

# Finite Volume Methods for Hyperbolic Problems

## TVD Methods and Limiters

- Slope limiters vs. flux limiters
- Total variation for scalar problems
- Proving TVD in flux-limiter form
- Design of TVD limiters
- Sweby Region

# High-Resolution methods

- Methods that give **good accuracy for smooth solutions**  
Clawpack methods: at best second-order accuracy
- **Do not have oscillations** around discontinuities  
Not only ugly but can lead to nonlinear instabilities

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Minimal numerical dissipation  
“**Shock capturing**” methods for nonlinear problems

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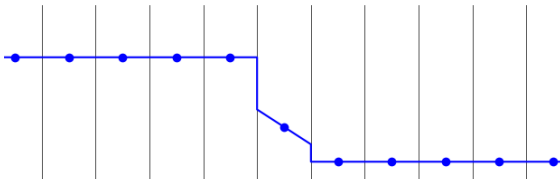
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Even sharper resolution of discontinuities

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**“Shock capturing”** methods for nonlinear problems
- Easy to combine with **adaptive mesh refinement** (AMR)  
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Even sharper resolution of discontinuities
- **Godunov-type methods** — based on Riemann solvers  
Wave-propagation algorithms with **“limiters”**

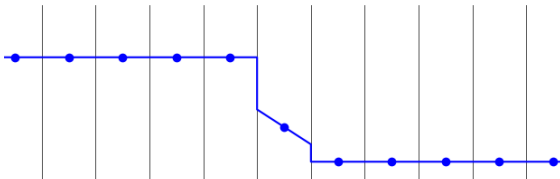
# Limiters can eliminate oscillations

Step function data with minmod slope:

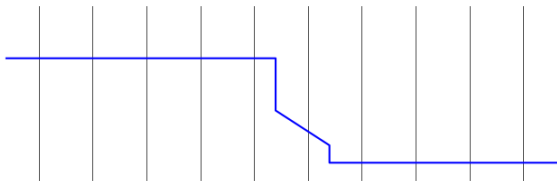


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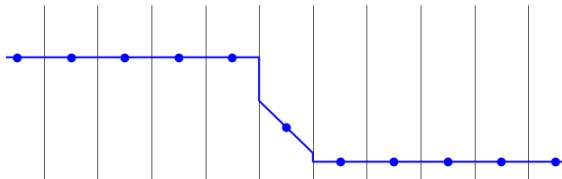


Evolving solution and averaging maintains monotonicity:



# Could make slope steeper and still be monotone

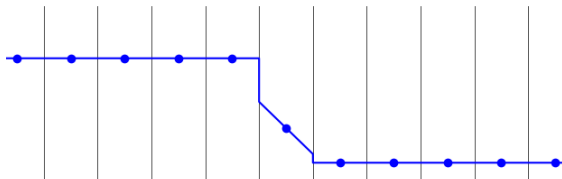
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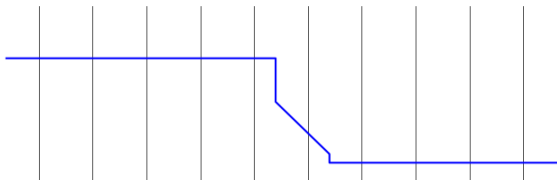


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Evolving solution and averaging maintains monotonicity:



# Monotonized centered (MC) limiter

Using the centered slope  $(Q_{i+1}^n - Q_{i-1}^n)/(2\Delta x)$  gives second-order accuracy ([Fromm's method](#)) but not monotonicity.

Limit this slope based on [twice](#) the one-sided slopes.

$$\sigma_i^n = \min\left(\left(\frac{Q_{i+1}^n - Q_{i-1}^n}{2\Delta x}\right), 2\left(\frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right), 2\left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}\right)\right).$$

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## Rationale:

- Where solution is smooth, centered slope is smaller and chosen, hence maintains accuracy.
- Near jumps in solution, don't expect second-order but want to resolve discontinuities as sharply as possible.

# TVD REA Algorithm

- 1 **Reconstruct** a piecewise **linear** function  $\tilde{q}^n(x, t_n)$  defined for all  $x$ , from the cell averages  $Q_i^n$ .

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for all } x \in \mathcal{C}_i$$

with the property that  $TV(\tilde{q}^n) \leq TV(Q^n)$ .

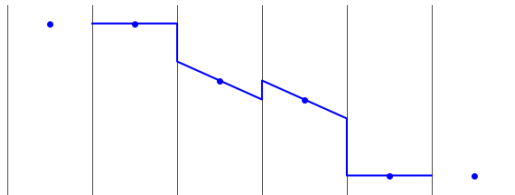
- 2 **Evolve** the hyperbolic equation exactly (or approximately) with this initial data to obtain  $\tilde{q}^n(x, t_{n+1})$  a time  $k$  later.
- 3 **Average** this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x} \int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1}) dx.$$

**Note:** Steps 2 and 3 are always TVD.

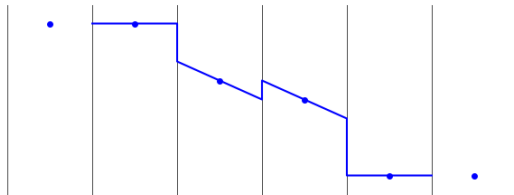
# MC slopes are **not** always a TVD reconstruction

Sample data with MC slope (twice that of minmod):

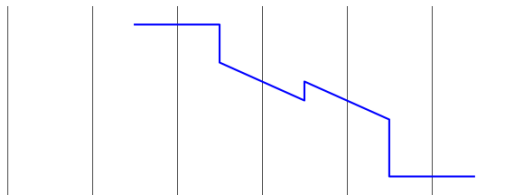


# MC slopes are **not** always a TVD reconstruction

Sample data with MC slope (twice that of minmod):



But evolving and averaging still maintains monotonicity (TVD):



# Slope limiters and flux limiters

Slope limiter formulation for advection:

$$\tilde{q}^n(x, t_n) = Q_i^n + \sigma_i^n(x - x_i) \quad \text{for } x_{i-1/2} \leq x < x_{i+1/2}.$$

Applying REA algorithm gives (for  $u > 0$ ):

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n) - \frac{1}{2} \frac{u\Delta t}{\Delta x} (\Delta x - u\Delta t) (\sigma_i^n - \sigma_{i-1}^n)$$

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Flux limiter formulation:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

with flux

$$F_{i-1/2}^n = uQ_{i-1}^n + \frac{1}{2}u(\Delta x - u\Delta t)\sigma_{i-1}^n$$



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$$F_{i-1/2}^n = uQ_{i-1}^n + \frac{1}{2}u(\Delta x - u\Delta t)\sigma_{i-1}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} u\tilde{q}(x_{i-1/2}, t) dt.$$

# Lax-Wendroff and flux limiters

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$$F_{i-1/2}^n = \begin{cases} uQ_{i-1}^n + \frac{1}{2}u(\Delta x - u\Delta t)\sigma_{i-1}^n & \text{if } u > 0, \\ uQ_i^n - \frac{1}{2}u(\Delta x + u\Delta t)\sigma_i^n & \text{if } u < 0. \end{cases}$$

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Flux limiter method: Replace  $\Delta Q_{i-1/2}^n$  by limited version  $\delta_{i-1/2}^n$

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# Flux limiters and wave limiters

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**For systems of equations:**

- Solve Riemann problem to decompose  $\Delta Q_{i-1/2}^n$  into waves

$$\Delta Q_{i-1/2} = \sum_p \mathcal{W}_{i-1/2}^p = \sum_p \alpha_{i-1/2}^p r^p$$

- Use wave propagation form of Godunov (first-order) update
- Apply limiters to waves to get  $\widetilde{\mathcal{W}}_{i-1/2}^p = \tilde{\alpha}_{i-1/2}^p r^p$
- Use limited waves in “second-order” corrections

# Flux limiters for scalar problem

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**Limiter based on the ratio**

$$\theta_{i-1/2}^n = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

where  $I$  denotes the cell in the upwind direction:

$$I = \begin{cases} i - 1 & \text{if } u > 0 \\ i + 1 & \text{if } u < 0. \end{cases}$$

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**Note that:**

- $\theta \approx 1 + \mathcal{O}(\Delta x)$  where the solution is smooth,
- $\theta < 0$  if slopes have different sign.



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**Desirable properties:**

- $\phi(\theta) = 0$  for  $\theta \leq 0$  (zero slope at extrema)
- $\phi(1) = 1$  so nearly using Lax-Wendroff where smooth

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Note that:

- $\phi(\theta) \equiv 0$  for all  $\theta \implies$  upwind method
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- $\phi(\theta) = \min\text{mod}(1, \theta) \implies$  Minmod method

# TVD flux limiter methods

For  $q_t + uq_x = 0$  with  $u > 0$  and  $\nu = u\Delta t/Dx$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (F_{i+1/2}^n - F_{i-1/2}^n)$$

$$\begin{aligned} F_{i-1/2}^n &= uQ_{i-1}^n + \frac{1}{2}u(1 - u\Delta t/\Delta x)\delta_{i-1/2}^n \\ &= uQ_{i-1}^n + \frac{1}{2}u(1 - \nu)[\phi(\theta_{i-1/2})(Q_i - Q_{i-1})] \end{aligned}$$

Can be written as:

$$Q_i^{n+1} = Q_i^n - \left[ \nu + \frac{1}{2}\nu(1 - \nu) \left( \frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2}) \right) \right] (Q_i^n - Q_{i-1}^n)$$

since  $Q_{i+1} - Q_i = (1/\theta_{i+1/2})(Q_i - Q_{i-1})$ .

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Use this part of [Theorem 6.1 \(Harten\)](#):

The method

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n (Q_i^n - Q_{i-1}^n)$$

is TVD provided  $0 \leq C_i^n \leq 1$  for all  $i$ , regardless of how these coefficients depend on  $Q^n$ ,  $\Delta x$ ,  $\Delta t$ .



# TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \quad TV(Q) = \sum |Q_{i+1} - Q_i|$$

Proof that method is TVD provided  $0 \leq C_i \leq 1$  for all  $i$ :

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= (Q_{i+1} - Q_i) - C_i(Q_{i+1} - Q_i) + C_{i-1}(Q_i - Q_{i-1}) \\ &= (1 - C_i)(Q_{i+1} - Q_i) + C_{i-1}(Q_{i+1} - Q_i) \end{aligned}$$

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$$|Q_{i+1}^{n+1} - Q_i^{n+1}| \leq (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}|$$

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$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \end{aligned}$$

# TVD flux limiter methods

$$Q_i^{n+1} = Q_i - C_{i-1}(Q_i - Q_{i-1}), \quad TV(Q) = \sum |Q_{i+1} - Q_i|$$

Proof that method is TVD provided  $0 \leq C_i \leq 1$  for all  $i$ :

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= (Q_{i+1} - Q_i) - C_i(Q_{i+1} - Q_i) + C_{i-1}(Q_i - Q_{i-1}) \\ &= (1 - C_i)(Q_{i+1} - Q_i) + C_{i-1}(Q_{i+1} - Q_i) \end{aligned}$$

$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq (1 - C_i)|Q_{i+1} - Q_i| + C_{i-1}|Q_i - Q_{i-1}| \\ \sum |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_{i-1}|Q_i - Q_{i-1}| \\ &\leq \sum (1 - C_i)|Q_{i+1} - Q_i| + \sum C_i|Q_{i+1} - Q_i| \\ &\leq \sum (1 - C_i + C_i)|Q_{i+1} - Q_i| = TV(Q^n) \end{aligned}$$

# TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[ \nu + \frac{1}{2}\nu(1 - \nu) \left( \frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2}) \right) \right] (Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$0 \leq \left[ \nu + \frac{1}{2}\nu(1 - \nu) \left( \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \right) \right] \leq 1$$

for all values of  $\theta_1$  and  $\theta_2$  (provided  $0 \leq \nu \leq 1$ ).

# TVD flux limiter methods

The method

$$Q_i^{n+1} = Q_i^n - \left[ \nu + \frac{1}{2}\nu(1 - \nu) \left( \frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2}) \right) \right] (Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$0 \leq \left[ \nu + \frac{1}{2}\nu(1 - \nu) \left( \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \right) \right] \leq 1$$

for all values of  $\theta_1$  and  $\theta_2$  (provided  $0 \leq \nu \leq 1$ ).

This is true if

$$-2 \leq \left( \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \right) \leq 2$$

for all values of  $\theta_1$  and  $\theta_2$

# TVD flux limiter methods

So the method

$$Q_i^{n+1} = Q_i^n - \left[ \nu + \frac{1}{2}\nu(1-\nu) \left( \frac{\phi(\theta_{i+1/2})}{\theta_{i+1/2}} - \phi(\theta_{i-1/2}) \right) \right] (Q_i^n - Q_{i-1}^n)$$

is TVD provided

$$-2 \leq \left( \frac{\phi(\theta_1)}{\theta_1} - \phi(\theta_2) \right) \leq 2$$

for all values of  $\theta_1$  and  $\theta_2$ .

Satisfied provided  $\phi(\theta)$  satisfies:

$$0 \leq \frac{\phi(\theta)}{\theta} \leq 2, \quad 0 \leq \phi(\theta) \leq 2,$$

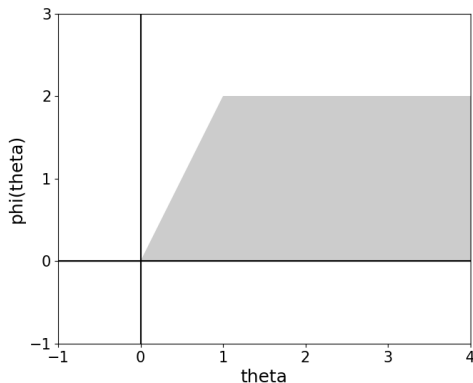
or

$$0 \leq \phi(\theta) \leq \min\text{mod}(2, 2\theta).$$



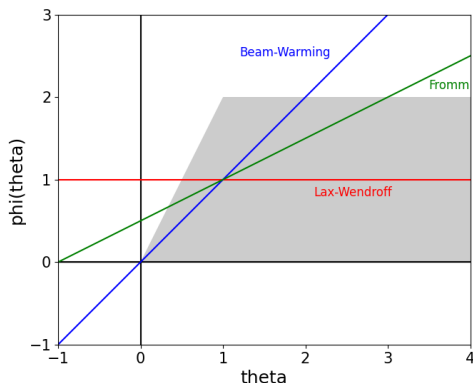
# Sweby diagram

If we plot  $\phi(\theta)$ , the curve must lie in the shaded region:



# Sweby diagram

If we plot  $\phi(\theta)$ , the curve must lie in the shaded region:



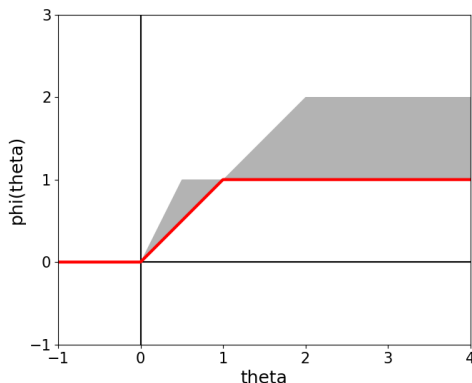
Standard second order methods go outside this region.

Recall we want  $\phi(1) = 1$  for good accuracy of smooth solutions.

# Sweby diagram

Sweby's investigation suggested best methods lie between Lax-Wendroff and Beam-Warming (and inside the TVD region).

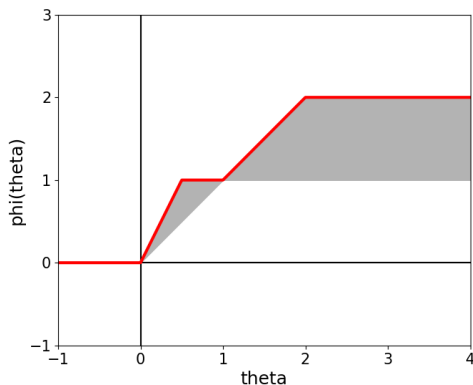
Sweby region:



$\phi(\theta) = \minmod(1, \theta)$  follows the lower limit of this region.

# Superbee method

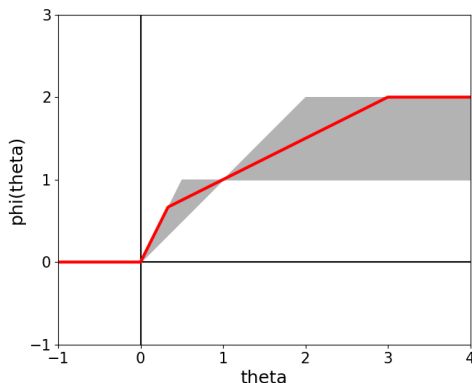
The **superbee** limiter follows the upper limit:



$$\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$$

# MC method

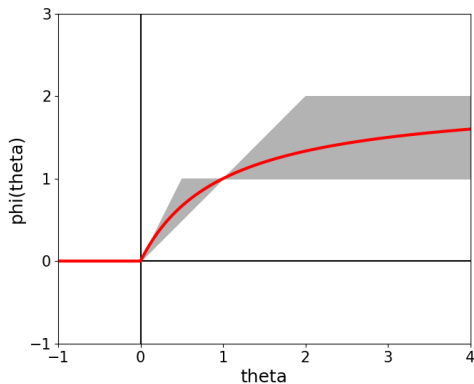
The **Monotonized Centered (MC)** limiter follows Fromm's method near  $\theta = 1$ , and is smooth at  $\theta = 1$ :



$$\phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$$

# van Leer method

The **van Leer** limiter is a smoother version of MC



$$\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}.$$

# Some popular limiters

## Linear methods:

$$\text{upwind : } \phi(\theta) = 0$$

$$\text{Lax-Wendroff : } \phi(\theta) = 1$$

$$\text{Beam-Warming : } \phi(\theta) = \theta$$

$$\text{Fromm : } \phi(\theta) = \frac{1}{2}(1 + \theta)$$

## High-resolution limiters:

$$\text{minmod : } \phi(\theta) = \text{minmod}(1, \theta)$$

$$\text{superbee : } \phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$$

$$\text{MC : } \phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$$

$$\text{van Leer : } \phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}$$