Numerical Methods for Partial Differential Equations
A.Y. 2023/2024

Exam fac-simile

Maximum score: 26. Duration: 2h 00m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded on WeBeep. All uploaded files (including source code) should be appropriately referenced when answering to questions on paper (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Let us consider the domain $\Omega = (0,1)^2$, with boundary $\partial\Omega = \Gamma_\mathrm{D} \cup \Gamma_\mathrm{N} = \cup_{i=0}^{3}\Gamma_i$, where $\Gamma_\mathrm{D} = \Gamma_0 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_\mathrm{N} = \Gamma_1 = \partial\Omega \backslash \Gamma_\mathrm{D}$. More formally:

$$
\begin{aligned}
\Gamma_0 &= \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 0\} \ , \\
\Gamma_1 &= \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 1\} \ , \\
\Gamma_2 &= \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 0\} \ , \\
\Gamma_3 &= \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 1\} \ .
\end{aligned}
$$

Let $\mathbf{n}$ denote the outward directed unit vector normal to $\partial\Omega$. The domain is reported in Figure 1.

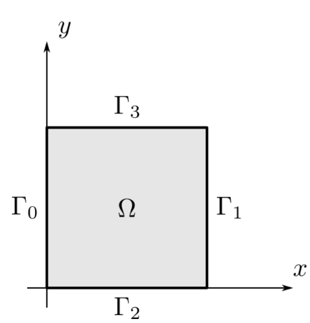

Figure 1: Domain Ω and boundary $\partial\Omega = \cup_{i=0}^{3}\Gamma_i$. Each boundary subset Γ_i corresponds to the tag i in the provided mesh files.

Let us consider the following strong problem: find $u : \Omega \to \mathbb{R}$ such that

$$
\begin{cases}
-\boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u) + \sigma u = f & \text{in } \Omega \ , \\
u = g & \text{on } \Gamma_\mathrm{D} \ , \\
\mu \boldsymbol{\nabla} u \cdot \mathbf{n} = 0 & \text{on } \Gamma_\mathrm{N} \ ,
\end{cases}
\tag{1}
$$

where $\mu \in (0, \infty)$, $\sigma \in [0, \infty)$, $f : \Omega \to \mathbb{R}$ and $g : \Gamma_\mathrm{D} \to \mathbb{R}$.

1.1. Write the weak formulation of problem (1), including the definitions of all terms involved, the choice of the function spaces and the derivation of the formulation.

1.2. Write the Galerkin-finite element approximation of problem (1), with Finite Elements of degree $r \geq 1$. Include the definitions of the function spaces, the basis functions, and the approximate solution.

1.3. Implement the Finite Element approximation of problem (1) in `deal.II`, with the following data:

$$
\begin{aligned}
\mu = \sigma &= 1\ , \\
f(x_1, x_2) &= \left(1 + \frac{\pi^2}{4}\right) \sin\left(\frac{\pi}{2} x_1\right) x_2\ , \\
g(x_1, x_2) &= \begin{cases} 0 & \text{on } \Gamma_0 \cup \Gamma_2\ , \\ \sin\left(\frac{\pi}{2} x_1\right) & \text{on } \Gamma_3\ . \end{cases}
\end{aligned}
$$

Use the mesh $\mathcal{T}_h$ of triangular finite elements of size $h = 0.1$, which can be found at the following link: `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh` (refer to Figure 1 for the boundary tags). Use finite elements built over the polynomial space $\mathbb{P}_2$. Comment the results and upload the source code.

1.4. Following the answer of previous point, suitably visualize the finite element solution u_h in Paraview and upload the corresponding picture file.

1.5. By knowing that the exact solution is $u(x_1, x_2) = \sin\left(\frac{\pi}{2} x_1\right) x_2$, compute the errors $\|u - u_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ for different values of the mesh size $h = 0.1, 0.05, 0.025$ and 0.0125, still using $\mathbb{P}_2$ FEs. Upload the source code and report the values of the errors.

1.6. Following the previous point, estimate the convergence orders of the errors with respect to h. Report the procedure used, compare the results with the theory, and critically discuss them. What are the expected theoretical convergence results by using $\mathbb{P}_r$ finite elements, for $r \geq 1$?

Exercise 2.

Let Ω be the same domain as in previous exercise. Consider the Stokes problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} & \text{in } \Omega\ , \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 & \text{in } \Omega\ , \\ \nu\dfrac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = \mathbf{0} & \text{on } \Gamma_1\ , \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_0\cup\Gamma_2\cup\Gamma_3\,; \end{cases} \tag{2}$$

where $\nu \in (0,\infty)$, $\mathbf{g} : (\Gamma_0\cup\Gamma_2\cup\Gamma_3) \to \mathbb{R}^2$.

2.1. Write the weak formulation of problem (2), including the definitions of all terms involved, the choice of the function spaces and the derivation of the formulation.

2.2. Write the Galerkin-finite element approximation of problem (2), with Finite Elements of degree $r \geq 1$ for $\mathbf{u}$ and of degree $q \geq 1$ for p. Include the definitions of the function spaces, the basis functions, and the approximate solution.

2.3. Formulate necessary and sufficient conditions for the existence and uniqueness of the solution of the Galerkin problem. Provide at least an example of a pair of velocity and pressure spaces that do not satisfy the inf-sup condition, together with one example that do satisfy the inf-sup condition. For the latter, indicate the expected error estimate.

2.4. Implement the Finite Element approximation of problem (1) in `deal.II`, with the following data: $\mu = 1$, $\mathbf{f} = \mathbf{0}$ and

$$\mathbf{g}(x_1,x_2) = \begin{cases} \mathbf{0} & \text{on } \Gamma_2\cup\Gamma_3\ , \\ \left(\dfrac{27}{4}x_2(1-x_2)^2, 0\right)^T & \text{on } \Gamma_1\ . \end{cases}$$

Use the mesh $\mathcal{T}_h$ of triangular finite elements of size $h = 0.1$, which can be found at the following link: `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh` (refer to Figure 1 for the boundary tags). Use finite elements built over the polynomial space $\mathbb{P}_2$ for velocity and $\mathbb{P}_1$. Comment the results and upload the source code.

2.5. Suitably visualize the finite element solutions $\mathbf{u}_h$ and p_h in Paraview. Upload the corresponding picture files. Visualize the field $\boldsymbol{\nabla}\cdot\mathbf{u}_h$, upload the resulting picture file and motivate the result.

2.6. Repeat points 4 and 5 above using $\mathbb{P}_1 - \mathbb{P}_1$ finite elements and motivate the result on the basis of the theory.

Exercise 3.

Consider the following elliptic problem:

$$\begin{cases} Lu = f & \text{in } \Omega = (0,1) \ , \\ u(0) = u(1) = 0 \ , \end{cases} \tag{3}$$

with $Lu = -u'' + \sigma u$ and $\sigma \in [0, \infty)$.

3.1. Let Ω be partitioned into the two non-overlapping subintervals $\Omega_1 = (0, \gamma)$ and $\Omega_2 = (\gamma, 1)$, for $\gamma \in (0, 1)$. Derive the Steklov-Poincaré (SP) problem on the interface $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$ associated with (3).

3.2. Explain the relation between (SP) and the Dirichlet-Neumann (DN) and Neumann-Neumann (NN) iterations to solve (3).

3.3. Discuss the convergence properties of DN and NN iterations.