18 INTEGRATION OF CONSTITUTIVE EQUATIONS

In the previous chapter, a detailed discussion was devoted to the solution of the nonlinear equilibrium equations that arise in the FE scheme. Returning to the standard iteration scheme given by (17.18), the out-of-balance forces $\boldsymbol{\psi}(\boldsymbol{a}^{i-1})$ given by (17.20) depend on the stress state $\boldsymbol{\sigma}^{i-1}$. Therefore, the determination of the stress state $\boldsymbol{\sigma}^{i-1}$ is fundamental for the iteration scheme.

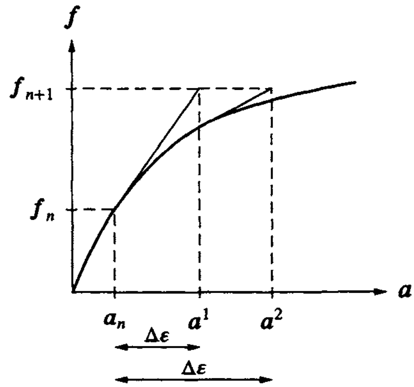

Figure 18.1: Illustration of integration limits.

Previously, we merely assumed, *a priori*, that this integration could be performed in an accurate and reliable manner. The present chapter is devoted to a discussion of how this integration is performed in practice. Recalling that $\boldsymbol{\varepsilon}_n = \boldsymbol{B}\boldsymbol{a}_n$ and $\boldsymbol{\varepsilon}^{i-1} = \boldsymbol{B}\boldsymbol{a}^{i-1}$, it is emphasized that

> *The integration of the constitutive equations is always performed from the last accepted equilibrium state $\boldsymbol{a}_n$ to the current state $\boldsymbol{a}^{i-1}$*

as discussed in relation to (17.24). With the notation $\Delta\varepsilon = \varepsilon^{i-1} - \varepsilon_n$, these integration limits are illustrated in Fig. 18.1.

For convenience, the integration limits ε_n and ε^{i-1} will now be denoted by 1 and 2 respectively. This implies that state 1 is the known state and we want to determine the stress $\sigma_{ij}^{(2)}$. While everything is known about state 1, the only thing we know about state 2 is the total strains $\varepsilon_{ij}^{(2)}$, which are provided by the solution of the global FE equations.

In the following, we will discuss numerical integration of the constitutive equations arising in elasto-plasticity and viscoplasticity. Emphasis is given to fundamental issues and for further discussion and results, we refer to Bathe (1996), Belytschko *et al.* (2000), Crisfield (1997), Simo and Hughes (1998) and Zienkiewicz and Taylor (1991).

18.1 Elasto-plasticity

First, we will consider elasto-plastic material behavior. It was shown in Section 10.1 that the fundamental equations are given by

$$\begin{aligned} \sigma_{ij} &= D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p) \\ K_\alpha &= K_\alpha(\kappa_\beta) \end{aligned} \tag{18.1}$$

where according to (10.14) the evolution laws for the non-associated case are given as

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \qquad \text{and} \qquad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial g}{\partial K_\alpha} \tag{18.2}$$

Moreover, for plastic behavior we require that

$$f(\sigma_{ij}, K_\alpha) = 0 \tag{18.3}$$

must hold. Evidently, it is the integration of the evolution equations (18.2) which is the topic of interest and the crucial point is that these evolution equations shall be solved on condition that $f = 0$ holds.

The question arises whether it is possible to integrate the constitutive equations exactly such that closed-form solutions can be obtained. Indeed, for some simple models this is possible and we refer to Krieg and Krieg (1977) for the elastic-perfectly plastic von Mises material. Later Yoder and Whirley (1984) extended the solution to linear and kinematic hardening and Ristinmaa and Tryding (1993) treated the Tresca and Coulomb materials. However, for more advanced models closed-form solutions cannot be obtained.

Two different ways to perform the integration numerically emerge. These two methods are here named the *indirect* and the *direct method* and they differ in how the yield condition during plastic loading is treated in the numerical solution procedure. In the literature, indirect methods are frequently referred to as explicit methods and direct methods as implicit methods, e.g. Zienkiewicz and Taylor (1991). It is felt that these names are somewhat misleading since both in the indirect as well as in the direct method explicit and implicit methods exist. As will be shown, the more natural names adopted here emerge from how the yield condition is treated.

In the indirect method, the consistency relation $\dot{f} = 0$ is used to reformulate the evolution equations whereas in the direct method, the condition $f = 0$ will be directly enforced.

Before we pursue this discussion, we will evaluate whether the strain increment from $\varepsilon_{ij}^{(1)}$ to $\varepsilon_{ij}^{(2)}$ results in development of plastic strains or not.

18.1.1 Loading and unloading criteria

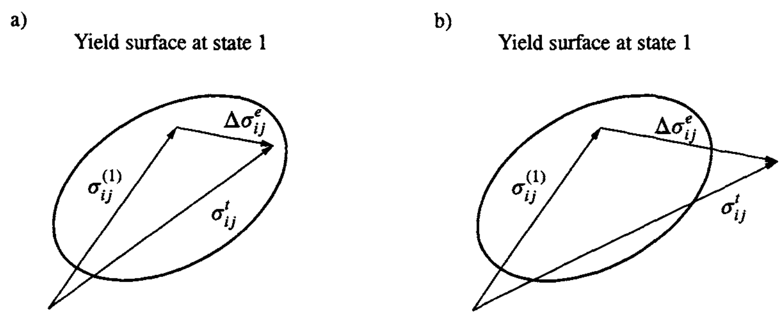

Figure 18.2: Yield surface at state 1 depicted in the stress space; illustration of the stresses $\sigma_{ij}^{(1)}$, the elastic stress increments $\Delta\sigma_{ij}^{e}$ and the trial stresses σ_{ij}^{t}; a) elastic response; b) elasto-plastic response.

From (18.1a) evaluated at state 2 and state 1, we obtain

$$\sigma_{ij}^{(2)} = D_{ijkl}(\varepsilon_{kl}^{(2)} - \varepsilon_{kl}^{p(2)}); \qquad \sigma_{ij}^{(1)} = D_{ijkl}(\varepsilon_{kl}^{(1)} - \varepsilon_{kl}^{p(1)})$$

Subtraction of these two equations gives

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{(1)} + D_{ijkl}\Delta\varepsilon_{kl} - D_{ijkl}\Delta\varepsilon_{kl}^{p} \tag{18.4}$$

where

$$\Delta\varepsilon_{kl} = \varepsilon_{kl}^{(2)} - \varepsilon_{kl}^{(1)}; \qquad \Delta\varepsilon_{kl}^{p} = \varepsilon_{kl}^{p(2)} - \varepsilon_{kl}^{p(1)}$$

In analogy with (10.36), we introduce the following notation

$$\Delta\sigma_{ij}^{e} = D_{ijkl}\Delta\varepsilon_{kl} \tag{18.5}$$

i.e. $\Delta\sigma^e_{ij}$ is the stress change that would occur if the step $\Delta\varepsilon_{kl}$ is purely elastic. Moreover, define the so-called *trial stresses* σ^t_{ij} by

Trial stresses

$$\boxed{\sigma^t_{ij} = \sigma^{(1)}_{ij} + \Delta\sigma^e_{ij}} \tag{18.6}$$

The stresses $\sigma^{(1)}_{ij}$, $\Delta\sigma^e_{ij}$ and σ^t_{ij} are illustrated in Fig. 18.2. With (18.5) and (18.6), (18.4) may be written as

$$\boxed{\sigma^{(2)}_{ij} = \sigma^t_{ij} - D_{ijkl}\Delta\varepsilon^p_{kl}} \tag{18.7}$$

It appears that if the step is purely elastic, then $\sigma^{(2)}_{ij} = \sigma^t_{ij}$.

With $\sigma^{(1)}_{ij}$ and $K^{(1)}_\alpha$ being the stresses and the hardening parameters at state 1, we introduce the notation

$$f^{(1)} = f(\sigma^{(1)}_{ij}, K^{(1)}_\alpha) \tag{18.8}$$

where $f^{(1)}$ is the value of the yield function for state 1. Since state 1 was accepted as a valid solution, we must have $f^{(1)} \leq 0$. If $f^{(1)} < 0$, then the stress state $\sigma^{(1)}_{ij}$ is located inside the yield surface and if $f^{(1)} = 0$ then $\sigma^{(1)}_{ij}$ is located on the yield surface.

Let us now evaluate the yield function f for the trial stresses σ^t_{ij} and the hardening parameters $K^{(1)}_\alpha$. In analogy with (18.8) we adopt the notation

$$f^t = f(\sigma^t_{ij}, K^{(1)}_\alpha)$$

It is recalled that during elastic response where $\dot{\lambda} = 0$ holds, no change of the hardening parameters occurs. If σ^t_{ij} denotes a stress state inside or on the yield surface valid for state 1, we will have $f^t \leq 0$. In this case, the trial stresses σ^t_{ij} are, in fact, the correct stresses at state 2, i.e. $\sigma^{(2)}_{ij} = \sigma^t_{ij}$ and we can then proceed directly in the iteration scheme of the global FE equations, i.e.

$$\boxed{f^t \leq 0 \quad \Rightarrow \quad \textit{elastic response} \quad \Rightarrow \quad \sigma^{(2)}_{ij} = \sigma^t_{ij}\ ; \quad K^{(2)}_\alpha = K^{(1)}_\alpha} \tag{18.9}$$

The situation (18.9) is illustrated in Fig. 18.2a).

Obviously, the situation where $f^t > 0$ means that the trial stress σ^t_{ij} is located outside the yields surface, cf. Fig. 18.2b). Since no stress state can be located outside the yield surface, we conclude that

$$\boxed{f^t > 0 \quad \Rightarrow \quad \textit{elasto-plastic response}} \tag{18.10}$$

Therefore, the trial stresses σ^t_{ij} that were determined assuming that the material responded in an elastic manner cannot be the stresses at state 2. In this case,

Box 18.1 Check for elasto-plastic response

· *Calculate trial stresses*

$$\sigma_{ij}^{t} = \sigma_{ij}^{(1)} + D_{ijkl}\Delta\varepsilon_{kl}$$

· *If* $f(\sigma^{t}, K_\alpha^{(1)}) \leq 0$

Elastic response

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t}\ ;\quad K_\alpha^{(2)} = K_\alpha^{(1)}$$

· *Else*

Elasto-plastic response, cf. Box 18.3 or Box 18.4

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t} - D_{ijkl}\Delta\varepsilon_{kl}^{p} \quad \textit{where} \quad \Delta\varepsilon_{kl}^{p} = \int_1^2 \frac{\partial g}{\partial \sigma_{kl}}\, d\lambda$$

we conclude that plastic strains will develop when the step $\Delta\varepsilon_{ij}$ is applied. The check for elasto-plastic loading is summarized in Box 18.1.

It is of interest that in this numerical approach, the elastic stress changes $\Delta\sigma_{ij}^{e}$ play a central role for the unloading/loading criteria given by (18.9) and (18.10) just like the elastic stress rates $\dot{\sigma}_{ij}^{e}$ do in the theoretical unloading/loading criteria, cf. (10.38).

When (18.10) is fulfilled, we know that plastic strains develop during the step $\Delta\varepsilon_{ij}$, but we do not know whether the entire step is elasto-plastic or if only part of the step creates plastic strains. This brings us to the concept of *contact stresses* to be discussed next.

18.1.2 Contact stresses

In the step $\Delta\varepsilon_{ij}$, the loading criterion (18.10) is assumed to hold and we therefore have to determine that part of the step which results in development of plastic strains. The situation is illustrated in Fig. 18.3).

During the step $\Delta\varepsilon_{ij}$, we will assume that the total strain varies linearly, i.e.

$$\varepsilon_{ij} = (1-\gamma)\varepsilon_{ij}^{(1)} + \gamma\varepsilon_{ij}^{(2)}\ ;\qquad 0 \leq \gamma \leq 1 \tag{18.11}$$

When $\gamma = 0$ we have $\varepsilon_{ij} = \varepsilon_{ij}^{(1)}$ whereas $\gamma = 1$ gives $\varepsilon_{ij} = \varepsilon_{ij}^{(2)}$. When determining the trial stress σ_{ij}^{t}, it was assumed that the material behaves in a linear elastic manner. Since the elastic stiffness tensor is constant for linear elasticity, the stress path between $\sigma_{ij}^{(1)}$ and σ_{ij}^{t} is a straight line in the stress space as illustrated in Fig. 18.3. This line will penetrate the yield surface valid at state 1 at

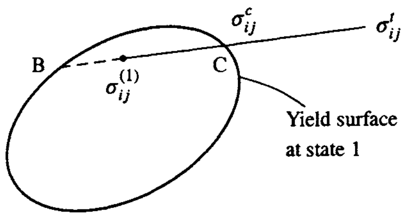

Figure 18.3: Stress space. The straight line between $\sigma_{ij}^{(1)}$ and σ_{ij}^t penetrates the yield surface at state 1 at the contact point with the contact stresses σ_{ij}^c.

the *contact point* C with the *contact stresses* σ_{ij}^c. As the stresses vary linearly between $\sigma_{ij}^{(1)}$ and σ_{ij}^t, the contact stresses are given by

$$\sigma_{ij}^c = (1-\gamma^c)\sigma_{ij}^{(1)} + \gamma^c \sigma_{ij}^t \quad ; \qquad 0 \le \gamma^c \le 1 \tag{18.12}$$

where the parameter γ^c is to be determined. As the material behaves elastically from $\sigma_{ij}^{(1)}$ to σ_{ij}^c, no changes occur in the hardening parameters. The contact stresses must therefore fulfill the yield criterion with the hardening parameters $K_\alpha^{(1)}$, i.e.

$$f^c = f(\sigma_{ij}^c, K_\alpha^{(1)}) = 0$$

Insertion of (18.12) provides

$$\boxed{f((1-\gamma^c)\sigma_{ij}^{(1)} + \gamma^c \sigma_{ij}^t,\ K_\alpha^{(1)}) = 0} \tag{18.13}$$

It is recalled that $\sigma_{ij}^{(1)}, \sigma_{ij}^t$ and $K_\alpha^{(1)}$ are known quantities and (18.13) then provides one equation with one unknown, the parameter γ^c. In general, the function f is nonlinear and any of the methods derived in the previous chapter for a set of nonlinear equations may evidently also be applied to the nonlinear equation given by (18.13); we may therefore refer to the standard iteration format given by (17.18). This scheme requires a first initial guess for γ^c, i.e. a starting value for γ^c. Recognizing the interpretation of $f^{(1)}$ given by (18.8) and f^t given by (18.10), we may interpolate linearly between these values to obtain

$$\text{starting value for} \quad \gamma^c = -\frac{f^{(1)}}{f^t - f^{(1)}}$$

where it is recalled that $f^t > 0$ and $f^{(1)} \le 0$. The contact problem to be solved is summarized in Box 18.2

Having determined the γ^c-value, the total strains ε_{ij}^c at the contact point are given by

$$\varepsilon_{ij}^c = (1-\gamma^c)\varepsilon_{ij}^{(1)} + \gamma^c \varepsilon_{ij}^{(2)} = \varepsilon_{ij}^{(1)} + \gamma^c \Delta\varepsilon_{ij}$$

Box 18.2 Calculation of contact stresses and strains.

· *Solve γ^c from*

$$f(\sigma_{ij}^c, K_\alpha^{(1)}) = 0$$

where

$$\sigma_{ij}^c = (1-\gamma^c)\sigma_{ij}^{(1)} + \gamma^c \sigma_{ij}^t$$

· *Calculate strains*

$$\varepsilon_{ij}^c = (1-\gamma^c)\varepsilon_{ij}^{(1)} + \gamma^c \varepsilon_{ij}^{(2)}$$

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that is, the strain increment $\gamma^c \Delta\varepsilon_{ij}$ will result in elastic response whereas the remaining increment $(1-\gamma^c)\Delta\varepsilon_{ij}$ will give rise to elasto-plastic response. Therefore, (18.7) takes the form

$$\boxed{\sigma_{ij}^{(2)} = \sigma_{ij}^t - D_{ijkl}\Delta\varepsilon_{kl}^p \quad \textit{where} \quad \Delta\varepsilon_{kl}^p = \int_c^2 \frac{\partial g}{\partial \sigma_{kl}} d\lambda} \tag{18.14}$$

Exact solution of the contact stresses

In some cases, an exact analytical solution of (18.13) is possible. This is the case for isotropic hardening of a von Mises and a Drucker-Prager material as well as for Tresca and Coulomb materials, cf. Bicanic (1989). For convenience, we shall only solve (18.13) in case of isotropic von Mises hardening. We therefore consider

$$(\frac{3}{2}s_{ij}^c s_{ij}^c)^{1/2} - \sigma_y(\kappa^{(1)}) = 0 \tag{18.15}$$

Due to (18.12) we have

$$s_{ij}^c = (1-\gamma^c)s_{ij}^{(1)} + \gamma^c s_{ij}^t = s_{ij}^{(1)} + \gamma^c s_{ij}^* \tag{18.16}$$

where

$$s_{ij}^* = s_{ij}^t - s_{ij}^{(1)} \; ; \qquad J_2^* = \frac{1}{2}s_{ij}^* s_{ij}^* \; ; \qquad J_2^{(1)} = \frac{1}{2}s_{ij}^{(1)} s_{ij}^{(1)}$$

Use of (18.16) in (18.15) then leads to

$$(\gamma^c)^2 J_2^* + \gamma^c s_{ij}^{(1)} s_{ij}^* + J_2^{(1)} - \frac{1}{3}\sigma_y^2(\kappa^{(1)}) = 0 \tag{18.17}$$

This quadratic equation provides two solutions for γ^c corresponding to the fact that line (18.12) will intersect the yield surface at two points, the contact point C and the point B, cf. Fig. 18.3. Obviously, the true solution of (18.17) will be the larger of the two solutions provided.

18.1.3 Indirect consideration of the yield condition

As already mentioned in the introduction, in the indirect method the consistency relation $\dot{f} = 0$ is used to reformulate the evolution equations given by (18.2). Enforcing $\dot{f} = 0$ and using (18.1) as well as (18.2), allow us to determine the plastic multiplier as

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl}$$

cf. also (10.23); the parameter A is then given by (10.24). Insertion of the expression above into the evolution equations (18.2) gives

$$\begin{aligned} \dot{\varepsilon}^p_{ij} &= \frac{1}{A}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\dot{\varepsilon}_{kl} \\ \dot{\kappa}_\alpha &= -\frac{1}{A}\frac{\partial g}{\partial K_\alpha}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\dot{\varepsilon}_{kl} \end{aligned} \tag{18.18}$$

With the assumption that the total strain varies linearly, cf. (18.11), the contact strain is calculated as

$$\varepsilon^c_{ij} = (1-\gamma^c)\varepsilon^{(1)}_{ij} + \gamma^c\varepsilon^{(2)}_{ij} \tag{18.19}$$

where γ^c is identified from (18.13); thus we have $\gamma \in [\gamma^c, 1]$ during plastic development. In this interval, it turns out to be of advantage to introduce a new variable $z \in [0, 1]$ such that

$$\gamma = (1-\gamma^c)z + \gamma^c$$

Insertion into (18.11) and taking advantage of (18.19), the strains in the elasto-plastic regime vary according to

$$\varepsilon_{ij} = (1-z)\varepsilon^c_{ij} + z\varepsilon^{(2)}_{ij} \qquad z \in [0, 1]$$

Moreover, from the above relation and (18.19) we find that

$$\frac{d\varepsilon_{kl}}{dz} = \varepsilon^{(2)}_{kl} - \varepsilon^c_{kl} = (1-\gamma^c)\Delta\varepsilon_{kl}$$

With these introductory remarks we are now ready to define our problem. Using the result above, (18.18) can be written as

$$\begin{aligned} \frac{d\varepsilon^p_{ij}}{dz} &= (1-\gamma^c)\frac{1}{A}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\Delta\varepsilon_{kl} \\ \frac{d\kappa_\alpha}{dz} &= -(1-\gamma^c)\frac{1}{A}\frac{\partial g}{\partial K_\alpha}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\Delta\varepsilon_{kl} \end{aligned} \tag{18.20}$$

Box 18.3 Stress calculation, indirect method.

- *Calculate contact stresses and strains*
  cf. Box 18.2
- *Use any ODE-solver to solve*

$$\frac{d\varepsilon_{ij}^p}{dz} = (1-\gamma^c)\frac{1}{A}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\Delta\varepsilon_{kl}$$

$$\frac{d\kappa_\alpha}{dz} = -(1-\gamma^c)\frac{1}{A}\frac{\partial g}{\partial K_\alpha}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\Delta\varepsilon_{kl}$$

with the integration limits $z \in [0, 1]$ *and the initial conditions*

$$\varepsilon_{ij}^p(z=0) = \varepsilon_{ij}^{p(1)}$$

$$\kappa_\alpha(z=0) = \kappa_\alpha^{(1)}$$

To obtain the complete set of constitutive equations, two additional relations must be supplemented

$$\begin{aligned} \sigma_{ij} &= D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p) \\ K_\alpha &= K_\alpha(\kappa_\beta) \end{aligned} \tag{18.21}$$

A glance at (18.20) reveals that we have obtained an initial-value problem with $6 + \alpha$ ordinary differential equations where the initial conditions are given by the values at the contact point. It is then clear that any solver for ordinary differential equations (ODEs) can be used in this integration procedure which is summarized in Box 18.3.

To facilitate a simple and natural notation that corresponds to the notation often found in textbooks dealing with ordinary differential equations, we introduce the following definitions

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{\varepsilon}^p \\ \boldsymbol{\kappa} \end{bmatrix} \quad \text{and} \quad \boldsymbol{f} = (1-\gamma^c)\begin{bmatrix} \frac{1}{A}\frac{\partial g}{\partial \boldsymbol{\sigma}}(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \boldsymbol{D}\Delta\boldsymbol{\varepsilon} \\ -\frac{1}{A}\frac{\partial g}{\partial \boldsymbol{K}}(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \boldsymbol{D}\Delta\boldsymbol{\varepsilon} \end{bmatrix}$$

It then follows that (18.20) can be written as

$$\frac{d\boldsymbol{y}}{dz} = \boldsymbol{f}(\boldsymbol{y}) \quad \text{with the initial conditions} \quad \boldsymbol{y}(0) = \boldsymbol{y}^c = \begin{bmatrix} \boldsymbol{\varepsilon}^{p(1)} \\ \boldsymbol{\kappa}^{(1)} \end{bmatrix} \tag{18.22}$$

With the solution strategy described above, the result is the plastic strains $\varepsilon_{ij}^{p(2)}$ and the internal variables $\kappa_\alpha^{(2)}$ estimated for the strains $\varepsilon_{ij}^{(2)}$. From Hooke's law

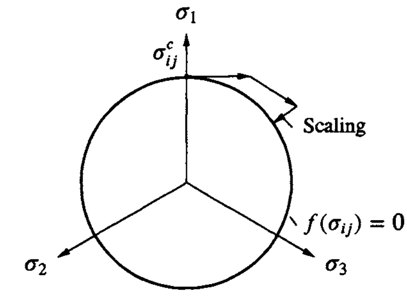

Figure 18.4: Illustration of the drift in Euler's method, ideal von Mises plasticity.

(18.21a), it is then easy to determine the corresponding stresses $\sigma_{ij}^{(2)}$ at state 2 and from (18.21b) we obtain the hardening parameters $K_\alpha^{(2)}$ at state 2.

There exists a large number of methods that can be used to solve (18.22). Since it is beyond the scope of this work to introduce and analyze the implication of all methods, only a very small fraction of the possible methods will be touched upon here.

In *single-step* methods, the entire step length $\Delta\varepsilon$ is applied in (18.22) and the most simple approach to solving these differential equations is the *Euler forward* method. A more accurate method, which involves the midpoint, has also been used; in the literature it goes under the names *second-order Runge-Kutta*, *explicit midpoint method* and *leap-frog method*, cf. Dahlquist and Björk (1974) and Zienkiewicz and Taylor (1991).

In *sub-stepping* methods, the entire step $\Delta\varepsilon$ is divided into sub-steps and any of the single-step methods can now be applied to each sub-step. A general discussion is provided by Gear (1971) and in a finite element context one may refer to Nayak and Zienkiewicz (1972), Bathe (1996) and Ottosen and Gunneskov (1986). The methods are rather straightforward to work with, but their intrinsic drawback is that eventually the stress state will drift away from the yield surface. Then, various scaling procedures are used by which the stresses are scaled back so that they fulfill the yield condition $f = 0$, cf. Potts and Gens (1985); this drift is illustrated in Fig. 18.4.

Adaptive sub-stepping methods are also available, cf. Dahlquist and Björk (1974). Here an error control is used to adjust the sub-step length. This type of integration technique belongs to the *embedded Runge-Kutta* methods, cf. Dormand and Prince (1980). Interesting results have been obtained for Tresca and Coulomb materials by Sloan (1987) and Sloan and Booker (1992) as well as for the delicate problem of combined plasticity and damage mechanics, cf. Wallin and Ristinmaa (2001).

In practice, however, these advanced methods are computationally expensive and most applications make use of the direct method to fulfill the yield condition; we will now address this issue in detail.

18.1.4 Direct consideration of the yield condition - Return methods

Any numerical integration scheme applied to (18.2) will imply that the plastic strains $\varepsilon_{ij}^{p(2)}$ and the hardening parameters $K_\alpha^{(2)}$ at state 2 can only be determined approximately, i.e. the stresses $\sigma_{ij}^{(2)}$ are only determined approximately. However, irrespective of these approximations we now introduce the requirement that the yield condition must be satisfied. This requirement is indeed fundamental, since the entire plasticity theory relies on the concept that the yield condition is fulfilled. By analogy, we may refer to the solution schemes for the nonlinear global FE equations where, irrespective of the other approximations introduced, we require that the equilibrium equations are fulfilled.

Let us therefore derive an integration scheme that fulfills the yield condition. First let us recall relation (18.14),

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t} - D_{ijkl}\Delta\varepsilon_{kl}^{p} \quad \text{where} \quad \Delta\varepsilon_{kl}^{p} = \int_c^2 \frac{\partial g}{\partial \sigma_{kl}}\, d\lambda \tag{18.23}$$

which is important for the following discussion.

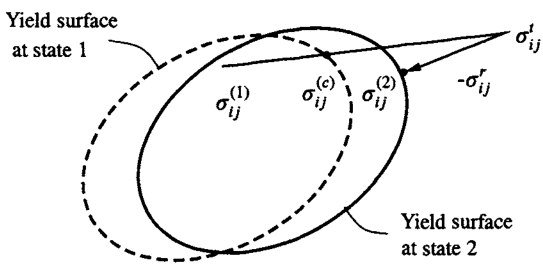

Figure 18.5: Illustration of return method.

We recall since state 1 and $\Delta\varepsilon_{kl}$ are known, $\Delta\sigma_{ij}^{e}$ and the trial stress σ_{ij}^{t} are known, cf. (18.6). Let us define the quantity σ_{ij}^{r}

$$\sigma_{ij}^{r} = D_{ijkl}\Delta\varepsilon_{kl}^{p}$$

which allows (18.23) to be written as

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t} - \sigma_{ij}^{r} \tag{18.24}$$

The format (18.24) illustrates why the integration schemes now considered are called ***return methods***. First, we calculate the trial stress σ_{ij}^{t} and then we return towards the yield surface by the amount $-\sigma_{ij}^r$ and achieve the sought stress state $\sigma_{ij}^{(2)}$ as illustrated in Fig. 18.5; this stress state is required to be located at the yield surface at state 2. The problem, however, is to determine σ_{ij}^r. For this purpose, the flow rule (18.23b) is considered with its specific integration limits

$$\Delta\varepsilon_{ij}^p = \int_{\lambda^{(1)}}^{\lambda^{(1)}+\Delta\lambda} \frac{\partial g}{\partial\sigma_{ij}} d\lambda \tag{18.25}$$

where $\lambda^{(2)} = \lambda^{(1)} + \Delta\lambda$ and where it is recalled that $\lambda^c = \lambda^{(1)}$ since no plastic strains develop along the path from state 1 to the contact point. The problem is that we only know the quantity $\partial g/\partial\sigma_{ij}$ at the contact point, but we may solve (18.25) approximately by writing

$$\Delta\varepsilon_{ij}^p = \int_c^2 \frac{\partial g}{\partial\sigma_{ij}} d\lambda \approx \left(\frac{\partial g}{\partial\sigma_{ij}}\right)^* \Delta\lambda \quad \text{where} \quad g = g(\sigma_{ij}, K_\alpha) \tag{18.26}$$

where the quantity $(\partial g/\partial\sigma_{ij})^*$ denotes $\partial g/\partial\sigma_{ij}$ evaluated at some state along the integration path from the contact point to state 2.

In addition to the plastic strains, we also need to determine the hardening parameters K_α that enter the yield function and the potential function. From (18.1b), it is recalled that the hardening parameters K_α depend on the internal parameters κ_α through

$$K_\alpha = K_\alpha(\kappa_\beta) \tag{18.27}$$

Moreover, the evolution law for κ_α is given by (18.2b), i.e.

$$\dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial g}{\partial K_\alpha}$$

Integration of this expression gives

$$\Delta\kappa_\alpha = -\int_c^2 \frac{\partial g}{\partial K_\alpha} d\lambda \approx -\left(\frac{\partial g}{\partial K_\alpha}\right)^* \Delta\lambda \tag{18.28}$$

where $\Delta\kappa_\alpha = \kappa_\alpha^{(2)} - \kappa_\alpha^{(1)}$ and the quantity $(\partial g/\partial K_\alpha)^*$ denotes $\partial g/\partial K_\alpha$ evaluated at some state along the integration path from the contact point to state 2. When $\Delta\kappa_\alpha$ has been determined from (18.28), (18.27) is used to determine the value $K_\alpha^{(2)}$ and we obtain

$$K_\alpha^{(2)} = K_\alpha(\kappa_\beta^{(1)} - \Delta\lambda\left(\frac{\partial g}{\partial K_\alpha}\right)^*)$$

Finally, the key-point in the integration method outlined above is that the yield condition is fulfilled, i.e. $\sigma_{ij}^{(2)}$ and $K_\alpha^{(2)}$ must therefore fulfill the yield criterion and

$$f(\sigma_{ij}^{(2)}, K_\alpha^{(2)}) = 0$$

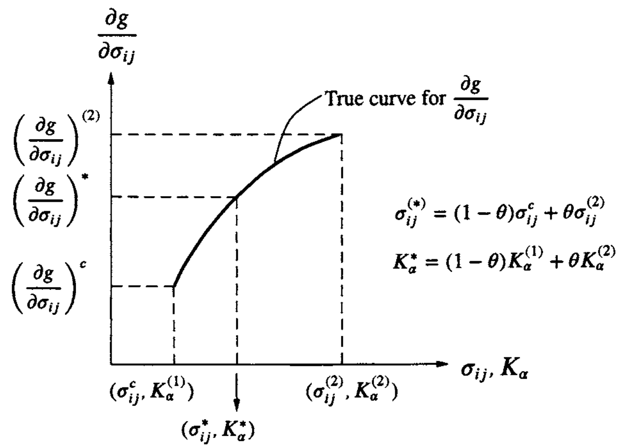

Figure 18.6: Illustration of generalized mid-point rule; note that $K_\alpha^{(1)} = K_\alpha^{(c)}$.

To determine $(\partial g/\partial \sigma_{ij})^*$ and $(\partial g/\partial K_\alpha)^*$ present in (18.26) and (18.28) respectively, two methods are available: the *generalized trapezoidal rule* and the *generalized mid-point rule*, cf. Ortiz and Popov (1985). Both methods employ information relating to the contact point and state 2. For convenience we will only focus on the generalized mid-point rule.

In the generalized mid-point rule, we interpolate σ_{ij} and K_α linearly between the contact point and state 2 and obtain

$$\boxed{\begin{aligned} \sigma_{ij}^* &= (1-\theta)\sigma_{ij}^c + \theta\sigma_{ij}^{(2)} \\ K_\alpha^* &= (1-\theta)K_\alpha^{(1)} + \theta K_\alpha^{(2)} \end{aligned} \quad \textit{generalized mid-point rule}} \qquad (18.29)$$

where $0 \leq \theta \leq 1$ and we note that $K_\alpha^{(c)} = K_\alpha^{(1)}$. With these values, we then determine $(\partial g/\partial \sigma_{ij})^*$ and $(\partial g/\partial K_\alpha)^*$ according to

$$\boxed{\begin{aligned} \left(\frac{\partial g}{\partial \sigma_{ij}}\right)^* &= \left.\frac{\partial g}{\partial \sigma_{ij}}\right|_{\sigma_{ij}^*, K_\alpha^*} \\ \left(\frac{\partial g}{\partial K_\alpha}\right)^* &= \left.\frac{\partial g}{\partial K_\alpha}\right|_{\sigma_{ij}^*, K_\alpha^*} \end{aligned} \quad \textit{generalized mid-point rule}} \qquad (18.30)$$

i.e. the quantities $\partial g/\partial \sigma_{ij}$ and $\partial g/\partial K_\alpha$ are evaluated for the variables given by σ_{ij}^* and K_α^*. The generalized mid-point rule is illustrated in Fig. 18.6.

We are now in a position to summarize the final equations of interest. Insertion of (18.26) into Hooke's law (18.23) gives

$$\sigma_{ij}^{(2)} = \sigma_{ij}^t - D_{ijkl}\left(\frac{\partial g}{\partial \sigma_{kl}}\right)^* \Delta\lambda \qquad (18.31)$$

Box 18.4 Stress calculation generalized mid-point rule.

- *Calculate contact stresses and strains*
  cf. Box 18.2
- *Solve σ_{ij}, K_α and $\Delta\lambda$ from*

$$\sigma^*_{ij} = (1-\theta)\sigma^c_{ij} + \theta\sigma^{(2)}_{ij}$$
$$K^*_\alpha = (1-\theta)K^{(1)}_\alpha + \theta K^{(2)}_\alpha$$
$$\sigma_{ij} = \sigma^t_{ij} - \Delta\lambda D_{ijkl}\left(\frac{\partial g}{\partial \sigma_{kl}}\right)^*$$
$$K_\alpha = K_\alpha(\kappa^{(1)}_\beta - \Delta\lambda\left(\frac{\partial g}{\partial K_\alpha}\right)^*)$$

subject to the constraint

$$f(\sigma_{ij}, K_\alpha) = 0$$

From (18.28) follows that

$$\kappa^{(2)}_\alpha = \kappa^{(1)}_\alpha - \left(\frac{\partial g}{\partial K_\alpha}\right)^* \Delta\lambda \tag{18.32}$$

whereas (18.27) gives

$$K^{(2)}_\alpha = K_\alpha(\kappa^{(2)}_\beta) \tag{18.33}$$

Moreover, the state $\sigma^{(2)}_{ij}$, $K^{(2)}_\alpha$ should fulfill the yield condition, i.e.

$$f(\sigma^{(2)}_{ij}, K^{(2)}_\alpha) = 0 \tag{18.34}$$

In the general case where $(\partial g/\partial\sigma_{ij})^*$ and $(\partial g/\partial K_\alpha)^*$ depend on both $\sigma^{(2)}_{ij}$ and $K^{(2)}_\alpha$ (i.e. $\kappa^{(2)}_\alpha$), equations (18.31)-(18.34) will provide a set of nonlinear equations that can be solved numerically to provide solutions for the unknowns $\sigma^{(2)}_{ij}$, $\kappa^{(2)}_\alpha$ and $\Delta\lambda$. (It is readily checked that the number of unknowns equals the number of equations=7+2α); these equations are summarized in Box 18.4.

Significant simplifications arise if we choose $\theta = 0$ and it appears that $(\partial g/\partial\sigma_{ij})^* = (\partial g/\partial\sigma_{ij})^c$ and $(\partial g/\partial K_\alpha)^* = (\partial g/\partial K_\alpha)^c$ only depend on the known quantities σ^c_{ij} and $K^{(1)}_\alpha$. This means that (18.31) and (18.33) may be inserted into (18.34) to provide one nonlinear equation with one unknown, the quantity $\Delta\lambda$. Unfortunately, the scheme $\theta = 0$ turns out to be unstable for large steps, cf. Ortiz and Popov (1985). Since the conditions at the known contact state are extrapolated to obtain the solution, one speaks of an *explicit scheme*: the method is also called a *Euler forward scheme*.

If $\theta = 1$ is chosen then we still have to work with the equations (18.31)-(18.34) simultaneously, but since $(\partial g/\partial \sigma_{ij})^* = (\partial g/\partial \sigma_{ij})^{(2)}$ and $(\partial g/\partial K_\alpha)^* = (\partial g/\partial K_\alpha)^{(2)}$ only depend on $\sigma_{ij}^{(2)}$ and $K_\alpha^{(2)}$ (i.e. $\kappa_\alpha^{(2)}$) and not on σ_{ij}^c we have established a scheme where determination of the contact stresses is unnecessary. This simplification is very convenient and since $\theta = 1$ turns out to provide a scheme that always is stable, cf. Ortiz and Popov (1985), it is very often used in practice.

Apart from the choice $\theta = 0$, we see that our scheme relies on the evaluation of the functions $(\partial g/\partial \sigma_{ij})^*$ and $(\partial g/\partial K_\alpha)^*$ at a state that is not known, *a priori*, and these schemes are therefore called *implicit schemes*. The choice $\theta = 1$ is the *fully implicit scheme* and this method is also called a *backward Euler scheme*.

We have advocated the choice $\theta = 1$, i.e. the fully implicit scheme, even though it generally results in a rather complex solution strategy. Some of the reasons for this choice have already been mentioned, but in addition, this scheme turns out to be very accurate. Let us therefore summarize the properties of this scheme

The fully implicit scheme is stable and accurate and it does not depend on the contact stresses

For the fully implicit scheme, consider now the solution procedure for the equation system defined by (18.31)-(18.34). Switching to a matrix format and using the definitions in Section 4.4 we find that the above set of equations is equivalent to

$$\begin{aligned} \boldsymbol{R}_\sigma &= \boldsymbol{\sigma} + \Delta\lambda \boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}} - \boldsymbol{\sigma}^t \\ \boldsymbol{R}_K &= \boldsymbol{K} - \hat{\boldsymbol{K}}(\boldsymbol{\kappa}^{(1)} - \Delta\lambda\frac{\partial g}{\partial \boldsymbol{K}}) \\ R_f &= f \end{aligned} \tag{18.35}$$

where the superscript (2) was suppressed and $\hat{\boldsymbol{K}}$ denotes the function whereas $\boldsymbol{K}$ is the variable. Defining a residual vector as $\boldsymbol{V} = [\boldsymbol{R}_\sigma, \boldsymbol{R}_K, R_f]^T$ and the vector containing the unknowns as $\boldsymbol{S} = [\boldsymbol{\sigma}, \boldsymbol{K}, \Delta\lambda]^T$, it is evident that $\boldsymbol{V}(\boldsymbol{S}) = \boldsymbol{0}$ defines the solution sought for.

Adopting the Newton-Raphson method, the iterative solution procedure is defined as

$$\boldsymbol{S}^{(i)} = \boldsymbol{S}^{(i-1)} - \left[\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{S}}^{(i-1)}\right]^{-1} \boldsymbol{V}^{(i-1)}$$

which is found by considering a Taylor series expansion of $\boldsymbol{V}$, cf. (17.28). The iteration procedure is stopped when the norm of $\boldsymbol{V}$ is sufficiently small. The iteration matrix can easily be identified as

$$\frac{\partial \boldsymbol{V}^{(i-1)}}{\partial \boldsymbol{S}} = \begin{bmatrix} \boldsymbol{I}_6 + \Delta\lambda \boldsymbol{D}\dfrac{\partial^2 g}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} & \Delta\lambda \boldsymbol{D}\dfrac{\partial^2 g}{\partial\boldsymbol{\sigma}\partial\boldsymbol{K}} & \boldsymbol{D}\dfrac{\partial g}{\partial\boldsymbol{\sigma}} \\ \Delta\lambda \boldsymbol{d}\dfrac{\partial^2 g}{\partial\boldsymbol{K}\partial\boldsymbol{\sigma}} & \boldsymbol{I}_\alpha + \Delta\lambda \boldsymbol{d}\dfrac{\partial^2 g}{\partial\boldsymbol{K}\partial\boldsymbol{K}} & \boldsymbol{d}\dfrac{\partial g}{\partial\boldsymbol{K}} \\ \dfrac{\partial f}{\partial\boldsymbol{\sigma}} & \dfrac{\partial f}{\partial\boldsymbol{K}} & 0 \end{bmatrix} \tag{18.36}$$

where $\boldsymbol{I}_6$ denotes the 6×6 unit matrix and $\boldsymbol{I}_\alpha$ the $\alpha \times \alpha$ unit matrix; moreover, $\boldsymbol{d} = \partial\hat{\boldsymbol{K}}/\partial\boldsymbol{\kappa}$. In the derivation above, a new matrix $\partial g^2/\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}$ emerges and a glance at the expression reveals that it should be defined in the same way as $\boldsymbol{D}^{-1}$, i.e.

$$\frac{\partial^2 g}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} = \begin{bmatrix} \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{11}\partial\sigma_{23}} \\ \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{22}\partial\sigma_{23}} \\ \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{33}\partial\sigma_{23}} \\ \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{12}\partial\sigma_{23}} \\ \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{13}\partial\sigma_{23}} \\ \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{11}} & \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{22}} & \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{33}} & \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{12}} & \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{13}} & \dfrac{\partial^2 g}{\partial\sigma_{23}\partial\sigma_{23}} \end{bmatrix} \tag{18.37}$$

where advantage is taken of the symmetry of the stress tensor when formulating the potential function g, cf. the discussion relating to (12.94). In that respect, a comparison of the expression above with (4.36) is illuminating. Evidently the matrix defined in (18.37) is symmetric.

The above method applies in general, but for certain models the equation set can be reduced considerably such that only one nonlinear scalar function needs to be considered in the numerical solution procedure. The steps to obtain this nonlinear equation are, usually, first to apply (18.35a) and (18.35b) and derive explicit expressions for $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\Delta\lambda)$ and $\boldsymbol{K} = \boldsymbol{K}(\Delta\lambda)$. These expressions are then used in the yield criterion, (18.35c), which now provides a nonlinear scalar function in $\Delta\lambda$. This approach will be illustrated below for von Mises plasticity.

Isotropic von Mises hardening - Radial return method

Let us first recall the fundamental equations for isotropic hardening of a von Mises material. Referring to Section 12.2, the yield criterion is given by

$$f = (\frac{3}{2}s_{ij}s_{ij})^{1/2} - \sigma_y(\kappa) = 0; \qquad \sigma_y = \sigma_{yo} + K(\kappa)$$

where κ is an internal variable, σ_y is the current yield stress and σ_{yo} the initial yield stress. Moreover, the effective stress is defined by

$$\sigma_{eff} = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}$$

and the yield criterion may then be written as

$$\sigma_{eff} - \sigma_y(\kappa) = 0 \tag{18.38}$$

We then obtain

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sigma_{eff}} \tag{18.39}$$

and the flow rule then provides

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}}$$

The effective plastic strain rate is defined by

$$\dot{\varepsilon}_{eff}^p = (\frac{2}{3}\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p)^{1/2}$$

and it follows that $\dot{\varepsilon}_{eff}^p = \dot{\lambda}$. Strain hardening is adopted, i.e. the evolution law becomes

$$\dot{\kappa} = \dot{\lambda} \quad \text{where} \quad \dot{\lambda} = \dot{\varepsilon}_{eff}^p \tag{18.40}$$

and the plastic modulus H is given by

$$H = \frac{d\sigma_y(\varepsilon_{eff}^p)}{d\varepsilon_{eff}^p}$$

where $\sigma_y(\varepsilon_{eff}^p)$ is a known function determined by experimental evidence. Finally, isotropic elasticity is assumed, i.e.

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \tag{18.41}$$

We will now apply the fully implicit integration scheme derived in the previous section, i.e. $\theta = 1$. From (18.26) and (18.39), we then obtain

$$\Delta\varepsilon_{ij}^{p} = \frac{3}{2}\frac{s_{ij}^{(2)}}{\sigma_{eff}^{(2)}}\Delta\lambda \tag{18.42}$$

where $\sigma_{eff}^{(2)}$ denotes σ_{eff} evaluated at state 2. From (18.31) and (18.41) follow that

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{t} - 3G\frac{s_{ij}^{(2)}}{\sigma_{eff}^{(2)}}\Delta\lambda \tag{18.43}$$

i.e.

$$\boxed{\sigma_{kk}^{(2)} = \sigma_{kk}^{t}} \tag{18.44}$$

This result is certainly not surprising, since the hydrostatic response of a von Mises material is purely elastic. Expressions (18.43) and (18.44) lead to

$$s_{ij}^{(2)} = \frac{s_{ij}^{t}}{1 + 3G\dfrac{\Delta\lambda}{\sigma_{eff}^{(2)}}} \tag{18.45}$$

Multiplying each side by itself results in

$$\sigma_{eff}^{(2)} = \frac{\sigma_{eff}^{t}}{1 + 3G\dfrac{\Delta\lambda}{\sigma_{eff}^{(2)}}}$$

i.e.

$$\sigma_{eff}^{(2)} = \sigma_{eff}^{t} - 3G\Delta\lambda \tag{18.46}$$

where σ_{eff}^{t} denotes σ_{eff} evaluated at the trial state. Insertion into (18.45) gives the result

$$\boxed{s_{ij}^{(2)} = (1 - 3G\frac{\Delta\lambda}{\sigma_{eff}^{t}})s_{ij}^{t}} \tag{18.47}$$

It appears that once the increment $\Delta\lambda$ has been identified, the expression above determines the deviatoric stress $s_{ij}^{(2)}$. Let us therefore determine the increment $\Delta\lambda$.

We first observe that the evolution law (18.40) provides

$$\boxed{\varepsilon_{eff}^{p(2)} = \varepsilon_{eff}^{p(1)} + \Delta\lambda} \tag{18.48}$$

Box 18.5 Radial return algorithm for isotropic von Mises plasticity.

- *Given:* $\varepsilon_{ij}^{(1)}$, $\varepsilon_{ij}^{p(1)}$, $\varepsilon_{eff}^{p(1)}$, $\sigma_{ij}^{(1)}$, *and* $\Delta\varepsilon_{ij}$
- *Calculate*

$$\sigma_{ij}^{t} = \sigma_{ij}^{(1)} + D_{ijkl}\Delta\varepsilon_{kl}$$

$$\sigma_{eff}^{t} = (\frac{3}{2}s_{ij}^{t}s_{ij}^{t})^{1/2}$$

- *Determine* $\Delta\lambda$ *from* $\sigma_{eff}^{t} - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{p(1)} + \Delta\lambda) = 0$
- *Calculate*

$$\varepsilon_{eff}^{p(2)} = \varepsilon_{eff}^{p(1)} + \Delta\lambda$$

$$\sigma_y^{(2)} = \sigma_y(\varepsilon_{eff}^{p(2)})$$

$$\sigma_{ij}^{(2)} = s_{ij}^{(2)} + \frac{1}{3}\sigma_{kk}^{(2)}\delta_{ij} \quad \textit{where} \quad s_{ij}^{(2)} = \frac{\sigma_y^{(2)}}{\sigma_{eff}^{t}}s_{ij}^{t}\,; \quad \sigma_{kk}^{(2)} = \sigma_{kk}^{t}$$

$$\varepsilon_{ij}^{p(2)} = \varepsilon_{ij}^{p(1)} + \Delta\varepsilon_{ij}^{p} \qquad \textit{where} \quad \Delta\varepsilon_{ij}^{p} = \frac{3}{2}\frac{\Delta\lambda}{\sigma_{eff}^{t}}s_{ij}^{t}$$

The yield criterion (18.38) evaluated at state 2 takes the form

$$\sigma_{eff}^{(2)} - \sigma_y^{(2)} = 0 \qquad \text{where} \qquad \sigma_y^{(2)} = \sigma_y(\varepsilon_{eff}^{p(2)}) \tag{18.49}$$

where $\sigma_y(\varepsilon_{eff}^{p(2)})$ is a known function. Insertion of (18.46) and (18.48) into the yield criterion (18.49) then gives the result

$$\boxed{\sigma_{eff}^{t} - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{p(1)} + \Delta\lambda) = 0} \tag{18.50}$$

In this equation everything but $\Delta\lambda$ is known and a (numerical) solution of (18.50) then provides the quantity $\Delta\lambda$; the strategies discussed in the previous chapter, and in particular the Newton-Raphson approach, can be adopted here. It appears that (18.50) states that the stress state $\sigma_{ij}^{(2)}$ together with the internal variable $\varepsilon_{eff}^{p(2)}$ satisfy the yield criterion and this is exactly the property we originally required by our solution scheme.

Solving (18.46) for $\Delta\lambda$, insertion into (18.47) gives

$$s_{ij}^{(2)} = \frac{\sigma_y^{(2)}}{\sigma_{eff}^{t}}s_{ij}^{t} \tag{18.51}$$

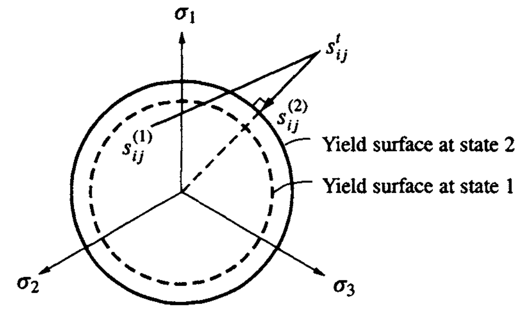

Figure 18.7: Illustration of radial return method for isotropic von Mises hardening.

Finally, inserting this expression into (18.42) we conclude that

$$\boxed{\Delta\varepsilon^p_{ij} = \frac{3}{2}\frac{\Delta\lambda}{\sigma^t_{eff}} s^t_{ij}} \tag{18.52}$$

that is, all quantities of interest can be determined in a very straightforward manner. Moreover, we recall that the yield condition is fulfilled and that we do not have to bother about determination of the contact stresses. Indeed, irrespective of the contact stresses, the results above hold.

The numerical scheme we have discussed was established by Wilkins (1964) and further investigated by Krieg and Krieg (1977) and provides a very accurate solution. The method is called the *radial return method*. This name follows directly from (18.51) showing that in the deviatoric plane, the stress $s^{(2)}_{ij}$ is located along the radial line given by the trial stress s^t_{ij}. This solution procedure is illustrated in Fig. 18.7, which supports the terminology of radial return method: first we estimate the trial stress s^t_{ij} and then we return radially to the stress state $s^{(2)}_{ij}$. The radial return algorithm is summarized in Box 18.5

Linear hardening

If $\sigma_y(\varepsilon^p_{eff})$ is a simple function, (18.50) even allows an analytical solution for $\Delta\lambda$. This is the case for linear hardening, where

$$\sigma_y = \sigma_{yo} + H\varepsilon^p_{eff}$$

and the plastic modulus H is constant. It follows that $\sigma_y(\varepsilon^{p(1)}_{eff} + \Delta\lambda) = \sigma_{yo} + H\varepsilon^{p(1)}_{eff} + H\Delta\lambda = \sigma^{(1)}_y + H\Delta\lambda$ and insertion into (18.50) gives the solution

$$\Delta\lambda = \frac{\sigma^t_{eff} - \sigma^{(1)}_y}{3G + H} \tag{18.53}$$

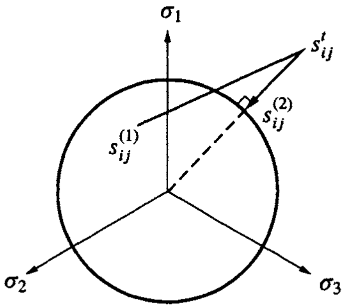

Figure 18.8: Illustration of radial return method for ideal von Mises plasticity.

Ideal von Mises plasticity

For ideal von Mises plasticity, the results simplify even more. Since σ_y is now a constant i.e. $\sigma_y = \sigma_{yo}$ and $H = 0$, (18.53) provides directly the explicit solution

$$\Delta\lambda = \frac{1}{3G}(\sigma^t_{eff} - \sigma_{yo})$$

and (18.51) becomes

$$s_{ij}^{(2)} = \frac{\sigma_{yo}}{\sigma^t_{eff}} s^t_{ij}$$

The plastic strains are still given by (18.52). Figure 18.8 shows the procedure for ideal plasticity.

18.1.5 Algorithmic tangent stiffness

It was mentioned previously in Section 17.4 that Newton-Raphson equilibrium iterations converge quickly. We will now scrutinize this property in more detail and find that in order to fully utilize this potential, the numerical procedure used for integration of the constitutive equations must be accounted for when establishing the tangential stiffness.

The essential property of the Newton-Raphson method is its *quadratic convergence*. A general discussion of this issue is given, for instance by Dahlquist and Björk (1974), and here we will merely mention the essential ingredients. Consider the nonlinear equation $f(x) = 0$ with the true solution $x = \alpha$. The error relating to the estimate x_i is defined by

$$\varepsilon_i = x_i - \alpha$$

It turns out that for a Newton-Raphson procedure, we have

$$\boxed{\begin{array}{l} \textit{Newton-Raphson procedure} \\ \textit{For } i \rightarrow \infty \quad |\varepsilon_{i+1}| = C|\varepsilon_i|^2 \ \textit{i.e. quadratic convergence} \end{array}} \qquad (18.54)$$

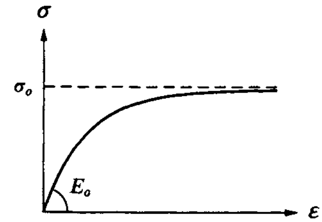

Figure 18.9: Uniaxial stress-strain curve.

where C is a positive constant (the so-called asymptotic error constant). It is evident that if the exponent is larger than two then a faster convergence is achieved whereas an exponent of less than two implies a slower convergence.

In order to maintain the result (18.54), the tangent used in the Newton-Raphson procedure must be the true tangent; if not, the quadratic convergence is lost. However, when a numerical algorithm is used for integration of the constitutive relations, it is the resulting relations that are used. Therefore, the constitutive relations as we see them are the ones that the numerical integration algorithm provides and in order that the Newton-Raphson procedure should maintain its quadratic convergence, the tangent stiffness must be derived from this algorithm. This tangent stiffness is called the *algorithmic tangent stiffness* or the *consistent tangent stiffness* and this important concept was introduced by Nagtegaal (1982) and Runesson and Booker (1982) and its ramifications were further investigated by Runesson and Samuelsson (1985) and Simo and Taylor (1985).

Let us first illustrate the problem by considering the simple uniaxial stress-strain relation given by

$$\dot{\sigma} = E_t \dot{\varepsilon} \quad \text{where} \quad E_t = E_0(1 - \frac{\sigma}{\sigma_0}) \tag{18.55}$$

Evidently, this constitutive relation can be integrated analytically to provide $\sigma = \sigma_0(1 - e^{-\frac{E_0}{\sigma_0}\varepsilon})$ and it is illustrated in Fig.18.9. Let us now integrate (18.55) numerically by means of the fully implicit method. We integrate from the last accepted equilibrium state defined by $\sigma^{(1)}$ and $\varepsilon^{(1)}$ up until the current state $\sigma^{(2)}$ and $\varepsilon^{(2)}$. We then obtain

$$\sigma^{(2)} - \sigma^{(1)} = E_t^{(2)}(\varepsilon^{(2)} - \varepsilon^{(1)})$$

i.e.

$$\sigma^{(2)} = \sigma^{(1)} + E_0(1 - \frac{\sigma^{(2)}}{\sigma_0})(\varepsilon^{(2)} - \varepsilon^{(1)}) ; \qquad \varepsilon^{(2)} - \varepsilon^{(1)} = \Delta\varepsilon \tag{18.56}$$

With $\sigma^{(1)}$ and $\varepsilon^{(1)}$ being constant quantities, we have $\sigma^{(2)} = g(\varepsilon^{(2)})$ and we will now establish the corresponding tangent stiffness.

This tangent stiffness can be identified if we consider an increment in $d\varepsilon^{(2)}$ and then observe the corresponding increment in $d\sigma^{(2)}$. Now this notation is somewhat awkward and, instead, we use the notation $\dot{\varepsilon}^{(2)}$ and $\dot{\sigma}^{(2)}$; occasionally we may use the terminology of differentiation which then should be understood in the manner described above. Thus from (18.56) we obtain

$$\dot{\sigma}^{(2)} = E_{ats}\dot{\varepsilon}^{(2)} \quad \text{where} \quad E_{ats} = \frac{E_0(1 - \frac{\sigma^{(2)}}{\sigma_0})}{1 + \frac{E_0}{\sigma_0}\Delta\varepsilon}$$

where E_{ats} denotes the algorithmic tangent stiffness. This tangent stiffness is the one that our numerical model experiences and it is the one that should be used in the Newton-Raphson procedure in order to maintain quadratic convergence. The so-called *continuum tangent stiffness* E_t is defined by (18.55) and we obtain

$$\frac{E_{ats}}{E_t} = \frac{1}{1 + \frac{E_0}{\sigma_0}\Delta\varepsilon}$$

Consequently, when $\Delta\varepsilon \to 0$ the algorithmic tangent stiffness approaches the continuum tangent stiffness, but otherwise $E_{ats}/E_t < 1$. For metals, we typically have $E_0/\sigma_0 \approx 500$ and with the strain increment $\Delta\varepsilon$ in the range $10^{-4} - 10^{-3}$ we obtain $E_{ats}/E_t \approx 0.67 - 0.95$. In conclusion, for strain increments encountered in practice, the algorithmic tangent stiffness is significantly smaller than the continuum tangent stiffness and the algorithmic tangent stiffness should be used in the Newton-Raphson procedure to maintain its quadratic convergence.

Derivation of $\boldsymbol{D}_{ats}$

With the above motivating example let us now tackle the general equations of elasto-plasticity. From the discussion above and from Box 17.2, it is evident that the algorithmic tangent stiffness is defined from

$$\dot{\boldsymbol{\sigma}}^{(2)} = \boldsymbol{D}_{ats}\dot{\boldsymbol{\varepsilon}}^{(2)} \quad \text{where} \quad \boldsymbol{D}_{ats} = \boldsymbol{D}_t^{i-1}$$

The numerical integration scheme most often used in practice is the fully implicit scheme, i.e. $\theta = 1$ in (18.29) and (18.30), which will be adopted here.

First, let us summarize the equations used in the integration of the constitutive relations; the stress-strain relation where the flow rule is used, the evolution law for the internal variable (for simplicity only one internal variable is assumed to exist) and the yield function. In matrix notation we then find from (18.7), (18.2) and (18.3)

$$\begin{aligned} &\boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^t - \Delta\lambda \boldsymbol{D} \left(\frac{\partial g}{\partial \boldsymbol{\sigma}}\right)^{(2)} \\ &\Delta\kappa = -\Delta\lambda \left(\frac{\partial g}{\partial K}\right)^{(2)} \qquad K^{(2)} = K(\kappa^{(2)}) \\ &f(\boldsymbol{\sigma}^{(2)}, K^{(2)}) = 0 \end{aligned} \tag{18.57}$$

where $\Delta\lambda = \lambda^{(2)} - \lambda^{(1)}$ and $\Delta\kappa = \kappa^{(2)} - \kappa^{(1)}$. Moreover, the gradient $\partial g/\partial \boldsymbol{\sigma}$ is defined according to (12.94). In the following, for simplicity we will make the restriction that the evolution function $\partial g/\partial K_\alpha$ only depends on the hardening parameter K.

To achieve a neat notation, the superscript (2) will be omitted and recalling that all quantities at state (1) are fixed, we obtain

$$\dot{\boldsymbol{\sigma}}^{(2)} = \dot{\boldsymbol{\sigma}}\,; \quad (\dot{\Delta\boldsymbol{\varepsilon}}) = \dot{\boldsymbol{\varepsilon}}^{(2)} = \dot{\boldsymbol{\varepsilon}}\,; \quad (\dot{\Delta\lambda}) = \dot{\lambda}^{(2)} = \dot{\lambda}\,; \quad (\dot{\Delta\kappa}) = \dot{\kappa}^{(2)} = \dot{\kappa}$$

Since $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}^{(1)} + \boldsymbol{D}\Delta\boldsymbol{\varepsilon}$, a differentiation of Hooke's law (18.57a) therefore results in

$$\boldsymbol{D}^{-1}\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}} - \Delta\lambda\frac{\partial^2 g}{\partial \boldsymbol{\sigma}\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} \tag{18.58}$$

Due to the existence of the last term, it can already be realized here that the derivations will not lead to the elasto-plastic continuum stiffness, cf. also (10.22).

From (18.58) we then obtain

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^a\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\boldsymbol{D}^a\frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{18.59}$$

where

$$\boldsymbol{D}^a = (\boldsymbol{D}^{-1} + \Delta\lambda\frac{\partial^2 g}{\partial \boldsymbol{\sigma}\partial \boldsymbol{\sigma}})^{-1} \tag{18.60}$$

The format obtained in (18.59) is of the same form as (10.22), however, the elastic stiffness is now modified and depends on the step length $\Delta\lambda$. It is also noted that if $\Delta\lambda = 0$ then (18.59) and (10.22) become identical.

The final task is then to obtain an expression for the plastic multiplier, which is achieved by differentiation of the yield function at state (2). From (18.57c) follows

$$(\frac{\partial f}{\partial \boldsymbol{\sigma}})^T\dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial K}\dot{K} = 0 \tag{18.61}$$

The last term requires a differentiation of K. From (18.57b) we obtain

$$\dot{K} = \frac{dK}{d\kappa}\dot{\kappa} = -\frac{dK}{d\kappa}(\dot{\lambda}\frac{\partial g}{\partial K} + \Delta\lambda\frac{\partial^2 g}{\partial K\partial K}\dot{K})$$

and solving for $\dot{K}$ results in

$$\dot{K} = d^a \dot{\lambda} \quad \text{where} \quad d^a = -(1 + \Delta\lambda \frac{dK}{d\kappa} \frac{\partial^2 g}{\partial K \partial K})^{-1} \frac{dK}{d\kappa} \frac{\partial g}{\partial K}$$

Use of this expression as well as (18.59) in (18.61) provides

$$\dot{\lambda} = \frac{1}{A^a} (\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \boldsymbol{D}^a \dot{\boldsymbol{\varepsilon}} \tag{18.62}$$

where

$$A^a = (\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \boldsymbol{D}^a \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial K} d^a \tag{18.63}$$

Finally, insertion of (18.62) into (18.59) provides the result sought for

Algorithmic tangent stiffness

$$\boxed{\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_{ats} \dot{\boldsymbol{\varepsilon}} \quad \text{where} \quad \boldsymbol{D}_{ats} = \boldsymbol{D}^a - \frac{1}{A^a} \boldsymbol{D}^a \frac{\partial g}{\partial \boldsymbol{\sigma}} (\frac{\partial f}{\partial \boldsymbol{\sigma}})^T \boldsymbol{D}^a} \tag{18.64}$$

Where $\boldsymbol{D}_{ats}$ defined above is the algorithmic tangent stiffness. Evidently it has the same format as $\boldsymbol{D}^{ep}$, but with the difference that it depends on $\Delta\lambda$, i.e. the size of the plastic load increment. In essence, the step $\Delta\lambda$ implies the following changes: $\boldsymbol{D} \rightsquigarrow \boldsymbol{D}^a$ and $A \rightsquigarrow A^a$. For a von Mises material, we will now identify these changes.

Isotropic linear hardening von Mises material

Let us now consider a von Mises material with isotropic linear hardening and derive an explicit form of the algorithmic stiffness. Certainly, we could use the general result (18.64), but it turns out to be simpler to adopt the following direct procedure.

The deviatoric stresses are given by (18.47), i.e.

$$s_{ij}^{(2)} = (1 - 3G \frac{\Delta\lambda}{\sigma_{eff}^t}) s_{ij}^t \tag{18.65}$$

where

$$s_{ij}^t = s_{ij}^{(1)} + 2G(e_{ij}^{(2)} - e_{ij}^{(1)}) \tag{18.66}$$

Moreover, the hydrostatic part of the stresses is not influenced by plasticity, i.e.

$$\sigma_{kk}^{(2)} = 3K\varepsilon_{kk}^{(2)} \tag{18.67}$$

Since linear hardening is assumed, we obtain from (18.53)

$$\Delta\lambda = \frac{\sigma_{eff}^t - \sigma_y^{(1)}}{H + 3G}$$

Use of this expression in (18.65) gives

$$s_{ij}^{(2)} = \frac{H + 3G\frac{\sigma_y^{(1)}}{\sigma_{eff}^t}}{H + 3G} s_{ij}^t \tag{18.68}$$

From (18.66) we obtain

$$\dot{s}_{ij}^t = 2G(\dot{\varepsilon}_{ij}^{(2)} - \frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}^{(2)}) \tag{18.69}$$

Moreover

$$\sigma_{eff}^t = (\frac{3}{2}s_{kl}^t s_{kl}^t)^{1/2} \quad \text{i.e.} \quad \dot{\sigma}_{eff}^t = \frac{3Gs_{kl}^t \dot{\varepsilon}_{kl}^{(2)}}{\sigma_{eff}^t} \tag{18.70}$$

Differentiation of (18.68) with respect to time and use of (18.69) and (18.70b) give

$$\dot{s}_{ij}^{(2)} = \frac{H + 3G\beta}{H + 3G} 2G(\dot{\varepsilon}_{ij}^{(2)} - \frac{1}{3}\delta_{ij}\dot{\varepsilon}_{kk}^{(2)}) - \frac{9G^2\beta}{H + 3G} \frac{s_{ij}^t s_{kl}^t}{(\sigma_{eff}^t)^2} \dot{\varepsilon}_{kl}^{(2)} \tag{18.71}$$

where the factor β is defined by

$$\beta = \frac{\sigma_y^{(1)}}{\sigma_{eff}^t}$$

It follows that $0 < \beta \leq 1$ and for infinitely small steps $\beta \to 1$. From (18.71) and (18.67) we finally achieve the result aimed at

Linear isotropic hardening von Mises material

$$\dot{\sigma}_{ij}^{(2)} = D_{ijkl}^{ats} \dot{\varepsilon}_{kl}^{(2)}$$

where

$$D_{ijkl}^{ats} = \frac{H + 3G\beta}{A^a} 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}] + K\delta_{ij}\delta_{kl} - \frac{9G^2\beta}{A^a} \frac{s_{ij}^t s_{kl}^t}{(\sigma_{eff}^t)^2}$$

where $A^a = A = H + 3G$

A comparison with (12.16) and (4.89) shows that the algorithmic stiffness approaches the continuum stiffness when $\beta \to 1$, as expected. Moreover, it appears that the factor β decreases the shear modulus from G to βG and it also decreases the plastic part of the stiffness tensor. It is also observed that in this specific case $A^a = A = H + 3G$.

18.2 Viscoplasticity

Having considered the integration of the constitutive relations for elasto-plasticity, the integration of the viscoplastic constitutive relations follows almost directly. As discussed in Chapter 15, creep modeling is a special case of viscoplasticity in which the elastic region disappears. Therefore, although not pursued here, the numerical treatment of creep is similar to that adopted for viscoplasticity.

Here we will focus on a method that will fit into the algorithms considered in Chapter 17. Other methods for solution of rate-dependent problems exist; these methods usually lead to both an additional force vector, due to the linearization of the constitutive relations, and a modified tangent stiffness, i.e. rate tangent stiffness; we refer to Zienkiewicz and Taylor (1991) and references therein and also to Peirce *et al.* (1984) for further information.

In Section 15.4 two different viscoplastic models were considered, the Perzyna model, (15.36), and the Duvaut-Lions model, (15.58). Except for the elasto-viscoplastic loading condition, the numerical treatment of these two models differs and will therefore be treated separately.

We have the constitutive relations

$$\begin{aligned} \sigma_{ij} &= D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{vp}) \\ K_\alpha &= K_\alpha(\kappa_\beta) \end{aligned} \tag{18.72}$$

Moreover, both models take advantage of a static yield function $f(\sigma_{ij}, K_\alpha)$. If $f \leq 0$ elastic response occurs whereas viscoplastic response is obtained if $f > 0$. Assuming that the loading is elastic allows for the definition of a trial stress and in correspondence with (18.6) we have

$$\sigma_{ij}^t = \sigma_{ij}^{(1)} + \Delta\sigma_{ij}^e \quad \text{where} \quad \Delta\sigma_{ij}^e = D_{ijkl}\Delta\varepsilon_{kl} \tag{18.73}$$

where $\Delta\varepsilon_{kl} = \varepsilon_{kl}^{(2)} - \varepsilon_{kl}^{(1)}$ and where the superscript (1) indicates the value of the quantity at the last known state in equilibrium and (2) the new sought state. Note that the total strains are known at state (2). By analogy with (18.7) we also have

$$\sigma_{ij}^{(2)} = \sigma_{ij}^t - D_{ijkl}\Delta\varepsilon_{kl}^{vp} \tag{18.74}$$

With (18.73) the loading condition used in the numerical treatment can be written as

$$f^t = f(\sigma_{ij}^t, K_\alpha^{(1)}) = \begin{cases} \leq 0 & \text{elastic response} \\ > 0 & \text{viscoplastic response} \end{cases}$$

If $f < 0$ the sought stresses are given by (18.73), i.e. $\sigma_{ij}^{(2)} = \sigma_{ij}^t$ and the new hardening parameters are given as $K_\alpha^{(2)} = K_\alpha^{(1)}$. It appears that the loading condition used in the numerical treatment becomes the same as for elasto-plasticity.

18.2.1 Perzyna viscoplasticity

First let us summarize the constitutive relations for Perzyna viscoplasticity. In addition to (18.72) we have from (15.36) and (15.37) the evolution laws

$$\begin{aligned} \dot{\varepsilon}_{ij}^{vp} &= \frac{\Phi(f)}{\eta}\frac{\partial g}{\partial \sigma_{ij}} \\ \dot{\kappa}_\alpha &= -\frac{\Phi(f)}{\eta}\frac{\partial g}{\partial K_\alpha} \end{aligned} \tag{18.75}$$

As indicated above it will be assumed that viscoplastic loading occurs.

In the numerical treatment, the fully implicit method will be used, i.e. the backward Euler rule. The corresponding numerical format of (18.75) is then given by

$$\begin{aligned} \Delta\varepsilon_{ij}^{vp} &= \int_{t^{(1)}}^{t^{(2)}} \frac{\Phi(f)}{\eta}\frac{\partial g}{\partial \sigma_{ij}} dt = \Delta\lambda \left(\frac{\partial g}{\partial \sigma_{ij}}\right)^{(2)} \\ \Delta\kappa_\alpha &= -\int_{t^{(1)}}^{t^{(2)}} \frac{\Phi(f)}{\eta}\frac{\partial g}{\partial K_\alpha} dt = -\Delta\lambda \left(\frac{\partial g}{\partial K_\alpha}\right)^{(2)} \end{aligned}$$

where $t^{(1)}$ and $t^{(2)}$ denote the time at the last known state in equilibrium and the current time respectively. Moreover, the definition

$$\Delta\lambda = \frac{\Phi(f^{(2)})}{\eta}\Delta t \tag{18.76}$$

was introduced. An interesting observation can now be made. According to (15.41) there exists an inverse function φ to Φ and (18.76) can then be written as

$$f^d = f^{(2)} - \varphi(\eta\frac{\Delta\lambda}{\Delta t})\,; \qquad f^d = 0$$

which is the numerical counterpart of the dynamic yield criterion, cf. (15.43).

The set of equations used in the numerical solution procedure can therefore be summarized as

$$\begin{aligned} \sigma_{ij}^{(2)} &= \sigma_{ij}^{t} - D_{ijkl}\Delta\varepsilon_{kl}^{vp} \\ K_\alpha^{(2)} &= K_\alpha(\kappa_\beta^{(2)}) \end{aligned} \tag{18.77}$$

with the evolution laws

$$\begin{aligned} \Delta\varepsilon_{ij}^{vp} &= \Delta\lambda \left(\frac{\partial g}{\partial \sigma_{ij}}\right)^{(2)} \\ \Delta\kappa_\alpha &= -\Delta\lambda \left(\frac{\partial g}{\partial K_\alpha}\right)^{(2)} \end{aligned} \tag{18.78}$$

and the dynamic yield criterion

$$f^{(2)} - \varphi(\eta \frac{\Delta\lambda}{\Delta t}) = 0 \tag{18.79}$$

For certain models it is possible to reduce the above set of equations dramatically, but for anisotropy, for instance, such a reduction is not possible. Let us therefore consider the general solution procedure for the equation system above. Switching to a matrix format using the definitions in Section 4.4 we find that the above set of equations is equivalent to

$$\begin{aligned} \boldsymbol{R}_\sigma &= \boldsymbol{\sigma} + \Delta\lambda \boldsymbol{D} \frac{\partial g}{\partial \boldsymbol{\sigma}} - \boldsymbol{\sigma}^t \\ \boldsymbol{R}_K &= \boldsymbol{K} - \hat{\boldsymbol{K}}(\boldsymbol{\kappa}^{(1)} - \Delta\lambda \frac{\partial g}{\partial \boldsymbol{K}}) \\ R_f &= f - \varphi(\eta \frac{\Delta\lambda}{\Delta t}) \end{aligned} \tag{18.80}$$

where the superscript (2) was suppressed and $\hat{\boldsymbol{K}}$ denotes the function whereas $\boldsymbol{K}$ is the variable. Defining a residual vector as $\boldsymbol{V} = [\boldsymbol{R}_\sigma, \boldsymbol{R}_K, R_f]^T$ and as this vector depends on the unknowns $\boldsymbol{S} = [\boldsymbol{\sigma}, \boldsymbol{K}, \Delta\lambda]^T$, it is evident that $\boldsymbol{V}(\boldsymbol{S}) = \boldsymbol{0}$ defines the solution sought for.

Adopting the Newton-Raphson method, the iterative solution procedure is defined as

$$\boldsymbol{S}^{(i)} = \boldsymbol{S}^{(i-1)} - \left[\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{S}}^{(i-1)}\right]^{-1} \boldsymbol{V}^{(i-1)} \tag{18.81}$$

which is found by considering a Taylor series expansion of $\boldsymbol{V}$, cf. (17.28). The iteration procedure is stopped when the norm of $\boldsymbol{V}$ is sufficiently small. The iteration matrix can easily be identified as

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{S}}^{(i-1)} = \begin{bmatrix} \boldsymbol{I}_6 + \Delta\lambda \boldsymbol{D} \frac{\partial^2 g}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} & \Delta\lambda \boldsymbol{D} \frac{\partial^2 g}{\partial \boldsymbol{\sigma} \partial \boldsymbol{K}} & \boldsymbol{D} \frac{\partial g}{\partial \boldsymbol{\sigma}} \\ \Delta\lambda \boldsymbol{d} \frac{\partial^2 g}{\partial \boldsymbol{K} \partial \boldsymbol{\sigma}} & \boldsymbol{I}_\alpha + \Delta\lambda \boldsymbol{d} \frac{\partial^2 g}{\partial \boldsymbol{K} \partial \boldsymbol{K}} & \boldsymbol{d} \frac{\partial g}{\partial \boldsymbol{K}} \\ \frac{\partial f}{\partial \boldsymbol{\sigma}} & \frac{\partial f}{\partial \boldsymbol{K}} & -\varphi' \frac{\eta}{\Delta t} \end{bmatrix} \tag{18.82}$$

where $\boldsymbol{I}_6$ denotes the 6×6 unit matrix, $\boldsymbol{I}_\alpha$ the $\alpha \times \alpha$ unit matrix, Moreover, $\boldsymbol{d} = \partial \hat{\boldsymbol{K}} / \partial \boldsymbol{\kappa}$ and $\varphi' = d\varphi(x)/dx$. It is interesting to compare (18.82) with its elasto-plastic counterpart (18.36) which reveals that the only difference is the term $\varphi' \eta / \Delta t$ and for $\eta = 0$, the two systems coincide, as expected.

As mentioned previously, it turns out that for certain models the equation system can be reduced to a single nonlinear equation with one unknown. In this approach, expression (18.80a) and (18.80b) can be reduced to obtain $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\Delta\lambda)$ and $\boldsymbol{K} = \boldsymbol{K}(\Delta\lambda)$. These two relations are then inserted into the dynamic yield criterion (18.80c) which then, usually, becomes a nonlinear scalar equation in $\Delta\lambda$. In the following, we will consider a situation where this reduction process is possible.

Isotropic von Mises hardening

The solution procedure (18.81) holds in general, but as mentioned above there are cases where the problem can be reduced to the solution of one nonlinear equation in the unknown $\Delta\lambda$.

Let us for example consider an isotropic hardening von Mises material where the static yield function is defined as

$$f = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2} - \sigma_y(\epsilon^{vp}_{eff})\ ; \qquad \sigma_y = \sigma_{yo} + K(\epsilon^{vp}_{eff}) \tag{18.83}$$

where the internal variable κ has been chosen as the effective viscoplastic strain ϵ^{vp}_{eff}, i.e. we have

$$\dot{\kappa} = \dot{\epsilon}^{vp}_{eff} = \left(\frac{2}{3}\dot{\epsilon}^{vp}_{ij}\dot{\epsilon}^{vp}_{ij}\right)^{1/2} = \frac{\Phi(f)}{\eta}$$

This result was achieved by using the flow rule (18.75a) which here becomes

$$\dot{\epsilon}^{vp}_{ij} = \frac{\Phi(f)}{\eta}\frac{3s_{ij}}{2\sigma_{eff}}\ ; \qquad \sigma_{eff} = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}$$

The task is now to make use of the set of equations defined by (18.77)-(18.79) and we will make an effort to reduce the set of equations as much as possible. For the elasto-plastic problem we eventually ended up with only one nonlinear equation, that is the yield criterion with $\Delta\lambda$ as the unknown. As will be shown for the viscoplastic model, the same reduction process leads to a single nonlinear equation, namely the dynamic yield criterion which has to be solved for the unknown quantity $\Delta\lambda$. By and large, the derivation follows the elasto-plastic situation. By using

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

insertion of (18.78a) into (18.77a) provides

$$\sigma^{(2)}_{ij} = \sigma^{t}_{ij} - 3G\frac{s^{(2)}_{ij}}{\sigma^{(2)}_{eff}}\Delta\lambda \tag{18.84}$$

for isotropic elasticity. From (18.84) it follows that

$$s_{ij}^{(2)} = \frac{s_{ij}^t}{1 + \dfrac{3G\Delta\lambda}{\sigma_{eff}^{(2)}}} ; \qquad \sigma_{kk}^{(2)} = \sigma_{kk}^t \tag{18.85}$$

Each side of (18.85a) is now multiplied by itself and this results in

$$\sigma_{eff}^{(2)} = \sigma_{eff}^t - 3G\Delta\lambda \tag{18.86}$$

i.e. the same set of equations is obtained as found for elasto-plasticity, cf. (18.44)-(18.46). Thus to calculate the stresses we have only to determine the value of $\Delta\lambda$ which will be determined by the dynamic yield function.

With (18.86), the static yield function (18.83) can be expressed as

$$f = \sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{vp(1)} + \Delta\lambda)$$

and insertion into (18.79) provides

$$\sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{vp(1)} + \Delta\lambda) - \varphi(\eta\frac{\Delta\lambda}{\Delta t}) = 0 \tag{18.87}$$

which is a scalar equation in the unknown $\Delta\lambda$. Typically, the Newton-Raphson method is used in the numerical solution of this nonlinear equation. It is interesting to note that if $\eta = 0$ is considered, i.e. rate-independent plasticity, (18.87) reduces to (18.50).

As a first example assume that the function φ is linear, i.e. $\varphi(x) = x$ which corresponds, essentially, to Hohenemser-Prager viscoplasticity (15.34) and with hardening also being considered. From (18.87) we then find

$$\sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{vp(1)} + \Delta\lambda) - \eta\frac{\Delta\lambda}{\Delta t} = 0$$

For linear hardening where $\sigma_y = \sigma_{yo} + H\varepsilon_{eff}^{vp}$, a closed-form solution can be obtained and it is given by

$$\Delta\lambda = \frac{\sigma_{eff}^t - \sigma_y^{(1)}}{3G + H + \frac{\eta}{\Delta t}}$$

Thus if $\eta = 0$ the above relation reduces to (18.53).

Let us finally consider the Cowper-Symonds model where use of (15.49) in (18.87) gives

$$\sigma_{eff}^t - 3G\Delta\lambda - \sigma_y(\varepsilon_{eff}^{vp(1)} + \Delta\lambda) - \left(\frac{1}{D}\frac{\Delta\lambda}{\Delta t}\right)^{1/p} \sigma_{yo} = 0$$

Note that $1/D$ has here assumed the role of η.

Having determined $\Delta\lambda$ the stresses are given by (18.85) and the plastic strains are found from (18.78a) as

$$\varepsilon_{ij}^{vp(2)} = \varepsilon_{ij}^{vp(1)} + \Delta\lambda \frac{3s_{ij}^t}{2\sigma_{eff}^t}$$

Algorithmic tangent stiffness for Perzyna viscoplasticity

In order to keep the algebra reasonably simple, we assume that only one hardening parameter K exists and that the quantity $\partial g/\partial K$ does not depend on the stresses $\boldsymbol{\sigma}$.

It is then interesting to note that the only difference between (18.80) and (18.35) relates to the static and dynamic yield functions. Thus for deriving the algorithmic tangent stiffness for Perzyna viscoplasticity we can take advantage of the derivations for the algorithmic tangent stiffness for elasto-plasticity.

Differentiation of the dynamic yield function (18.80c) results in

$$\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial K}\dot{K} - \varphi' \frac{\eta}{\Delta t}\Delta\dot{\lambda} = 0$$

and apart from the last term, this expression corresponds to the consistency relation (18.61) in plasticity. Therefore, similar to (18.62) we obtain

$$\Delta\dot{\lambda} = \frac{1}{A^{vp}}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}^a \dot{\boldsymbol{\varepsilon}}$$

where

$$A^{vp} = A^a + \varphi' \frac{\eta}{\Delta t}$$

Compared with elasto-plasticity, this is the only change and we then obtain the algorithmic tangent stiffness $\boldsymbol{D}_{ats}$ for viscoplasticity directly from (18.64) according to

Algorithmic tangent stiffness
$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_{ats}\dot{\boldsymbol{\varepsilon}}$ where $\boldsymbol{D}_{ats} = \boldsymbol{D}^a - \frac{1}{A^{vp}}\boldsymbol{D}^a \frac{\partial g}{\partial \boldsymbol{\sigma}}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}^a$

It emphasized that the only change compared with the elasto-plastic format (18.64) is that A^a is replaced by A^{vp}.

18.2.2 Duvaut-Lions viscoplasticity

As the starting point we will summarize the constitutive relations for Duvaut-Lions viscoplasticity. From (15.58) we have the evolution laws

$$\begin{aligned} \dot{\varepsilon}_{ij}^{vp} &= \Lambda C_{ijkl}(\sigma_{kl} - \bar{\sigma}_{kl}) \\ \dot{\kappa}_{\alpha} &= -\Lambda c_{\alpha\beta}(K_{\beta} - \bar{K}_{\beta}) \end{aligned} \tag{18.88}$$

where $(\bar{\sigma}_{ij}, \bar{K}_{\alpha})$ is the closest-point-projection of $(\sigma_{ij}, K_{\alpha})$ on the static yield surface $f(\bar{\sigma}_{ij}, \bar{K}_{\alpha}) = 0$. Moreover, $c_{\alpha\beta}$ is the inverse matrix to $d_{\alpha\beta} = \partial K_{\alpha}/\partial \kappa_{\beta}$ and $c_{\alpha\beta}$, as well as C_{ijkl}, are considered as constant quantities. The solution to the closest-point-projection is given by (15.57) and summarized here

$$\begin{aligned} &-C_{ijkl}(\sigma_{kl} - \bar{\sigma}_{kl}) + \mu\frac{\partial f}{\partial \bar{\sigma}_{ij}} = 0 \\ &-c_{\alpha\beta}(K_{\beta} - \bar{K}_{\beta}) + \mu\frac{\partial f}{\partial \bar{K}_{\alpha}} = 0 \\ &f(\bar{\sigma}_{ij}, \bar{K}_{\alpha}) = 0 \end{aligned} \tag{18.89}$$

where μ is a Lagrangian multiplier. Before proceeding it turns out that instead of (18.72) it is of advantage to use (15.54), i.e.

$$\dot{K}_{\alpha} = d_{\alpha\beta}\dot{\kappa}_{\beta} \tag{18.90}$$

as a more structured set of equations then will be obtained; here $d_{\alpha\beta}$ is the inverse of $c_{\alpha\beta}$.

Applying the backward Euler rule to (18.88) results in

$$\begin{aligned} \Delta\varepsilon_{ij}^{vp} &= \Lambda^{(2)}\Delta t C_{ijkl}(\sigma_{kl}^{(2)} - \bar{\sigma}_{kl}) \\ \Delta\kappa_{\alpha} &= -\Lambda^{(2)}\Delta t c_{\alpha\beta}(K_{\beta}^{(2)} - \bar{K}_{\beta}) \end{aligned}$$

Taking advantage of the above relations in (18.74) and (18.90), where the backward Euler rule is applied, provides after some manipulations

$$\begin{aligned} \sigma_{ij}^{(2)} &= \frac{1}{1 + \Lambda^{(2)}\Delta t}(\sigma_{ij}^{t} + \Lambda^{(2)}\Delta t\bar{\sigma}_{ij}) \\ K_{\alpha}^{(2)} &= \frac{1}{1 + \Lambda^{(2)}\Delta t}(K_{\alpha}^{(1)} + \Lambda^{(2)}\Delta t\bar{K}_{\alpha}) \end{aligned} \tag{18.91}$$

Insertion into (18.89) results in

$$\begin{aligned} \bar{\sigma}_{ij} &= \sigma_{ij}^{t} - \Delta\lambda D_{ijkl}\frac{\partial f}{\partial \bar{\sigma}_{kl}} \\ \bar{K}_{\alpha} &= K_{\alpha}^{(1)} - \Delta\lambda d_{\alpha\beta}\frac{\partial f}{\partial \bar{K}_{\beta}} \\ f(\bar{\sigma}_{ij}&, \bar{K}_{\alpha}) = 0 \end{aligned} \tag{18.92}$$

where

$$\Delta\lambda = (1 + \Lambda^{(2)}\Delta t)\mu$$

Following Simo *et al.* (1988) and considering rate-independent plasticity, an interesting observation can now be made. Let $\bar{\sigma}_{ij}$ and $\bar{K}_\alpha$ be considered as the final stress state and final hardening parameters respectively, it then appears that (18.92) expresses the fully implicit scheme. The solution of the above system provides $(\bar{\sigma}_{ij}, \bar{K}_\alpha, \Delta\lambda)$ and, finally, from (18.91) the current state is obtained. For a more detailed discussion of the numerical implication as well as for the derivation of the algorithmic tangential stiffness matrix we refer to Simo *et al.* (1988), Simo and Hughes (1998) and Runesson *et al.* (1999).

Isotropic von Mises hardening

Considering associated plasticity and linear hardening, the static yield function is given by

$$f = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2} - \sigma_y(\varepsilon^{vp}_{eff}) \; ; \qquad \sigma_y = \sigma_{yo} + H\varepsilon^{vp}_{eff} \tag{18.93}$$

where the internal variable is taken as $\kappa = \varepsilon^{vp}_{eff}$ the effective viscoplastic strain.

As discussed above, when $(\bar{\sigma}_{ij}, \bar{K}_\alpha)$ is replaced with $(\sigma^{(2)}_{ij}, K^{(2)}_\alpha)$, (18.92) is the same set of equations that is found in rate-independent plasticity for the fully implicit scheme. From (18.47) the solution can therefore be written down directly

$$\begin{aligned}
\bar{s}_{ij} &= (1 - 3G\frac{\Delta\lambda}{\sigma^t_{eff}})s^t_{ij} \\
\bar{\sigma}_{kk} &= \sigma^t_{kk} \\
\bar{K} &= K^{(1)} + H\Delta\lambda
\end{aligned}$$

Insertion into the static yield criterion (18.93) results in

$$f = \sigma^t_{eff} - 3G\Delta\lambda - (\sigma_{yo} + K^{(1)} + H\Delta\lambda) = 0$$

with the solution

$$\Delta\lambda = \frac{\sigma^t_{eff} - (\sigma_{yo} + K^{(1)})}{3G + H}$$

The final state is then achieved directly from (18.91). If we, for instance, choose $\Lambda = G/\eta$ as well as $H = 0$, that is, ideal plasticity, it can be shown that the model corresponds to Hohenemser-Prager viscoplasticity, cf. (15.34).