

Derivation of the Helfrich Free Energy

1 Introduction

The Helfrich free energy is a continuum description of the bending elasticity of lipid membranes and other flexible two-dimensional surfaces. It is particularly useful for understanding membrane shape fluctuations, vesicle morphology, and the mechanical responses of lipid bilayers under various constraints.

In this derivation, we start from symmetry and scaling arguments. The goal is to identify the most general form of a curvature-dependent free energy functional that is compatible with the symmetry and geometry of a two-dimensional manifold embedded in three-dimensional space. The final form of the Helfrich free energy captures the essential physics of bending and can be written in terms of the mean curvature H and Gaussian curvature K of the membrane surface.

2 Preliminaries

2.1 Membrane Geometry

Consider a closed membrane modeled as a two-dimensional surface without edges, parameterized by coordinates (u, v) , and embedded in three-dimensional Euclidean space. Let the surface be described by a vector function:

$$\mathbf{X}(u, v) : (u, v) \mapsto \mathbb{R}^3.$$

The metric on the surface is given by:

$$g_{\alpha\beta} = \frac{\partial \mathbf{X}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{X}}{\partial u^\beta},$$

where $\alpha, \beta \in \{1, 2\}$ and $(u^1, u^2) = (u, v)$.

The infinitesimal area element on the surface is:

$$dA = \sqrt{\det(g_{\alpha\beta})} du dv.$$

The curvature of the surface is described by two principal curvatures, c_1 and c_2 . From these, one defines:

$$H = \frac{c_1 + c_2}{2}, \quad K = c_1 c_2.$$

Here, H is the mean curvature and K is the Gaussian curvature.

2.2 Symmetry and Invariance Arguments

A lipid bilayer membrane is a thin fluid sheet with no intrinsic reference curvature (unless deformations break symmetries by, for example, introducing a spontaneous curvature). Physically, the free energy should be invariant under:

- Rotations and translations in three-dimensional space.
- Reparameterizations of the surface coordinates (u, v) .

These symmetries strongly constrain the possible terms that can appear in the free energy. The simplest scalar invariants constructed from the geometry of the surface are H and K . Because the membrane free energy should not depend on directions in the embedding space, one expects the free energy density to be a scalar constructed from H and K .

3 Constructing the Free Energy Functional

The key insight is that the free energy should be an integral over the surface of a local free energy density that depends on curvature invariants:

$$\mathcal{F} = \int dA f(H, K).$$

We must now determine the form of $f(H, K)$ based on symmetry and the principle of simplicity (lowest-order terms in curvature).

3.1 Lowest-Order Expansion in Curvature

For small curvature deformations, one expands $f(H, K)$ in powers of H and K :

$$f(H, K) = \underbrace{f_0}_{\text{constant term}} + \underbrace{f_1 H}_{\text{linear term}} + \underbrace{f_2 H^2}_{\text{quadratic term}} + \underbrace{f_3 K}_{\text{Gaussian term}} + \dots$$

Since we describe a symmetric bilayer membrane with no preferred direction for bending initially, there should be no linear term in H unless an intrinsic or spontaneous curvature c_0 is introduced. Thus, if the membrane is symmetrical (no preferred curvature), the lowest-order form that respects these constraints is:

$$f(H, K) = \frac{\kappa}{2}(2H - c_0)^2 + \bar{\kappa}K + \text{constant},$$

where:

- κ : bending rigidity coefficient.
- c_0 : spontaneous curvature (could vanish for symmetric bilayers).
- $\bar{\kappa}$: Gaussian curvature modulus.

The constant term only shifts the overall free energy and does not affect shape, thus it can be dropped for shape optimization. Typically, one writes:

$$f(H, K) = \frac{\kappa}{2}(2H - c_0)^2 + \bar{\kappa}K.$$

3.2 Final Form of the Helfrich Free Energy

Combining all of the above, the Helfrich free energy can be written as:

$$\mathcal{F} = \int dA \left[\frac{\kappa}{2} (2H - c_0)^2 + \bar{\kappa} K \right].$$

For a closed vesicle with a fixed topology, the integral of the Gaussian curvature term K is related to the Euler characteristic χ of the surface by the Gauss-Bonnet theorem:

$$\int dA K = 2\pi\chi.$$

Since χ is a topological invariant, the Gaussian term contributes only a constant shift to the free energy. Therefore, for fixed topology, the shape depends primarily on minimizing the bending term:

$$\mathcal{F} = \int dA \frac{\kappa}{2} (2H - c_0)^2 + \text{constant}.$$

This is the classic Helfrich free energy functional for bending elasticity of fluid membranes.

4 Conclusion

The Helfrich free energy emerges from a combination of symmetry arguments, invariance, and a low-order expansion in terms of curvature invariants. It encapsulates the bending rigidity and the role of spontaneous curvature, and serves as a cornerstone model in the theoretical description of lipid bilayers and other flexible surfaces in soft matter physics.