CARTIER DUAL EXAMPLES

VASILY ILIN, ALEXANDER WAUGH

Problem 1. *Let k be a field of characteristic p, and $\alpha = \mathbb{G}_{a(1)}$ denote the affine group scheme represented by $A = k[x]/x^p$, so that $\alpha(R) = \mathrm{Hom}(A, R) \cong \{a \in R \mid a^p = 0\}$ with addition inherited from R. Then α is self-dual, i.e. $A^D \cong A$ as Hopf k-algebras.*

Proof. First recall the definition of a Hopf algebra dual: $A^D = \mathrm{Hom}_{k\text{-vector spaces}}(A, k)$ with multiplication and co-multiplication given by the duals of co-multiplication and multiplication in A. As k-vector spaces A and A^D are isomorphic (both are p-dimensional), but we need to find the isomorphism that preserves the Hopf-algebra structure.

Recall the Hopf algebra structure on A:

$$m : A \otimes A \to A,\ \Delta : A \to A \otimes A$$

are given by $m(x^i \otimes x^j) = x^{i+j}$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$. Since Δ is an algebra homomorphism, we have the following useful expression

$$\Delta(x^i) = \Delta(x)^i = (x \otimes 1 + 1 \otimes x)^i = \sum_{j=0}^{i} \binom{i}{j} x^j \otimes x^{i-j}.$$

Let us now compute $m^D : A^D \to A^D \otimes A^D$ – the dual of multiplication on A. It will serve as the co-multiplication in A^D. Fix basis $\{1, x, \dots, x^{p-1}\}$ of A and its dual basis $\{\varepsilon_i\}_{i=0,\dots,p-1}$ so that $\varepsilon_i(x^j) = \delta(i,j)$. For $0 \le i, a, b \le p-1$

$$m^D(\varepsilon_i)(x^a \otimes x^b) = \varepsilon_i \circ m(x^a \otimes x^b) = \varepsilon_i(x^{a+b}) = \delta(i, a+b) = \sum_{j=0}^{i} \varepsilon_j \otimes \varepsilon_{i-j}(x^a \otimes x^b).$$

In other words, co-multiplication in A^D is given by $\Delta_{A^D}(\varepsilon_i) = \sum_{j=0}^{i} \varepsilon_j \otimes \varepsilon_{j-i}$, which looks similar to the co-multiplication formula in A but without the binomial coefficient $\binom{i}{j}$. This suggests an isomorphism $A \to A^D$ that takes x^i to $\lambda_i \varepsilon_i$ for some non-zero scalar λ_i.

To determine λ_i we compute $\Delta^D : A^D \otimes A^D \to A^D$ – the dual of co-multiplication on A, which serves as multiplication on A^D. For $0 \le i, a, b \le p-1$

$$\begin{aligned}\Delta^D(\varepsilon_a \otimes \varepsilon_b)(x^i) &= (\varepsilon_a \otimes \varepsilon_b) \circ \Delta(x^i) = (\varepsilon_a \otimes \varepsilon_b)\left(\sum_{j=0}^{i} \binom{i}{j} x^j \otimes x^{i-j}\right) \\ &= \sum_{j=0}^{i} \binom{i}{j} \varepsilon_a(x^j)\varepsilon_b(x^{i-j}) = \binom{a+b}{a} \delta(i, a+b) = \binom{a+b}{a} \varepsilon_{a+b}(x^i).\end{aligned}$$

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In other words, multiplication in A^D is given by $m_{A^D}(\varepsilon_a \otimes \varepsilon_b) = \binom{a+b}{a}\varepsilon_{a+b}$. In particular,

$$\begin{aligned}
\varepsilon_1^2 &= m(\varepsilon_1 \otimes \varepsilon_1) = 2\varepsilon_2,\\
\varepsilon_1^3 &= m(2\varepsilon_2 \otimes \varepsilon_1) = 6\varepsilon_3,\\
&\vdots\\
\varepsilon_1^n &= m((n-1)!\varepsilon_{n-1} \otimes \varepsilon_1) = n!\varepsilon_n.
\end{aligned}$$

We now see that defining the isomorphism $\phi : A \to A^D$ by $\phi(x^n) = n!\varepsilon_n$ is a natural choice. Indeed, ϕ is a k-algebra homomorphism since

$$\phi(x^n x^m) = \phi(x^{n+m}) = (n+m)!\varepsilon_{n+m}$$

and

$$\phi(x^n)\phi(x^m) = n!m!\varepsilon_n\varepsilon_m = n!m!\binom{n+m}{n}\varepsilon_{n+m} = (n+m)!\varepsilon_{n+m}.$$

And ϕ is a Hopf algebra morphism since

$$\begin{aligned}
(\phi\otimes\phi)(\Delta(x^i)) = (\phi\otimes\phi)\left(\sum_{j=0}^{i}\binom{i}{j}x^j \otimes x^{i-j}\right) &= \sum_{j=0}^{i}\binom{i}{j}\phi(x^j)\otimes\phi(x^{i-j})\\
&= \sum_{j=0}^{i}\binom{i}{j}j!(i-j)!\varepsilon_j \otimes \varepsilon_{i-j} = i!\sum_{j=0}^{i}\varepsilon_j \otimes \varepsilon_{i-j}
\end{aligned}$$

and

$$\Delta(\phi(x^i)) = \Delta(i!\varepsilon_i) = i!\sum_{j=0}^{i}\varepsilon_j \otimes \varepsilon_{j-i}.$$

Thus, $\phi : A \to A^D$ is a k-vector space isomorphism (taking a basis to a basis) that preserves the Hopf algebra structure. Hence, it is an isomorphism of Hopf algebras. By the Yoneda lemma, the affine group schemes α and α^D represented by A and A^D are isomorphic as well.

□

Problem 2. *Let k be a field of characteristic p, and $\mathbb{G}_{a(2)}$ denote the affine group scheme represented by $A = k[x]/x^{p^2}$. The Cartier dual of $\mathbb{G}_{a(2)}$ is $\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}$.*

Proof. For convenience, let $B = k[\mathbb{G}_{a(1)}] = k[x]/x^p$. Then $C = k[\mathbb{G}_{a(1)} \times \mathbb{G}_{a(1)}] \cong B \otimes B \cong k[x,y]/(x^p, y^p)$ as Hopf algebras. Under these identifications, the collection $\{x^i y^j \mid 0 \le i, j < p\}$ is a vector space basis for C. It is then easy to describe the Hopf algebra structure on C:

$$m_C : C \otimes C \to C, \qquad x^i y^j \otimes x^k y^l \mapsto x^{i+k} y^{j+l}$$
$$\Delta_C : C \to C \otimes C, \qquad x^i y^j \mapsto \Delta_B(x^i)\Delta_B(y^j).$$

To complete the proof we will compute A^D and give an isomorphism of Hopf algebras $C \to A^D$. First notice that $A^D = \mathrm{Hom}_{k-vectorspaces}(A, k)$ is a p^2 dimensional vector space. So A^D and C are isomorphic as vector spaces. To compute co-multiplication on A^D, let $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{p^2-1}\}$ be the basis for A^D dual to $\{1, x, x^2, \ldots, x^{p^2-1}\}$. Then,

$$\begin{aligned} \Delta_{A^D}(\varepsilon_r)(x^i \otimes x^j) &= \varepsilon_r(x^{i+j}) \\ &= \begin{cases} 1, & i+j=r \\ 0, & \text{else} \end{cases}. \end{aligned}$$

So $\Delta_{A^D}(\varepsilon_r) = \sum_{i+j=r} \varepsilon_i \otimes \varepsilon_j$. For multiplication, recall that $\Delta_A(x) = x \otimes 1 + 1 \otimes x$. Then,

$$\begin{aligned} m_{A^D}(\varepsilon_i \otimes \varepsilon_j)(x^r) &= (\varepsilon_i \otimes \varepsilon_j)\Delta_A(x)^r \\ &= (\varepsilon_i \otimes \varepsilon_j) \sum_{k=0}^{r} \binom{r}{k} x^k \otimes x^{r-k} \\ &= \begin{cases} \binom{i+j}{i}, & i+j=r \\ 0, & \text{else} \end{cases}, \end{aligned}$$

implying $m_{A^D}(\varepsilon_i \otimes \varepsilon_j) = \binom{i+j}{i}\varepsilon_{i+j}$.

To motivate the Hopf algebra isomorphism between A^D and C, we recall Lucas' theorem: for non-negative integers m and n and a prime p,

$$\binom{m}{n} = \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p,$$

where

$$m = m_k p^k + m_{k-1}p^{k-1} + \cdots + m_1 p + m_0,$$

and

$$n = n_k p^k + n_{k-1}p^{k-1} + \cdots + n_1 p + n_0.$$

In our case, both i and j range between zero and $p^2 - 1$. So we may write them as

$$i = i_1 p + i_0, \qquad j = j_1 p + j_0$$

where $0 \le i_k, j_k < p$ for $k = 0, 1$. If $i_0 + j_0 \ge p$ or $i_1 + j_1 \ge p$, then the binomial coefficient $\binom{i+j}{i}$ is zero. This is exactly the same behavior we observe when multiplying $x^{i_1}y^{i_0}$ and $x^{j_1}y^{j_0}$ in C. This suggests a map $\phi : C \to A^D$ which sends $x^i y^j$ to $\lambda_{i,j}\varepsilon_{ip+j}$, where $\lambda_{i,j} \ne 0$. Since

$k[x]/(x^p) \subset C$ and $k[y]/(y^p) \subset C$ as Hopf algebras, we should take $\lambda_{i,0} = i!$ and $\lambda_{0,j} = j!$ just as in the previous problem. For the mixed terms $x^i y^j$, notice

$$\phi(x^i)\phi(y^j) = i!j!\varepsilon_{ip}\varepsilon_j = i!j!\binom{ip+j}{ip}\varepsilon_{ip+j} = i!j!\binom{i}{i}\binom{j}{0}\varepsilon_{ip+j} = i!j!\varepsilon_{ip+j}.$$

So we can take $\lambda_{i,j} = i!j!$. This map respects multiplication since

$$\begin{aligned}
\phi(x^i)\phi(y^j) &= i!j!\binom{ip+j}{ip}\varepsilon_{ip+j} = i!j!\binom{i}{i}\binom{j}{0}\varepsilon_{ip+j} = \phi(x^i y^j),\\
\phi(x^i)\phi(x^k) &= i!k!\binom{(i+k)p}{ip}\varepsilon_{(i+k)p} = i!k!\binom{i+k}{i}\varepsilon_{(i+k)p} = \phi(x^{i+k}),\\
\phi(y^j)\phi(y^l) &= j!l!\binom{j+l}{j}\varepsilon_{j+l} = (j+l)!\varepsilon_{j+l} = \phi(y^{j+l}).
\end{aligned}$$

It also respects co-multiplication:

$$\begin{aligned}
(\phi\otimes\phi)\Delta_C(x^r y^s) &= \phi\otimes\phi\left[\left(\sum_{i+j=r}\binom{i+j}{i}x^i\otimes x^j\right)\left(\sum_{k+l=s}\binom{k+l}{k}y^k\otimes y^l\right)\right]\\
&= \left(\sum_{i+j=r}(i+j)!\varepsilon_{ip}\otimes\varepsilon_{jp}\right)\left(\sum_{k+l=s}(k+l)!\varepsilon_k\otimes\varepsilon_l\right)\\
&= \Delta_{A^D}(\phi(x^r)\phi(y^s))\\
&= \Delta_{A^D}\phi(x^r y^s),
\end{aligned}$$

where the third line holds because Δ_{A^D} is an algebra homomorphism. $\square$

Remark. *The above result generalizes to* $(\mathbb{G}_a(n))^D = \prod_{i=1}^n \mathbb{G}_{a(1)}$ *with the dual of* $k[x]/x^{p^n}$ *being* $k[x_1,\ldots,x_n]/(x_1^p,\ldots,x_n^p)$.

Problem 3. *Fix a field k. Let μ_n and $k\mathbb{Z}/n_{\text{const}}$ denote the group schemes represented by $k[x]/(x^n-1)$ and $k^{\mathbb{Z}/n} := \mathrm{Hom}_{k-vectorspaces}(k\mathbb{Z}/n, k)$. These group schemes are Cartier dual to each other.*

Proof. For convenience, let $A = k[x]/(x^n-1)$ and $B = k^{\mathbb{Z}/n}$. Recall the multiplication and co-multiplication structures on A:

$$m_A : A \otimes A \to A, \qquad x^i \otimes x^j \mapsto x^{i+j \mod n},$$
$$\Delta_A : A \to A \otimes A, \qquad x \mapsto x \otimes x.$$

To describe the multiplication and co-multiplication for B, fix a (vector space) basis $\{e_0, \ldots, e_{n-1}\}$ for $k\mathbb{Z}/n$ where $e_k = k \in \mathbb{Z}/n$, and denote the dual basis by $\{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$. Then,

$$m_B : B \otimes B \to B, \qquad \varepsilon_i \otimes \varepsilon_j \mapsto \varepsilon_i \varepsilon_j,$$
$$\Delta_B : B \to B \otimes B, \qquad \varepsilon_r \mapsto \sum_{e_i e_j = e_r} \varepsilon_i \otimes \varepsilon_j.$$

We must exhibit a Hopf-algebra isomorphism $A^D \to B$. That is, a k-vector space isomorphism which is compatible with the multiplication and co-multiplication structures on A^D and B. Fix a (vector space) basis $\{1, x, x^2, \ldots, x^{n-1}\}$ for A and let $\{\chi_0, \chi_1, \chi_2, \ldots, \chi_{n-1}\}$ be the dual basis. For $0 \le i, j \le n-1$,

$$\begin{aligned} m_{A^D}(\chi_i \otimes \chi_j)(x^k) &= \Delta_A^D(\chi_i \otimes \chi_j)(x^k) \\ &= (\chi_i \otimes \chi_j)\Delta_A(x^k) \\ &= (\chi_i \otimes \chi_j)(x^k \otimes x^k) \\ &= \begin{cases} 1, & i = j = k \\ 0, & \text{else} \end{cases}. \end{aligned}$$

It is then easy to see that $m_{A^D}(\chi_i \otimes \chi_j) = \chi_i \chi_j$. Similarly,

$$\begin{aligned} \Delta_{A^D}(\chi_r)(x^i \otimes x^j) &= m_A^D(\chi_r)(x^i \otimes x^j) \\ &= \chi_r(x^{i+j \mod n}) \\ &= \begin{cases} 1, & i + j \equiv r, \\ 0, & \text{else} \end{cases}. \end{aligned}$$

While somewhat harder to see, it is easily verified that $\Delta_{A^D}(\chi_r) = \sum_{i+j\equiv r} \chi_i \otimes \chi_j$. Let $\phi : A^D \to B$ be the vector space isomorphism sending χ_i to ε_i. This is a map of Hopf-algebras:

$$(\phi \otimes \phi)\Delta_{A^D}(\chi_r) = (\phi \otimes \phi)(\sum_{i+j\equiv r} \chi_i \otimes \chi_j) = \sum_{e_i e_j = e_r} \varepsilon_i \otimes \varepsilon_j = \Delta_B \phi(\chi_r)$$

and

$$\phi m_{A^D}(\chi_i \otimes \chi_j) = \phi(\chi_i \chi_j) = \varepsilon_i \varepsilon_j = m_B(\phi \otimes \phi)(\chi_i \otimes \chi_j).$$

One can do a similar computation showing $B^D \cong A$ as Hopf-algebras to prove $(k\mathbb{Z}/n_{\text{const}})^D$ is isomorphic to μ_n. $\square$