Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Laboratory 09
Finite Element method for the Stokes problem

Exercise 1.

Let $\Omega \subset \mathbb{R}^3$ be the domain shown in Figure 1. Let us consider the stationary Stokes problem:

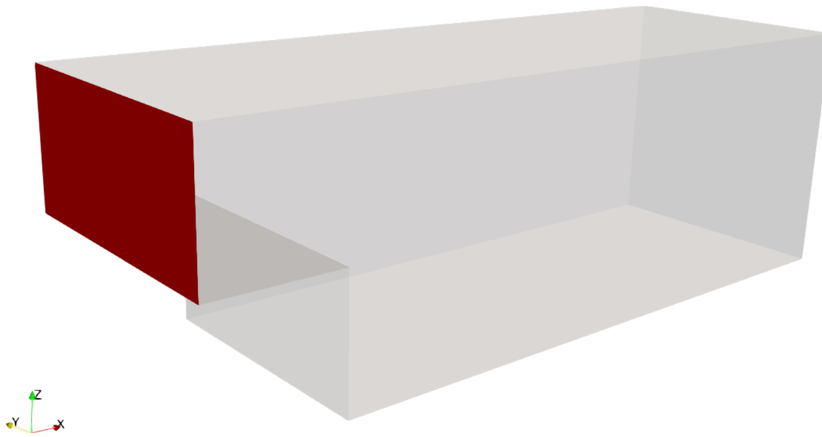
$$\begin{cases} -\nu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} & \text{in } \Omega, & (1a) \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 & \text{in } \Omega, & (1b) \\ \mathbf{u} = \mathbf{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, & (1c) \\ \nu(\boldsymbol{\nabla}\mathbf{u})\mathbf{n} - p\mathbf{n} = -p_{\text{out}}\mathbf{n} & \text{on } \Gamma_{\text{out}}, & (1d) \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\text{wall}}, & (1e) \end{cases}$$

where $\mathbf{u} : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}$ are the velocity and pressure fields of a viscous, incompressible fluid, $\nu = 1\,\text{m}^2/\text{s}$, $\mathbf{f} = \mathbf{0}$ m/s^2, $\mathbf{u}_{\text{in}} = (-\alpha\, y\,(2-y)\,(1-z)(2-z)\,\text{m/s}, 0\,\text{m/s}, 0\,\text{m/s})^T$, $\alpha = 1/(\text{m}^3\cdot\text{s})$, $p_{\text{out}} = 10\,\text{Pa}$.

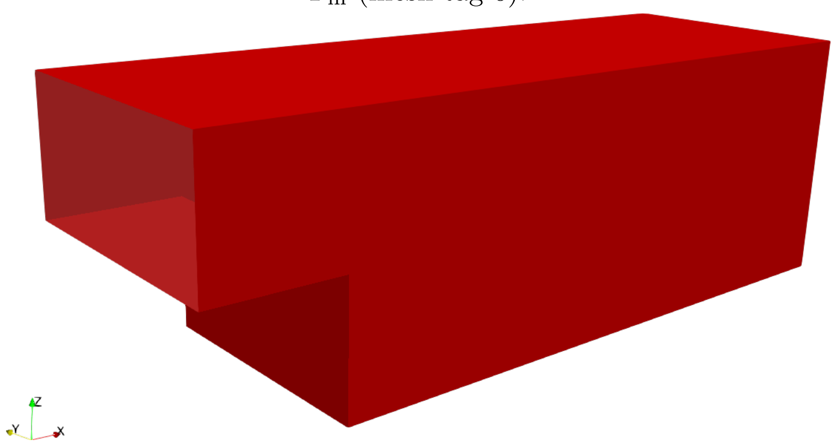
1.1. Derive the weak formulation of the problem.

1.2. Derive the finite element formulation to problem (1).

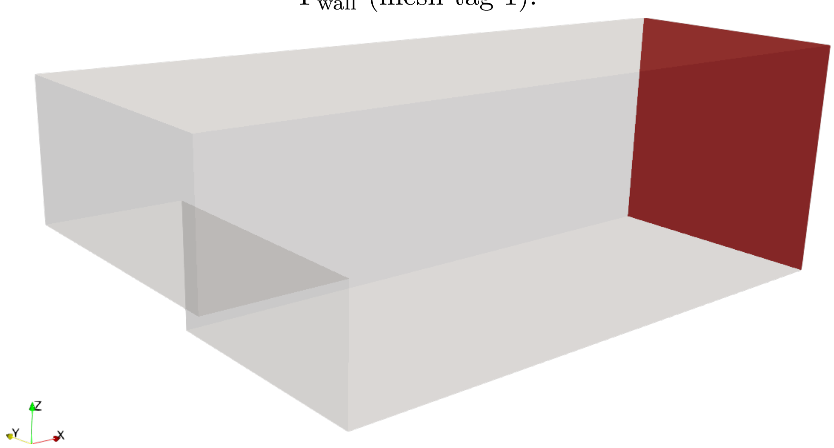
1.3. Implement a finite element solver for (1) and compute its numerical solution using the mesh `mesh/mesh-step-5.msh` (whose boundary tags are described in Figure 1).



Γ_{in} (mesh tag 0).



Γ_{wall} (mesh tag 1).



Γ_{out} (mesh tag 2).

Figure 1: Domain and partition of its boundary.