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Efficient topology optimization with linearized buckling using reduced order modelling

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Abstract Keywords Topology optimization, Buckling, Reduced order modelling, Combined approximations

The simplest and frequently considered optimization problem is compliance minimization under a volume constraint. This formulation favors

1 Introduction

Topology optimization (TO) is a useful tool for designing structural components and materials. ~~Frequently, the compliance is optimized, which results in~~ designs consisting of slender members, in compression and tension. Unfortunately, in practical applications these structures are inherently prone to buckle, since stiffness and stability are two competing concepts. The most common way of accommodating for stability in optimization is to perform a linearized buckling analysis, and to constrain the critical buckling load factor (Neves et al. [18], Dalklint et al. [6], Gao and Ma [10], Ferrari and Sigmund [8]).

LaTex fixar detta själv så jag misstänker att du skriver in detta.

Typically, the optimization problem is solved using a nested approach by implicitly enforcing the equilibrium equations, which constitute the majority of the computational cost. This cost is further increased when buckling is considered, since the buckling load factors are computed from a generalized eigenvalue problem. Eigenvalue problems are notoriously difficult to solve and scale poorly with the problem size, in comparison to linear systems of equations which in the best case scale linearly with the number of unknowns. This cost becomes prohibitive for large to medium scale problems, which makes the technique unappealing for industrial applications.

Skriv att flera hundra designuppdateringar normalt behöver göras.

Recently, efforts have been made to extend TO with buckling constraints to large scale problems. Dunning et al. [7], presented a Block-Jacobi CG solver with a shift-and invert strategy based on information from previous design iterations. While this method was shown to be efficient it still requires a matrix factorization at each design update. Ferrari and Sigmund [9], instead reused the multigrid preconditioner from the initial equilibrium problem to solve the eigenvalue problem on a coarser finite element mesh. The coarse-grid eigenvectors were then projected onto a finer mesh and refined using inverse iteration. This reduced the cost for solving the eigenvalue problems while retaining a good approximation of the eigen modes. Also worth noting is the work by Bian and Fang [3], where an assembly-free eigenvalue solver was used to solve the eigenvalue problem on a grid of several million elements.

A number of methods based on reduced order modeling (ROM) have been proposed to accelerate topology optimization. In ROM the equilibrium equations are projected onto a reduced solution space which is much smaller than the high-fidelity problem. The efficacy of a ROM is ultimately decided by the cost of generating the reduced space. One approach is to base the reduced space on previous solutions to the equilibrium equations, and update the space when new solution vectors become available [11,5]. Kang et al. ~~in~~ [13] solved the eigenvalue problem concurrently with the optimization problem, improving the accuracy sequentially.

Address(es) of author(s) should be given

An attractive ROM-technique for optimization problems is reanalysis, in which the system matrix's factorization in a design iteration is used to build a reduced space for subsequent iterations. The use of reanalysis to solve linear systems was first discussed by Kirsch [14]. More recently, reanalysis has been applied to topology optimization. Kirsch showed ~~in~~ [16] that a ROM using reanalysis is equivalent to a preconditioned conjugate gradient method for linear systems of equations. Amir [1] studied reanalysis for compliance problems and discussed the idea of a consistent sensitivity analysis, i.e. where the inaccuracies in the ROM are taken into account. Senne et al [19] extended reanalysis to problems with geometric and material nonlinearities. Bogomolny [4] extended reanalysis to vibrational problems, wherein a reduced order model was built for each sought eigenpair. For the generalized eigenvalue problem the basis vectors must be orthogonalized to ensure that the correct eigenpair is approximated. Bogomolny also derives a consistent sensitivity analysis for the vibrational problem.

In this paper, we aim to show that ROM using Combined Approximation can effectively be implemented for buckling problems. What makes this work unique is that CA will be used for the linear analysis, generalized eigenproblem and the adjoints.

2 Preliminaries

sec:prelim

We consider the finite element discretized linear system

reference

$$\mathbf{K}_L \mathbf{u}_o = \mathbf{F}_o, \tag{1}$$

mech

in combination with appropriate boundary conditions to describe our mechanical system. In (1), ~~we define~~ the symmetric and positive definite linear stiffness matrix $\mathbf{K}_L \in \mathbb{R}^{n \times n}$, the displacement vector $\mathbf{u}_o \in \mathbb{R}^n$ and the reference load vector $\mathbf{F}_o \in \mathbb{R}^n$ where n is the number of degrees-of-freedom.

$_o$ is the

To solve (1), we utilize the Cholesky decomposition of $\mathbf{K}_L$ in combination with forward/backward substitutions, i.e.

$$\mathbf{K}_L = \mathbf{U}^T \mathbf{U}, \tag{2}$$

mechFac01

where $\mathbf{U} \in \mathbb{R}^{n \times n}$ is upper triangular. After solving (1) for $\mathbf{u}_o$ using (2), we compute the symmetric and indefinite stress stiffness matrix $\mathbf{K}_G(\mathbf{u}_o) \in \mathbb{R}^{n \times n}$, which enables the generalized eigenvalue problem

Ref. vill var man hittar K_L & K_G

$$\left(\mathbf{K}_L + \lambda_j \mathbf{K}_G(\mathbf{u}_o)\right) \boldsymbol{\phi}_j = \mathbf{0}, \quad j \in \mathbb{N}_n, \tag{3}$$

lineig01

to be solved for the eigenpairs $(\lambda_j, \boldsymbol{\phi}_j) \in (\mathbb{R} \times \mathbb{R}^n)$. However, since $\mathbf{K}_G$ is indefinite, it is convenient to pose (3) as

$$\left(\mathbf{K}_G(\mathbf{u}_o) - \mu_j \mathbf{K}_L\right) \boldsymbol{\phi}_j = \mathbf{0}, \quad j \in \mathbb{N}_n, \tag{4}$$

lineig02

where $\mu_j = -\frac{1}{\lambda_j}$ and we assume that $\boldsymbol{\phi}_i^T \mathbf{K}_L \boldsymbol{\phi}_j = \delta_{ij}$. Solving the generalized eigenvalue problem in (3) or (4), is referred to as *linearized buckling analysis*, wherefore the eigenvalues λ_j are denoted the critical load factors, so that the critical load is $\mathbf{F}_i = \lambda_i \mathbf{F}_o$. The eigenmode associated with the critical load $\mathbf{F}_i$ is $\boldsymbol{\phi}_i$.

3 Reduced order modelling

sec:CA

The basic idea of reduced order modelling (ROM) is to approximate solutions to linear systems using a reduced basis, i.e.

$$\tilde{\mathbf{u}} \approx \alpha_1 \hat{\mathbf{u}}_1 + \alpha_2 \hat{\mathbf{u}}_2 + \ldots \alpha_m \hat{\mathbf{u}}_m \tag{5}$$

rom1

where the number of bases, m, is significantly smaller that the number of dimension of the full problem, i.e. $m \ll n$. The success of the ROM method relies on the choice of the basis vectors, $\hat{\mathbf{u}}_i$ and in this work we will use CA to establish the basis vectors.

The idea of CA is to efficiently find approximate solutions $\tilde{\mathbf{u}} \approx \mathbf{u}$ to (1), and $(\tilde{\mu}_j, \tilde{\boldsymbol{\phi}}_j) \approx (\mu_j, \boldsymbol{\phi}_j)$ to (4), which are sufficiently accurate. Let us start by considering the approximate solution of (1).

In the combined approximations (CA) approach, we utilize the factorization of $\mathbf{K}_L$ in a previous design iteration, to compute the displacement in the current design iteration, i.e.

$$(\mathbf{K}_L + \Delta\mathbf{K}_L)\,\mathbf{u} = \mathbf{F}, \quad \text{osv.} \tag{6}$$ mech02

where $\Delta\mathbf{K}_L$ is the change in stiffness due to the design change. Multiplying (6) by $\mathbf{K}_L^{-1}$ yields

$$\left(\mathbf{1} + \mathbf{K}_L^{-1}\Delta\mathbf{K}_L\right)\mathbf{u} = \mathbf{K}_L^{-1}\mathbf{F}. \tag{7}$$ mech03

can be expressed?

It can be shown that the inverse of $(\mathbf{1} + \mathbf{K}_L^{-1}\Delta\mathbf{K}_L)$ exists as a power series if $\left\|\mathbf{K}_L^{-1}\Delta\mathbf{K}_L\right\|_2 < 1$, hence

$$(\mathbf{1} + \mathbf{B})^{-1} = \sum_{k=0}^{\infty}(-\mathbf{B})^k, \quad \text{where} \quad \mathbf{B} = \mathbf{K}_L^{-1}\Delta\mathbf{K}_L. \tag{8}$$ mech04

The solution to (7) can therefore be expressed as

$$\mathbf{u} = \left(\mathbf{1} - \mathbf{B} + \mathbf{B}^2 - \ldots\right)\mathbf{K}_L^{-1}\mathbf{F}. \tag{9}$$ mech05

The idea of CA is to truncate the infinite power series (8) at e.g. $s \ll n$, and assume that the approximative solution is spanned by this basis, i.e.

$$\tilde{\mathbf{u}} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + \ldots + y_s\mathbf{u}_s = \mathbf{R}\mathbf{y}, \tag{10}$$ uapprox

where $\mathbf{R} \in \mathbb{R}^{n\times s}$ contains the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_s$ and $\mathbf{y} \in \mathbb{R}^s$ the coefficients. The basis vectors are obtained recursively from (9) as

solution 6

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{K}_L^{-1}\mathbf{F}, \\ \mathbf{u}_i &= -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s, \\ \mathbf{t}_i &= \mathbf{u}_i(\mathbf{u}_i^T\mathbf{u}_i)^{-1/2}, \quad i = 1, ..., s, \end{aligned} \tag{11}$$ basisvec

where $\mathbf{u}_1$ is known from the previous optimization step. Now, replacing the true solution $\mathbf{u}$ with the approximative (10) in (1) and premultiplying both sides with $\mathbf{R}^T$ yields

$$\mathbf{K}_L^R\mathbf{y} = \mathbf{F}^R, \tag{12}$$ redSys

where $\mathbf{K}_L^R = \mathbf{R}^T\mathbf{K}_L\mathbf{R}$ and $\mathbf{F}^R = \mathbf{R}^T\mathbf{F}$. The reduced linear system (12) has $s \ll n$ unknowns $\mathbf{y}$.

If we employ Gram-Schmidt orthogonalization of the basis vectors with respect to $\mathbf{K}_L$ before solving (12), we are able to obtain a fully decoupled system of equations. Let the orthonormalized vectors be denoted $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_s$, then

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \\ \mathbf{v}_i &= \mathbf{u}_i - \sum_{j=1}^{i-1}\left(\mathbf{u}_i^T\mathbf{K}_L\mathbf{v}_j\right)\mathbf{v}_j. \end{aligned} \tag{13}$$ ortho01

Now, let $\tilde{\mathbf{u}} = \mathbf{V}\mathbf{z}$, wherefore (1) can be approximated as

$$\mathbf{z} = \mathbf{V}^T\mathbf{F}, \tag{14}$$ redSysOrtho

and we find the approximative solution $\tilde{\mathbf{u}} = \mathbf{V}(\mathbf{V}^T\mathbf{F})$. This ~~results in that~~ orthogonalization procedure removes any potential issues ~~with~~ due to ill-conditioning of $\mathbf{K}_L^R$ ~~vanishes when solving for the approximate solution~~.

Skriv ut hur resulterande systemen ser ut

3.1 Reanalysis of the buckling problem

already established

Let us now turn to the approximative solution of (4). Again, we utilize the factorization of $\mathbf{K}_L$ in a previous design iteration, to compute the eigenpairs in the current design iteration, i.e.

$$\mathbf{K}_G \boldsymbol{\phi}_j = \mu_j \left(\mathbf{K}_L + \Delta\mathbf{K}_L\right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \qquad (15) \quad \boxed{\texttt{eig01}}$$

Premultiplication of (15) by $\mathbf{K}_L^{-1}$ yields gillar inte "exist"

$$\mathbf{K}_L^{-1}\mathbf{K}_G \boldsymbol{\phi}_j = \mu_j \left(\mathbf{1} + \mathbf{B}\right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \qquad (16) \quad \boxed{\texttt{eig02}}$$

As before, the inverse of $(\mathbf{1} + \mathbf{B})$ exists as a power series if $\|\mathbf{B}\| < 1$, i.e.

$$\boldsymbol{\phi}_j = \frac{1}{\mu_j}\left(\mathbf{1} - \mathbf{B} + \mathbf{B}^2 - \ldots\right) \mathbf{K}_L^{-1}\mathbf{K}_G \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n, \qquad (17) \quad \boxed{\texttt{eig03}}$$

An approximation to $\boldsymbol{\phi}_j$, i.e. $\tilde{\boldsymbol{\phi}}_j$, is obtained by truncating the infinite sum at term $s \ll n$ and substituting $\boldsymbol{\phi}_j$ for $\bar{\boldsymbol{\phi}}_j$ the j:th eigenvector corresponding to the matrix $\mathbf{K}_L$, such that

Ej def?

$$\tilde{\boldsymbol{\phi}}_j = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + \ldots + y_s\mathbf{u}_s, \qquad (18) \quad \boxed{\texttt{phiapprox}}$$

where the basis vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ are obtained from (17) as

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{K}_L^{-1}\mathbf{K}_G \bar{\boldsymbol{\phi}}_j, \\
\mathbf{u}_i &= -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s, \\
\mathbf{t}_i &= \mathbf{u}_i \left(\mathbf{u}_i^T \mathbf{K}_L \mathbf{u}_i\right)^{-1/2}, \quad i = 1, ..., s.
\end{aligned} \qquad (19) \quad \boxed{\texttt{phibasisvec}}$$

To obtain an improved set of approximated eigenvectors, we employ deflation, i.e. Gram-Schmidt orthogonalization. In this way, we rather use

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{K}_L^{-1}\mathbf{K}_G \bar{\boldsymbol{\phi}}_j, \\
\mathbf{u}_i &= -\mathbf{B}\mathbf{t}_{i-1}, \quad i = 2, ..., s, \\
\mathbf{t}_i &= \mathbf{u}_i \left(\mathbf{u}_i^T \mathbf{K}_L \mathbf{u}_i\right)^{-1/2}, \quad i = 1, ..., s, \\
\mathbf{v}_i &= \mathbf{t}_i - \sum_{j=1}^{k-1} \left(\tilde{\boldsymbol{\phi}}_j^T \mathbf{K}_L \mathbf{t}_i\right) \tilde{\boldsymbol{\phi}}_j, \quad i = 1, ..., s.
\end{aligned} \qquad (20) \quad \boxed{\texttt{phibasisvec2}}$$

Here we show the equations that lead to CA for the buckling problem - how are linear systems approximated, how are EV systems approximated, e.g. [14,15].

4 Design representation, filtration and thresholding

The goal of our topology optimization is to optimally distribute a linear elastic material in the design domain, $\Omega \in \mathbb{R}^3$, which is quantified by the continuous non-dimensional volume fraction field $z : \Omega \to [0, 1]$. A well-posed optimization problem is obtain using restriction, by which fine-scale oscillations of z are penalized via the Helmholtz PDE-filter, cf. [17]. In this way, z is replaced by the smooth field $\nu : \Omega \to [0, 1]$. To limit regions wherein $\nu(\boldsymbol{X}) \in (0, 1)$, we use thresholding ([12], [22]) and penalization ([2]), such that

$$\begin{aligned}
&\mathbf{D} = \chi_L(\bar{\nu})\mathbf{D}_o \quad \text{and} \quad \mathbf{G} = \chi_G(\bar{\nu})\mathbf{G}_o, \\
&\text{where} \quad \chi_L(\bar{\nu}) = \delta_o + \bar{\nu}^p(1 - \delta_o), \quad \chi_G(\bar{\nu}) = \bar{\nu}^p, \\
&\text{and} \quad \bar{\nu} = H_{\beta,\eta}(\nu) = \frac{\tanh(\beta\eta) + \tanh(\beta(\nu - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}.
\end{aligned} \qquad (21) \quad \boxed{\texttt{matInter}}$$

Kanske skriva ut PDE'n + lengthscale

Skriv varför du gjort detta val. (tar bort oönskade moder i void regionen)

In (21), β and η are numerical parameters defined such that $\lim_{\beta\to\infty} H_{\beta,\eta}(\nu) = u_s(\nu-\eta)$, where u_s is the unit step function. Increasing values of $p \in \mathbb{R}^+$ enforce increasing levels of penalization, and $\delta_o \in \mathbb{R}^+$ is the ersatz material stiffness scaling. Henceforth, we consider

$$\begin{aligned} \mathbf{K}_L(\rho) &= \sum \int_\Omega \chi_L(\rho)\mathbf{B}^T\mathbf{D}\mathbf{B}\,dV \\ \mathbf{K}_G(\rho,\mathbf{u}(\rho)) &= \sum \int_\Omega \chi_G(\rho)\mathbf{Y}^T\mathbf{G}(\mathbf{u})\mathbf{Y}\,dV, \end{aligned} \tag{22}$$ Kg

where $\mathbf{B}$ and $\mathbf{G}$ contains the gradient of the element shape functions, $\mathbf{Y}$ is a symmetric block matrix of the stresses and $\mathbf{D}$ is the Voigt representation of the material tangent.

4.1 Aggregation

It is well known that the min-function is not differentiable. For this reason the critical buckling load facotor is approximated using the p-norm (see Torii et al [21]), which aggregates the n_{eval} lowest buckling load factors. In practice, we instead put an upper bound on the reciprocal value $\gamma = \lambda^{-1}$. The constraint on the critical buckling load factor is stated as

Konflikt med p i simp.

$$\gamma_c = \max \gamma_k \approx \left(\sum_{i=1}^{n_{\text{eval}}} \gamma_i^p\right)^{1/p} \le \gamma^*. \tag{23}$$ eq:pnorm

p → q?

Skriv lite fylligare här.

Vad händer om inte 23 används?

5 Optimization formulation

We use the topology optimization to solve either of the two optimization problems presented below. The first optimization problem is

$$(\mathbb{TO}_1)\begin{cases} \min\limits_{\mathbf{z}} & g_o = \gamma = \dfrac{1}{\lambda}, \\ \text{s.t} & \begin{cases} g_1 = \displaystyle\int_\Omega \bar{\nu}\,dV - V_{max} \le 0, \\ 0 \le \mathrm{z}_e \le 1, \quad e = 1, ..., N_e, \end{cases} \end{cases} \tag{24}$$ TO1

in which we seek to maximize the smallest buckling load factor λ by minimizing the reciprocal γ, subject to a standard upper bound constraint on the structural volume. The second optimization problem is

$$(\mathbb{TO}_2)\begin{cases} \min\limits_{\mathbf{z}} & g_o = \mathbf{F}^T\mathbf{u}, \\ \text{s.t} & \begin{cases} g_1 = \displaystyle\int_\Omega \bar{\nu}\,dV - V_{max} \le 0, \\ g_2 = \gamma \le \gamma^\star, \\ 0 \le \mathrm{z}_e \le 1, \quad e = 1, ..., N_e, \end{cases} \end{cases} \tag{25}$$ TO2

which is a standard compliance minimization problem subject to an lower bound constraint on the smallest buckling load factor and a maximum volume constraint.

- ~~Minimize $\gamma = \frac{1}{\lambda}$ s.t. volume~~
- ~~Minimize compliance s.t. $\gamma \le \gamma^\star$ and volume~~

ej def.

brukar inte skriva så här

5.1 Sensitivity analysis

When computing the sensitivity of a non self-adjoint displacement dependent function, the implicit sensitivity of the displacements with respect to the design variables appears. In this work, we annihilate this implicit sensitivity through the adjoint method. To exemplify this process, we introduce the augmented version of a general function $g = g(\boldsymbol{\nu}(\mathbf{z}), \mathbf{u}(\boldsymbol{\nu}(\mathbf{z})))$ as

$$g := g + \boldsymbol{\Lambda}^T \mathbf{r} - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \mathbf{r}_{\boldsymbol{\nu}}, \tag{26}$$

gensens01

where $\mathbf{r} = \mathbf{K}_L \mathbf{u} - \mathbf{F}$ and $\mathbf{r}_{\boldsymbol{\nu}} = \mathbf{K}_{\boldsymbol{\nu}} \boldsymbol{\nu} - \mathbf{F}_{\boldsymbol{\nu}} \mathbf{z}$. The sensitivity of (26) with respect to $\mathbf{z}$ is

$$\frac{\partial g}{\partial \mathbf{z}} = \frac{\partial g}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} + \frac{\partial g}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} - \boldsymbol{\Lambda}^T \left(\frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} + \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} \right) - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \left(\frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} + \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}} \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}} \right), \tag{27}$$

gensens02

which can be rearranged to

$$\frac{\partial g}{\partial \mathbf{z}} = -\boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} + \left[\frac{\partial g}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}} + \left(\frac{\partial g}{\partial \mathbf{u}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \right) \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \right] \frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}}. \tag{28}$$

gensens03

Here, we find that the implicit $\frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}}$ and $\frac{\partial \boldsymbol{\nu}}{\partial \mathbf{z}}$ sensitivities are annihilated by sequentially solving the coupled linear systems

$$\begin{aligned} \mathbf{K}_L \boldsymbol{\Lambda} &= \frac{\partial g}{\partial \mathbf{u}}, \\ \mathbf{K}_{\boldsymbol{\nu}} \boldsymbol{\Lambda}_{\boldsymbol{\nu}} &= \frac{\partial g}{\partial \boldsymbol{\nu}} - \boldsymbol{\Lambda}^T \frac{\partial \mathbf{r}}{\partial \boldsymbol{\nu}}, \end{aligned} \tag{29}$$

gensen04

which yields the final expression for the sensitivity of g as

$$\frac{\partial g}{\partial \mathbf{z}} = -\boldsymbol{\Lambda}_{\boldsymbol{\nu}}^T \frac{\partial \mathbf{r}_{\boldsymbol{\nu}}}{\partial \mathbf{z}} \tag{30}$$

gensens05

When seeking the sensitivity of the smallest buckling load factor, we let $g = \mu_j$ and compute $\frac{\partial g}{\partial \mathbf{u}}$ and $\frac{\partial g}{\partial \boldsymbol{\nu}}$ from the differentiations of (4), i.e.

$$\begin{aligned} \frac{\partial \mu_j}{\partial \mathbf{u}} &= \boldsymbol{\phi}_j^T \frac{\partial \mathbf{K}_G}{\partial \boldsymbol{\nu}} \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n, \\ \frac{\partial \mu_j}{\partial \boldsymbol{\nu}} &= \boldsymbol{\phi}_j^T \left(\frac{\partial \mathbf{K}_G}{\partial \boldsymbol{\nu}} - \mu_j \frac{\partial \mathbf{K}_L}{\partial \boldsymbol{\nu}} \right) \boldsymbol{\phi}_j, \quad j \in \mathbb{N}_n. \end{aligned} \tag{31}$$

bucklingsens01

5.2 Consistent sensitivity analysis

The sensitivities in the previous section assume that the initial equilibrium (1) and buckling problem (3) are solved exactly. However, in this ~~paper~~ work they are solved using a ROM desribed in section 3, meaning there is a residual $\mathbf{y} = \mathbf{f} - \mathbf{K}_L \mathbf{u}$ and $\mathbf{y} = (\mathbf{K}_G - \mu_j \mathbf{K}_L) \boldsymbol{\phi}_j$. These inaccuracies can be accounted for (see Amir [1] and Bogomolny [4]) at the cost of introducing more adjoint vectors which require backward and forward substitution of the factorized stiffness matrix. Although this might be mathematically satisfying the consistent sensitivities ultimately reflect the approximate problem, not the exact problem. Meaning they are accurate with regard to the inaccurate problem. We aim to show that if the equilibrium equations are solved accurately enough using the sensitivities of the exact problem will lead to the correct design.

sec:examples

6 Numerical examples

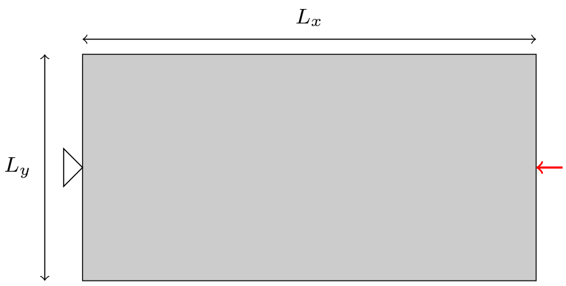
Finns tidigare

?

We demonstrate the reduced order model on three different geometries, see figures 1, 3 and 5. The optimization problem (25) is solved using Svanbergs MMA solver, with default parameters see [20] for details. A PDE-filter of Helmholtz type is used to control the length scale, following [17], with a filter radius of $0.1 \times \min(L_x, L_y)$ Young's modulus and Poisson's ratio are set to $\times 10^5$ and $\nu = 0.3$. ~~A~~ a continuation ~~scheme~~ is used for the material penalization in (21), where $p = 2$ initially and is incremented by 0.5 every 20 iterations until $p = 6$, and $\delta_0 = 1 \times 10^{-6}$. A constraint on the critical buckling load factor is enforced by aggregating the 6 lowest load factors with the p-norm. The penalty exponent is set to $p = 8$ initially and is amplified by 4 once $p = 6$ for the material penalization. Finally, the filtered densities are penalized using a smooth Heaviside approximation according to (21), where $\beta = 6$ and $\eta = 0.5$.

6.1 Axially loaded rod

This is where the geometry would be described. The volume fraction is set to 50%.



Vad är simp och
vad är p-norm?

Fig. 1 Geometry of column fig:geometry_colu

Fig. 2 Solution using exact analysis. Yellow indicates a filled element, blue void. fig:column_ref

6.2 Spire structure

This is where the geometry would be described. The volume fraction is set to 35%.

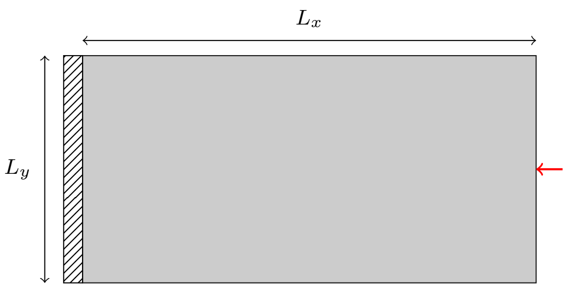


Fig. 3 Geometry of spire fig:geometry_spi

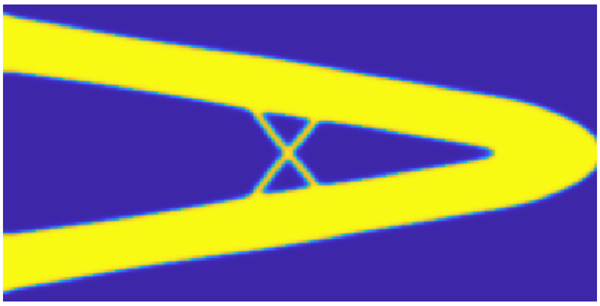


Fig. 4 Solution using exact analysis.

fig:spire_ref

6.3 V structure

This is where the geometry would be described. The volume fraction is set to 18%.

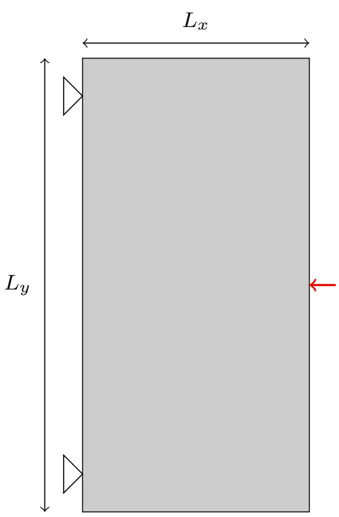


Fig. 5 Geometry of twobar

fig:geometry_twol

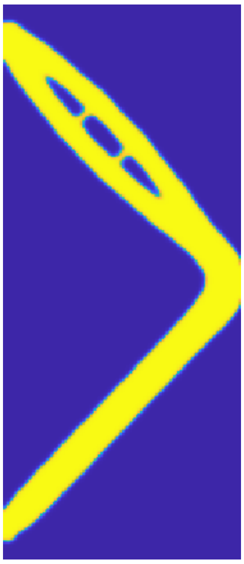


Fig. 6 Solution using exact analysis. fig:twobar_ref

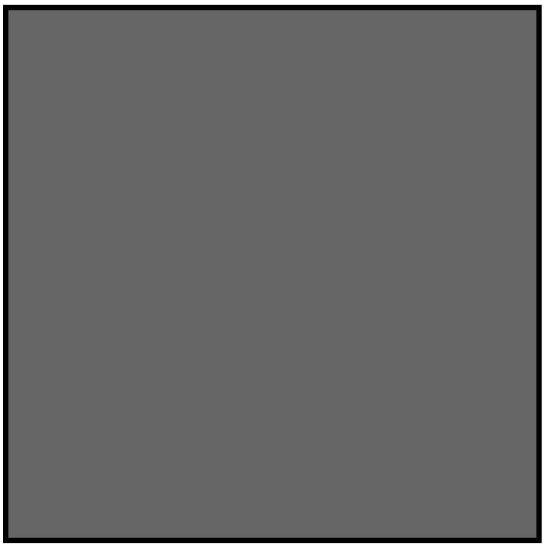


Fig. 7 Please write your figure caption here fig:1

Table 1 Please write your table caption here tab:1

first	second	third
number	number	number
number	number	number

7 Conclusions

sec:conclusions

Acknowledgements

Replication of results

All MATLAB code is provided as supplementary material.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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