Exercises Week 3

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Exercise 1

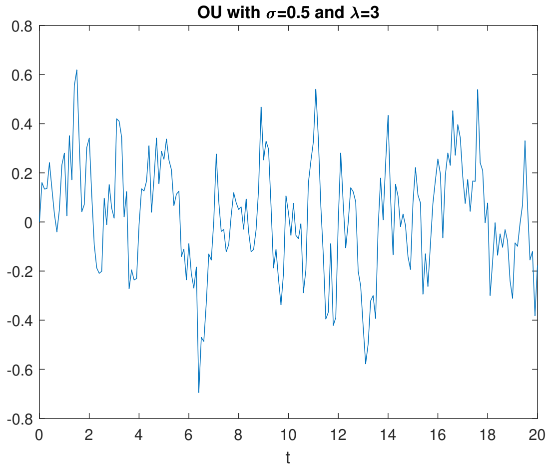

Figure 1: One realisation of Ornstein-Uhlenbeck.

Ornstein-Uhlenbeck process can be discretized as follows:

$$X(t+1) = X(t) - \lambda X(t)h + \sigma\sqrt{h}W(t), \tag{1}$$

where h is the time step (= dt) and W(t) is white noise $W \sim \mathcal{N}(0, I)$. Equation (1) can be further formulated:

$$\begin{aligned} X(t+1) &= (1-\lambda h)X(t) + \sigma\sqrt{h}W(t) \\ X(t+1) - (1-\lambda h)X(t) &= \sigma\sqrt{h}W(t) \\ X(t+1) - \lambda' X(t) &= \sigma\sqrt{h}W(t) \\ \mathbf{LX} &= \sigma\sqrt{h}\mathbf{W}, \end{aligned}$$

where $\mathbf{X}$ is vector containing all the X values and $\mathbf{W}$ is a vector containing all the white noise values.

$\mathbf{L}$ is a $n \times n$ matrix:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \dots & & & & 0 \\ -\lambda' & 1 & 0 & \dots & & & 0 \\ 0 & -\lambda' & 1 & 0 & \dots & & 0 \\ . & . & . & . & & & \\ . & . & . & . & . & & \\ 0 & . & . & 0 & -\lambda' & 1 & 0 \\ 0 & . & . & . & . & -\lambda' & 1 \end{bmatrix}.$$

Finally X can be solved from the matrix equation

$$\mathbf{X} = \mathbf{L}^{-1}\sigma\sqrt{h}\mathbf{W},$$

from which we can notice that $\mathbf{X} \sim \mathcal{N}(0, \mathbf{C})$, where $\mathbf{C} = \sigma^2 h \mathbf{L}^{-T}\mathbf{L}^{-1}$.

Now we can write the posterior of λ without prior:

$$p(\lambda|\mathbf{X}, \sigma) \propto \frac{1}{\sqrt{(2\pi)^n |\mathbf{C}|}} \exp\left(-\frac{1}{2}\mathbf{X}^T\mathbf{C}^{-1}\mathbf{X}\right).$$

For λ estimation DRAM was used with 50 000 simulations. From the figures (3), (2) and (4) the chain, histogram and ACF of estimated λ can be seen.

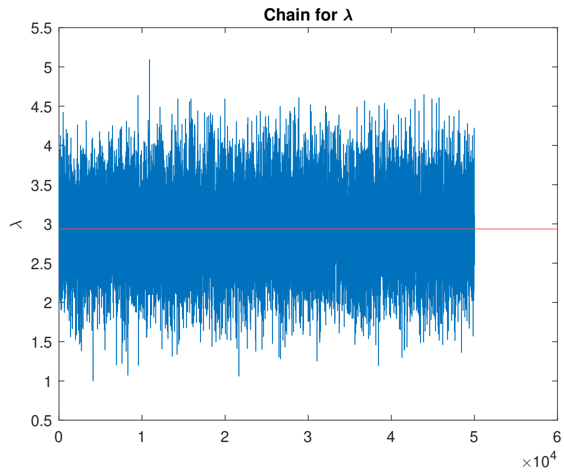

Figure 2: Markov chain of λ estimations.

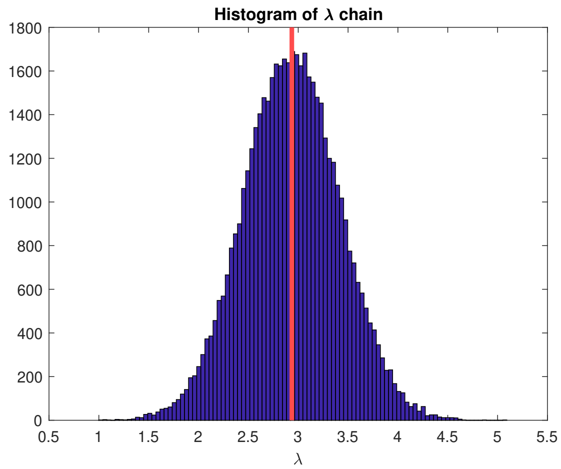

Figure 3: Histogram of λ chain.

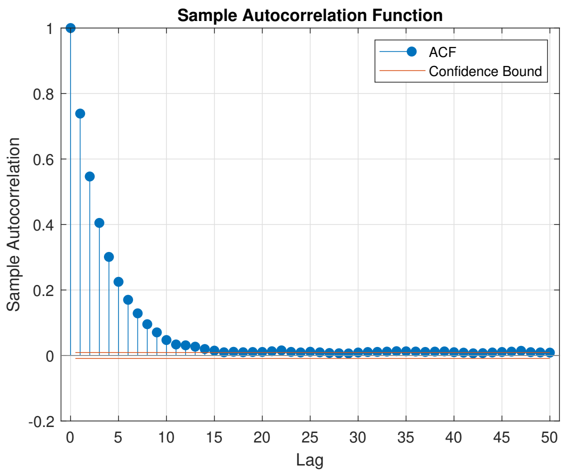

Figure 4: ACF of λ chain.

Exercise 2

Gaussian prior

Gaussian prior $\lambda \sim \mathcal{N}(0, 1)$ was used. From the figures (5) and (6) the chain and histograms of λ estimations can be seen. When comparing the histograms of the figure (3) and (6) it can be noticed that since the mean and variance are not optimal for Gaussian prior the estimation is forced closer to zero. If the variance of the prior would be increased probably the estimation also would be closer to correct lambda ($\lambda = 3$).

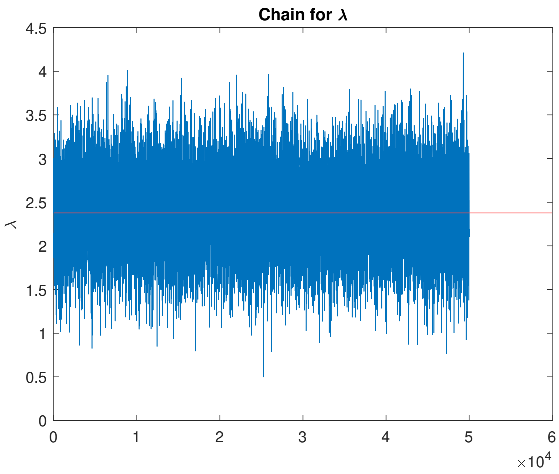

Figure 5: λ chain with Gaussian prior.

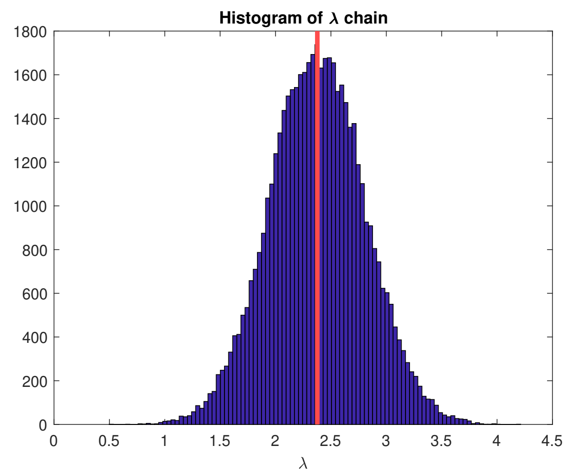

Figure 6: Histogram of λ chain with Gaussian prior.

Logarithmic transformation prior

Prior $\log(\lambda) \sim \mathcal{N}(0, 1)$ produced results which can be seen in the figures (7) and (8). When comparing the results to Gaussian prior it can be seen that with logarithmic transformation prior the estimation is closer to correct value although the same variance and mean was used than with Gaussian prior.

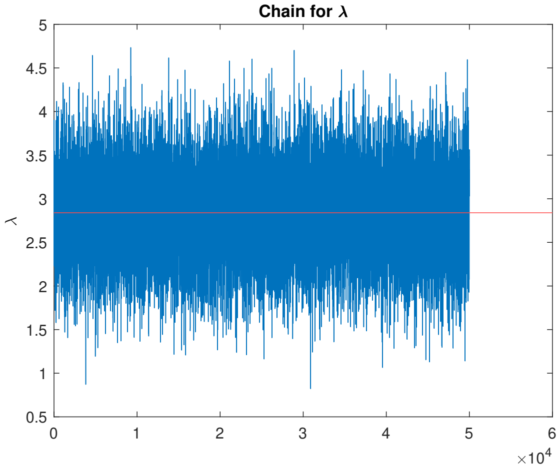

Figure 7: λ chain with logarithmic transformation prior.

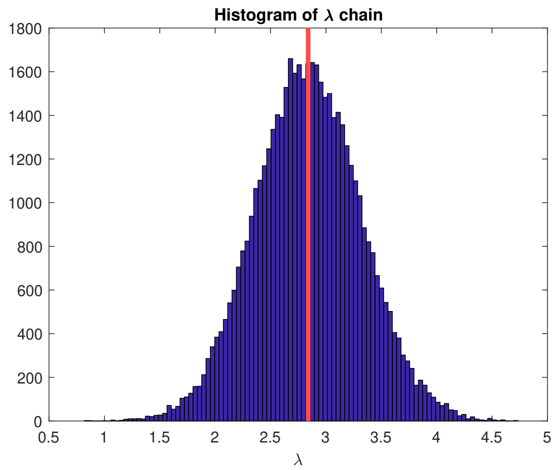

Figure 8: Histogram of λ chain with logarithmic transformation prior.

Uniform prior

The range $[-5, 5]$ was used for λ with uniform prior. The results can be seen in the figures (9) and (10).

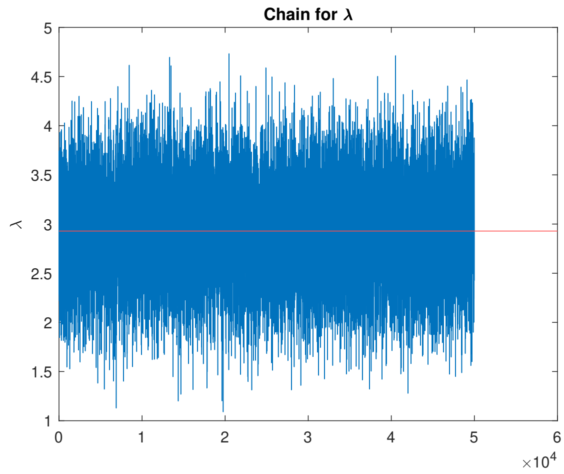

Figure 9: λ chain with uniform prior.

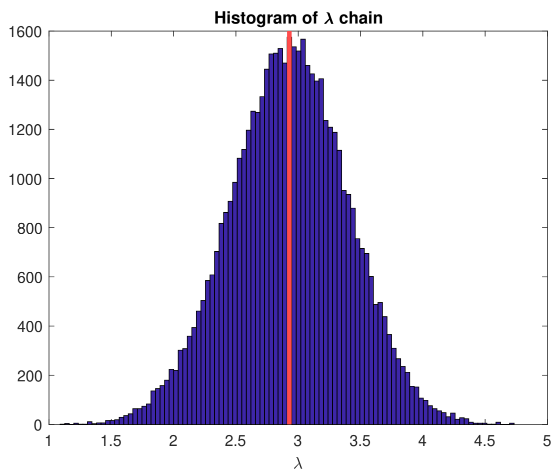

Figure 10: Histogram of λ chain with uniform prior.

Exercise 3

Estimating σ

Gaussian prior $\mathcal{N}(0, 1)$ for σ was used for estimation. The results can be seen in the figures (11) and (12). It can bee noted that the estimations are not as good as for λ.

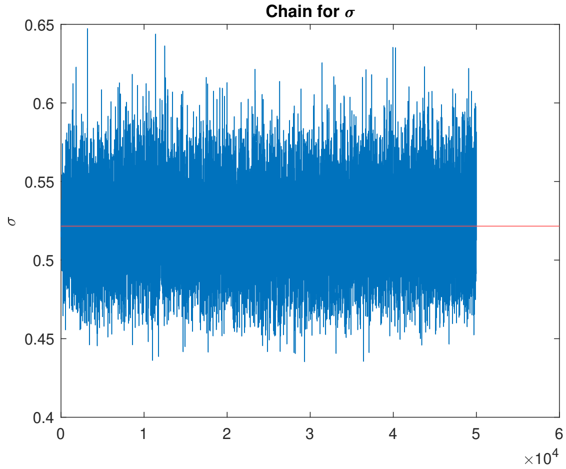

Figure 11: Chain for σ.

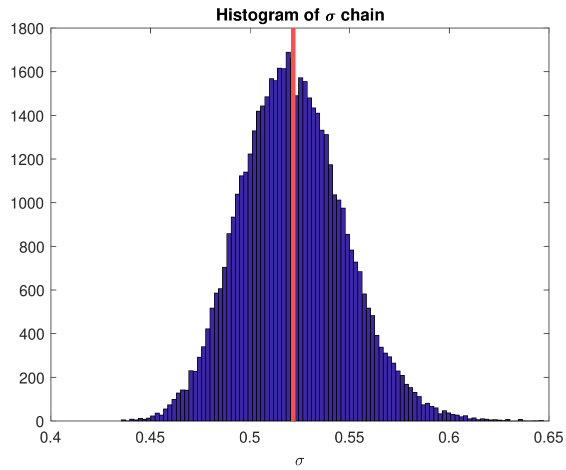

Figure 12: Histogram for σ.

Estimating λ and σ

For estimating both σ and λ Gaussian prior $\mathcal{N}([0, 0], [5, 1])$ was used. The results of 70 000 simulations can be seen in the figures (13) and (14). Joint density of the parameters can be seen in the figure (15).

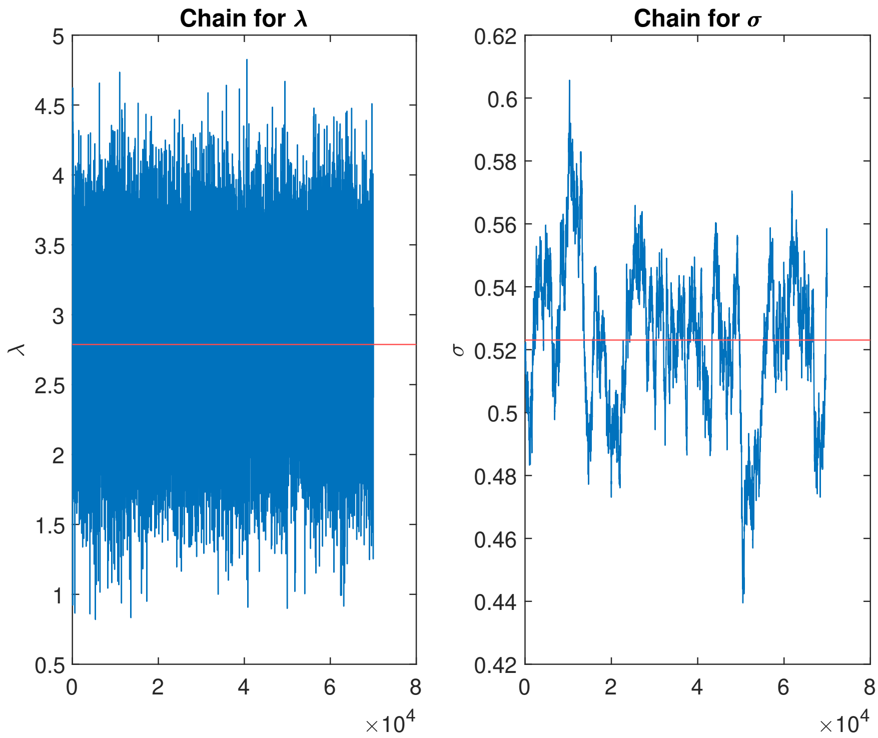

Figure 13: Chain for σ and λ.

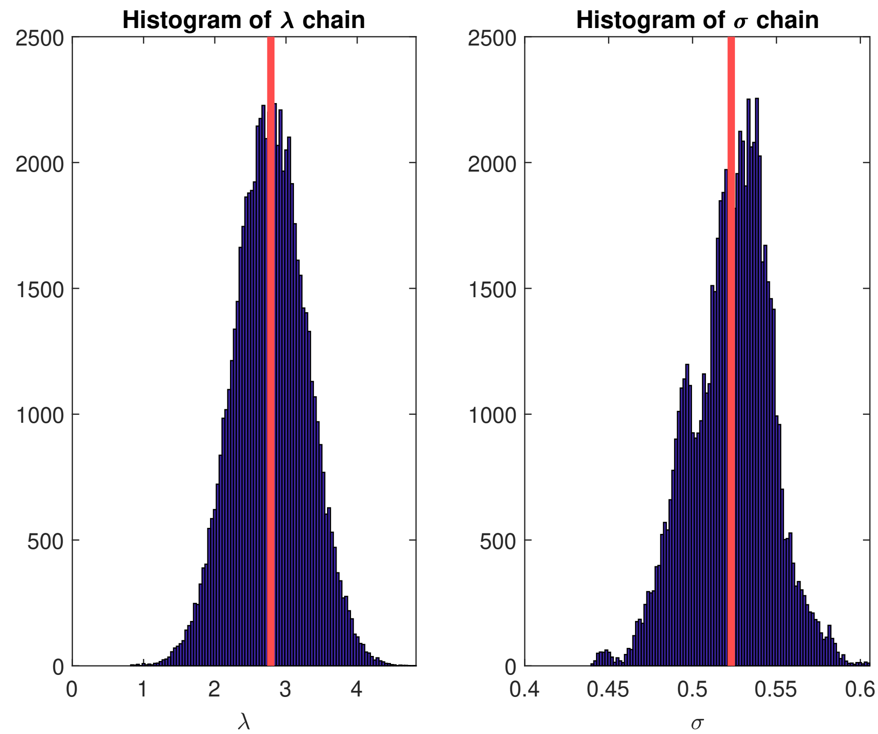

Figure 14: Histogram for σ and λ.

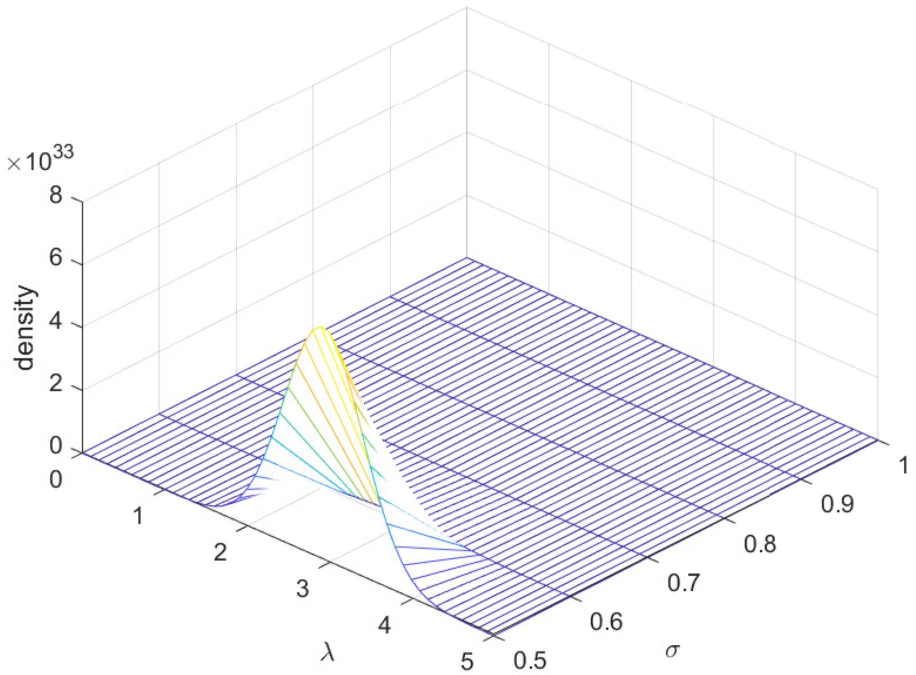

Figure 15: Joint density of σ and λ.