Exercises Week 5

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Exercise 1

The following derivation follows the Kalman filter derivation given in the book (Sarkka (2013)). Process and measurement models can be expressed as

$$x_k = Ax_{k-1} + q_{k-1}, \qquad q_{k-1} \sim \mathcal{N}(m_q, Q) \tag{1}$$
$$y_k = Hx_k + r_k, \qquad r_k \sim \mathcal{N}(m_r, R). \tag{2}$$

Since the noises has non-zero mean the models can be rewritten as

$$x_k = Ax_{k-1} + m_q + (q_{k-1} - m_q), \tag{3}$$
$$y_k = Hx_k + m_r + (r_k - m_r), \tag{4}$$

where $(q_k - m_q) \sim \mathcal{N}(0, Q)$ and $(r_k - m_r) \sim \mathcal{N}(0, R)$. Now the process and measurement models can be evaluated in closed form with the resulting Gaussian distributions

$$\begin{aligned}
&p(x_k|y_{1:k-1}) = \mathcal{N}(x_k|m_k^-, P_k^-)\\
&p(x_k|y_{1:k}) = \mathcal{N}(x_k|m_k, P_k)\\
&p(y_k|y_{1:k-1}) = \mathcal{N}(y_k|Hm_k^-, S_k).
\end{aligned}$$

The parameters for the distributions can be calculated following the calculation rules for joint distribution covariance as follows

$$\begin{aligned}
&m_k^- = Am_{k-1} + m_q,\\
&P_k^- = cov(m_k^-) + Q_{k-1} = cov(Am_{k-1} + m_q) + Q_{k-1} = AP_{k-1}A^T + Q_{k-1},\\
&S_k = cov(Hm_k^- + m_r) + R_k = HP_k^- H^T + R_k.\\
&v_k = y_k - H_k m_k^- - m_r,\\
&K_k = P_k^- H_k^T S_k^{-1},\\
&m_k = m_k^- + K_k v_k,\\
&P_k = P_k^- - K_k S_k K_k^T.
\end{aligned}$$

Exercise 2

In the figure (1) is the Kalman filter implementation for Gaussian random walk.

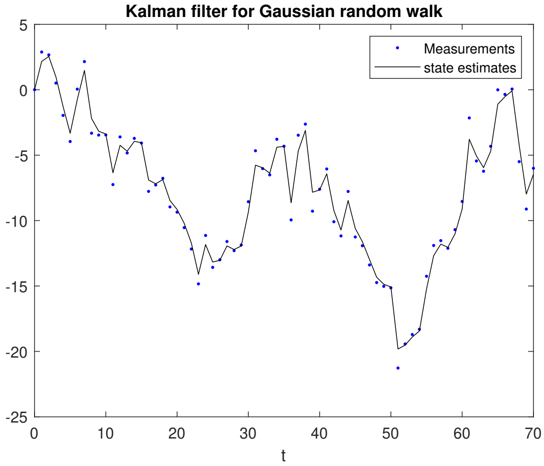

Figure 1: Exercise 2

Exercise 3

The forced spring mass system can be modeled with the equation

$$mx'' + cx' + kx = F(x). \tag{5}$$

By defining $v = x'$ the equation can be transformed into first order ODE

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = -\frac{c}{m}v - \frac{k}{m}x + \frac{1}{m}F(x). \end{cases} \tag{6}$$

The equation (6) can be expressed in matrix from as

$$X' = AX + BZ + q, \tag{7}$$

where $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$, $X = \begin{bmatrix} x \\ v \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix}$ and $Z = \begin{bmatrix} 1 \\ F(x) \end{bmatrix}$. The equation (7) can be discretized as follows

$$\begin{aligned} \frac{X_{i+1} - X_i}{h} &= AX_i + BZ_i + q_i \\ X_{i+1} &= AX_ih + X_i + BZ_ih + hq_i \\ X_{i+1} &= (I + Ah)X_i + BZ_ih + hq_i, \end{aligned}$$

where h is the time step and $q_i \sim \mathcal{N}(\mu_q, Q)$ is process noise. With discretized X_{i+1} equation the Kalman filter state and measurement equations can be expressed as

$$\begin{aligned} X_{i+1} &= (I + Ah)X_i + BZ_ih + hq_i \\ Y_{i+1} &= X_{i+1} + r_{i+1}, \end{aligned}$$

where $r \sim \mathcal{N}(\mu_r, R)$ is measurement noise.

The estimation and the measurements can be seen in the figure (2). R^2 value was used for the goodness of fit between the true states and state estimations which is 0.8125 for the corresponding estimate.

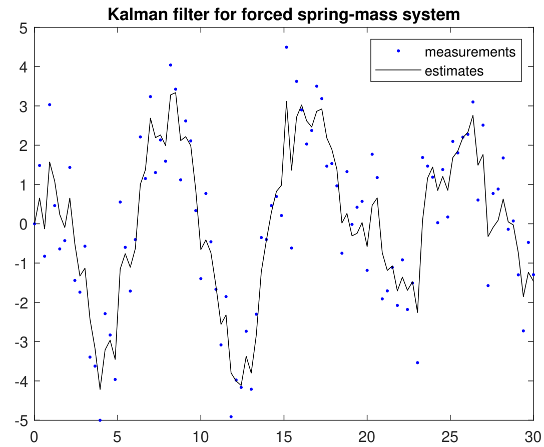

Figure 2: Exercise 3

Exercise 4

The figure (3) contains the extended Kalman filter implementation for the pendulum model of the book (Sarkka (2013)).

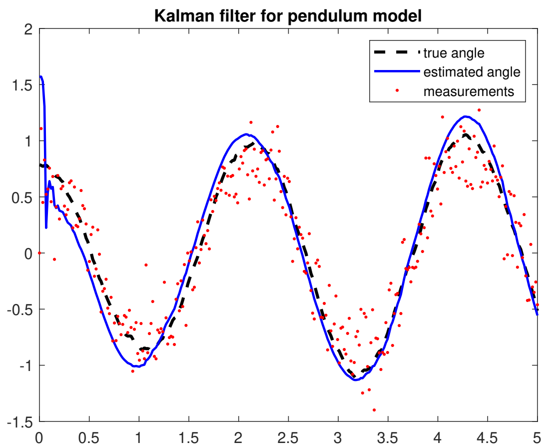

Figure 3: Exercise 4

References

Sarkka, S. (2013). *Bayesian filtering and smoothing.* Institute of Mathematical Statistics textbooks ; 3. Cambridge University Press, Cambridge.