Traffic flow modelling

Project in Applied Mathematics

Author:
Dahlberg, Vilmer
vi8808da-s@student.lu.se

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Den är ju även diskontinuerlig visar det sig... Modelleringen görs med ekvationen brukar man säga.

1 Introduction

In this project, traffic flow in one dimension is studied. Traffic flow is modelled by a continuous function representing the concentration of cars, which is governed by a conservation law. The flux of cars determines the solution to the conservation law, and serves as an important analytical tool. After some mathematical tools are introduced, a set of model problems are solved geometrically and their solutions are interpreted both mathematically and physically.

affinely (en linjär funktion går genom origo)

Let $u(x,t)$ denote the concentration of cars at position x and time t, and let $u_{\max}$ denote the maximum possible concentration of cars. In this project, the velocity of the cars, denoted by $v = v(u)$, is assumed to depend linearly on the concentration of cars. If the concentration of cars at a certain point is zero, the cars are free to move at the maximum velocity, meaning $v(0) = v_0$. If however, the concentration is the maximum allowed concentration at any point, the cars are in a stand-still and their velocity is zero.

$$v(u) = v_0 \left(1 - \frac{u}{u_{\max}}\right)$$

The flow of cars from any point is then given by the product of the concentration of cars at that point, and their velocity, or

$$f(u) = v(u)u.$$

Given the flux function above, the two following situations are studied.

A queue is formed behind a traffic light, where the concentration behind the queue is $u_{\max}$, and the concentration in front of the queue is 0. The traffic light turns green and cars are free to move. What is the concentration of cars over time? What is the path of a car that starts behind the queue?

Cars are moving at a constant speed of v_i [km/h] behind a queue which has the length L [km]. The concentration of cars is zero in front of the queue. The traffic light turns green and cars are free to move. What is the concentration of cars over time? What is the maximum time a car must wait before it can start moving? How long time does it take for the traffic to flow freely again?

2 Theory and mathematical tools

2.1 Deriving the balance equation

Recall that the concentration of cars at position x and time t is denoted by $u(x,t)$, and that the flow of cars for a given concentration is

$$f(u) = v_0 u \left(1 - \frac{u}{u_{\max}}\right), \tag{1}$$

where v_0 is the maximum allowed velocity and $u_{\max}$ is the maximum possible concentration of cars. Now, consider the amount of cars in some interval $[x_a, x_b]$, which is given by the integral

$$\int_{x_a}^{x_b} u dx.$$

Assuming there are no sinks or sources, the change in cars in the interval is given by the flow in at x_a minus the flow out from x_b, or

$$\frac{\partial}{\partial t}\int_{x_a}^{x_b} u dx = -f\big|_{x_b} + f\big|_{x_a}. \tag{2}$$

One integral can be formed by applying the fundamental theorem of calculus to the right hand side and moving the derivative on the left hand side inside the integral (assuming that the concentration is $\mathcal{C}^1$),

$$\int_{x_a}^{x_b} \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} dx = 0.$$

This should hold for any length of the interval, and thus the integrand must be zero (assuming it is continuous). The partial differential equation that describes the concentration of cars over time is hence given by

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0.$$

In this application the flux is a function of the concentration such that $f = f(u(x,t))$, and we can write

$$\frac{\partial u}{\partial t} + f' \frac{\partial u}{\partial x} = 0. \tag{3}$$

2.2 Characteristic Lines

The differential equation (3) can be solved geometrically by studying so called characteristic lines which are level curves of u in x-t plane, or

$$u(x(t), t) = U_0.$$

The equation for the curve $x = x(t)$ is found by differentiating the expression above with respect to t and using the conservation law (3)

$$0 = \frac{\partial u}{\partial x} x'(t) + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\big(x'(t) - f'(U_0)\big) = 0.$$

Thus, the speed of the curve is either given by $f'(U_0)$, or the function is constant in space which implies that the function is also constant in time as it satisfies (3). Given some initial data the solution can be constructed geometrically by solving the following set of equations

$$\begin{cases} x(t) = f'(U_0)t + x_0 \\ u(x(t), t) = u(x_0, t_0) = U_0. \end{cases} \tag{4}$$

2.3 Shock Waves

Using the implicit function theorem it can be shown that discontinuous solutions to the conservation law may appear, even for differentiable initial data (meaning $u(x, t_0) \in \mathcal{C}^1$). The remedy for this is to generalize the solution to accept so called weak solutions, see S. Diehl (1996) [1] for details. The speed of the shock, or discontinuity, is described by the local conservation law (2). Let u^- be the concentration to the left of the shock (on the x-axis), and u^+ be the concentration to the right. The conservation law (2) then gives

$$\begin{aligned}
&- f(u(x_b,t)) + f(u(x_a,t)) = \frac{\partial}{\partial t}\int_a^b u dx = \frac{\partial}{\partial t}\left(\int_a^{x(t)} u dx + \int_{x(t)}^b u dx\right) = \\
&= \int_a^{x(t)} \frac{\partial u}{\partial t} dx + x'(t) u(x^-(t), t) + \int_{x(t)}^b \frac{\partial u}{\partial t} dx - x'(t) u(x^+(t), t) = \left[\frac{\partial u}{\partial t} = -\frac{\partial f}{\partial x}\right] = \\
&= -f(u(x^-(t),t)) + f(u(x_a), t) - f(u(x_b,t)) + f(u(x^+(t),t)) + x'(t)(u^- - u^+) \\
&= -f(u(x_b,t)) + f(u(x_a,t)) + f(u^+) - f(u^-) - x'(t)(u^+ - u^-).
\end{aligned}$$

Meaning the speed of the discontinuity fulfills the so called jump condition

This means that

$$x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

Figur av f(u) och tolkning av hoppvillkoret?

2.4 Entropy condition

For some initial data $u_0 = u(x, t_0)$ there may exist several ways to construct a geometrical solution. A condition to ensure uniqueness is needed. A condition known as the entropy condition can be derived by studying an augmented problem where some diffusion is added to the conservation law (3), for more details see S. Diehl (1996). The entropy condition bounds the shock from above and below, and is given by the equation below for strictly convex and concave flux functions.

the following inequalities

$$f'(u^-) > x'(t) > f'(u+) \tag{5}$$

For the flux function used in this problem given by 1 we can re-write the entropy condition to (1)

$$v_0(1 - 2\frac{u^-}{u_{\max}}) > x'(t) > v_0(1 - 2\frac{u^+}{u_{\max}}) \iff u^- < \frac{u_{\max}}{2}\left(1 - \frac{x'(t)}{v_0}\right) < u^+.$$

If the speed of the wave is ignored the inequality can be re-written as

(5)

$$u^- < u^+. \tag{6}$$

Meaning for any admissible shock wave the concentration in front of the wave must be larger than the concentration behind the wave.

3 Problem solving

In the problems below we study the balance equation 3 with the flux function 1 for different initial conditions.

Problem 1

In the first problem, there is a traffic light at $x = 0$ stopping the flow of cars. At time t_0 the light turns green and the flow of cars is returned to normal. The goal is to compute the concentration of cars over time, and to derive an expression for the path of a car starting at $x_0 < 0$. The maximum concentration of cars is 100 [cars/km], the maximum velocity is 100 [km/h], and the initial condition is plotted below.

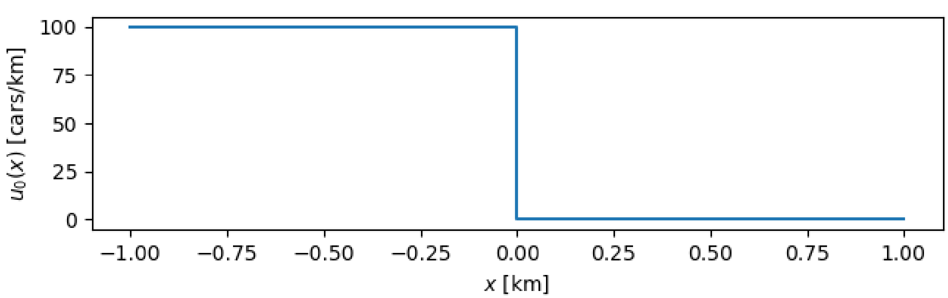

Figure 1: Initial condition for the first problem.

Solving for the concentration

The problem can be solved by using the idea of characteristic lines, equation 4, where the derivative of the flux is given by $f' = v_0\left(1 - 2\frac{u}{u_{\max}}\right) = 100\left(1 - \frac{u}{50}\right)$. The shape of the initial distribution motivates dividing the solution into two cases, defined by the line where $x = 0$.

case : $x_0 < 0$

$f'(u_0) = 100\,(1 - 100/50) = -100$

$x(t) = f'(u_0)t + x_0 = -100t + x_0$

$u(x(t), t) = u(x_0, t_0) = 100$

case : $x_0 > 0$

$f'(u_0) = 100\,(1 - 0) = 100$

$x(t) = f'(u_0)t + x_0 = 100t + x_0$

$u(x(t), t) = u(x_0, t_0) = 0$

The solution can be plotted as characteristic lines in x-t plane, see the figure below.

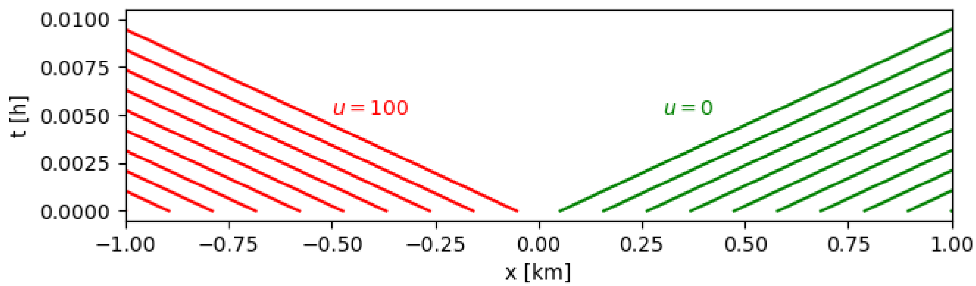

Figure 2: Characteristic lines.

The characteristic lines cannot be drawn in the interval $-100t < x < 100t$, due to the discontinuity at $x = 0$, and so we have no solution there. From the entropy condition 5 we know that there may not be a shock here since the shock speed cannot fulfill the condition $-100 > x'(t) > 100$. To derive what should fill the empty space, look at a slightly modified initial condition where the discontinuity is replaced by a line going from $-\epsilon$ to ϵ, see the figure below.

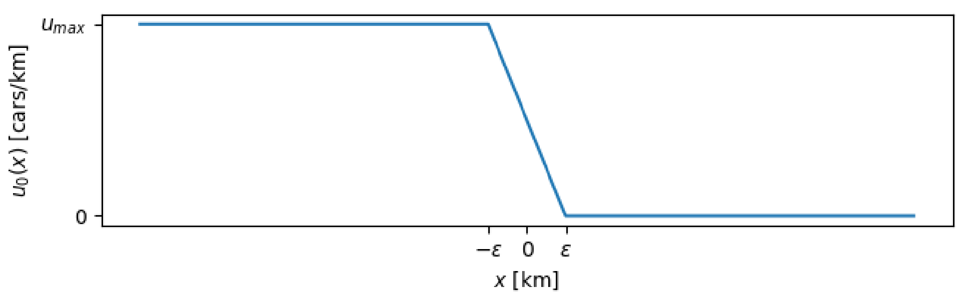

Figure 3: Slightly modified initial condition.

The solution in the interval $|x| > \epsilon$ is the same as before, thus study the interval $|x| < \epsilon$, where $u_0(x) = -\frac{u_{\max}}{2\epsilon}(x + \epsilon) + u_{\max}$.

case : $|x| < \epsilon$

$$f'(u_0) = v_0\left(1 - 2\frac{-\frac{u_{\max}}{2\epsilon}(x_0 + \epsilon) + u_{\max}}{u_{\max}}\right) = v_0\left(1 - \left(\frac{-1}{\epsilon}(x_0 + \epsilon) + 2\right)\right) =$$

$$= v_0\left(\frac{x_0 + \epsilon}{\epsilon} - 1\right) = \frac{v_0 x_0}{\epsilon}$$

$$x(t) = \frac{v_0 x_0}{\epsilon}t + x_0 = x_0\left(\frac{v_0 t}{\epsilon} + 1\right) \iff x_0 = \frac{x(t)\epsilon}{(v_0 t + \epsilon)}$$

$$u(x(t), t) = -\frac{u_{\max}}{2\epsilon}\left(\frac{x(t)\epsilon}{v_0 t + \epsilon} + \epsilon\right) + u_{\max} = \frac{u_{\max}}{2}\left(1 - \frac{x(t)}{v_0 t + \epsilon}\right)$$

Then, if we let $\epsilon \longrightarrow 0$ the initial condition in figure 3 converges to the one in figure 1, and the solution converges to $u_{\text{fan}}(x, t) = \frac{u_{\max}}{2}\left(1 - \frac{x}{v_0 t}\right)$. To test that this is a valid solution, insert u_{fan}

into the conservation law.

$$(u_{\text{fan}})'_t = \frac{u_{\max} x}{2v_0 t^2} \quad (u_{\text{fan}})'_x = -\frac{u_{\max}}{2v_0 t}$$

$$f' = v_0 \left(1 - 2\frac{\frac{u_{\max}}{2}\left(1 - \frac{x}{v_0 t}\right)}{u_{\max}} \right) = v_0 \left(1 - 1 + \frac{x}{v_0 t}\right) = \frac{x}{t_v 0}$$

$$(u_{\text{fan}})'_t + f'(u_{\text{fan}}))(u_{\text{fan}})'_x = \frac{u_{\max} x}{2v_0 t^2} - \frac{u_{\max}}{2v_0 t}\frac{x}{t} = 0$$

The entropy condition ensures a unique solution, so u_{fan} must be the solution we're seeking. Thus, the solution to the IVP is given by

we are (skriv alltid ut orden i vetenskaplig text)

$$u(x,t) = \begin{cases} 100 & \text{if } x < -100t \\ 50 - \frac{x}{2t} & \text{if } |x| < 100t \\ 0 & \text{if } x > 100t. \end{cases}$$

and the characteristic lines are plotted below.

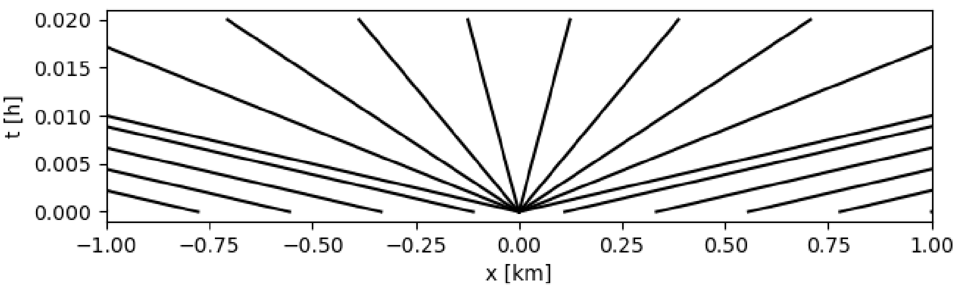

Figure 4: Characteristic lines, full solution.

Solving for the car's path

Since u is known, the path of a car starting at $x_0 < 0$ can be studied, which is given by the relation below.

$$\begin{cases} x'(t) = v(u) = 100\left(1 - \frac{u}{100}\right) = 100 - u \\ x(0) = x_0 < 0. \end{cases}$$

where u=u(x(t),t)

Up to $t_i = \frac{-x_0}{100}$ the concentration is 100 which means the velocity is zero. In other words $x(t) = x_0$ for $t \in [0, t_i]$. Then for $t > t_i$ the concentration is $u = 50\left(1 - \frac{x}{100t}\right)$, which gives the initial value problem

$$\begin{cases} x'(t) = 100 - 50\left(1 - \frac{x}{100t}\right) = 50 + \frac{x}{2t} \\ x(\frac{-x_0}{100}) = x_0 \end{cases} \iff \begin{cases} x' - \frac{x}{2t} = 50 \\ x(\frac{-x_0}{100}) = x_0. \end{cases}$$

\ln

The IVP can be solved using integrating factor, which is given by $f(t) = e^{G(t)}$ where $G(t) = \int g(t)dt = \int \frac{-1}{2t}dt = -\frac{ln(t)}{2}$, thus $f(t) = e^{-\frac{ln(t)}{2}} = t^{-1/2}$. Multiplication of the integrating factor gives

$$x' t^{-1/2} - \frac{x}{2t}t^{-1/2} = 50t^{-1/2} \iff \left(xt^{-1/2}\right)'_t = 50t^{-1/2}$$

Integration with respect to time gives

$$\left(xt^{-1/2}\right) = 100t^{1/2} + C \iff x = 100t + Ct^{1/2}.$$

Next, the constant can be eliminated by insertion of the initial condition.

$$\begin{cases} x\left(\frac{-x_0}{100}\right) = x_0 \\ x\left(\frac{-x_0}{100}\right) = -x_0 + C\sqrt{-x_0/100} \end{cases} \iff C = 2x_0\sqrt{\frac{-100}{x_0}} = 20x_0\sqrt{\frac{1}{x_0}}$$

Since $x_0 < 0$ we can use $x_0 = -|x_0|$, and rewrite

$$C = -20|x_0|\sqrt{\frac{-1}{x_0}} = -20\sqrt{x_0^2}\sqrt{\frac{-1}{x_0}} = -20\sqrt{x_0}$$

Finally the path of the car is given by the expression below.

$$x(t) = \begin{cases} x_0 & \text{if } 0 < t \leq \frac{-x_0}{100} \\ 100t - 20\sqrt{-x_0 t} & \text{if } t > \frac{-x_0}{100}. \end{cases}$$

Some paths are plotted below

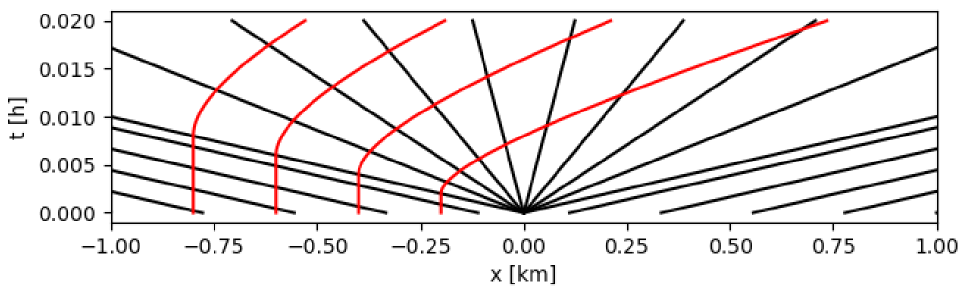

Figure 5: Characterisitc lines and the paths of some cars starting at $x < 0$ is plotted.

Problem 2

In the next problem, cars are moving on a highway at the speed $0 \leq v_i \leq v_0$ [km/h]. At some time an accident occurs at $x = L > 0$ [km] which stops the flow of cars, and results in a queue extending to $x = 0$ where $u = u_{\max}$ [cars/km]. At some time the flow of cars starts up again, solve for the concentration of the cars. The concentration at $x < 0$ denoted by u_i can be expressed in terms of the velocity v_i, $u_i = u_{\max}\left(1 - \frac{v_i}{v_0}\right)$. The initial distribution is plotted below.

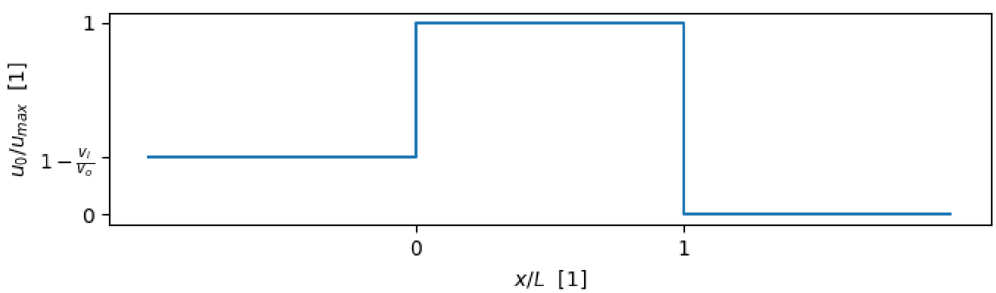

Figure 6: Initial distribution of second problem.

Start by finding the characteristic lines, and recall that the derivative of the flux function is $f' = v_0\left(1 - 2\frac{u}{u_{\max}}\right)$.

There will be a discontinuity starting at $x = 0$ due to the characteristics on the left moving at the speed $(2v_i - v_0) \geq 0$ [km/h] and characteristics to the right moving at $-v_0$ [km/h]. According to the jump condition the speed of the discontinuity is

$$x'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{v(u^+)u^+ - v(u^-)u^-}{u^+ - u^-} = \frac{0 \cdot u_{\max} - v_i u_i}{u_{\max} - u_i}$$
$$= \frac{-v_i u_{\max}\left(1 - \frac{v_i}{v_0}\right)}{u_{\max} - u_{\max}\left(1 - \frac{v_i}{v_0}\right)} = \frac{-v_i(v_0 - v_i)}{v_0 - v_0 + v_i} = -(v_0 - v_i)$$

case: $x < 0$

$$f'(u_0) = v_0\left(1 - 2(1 - \frac{v_i}{v_0})\right)$$
$$= (2v_i - v_0)$$
$$x = (2v_i - v_0)t + x_0$$
$$u = u_i$$

case: $0 < x < L$

$$f'(u_0) = v_0(1 - 2\frac{u_{\max}}{u_{\max}})$$
$$= -v_0$$
$$x = -v_0 t + x_0$$
$$u = u_{\max}$$

case: $x > L$

$$f'(u_0) = v_0(1 - 2\frac{0}{u_{\max}})$$
$$= v_0$$
$$x = v_0 t + x_0$$
$$u = 0$$

which is a valid discontinuity according to the entropy condition. Next, a shifted fan wave, similar to the one used in the previous problem, will be added in the empty region starting at $t = 0, x = L$.

$$u_{\text{fan}} = \frac{u_{\max}}{2}\left(1 - \frac{x - L}{v_0 t}\right).$$

Before continuing with the solution, study the characteristic lines drawn so far.

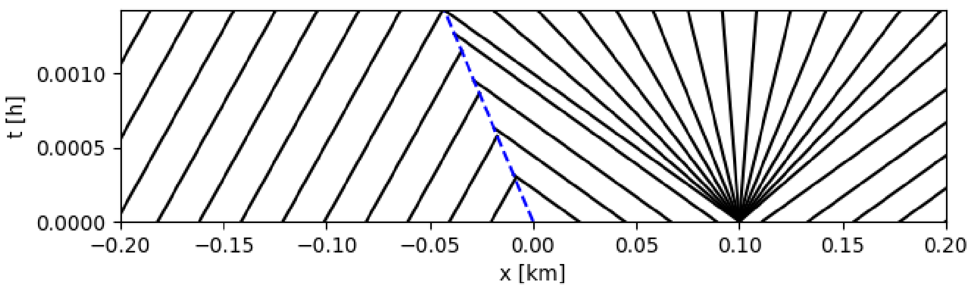

Figure 7: Characteristic lines so far, with the shock plotted in blue.

The characteristic lines moving at speed $-v_0$ from the interval $x \in [0, L]$ collide with characteristic lines from $x < 0$ and form the shock. At a critical time $t = t_c$ the last characteristic line from the interval $[0, L]$ is eliminated. The critical time (t_c) and the critical position (x_c) can be found by solving the system below.

$$\begin{cases} x_{\text{shock}}(t_c) = x_c \\ x_{\text{char}}(t_c) = x_c \\ x_{\text{shock}}(t_c) = -(v_0 - v_i)t_c \\ x_{\text{char}}(t_c) = L - v_0 t_c \end{cases} \iff \begin{cases} -(v_0 - v_i)t_c = L - v_0 t_c \\ x_c = -(v_0 - v_i)t_c \end{cases} \iff \begin{cases} t_c = \frac{L}{v_i} \\ x_c = -\frac{L}{v_i}(v_0 - v_i) \end{cases} \tag{7}$$

After $t = t_c$ characteristic lines from the fan wave with the concentration $u = u_{\text{fan}}$ collide with the characteristic lines with the remaining concentration $u = u_i$. Since $u_{\text{fan}} \neq u_{\max}$ the speed of the shock wave changes. Begin by simplifying $f(u^+),\ f(u^-),\ u^+ - u^-$

$$\begin{aligned} f(u^+) &= v(u^+)u^+ = v_0\left(1 - \frac{u_{\max}}{2u_{\max}}\left(1 - \frac{x - L}{v_0 t}\right)\right)\frac{u_{\max}}{2}\left(1 - \frac{x - L}{v_0 t}\right) \\ &= \frac{v_0 u_{\max}}{4}\left(1 + \frac{x - L}{v_0 t}\right)\left(1 - \frac{x - L}{v_0 t}\right) = \frac{u_{\max}}{4v_0}\left(v_0 - \left(\frac{x - L}{t}\right)^2\right) \\ f(u^-) &= v_i u_i = v_i u_{\max}(1 - \frac{v_i}{v_0}) = \frac{4v_i u_{\max}}{4v_0}(v_0 - v_i) \\ u^+ - u^- &= \frac{u_{\max}}{2}\left(1 - \frac{x - L}{v_0 t}\right) - u_i = \frac{u_{\max}}{2}\left(1 - \frac{x - L}{v_0 t}\right) - u_{\max}(1 - \frac{v_i}{v_0}) \\ &= \frac{2u_{\max}}{4v_0}\left(v_0 - \frac{x - L}{t} - 2(v_0 - v_i)\right) \end{aligned}$$

Inserting the expressions above into the equation for the shock speed yields

$$x'(t) = \frac{\frac{u_{\max}}{4v_0}\left(v_0 - \left(\frac{x-L}{t}\right)^2\right) - \frac{4v_i u_{\max}}{4v_0}(v_0 - v_i)}{\frac{2u_{\max}}{4v_0}\left(v_0 - \frac{x-L}{t} - 2(v_0 - v_i)\right)} = \frac{v_0 - \left(\frac{x-L}{t}\right)^2 - 4v_i(v_0 - v_i)}{2\left((2v_i - v_0) - \left(\frac{x-L}{t}\right)\right)}$$
$$= \frac{(2v_i - v_0)^2 - \left(\frac{x-L}{t}\right)^2}{2\left((2v_i - v_0) - \left(\frac{x-L}{t}\right)\right)} = \frac{1}{2}\left((2v_i - v_0) + \left(\frac{x-L}{t}\right)\right).$$

The path of the shock is given by the IVP

$$\begin{cases} x' - \frac{1}{2}xt^{-1} = \frac{1}{2}(2v_i - v_0) - \frac{1}{2}Lt^{-1} \\ x\left(\frac{L}{v_i}\right) = -\frac{L}{v_i}(v_0 - v_i). \end{cases}$$

Multiplying by the integrating factor $t^{-1/2}$ followed by integration, the first line can be re-written as

$$\left(xt^{-1/2}\right)' = \frac{1}{2}(2v_i - v_0)t^{-1/2} - \frac{1}{2}Lt^{-3/2} \iff$$
$$x = t^{1/2}\left((2v_i - v_0)t^{1/2} + Lt^{-1/2} + C\right) = (2v_i - v_0)t + L + Ct^{-1/2}.$$

The integrating constant can be eliminated by inserting the initial condition

$$\begin{cases} x\left(\frac{L}{v_i}\right) = -\frac{L}{v_i}(v_0 - v_i) \\ x\left(\frac{L}{v_i}\right) = (2v_i - v_0)\frac{L}{v_i} + L + C\sqrt{\frac{L}{v_i}} \end{cases} \implies$$

$$-\frac{L}{v_i}(v_0 - v_i) = (2v_i - v_0)\frac{L}{v_i} + L + C\sqrt{\frac{L}{v_i}} \iff$$
$$C\sqrt{\frac{L}{v_i}} = -2L \iff C = -2\sqrt{v_i L}$$

The path of the shock is then given by

$$x_{\text{shock}}(t) = \begin{cases} -(v_0 - v_i)t & \text{if } 0 < t \leq \frac{L}{v_i} \\ (2v_i - v_0)t - 2\sqrt{v_i L t} + L & \text{if } t > \frac{L}{v_i} \end{cases}$$

Now, the solution to the conservation law can be expressed.

$$u(x,t) = \begin{cases} u_i = u_{\max}\left(1 - \frac{v_i}{v_0}\right) & \text{if } x < x_{\text{shock}}(t) \\ u_{\max} & \text{if } x_{\text{shock}}(t) < x < -v_0 t \\ u_{\text{fan}} = \frac{u_{\max}}{2}\left(1 - \left(\frac{x-L}{v_0 t}\right)\right) & \text{if } -v_0 t < x < v_0 t \\ 0 & \text{if } x > v_0 t \end{cases}$$

Below are some plots of the characteristic lines.

Mycket bra figurer, men t_f är väl tyvärr olika i de två.

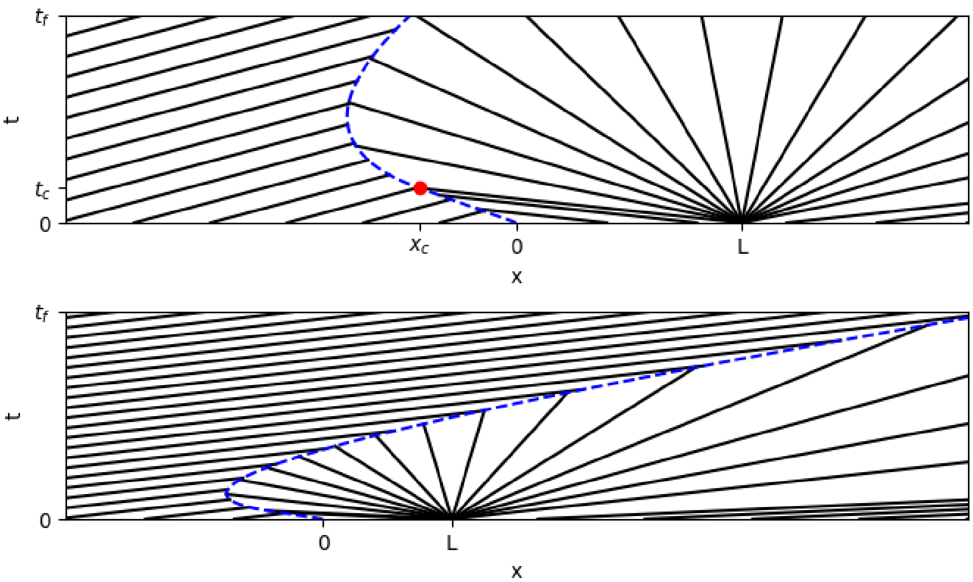

Figure 8: Characteristic lines of final solution.

Analyzing the results

Begin by analyzing the entropy condition, which requires that for any admissible shock wave the concentration in front of the wave must be larger than the concentration behind the wave. In the first problem the initial condition was a step function, see figure 1, which is a discontinuity where the concentration behind the wave is larger than the concentration in front of the wave. Physically one would expect the cars to move forward and smooth out the discontinuity, which is what solution predicts and what the entropy condition requires. Another solution, if the entropy condition is ignored, is that the cars would just stand still, and the discontinuity would not move. This solution does not make sense physically, and is discarded by the entropy condition.

This solution satisfies the PDE away from the discontinuity and the jump condition at the discontinuity.

Next study the solution to the second problem, how long time does it take for the traffic flows free again, and what is the longest wait time for a car? The first question is easy to answer. The traffic flows free once the maximum concentration is less than u_{max}, which is at the critical time t_c given by equation 7.

förklara annorlunda

The second question is more intricate, so study the plot below.

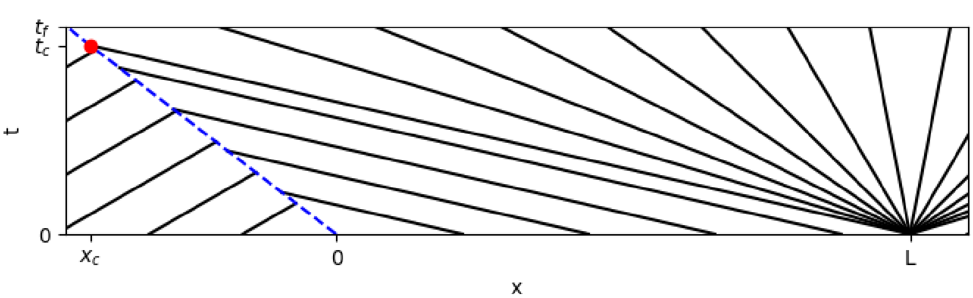

Figure 9: Zoomed in plot of the characteristic lines of the solution to the second problem.

Är x>0 i hela triangeln?

The queue is in the triangle where $u = u_{max}$, which are points $x, t > 0$ such that $-(v_0 - v_i)t < x < L - v_0 t$. Let $A = (0, 0), B = (x_c, t_c)$, and $C = (L, 0)$ be the vertices of that triangle. The wait time for any car is then the length of the vertical line from the point x_0 at the bottom of the triangle, to the top of the triangle. Let $D = (0, L/v_0)$, the longest wait time is then given by the length of the line $\vec{AD}$. To prove this take any line $\vec{A'D'}$ to the right of and parallel to the line $\vec{AD}$. The triangles $\triangle ADC$ and $\triangle A'D'C$ are congruent since they share the angle $\angle C$ and a right angle. The lengths of the sides of the triangles are then related by

$$\frac{AD}{A'D'} = \frac{AC}{A'C}.$$

Since the line $\vec{A'D'}$ is to the right of $\vec{AD}$, the length of $\vec{A'C}$ is less than the length of $\vec{AC}$. Thus the length of $\vec{A'D'}$ is less than the length of $\vec{AD}$. Next, take any line $\vec{A''D''}$ to the left of and parallel to the line $\vec{AD}$ where the point A'' is on the line $x = -40t$. The angles $\angle BA''D''$ and $\angle BAD$ are equilateral since the lines $\vec{A''D''}$ and $\vec{AD}$ are parallel. Thus the triangles $\triangle BA''D''$ and $\triangle BAD$ are congruent, and the same argument used before can be used to prove that the line $\vec{A''D''}$ is shorter than the line $\vec{AD}$. This proves that the longest wait time for a car is $|\vec{AD}| = \frac{L}{v_0}$.

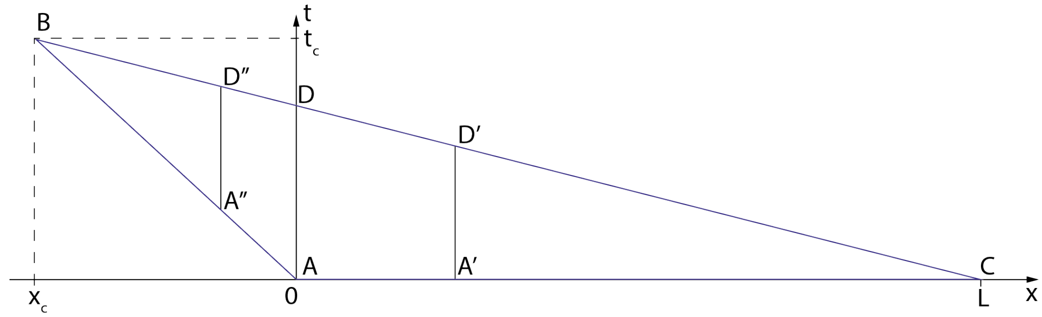

Figure 10: Geometrical proof of the longest wait time.

4 Conclusion

It is possible to model one dimensional traffic flow using a conservation law and flux function, and to solve these problems geometrically using characteristic lines. We also saw that often we cannot find continuous a unique solution, and must allow for weak solutions. To get a unique solution we must have the entropy condition, which discards physically unreasonable solutions. Furthermore, discontinuities occur naturally as a result of the flux function, but using the jump condition these discontinuities can be resolved. After a solution is found using the above techniques the solution can be analyzed in terms of traffic terminology and questions such as the longest wait time can be answered.

References

[1] Stefan Diehl. Introduction to the scalar non-linear conservation law, 1996.