7 HYPO - ELASTICITY

Let us postulate a constitutive relation between the time-derivative of the stress tensor and the time-derivative of the strain tensor in the form

$$\dot{\sigma}_{ij} = f_{ij}(\sigma_{kl}, \dot{\varepsilon}_{mn}) \tag{7.1}$$

Here a dot denotes the time-derivative, i.e. $\dot{\sigma}_{ij} = d\sigma_{ij}/dt$, where t is the time. From (7.1) appears that the time-derivative of the stress tensor is assumed to depend not only on the time-derivative of the strain tensor, but also of the current stress tensor. Thus, (7.1) provides a very general constitutive relation.

Since (7.1) is given in rate form, an integration along the load path is necessary in order to obtain the current stresses or strains. The constitutive relation (7.1) therefore expresses a material, where the response depends on the history. In the remaining part of the book, we will be concerned with such kinds of constitutive relations.

Expression (7.1) is clearly of the form investigated in (6.18) and replacing $\boldsymbol{M}$ by $\dot{\sigma}_{ij}$, $\boldsymbol{N}$ by σ_{ij} and $\boldsymbol{P}$ by $\dot{\varepsilon}_{ij}$ and using the result of (6.24), it appears that (7.1) is equivalent with

$$\begin{aligned}\dot{\sigma}_{ij} = \ & \alpha_1\delta_{ij} + \alpha_2\sigma_{ij} + \alpha_3\sigma_{ik}\sigma_{kj} + \alpha_4\dot{\varepsilon}_{ij} + \alpha_5\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} \\ & +\alpha_6(\sigma_{ik}\dot{\varepsilon}_{kj} + \dot{\varepsilon}_{ik}\sigma_{kj}) + \alpha_7(\sigma_{ik}\sigma_{kl}\dot{\varepsilon}_{lj} + \dot{\varepsilon}_{ik}\sigma_{kl}\sigma_{lj}) \\ & +\alpha_8(\sigma_{ik}\dot{\varepsilon}_{kl}\dot{\varepsilon}_{lj} + \dot{\varepsilon}_{ik}\dot{\varepsilon}_{kl}\sigma_{lj})\end{aligned} \tag{7.2}$$

where $\alpha_1 \ldots \alpha_8$ are scalar functions of the ten invariants defined in (6.12) - (6.15), i.e.

$$\begin{aligned}\alpha_i = \alpha_i[& tr\sigma,\ tr(\sigma^2),\ tr(\sigma^3),\ tr\dot{\varepsilon},\ tr(\dot{\varepsilon}^2),\ tr(\dot{\varepsilon}^3),\ tr(\sigma\dot{\varepsilon}), \\ & tr(\sigma\dot{\varepsilon}^2),\ tr(\sigma^2\dot{\varepsilon}),\ tr(\sigma^2\dot{\varepsilon}^2)]\end{aligned} \tag{7.3}$$

7.1 Time-independent response

We now require that (7.1) corresponds to a time-independent constitutive relation, i.e. the same response is obtained irrespective of the loading rate. This can only be accomplished if (7.2) is a homogeneous function of time, i.e. the only way time can appear in (7.2) is through the factor $1/dt$ present in all terms of (7.2), cf. the discussion following (6.30). This, in turn, implies that (7.1) can be written as

$$\boxed{\dot{\sigma}_{ij} = D_{ijkl}(\sigma_{mn})\dot{\varepsilon}_{kl} \quad \text{or} \quad d\sigma_{ij} = D_{ijkl}(\sigma_{mn})d\varepsilon_{kl}} \tag{7.4}$$

where the latter form clearly shows that time does not influence the response of the material.

To achieve the form given by (7.4), each term on the right-hand side of (7.2) must only contain the time-derivative of the strain tensor to the power of one. We immediately conclude that

$$\alpha_5 = \alpha_8 = 0 \tag{7.5}$$

Let us next consider the remaining terms in (7.2). Starting with the term containing α_1 that must contain the time-derivative of the strain tensor to the power of one, we conclude from (7.3) that the most general form is obtained by writing α_1 as

$$\begin{aligned}\alpha_1 = \; & \beta_1[tr\sigma,\ tr(\sigma^2),\ tr(\sigma^3)]tr\dot{\varepsilon} + \beta_4[tr\sigma,\ tr(\sigma^2),\ tr(\sigma^3)]tr(\sigma\dot{\varepsilon}) \\ & +\beta_8[tr\sigma,\ tr(\sigma^2),\ tr(\sigma^3)]tr(\sigma^2\dot{\varepsilon})\end{aligned}$$

where the reason for the notation β_1, β_4 and β_8 will appear later on. As $tr\sigma$, $tr\sigma^2$ and $tr\sigma^3$ are equivalent with the three invariants I_1, I_2 and I_3 of the stress tensor, we can write the above relation for α_1 as

$$\alpha_1 = \beta_1\dot{\varepsilon}_{kk} + \beta_4\sigma_{mn}\dot{\varepsilon}_{nm} + \beta_8\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl} \tag{7.6}$$

where β_1, β_4 and β_8 are functions of the three invariants of the stress tensor. Likewise, for the terms in (7.2), that involve α_2 and α_3, we conclude that

$$\alpha_2 = \beta_3\dot{\varepsilon}_{kk} + \beta_7\sigma_{mn}\dot{\varepsilon}_{nm} + \beta_{11}\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl} \tag{7.7}$$

and

$$\alpha_3 = \beta_6\dot{\varepsilon}_{kk} + \beta_{10}\sigma_{mn}\dot{\varepsilon}_{nm} + \beta_{12}\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl} \tag{7.8}$$

where the β−parameters may depend on the stress invariants. The terms in (7.2) that involve α_4, α_6 and α_7 already contain the time-derivative of the strain tensor to the power of one. We shift the notation and write

$$\alpha_4 = \beta_2 \qquad \alpha_6 = \beta_5 \qquad \alpha_7 = \beta_9 \tag{7.9}$$

where also β_2, β_5 and β_9 may only depend on the three invariants of the stress tensor.

With (7.5)-(7.9) in (7.2), we then achieve the following most general time-independent incremental constitutive relation

$$\boxed{\begin{aligned}\dot{\sigma}_{ij} = {} & \beta_1\dot{\varepsilon}_{kk}\delta_{ij} + \beta_2\dot{\varepsilon}_{ij} + \beta_3\dot{\varepsilon}_{kk}\sigma_{ij} + \beta_4\sigma_{mn}\dot{\varepsilon}_{mn}\delta_{ij} \\ & +\beta_5(\sigma_{ik}\dot{\varepsilon}_{kj} + \dot{\varepsilon}_{ik}\sigma_{kj}) + \beta_6\dot{\varepsilon}_{mm}\sigma_{ik}\sigma_{kj} + \beta_7\sigma_{mn}\dot{\varepsilon}_{nm}\sigma_{ij} \\ & +\beta_8\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl}\delta_{ij} + \beta_9(\sigma_{ik}\sigma_{kl}\dot{\varepsilon}_{lj} + \dot{\varepsilon}_{ik}\sigma_{kl}\sigma_{lj}) \\ & +\beta_{10}\sigma_{mn}\dot{\varepsilon}_{nm}\sigma_{ik}\sigma_{kj} + \beta_{11}\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl}\sigma_{ij} \\ & +\beta_{12}\sigma_{lm}\sigma_{mn}\dot{\varepsilon}_{nl}\sigma_{ik}\sigma_{kj}\end{aligned}} \tag{7.10}$$

where $\beta_1 \ldots \beta_{12}$ may depend on the three stress invariants. In a symbolic form, this expression may be written as (7.4).

The constitutive relation given by (7.4) or (7.10) is the most general so-called *hypo-elastic model.* As this relation is given in an incremental fashion, the corresponding material response will, in general, be *path-dependent*, i.e. the stresses corresponding to a specific strain state will generally depend on the strain path which led to that strain state. Hypo-elasticity was introduced by Truesdell (1955a,b).

It is evident that a material obeying the constitutive relation (7.4) will behave in a nonlinear fashion. However, (7.4) describes a material that is incrementally linear, since a doubling of the strain increment $d\varepsilon_{kl}$ will double the stress increment $d\sigma_{ij}$.

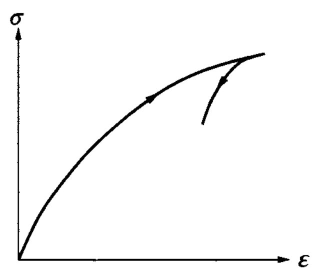

Figure 7.1: Hypo-elastic behavior; the incrementally reversible behavior is seen.

In (7.4), $D_{ijkl}(\sigma_{mn})$ is considered to be a continuous function of the stresses and this is a characteristic feature of hypo-elastic behavior. The word 'elasticity' is used (Prager (1961), pp. 142-145), since just like infinitely small strain changes $d\varepsilon_{kl}$ result in infinitely small stress changes $d\sigma_{ij}$, so do infinitely small strain changes $-d\varepsilon_{kl}$ result in infinitely small stress changes $-d\sigma_{ij}$. Thus, for infinitely small changes, the behavior is reversible, as it is for ordinary elastic behavior. In general however, the response for finite changes is path dependent, since an integration is required to obtain the total stresses from a given strain history, thus explaining the term 'hypo' meaning 'to a lesser degree', cf. Fig. 7.1. Since the response is path-dependent, it follows that the behavior in unloading will be different from that of loading (apart from incremental loading and unloading).

Later, in Chapter 10, we shall see that for general elasto-plastic behavior, an equation similar to (7.4) applies, where D_{ijkl} now corresponds to the so-called elasto-plastic stiffness tensor. However, in this case D_{ijkl} is not a continuous function of the stresses, as it changes in an abrupt manner depending on whether plastic loading or elastic unloading occurs, hereby causing the irreversibility also for infinitely small changes that is intimately connected with plastic behavior.

If $\beta_3 = \beta_4 = \ldots = \beta_{12} = 0$, then (7.10) reduces to

$$\dot{\sigma}_{ij} = \beta_1 \dot{\varepsilon}_{kk} \delta_{ij} + \beta_2 \dot{\varepsilon}_{ij}$$

which is called a *hypo-elastic material of grade zero*, as the stress tensor only enters to a power of zero on the right-hand side. Letting $\beta_1 = \lambda$ and $\beta_2 = 2\mu$, where λ and μ are the Lamé constants, we recover the usual Hooke's law in an incremental form, cf. (4.81). Likewise, a material is said to be *hypo-elastic of grade one*, if only powers of σ_{ij} up to one are allowed. A comparison with (7.10) shows that a hypo-elastic material of grade one includes the first five terms, whereas $\beta_6 = \ldots = \beta_{12} = 0$ must apply. It appears that the most general form of (7.10) corresponds to a hypo-elastic material of grade four.

The general conditions for which (7.10) represents elastic behavior, i.e. when the relation can be integrated to allow the total current stresses to be related to the total current strains independent of the load history, have been investigated by Bernstein (1960) and Coon and Evans (1971). In this case $d\sigma_{ij} = D_{ijkl}(\sigma_{st}) d\varepsilon_{kl}$ must be a perfect differential, and the integrability conditions therefore become (similar to (4.3a) and (4.9))

$$\frac{\partial D_{ijkl}}{\partial \varepsilon_{mn}} = \frac{\partial D_{ijmn}}{\partial \varepsilon_{kl}}$$

or

$$\frac{\partial D_{ijkl}}{\partial \sigma_{pq}} \frac{\partial \sigma_{pq}}{\partial \varepsilon_{mn}} = \frac{\partial D_{ijmn}}{\partial \sigma_{pq}} \frac{\partial \sigma_{pq}}{\partial \varepsilon_{kl}}$$

where D_{ijkl} can be established from (7.10). The consequences of this restriction are discussed in detail in the references mentioned above.

Finally, let us evaluate the concept of *failure conditions*. For uniaxial loading, we then assume that the stress-strain curve exhibits a peak stress (i.e. a 'failure' stress) as shown in Fig. 7.2. At the peak stress, it is possible to change the strain without changing the stress, i.e. $d\varepsilon \neq 0$ and $d\sigma = 0$.

Following Coon and Evans (1972), let us generalize this observation to the constitutive relation (7.4). Using the technique discussed in Section 4.4, the matrix formulation of (7.4) becomes

$$d\boldsymbol{\sigma} = \boldsymbol{D} d\boldsymbol{\varepsilon}$$

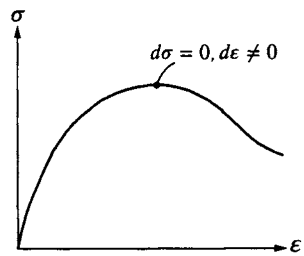

Figure 7.2: Uniaxial loading; condition at peak stress.

If this constitutive relation implies that peak stresses - i.e. 'failure' stresses - exist, then at peak we must have that $d\boldsymbol{\varepsilon} \neq \mathbf{0}$ implies $d\boldsymbol{\sigma} = \mathbf{0}$. The condition for failure therefore becomes that the homogeneous equation system

$$\boldsymbol{D}d\boldsymbol{\varepsilon} = \mathbf{0}$$

must possess a non-trivial solution $d\boldsymbol{\varepsilon} \neq \mathbf{0}$. This is only possible if

$$\det \boldsymbol{D} = 0 \tag{7.11}$$

i.e. the constitutive relation (7.4) inherently contains a condition for failure, if such peak stresses exist, and this condition is given by (7.11). The relevance of this approach for establishing the failure conditions for concrete using hypo-elasticity was discussed by Coon and Evans (1972).

Specific hypo-elastic models suggested for concrete and soil are evaluated by Chen and Saleeb (1982) and Desai and Siriwardane (1984).