Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Written exam - September 2, 2024

Maximum score: 26. Duration: 2h 00m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Consider the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu u'' + \kappa u = f & 0 < x < 1,\ t > 0\ , \\ u(x = 0, t) = 0 & t > 0\ , \\ u'(x = 1, t) = 0 & t > 0\ , \\ u(x, t = 0) = u_0(x) & 0 < x < 1\ , \end{cases} \tag{1}$$

where μ is a positive constant, κ a non-negative constant, u_0 and f two given functions.

1.1. [*2 pt*] Approximate (1) by the piecewise linear finite element method in space and the backward Euler method in time.

1.2. [*2 pt*] Report the stability property and the convergence property of the method introduced at point 1.1, with the corresponding error estimate.

1.3. [*2 pt*] For every time step t^n, consider the problem obtained at $t = t^n$ and propose a *Dirichlet-Neumann* (DN) iterative method for its solution, after partitioning the interval $[0, 1]$ into $[0, L] \cup [L, 1]$, for a suitable $0 < L < 1$.

1.4. [*1 pt*] Discuss the convergence properties of the DN method introduced at point 1.3.

1.5. [*3 pt*] Using `deal.II`, implement a solver for the DN method introduced at point 1.3, applied to the first time discretization step (i.e. to compute u^1 at time $t^1 = \Delta t$). Use the following data:

$$\mu = 1\ , \qquad \kappa = 1\ , \qquad u_0(x) = 0\ , \qquad f = 1\ .$$

Set $L = 0.5$ and use a relaxation coefficient of $\lambda = 0.5$. Use a mesh of size $h = 0.1$, generated using `deal.II`'s internal functions. Consider a time step $\Delta t = 0.1$. Upload the source code of the solver, and report the stopping criterion used for DN iterations.

1.6. [*2 pt*] Report the number of DN iterations needed for convergence, and plot the solution u^1 in space. **Hint:** combine the Paraview filters "Group datasets" and "Plot over line".

1.7. [*4 pt*] Implement a solver for the time-dependent problem (1), using the solver implemented at point 1.5 at every time step. Solve the problem up to a final time $T = 1$, with the same data and discretization as point 1.5. Upload the source code of the solver.

1.8. [*2 pt*] Plot the solution obtained at point 1.8 as a function of time at $x = 0.5$. **Hint:** only save the final solution of the DN algorithm, then combine the Paraview point selection tool and the filter "Plot selection over time".

Exercise 2.

Consider the generalized Stokes problem:

$$\begin{cases} \alpha\mathbf{u} - \mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} & \text{in } \Omega \subset \mathbb{R}^2\,, \\ \text{div }\mathbf{u} = 0 & \text{in } \Omega \subset \mathbb{R}^2\,, \\ \mathbf{u} = \phi & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

where α and μ are two positive constants, f and ϕ two given funtions.

2.1. [*2 pt*] Provide the weak formulation of problem (2).

2.2. [*2 pt*] Approximate the problem obtained at point 2.1 by the Taylor-Hood elements of degree 3 for the velocity components and 2 for the pressure. Write the finite element subspaces.

2.3. [*2 pt*] Discuss the stability and convergence of the problem obtained at point 2.2 and provide the associated stability estimate and error estimate.

2.4. [*2 pt*] Provide the algebraic formulation of the problem obtained at point 2.2 and indicate a possible solution algorithm for the associated linear algebraic system.