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Efficient topology optimization with linearized buckling using reduced order modelling

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Abstract Keywords Topology optimization, Buckling, Reduced order modelling, Combined approximations

1 Introduction

Frequently, the

Topology optimization (TO) is a useful tool ~~used to~~ for ~~quickly generate initial~~ designs ing ~~for macroscopic~~ structural components and ~~micro-structures~~ materials. Compliance ~~optimization under~~ minimization subject to a volume constraint~~s~~ is ~~a widespread application~~ considered. ~~, which can be attributed to its simplicity both computationally and implementation-wise~~. However, in practical applications, also stability can be of crucial importance ~~structures are required to meet certain stability constraints; for example they must be resistant to buckling~~. ~~In some examples these~~ Inherently, stiffness and stability are two competing concepts, wherefore stability considerations needs to be explicitly included in the optimization formulation ~~optimizing a structure for compliance can make it susceptible to buckling~~. The most common way of ~~introducing buckling constraints~~ analyzing stability is to perform a linearized buckling analysis. ~~The structure is subject to a trial load, the critical load factor can then be computed, which determines how much the trial load can be scaled before buckling occurs.~~

Vad vill du säga med den här meningen?

Vet inte om detta stycke tillför så mycket till denna artikel

Although optimization of truss and beam structures with buckling constraints has seen much interest, many issues have been encountered when considering continuum structures. Optim~~al~~ized designs of continuum structures with buckling constraints was first presented by Neves et al in [18], but has since encountered many problems. Both buckling and vibration problems suffer from issues relating to non-physical (or pseudo) buckling/eigen modes appearing in low-density regions (see for example Dalklint [6]). This problem can be alleviated in vibrational problems by penalizing the mass in low density regions (see for example Du and Olhoff in [7]) and similar schemes for buckling problems have been suggested by Gao and Ma [11]. While this approach can be successful, it requires fine-tuning of penalization parameters. Yet another issue regards the approximation of the stress field, discussed by Ferrari and Sigmund in [9]. While the traditional four-node element is popular in TO, it poorly approximates the stress field when the spatial resolution is low. This can cause severe issues when thin bar elements are formed, which requires either the use of higher order elements or methods that filter out local buckling modes.

Typically, the optimization problem is solved using a nested approach by implicitly enforcing the equilibrium equations, which constitute the majority of the computational cost. This cost is further increased when buckling is considered, since then a ~~amplified for buckling problems due the arising~~ generalized eigenvalue problem must be solved. Eigenvalue problems ~~, which~~ are notoriously difficult to solve and scale poorly with the problem size, in comparison to linear ~~equilibrium~~ systems of equations. This cost becomes prohibitive for large to medium scale problems in 3D making the technique unappealing for industrial applications. Källa? (2D också väl?

Recently, efforts have been made to extend TO with buckling constraints to ~~a~~ large ~~scale setting~~ scale problems. Dunning ~~and coauthors~~ et. al ~~in~~ [8], presented a Block-Jacobi CG solver with a shift-and invert strategy based on information from previous design iterations. While this method was shown to be

Address(es) of author(s) should be given

Jag hade nog bara radat upp referenserna i slutet av föregående stycke, dvs "... is to perform a linearized buckling analysis (Neves et. al Dalklint et. al, Gao and Ma...)

Du kan dock använda delar av texten till referenser senare i artikeln rsâ slång ej.

Med vad?

approaches the problem by reusing

successful it still requires ~~a~~ matrix factorization at each design update. Ferrari and Sigmund ~~take a different approach in~~ [10], ~~where they reuse~~ the multigrid preconditioner from the initial ~~load~~ equilibrium problem to solve the eigenvalue problem on ~~the~~ a coarser grid. The coarse-scale eigenvectors are then projected onto the fine grid and are refined using inverse iteration. This approach ~~both~~ was shown to reduced the cost ~~of the~~ for solving eigenvalue problems, and filtered out pseudo buckling modes, while retaining a good approximation of the global modes. Also worth noting is the work by Bian and Fan [3], where an assembly-free eigenvalue solver is used to solve the eigenvalue problem on a grid of several million elements ~~[8]~~.

A number of methods based on reduced order modeling (ROM) have been proposed to accelerate topology optimization. In ROM, the displacements or buckling modes are approximated in a reduced solution space. ~~the key ingredient determining~~ The efficacy of a ROM is ~~how the~~ determined by the generation of the reduced space ~~is generated.~~ One approach is to build ~~a~~ the reduced space from previous solutions to the equilibrium equations, which are updated as new solution vectors become available [12,5]. Worth noting is the work by Kang et al in [14]. Vad gör de?

~~Reanalysis differs from the methods above in the sense that~~ the system matrix's factorization is used to build a reduced space for subsequent iterations. The use of reanalysis to solve linear systems was ~~These methods were~~ first discussed by Kirsch ~~in a general setting in~~ [15]. More recently, reanalysis has ~~but have since~~ been applied to topology optimization. Amir studied reanalysis for compliance problems ~~in~~ [1], and ~~where it can be~~ the method was shown ~~to be~~ equivalent to a preconditioned conjugate-gradient method. Furthermore, ~~Amir also discusses~~ the idea of a consistent sensitivity analysis, i.e. where the inaccuracies in the ROM are taken into account was discussed [4]. Bogomolny extended reanalysis to vibrational problems in [4], wherein a reduced order model is ~~computed~~ built for each sought eigenpair. For the generalized eigenvalue problem the basis vectors must be orthogonalized to ensure that the correct eigenpair is approximated. Bogomolny also derives a consistent sensitivity analysis for the vibrational problem.

In this paper, we aim to show that ROM using Combined Approximation can effectively be implemented for buckling problems. What makes this work unique is that CA will be used for the linear analysis, generalized eigenproblem and the adjoints.

Another form of ROM is reanalysis, in which

2 Preliminaries

sec:prelim

We consider the finite element discretized linear system

$$\mathbf{K}_L \mathbf{u}_o = \mathbf{F}_o, \tag{1}$$

mech

in combination with appropriate boundary conditions to describe our mechanical system. In (1), we define the symmetric and positive definite linear stiffness matrix $\mathbf{K}_L \in \mathbb{R}^{n\times n}$, the displacement vector $\mathbf{u}_o \in \mathbb{R}^n$ and the reference load vector $\mathbf{F}_o \in \mathbb{R}^n$, where n is the number of degrees-of-freedom. To solve (1), we utilize the Cholesky decomposition of $\mathbf{K}_L$ in combination with forward/backward substitutions, i.e.

$$\mathbf{K}_L = \mathbf{U}^T \mathbf{U}, \tag{2}$$

mechFac01

where $\mathbf{U} \in \mathbb{R}^{n\times n}$ is upper triangular. After solving (1) for $\mathbf{u}_o$ using (2), we compute the symmetric and indefinite stress stiffness matrix $\mathbf{K}_G(\mathbf{u}_o) \in \mathbb{R}^{n\times n}$, which enables the generalized eigenvalue problem

$$\left(\mathbf{K}_L + \lambda_j \mathbf{K}_G(\mathbf{u}_o)\right) \boldsymbol{\phi}_j = \mathbf{0}, \quad j \in \mathbb{N}_n, \tag{3}$$

lineig01

to be solved for the eigenpairs $(\lambda_j, \boldsymbol{\phi}_j) \in (\mathbb{R}\times\mathbb{R}^n)$. However, since $\mathbf{K}_G$ is indefinite, it is convenient to pose (3) as

$$\left(\mathbf{K}_G(\mathbf{u}_o) - \mu_j \mathbf{K}_L\right) \boldsymbol{\phi}_j = \mathbf{0}, \quad j \in \mathbb{N}_n, \tag{4}$$

lineig02

where $\mu_j = -\frac{1}{\lambda_j}$ and we assume that $\boldsymbol{\phi}_i^T \mathbf{K}_L \boldsymbol{\phi}_j = \delta_{ij}$. Solving the generalized eigenvalue problem in (3) or (4), is referred to as *linearized buckling analysis*, wherefore the eigenvalues λ_j are denoted the critical load factors, so that the critical load is $\mathbf{F}_i = \lambda_i \mathbf{F}_o$. The eigenmode associated with the critical load $\mathbf{F}_i$ is $\boldsymbol{\phi}_i$.

3 Reduced order modelling

sec:CA

The basic idea of reduced order modelling (ROM) is to approximate solutions to linear systems using a reduced basis, i.e.

$$\tilde{\mathbf{u}} \approx \alpha_1 \hat{\mathbf{u}}_1 + \alpha_2 \hat{\mathbf{u}}_2 + ...\alpha_m \hat{\mathbf{u}}_m \tag{5}$$ rom1

where the number of bases, m, is significantly smaller that the number of dimension of the full problem, i.e. $m \ll n$. The success of the ROM method relies on the choice of the basis vectors, $\hat{\mathbf{u}}_i$ and in this work we will use CA to establish the basis vectors.

The idea of CA is to efficiently find approximate solutions $\tilde{\mathbf{u}} \approx \mathbf{u}$ to (1), and $(\tilde{\mu}_j, \tilde{\boldsymbol{\phi}}_j) \approx (\mu_j, \boldsymbol{\phi}_j)$ to (4), which are sufficiently accurate. Let us start by considering the approximate solution of (1). In the combined approximations (CA) approach, we utilize the factorization of $\mathbf{K}_L$ in a previous design iteration, to compute the displacement in the current design iteration, i.e.

$$\left(\mathbf{K}_L + \Delta\mathbf{K}_L\right)\mathbf{u} = \mathbf{F}, \tag{6}$$ mech02

where $\Delta\mathbf{K}_L$ is the change in stiffness due to the design change. Multiplying (6) by $\mathbf{K}_L^{-1}$ yields

$$\left(\mathbf{1} + \mathbf{K}_L^{-1}\Delta\mathbf{K}_L\right)\mathbf{u} = \mathbf{K}_L^{-1}\mathbf{F}. \tag{7}$$ mech03

It can be shown that the inverse of $(\mathbf{1} + \mathbf{K}_L^{-1}\Delta\mathbf{K}_L)$ exists as a power series if $\left\|\mathbf{K}_L^{-1}\Delta\mathbf{K}_L\right\|_2 < 1$, hence

$$(\mathbf{1} + \mathbf{B})^{-1} = \sum_{k=0}^{\infty}(-\mathbf{B})^k, \quad \text{where} \quad \mathbf{B} = \mathbf{K}_L^{-1}\Delta\mathbf{K}_L. \tag{8}$$ mech04

The solution to (7) can therefore be expressed as

$$\mathbf{u} = \left(\mathbf{1} - \mathbf{B} + \mathbf{B}^2 - \ldots\right)\mathbf{K}_L^{-1}\mathbf{F}. \tag{9}$$ mech05

The idea of CA is to truncate the infinite power series (8) at e.g. $s \ll n$, and assume that the approximative solution is spanned by this basis, i.e.

$$\tilde{\mathbf{u}} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + \ldots + y_s\mathbf{u}_s = \mathbf{R}\mathbf{y}, \tag{10}$$ uapprox

where $\mathbf{R} \in \mathbb{R}^{n\times s}$ contains the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_s$ and $\mathbf{y} \in \mathbb{R}^s$ the coefficients. The basis vectors are obtained recursively from (9) as

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{K}_L^{-1}\mathbf{F}, \\ \mathbf{u}_i &= -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s, \\ \mathbf{t}_i &= \mathbf{u}_i(\mathbf{u}_i^T\mathbf{u}_i)^{-1/2}, \quad i = 1, ..., s, \end{aligned} \tag{11}$$ basisvec

where $\mathbf{u}_1$ is known from the previous optimization step. Now, replacing the true solution $\mathbf{u}$ with the approximative (10) in (1) and premultiplying both sides with $\mathbf{R}^T$ yields

$$\mathbf{K}_L^R\mathbf{y} = \mathbf{F}^R, \tag{12}$$ redSys

where $\mathbf{K}_L^R = \mathbf{R}^T\mathbf{K}_L\mathbf{R}$ and $\mathbf{F}^R = \mathbf{R}^T\mathbf{F}$. The linear system (12) has $s \ll n$ unknowns $\mathbf{y}$.

If we employ Gram-Schmidt orthogonalization of the basis vectors with respect to $\mathbf{K}_L$ before solving (12), we are able to obtain a fully decoupled system of equations. Let the orthonormalized vectors be denoted $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_s$, then

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_i &= \mathbf{u}_i - \sum_{j=1}^{i-1}\left(\mathbf{u}_i^T\mathbf{K}_L\mathbf{v}_j\right)\mathbf{v}_j. \end{aligned} \tag{13}$$ ortho01

Now, let $\tilde{\mathbf{u}} = \mathbf{V}\mathbf{z}$, wherefore (1) can be approximated as

$$\mathbf{z} = \mathbf{V}^T\mathbf{F}, \tag{14}$$ redSysOrtho

and we find the approximative solution $\tilde{\mathbf{u}} = \mathbf{V}(\mathbf{V}^T\mathbf{F})$. This results in that any potential issues with ill-conditioning of $\mathbf{K}_L^R$ vanishes when solving for the approximate solution.

3.1 Reanalysis of the buckling problem

Let us now turn to the approximative solution of (4). Again, we utilize the factorization of $\mathbf{K}_L$ in a previous design iteration, to compute the eigenpairs in the current design iteration, i.e.

$$\mathbf{K}_G \boldsymbol{\phi}_j = \mu_j \left(\mathbf{K}_L + \varDelta\mathbf{K}_L\right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \tag{15}$$

eig01

Premultiplication of (15) by $\mathbf{K}_L^{-1}$ yields

$$\mathbf{K}_L^{-1}\mathbf{K}_G \boldsymbol{\phi}_j = \mu_j \left(\mathbf{1} + \mathbf{B}\right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \tag{16}$$

eig02

As before, the inverse of $(\mathbf{1} + \mathbf{B})$ exists as a power series if $\|\mathbf{B}\| < 1$, i.e.

$$\boldsymbol{\phi}_j = \frac{1}{\mu_j}\left(\mathbf{1} - \mathbf{B} + \mathbf{B}^2 - \ldots\right) \mathbf{K}_L^{-1}\mathbf{K}_G \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n, \tag{17}$$

eig03

An approximation to $\boldsymbol{\phi}_j$, i.e. $\tilde{\boldsymbol{\phi}}_j$, is obtained by truncating the infinite sum at term $s \ll n$ and substituting $\boldsymbol{\phi}_j$ for $\bar{\boldsymbol{\phi}}_j$ the j:th eigenvector corresponding to the matrix $\mathbf{K}_L$, such that

$$\tilde{\boldsymbol{\phi}}_j = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + \ldots + y_s\mathbf{u}_s, \tag{18}$$

phiapprox

where the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ are obtained from (17) as

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{K}_L^{-1}\mathbf{K}_G \bar{\boldsymbol{\phi}}_j,\\
&\mathbf{u}_i = -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s,\\
&\mathbf{t}_i = \mathbf{u}_i\left(\mathbf{u}_i^T\mathbf{K}_L\mathbf{u}_i\right)^{-1/2}, \quad i = 1, ..., s.
\end{aligned} \tag{19}$$

phibasisvec

To obtain an improved set of approximated eigenvectors, we employ deflation, i.e. Gram-Schmidt orthogonalization. In this way, we rather use

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{K}_L^{-1}\mathbf{K}_G \bar{\boldsymbol{\phi}}_j,\\
&\mathbf{u}_i = -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s,\\
&\mathbf{t}_i = \mathbf{u}_i\left(\mathbf{u}_i^T\mathbf{K}_L\mathbf{u}_i\right)^{-1/2}, \quad i = 1, ..., s,\\
&\mathbf{v}_i = \mathbf{t}_i - \sum_{j=1}^{k-1}\left(\tilde{\boldsymbol{\phi}}_j^T\mathbf{K}_L\mathbf{t}_i\right)\tilde{\boldsymbol{\phi}}_j, \quad i = 1, ..., s.
\end{aligned} \tag{20}$$

phibasisvec2

Here we show the equations that lead to CA for the buckling problem - how are linear systems approximated, how are EV systems approximated, e.g. [15,16].

4 Design representation, filtration and thresholding

The goal of our topology optimization is to optimally distribute a linear elastic material in the design domain, $\varOmega \in \mathbb{R}^3$, which is quantified by the continuous non-dimensional volume fraction field $z : \varOmega \to [0, 1]$. A well-posed optimization problem is obtain using restriction, by which fine-scale oscillations of z are penalized via the Helmholtz PDE-filter, cf. [17]. In this way, z is replaced by the smooth field $\nu : \varOmega \to [0, 1]$. To limit regions wherein $\nu(\boldsymbol{X}) \in (0, 1)$, we use thresholding ([13], [21]) and penalization ([2]), such that

$$\begin{aligned}
&\mathbf{D} = \chi_L(\bar{\nu})\mathbf{D}_o \quad \text{and} \quad \mathbf{G} = \chi_G(\bar{\nu})\mathbf{G}_o,\\
&\text{where} \quad \chi_L(\bar{\nu}) = \delta_o + \bar{\nu}^p(1 - \delta_o), \quad \chi_G(\bar{\nu}) = \bar{\nu}^p,\\
&\text{and} \quad \bar{\nu} = H_{\beta,\eta}(\nu) = \frac{\tanh(\beta\eta) + \tanh(\beta(\nu - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}.
\end{aligned} \tag{21}$$

matInter

In (21), β and η are numerical parameters defined such that $\lim_{\beta\to\infty} H_{\beta,\eta}(\nu) = u_s(\nu - \eta)$, where u_s is the unit step function. Increasing values of $p \in \mathbb{R}^+$ enforce increasing levels of penalization, and $\delta_o \in \mathbb{R}^+$ is the ersatz material stiffness scaling. Henceforth, we consider

$$
\begin{aligned}
\mathbf{K}_L(\rho) &= \sum \int_\Omega \chi_L(\rho) \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \\
\mathbf{K}_G(\rho, \mathbf{u}(\rho)) &= \sum \int_\Omega \chi_G(\rho) \mathbf{Y}^T \mathbf{G}(\mathbf{u}) \mathbf{Y} \, dV,
\end{aligned}
\tag{22}
$$

Kg

where $\mathbf{B}$ and $\mathbf{G}$ contains the gradient of the element shape functions, $\mathbf{Y}$ is a symmetric block matrix of the stresses and $\mathbf{D}$ is the Voigt representation of the material tangent.

Don't forget to show the aggregation of eigenvalues using a p-norm, following Torii [20, 9]

5 Optimization formulation

We use the topology optimization to solve either of the two optimization problems presented below. The first optimization problem is

$$
(\mathbb{TO}_1) \begin{cases} \min\limits_{\mathbf{z}} \quad g_o = \gamma = \dfrac{1}{\lambda}, \\ \text{s.t} \quad \begin{cases} g_1 = \displaystyle\int_\Omega \bar{\nu} \, dV - V_{max} \le 0, \\ 0 \le \mathsf{z}_e \le 1, \quad e = 1, ..., N_e, \end{cases} \end{cases}
\tag{23}
$$

TO1

in which we seek to maximize the smallest buckling load factor λ by minimizing the reciprocal γ, subject to a standard upper bound constraint on the structural volume. The second optimization problem is

$$
(\mathbb{TO}_2) \begin{cases} \min\limits_{\mathbf{z}} \quad g_o = \mathbf{F}^T \mathbf{u}, \\ \text{s.t} \quad \begin{cases} g_1 = \displaystyle\int_\Omega \bar{\nu} \, dV - V_{max} \le 0, \\ g_2 = \gamma \le \gamma^\star, \\ 0 \le \mathsf{z}_e \le 1, \quad e = 1, ..., N_e, \end{cases} \end{cases}
\tag{24}
$$

TO2

which is a standard compliance minimization problem subject to an lower bound constraint on the smallest buckling load factor and a maximum volume constraint.

- Minimize $\gamma = \frac{1}{\lambda}$ s.t. volume
- Minimize compliance s.t. $\gamma \le \gamma^\star$ and volume

5.1 Sensitivity analysis

When computing the sensitivity of a non self-adjoint displacement dependent function, the implicit sensitivity of the displacements with respect to the design variables appears. In this work, we annihilate this implicit sensitivity through the adjoint method. To exemplify this process, we introduce the augmented version of a general function $g = g(\boldsymbol{\nu}(\mathbf{z}), \mathbf{u}(\boldsymbol{\nu}(\mathbf{z})))$ as

$$
g := g - \boldsymbol{\Lambda}^T \mathbf{r} - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \mathbf{r}_{\boldsymbol{\nu}},
\tag{25}
$$

gensens01

where $\mathbf{r} = \mathbf{K}_L \mathbf{u} - \mathbf{F}$ and $\mathbf{r}_{\boldsymbol{\nu}} = \mathbf{K}_{\boldsymbol{\nu}} \boldsymbol{\nu} - \mathbf{F}_{\boldsymbol{\nu}} \mathbf{z}$. The sensitivity of (25) with respect to $\mathbf{z}$ is

$$
\frac{\partial g}{\partial \mathbf{z}} = \frac{\partial g}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} + \frac{\partial g}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} - \boldsymbol{\Lambda}^T \left(\frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} + \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} \right) - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \left(\frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} + \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} \right),
\tag{26}
$$

gensens02

which can be rearranged to

$$\frac{\partial g}{\partial \mathbf{z}} = -\boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} + \left[\frac{\partial g}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}} + \left(\frac{\partial g}{\partial \mathbf{u}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \right) \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \right] \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}}. \tag{27}$$

gensens03

Here, we find that the implicit $\frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}}$ and $\frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}}$ sensitivities are annihilated by sequentially solving the coupled linear systems

$$\begin{aligned} \mathbf{K}_L \boldsymbol{\Lambda} &= \frac{\partial g}{\partial \mathbf{u}}, \\ \mathbf{K}_{\boldsymbol{\nu}} \boldsymbol{\Lambda}_{\boldsymbol{\nu}} &= \frac{\partial g}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}}, \end{aligned} \tag{28}$$

gensen04

which yields the final expression for the sensitivity of g as

$$\frac{\partial g}{\partial \mathbf{z}} = -\boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} \tag{29}$$

gensens05

When seeking the sensitivity of the smallest buckling load factor, we let $g = \mu_j$ and compute $\frac{\partial g}{\partial \mathbf{u}}$ and $\frac{\partial g}{\partial \boldsymbol{\nu}}$ from the differentiations of (4), i.e.

$$\begin{aligned} \frac{\partial \mu_j}{\partial \mathbf{u}} &= \boldsymbol{\phi}_j^T \frac{\partial \mathbf{K}_G}{\partial \boldsymbol{\nu}} \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n, \\ \frac{\partial \mu_j}{\partial \boldsymbol{\nu}} &= \boldsymbol{\phi}_j^T \left(\frac{\partial \mathbf{K}_G}{\partial \boldsymbol{\nu}} - \mu_j \frac{\partial \mathbf{K}_L}{\partial \boldsymbol{\nu}} \right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \end{aligned} \tag{30}$$

bucklingsens01

- Consistent sensitivity is favorable, but inconsistent approaches can be efficient.
- Show the challenge of orthogonalization with evolving modes.

6 Numerical examples

sec:examples

We demonstrate the reduced order model on three different geometries, see figures 1, 2 and 3. The optimization problem (24) is solved using Svanbergs MMA solver, with default parameters see [19] for details. A PDE-filter of Helmholtz type is used to control the length scale, following [17], with a filter radius of $0.1 \times \min(L_x, L_y)$. Young's modulus is set to 2×10^5, Poisson's ratio is set to $\nu = 0.3$. A continuation scheme is used for the material penalization in (21), where $p = 2$ initially and is incremented by 0.5 every 20 iterations until $p = 6$, and $\delta_0 = 1 \times 10^{-6}$. A constraint on the critical buckling load factor is enforced by aggregating the 6 lowest load factors with the p-norm. The penalty exponent is set to $p = 8$ initially and is amplified by 4 once $p = 6$ for the material penalization. Finally, the filtered densities are penalized using a smooth Heaviside approximation according to (21), where $\beta = 6$ and $\eta = 0.5$.

6.1 Axially loaded rod

Fig. 1 Geometry of column

fig:geometry_colu

6.2 Spire structure

Fig. 2 Geometry of spire

fig:geometry_spi

6.3 V structure

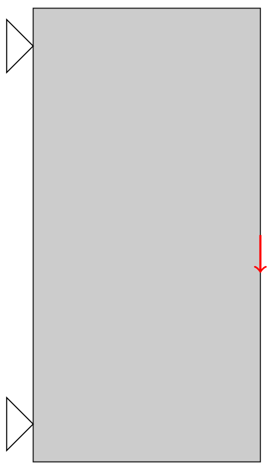

Fig. 3 Geometry of twobar

fig:geometry_twob

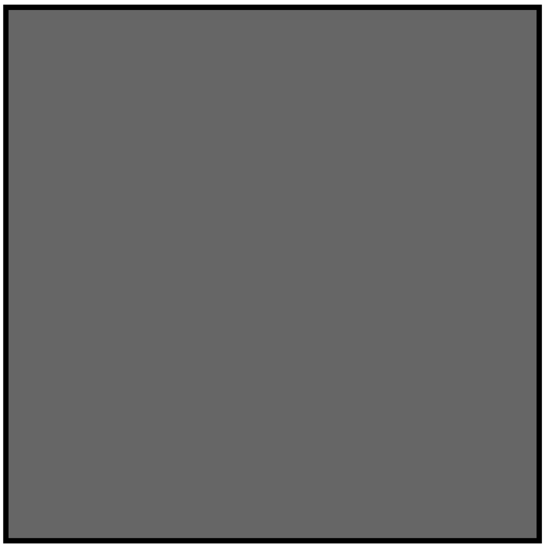

Fig. 4 Caption fig:column_resul

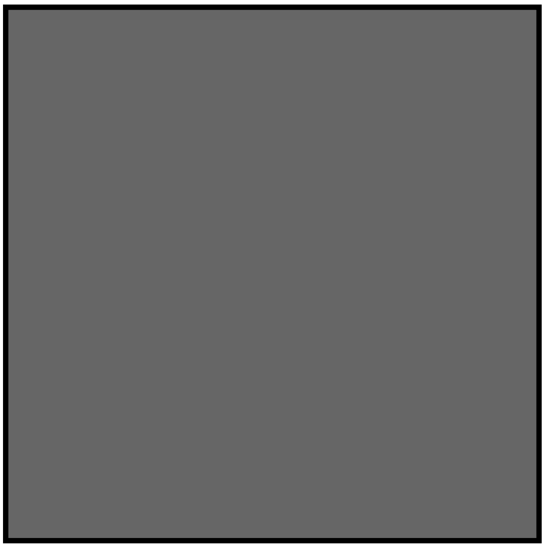

Fig. 5 Please write your figure caption here fig:1

Table 1 Please write your table caption here tab:1

first	second	third
number	number	number
number	number	number

7 Conclusions

sec:conclusions

Acknowledgements

Replication of results

All MATLAB code is provided as supplementary material.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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