Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Laboratory 08

Finite Element method for the heat equation: convergence analysis and a nonlinear time dependent problem

Exercise 1.

Let $\Omega = (0,1)^3$ be the unit cube and $T = 1$. Let us consider the following time-dependent problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u) = f & \text{in } \Omega \times (0,T), & \text{(1a)} \\ u = g & \text{on } \partial\Omega \times (0,T)\ , & \text{(1b)} \\ u = u_0 & \text{in } \Omega \times \{0\}, & \text{(1c)} \end{cases}$$

with $\mu = 1$.

1.1. Knowing that the exact solution to problem (1) is

$$u_{\text{ex}}(\mathbf{x}, t) = \sin(5\pi t)\sin(2\pi x)\sin(3\pi y)\sin(4\pi z)\ ,$$

where $\mathbf{x} = (x,y,z)^T$, compute the forcing term f, the boundary datum g and the initial condition u_0.

1.2. Starting from the code of Laboratory 07, implement a finite element solver for problem (1) with the data computed at Point 1. Then, implement a method `double Heat::compute_error(const VectorTools::NormType & norm_type)` that computes the L^2 and H^1 norms of the error at the final time T between the numerical and the exact solution.

1.3. Consider the mesh `mesh/mesh-cube-10.msh` (corresponding to a mesh size $h = 0.1$). Solve the problem (1) with the implicit Euler method (setting $\theta = 1$), with time steps $\Delta t = 0.25, 0.125, 0.0625, 0.03125, 0.015625$ and linear finite elements (degree $r = 1$). Compute the error L^2 and H^1 norms of the error at the final time T and estimate their convergence order with respect to Δt.

1.4. Repeat the previous point with quadratic polynomials (degree $r = 2$).

1.5. Repeat Points 3 and 4 using the mesh `mesh/mesh-cube-20.msh` and using the Crank-Nicolson method (i.e. setting $\theta = \frac{1}{2}$).

Exercise 2.

Let $\Omega = (0,1)^3$ be the unit cube and $T = 1$. Let us consider the following nonlinear, time-dependent problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \boldsymbol{\nabla} \cdot \left(\left(\mu_0 + \mu_1 u^2 \right) \boldsymbol{\nabla} u \right) = f & \text{in } \Omega \times (0,T), \quad (2a) \\ u = g & \text{on } \partial\Omega \times (0,T)\ , \quad (2b) \\ u = u_0 & \text{in } \Omega \times \{0\}, \quad (2c) \end{cases}$$

with $\mu_0 = 0.1$, $\mu_1 = 1$, $g(\mathbf{x},t) = 0$, $u_0(\mathbf{x}) = 0$ and

$$f(\mathbf{x},t) = \begin{cases} 2 & \text{if } t < 0.25\ , \\ 0 & \text{if } t \geq 0.25\ . \end{cases}$$

2.1. Implement in `deal.II` a finite element solver for problem (2), using the implicit Euler method for time discretization and Newton's method for linearization. Then, compute the solution using the mesh `mesh/mesh-cube-20.msh`, with linear finite elements (degree $r = 1$) and $\Delta t = 0.05$.