Numerical Methods for Partial Differential Equations
A.Y. 2024/2025

Laboratory 04
Finite Element method for the diffusion-reaction equation in 2D: convergence analysis

Exercise 1.

Let $\Omega = (0,1) \times (0,1)$, and let us consider the following diffusion-reaction problem with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\boldsymbol{\nabla} \cdot (\mu \, \boldsymbol{\nabla} u) + \sigma u = f & \mathbf{x} \in \Omega, \quad (1a) \\ u = 0 & \text{on } \partial\Omega, \quad (1b) \end{cases}$$

where $\mathbf{x} = (x, y)^T$, $\mu(\mathbf{x}) = 1$, $\sigma = 1$ and

$$f(\mathbf{x}) = (20\pi^2 + 1)\sin(2\pi x)\sin(4\pi y) \ .$$

The exact solution to this problem is

$$u_{\text{ex}}(x, y) = \sin(2\pi x)\sin(4\pi y) \ .$$

1.1. Write the weak formulation, the Galerkin formulation and the finite element formulation of (1).

1.2. Starting from the code of Laboratory 3, implement a finite element solver for problem (1). The solver should read the mesh from file (four differently refined meshes are provided as `mesh/mesh-square-*.msh`).

1.3. Using the four meshes provided, study the convergence of the solver for polynomials of degree $r = 1$ and of degree $r = 2$. Plot the error in the L^2 and H^1 norms against h, knowing that for every mesh file `mesh/mesh-square-`N`.msh`, the mesh size equals $h = 1/N$.

Exercise 2.

Let Ω be the domain depicted in Figure 1, contained in the files `mesh/mesh-u-*.msh`. The boundaries of the mesh are labelled as shown in Figure 1 (i.e. Γ_0 is labelled 0, Γ_1

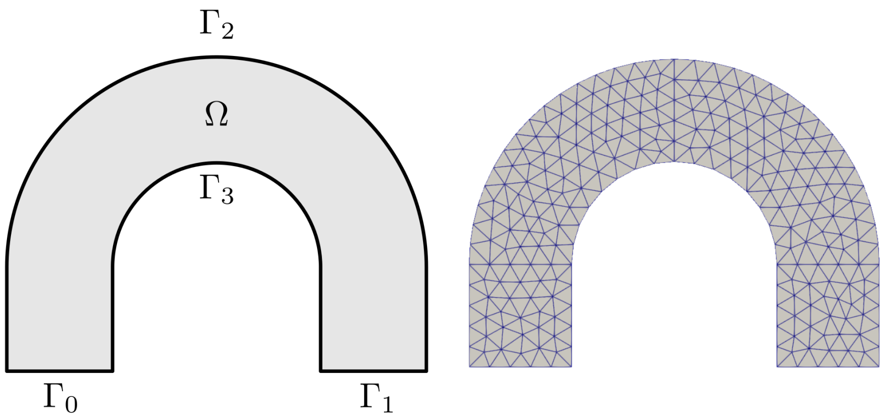

Figure 1: Domain for Exercise 2 (left), and a triangular mesh over it (right), corresponding to the file `mesh/mesh-u-5.msh`.

is labelled 1, and so on). Consider the problem:

$$\begin{cases} -\boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} u) = 0 & \text{in } \Omega \,, \\ u = 0 & \text{on } \Gamma_0 \,, \\ u = 1 & \text{on } \Gamma_1 \,, \\ \mu \boldsymbol{\nabla} u \cdot \mathbf{n} = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \,. \end{cases}$$

2.1. Starting from the previously implemented code, solve (2) using linear finite elements.

Possibilities for extension

Space adaptivity One of the key features of `deal.II` is it support of *space adaptivity*, i.e. refining the mesh only in regions where the solution is less accurate (i.e. where an *a posteriori error estimate* indicates that the solution is inaccurate). This allows to strike an excellent compromise between accuracy and problem size (and thus computational cost). Based on `deal.II`'s step 6 tutorial (`https://dealii.org/9.5.0/doxygen/deal.II/step_6.html`), modify the code of Exercise 1 to use adaptive space refinement. Compare the results with those of Exercise 1 in terms of error against the number of degrees of freedom. Beware that, as of now, adaptivity is only supported for quadrilateral (or hexahedral) meshes.