathcal{F}^\bullet)$ exists and we have the identity

$$\tau^{\leq n}(\mathrm{Tot}(\mathcal{F}^\bullet)) \simeq \tau^{\leq n}(\mathrm{Tot}^{\leq n+1}(\mathcal{F}^\bullet))$$

where $\mathrm{Tot}^{\leq n+1}(\mathcal{F}^\bullet)$ denotes the partial totalization, i.e. the limit over $\Delta_{\leq n+1}$.

Proof of Proposition 8.4. We have

$$L(\mathrm{Tot}(\mathcal{F}^\bullet)) \xleftarrow{\sim} L(\mathrm{colim}_n(\tau^{\leq n}(\mathrm{Tot}(\mathcal{F}^\bullet)))) \simeq \mathrm{colim}_n L(\tau^{\leq n}(\mathrm{Tot}(\mathcal{F}^\bullet))) \simeq \mathrm{colim}_n L(\tau^{\leq n}(\mathrm{Tot}^{\leq n+1}(\mathcal{F}^\bullet)))$$

where the first step uses the right-completeness of the t-structure of $\mathcal{D}$, the second uses the fact that all functors of DG categories are continuous, and the third uses Lemma 8.5. We may continue this chain of equivalences to obtain

$$L(\mathrm{Tot}(\mathcal{F}^\bullet)) \simeq \mathrm{colim}_n(\tau^{\leq n}(\mathrm{Tot}^{\leq n+1}(L\mathcal{F}^\bullet))) \simeq \mathrm{colim}_n(\tau^{\leq n}(\mathrm{Tot}(L\mathcal{F}^\bullet)))$$

where the first step follows from the fact that L is t-exact and commutes with finite limits and the second equivalence follows from Lemma 8.5. In particular, by the right-completeness of the t-structure of $\mathcal{D}$, we see that L preserves these totalizations, as desired. $\square$

Corollary 8.6. Let $\mathcal{C}, \mathcal{D}$ be DG categories equipped t-structures for which the t-structure on $\mathcal{D}$ is right-complete, and assume that $L : \mathcal{C} \to \mathcal{D}$ is a t-exact functor which admits a right adjoint R. Then if L is conservative on $\mathcal{C}^{\heartsuit}$, the restricted functor $\mathcal{C}^+ \to \mathcal{D}^+$ is comonadic.

Proof. Because L is a t-exact functor, its right adjoint is left t-exact. Given an object in the eventually coconnective subcategory $C \in \mathcal{C}^+$, by the t-exactness of L we may assume $C \in \mathcal{C}^{\geq 0}$. These two facts imply the totalization of the cosimplicial object $\mathcal{F}^\bullet := (RL)^\bullet(C)$ is a cosimplicial object in $\mathcal{C}^{\geq 0}$. In particular, this totalization exists by the right-completeness assumption on C and by Lemma 8.5, and is preserved by L by Proposition 8.4. $\square$

9. Proof of Theorem 1.13

In this section, we prove Theorem 1.13. We first identify the two categories as DG categories in Section 9.1. Then, in Section 9.2, after some preliminary categoral recollections we prove this equivalence can be equipped with a monoidal structure by relating both categories of Theorem 1.13 to the category $\underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*))$.

9.1. **Identification of Theorem 1.13 as DG Categories.** Let $s_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \to \mathrm{IndCoh}(\mathfrak{t}^*)$ be the pushforward associated to the projection map $s : \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^* \to \mathfrak{t}^*$, and, as before, we denote the canonical functor $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}} \to \mathrm{IndCoh}(\mathfrak{t}^*)$ given by the composite $\mathrm{Av}_!^{\psi}\mathrm{oblv}^N$, Proposition 1.7, and the Mellin transform also by $\mathrm{Av}_!^{\psi}$. We first state the eventually coconnective version of Theorem 1.13.

Theorem 9.1. We have the following:

1. The functor $s_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+ \to \mathrm{IndCoh}(\mathfrak{t}^*)^+$ is comonadic.
2. The functor $\mathrm{Av}_!^{\psi} : \mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w}),\,+}_{\mathrm{nondeg}} \to \mathrm{IndCoh}(\mathfrak{t}^*)^+$ is comonadic.
3. The coalgebras given by s_*^{IndCoh} and $\mathrm{Av}_!^{\psi}$ are canonically isomorphic.

We will prove Theorem 9.1 in Section 9.1.2 and Section 9.1.3, and show how it implies the non-monoidal version of Theorem 1.13 in Section 9.1.4. To prove Theorem 9.1, we will use Corollary 8.6, which requires us to argue that the t-structure on $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+$ is right-complete. We will argue more generally that the t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is right-complete for any ind-scheme $\mathcal{X}$ in Section 9.1.1.

9.1.1. *Right-Completeness of t-Structure of Ind-Coherent Sheaves on Ind-Scheme.* Let $\mathcal{X}$ denote any ind-scheme. Recall the standard t-structure on $\mathrm{IndCoh}(\mathcal{X})$ characterized by the property that $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ contains precisely those $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ for which $i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}$ for any closed subscheme $X \overset{i}{\hookrightarrow} \mathcal{X}$, and which is compatible with filtered colimits [GR17a, Chapter 4, Section 1.2].

Proposition 9.2. The t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is right-complete.

Proof. Given some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, let $\phi^{\mathcal{F}} : \mathcal{F} \to \mathrm{colim}_n \tau^{\leq n}\mathcal{F}$ denote the canonical map, and let $\mathcal{K}$ denote the fiber of $\phi^{\mathcal{F}}$. Since the t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is compatible with filtered colimits, we see that $\tau^{\leq n_0}(\phi^{\mathcal{F}})$ is an equivalence for all $n_0 \in \mathbb{Z}$, and that, in particular, $\mathcal{K} \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$.

We first prove this in the case where $\mathcal{X}$ is itself a scheme X. In this case, we have a t-exact equivalence $\Psi_X : \mathrm{IndCoh}(X)^{\geq 0} \xrightarrow{\sim} \mathrm{QCoh}(X)^{\geq 0}$. Because Ψ_X is exact, we see that $\Psi_X(\mathcal{K}) \simeq \mathrm{fib}(\Psi_X(\phi^{\mathcal{F}}))$. Because Ψ is continuous and t-exact, we obtain a canonical identification $\Psi_X(\phi^{\mathcal{F}}) \simeq \phi^{\Psi_X(\mathcal{F})}$. However, $\phi^{\Psi_X(\mathcal{F})}$ is an equivalence since the t-structure on $\mathrm{QCoh}(X)$ is right-complete [GR17a, Chapter 3, Corollary 1.5.7]. Thus $\Psi_X(\mathcal{K}) \simeq 0$, and since Ψ_X is in particular conservative on $\mathrm{IndCoh}(X)^{\geq 0}$, we see that $\mathcal{K} \simeq 0$ in this case.

We now assume the result of Proposition 9.2 for schemes. For any closed subscheme $i : X \hookrightarrow \mathcal{X}$, we therefore obtain equivalences

$$i^!(\mathcal{K}) \xrightarrow{\phi^{i^!(\mathcal{F})}} \mathrm{colim}_n \tau^{\leq n} i^!(\mathcal{K}) \simeq \mathrm{colim}_n \tau^{\leq n} i^!(\tau^{\leq n}\mathcal{K}) \simeq \mathrm{colim}_n \tau^{\leq n} i^!(0)$$

where the first map is an equivalence t-structure on $\mathrm{IndCoh}(X)$ is right-complete, the second step uses the fact that $i^!$ is right t-exact, and the third equivalence is a direct consequence of the above fact that $\tau^{\leq n}(\phi^{\mathcal{F}})$ is an equivalence for all n. $\square$

9.1.2. *Conservativity of IndCoh Side.* We have seen in Corollary 2.42 that $L_{\mathcal{D}} = \mathrm{Av}_!^{\psi}$ is t-exact, and $L_I = \pi_*^{\mathrm{IndCoh}}$ is t-exact by Proposition 3.20. Therefore, by Corollary 8.6, to prove points (1) and (2) of Theorem 9.1 it suffices to verify that both functors are conservative, since the t-structure on both categories are right-complete by Proposition 9.2 and Proposition 5.18. The functor $\mathrm{Av}_!^{\psi}$ is conservative on the eventually coconnective subcategory by construction, so we prove the analogous conservativity for π_*^{IndCoh} in this subsection.

We first prove a preliminary result regarding a locally almost finite type ind-scheme[5] $\mathcal{X}$ with colimit presentation ${}^{cl}\mathcal{X} = \mathrm{colim}_{\alpha} X_{\alpha}$ and associated closed embeddings $i_{\alpha} : X_{\alpha} \hookrightarrow X$. This allows us to place a t-structure on the category $\mathrm{IndCoh}(\mathcal{X})$, see Chapter 3 of [GR17b].

Corollary 9.3. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. Then there exists some closed subscheme $X := X_{\alpha} \overset{i}{\hookrightarrow} \mathcal{X}$ such that $H^0(i^!(\mathcal{F}))$ is nonzero.

Proof. Pick a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. By [GR17b, Chapter 3, Corollary 1.2.7], there exists some closed subscheme $X_{\alpha} \overset{i}{\hookrightarrow} \mathcal{X}$ and some $\mathcal{G} \in \mathrm{Coh}(X)^{\heartsuit}$ such that the map $i_{S,*}^{\mathrm{IndCoh}}(\mathcal{G}) \to \mathcal{F}$ is nonzero, so that the space $\mathrm{Hom}_{\mathrm{IndCoh}(\mathcal{X})}(i_{S,*}^{\mathrm{IndCoh}}(\mathcal{G}), \mathcal{F})$ is a discrete space (as both objects lie in the heart of a t-structure) with more than one point. Therefore, by adjunction, the same holds for $\mathrm{Hom}_{\mathrm{IndCoh}(X_{\alpha})}(\mathcal{G}, i^!\mathcal{F})$. However, since i_*^{IndCoh} is t-exact, its right adjoint is left t-exact, and so we see that this implies that there exists a nonzero map $\mathcal{G} \to \tau^{\leq 0} i^! \mathcal{F} \simeq H^0(i^!\mathcal{F})$ which obviously implies our claim. ☐

Corollary 9.4. Assume $q : \mathcal{X} \to Y$ is a map from an ind-affine scheme $\mathcal{X}$ to a scheme Y. Then q_*^{IndCoh} is conservative on $\mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$.

Proof. Pick $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\heartsuit}$. Note that, by Corollary 9.3, there exists some closed subscheme $i : X \hookrightarrow \mathcal{X}$ for which $H^0(i^!(\mathcal{F}))$ is nonzero, and, by ind-affineness, we may assume X is affine. Let π denote the composite $q \circ i : X \to Y$. Then we have that $H^0(\pi_*^{\mathrm{IndCoh}} i^!(\mathcal{F})) \simeq H^0(q_*^{\mathrm{IndCoh}} i_*^{\mathrm{IndCoh}} i^!(\mathcal{F}))$ is a subobject of $H^0(q_*^{\mathrm{IndCoh}}(\mathcal{F}))$, as q_*^{IndCoh} is ind-affine and thus is t-exact by [GR17b, Chapter 3, Lemma 1.4.9]. However, we see that π_*^{IndCoh} is conservative (it is the pushforward of an affine morphism) and so $H^0(\pi_*^{\mathrm{IndCoh}} i^!(\mathcal{F}))$ is nonzero, and therefore so too is $H^0(q_*^{\mathrm{IndCoh}}(\mathcal{F}))$. ☐

Of course, as a special case of this, we obtain our desired conservativity:

Corollary 9.5. The functor $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \to \mathrm{IndCoh}(\mathfrak{t}^*)$ is conservative when restricted to the full subcategory $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^{\heartsuit}$.

We now record a consequence of Corollary 9.4 for later use.

Corollary 9.6. The functor $\overline{s}_*^{\mathrm{IndCoh}}$ is conservative.

Proof. It suffices to show that the functor $\overline{s}^! \overline{s}_*^{\mathrm{IndCoh}}$ is conservative. By base change (Corollary 3.18), it suffices to show that $t_*^{\mathrm{IndCoh}} s^!$ is conservative on $\mathrm{IndCoh}(\mathfrak{t}^*)$. For a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)$, there exists some i for which $H^i(\mathcal{F})$ is nonzero. Since $t_*^{\mathrm{IndCoh}} s^!$ is t-exact (Proposition 3.21(2)), we see that $H^i(t_*^{\mathrm{IndCoh}} s^! \mathcal{F}) \cong t_*^{\mathrm{IndCoh}} s^! H^i(\mathcal{F})$ and so we may assume $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^*)^{\heartsuit}$. For such an $\mathcal{F}$, we have $s^!(\mathcal{F}) \in \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^{\heartsuit}$ by Proposition 3.20 and is nonzero, since for example $\Delta^! s^!(\mathcal{F}) \simeq \mathcal{F}$ for Δ the diagonal map. Thus by Corollary 9.4 we see that $t_*^{\mathrm{IndCoh}} s^!(\mathcal{F})$ is nonzero, as required. ☐

9.1.3. *Identification of Coalgebras.* Now we carry out the explicit identification of the coalgebras given by Barr-Beck and Theorem 9.1. Note that, in the notation of Section 4.5, we have:

$$t_*^{\mathrm{IndCoh}}(t^!(\omega_{\mathfrak{t}^*})) \simeq \overline{s}_* s^! \overline{s}^! (\omega_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}}) \simeq \overline{s}^! \overline{s}_*^{\mathrm{IndCoh}} \overline{s}^! (\omega_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}}) \simeq \mathrm{Av}_*^{N_r, (T_r, w)} \mathrm{Av}_!^{N_{\ell}^-, \psi} \mathrm{Av}_*^{N_{\ell}, (T_{\ell}, w)}(\delta_{\psi}) \tag{14}$$

where the subscripts ℓ and r refer to the left and right averaging, and the last step follows from Theorem 1.4. Continuing this chain of equivalences, and using the fact that left and right averaging canonically commute, we obtain:

$$t_*^{\mathrm{IndCoh}}(t^!(\omega_{\mathfrak{t}^*})) \simeq \mathrm{Av}_!^{N_{\ell}^-, \psi} \mathrm{Av}_*^{N_{\ell}, (T_{\ell}, w)} \mathrm{Av}_*^{N_r, (T_r, w)}(\delta_{\psi}) \simeq \mathrm{Av}_!^{N_{\ell}^-, \psi} \mathrm{Av}_*^{N_{\ell}, (T_{\ell}, w)}(\delta_1^{T, w}) \tag{15}$$

[5]In the notation of [GR17b], $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$.

where $\delta_1^{T,w} \in \mathcal{D}(T)^{T,w}$ refers to the skyscraper sheaf at the identity with the trivial T-representation structure, i.e. the essential image of $i_*^{\mathrm{IndCoh}}(k_{triv})$ under the functor $\mathrm{Rep}(T) \simeq \mathrm{Vect}^{T,w} \to \mathcal{D}(T)^{T,w}$. Similar analysis shows that this is an isomorphism of coalgebras, where, since the composite functors in (14) and (15) are all t-exact (using Corollary 2.42, Theorem 2.43, and Proposition 3.21), this is a property of this identfication and not additional structure.

Remark 9.7. The structure of the comonad $\mathrm{Av}_*^N\mathrm{Av}_!^{\psi}$ was previously known on the full subcategory of B-bimonodromic objects of $\mathcal{D}(N\backslash G/N)^{\heartsuit}_{\mathrm{nondeg}}$, see [Bez16, Section 5.1].

9.1.4. *Identification of Compact Objects.* Note that we have seen in Corollary 5.42 that $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}}$ has a canonical set of compact generators labeled by $\tilde{W}^{\mathrm{aff}}$ given by the set $\{\delta_{\mathcal{D}}w : w \in \tilde{W}^{\mathrm{aff}}\}$, where $\delta_{\mathcal{D}}$ denotes the monoidal unit of $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}}$. We obtain a similar description for $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ and use it to complete the proof of the non-monoidal version of Theorem 1.13:

Proposition 9.8. The category $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ has a canonical set of compact generators given by $\{\delta w : w \in \tilde{W}^{\mathrm{aff}}\}$, where $\delta := i_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*})$ and $i : \mathfrak{t}^* \hookrightarrow \mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$ the diagonal map, so that δ is the monoidal unit.

Proof. These objects are compact since the IndCoh pushforward by a closed embedding is a left adjoint with a continuous right adjoint, and thus preserves compact objects. We now show this set generates; fix a nonzero $\mathcal{F} \in \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$. There exists some finite subset $S \subseteq \tilde{W}^{\mathrm{aff}}$ so that $i_S^!(\mathcal{F})$ is nonzero. Note also that that for each finite $S \subseteq \tilde{W}^{\mathrm{aff}}$, the map $\coprod_{w\in S} \mathfrak{t}^* \to \Gamma_S$ is surjective at the level of geometric points, and so in particular, by [GR17a, Proposition 6.2.2], there exists some $w \in \tilde{W}^{\mathrm{aff}}$ such that $i_w^!(\mathcal{F})$ is nonzero. Therefore $\underline{\mathrm{Hom}}(\omega_{\mathfrak{t}^*}, i_w^!(\mathcal{F})) \simeq \underline{\mathrm{Hom}}(i_{w,*}^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}), \mathcal{F}) \simeq \underline{\mathrm{Hom}}(\delta w, \mathcal{F})$ is nonzero. ☐

Proof of Non-Monoidal Version of Theorem 1.13. We have seen in Proposition 9.8 and Corollary 5.42 that both categories $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ and $\mathcal{D}(N\backslash G/N)^{T\times T,w}$ are generated by the objects δw, where δ is the monoidal unit in the respective category and $w \in \tilde{W}^{\mathrm{aff}}$. Each of the functors $t_*^{\mathrm{IndCoh,enh}}$ and $\mathrm{Av}_!^{\psi,\mathrm{enh}}$ sends the monoidal unit to equivalent comodules, as we have seen in Section 9.1.3. Therefore, because the functors $t_*^{\mathrm{IndCoh,enh}}$ and $\mathrm{Av}_!^{\psi,\mathrm{enh}}$ are both $\tilde{W}^{\mathrm{aff}}$-equivariant, we see that these functors identify a collection of compact generators. Thus these functors identify the (not necessarily cocomplete) subcategories of the compact objects of $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ and the compact objects of $\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}}$, since both categories may be identified as the objects Karoubi-generated by the compact generators [DG, Corollary 1.4.6]. Both categories are compactly generated and thus may be identified with the ind-completion of their compact objects [DG, Corollary 1.3.4], and therefore are equivalent as DG categories. ☐

9.2. **Monoidality.** Let $\mathcal{E}$ denote the category $\underline{\mathrm{End}}_{\mathrm{IndCoh}(\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}})}(\mathrm{IndCoh}(\mathfrak{t}^*))$ (or, equivalently by Theorem 1.4, $\underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathcal{D}(N_\psi^-\backslash G/N)^{T,w})$. Note that we have a canonical, monoidal functor

$$F_I : \mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \to \mathcal{E}$$

given by the formalism of convolution [GR17a, Chapter 5.5]. We similarly have a monoidal functor $F_{\mathcal{D}}$ given by the composite

$$\mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}} \simeq \underline{\mathrm{End}}_G(\mathcal{D}(G/N)^{T,w}_{\mathrm{nondeg}}) \to \underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathcal{D}(N_\psi^-\backslash G/N)^{T,w}) \simeq \mathcal{E}$$

where the left equivalence is given by Theorem 2.10 and Remark 2.11, the right arrow is given by tensoring with the $\mathcal{D}(G), \mathcal{H}_\psi$ bimodule $\mathcal{D}(N_\psi^-\backslash G)$, and the right equivalence is given again by Theorem 1.4.

We will give our equivalence of categories $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^*/\!\!/\tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \xrightarrow{\sim} \mathcal{D}(N\backslash G/N)^{(T\times T,\text{ w})}_{\mathrm{nondeg}}$ monoidal structure by relating both categories to the category $\mathcal{E}$. We first give some general results on computing limits, and then in Section 9.2.3 we show that the evaluation functor $\mathcal{E} \to \mathrm{IndCoh}(\mathfrak{t}^*)$ is comonadic. Using this, we show how to equip the equivalence of Theorem 1.13 with a monoidal structure in Section 9.2.4.

9.2.1. *Reminders on Lax Limits.* We summarize the following useful proposition on computing limits in a limit of categories in DGCat, see [AG18, Section 4.1] for more information. Assume $I \to \mathrm{DGCat}$ is a diagram of categories for some index ∞-category I, which we denote $i \mapsto \mathcal{C}_i$, and let $\lim \mathcal{C}_i$ denote a limit category. We recall the notion of a *lax-limit category* lax-lim $\mathcal{C}_i$, which is defined using co-Cartesian fibrations and, in particular, whose objects consist of objects $\mathcal{F}_i \in \mathcal{C}_i$ for each $i \in I$ and for every map $i_1 \xrightarrow{\alpha} i_2$ in I, the corresponding map $\Phi_\alpha(\mathcal{F}_{i_1}) \to \mathcal{F}_{i_2}$.

Proposition 9.9. Assume $I \to \mathrm{DGCat}, i \mapsto \mathcal{C}_i$ is defined as above.

- [AG18, Section 4.1.1] There is a natural, fully faithful functor $\lim \mathcal{C}_i \hookrightarrow \text{lax-lim } \mathcal{C}_i$, and an object is in the essential image if and only if the associated maps $\Phi_\alpha(\mathcal{F}_{i_1}) \to \mathcal{F}_{i_2}$ are equivalences for all α.
- [AG18, Section 4.1.8] For each $i \in I$, the natural evaluation functor $\mathrm{ev}_i : \text{lax-lim } \mathcal{C}_i \to \mathcal{C}_i$ admits a left adjoint, and in particular commutes with limits.

Corollary 9.10. Assume we are given a diagram $J \to \lim {}_i\mathcal{C}_i$, which we write $j \mapsto \mathcal{F}_{j,i} \in \mathcal{C}_i$, such that for each j and for each map $i_1 \xrightarrow{\alpha} i_2$ in I, the corresponding map $\Phi_\alpha(\mathcal{F}_{j,i_1}) \to \mathcal{F}_{j,i_2}$ is an equivalence. Then the corresponding limit is computed termwise.

Proof. The condition that each corresponding map $\Phi_\alpha(\mathcal{F}_{j,i_1}) \to \mathcal{F}_{j,i_2}$ is an equivalence implies that the limit over our J-shaped diagram, computed in the category $\text{lax-lim } \mathcal{C}_i$, lies in the category $\lim \mathcal{C}_i$. Since the evaluation functor is a right adjoint, it commutes with limits, thus giving our claim. ☐

9.2.2. *Nilpotent Towers and Effective Limits.* In this section, we recall the DG-analogue of ideas of Akhil Mathew (see, for example, [Mat18, Subsection 2.3]) which will be used later. For this subsection, fix two DG categories $\mathcal{C}, \mathcal{D}$.

Definition 9.11. Assume we are given a *tower* in $\mathcal{C}$, or, equivalently, a sequence $... \to \mathcal{F}^2 \to \mathcal{F}^1 \to F^0$ in $\mathcal{C}$. We say this tower is *weakly nilpotent* if for all $n \in \mathbb{N}^{\geq 0}$ there exists an N such that for all $m \geq N$, the natural map $\mathcal{F}^{m+n} \to \mathcal{F}^n$ is nullhomotopic.

Definition 9.12. Let $\mathcal{C}$ be some DG category (or, more generally, any stable ∞-category, and fix some $\mathcal{F} \in \mathcal{C}$.

(1) Let $\mathcal{F}_\bullet := (... \to \mathcal{F}_1 \to \mathcal{F}_0)$ be a tower in $\mathcal{C}$, and let $\underline{\mathcal{F}}$ denote the constant tower. We say the map of towers $\underline{\mathcal{F}} \to \mathcal{F}_\bullet$ forms an *effective limit* (or, more informally, the maps $\mathcal{F} \to \mathcal{F}_\bullet$ *form an effective limit*) if the tower $n \mapsto \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent.
(2) Let $S^\bullet$ denote some cosimplicial object of a category $\mathcal{C}$ and temporarily denote by $\mathcal{F}^\bullet$ the constant cosimplicial object. We say the map of cosimplicial objects $\mathcal{F}^\bullet \to S^\bullet$ (or, more informally, the maps $\mathcal{F} \to S^\bullet$) *form an effective limit* if the maps $\mathcal{F} \to \mathrm{Tot}^{\leq n}(S^\bullet)$ form an effective limit.

Remark 9.13. By definition, a tower in $\mathcal{C}$ is an object of the $(\infty, 1)$-category of functors $\mathrm{Fun}(\mathbb{Z}^{op}_{\geq 0}, \mathcal{C})$. Since colimits in functor categories are computed termwise, the cokernel of a map of towers is the tower of cokernels. Note also that if the tower $n \mapsto \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent, then its limit is zero.

We therefore see that, if the tower $n \mapsto \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_n)$ is weakly nilpotent, the canonical map $\mathcal{F} \simeq \lim(\underline{\mathcal{F}}) \to \lim_n \mathcal{F}_n$ is an equivalence, so the term 'effective limit' is justified. By abuse of notation, we sometimes say that the maps $\mathcal{F} \to \mathcal{F}^i$ form an effective limit.

We now record a basic property of effective limits, see [Mat18, Proposition 2.20]:

Lemma 9.14. Let $F : \mathcal{C} \to \mathcal{D}$ is some exact functor of stable ∞-categories (which is always satisfied if F is a map in $\mathrm{DGCat}^k_{\mathrm{cont}}$), and let $\mathcal{F} \to \mathcal{F}^i$ be a compatible family of maps as in Definition 9.12. Then if the maps $\mathcal{F} \to \mathcal{F}^i$ form an effective limit in $\mathcal{C}$, then the maps $F(\mathcal{F}) \to F(\mathcal{F}^i)$ form an effective limit in $\mathcal{D}$.

Proof. By definition of effective limit, the tower of cofibers given by $C_i := \mathrm{cofib}(\mathcal{F} \to \mathcal{F}_i)$ is weakly nilpotent. By the definition of exactness, F commutes with finite colimits, so that $F(C_i) \simeq \mathrm{cofib}(F(\mathcal{F}) \to F(\mathcal{F}_i))$. Therefore, since F preserves the class of maps which are equivalent to the 0 map, our claim follows, since exact functors preserve the zero object. ☐

9.2.3. *An Intermediate Comonadic Category.* This subsection will be devoted to the proof of the following Proposition:

Proposition 9.15. The functor $\mathrm{oblv}^{\mathcal{H}_\psi} : \underline{\mathrm{End}}_{\mathcal{H}_\psi}(\mathrm{IndCoh}(\mathfrak{t}^*)) \to \underline{\mathrm{End}}(\mathrm{IndCoh}(\mathfrak{t}^*))$ is comonadic.

We prove this after showing the following lemma and deducing a corollary from it.

Lemma 9.16. The maps $\mathrm{id}_{\mathrm{IndCoh}(\mathfrak{t}^*)} \to (\overline{s}^! \overline{s}_*^{\mathrm{IndCoh}})^{\bullet+1}$ form an effective limit.

Proof. Using the identification $\text{ev}_{\omega_{\mathfrak{t}^*}} : \underline{\text{End}}(\text{IndCoh}(\mathfrak{t}^*)) \xrightarrow{\sim} \text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)$ this claim is equivalent to the claim that the maps $\omega_{\mathfrak{t}^*} \to (\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ form an effective limit. Let $c_n : \omega_{\mathfrak{t}^*} \to \text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ denote the canonical map for each n. We claim that for each n, we have that $\tau^{\leq n-1}c_n$ is an equivalence. To see this, note that by the conservativity of $\overline{s}_*^{\text{IndCoh}}$ (Corollary 9.6) it suffices to show that $\overline{s}_*^{\text{IndCoh}}\tau^{\leq n-1}c_n$ is an equivalence. The t-exactness of $\overline{s}_*^{\text{IndCoh}}$ (Proposition 3.20) allows us to identify this map with $\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}c_n$. Since $\overline{s}_*^{\text{IndCoh}}$ is exact, it commutes with finite limits, so we may furthermore identify $\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}c_n$ with the map

$$\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}\omega_{\mathfrak{t}^*} \to \tau^{\leq n-1}\text{Tot}^{\leq n}\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$$

and by Lemma 8.5 we may further identify this map with the canonical map

$$\tau^{\leq n-1}\overline{s}_*^{\text{IndCoh}}\omega_{\mathfrak{t}^*} \to \tau^{\leq n-1}\text{Tot}(\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*}))$$

using the fact that $(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{j+1}(\omega_{\mathfrak{t}^*})$ lies in the heart for every $j \in \mathbb{Z}^{\geq 0}$, see Proposition 3.21(2). However, this map is an equivalence since the cosimplicial object $\overline{s}_*^{\text{IndCoh}}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ is split by $\overline{s}_*^{\text{IndCoh}}$. We therefore see that $\tau^{\leq n-1}c_n$ is an equivalence.

We also have that $\text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ is a totalization of objects in the heart of a category equivalent to A-mod for some classical ring A, again using the exactness of Proposition 3.21(2). We thus see see $\text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}(\omega_{\mathfrak{t}^*})$ lies in cohomological degree $[0, n]$. Therefore, if K^n denotes the cofiber of the map c_n, this cofiber is concentrated in degree n since $\tau^{\leq n-1}c_n$ is an equivalence. In particular, we may choose $N \gg 0$ so that the space $\text{Hom}_{\text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)}(K^{N+n}, K^n)$ is connected (by the finite cohomological dimension of the t-structure on $\text{IndCoh}(\mathfrak{t}^* \times \mathfrak{t}^*)$), so the maps from the identity to the tower of partial totalizations of our cosimplicial object form an effective limit by definition. ☐

Corollary 9.17. For any $i \in \mathbb{Z}^{\geq 1}$, the maps $\text{id} \to (\text{id}_{\mathcal{H}_\psi^{\otimes i-1}} \otimes \overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ form an effective limit.

Proof. First note that the functor

$$\text{id}_{\mathcal{H}_\psi^{\otimes i}} \otimes - : \underline{\text{End}}(\text{IndCoh}(\mathfrak{t}^*)) \to \underline{\text{End}}(\mathcal{H}_\psi^{\otimes i-1} \otimes \text{IndCoh}(\mathfrak{t}^*))$$

is exact since it is continuous [Lur17][Proposition 1.1.4.1]. Thus this functor preserves effective limits by Lemma 9.14, so we see that the maps

$$\text{id} \to \text{id}_{\mathcal{H}_\psi^{i-1}} \otimes \text{Tot}^{\leq n}(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$$

form an effective limit. Our claim then follows from the fact that the functor $\text{id}_{\mathcal{H}_\psi^{i-1}} \otimes -$ is exact and thus commutes with finite limits. ☐

Proof of Proposition 9.15. We identify $\underline{\text{End}}_{\mathcal{H}_\psi}(\text{IndCoh}(\mathfrak{t}^*))$ as the limit $\lim_\Delta(\underline{\text{Hom}}(\mathcal{H}_\psi^\bullet \otimes \text{IndCoh}(\mathfrak{t}^*), \text{IndCoh}(\mathfrak{t}^*)))$. Let $F_\bullet$ denote an object of this limit category, and let I denote the inclusion into the lax-limit category $\text{lax-lim}_\Delta\underline{\text{Hom}}(\mathcal{H}_\psi^\bullet \otimes \text{IndCoh}(\mathfrak{t}^*), \text{IndCoh}(\mathfrak{t}^*))$. We compute the limit $\lim_\Delta I(F(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1})$ and show it lies in the subcategory $\lim_\Delta\underline{\text{Hom}}(\mathcal{H}_\psi^\bullet \otimes \text{IndCoh}(\mathfrak{t}^*), \text{IndCoh}(\mathfrak{t}^*))$. Fix some map in Δ, say $i_1 \xrightarrow{\alpha} i_2$, and let

$$\Phi_\alpha : \underline{\text{Hom}}(\mathcal{H}_\psi^{i_1} \otimes \text{IndCoh}(\mathfrak{t}^*), \text{IndCoh}(\mathfrak{t}^*)) \to \underline{\text{Hom}}(\mathcal{H}_\psi^{i_2} \otimes \text{IndCoh}(\mathfrak{t}^*), \text{IndCoh}(\mathfrak{t}^*))$$

denote the canonical map obtained by pullback. Since this map is exact and the maps $\text{id} \to (\text{id}_{\mathcal{H}_\psi} \otimes \overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}$ form an effective limit (Corollary 9.17), by Lemma 9.14 we obtain a canonical equivalence

$$\Phi_\alpha \lim_\Delta IF_{\bullet,i_1} \xrightarrow{\sim} \lim_\Delta \Phi_\alpha IF_{\bullet,i_1}$$

where we denote $IF_{\bullet,i_1}$ the object $\text{ev}_{i_1}(IF_{\bullet,i_1})$. We obtain an equivalence

$$\lim_\Delta \Phi_\alpha IF_{\bullet,i_1} \xrightarrow{\sim} \lim_\Delta IF_{\bullet,i_2}$$

since $F_{\bullet,i_1}$ lies in the limit category. Thus by Proposition 9.9 we see that the canonical map

$$I(\lim_\Delta F(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1}) \to \lim_\Delta I(F(\overline{s}^! \overline{s}_*^{\text{IndCoh}})^{\bullet+1})$$

is an equivalence. In particular, this limit is computed termwise by Proposition 9.9(2). Thus we see that our left adjoint $ev_0 \simeq \text{oblv}^{\mathcal{H}_\psi}$ satisfies Theorem 8.3(3), and thus the Barr-Beck-Lurie theorem gives the desired comonadicity. ☐

9.2.4. *Identification of Monoidal Categories.* In this section, we finish the proof of Theorem 1.13. Note that we now have:

Theorem 9.18. If S denotes the coalgebra $\mathrm{Av}_!^{\psi}\mathrm{Av}_*^N(\omega_{\mathfrak{t}^*}) \simeq \overline{s}_*^{\mathrm{IndCoh}}\overline{s}^!(\omega_{\mathfrak{t}^*})$, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+ \arrow[r, "\pi_*^{\mathrm{IndCoh,enh}}"] \arrow[d] & S\text{-Comod}(\mathrm{IndCoh}(\mathfrak{t}^*)^+) \arrow[d] & \mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w}),\ +}_{\mathrm{nondeg}} \arrow[l, "\mathrm{Av}_!^{\psi,\mathrm{enh}}"'] \arrow[d] \\
\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \arrow[r, "F_I"] & \mathcal{E} & \mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w})}_{\mathrm{nondeg}} \arrow[l, "F_{\mathcal{D}}"']
\end{tikzcd}$$

where the middle arrow is given by the composite $S\text{-Comod}(\mathrm{IndCoh}(\mathfrak{t}^*)^+) \hookrightarrow S\text{-Comod}(\mathrm{IndCoh}(\mathfrak{t}^*)) \xleftarrow{\sim} \mathcal{E}$ (see Proposition 9.15) and the other the vertical arrows are the obvious inclusions.

Note that each of F_I and $F_{\mathcal{D}}$ are monoidal, and we have seen that the top two arrows are comonadic. We further claim that the categories $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+$ and $\mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w}),\ +}_{\mathrm{nondeg}}$ are monoidal subcategories. This directly follows from the following proposition:

Proposition 9.19. The convolution structure on $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ and $\mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w})}_{\mathrm{nondeg}}$ preserve the respective eventually coconnective subcategories.

Proof. Convolution is given by the pullback by a closed embedding and the pushforward of $(\mathrm{id}, \overline{s}, \mathrm{id})$: $\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^* \to \mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}} \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*$. We have seen that the latter is exact by Proposition 3.20, and the former is exact since the pullback by a closed embedding is a right adjoint to a t-exact functor and therefore left t-exact. Thus convolution preserves the eventually coconnective subcategory $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)^+$.

For $\mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w})}_{\mathrm{nondeg}}$, we note that the convolution is given by the composite of a forgetful functor and an averaging functor, which are t-exact and left t-exact (again as the right adjoint of a right t-exact functor is left t-exact) respectively, so the claim follows. □

We have therefore constructed a monoidal equivalence of the respective eventually coconnective subcategories. Since the convolution structures commute with colimits, all compact objects are eventually coconnective, and our categories $\mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w})}_{\mathrm{nondeg}}$ and $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*)$ are compactly generated, we in particular can equip our equivalence $\mathrm{IndCoh}(\mathfrak{t}^* \times_{\mathfrak{t}^* /\!/ \tilde{W}^{\mathrm{aff}}} \mathfrak{t}^*) \xrightarrow{\sim} \mathcal{D}(N\backslash G/N)^{(T\times T,\ \mathrm{w})}_{\mathrm{nondeg}}$ with a monoidal structure, thus completing the proof of monoidality and thus the proof of Theorem 1.13. □

10. The Nondegenerate Horocycle Functor

We consider the category $\mathcal{D}(G \times G)$ as a right $G \times G$ category and let Ψ denote the composite functor

$$\mathcal{D}(G \times G)^{\Delta_G} \xrightarrow{\mathrm{oblv}^{\Delta_G}_{\Delta_B}} \mathcal{D}(G \times G)^{\Delta_B} \xrightarrow{\mathrm{Av}_*^{N\times N}} \mathcal{D}(G/N \times G/N)^{\Delta_T}$$

where the group Δ_G denotes the diagonal copy of G and the rightmost functor is induced by the averaging functor of Theorem 2.10. Let $J^!$ denote the quotient functor $\mathcal{D}(G/N \times G/N) \to \mathcal{D}(G/N \times G/N)_{\mathrm{nondeg}}$ which projects onto the nondegenerate subcategory, again taken with respect to the right action. Since this nondegenerate category is closed under the action of $T \times T$, we may equivalently view $J^!$ as a functor $\mathcal{D}(G/N \times G/N)^{\Delta_T} \to \mathcal{D}(G/N \times G/N)^{\Delta_T}_{\mathrm{nondeg}}$.

Theorem 10.1. Fix a simple coroot α. With the above notation, we have the following:

(1) The functor $J^!\Psi : \mathcal{D}(G \times G)^{\Delta_G} \to \mathcal{D}(G/N \times G/N)^{\Delta_T}$ lifts to a functor
$$\tilde{\Psi} : \mathcal{D}(G \times G)^{\Delta_G} \to \mathcal{D}(G/N \times G/N)^{\Delta_T \rtimes W}_{\mathrm{nondeg}}.$$

(2) The action of the Klein four group $\langle s_\alpha \times s_\alpha \rangle$ on $\mathcal{D}(G/N \times G/N)^{\Delta_T \mathbb{G}_m^\alpha}_{\mathrm{nondeg}}$ is trivial.

(3) The composite

$$\mathcal{D}(G\times G)^{\Delta_G} \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\mathrm{oblv}^W_{\langle s_\alpha\rangle}\tilde{\Psi}} \mathcal{D}(G/N\times G/N)^{\Delta_T\mathbb{G}_m^\alpha,\langle s_\alpha\rangle}_{\mathrm{nondeg}} \simeq \mathcal{D}(G/N\times G/N)^{\Delta_T\mathbb{G}_m^\alpha}_{\mathrm{nondeg}}\otimes \mathrm{Rep}\langle s_\alpha\rangle \tag{16}$$

where the second equivalence is given by (2), lies entirely in the summand indexed by the trivial representation.

(4) If $\mathcal{F}\in\mathcal{D}(G)^G\simeq\mathcal{D}(G\times G)^{\Delta_G^\ell\times\Delta_G^r}$ is very central, then the sheaf

$$\tilde{\Psi}(\mathcal{F})\in\mathcal{D}(N\backslash G/N)^{\Delta_{T\rtimes W}}_{\mathrm{nondeg}}\simeq\mathcal{D}((G/N\times G/N)/T)^{\Delta_G}$$

has the property that the canonical $\langle s_\alpha\rangle$-representation on $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\tilde{\Psi}(\mathcal{F})$ is trivial.

The final point of Theorem 10.1 should be compared to the condition of satisfying Coxeter descent given by the final point of Proposition 4.23. Note that we may identify $\mathrm{hc}:=\Psi^{\Delta_G^\ell}$.

Proof. Note that the functor Ψ itself is G-equivariant and by Remark 2.11 it suffices to construct a lift of $\mathrm{hc}(\delta_{\Delta_G})$. However, hc is monoidal. This fact as well known, and can be seen by identifying the monoidal functor

$$\mathcal{D}(G)^G\simeq\underline{\mathrm{End}}_{G\times G}(\mathcal{D}(G))\xrightarrow{\mathrm{Av}_*^N}\underline{\mathrm{End}}_{G\times T}(\mathcal{D}(G/N))\simeq\mathcal{D}(N\backslash G/N)^T \tag{17}$$

with hc. Therefore, we see that we may identify $\Psi(\delta_{\Delta_G})$ (with its left Δ_G-equivariance) in the category $\mathcal{D}(N\backslash G/N)^T$ with the monoidal unit equipped with its canonical T-equivariance. In particular, $J^!\Psi(\delta_{\Delta_G})$ may be identified with $J^!(\delta_1)$, where $\delta_1\in\mathcal{D}(N\backslash G/N)^T$ is the monoidal unit. Under the equivalence Theorem 1.13, the sheaf $J^!(\delta_1)$ corresponds to the pushforward $\Delta_*^{\mathrm{IndCoh}}(\omega_{\mathfrak{t}^*}/X^\bullet(T))$. In particular, this sheaf is equivariant with respect to the diagonal W-action. Thus we see that $J^!\Psi(\delta_{\Delta_G})\in\mathcal{D}(G/N\times G/N)^{T,\Delta_G}_{\mathrm{nondeg}}\simeq\mathcal{D}(N\backslash G/N)^T_{\mathrm{nondeg}}$ may be equipped with a canonical W-equivariant structure, showing (1). Point (2) follows directly from Corollary 5.45.

To show (3), note that we have identifications

$$\mathrm{oblv}^W_{\langle s_\alpha\rangle}J^!\Psi(\delta_{\Delta_G})\simeq\mathrm{oblv}^W_{\langle s_\alpha\rangle}J^!\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\Psi(\delta_{\Delta_G}) \tag{18}$$

since the quotient functor is $T\times T$-equivariant. Under the Mellin transform, the sheaf $\mathrm{Av}_*^{\mathbb{G}_m^\alpha}\Psi(\delta_{\Delta_G})\in\mathcal{D}(T/\mathbb{G}_m^\alpha)$ corresponds to the monoidal unit. In particular, (18) gives that the integral kernel of the $G\times G$-equivariant functor given by (16) lies in the full $G\times G$-subcategory indexed by the trivial representation, establishing (3). Finally, (4) is a special case of (3). □

We conclude by recording expectations of how these results may be extended:

Remark 10.2. We expect that similar methods yield equivalences of categories

$$\mathfrak{g}\text{-mod}^N_{\mathrm{nondeg}}\simeq\mathrm{IndCoh}(\mathfrak{t}^*/\!\!/W\times_{\mathfrak{t}^*/\!\!/W^{\mathrm{aff}}}\mathfrak{t}^*/X^\bullet(T))$$

compatible with the T action and the $\mathrm{IndCoh}(\mathfrak{t}^*/\!\!/W)$ action and a monoidal equivalence

$$\mathcal{HC}_{\mathrm{nondeg}}:=\underline{\mathrm{End}}_G(\mathcal{D}(G)^{G,w}_{\mathrm{nondeg}})\simeq\mathrm{IndCoh}(\mathfrak{t}^*/\!\!/W\times_{\mathfrak{t}^*/\!\!/W^{\mathrm{aff}}}\mathfrak{t}^*/\!\!/W).$$

Remark 10.3. Note that the integral kernel of the composite $ch\circ hc$ on the level of nondegenerate categories canonically acquires a W-representation structure, and the sheaf $ch\circ hc(\delta)$ is known to be the Springer sheaf. This sheaf has endomorphisms which may be identified with the group ring of W–for a recent survey of this, see [BZCHN21]. Therefore we expect that, at least on the level of nondegenerate categories, the functor $\tilde{hc}:\mathcal{D}(G)^G_{\mathrm{nondeg}}\to\mathcal{D}(N\backslash G/N)^{T\rtimes W}_{\mathrm{nondeg}}$ is fully faithful.

Of course, one did not need to pass to nondegenerate categories to obtain that the composite $ch\circ hc$ is given by convolution with the Springer sheaf. Therefore, one might hope that the functor hc factors through some subcategory $\mathcal{D}\hookrightarrow\mathcal{D}(N\backslash G/N)^T$ which acquires a W-action, giving rise to a fully faithful functor $\tilde{hc}:\mathcal{D}(G)^G\hookrightarrow\mathcal{D}^W$. We do not yet know what to make of this.

Appendix A. Union of Graphs of Closed Subsets of the Affine Weyl Group

The main result of this section is Corollary A.21, which states that the union of a finite closed subset of W^{aff} (see Definition A.1) is flat over the source. To do this, we compute an extension of the Borel isomorphism to the union of graphs of a closed subset of W in Appendix A.1. Then, in Appendix A.2, we prove Corollary A.21 and review some basic notions on the coinvariant algebra.

A.1. **Generalization of the Borel Isomorphism.** A celebrated theorem of Borel identifies the cohomology of the flag variety $H^*(G^\vee/B^\vee)$ with the coinvariant algebra $C := \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})^W$. Given a $v \in W$, one can consider the (closed) Schubert variety $X_v \hookrightarrow G^\vee/B^\vee$. The functoriality of cohomology implies that there is a surjective ring map $H^*(G^\vee/B^\vee) \to H^*(X_v)$. We wish to determine a similar isomorphism for the graded ring $H^*(X_v)$. In fact, we will generalize this theorem to the cohomology of unions of Schubert cells $H^*(X_S)$ determined by *closed* subsets of the Weyl group:

Definition A.1. If $\tilde{W}$ is a Coxeter group, we say a subset $S \subseteq \tilde{W}$ is *closed* if $w \in S$ and $w' \leq w$ implies $w' \in S$.

To state our desired extension to closed subsets of the Weyl group, we first obtain the following alternate description of C. Consider the scheme $\mathfrak{t}^* \times \mathfrak{t}^*$, and let graph$(w)$ denote the closed subscheme cut out by the ideal $I_{\mathrm{graph}(w)}$, defined in turn to be the ideal generated by elements of the form $wp \otimes 1 - 1 \otimes p$ for $p \in \mathrm{Sym}(\mathfrak{t})$. Set $I_W := \cap_{w \in W} I_{\mathrm{graph}(w)}$, and set J_W to be the ideal generated by I_W and $\mathrm{Sym}(\mathfrak{t})_+ \otimes \mathrm{Sym}(\mathfrak{t})$. Similarly, assume we are given a closed subset $S \subseteq W$. Set $I_S := \cap_{w \in S} I_{\mathrm{graph}(w)}$, and set J_S to be the ideal generated by I_S and $\mathrm{Sym}(\mathfrak{t})_+ \otimes \mathrm{Sym}(\mathfrak{t})$. Note that we obtain a canonical map

$$\Phi : \mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t}) \to \mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/I_W$$

We now prove the following proposition, which says that the product $\mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ W} \mathfrak{t}^*$ may be identified with the union of graphs of W:

Proposition A.2. The map Φ is an isomorphism.

Proof. We note that Φ is in particular a map of $\mathrm{Sym}(\mathfrak{t})$-modules. We see that by base changing the quotient map $\mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ W$ by itself, by Remark 3.8 the $\mathrm{Sym}(\mathfrak{t})$-module $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t})$ is finite flat. The fact that $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t})$ is finite implies that $\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/I_W$ is also finite as a $\mathrm{Sym}(\mathfrak{t})$-module. We also see that W acts freely on a dense open subset of $\mathfrak{t}^*$, so that for a dense open subset of $\mathfrak{t}^*$, the fiber of the map $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/I_W) \to \mathfrak{t}^*$ has degree W. This in particular implies that all fibers of the $\mathrm{Sym}(\mathfrak{t})$-module $\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/I_W$ have rank no less than $|W|$, since degree of a finite morphism is a upper semicontinuous function on the target. However, we also see that each fiber at some point x admits a surjection from the fiber of $\mathrm{Sym}(\mathfrak{t}) \otimes_{\mathrm{Sym}(\mathfrak{t})^W} \mathrm{Sym}(\mathfrak{t})$ at the point, which has rank precisely $|W|$ since $\mathrm{Sym}(\mathfrak{t})$ is a free rank $\mathrm{Sym}(\mathfrak{t})^W$-module of rank W by Remark 3.8. □

Remark A.3. We temporarily assume that $G_{\mathbb{Z}}$ is an adjoint type Chevalley group scheme defined over the integers with maximal torus $T_{\mathbb{Z}}$. An identical result to Proposition A.2 with a similar proof (which will not be used below) also holds for the action of the Weyl group W on the torus $T_{\mathbb{Z}}$. The analogue of Remark 3.8 is the Pittie-Steinberg theorem [Ste75], which says that for such $G_{\mathbb{Z}}$ that $\mathcal{O}(T_{\mathbb{Z}}^\vee)$ is a free $\mathcal{O}(T_{\mathbb{Z}}^\vee)^W$-module of rank $|W|$.

Remark A.4. Note that we work with the union of graphs given by the intersection of ideals, not the product. For example, if $\mathfrak{g} = \mathfrak{sl}_3$, we may pick a simple reflection $s \in W$ and choose coordinates on $\mathfrak{t}^*$ so that $\mathfrak{t}^* \simeq \mathrm{Spec}(k[h,p])$ where $s(h) = -h$ and $s(p) = p$. The intersection $I_1 \cap I_s$ contains the degree 1 polynomial $p \otimes 1 - 1 \otimes p$, whereas the product $I_1 I_s$ is generated by degree two polynomials.

In particular, Proposition A.2 shows that one can identify $C \cong \mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_W$. For a closed subset $S \subseteq W$, let X_S denote the closed subvariety of $G^\vee/B^\vee$ given by the union of the Schubert cells labelled by $w \in S$. We now show the following:

Proposition A.5. Fix a closed subset $S \subseteq W$. There is an isomorphism $H^*(X_S) \simeq \mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_S$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})/J_W \arrow[r, "\sim"] \arrow[d] & H^*(G^\vee/B^\vee) \arrow[d] \\
\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})/J_S \arrow[r, "\sim"'] & H^*(X_S)
\end{tikzcd}$$

where the vertical maps are the canonical quotient maps and the top arrow is the Borel isomorphism.

Remark A.6. After this appendix was written, we learned of the paper [Car92] which, in Theorem 4.3 and Corollary 4.4, which proves a similar result to Proposition A.5. The proof below shows that many of the methods and results of [Car92] and [BGG73b] need not require a choice of regular element of $\mathfrak{t}(\mathbb{Q})$.

Remark A.7. An alternate description of the rings $\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})/J_S$ in the case $G = \mathrm{GL}_n$ and S is the closure of some Weyl group element is computed in [ALP92]. We thank Victor Reiner for making us aware of this reference.

A.1.1. *Results on Demazure Operators.* We first will recall some definitions and results of [BGG73a] and [BGG73b].

Definition A.8. For a simple reflection $s \in W$ associated to a coroot α, define the vector space map $D_s : \mathrm{Sym}(\mathfrak{t}) \to \mathrm{Sym}(\mathfrak{t})$ via $D_s(f) := \frac{f-sf}{\alpha}$. For a $w \in W$, choose a reduced expression $w = s_1...s_r$ and set $D_w := D_{s_1}...D_{s_r}$. These D_w are known as *Demazure operators.*

Let w_0 denote the longest element of the Weyl group W with repsect to some ordering, and let $\ell := \ell(w_0)$. We now recall the following theorem:

Theorem A.9. [BGG73b] We have the following:

(1) The Demazure operators are well defined and independent of reduced expression.
(2) If $s_1, ..., s_p$ is not a simple expression, then $D_{s_1}...D_{s_p}$ vanishes.
(3) The Poincaré dual class to the Schubert variety $[X_1] \in H_0(G/B)$ maps to $\rho^\ell/\ell!$ in the coinvariant algebra, and if $S \subseteq W$ is closed, the vector space $H^*(X_S)$ has a basis given by the $D_u(\rho^{\vee,\ell}/\ell!)$ for which $w_0u^{-1} \in S$.

Remark A.10. To translate between point (3) of Theorem A.9 and Theorem 3.15 in [BGG73b], we note that the vector space $H^*(X_S)$ also, in the notation of [BGG73b], has basis P_w for which $w \in S$. The notation P_w will not be used outside this remark.

A.1.2. *Proof of Proposition A.5.* In this subsection, we prove Proposition A.5. To prove Proposition A.5, we will first determine a specific element of $\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})$ which projects to a nonzero homogeneous element of degree $\ell := \ell(w_0)$ under the composite $\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t}) \twoheadrightarrow \mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})/J_W \simeq H^*(G/B)$.

Proposition A.11. There exists a polynomial $F(x, y) \in \mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})$ such that

(1) $F(x, vx) = 0$ if $v \neq w_0$,
(2) $F(x, w_0x) = \prod \gamma(x)$, where γ varies over the positive coroots, and
(3) $F(x, y) \neq 0$ in the coinvariant algebra $\mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})/J_W$.

To prove Proposition A.11, we will set the following notation, closely following the notation and proof of [BGG73b, Theorem 3.15].

Proposition A.12. There exists some polynomial $Q(x, y) \in \mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})$ of y-degree $\ell(w_0)$ for which $Q(x, wx) = 0$ for $w \neq w_0$ and such that $Q(x, w_0x)$ is generically nonvanishing.

Proof. There exists some polynomial $Q' \in \mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})$ such that $Q'(x, wx) = 0$ for $w \neq wx$ and $Q'(x, w_0x)$ generically does not vanish. Set $R_{1\times w_0}(y) := (1 \otimes \rho)^\ell/\ell!$, and for $w \neq w_0$ set $R_{1\times w} := D_{1\times w_0w^{-1}}(R_{1\times w_0})$. These give minimal degree lifts of the basis of the coinvariant algebra labeled by the Schubert cells by Theorem A.9. In particular, there exist polynomials $g_w(x, y) \in \mathrm{Sym}(\mathfrak{t}\times\mathfrak{t})^{1\times W}$ such that $Q'(x, y) = \sum_{w\in W} g_w(x, y)R_{1\times w}(y)$. Set $Q(x, y) := \sum_{w\in W} g_w(x, x)R_{1\times w}(y)$. Then since

$$Q(x, vx) := \sum_{w \in W} g_w(x, x) R_{1 \times w}(vx) = \sum_{w \in W} g_w(x, vx) R_{1 \times w}(vx) = Q'(x, vx),$$

we see that $Q(x, wx) = 0$ for $w \neq w_0$ and $Q(x, w_0 x)$ is generically nonvanishing. $\square$

Choose such a $g_w(x, y)$, $R_{1\times w}(y)$, and Q as in the proof of Proposition A.12. Unfortunately, such a Q need not satisfy condition (3) of Proposition A.11, even if Q' does. Therefore, we will need to modify our choice of Q further. To this end, choose any reduced expression $w_0 = s_{\alpha_1} ... s_{\alpha_r}$ labelled by coroots α_i. Given this decomposition, set $w_i := s_{\alpha_i} ... s_{\alpha_1}$, $v_i := s_{\alpha_{i+1}} ... s_{\alpha_r}$, $Q_i := D_{1 \times v_i} Q$, $\gamma_1 := \alpha_1$, and, for $i > 1$, we set[6] $\gamma_i := w_{i-1}^{-1}(\alpha_i)$.

Lemma A.13. For any Q satisfying the conditions of Proposition A.12 and any reduced expression for w_0, in the notation above, each polynomial Q_i has y-degree i, $Q_i(x, w_i x) \prod_{r \geq j > i} \gamma_j(x) = (-1)^{(r-i)} Q(x, w_0 x)$ and $Q_i(x, wx) = 0$ if $w \not\geq w_i$.

Proof. This proof closely follows the proof of the Lemma below [BGG73b, Theorem 3.15]; we include the details for the reader's convenience. We proceed by backward induction on i. Note that when $i = \ell(w_0)$, we see that by assumption $Q_i(x, wx) = Q(x, wx)$ so the claim follows trivially from Proposition A.12.

Now assume that the lemma has been proved for Q_i for some $i > 0$. Then we obviously have the y-degree of Q_{i-1} is $i - 1$. Furthermore, we compute that for any $w \in W$, we have

$$Q_{i-1}(x, wx) = \frac{Q_i(x, wx) - Q_i(x, s_{\alpha_{i-1}} wx)}{\alpha_i(wx)}. \tag{19}$$

In particular, if $w = w_{i-1}$, we see that by our inductive hypothesis, $Q_i(x, wx) = 0$, and furthermore that $\alpha_i(wx) = (w_{i-1}^{-1} \alpha_i)(x) = \gamma_i(x)$. Therefore, we see that in this case, we have

$$Q_{i-1}(x, wx) = -\frac{Q_i(x, s_{\alpha_{i-1}} wx)}{\gamma_i(x)}$$

and so by induction we have

$$Q_{i-1}(x, w_{i-1} x) \prod_{r \geq j > i-1} \gamma_j(x) = (-1)^{(r-i)} Q(x, w_0 x).$$

Finally, if $w \not\geq w_{i-1}$ [BGG73b, Corollary 2.6] implies that $w \not\geq w_i$ and $s_{\alpha_i} w \not\geq w_i$. By induction we see that both terms in the numerator of Eq. (19) vanish so our claim is proved. $\square$

We note the following corollary of Lemma A.13:

Corollary A.14. We have $g_{w_0}(x, x) \prod_\gamma \gamma(x) = (-1)^\ell Q(x, w_0 x)$, and furthermore for all $w \in W$, $Q(x, w_0 x)$ divides $g_w(x, x) \prod_\gamma \gamma(x)$.

Proof. The first statement is a direct application of the $i = 0$ claim of Lemma A.13. We also use it as the base case of for the second statement, which we prove by backwards induction on $\ell(w)$ for $w \in W$. Fix $w \in W$, and set $\tilde{w} := w_0 w w_0$. Apply $D_{1 \times \tilde{w}}$ to the equality $Q(x, y) := \sum_{u \in W} g_u(x, x) R_{1 \times u}(y)$ to obtain

$$D_{1 \times \tilde{w}}(Q(x, y)) = \sum_{\{u : \ell(\tilde{w} w_0 u^{-1}) = \ell(\tilde{w}) + \ell(w_0 u^{-1})\}} g_u(x, x) D_{1 \times \tilde{w} w_0 u^{-1}}(R_{1 \times w_0})(y) \tag{20}$$

where the other terms vanish by (2) of Theorem A.9. In particular, the set $\{u : \ell(\tilde{w} w_0 u^{-1}) = \ell(\tilde{w}) + \ell(w_0 u^{-1})\}$ contains a unique element of minimal length, namely $u = w_0 \tilde{w} w_0 = w$, because the element $y \in W$ of largest length for which $\ell(\tilde{w} y) = \ell(\tilde{w}) + \ell(y)$ is $y = \tilde{w}^{-1} w_0$. Multiply both sides of Eq. (20) by $\prod_\gamma \gamma(x)$. By Lemma A.13, we have that $Q(x, w_0 x)$ divides the left hand side, and by induction, we have that $Q(x, w_0 x)$ divides all terms in the right hand side except the term $g_w(x, x) R_{1 \times w}(y)$, so therefore it divides $g_w(x, x)$. $\square$

Proof of Proposition A.11. Set $F(x, y) = \frac{Q(x,y) \prod \gamma(x)}{Q(x, w_0 x)}$, which is well defined by Corollary A.14. Then by construction, F satisfies (2), and F satisfies (1) because Q does. Furthermore, since the coefficient on $R_{1 \times w_0}$ on F is $(-1)^\ell$ in the coinvariant algebra, (3) is satisfied, thus completing the proof of Proposition A.11. $\square$

[6]Note that the notation of [BGG73b, Theorem 3.15] contains a typographic error: in their version, α_1 should be replaced with α_i. This is reflected in their [BGG73b, Lemma 2.2], which is appealed to in the proof of [BGG73b, Theorem 3.15]

Corollary A.15. If $Z \subseteq W$ is a closed subset, the underlying vector space of $\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_Z$ has basis given by the images of $D_{1\times u}(F)$ for which $w_0u^{-1} \in Z$.

Proof. Consider the map $\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_W \twoheadrightarrow \mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_Z$. Since, generically in $\mathfrak{t}$, the union of $|Z|$ graphs will have $|Z|$ points lying above them, we have that $\dim_k(\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_Z) \geq |Z|$.

On the other hand, assume that $w_0u^{-1} \notin Z$. We will show that $D_{1\times u}(F)(tx', x') = 0$ for all $t \in Z$. In particular, by Proposition A.11, these elements are all linearly independent, and so by showing this we will obtain the opposite inequality $\dim_k(\mathrm{Sym}(\mathfrak{t} \times \mathfrak{t})/J_Z) \leq |Z|$. Choose a reduced word decomposition for w_0u^{-1} and of u to obtain a reduced word decomposition of $w_0 = (w_0u^{-1})u$. Apply Lemma A.13 (where, in the notation of the lemma, i denotes the length of $w_0u^{-1}, w_i = uw_0, v_i = u$, and $w = t$) to see that $D_{1\times u}(F)(x, t^{-1}x) = 0$ if $t^{-1} \ngeq uw_0$, where we make the coordinate change $x := tx'$. Since inversion preserves ordering, we see that this is equivalent to the condition that $t \ngeq w_0u^{-1}$. However, by assumption, $t \in Z$ and Z is closed, so $t \ngeq w_0u^{-1}$, and so our desired vanishing holds. □

Setting $Z = S$ in Corollary A.15, this precisely matches the description of Theorem A.9, which therefore proves Proposition A.5.

A.2. **A Corollary for Graphs of the Affine Weyl Group.** The main content of this subsection is Corollary A.21, where we show that the union of graphs of any finite closed subset of W^{aff} is finite-flat. This will be used to show that the union of *all* graphs of W^{aff} is ind-finite flat, see Proposition 3.12. We prove this after recalling some results on the integral Weyl group in Appendix A.2.1.

A.2.1. *The Integral Weyl Group.* For a fixed $x \in \mathfrak{t}^*(k)$, let $[x]$ denote the image of this k-point in the quotient $\mathfrak{t}^*/\Lambda$. The following is essentially shown in the proof of Satz 1.3 of [Jan79]; we recall some details for the convenience of the reader:

Proposition A.16. Fix some $x \in \mathfrak{t}^*(k)$. The following subgroups of W are identical:

1. The image $\overline{W_x^{\mathrm{aff}}}$ of the stabilizer W_x^{aff} of x under the W^{aff}-action on $\mathfrak{t}^*$ under the quotient map $W^{\mathrm{aff}} \to W^{\mathrm{aff}}/\mathbb{Z}\Phi \cong W$.
2. The subgroup $W_{[x]} := \{w \in W : wx - x \in \mathbb{Z}\Phi\}$.
3. The subgroup $W_{[x]}^{\bullet} := \{w \in W : w \cdot x - x \in \mathbb{Z}\Phi\}$.
4. The subgroup $W^x := \langle s_\alpha : \langle x, \alpha^\vee\rangle \in \mathbb{Z}\}$, where α varies over the set of roots Φ.

Furthermore, the group W^x is a Weyl group of the root system whose roots are $\Phi_{[x]} := \{\alpha \in \Phi : \langle x, \alpha^\vee\rangle \in \mathbb{Z}\}$.

Proof. We first show that $\overline{W_x^{\mathrm{aff}}} = W_{[x]}$. Choose a $\mathbb{Q}$-basis for k, which induces a direct sum decomposition for the $\mathbb{Q}$-vector space $\mathfrak{t}^*(k)$. In particular, we may write $x = \sum_{i=0}^d q_ix_i$, where x_0 lies in the $\mathbb{Q}$-span of the roots and the x_i for $i > 0$ do not and $q_i \in \mathbb{Q}$. Note that W preserves this direct sum decomposition.

Assume $w \in \overline{W_x^{\mathrm{aff}}}$. Then there is some $\mu \in \mathbb{Z}\Phi$ $(\tau_\mu, w) \in W^{\mathrm{aff}}$ which fixes x, where the τ_μ denotes translation by μ. Using the W-invariant decomposition above, we see that if $i > 0$, $wx_i = x_i$ and that $\tau_\mu wx_0 = x_0$. In particular, $wx - x = wx_0 - x_0 = -\mu$ lies in the root lattice, so we see $w \in W_{[x]}$. Conversely, assume that $w \in W_{[x]}$. Then because $wx - x \in \mathbb{Z}\Phi$, the direct sum decomposition above implies that $wx_i = x_i$ for all $i > 0$ and $wx_0 - x_0 = \nu$ for some $\nu \in \mathbb{Z}\Phi$. This in particular implies that $(\tau_{-\nu}, w)x = x$, so that $(\tau_{-\nu}, w) \in W_x^{\mathrm{aff}}$ and thus $w \in \overline{W_x^{\mathrm{aff}}}$.

To show $W_{[x]}^{\bullet} = W_{[x]}$, we first note that for $w \in W$ and $x \in \mathfrak{t}^*(k)$, $w \cdot x - x = wx - x + w\rho - \rho$. Therefore, our desired equality follows from the fact that $w\rho - \rho$ lies in the root lattice. In fact, this is more generally true for any $\nu \in \Lambda$ and any $w \in W$. This is because if w is a simple reflection associated to a root α because $w\nu - \nu = -\langle \alpha^\vee, \nu\rangle\alpha$, which lies in the root lattice by the definition of Λ, and follows for general $w \in W$ by writing $w = s_1...s_r$ and noting that

$$w\nu - \nu = (s_1(s_2...s_r\nu) - s_2...s_r\nu) + (s_2(s_3...s_r\nu) - s_3...s_r\nu) + ... + (s_1\nu - \nu)$$

and that the Weyl group preserves Λ.

Finally, the equality $W^x = W_{[x]}$ and the final statement are precisely the content of [Jan79, Satz 1.3]. □

Definition A.17. For a fixed $x \in \mathfrak{t}^*(k)$, we will refer to the group $W_{[x]}$ as the *integral Weyl group* associated to x.

Remark A.18. Note that we may also realize the integral Weyl group associated to x as a subgroup of W^{aff}. In other words, in the notation of Proposition A.16, we have that the quotient map induces an isomorphism $W_x^{\mathrm{aff}} \cong \overline{W_x^{\mathrm{aff}}}$. This follows because if $w \in \overline{W_x^{\mathrm{aff}}}$, there exists a unique $\mu \in \mathbb{Z}\Phi$ such that $(\tau_\mu, w)x = x$; the existence is given in the second paragraph of the proof of Proposition A.16 by taking $\mu := -\nu$, and the uniqueness follows trivially since $wx + \mu_1 = wx + \mu_2$ if and only if $\mu_1 = \mu_2$.

A.2.2. *The Coinvariant Algebra for a Stabilizer Group.* Fix an $x \in \mathfrak{t}^*(k)$ and let $[x]$ be as in Appendix A.2.1. We record the following definition.

Definition A.19. The *coinvariant algebra* $C_{[x]}$ for $W_{[x]}$ action on $\mathrm{Sym}(\mathfrak{t})$ given by reflections about the origin is defined to be the quotient $\mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})_+^{W_{[x]}}$.

Remark A.20. Note we may also define the coinvariant algebra of a given $x \in \mathfrak{t}^*(K)$ as follows. We temporarily let $\mathrm{Sym}(\mathfrak{t})_{x,+}$ denote those polynomials in $\mathrm{Sym}(\mathfrak{t})$ which vanish at x, which obtains an action of W_x^{aff}. We can therefore define the quotient $C_x := \mathrm{Sym}(\mathfrak{t})/\mathrm{Sym}(\mathfrak{t})_{x,+}^{W_x^{\mathrm{aff}}}$. Then in particular the map induced by translation by x, i.e. $\mathfrak{t}^* \xrightarrow{\tau_x} \mathfrak{t}^*$, induces an isomorphism $C_{[x]} \xleftarrow{\sim} C_x$.

A.2.3. *The Union of Graphs of a Finite Closed Subset of W^{aff} is Finite Flat.* We record the following application of Proposition A.5 to be used in Section 4.5:

Corollary A.21. Let $S \subseteq W^{\mathrm{aff}}$ denote a finite, closed subset of the affine Weyl group W^{aff}, and let $\pi_S : \Gamma_S \hookrightarrow \mathfrak{t}^* \times \mathfrak{t}^*$ denote the union of graphs of those $w \in S$, cut out by the intersection of ideals (see Remark A.4). Then the projection map onto the first factor $\Gamma_S \to \mathfrak{t}^*$ is finite flat.

We prove this after proving the following lemma:

Lemma A.22. Fix some finite subset $S \subseteq W^{\mathrm{aff}}$, and fix some $\lambda \in \mathfrak{t}^*(K)$ for K a (classical) field. Then, if $W_\lambda^{\mathrm{aff}} \leq W^{\mathrm{aff}}$ denotes the stabilizer of λ, the coproduct of inclusions induces a canonical isomorphism:

$$\Gamma_S \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \simeq \coprod_{wW_\lambda^{\mathrm{aff}} \in W^{\mathrm{aff}}/W_\lambda^{\mathrm{aff}}} (\Gamma_{S \cap wW_\lambda^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K))$$

Proof. Enumerate the W^{aff}-orbit of λ as $\{\lambda_i\}_{i \in \mathbb{N}}$, we can partition the set $S = \cup_i S_i$ where all $w_i \in S_i$ have the property that $w_i\lambda = \lambda_i$. Let Z_S denote the closed subscheme of $\mathfrak{t}^*$ given by $\bigcup_{i,j \in \mathbb{N}, i \neq j} \bigcup_{u \in S_i, v \in S_j} \{w_i = w_j\}$; since S is finite, Z_S can be expressed as a finite union of nonempty Zariski closed subsets. Furthermore, we see that λ factors through the open complement U_s of Z_S. Therefore, since $\Gamma_S \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \simeq (\Gamma_S \times_{\mathfrak{t}^*} U_S) \times_{U_S} \mathrm{Spec}(L)$ and, by definition of U_S, we have that $(\Gamma_S \times_{\mathfrak{t}^*} U_S)$ can be written as a disjoint union of subschemes indexed by each S_i, we see that our induced map is an isomorphism. □

Proof of Corollary A.21. The map π_S is trivially finite, so it remains to show that π_S is flat. A standard result in commutative algebra states that, since π_S is finite, it is enough to show that for all points x in $\mathfrak{t}$, the length of the fiber at x as an $\mathcal{O}_{\Gamma_S}$ module is independent of x. Choose some field-valued $x \in \mathfrak{t}^*(L)$. We enumerate the distinct L-points in the fiber of x, say, $(x, y_1), ..., (x, y_m) \in \Gamma_S$, and write $y_i = w_ix$ for the minimal such $w_i \in W^{\mathrm{aff}}$. Write $W^{\mathrm{aff}} = MW_x^{\mathrm{aff}}$ where M denotes the set of minimal elements of each coset in $W^{\mathrm{aff}}/W_x^{\mathrm{aff}}$. We thus have an isomorphism given by Lemma A.22

$$\Gamma_S \times_{\mathfrak{t}^*} \{x\} \cong \coprod_i (\Gamma_{w_iW_x^{\mathrm{aff}} \cap S} \times_{\mathfrak{t}^*} \{x\})$$

where i ranges over a finite index set and $w_i \in M$. Note that each $\Gamma_{w_iW_x^{\mathrm{aff}} \cap S} \times_{\mathfrak{t}^*} \{x\}$ is isomorphic via left multiplication by w_i^{-1} to the fiber $\Gamma_{S'} \times_{\mathfrak{t}^*} *$ for some subset $S' \subseteq W_x^{\mathrm{aff}}$.

Furthermore, we claim this S' is a closed subset of W_x^{aff}. To see this, recall the canonical isomorphism $W_x^{\mathrm{aff}} \cong \overline{W_x^{\mathrm{aff}}} = W^x$ given in Remark A.18 and Proposition A.16. If $u \in S'$ and $u' \leq^x u$ (where $\leq^x$ refers to the ordering on the Coxeter group W^x), we have that by [Lus94, Lemma 2.5], $w_iu' \leq w_iu$, so that the fact that S is closed gives that $u' \in S'$, and so S' is closed in W^x. Therefore we can apply Corollary A.15 to see that the total length of the fiber is $\sum_{i=1}^m |S \cap w_iW_x| = |S|$, which is independent of x. □

A.2.4. *Explicit Computations for the Affine Weyl Group.* Fix some field K/k and let $x \in \mathfrak{t}^*(K)$. This subsection uses some of the notation of Section 3.

Lemma A.23. The multiplication map induces a left W^{aff}-equivariant isomorphism

$$\eta : W^{\mathrm{aff}} \overset{W_x^{\mathrm{aff}}}{\times} (\Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)) \xrightarrow{\sim} \Gamma_{W^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)$$

where $\Gamma_{W_x^{\mathrm{aff}}}$ is the union of graphs of the subgroup $W_x^{\mathrm{aff}} \leq W^{\mathrm{aff}}$, which admits a map $\Gamma_{W_x^{\mathrm{aff}}} \to \mathfrak{t}^*$ given by the projection $(wx, x) \mapsto x$.

Proof. Using the results of Lemma A.22 for every finite subset $S \subseteq W^{\mathrm{aff}}$, we obtain an isomorphism

$$\Gamma_{W^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \simeq \bigsqcup (\Gamma_{wW_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K))$$

where the right hand side ranges over the cosets of $W^{\mathrm{aff}}/W_x^{\mathrm{aff}}$. Therefore we may check that the mutliplication map induces an isomorphism at each open subset $\Gamma_{wW_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)$. However, we see that taking the fiber product of the above multiplication map by this open subset, we obtain the map

$$wW_x^{\mathrm{aff}} \overset{W_x^{\mathrm{aff}}}{\times} (\Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)) \to \Gamma_{wW_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)$$

which is an isomorphism. Furthermore, η is W^{aff}-equivariant because the multiplication map is W^{aff}-equivariant. □

Corollary A.24. We have canonical isomorphisms

$$X^\bullet(T)\backslash \mathfrak{t}^* \times_{\mathfrak{t}^* /\!\!/ \tilde{W}^{\mathrm{aff}}} \mathrm{Spec}(K) \simeq X^\bullet(T)\backslash \Gamma_{\tilde{W}^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \xrightarrow{\sim} W \overset{W_{[x]}}{\times} \mathrm{Spec}(C_{W_{[x]}})$$

where $C_{[x]}$ denotes the coinvariant algebra for the integral Weyl group (Definition A.17) of x.

Proof. The first equivalence follows from direct application of Proposition 3.14; we now show the second. Note that the inclusion $W^{\mathrm{aff}} \hookrightarrow \tilde{W}^{\mathrm{aff}}$ induces an isomorphism $\mathbb{Z}\Phi\backslash W^{\mathrm{aff}} \xrightarrow{\sim} X^\bullet(T)\backslash \tilde{W}^{\mathrm{aff}}$. Therefore we obtain canonical isomorphisms

$$X^\bullet(T)\backslash \Gamma_{\tilde{W}^{\mathrm{aff}}} := X^\bullet(T)\backslash \tilde{W}^{\mathrm{aff}} \overset{W^{\mathrm{aff}}}{\times} \Gamma_{W^{\mathrm{aff}}} \xleftarrow{\sim} \mathbb{Z}\Phi\backslash W^{\mathrm{aff}} \overset{W^{\mathrm{aff}}}{\times} \Gamma_{W^{\mathrm{aff}}} \simeq \mathbb{Z}\Phi\backslash \Gamma_{W^{\mathrm{aff}}}$$

over $\mathfrak{t}^*$ with respect to the (right) projection map. Furthermore, by Lemma A.23, we see:

$$\mathbb{Z}\Phi\backslash \Gamma_{W^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \xleftarrow{\sim} \mathbb{Z}\Phi\backslash W^{\mathrm{aff}} \overset{W_x^{\mathrm{aff}}}{\times} (\Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K))$$

so that, composing with the quotient map $\mathbb{Z}\Phi\backslash W^{\mathrm{aff}} \xrightarrow{\sim} W$, we see the quotient map induces an isomorphism

$$\mathbb{Z}\Phi\backslash W^{\mathrm{aff}} \overset{W_x^{\mathrm{aff}}}{\times} (\Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)) \xrightarrow{\sim} W \overset{W_{[x]}}{\times} \Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K)$$

where we identify the image of the stabilizer W_x^{aff} with the integral Weyl group as in Remark A.18. Therefore, since we may identify

$$W \overset{W_{[x]}}{\times} \Gamma_{W_x^{\mathrm{aff}}} \times_{\mathfrak{t}^*} \mathrm{Spec}(K) \simeq W \overset{W_{[x]}}{\times} \mathrm{Spec}(C_x) \cong W \overset{W_{[x]}}{\times} \mathrm{Spec}(C_{[x]})$$

where the first isomorphism follows from the definition of $C_{[x]}$ and the second isomorphism is given in Remark A.20, we obtain our claim. □

Appendix B. Twisted D-Modules are Derived Category of the Heart

We fix some $(W, \cdot)$-antidominant regular $\lambda \in \mathfrak{t}^*(k)$. We now show in Proposition B.2 that $\mathcal{D}(G/_\lambda B)^N$ is the derived category of its heart. Here, by derived category, we mean the canonical DG structure on the derived category of its heart in the sense of [Lur17, Definition 1.3.5.8]. To define the (unbounded) derived category of an abelian category $\mathcal{A}$ in this sense, we must first argue that $\mathcal{A}$ is a Grothendieck abelian category.

Lemma B.1. The category $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ is a Grothendieck abelian category.

Proof. Note that t-structure on $\mathcal{D}(G)$ is compatible with filtered colimits [GR, Section 4.3.2]. In particular, the connective (respectively, coconnective) objects can be identified with the ind-completion of subcategory of connective (respectively, coconnective) objects which are compact [GR17a, Chapter 4, Lemma 1.2.4], and so we also see that the t-structure on $\mathcal{D}(G)$ is accessible. Therefore, since the forgetful functor $\text{oblv} : \mathcal{D}(G/_\lambda B)^N \to \mathcal{D}(G)$ is t-exact and conservative, we also see that the t-structure on $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ is compatible with filtered colimits and accessible. Therefore $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ is a Grothendieck abelian category by [Lur17, Remark 1.3.5.23]. □

Proposition B.2. The category $\mathcal{D}(G/_\lambda B)^N$ is the unbounded derived category of its heart.

This entire appendix will be devoted to the proof of Proposition B.2. The strategy will proceed as follows: in Appendix B.1, we show that the bounded-below derived categories agree. Then, in Appendix B.2 and Appendix B.3, we show that both categories of Proposition B.2 are left-complete, thus proving Proposition B.2.

B.1. **Bounded By Below Twisted D-Modules are Bounded Derived Category of Heart.** In Corollary B.6, we will show that the bounded below variant of Proposition B.2 holds. We will do this via the following general claim:

Lemma B.3. [Lur17, Proposition 1.3.3.7, Dual Version] Assume we are given a stable ∞-category $\mathcal{C}$ equipped with a right-complete t-structure for which $\mathcal{C}^\heartsuit$ has enough injectives. Then we have an induced t-exact functor $\mathcal{D}^+(\mathcal{C}^\heartsuit) \to \mathcal{C}^+$, and this functor is an equivalence if and only if for any object $I \in \mathcal{C}^\heartsuit$ which is injective at the abelian categorical level (i.e. the functor $\text{Hom}_{\mathcal{C}^\heartsuit}(-, I)$ is an exact functor of abelian categories), and $Y \in \mathcal{C}^\heartsuit$, there exists some $Z \in \mathcal{C}^\heartsuit$ and an epimorphism $Z \twoheadrightarrow Y$ such that $\text{Ext}^i_{\mathcal{C}}(Z, I) \cong 0$ for all $i > 0$.

We first show this Ext vanishing for objects of $\mathcal{O}_\lambda$:

Lemma B.4. Assume $\mathcal{F} \in \mathcal{D}(G/_\lambda B)^{N,c,\heartsuit}$ and $J \in \mathcal{D}(G/_\lambda B)^{N,c,\heartsuit}$ is injective. Then $\underline{\text{Hom}}_{\mathcal{D}(G/_\lambda B)^N}(\mathcal{F}, J)$ is concentrated in degree 0.

Proof. The category $\mathcal{D}(G/_\lambda B)^{N,c,\heartsuit}$ is of finite length, and each object admits a filtration by simple objects of $\mathcal{D}(G/_\lambda B)^{N,c,\heartsuit}$. Each of these simples is given by the intermediate extension of the generator of $\mathcal{D}(N\backslash NwB)^{B_\lambda,\heartsuit}$. We will denote this intermediate extension by L_w. We show, by induction on the ordering in W, that for a fixed $w \in W$ any object which admits a filtration by simples L_v for $v \leq w$ has this Ext vanishing property. We recall that our injective object J admits a filtration by costandard objects ∇_w, i.e. those objects which are the $(*, dR)$-pushforward of a generator of $\mathcal{D}(N\backslash NwB)^{B_\lambda}$ (in $\mathcal{O}_\lambda$, for example, this is the Verdier dual version of [Hum08, Theorem 3.10]).

For the base case, when $\ell(w) = 1$ (and therefore $w = 1$), we let M be any object which admits a filtration whose subquotients are all of the form L_1. Then, since $\Delta_1 \xrightarrow{\sim} L_1$, any such M is the direct sum of the Δ_1. Therefore our desired Ext vanishing follows from the fact that there are no extensions between any Δ_1 and objects which admit filtrations whose subquotients are ∇_w (which, in turn, follows by induction on the length of a filtration and the fact that $i_1^!(\nabla_w) \simeq 0$ if $w \neq 1$). Finally, for a general $w \in W$, we have a short exact sequence

$$0 \to S \to \Delta_w \to L_w \to 0$$

where S admits a filtration of objects L_v for $v < w$. We then obtain an exact sequence

$$\text{Ext}^{j-1}(\Delta_w, J) \to \text{Ext}^{j-1}(S, J) \to \text{Ext}^j(L_w, J) \to \text{Ext}^j(\Delta_w, J)$$

for any $j > 0$. Note that the last term in this sequence vanishes because, in $\mathcal{D}(G/_\lambda B)^N$, there are no extensions between any Δ_w and objects which admit filtrations whose subquotients are ∇_w. If $j = 1$, then, we may identify the third term as the cokernel of the map $\text{Hom}(\Delta_w, J) \to \text{Hom}^{j-1}(S, J)$, where the maps are equivalently taken in $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$, a full subcategory of $\mathcal{D}(G/_\lambda B)^N$. In particular, the injectivity of J shows that this map is a surjection so the cokernel vanishes. If $j > 1$, we similarly obtain that $\text{Ext}^{j-1}(\Delta_w, J)$ vanishes and so that our inductive hypothesis shows that $\text{Ext}^j(L_w, J) \xleftarrow{\sim} \text{Ext}^{j-1}(S, J)$ vanishes, since S admits a filtration by simple objects L_v for $v < w$. □

We now use the following general lemma to reduce to the above computations.

Lemma B.5. Let $\mathcal{A}'$ be an abelian category closed under subquotients in its ind-completion $\mathcal{A}$ with enough injectives such that every injective object of $\mathcal{A}'$ is a finite direct product of indecomposable injective objects. Then, for every injective object $I \in \mathcal{A}$, there exists some injective $M \in \mathcal{A}$ such that $I \times M \cong \prod_n I_n$ for $I_n \in \mathcal{A}'$ indecomposable injectives.

Proof. Assume $I \in \mathcal{A}$ is some injective object. By assumption and Lemma 2.51, we can write I as an increasing union of objects of $\mathcal{A}'$, say $I = \cup_n A_n$. By assumption that $\mathcal{A}'$ has enough injectives, we may choose an injection $A_n \hookrightarrow I_n$ into an injective object $I_n \in \mathcal{A}'$. Note that $\mathcal{A}$ is a Grothendieck abelian category, so I_n is an injective object of $\mathcal{A}$ and so the inclusion map $i_n : A_n \subseteq I$ extends to a map $f_n : I \to I_n$.

Consider the canonical induced map $f : I \to \prod_n I_n$ with projection maps $pr_m : \prod_n I_n \to I_m$. The map f is injective, since if K denotes the kernel of f, then $\mathrm{pr}_n f|_{A_n \cap K} = i_n|_{A_n \cap K}$ so $A_n \cap K = 0$ for all n, and thus $K = \cup_n (A_n \cap K) = 0$. Write each I_n as a product of indecomposable injectives and relabel, if necessary, so that each I_n is an indecomposable injective object; we may do this since every object of $\mathcal{A}'$ is a finite direct product of indecomposable injective objects. However, $\prod_n I_n$ is injective since the property of being an injection is closed under products, and so the injective map $I \overset{f}{\hookrightarrow} \prod_n I_n$ splits, using:

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & \prod_n I_n \arrow[dl, "\exists"] \\ I & \end{tikzcd}$$

and so we can write $\prod_n I_n \cong I \times M$ for some $M \in \mathcal{A}$. The fact that $\prod_n I_n$ and I are both injective implies that M is, which follows from the fact that a product in an abelian category is injective only if each factor is. $\square$

Corollary B.6. The induced functor from the left-bounded derived category of the heart of $\mathcal{D}(G/_\lambda B)^N$ to the category $\mathcal{D}(G/_\lambda B)^{N,+}$ is an equivalence.

Proof. By Lemma B.3, it suffices to show that $\underline{\mathrm{Hom}}_{\mathcal{D}(G/_\lambda B)^N}(\mathcal{F}, I)$ is concentrated in degree 0 if $\mathcal{F}, I \in \mathcal{D}(G/_\lambda B)^N$ and I is injective. However, the category $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ satisfies the hypotheses of Lemma B.5, so we can find some injective M of $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$ for which

$$\underline{\mathrm{Hom}}(\mathcal{F}, I) \times \underline{\mathrm{Hom}}(\mathcal{F}, M) \simeq \underline{\mathrm{Hom}}(\mathcal{F}, I \times M) \simeq \prod \underline{\mathrm{Hom}}(M, I_n)$$

for I_n compact injective objects of $\mathcal{D}(G/_\lambda B)^{N,\heartsuit}$. Therefore, it suffices to show that I is a compact injective object. Furthermore, by compact generation of $\mathcal{D}(G/_\lambda B)^N$, we may write $\mathcal{F}$ as a filtered colimit of compact objects. Since the t-structure on Vect is compatible with filtered limits, to show that $\underline{\mathrm{Hom}}(\mathcal{F}, I)$ is concentrated in degree 0, it suffices to assume that $\mathcal{F}$ is compact. The claim then follows from Lemma B.4. $\square$

B.2. **Left-Completeness of Derived Category.** Let $\mathcal{O}_\lambda := \mathcal{D}(N\backslash G/_\lambda B)^\heartsuit$. Note that the category $\mathcal{D}^b(\mathcal{O}_\lambda)$ is canonically equipped with a t-structure, so that $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}_\lambda))$ is equipped with a canonical t-structure such that the inclusion functor $\mathcal{D}^b(\mathcal{O}_\lambda)) \hookrightarrow \mathrm{Ind}(\mathcal{D}^b(\mathcal{O}_\lambda))$ is t-exact and the heart of the t-structure is $\mathrm{Ind}(\mathcal{O}_\lambda)$, see e.g. [GR17a, Chapter 4, Lemma 1.2.4].

Remark B.7. We note in passing that the ind-completion of any ordinary (1,1)-category is always an ordinary (1,1)-category. This is because, by definition, the ind-completion of an ordinary category C is defined as the full subcategory of presheaves on C, i.e. $\mathrm{Fun}(C^{op}, \mathrm{Spc})$ containing the representable functors and closed under filtered colimits. In particular, since all representable objects map to discrete spaces (i.e. sets) and discrete spaces are closed under filtered colimits, we see that the classical definition of the ind-completion of C agrees with the higher categorical definition as in [Lur09, Section 5.3].

We now claim the following:

Lemma B.8. If $\mathcal{O}$ is some small abelian category of finite cohomological dimension with a compact projective generator, then we have an equivalence of DG categories $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O})) \simeq \mathcal{D}(\mathrm{Ind}(\mathcal{O}))$, where the right hand category denotes the canonical DG model on the derived ∞-category of the Grothendieck abelian category $\mathrm{Ind}(\mathcal{A})$ given by the ind-completion of $\mathcal{A}$.

Proof of Lemma B.8. Let $\mathcal{G} \in \mathcal{O}$ denote the compact projective generator. Then every object $M \in \mathcal{O}$ admits a finite length projective resolution, so the smallest subcategory containing $\mathcal{G}$ and all finite colimits is $\mathcal{D}^b(\mathcal{O})$. Since, by definition, every object of $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))$ is a filtered colimit of objects of $\mathcal{D}^b(\mathcal{O})$, we see that the smallest subcategory containing $\mathcal{G}$ and all colimits is $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))$ itself. Therefore, the functor

$$\underline{\mathrm{Hom}}_{\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))}(\mathcal{G}, -) : \mathrm{Ind}(\mathcal{D}^b(\mathcal{O})) \to E\text{-mod}$$

where $E := \underline{\mathrm{End}}_{\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))}(\mathcal{G})$, gives an equivalence of categories by Barr-Beck. Furthermore, the fact that $\mathcal{G}$ is a projective object implies that E is a discrete (i.e. classical) algebra and that this equivalence is t-exact. Thus $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))$ is equivalent to a category which is the unbounded derived category of its heart by a t-exact functor. We therefore see that $\mathrm{Ind}(\mathcal{D}^b(\mathcal{O}))$ is itself the unbounded derived category of its heart. $\square$

By [AGK$^+$20, Theorem B.1.3], we therefore see:

Corollary B.9. The unbounded derived category of the ind-completion of $\mathcal{O}_\lambda$ is left-complete in its t-structure.

B.3. **Identification of Derived Categories.** Recall in Corollary B.9 that we showed the unbounded derived category of $\mathcal{D}(N\backslash G/_\lambda B)^\heartsuit$ is left-complete. Therefore, as we have identified the bounded by below subcategories in Corollary B.6, to identify the unbounded derived categories it suffices to show the following claim:

Proposition B.10. The t-structure on $\mathcal{D}(G/_\lambda B)^N$ is left-complete.

We will prove Proposition B.10 after showing the following general lemma:

Lemma B.11. Let $R : \mathcal{C} \to \mathcal{D}$ denote a conservative functor between categories with t-structures such that R commutes with limits (i.e. is a right adjoint) and has bounded cohomological amplitude. Then if the t-structure on $\mathcal{D}$ is left-complete, then the t-structure on $\mathcal{C}$ is also left-complete.

Proof. Let $\mathcal{F} \in \mathcal{C}$. We wish to show the canonical map $\mathcal{F} \to \mathrm{lim}_m \tau^{\geq m}(\mathcal{F})$ is an equivalence, where m varies over the nonpositive integers. By conservativity of R and the fact that R commutes with limits, we see that we may equivalently show the canonical map $R(\mathcal{F}) \to \mathrm{lim}_m R(\tau^{\geq m}\mathcal{F})$ is an equivalence. However, note that

$$\mathrm{lim}_m R(\tau^{\geq m}\mathcal{F}) \simeq \mathrm{lim}_m \mathrm{lim}_n \tau^{\geq n} R(\tau^{\geq m}\mathcal{F}) \simeq \mathrm{lim}_n \mathrm{lim}_m \tau^{\geq n} R(\tau^{\geq m}\mathcal{F})$$

where the first expression uses the left-completeness of t and the second uses the fact that limits commute with limits. Now let $[p, q]$ denote the cohomological amplitude of R for $p, q \in \mathbb{Z}$. we see that for any fixed n we have that if $m < n + p$ then

$$\tau^{\geq n} R(\tau^{\geq m}\mathcal{F}) \simeq \tau^{\geq n} R(\tau^{\geq n+p}\mathcal{F}) \simeq \tau^{\geq n} R(\mathcal{F})$$

where both steps follow from the definition of bounded cohomological amplitude. Therefore, continuing the above chain of equivalences, we see

$$\mathrm{lim}_m R(\tau^{\geq m}\mathcal{F}) \simeq \mathrm{lim}_n \tau^{\geq n} R(\mathcal{F}) \xleftarrow{\sim} R(\mathcal{F})$$

where the last step follows by the left-completeness of $\mathcal{D}$, as desired. $\square$

Proof of Proposition B.10. By Lemma B.11, we may exhibit a finite set of compact generators of finite cohomological amplitude, since then we may take the direct sum $\mathcal{G}$ and set $R := \underline{\mathrm{Hom}}_{\mathcal{D}(G/_\lambda B)^N}(\mathcal{G}, -)$. However, we make take our compact generators as the standard objects $\Delta_w \in \mathcal{D}(G/_\lambda B)^{N,\heartsuit}$. Then maps from each standard object are given by a !-restriction by a locally closed map of smooth schemes, which in particular has finite cohomological amplitude [HTT08, Proposition 1.5.13, Proposition 1.5.14]. $\square$

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