Numerical Methods for Partial Differential Equations
A.Y. 2023/2024

Written exam - January 15, 2024

Maximum score: 26. Duration: 2h 00m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded on WeBeep. All uploaded files (including source code) should be appropriately referenced when answering to questions on paper (e.g. "The solution is depicted in `exercise-1-plot.png`").

Exercise 1.

Consider the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + L\,u = f & \mathbf{x} \in \Omega \subset \mathbb{R}^2,\ t > 0\ , \\ u(\mathbf{x}, t) = 0 & \mathbf{x} \in \partial\Omega,\ t > 0\ , \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega\ , \end{cases} \tag{1}$$

with

$$L\,u = -\mu \Delta u + \mathbf{b} \cdot \boldsymbol{\nabla} u\ ,$$

where $\mu > 0$ and $\mathbf{b} \in \mathbb{R}^2$ are given constants, $f \in L^2(\Omega \times (0, +\infty))$ and $u_0 \in L^2(\Omega)$ are given functions.

1.1. [*1 pt*] Write the weak formulation of problem (1) in the following form:

$$\begin{aligned} &\forall t > 0\ ,\ \text{find } u \in V \text{ such that} \\ &\int_\Omega \frac{\partial u}{\partial t} v d\mathbf{x} + a(u, v) = \int_\Omega f v d\mathbf{x} \qquad \forall v \in V \\ &\text{with } u|_{t=0} = u_0\ , \end{aligned} \tag{2}$$

providing the precise definition of V and $a(\cdot, \cdot)$ in the current case.

1.2. [*1 pt*] Discuss the coercivity property of $a(\cdot, \cdot)$.

1.3. [*2 pt*] Approximate (2) using piecewise linear finite elements in space and the Crank-Nicolson method in time. Write the corresponding formulation.

1.4. [*3 pt*] When $f = 0$, discuss the stability property and the expected error estimate of the approximation introduced at point 1.3.

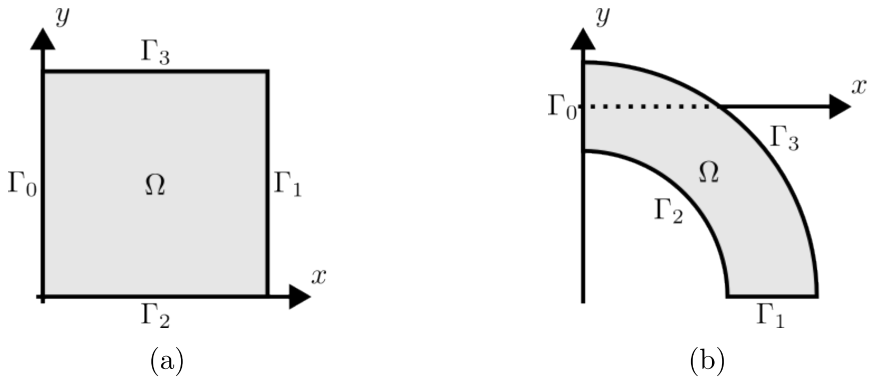

Figure 1: Each boundary subset Γ_i corresponds to the tag i in the provided mesh files. (a) Domain for Exercise 1. (b) Domain for Exercise 2.

1.5. [*1 pt*] Consider the domain Ω depicted in Figure 1a. Assume that the exact solution to (1) is the function

$$u_{\text{ex}}(\mathbf{x}, t) = \sin(2\pi t)\,\sin(\pi x)\,\sin(2\pi y)\ ,$$

and that $\mu = 1$, $\mathbf{b} = (0.1, 0.2)^T$. Determine the functions f and u_0 so that u_{ex} is a solution to (1).

1.6. [*4 pt*] Using `deal.II`, implement a finite element solver for (2), with the data determined at point 1.5. Use the mesh $\mathcal{T}_h$ of triangular finite elements of size $h = 0.1$, which can be found at the following link: `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh` (refer to Figure 1a for the boundary tags). Use a time discretization step $\Delta t = 0.05$, and solve the problem up to a final time $T = 1.0$. Upload the source code of the solver, a screenshot of the solution at the final time T and a plot of the final solution along the line $x = 0.25$.

(**Hint:** from within an object derived from `Function<dim>`, the current time can be accessed through the `get_time` method.)

1.7. [*3 pt*] With the data determined at point 1.5, perform a convergence study for the implemented solver, computing the $L^2(\Omega)$ norm of the error at the final simulation time. Upload the source code, report the estimated convergence orders, and discuss the results in light of the theory.

Exercise 2.

Consider the Stokes equations:

$$\begin{cases} -\mu\Delta\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f} & \mathbf{x} \in \Omega \subset \mathbb{R}^2 \,, \\ \operatorname{div}\, \mathbf{u} = 0 & \mathbf{x} \in \Omega \\ \mu\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{0} & \mathbf{x} \in \partial\Omega \,. \end{cases} \tag{3}$$

2.1. [*2 pt*] Provide the weak formulation for problem (3).

2.2. [*2 pt*] Approximate the formulation established at point 2.1 using the Taylor-Hood finite elements, and write the corresponding finite element spaces for both velocity and pressure.

2.3. [*2 pt*] Discuss the well-posedness (i.e. existence and uniqueness of solutions), stability and error estimate of the numerical problem obtained at point 2.2.

2.4. [*2 pt*] For a generic approximation of problem (3), formulate the inf-sup condition and explain its relation with the existence of pressure spurious modes.

2.5. [*3 pt*] Consider the curved pipe domain of Figure 1b. Consider the following boundary conditions (replacing those of (3)):

$$\begin{cases} \mathbf{u} = \mathbf{0} & \mathbf{x} \in \Gamma_2 \cup \Gamma_3 \,, \\ \mathbf{u} = \left(\dfrac{1}{4} - y^2, 0\right)^T & \mathbf{x} \in \Gamma_0 \,, \\ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = 0 & \mathbf{x} \in \Gamma_1 \,, \end{cases}$$

and set $\mathbf{f} = \mathbf{0}$ and $\mu = 1$. Using `deal.II`, implement a solver for problem (3). Use the "pipe" mesh available at `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh`. Upload the source code of the solver and a plot of the solution over the domain Ω.