Numerical Methods for Partial Differential Equations
A.Y. 2023/2024

Written exam - June 10, 2024

Maximum score: 26. Duration: 2h 00m.

Answers to questions must be provided on paper (handwritten). Source code and picture files should be uploaded to the dedicated WeBeep folder. **All uploaded files (including source code) must be appropriately referenced when answering to questions on paper** (e.g. “The solution is depicted in `exercise-1-plot.png`”).

Exercise 1.

Consider the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = 0 & 0 < x < 1,\ t > 0\ , \\ u(x = 0, t) = \alpha & t > 0\ , \\ \dfrac{\partial u}{\partial x}(x = 1, t) = \beta & t > 0\ , \\ u(x, t = 0) = u_0(x) & 0 < x < 1\ , \end{cases} \tag{1}$$

where

$$Lu = -\frac{\partial^2 u}{\partial x^2} + ku$$

and $k \geq 0$, α and β are three constants.

1.1. [*2 pt*] Write the weak formulation of (1), under the form $\forall t > 0$, find $u \in V$ such that

$$\int_0^1 \frac{\partial u}{\partial t}\, v\, dx + a(u, v) = F(v) \qquad \forall v \in V\ ,$$

with $u|_{t=0} = u_0$. Provide the correct definition of V, $a(u, v)$, $F(v)$.

1.2. [*1 pt*] Discuss the coercivity properties of $a(\cdot, \cdot)$.

1.3. [*2 pt*] Write the fully discrete approximation of (1) by using piecewise linear finite elements in space and the implicit backward Euler method in time.

1.4. [*3 pt*] When $\alpha = 0$, $\beta = 0$ and $k = 0$, discuss the stability of the scheme at point 1.3 and the expected error estimate.

1.5. [*3 pt*] Using `deal.II`, implement a solver for problem (1). Use the internal `deal.II` functions for generating the mesh. Consider the following data:

$$k = 6 \ , \qquad \alpha = e^{-2t} \ ,$$
$$\beta = 2e^{2-2t} \ , \qquad u_0(x) = e^{2x} \ .$$

Consider the backward Euler method for time discretization, and solve up to a final time $T = 1$ with a time step $\Delta t = 0.05$. Use finite elements of degree 1 for space discretization, over a mesh with $N = 20$ elements. Upload the source code of the solver.

1.6. [*2 pt*] Plot and upload the solution as a function of space at the final time $T = 1$, and the solution as a function of time at the point $x = 0.5$.

1.7. [*2 pt*] The exact solution to problem (1) with the data of point 1.5 is given by

$$u_{\text{ex}}(x, t) = e^{2(x-t)} \ .$$

Knowing this, analyze the spatial convergence of the solver, considering meshes with $N = 5, 10, 20, 40$ elements and time steps $\Delta t = 0.05$ and $\Delta t = 0.005$. Report the results and discuss them in light of the theory.

Exercise 2.

Consider the problem

$$\begin{cases} -\mu\Delta\mathbf{u} + \alpha\mathbf{u} + \boldsymbol{\nabla} p = 0 & \text{in } \Omega \ , \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \subset \mathbb{R}^2 \ , \\ \mathbf{u} = \mathbf{A} & \text{on } \Gamma_{\text{D}} \subset \partial\Omega \ , \\ \mu\dfrac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = \mathbf{B} & \text{on } \Gamma_{\text{N}} \subset \partial\Omega \ , \end{cases} \tag{2}$$

where α, $\mathbf{A}$ and $\mathbf{B}$ are three constants, $\Gamma_{\text{D}} \cup \Gamma_{\text{N}} = \partial\Omega$, $\Gamma_{\text{D}} \cap \Gamma_{\text{N}} = \emptyset$, $\Gamma_{\text{N}} \neq \emptyset$.

2.1. [*3 pt*] Provide the weak formulation of problem (2).

2.2. [*1 pt*] Approximate the weak formulation of problem (2) by using the Taylor-Hood finite elements of degree $k \geq 2$ for the velocity component $\mathbf{u}$ and degree $k - 1$ for the pressure component p.

2.3. [*1 pt*] Is the approximation at point 2.2. stable? Explain why.

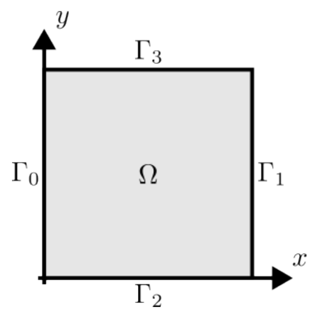

Figure 1: Domain for Exercise 2. Each boundary subset Γ_i corresponds to the tag i in the provided mesh files.

2.4. [*2 pt*] Write the expected error estimate for both velocity and pressure of the approximation at point 2.2. Do you expect spurious modes for this approximation? Motivate your answer.

2.5. [*3 pt*] Using `deal.II`, implement a finite element solver for problem (2). Consider the square domain of Figure 1, and the following data:

$$\Gamma_{\mathrm{D}} = \Gamma_0 \cup \Gamma_2 \cup \Gamma_3 \,, \qquad \Gamma_{\mathrm{N}} = \Gamma_1 \,,$$
$$\mu = 1 \,, \qquad \alpha = 1 \,,$$
$$\mathbf{A} = \begin{cases} (1,0)^T & \text{on } \Gamma_0 \,, \\ (0,0)^T & \text{elsewhere} \,, \end{cases} \qquad \mathbf{B} = \mathbf{0} \,.$$

Use the mesh $\mathcal{T}_h$ of triangular finite elements of size $h = 0.1$, which can be found at the following link: `https://github.com/michelebucelli/nmpde-labs-aa-23-24/tree/main/examples/gmsh` (refer to Figure 1 for the boundary tags). Consider finite elements of degree 2 and 1 for velocity and pressure respectively. Upload the source code of the solver.

2.6. [*1 pt*] Plot and upload the solution.