Numerical Methods for Partial Differential Equations
A.Y. 2023/2024

Written exam - February 5, 2024

Maximum score: 26. Duration: 2h 00m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded on WeBeep. All uploaded files (including source code) should be appropriately referenced when answering to questions on paper (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Consider the problem

$$\begin{cases} Lu = f & \text{in } \Omega \subset \mathbb{R}^2 \,, \\ u = g & \text{on } \Gamma_\mathrm{D} \,, \\ \mu \dfrac{\partial u}{\partial \mathbf{n}} - (\mathbf{b} \cdot \mathbf{n})\, u = \phi & \text{on } \Gamma_\mathrm{N} \,, \end{cases} \tag{1}$$

where $\Gamma_\mathrm{D} \cup \Gamma_\mathrm{N} = \partial\Omega$,

$$Lu = -\mathrm{div}(\mu \boldsymbol{\nabla} u) + \mathrm{div}(\mathbf{b}u) + \sigma u \,,$$

μ is a positive constant, σ a non-negative constant, $\mathbf{b} \in \mathbb{R}^2$ a constant vector, and f, g and ϕ are given functions.

1.1. [*2 pt*] Write the weak formulation of (1), with the right choice of spaces.

1.2. [*3 pt*] **Prove** that the weak formulation has a unique solution and specify which conditions should be required on the problem data (that is, μ, $\mathbf{b}$, σ, g and ϕ).

1.3. [*2 pt*] Approximate the weak formulation obtained at point 1.2 by piecewise-quadratic finite elements and provide (without proof) the results of stability and convergence, with the proper norms.

1.4. [*1 pt*] If, in the formulation of point 1.3, the mesh size h is reduced by a factor 5 (that is, it becomes $h/5$), by which factor will the error be reduced?

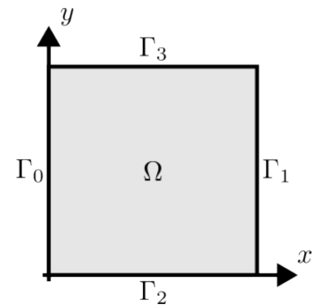

Figure 1: Domain for Exercise 1. Each boundary subset Γ_i corresponds to the tag i in the provided mesh files.

1.5. [*4 pt*] Using `deal.II`, implement a solver for problem (1). Consider the square domain of Figure 1, and the following data:

$$
\begin{aligned}
&\Gamma_{\mathrm{D}} = \Gamma_0 \cup \Gamma_2 \,, && \Gamma_{\mathrm{N}} = \Gamma_1 \cup \Gamma_3 \,, \\
&\mu = 1 \,, && \sigma = 1 \,, \\
&\mathbf{b} = (-1, -1)^T \,, && g(x, y) = \sin(3\pi x) + \sin(2\pi y) \,, \\
&\phi(x, y) = 1 \,, && f(x, y) = 0 \,.
\end{aligned}
$$

Use the mesh $\mathcal{T}_h$ of triangular finite elements of size $h = 0.1$, which can be found at the following link: `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh` (refer to Figure 1 for the boundary tags). Consider finite elements of degree $r = 2$. Upload the source code of the solver, a plot of the solution and a plot of the solution along the line $x = 0.25$.

1.6. [*3 pt*] Assume now to compute new definitions for functions f, g and ϕ such that the exact solution to (1) is $u_{\mathrm{ex}}(x, y) = x + y$. What is the expected error between u_{ex} and the approximate solution u_h?

Exercise 2.

Consider the problem

$$
\begin{cases}
Lu = f & 0 < x < 1 \,, \\
u(0) = u(1) = 0 \,,
\end{cases}
\tag{2}
$$

where $Lu = -u'' + \alpha u$ and $\alpha > 0$.

2.1. [*1 pt*] Write the (additive) Dirichlet-Neumann (DN) and the Neumann-Neumann (NN) iterations for problem (2), where the domain $\Omega = (0, 1)$ is split into the subdomains $\Omega_1 = (0, \gamma)$ and $\Omega_2 = (\gamma, 1)$ for a suitable $\gamma \in (0, 1)$.

2.2. [*2 pt*] After approximating (2) by piecewise linear finite elements, formulate both (DN) and (NN) iterations as preconditioned iterative methods.

2.3. [*1 pt*] Discuss the convergence properties of the (DN) and (NN) iterative schemes obtained at point 2.2, for a partition of Ω into two subdomains.

2.4. [*3 pt*] Report the rate of convergence of the two methods and comment on their properties of scalability and optimality in the case of partitions with $M > 2$ subdomains.

2.5. [*2 pt*] Using `deal.II`, implement a solver that uses the DN algorithm to solve (2). Set $\alpha = 1$ and $\gamma = 0.75$. Report the stopping criteria used and number of iterations needed for convergence. Upload the source code and a plot of the solution at the first and at the final iteration. **Hint:** combine the "Group datasets" and "Plot over line" ParaView filters.

2.6. [*2 pt*] Repeat previous point setting $\gamma = 0.25$. Discuss the convergence of the DN algorithm in this case, and propose a possible strategy to improve it.