6 REPRESENTATION THEOREMS

Establishment of constitutive relations for nonlinear material behavior is in general not trivial and a number of methods exists. Often, the only initial information available is the knowledge that a certain quantity depends on some other quantities, but the explicit constitutive relation is unknown. Evidently, knowledge of experimental results provides important information, but experimental data are often of a form that only enables one to determine the explicit constitutive relation for particular load paths - uniaxial loading, for instance. The question of determination of the explicit constitutive relation for arbitrary load paths then remains. In recent years, increasing use has been made of certain powerful theorems, which enables one to determine the most general explicit form of constitutive relations. These theorems are based on so-called *representation theorems*, which determine the most general forms of various scalar and tensor functions that satisfy both the coordinate invariance principle as well as the material symmetry in question.

To illustrate this approach, we may again consider Cauchy-elasticity where it is known that a one-to-one relation exists between strains and stresses, i.e.

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}) \tag{6.1}$$

where $\boldsymbol{g}$ is the response function. The material is assumed to be isotropic and we want to investigate whether it is possible to obtain a more explicit form for (6.1). Obviously, by performing a number of experiments, it is possible to derive some information of a more explicit form for (6.1). However, as shown in the previous chapter, just the coordinate invariance principle and the assumption of isotropy alone allow us to conclude the most general explicit form of (6.1). The result was given by (5.16) and below we will recall the essential steps as they provide an example of a result given in terms of a representation theorem.

In another coordinate system, the constitutive relation (6.1) reads

$$\boldsymbol{\varepsilon}' = \boldsymbol{g}^*(\boldsymbol{\sigma}') \tag{6.2}$$

Suppose that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ correspond to $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$ respectively, but just measured in another coordinate system. Then we have

$$\boldsymbol{\varepsilon}' = \boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T \quad ; \qquad \boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T \tag{6.3}$$

The coordinate invariance principle, (5.10), states that the material response is independent of the coordinate system. Hence, the predictions provided by (6.1) and (6.2) must be identical. With (6.3a) in (6.2), we therefore have

$$\boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T = \boldsymbol{g}^*(\boldsymbol{\sigma}')$$

and use of (6.1) leads to

$$\boldsymbol{A}\boldsymbol{g}(\boldsymbol{\sigma})\boldsymbol{A}^T = \boldsymbol{g}^*(\boldsymbol{\sigma}') \qquad \text{coordinate invariance} \tag{6.4}$$

Referring to (5.13), the assumption of isotropy means that the response function is the same in all coordinate systems, i.e.

$$\boldsymbol{g}^*(\boldsymbol{\sigma}') = \boldsymbol{g}(\boldsymbol{\sigma}') \qquad \text{isotropy} \tag{6.5}$$

Insertion of (6.5) and (6.3b) in (6.4) then leads to

$$\boldsymbol{A}\boldsymbol{g}(\boldsymbol{\sigma})\boldsymbol{A}^T = \boldsymbol{g}(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T) \qquad \text{coordinate invariance + isotropy}$$

which means that the response function $\boldsymbol{g}$ is an *isotopic second-order tensor function.* In Chapter 5, we showed that this requirement implies that the most general format of (6.1) is provided by (5.16), i.e.

$$\boldsymbol{\varepsilon} = \alpha_1\boldsymbol{I} + \alpha_2\boldsymbol{\sigma} + \alpha_3\boldsymbol{\sigma}^2 \tag{6.6}$$

and this representation theorem is evidently of a very powerful nature.

Representation theorems are available for a number of relations of which (6.6) only represents a very simple example. In general, representation theorems turn out to be rather difficult to prove, but since they are often of great help when establishing constitutive theories, we will present below some useful results. In the literature, there has been a long standing debate as to what should be considered as correct representation theorems. A concise theoretical formulation was established by Rivlin and Ericksen (1955) and Truesdell (1955a) and (6.6) is in the literature often referred to as the *Rivlin-Ericksen representation theorem.* Later, important contributions were provided by, for instance, Wang (1970), Smith (1971), Spencer (1971) and Boehler (1977). The review articles of Murakami and Sawczuk (1981) and Zheng (1994) contain comprehensive discussions of various results and their historical development.

In the following, we shall consider scalar functions as well as isotropic tensor functions of second order that depend on certain quantities and we will present the corresponding representation theorems. To keep the formulation as general as possible let

$\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S} =$	*symmetric second-order tensors*
$H_\alpha =$	*scalar quantities* ($\alpha = 1, 2, \ldots$)

We observe that in another coordinate system $\boldsymbol{x}'$, the scalars H_α are unchanged whereas $\boldsymbol{M}'$, $\boldsymbol{N}'$, $\boldsymbol{P}'$ and $\boldsymbol{S}'$ in accordance with (1.41) become

$$\boldsymbol{M}' = \boldsymbol{A}\boldsymbol{M}\boldsymbol{A}^T\,; \quad \boldsymbol{N}' = \boldsymbol{A}\boldsymbol{N}\boldsymbol{A}^T\,; \quad \boldsymbol{P}' = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T\,; \quad \boldsymbol{S}' = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^T \tag{6.7}$$

Representation theorems that even involve, for instance, vectors and anti-symmetric second-order tensors are also available and the reader may consult the literature mentioned above on this subject.

6.1 Scalar functions

Consider some quantity which is assumed to be a scalar, i.e. an invariant. Let us further assume that this quantity in some coordinate system is given by

$$\boxed{g = g(\boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S}, H_\alpha)} \tag{6.8}$$

where g expresses some property of the material and it is therefore a response function. Since the quantity g is a scalar, this implies that it takes the same value in all coordinate systems; it is an invariant. In the x_i-coordinate system, we have $g = g(\boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S}, H_\alpha)$ whereas in another coordinate system x'_i, we have $g^* = g^*(\boldsymbol{N}', \boldsymbol{P}', \boldsymbol{S}', H'_\alpha)$. In accordance with the discussion following (5.7), the response function is denoted g in the x_i-coordinate system and g^* in the x'_i-coordinate system. Since the quantity takes the same value in all coordinate systems, we have

$$g(\boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S}, H_\alpha) = g^*(\boldsymbol{N}', \boldsymbol{P}', \boldsymbol{S}', H'_\alpha) \qquad \text{coordinate invariance} \tag{6.9}$$

Assume that the material is isotropic. Isotropy means that the response function is the same in all coordinate systems, cf. (5.13). This implies that

$$g^*(\boldsymbol{N}', \boldsymbol{P}', \boldsymbol{S}', H'_\alpha) = g(\boldsymbol{N}', \boldsymbol{P}', \boldsymbol{S}', H'_\alpha) \qquad \text{isotropy}$$

Insertion of this expression in (6.9) and use of (6.7) then leads to

$$\boxed{g(\boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S}, H_\alpha) = g(\boldsymbol{A}\boldsymbol{N}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^T, H_\alpha) \quad \textit{coord. inv.+iso.}} \tag{6.10}$$

A scalar function which fulfills this expression is called an *isotropic scalar tensor function*. It is evident that if $\boldsymbol{N}$, $\boldsymbol{P}$ and $\boldsymbol{S}$ enter the function g in terms of invariants, then (6.10) is fulfilled. Moreover, referring to Zheng (1994), for instance, it turns out that the most general form of g, that fulfills requirement (6.10), is given by

$$\boxed{\begin{aligned} g = g(&I_{1N}, I_{2N}, I_{3N}, I_{1P}, I_{2P}, I_{3P}, I_{1S}, I_{2S}, I_{3S}, \\ &J_1^o, J_2^o, J_3^o, J_4^o, J_5^o, J_6^o, J_7^o, J_8^o, J_9^o, J_{10}^o, J_{11}^o, J_{12}^o, J_{13}^o, H_\alpha) \end{aligned}} \tag{6.11}$$

where the three invariants of $\boldsymbol{N}$, as usual, are given by

$$I_{1N} = tr\,\boldsymbol{N}\,; \quad I_{2N} = \frac{1}{2} tr\,(\boldsymbol{N}^2)\,; \quad I_{3N} = \frac{1}{3} tr\,(\boldsymbol{N}^3) \tag{6.12}$$

the three invariants of $\boldsymbol{P}$ are given by

$$I_{1P} = tr\,\boldsymbol{P}\,; \quad I_{2P} = \frac{1}{2} tr\,(\boldsymbol{P}^2)\,; \quad I_{3P} = \frac{1}{3} tr\,(\boldsymbol{P}^3) \tag{6.13}$$

and, likewise, the three invariants of $\boldsymbol{S}$ are given by

$$I_{1S} = tr\,\boldsymbol{S}\,;\quad I_{2S} = \frac{1}{2}tr\,(\boldsymbol{S}^2)\,;\quad I_{3S} = \frac{1}{3}tr\,(\boldsymbol{S}^3) \tag{6.14}$$

However, the so-called *joint invariants* are given by

$$\boxed{\begin{array}{ll} J_1^o = tr\,(\boldsymbol{NP})\,; & J_2^o = tr\,(\boldsymbol{NP}^2) \\ J_3^o = tr\,(\boldsymbol{N}^2\boldsymbol{P})\,; & J_4^o = tr\,(\boldsymbol{N}^2\boldsymbol{P}^2) \\ J_5^o = tr\,(\boldsymbol{NS})\,; & J_6^o = tr\,(\boldsymbol{NS}^2) \\ J_7^o = tr\,(\boldsymbol{N}^2\boldsymbol{S})\,; & J_8^o = tr\,(\boldsymbol{N}^2\boldsymbol{S}^2) \\ J_9^o = tr\,(\boldsymbol{PS})\,; & J_{10}^o = tr\,(\boldsymbol{PS}^2) \\ J_{11}^o = tr\,(\boldsymbol{P}^2\boldsymbol{S})\,; & J_{12}^o = tr\,(\boldsymbol{P}^2\boldsymbol{S}^2) \\ J_{13}^o = tr\,(\boldsymbol{NPS}) & \end{array}} \tag{6.15}$$

In index notation (6.12) - (6.15) read

$$\begin{aligned} I_{1N} &= N_{kk}\,; \quad I_{2N} = \frac{1}{2}N_{kl}N_{lk}\,; \quad I_{3N} = \frac{1}{3}N_{kl}N_{lm}N_{mk} \\ I_{1P} &= P_{kk}\,; \quad I_{2P} = \frac{1}{2}P_{kl}P_{lk}\,; \quad I_{3P} = \frac{1}{3}P_{kl}P_{lm}P_{mk} \\ I_{1S} &= S_{kk}\,; \quad I_{2S} = \frac{1}{2}S_{kl}S_{lk}\,; \quad I_{3S} = \frac{1}{3}S_{kl}S_{lm}S_{mk} \end{aligned}$$

and

$$\begin{array}{ll} J_1^o = N_{kl}P_{lk}\,; & J_2^o = N_{kl}P_{lm}P_{mk} \\ J_3^o = N_{kl}N_{lm}P_{mk}\,; & J_4^o = N_{kl}N_{lm}P_{mn}P_{nk} \\ J_5^o = N_{kl}S_{lk}\,; & J_6^o = N_{kl}S_{lm}S_{mk} \\ J_7^o = N_{kl}N_{lm}S_{mk}\,; & J_8^o = N_{kl}N_{lm}S_{mn}S_{nk} \\ J_9^o = P_{kl}S_{lk}\,; & J_{10}^o = P_{kl}S_{lm}S_{mk} \\ J_{11}^o = P_{kl}P_{lm}S_{mk}\,; & J_{12}^o = P_{kl}P_{lm}S_{mn}S_{nk} \\ J_{13}^o = N_{kl}P_{lm}S_{mk} & \end{array}$$

It follows that a total of 22 invariants appears in (6.11). As these invariants are independent of each other and all other invariants of $\boldsymbol{N}$, $\boldsymbol{P}$ and $\boldsymbol{S}$ can be expressed in terms of the 22 invariants defined, these invariants provide a so-called *minimal function basis*. Occasionally, one uses the phrase that these 22 invariants comprise an *irreducible set of invariants*. We also observe that as the number of variables in the g function increases, the number of irreducible invariants increases dramatically: the function $g = g(\boldsymbol{N})$ implies three invariants, $g = g(\boldsymbol{N}, \boldsymbol{P})$ implies ten invariants whereas $g = g(\boldsymbol{N}, \boldsymbol{P}, \boldsymbol{S})$ implies 22 invariants.

As an example of the use of (6.8) and (6.11) assume that the strain energy for an isotropic material is given in the form $W = W(\varepsilon_{ij})$. A comparison with (6.8) shows that $N_{ij} = \varepsilon_{ij}$ and $P_{ij} = S_{ij} = H_\alpha = 0$. In this case (6.11) provides

$$W = W(I_{1\varepsilon}, I_{2\varepsilon}, I_{3\varepsilon})$$

and noting that $I_{1\varepsilon} = \tilde{I}_1$, $I_{2\varepsilon} = \tilde{I}_2$ and $I_{3\varepsilon} = \tilde{I}_3$, cf. (6.12) and (2.51), we have obtained a form that precisely corresponds to (4.65).

Let us next determine the expression $\partial g / \partial N_{ij}$ and let us for simplicity assume that $g = g(\boldsymbol{N}, \boldsymbol{P})$. From (6.11) we find that

$$\begin{aligned} \frac{\partial g}{\partial N_{ij}} &= \frac{\partial g}{\partial I_{1N}} \frac{\partial I_{1N}}{\partial N_{ij}} + \frac{\partial g}{\partial I_{2N}} \frac{\partial I_{2N}}{\partial N_{ij}} + \frac{\partial g}{\partial I_{3N}} \frac{\partial I_{3N}}{\partial N_{ij}} \\ &+ \frac{\partial g}{\partial J_1^o} \frac{\partial J_1^o}{\partial N_{ij}} + \frac{\partial g}{\partial J_2^o} \frac{\partial J_2^o}{\partial N_{ij}} + \frac{\partial g}{\partial J_3^o} \frac{\partial J_3^o}{\partial N_{ij}} + \frac{\partial g}{\partial J_4^o} \frac{\partial J_4^o}{\partial N_{ij}} \end{aligned}$$

i.e.

$$\boxed{\begin{aligned} & g = g(\boldsymbol{N}, \boldsymbol{P}) \qquad \textit{implies} \\ & \frac{\partial g}{\partial N_{ij}} = \phi_1 \delta_{ij} + \phi_2 N_{ij} + \phi_3 N_{ik} N_{kj} + \phi_4 P_{ij} + \phi_5 P_{ik} P_{kj} \\ & \qquad + \phi_6 (N_{ik} P_{kj} + P_{ik} N_{kj}) + \phi_7 (N_{ik} P_{kl} P_{lj} + P_{ik} P_{kl} N_{lj}) \end{aligned}} \tag{6.16}$$

where $\phi_1 \ldots \phi_7$ are given by $\partial g / \partial I_{1N} \ldots \partial g / \partial J_4^o$ and where we therefore have the constraints

$$\frac{\partial \phi_i}{\partial I_j} = \frac{\partial \phi_j}{\partial I_i} \tag{6.17}$$

In this expression i goes from 1 to 7 and the notation $I_1 \ldots I_7 = I_{1N} \ldots J_4^o$ has been used for convenience. The ϕ-quantities in (6.16) depend, in general, on ten of the invariants defined by (6.12)-(6.15).

As an example, consider hyper-elasticity and let g be chosen as the strain energy $W(\varepsilon_{ij})$ i.e. $N_{ij} = \varepsilon_{ij}$ and $P_{ij} = S_{ij} = H_\alpha = 0$. In this case (6.16) reduces to the previous expressions given by (4.8) and (4.71) where we notice that the constraints (6.17) are similar to (4.72).

6.2 Second-order tensor functions

Let us assume that the quantity $\boldsymbol{M}$ in some coordinate system depends on $\boldsymbol{N}$, $\boldsymbol{P}$ and H_α through the tensor function $\boldsymbol{f}$ of second order, i.e.

$$\boxed{\boldsymbol{M} = \boldsymbol{f}(\boldsymbol{N}, \boldsymbol{P}, H_\alpha)} \tag{6.18}$$

We assume that (6.18) expresses some constitutive relation, i.e. $\boldsymbol{f}$ is a response function. In another coordinate system $\boldsymbol{x}'$, this relation reads

$$\boldsymbol{M}' = \boldsymbol{f}^*(\boldsymbol{N}', \boldsymbol{P}', H_\alpha) \tag{6.19}$$

The coordinate invariance principle states that (6.18) and (6.19) must describe the same physical property that is just measured in different coordinate systems. Therefore, use of (6.7) on the left-hand side of (6.19) gives

$$\boldsymbol{A}\boldsymbol{M}\boldsymbol{A}^T = \boldsymbol{f}^*(\boldsymbol{N}', \boldsymbol{P}', H_\alpha)$$

which with (6.18) takes the form

$$\boldsymbol{A}\boldsymbol{f}(\boldsymbol{N}, \boldsymbol{P}, H_\alpha)\boldsymbol{A}^T = \boldsymbol{f}^*(\boldsymbol{N}', \boldsymbol{P}', H_\alpha) \qquad \text{coordinate invariance} \tag{6.20}$$

Assuming the material to be isotropic, this means according to (5.13) that the response function is the same in all coordinate systems, i.e.

$$\boldsymbol{f}^*(\boldsymbol{N}', \boldsymbol{P}', H_\alpha) = \boldsymbol{f}(\boldsymbol{N}', \boldsymbol{P}', H_\alpha) \qquad \text{isotropy}$$

Insertion of this expression in (6.20) and observation of (6.7) then imply

$$\boxed{\boldsymbol{A}\boldsymbol{f}(\boldsymbol{N}, \boldsymbol{P}, H_\alpha)\boldsymbol{A}^T = \boldsymbol{f}(\boldsymbol{A}\boldsymbol{N}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T, H_\alpha) \quad \textit{coord. inv.+iso.}} \tag{6.21}$$

A second-order tensor function which fulfills this expression is called an *isotropic second-order tensor function.*

In order to fulfill (6.21), it turns out, cf. for instance, Zheng (1994), that the most general form of (6.18) is provided by

$$\boxed{\boldsymbol{M} = \sum_{i=1}^{8} \alpha_i \boldsymbol{G}_i} \tag{6.22}$$

where α_i are scalar functions of H_α as well as of the ten invariants formed by $\boldsymbol{N}$ and $\boldsymbol{P}$, i.e. the ten invariants defined in (6.12)-(6.15) and where the $\boldsymbol{G}_i$-tensors are given by

$$\begin{aligned} &\boldsymbol{G}_1 = \boldsymbol{I}\,; \ \ \boldsymbol{G}_2 = \boldsymbol{N}\,; \ \ \boldsymbol{G}_3 = \boldsymbol{N}^2\,; \ \ \boldsymbol{G}_4 = \boldsymbol{P}\,; \ \ \boldsymbol{G}_5 = \boldsymbol{P}^2 \\ &\boldsymbol{G}_6 = \boldsymbol{N}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}\,; \ \ \boldsymbol{G}_7 = \boldsymbol{N}^2\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}^2\,; \\ &\boldsymbol{G}_8 = \boldsymbol{N}\boldsymbol{P}^2 + \boldsymbol{P}^2\boldsymbol{N} \end{aligned} \tag{6.23}$$

The G_i-matrices (tensors) are often termed *tensor generators* since M according to (6.22) is generated by the G−matrices. Using (6.23), (6.22) can be written as

$$\boxed{\begin{aligned} \boldsymbol{M} = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{N} + \alpha_3 \boldsymbol{N}^2 + \alpha_4 \boldsymbol{P} + \alpha_5 \boldsymbol{P}^2 + \alpha_6(\boldsymbol{N}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}) \\ + \alpha_7(\boldsymbol{N}^2\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}^2) + \alpha_8(\boldsymbol{N}\boldsymbol{P}^2 + \boldsymbol{P}^2\boldsymbol{N}) \end{aligned}} \tag{6.24}$$

According to (6.18) $\boldsymbol{f} = \boldsymbol{M}$ and it is easy to check that with $\boldsymbol{f}$ given by (6.24), (6.21) is fulfilled; however, to show that solution (6.24) is, in fact, the most general solution is much more involved and we refer to the previously mentioned literature for available proofs.

As an example consider Cauchy-elasticity and let $\boldsymbol{M}=\boldsymbol{\varepsilon}$, $\boldsymbol{N}=\boldsymbol{\sigma}$ and $\boldsymbol{P}=H_\alpha=0$. In this case (6.24) reduces to our well-known form given by (6.6). We recall that the scalars $\alpha_1 \ldots \alpha_8$ in (6.24) are completely arbitrary.

It is of interest to observe that the formats (6.16) and (6.24) are almost identical, except that the term corresponding to α_7 does not appear in (6.16). Moreover, the α_i-functions in (6.24) are completely arbitrary functions dependent on the ten invariants in (6.12)-(6.15), whereas the ϕ_i-functions in (6.16) are restricted by the constraints (6.17). This is completely analogous with the difference between Cauchy- and hyper-elasticity.

It is emphasized, that the results above provide very powerful tools when constructing constitutive models and that modern constitutive theories make increasing use of these results. Quite often this approach is referred to as the *tensor function approach*. In the next chapter, we shall see how these results can be used to establish the most general incremental time-independent constitutive relation. Before this is done, we shall demonstrate some further examples of the usefulness of the representation theorems (6.11) and (6.24).

6.3 Thermoelasticity

As a first example, assume that the strains depend on the stresses as well as the temperature difference ΔT measured from some reference state where nò thermal expansion exists, i.e. we assume that

$$\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl}, \Delta T)$$

A comparison with (6.18) shows that $M_{ij} = \varepsilon_{ij}$, $N_{ij} = \sigma_{ij}$, $P_{ij} = 0$ and $H_\alpha = \Delta T$, i.e. (6.24) provides

$$\varepsilon_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \sigma_{ij} + \alpha_3 \sigma_{ik} \sigma_{kj} \tag{6.25}$$

where α_1, α_2 and α_3 may depend on the stress invariants σ_{kk}, $\sigma_{kl}\sigma_{lk}$, $\sigma_{kl}\sigma_{lm}\sigma_{mk}$ as well as on ΔT. Let us choose

$$\alpha_1 = -\frac{\nu}{E}\sigma_{kk} + \alpha \Delta T\,; \qquad \alpha_2 = \frac{1+\nu}{E}\,; \qquad \alpha_3 = 0$$

then (6.25) reduces to

$$\boxed{\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^o}$$

where

$$\boxed{\varepsilon_{ij}^e = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \; ; \qquad \varepsilon_{ij}^o = \alpha \Delta T \delta_{ij}}$$

A comparison with (4.91) shows that ε_{ij}^e are the elastic strains determined by the usual Hooke formulation and a comparison with (4.63) shows that ε_{ij}^o represents the thermal strains and we have then recovered the constitutive law for isotropic thermoelasticity.

The thermoelasticity presented above is uncoupled in the sense that the stress state does not influence the thermal strains. However, for some materials, for instance concrete at high temperatures, it turns out that the thermal strains depend on the stress state. Also in that complex case, the representation theorem (6.24) may be used in an elegant manner to obtain a suitable constitutive law, as demonstrated by Sawzcuk (1984) and Thelandersson (1987).

6.4 Viscoelasticity

Let us next turn our attention to materials where the response depends not only on the loading itself, but also on the duration of the loading. In that case one speaks of *time-dependent behavior* or *creep* and we shall return to this subject in Chapter 14.

Quite a number of creep models are constructed on the basis of certain combinations of linear springs and dashpots. These components are illustrated in Fig. 6.1 and the uniaxial constitutive relations are

$$\sigma^e = E\varepsilon^e \; ; \qquad \sigma^v = \eta \dot{\varepsilon}^v \tag{6.26}$$

where superscripts e and v refer to elastic and *viscous* behavior respectively. Moreover, a dot denotes the time derivative, i.e. $\dot{\varepsilon}^v = d\varepsilon^v/dt$ and η is the *viscosity coefficient* [Pa·s]. The time derivative is also referred to as the *rate*. We note that the dashpot responds as a rigid member for a sudden application of the load.

As an example, consider the so-called *Maxwell model* illustrated in Fig. 6.2a). Based on (6.26) and

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^v \; ; \qquad \sigma = \sigma^e = \sigma^v \tag{6.27}$$

it is readily shown that the constitutive relation for a Maxwell model reads

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{6.28}$$

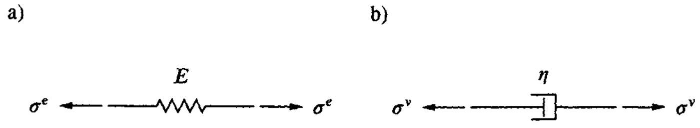

Figure 6.1: a) linear spring; b) dashpot.

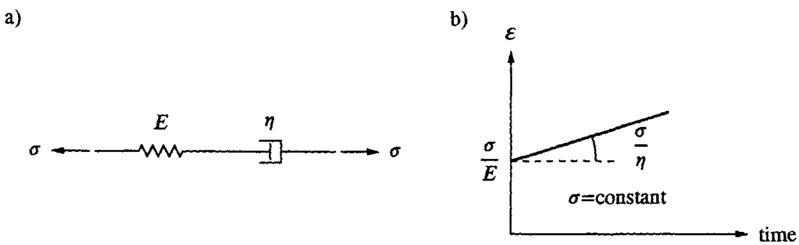

Figure 6.2: Maxwell model; a) configuration; b) response for sudden applied constant load σ.

This constitutive relation is of interest for two reasons.

First, the constitutive relation is formulated in a rate form. This means that the current strain must be obtained by an integration over the stress history; i.e. the current strain is not given as a function of the current stress, but rather as a function of the stress history. To show this explicitly, multiplication of (6.28) by dt and integration from the unloaded state up to the current state gives

$$\varepsilon = \frac{\sigma}{E} + \frac{1}{\eta}\int_0^t \sigma dt \qquad \text{where} \quad \sigma = \sigma(t) \tag{6.29}$$

where t is the current time and it is assumed that the stress is given as a function of time, i.e. $\sigma = \sigma(t)$. As an example, suppose that the stress history is a sudden application of a constant stress. In that case (6.29) gives

$$\varepsilon = \frac{\sigma}{E} + \frac{\sigma t}{\eta} \tag{6.30}$$

as illustrated in Fig. 6.2b). It appears that the current strain not only depend on the stress, but also on the load duration and, evidently, this is why one speaks of time-dependent behavior or creep.

The second point of interest is the question why (6.28) models a time-dependent behavior. This hinges on the fact that the quantity dt enters the constitutive relation in an inhomogeneous fashion. As an example of a homogeneous equation in dt, suppose that we are faced with the constitutive relation

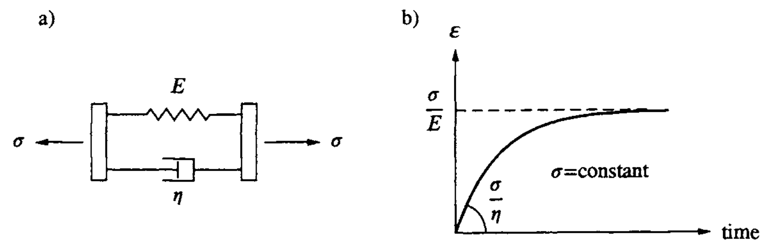

Figure 6.3: Kelvin model; a) configuration; b) response for sudden applied constant load σ.

$\dot{\varepsilon} = \dot{\sigma}/E + k\sigma\dot{\sigma}$, where k is some material parameter. Multiplication by dt gives $d\varepsilon = d\sigma/E + k\sigma d\sigma$ and it is apparent that the constitutive relation is time-independent.

As another example, the so-called *Kelvin model* is illustrated in Fig 6.3a). Based on (6.26) and

$$\sigma = \sigma^e + \sigma^v \; ; \qquad \varepsilon = \varepsilon^e = \varepsilon^v \tag{6.31}$$

the constitutive relation is easily shown to have the following form

$$\sigma = E\,\varepsilon + \eta\dot{\varepsilon} \tag{6.32}$$

The response for a sudden applied constant load is easily shown to be $\varepsilon = \frac{\sigma}{E}(1 - e^{-E/\eta\, t})$ and it is illustrated in Fig. 6.3b). We observe that if the stress is doubled then, according to (6.28) and (6.32), the strain is also doubled. Due to this linearity, the creep models considered here are examples of so-called *viscoelasticity*.

We will now see how representation theorems allow us to obtain generalized forms of (6.28) and (6.32) applicable to general stress states.

Consider first the Maxwell model (6.28). Generalizing (6.27), we have

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^v_{ij} \; ; \qquad \sigma_{ij} = \sigma^e_{ij} = \sigma^v_{ij} \tag{6.33}$$

i.e. the total strain rate is split into an elastic strain rate and a viscous strain rate. In accordance with (6.26), we take

$$\varepsilon^e_{ij} = \varepsilon^e_{ij}(\sigma_{kl}) \; ; \qquad \dot{\varepsilon}^v_{ij} = \dot{\varepsilon}^v_{ij}(\sigma_{kl})$$

Starting with the elastic part ε^e_{ij}, we choose in (6.18) $M_{ij} = \varepsilon^e_{ij}$, $N_{ij} = \sigma_{ij}$, $P_{ij} = H_\alpha = 0$ and (6.24) then provides

$$\varepsilon^e_{ij} = \alpha_1\delta_{ij} + \alpha_2\sigma_{ij} + \alpha_3\sigma_{ik}\sigma_{kj} \tag{6.34}$$

where α_1, α_2 and α_3 may depend on the stress invariants. Since a linear theory is aimed at, we must choose $\alpha_3 = 0$. Moreover, led by (4.91), we choose α_1 and α_2 to be in accordance with that expression, i.e.

$$\alpha_1 = -\frac{\nu}{2G(1+\nu)}\sigma_{kk}\ ; \qquad \alpha_2 = \frac{1}{2G}\ ; \qquad \alpha_3 = 0$$

It then follows that (6.34) becomes

$$\varepsilon^e_{ij} = -\frac{\nu}{2G(1+\nu)}\sigma_{kk}\delta_{ij} + \frac{1}{2G}\sigma_{ij} \tag{6.35}$$

Turning to the viscoelastic part $\dot{\varepsilon}^v_{ij} = \dot{\varepsilon}^v_{ij}(\sigma_{kl})$ similarly to (6.34), we obtain

$$\dot{\varepsilon}^v_{ij} = \beta_1\delta_{ij} + \beta_2\sigma_{ij} + \beta_3\sigma_{ik}\sigma_{kj} \tag{6.36}$$

where β_1, β_2 and β_3 may depend on the stress invariants. Again we choose $\beta_3 = 0$ in order to obtain a linear relation. Moreover, we choose

$$\beta_1 = -\frac{\sigma_{kk}}{\eta_1}\ ; \qquad \beta_2 = \frac{1}{\eta_2}\ ; \qquad \beta_3 = 0$$

where η_1 and η_2 are viscosity coefficients. Expression (6.36) then takes the form

$$\dot{\varepsilon}^v_{ij} = -\frac{\sigma_{kk}}{\eta_1}\delta_{ij} + \frac{\sigma_{ij}}{\eta_2} \tag{6.37}$$

It appears the $\dot{\varepsilon}^v_{kk} = (-\frac{3}{\eta_1} + \frac{1}{\eta_2})\sigma_{kk}$. It is an experimental experience that the viscous strain is often incompressible and this is obtained by choosing $\eta_1 = 3\eta_2$. In the general case, however, combination of (6.33a), (6.35) and (6.37) gives

$$\boxed{\dot{\varepsilon}_{ij} = \frac{1}{2G}(\dot{\sigma}_{ij} - \frac{\nu}{1+\nu}\dot{\sigma}_{kk}\delta_{ij}) - \frac{\sigma_{kk}}{\eta_1}\delta_{ij} + \frac{\sigma_{ij}}{\eta_2} \qquad \textit{Maxwell}} \tag{6.38}$$

The generalization of the uniaxial Maxwell model (6.28) is then given by (6.38) where we put $\eta_1 = 3\eta_2$ if, as often is the case, the viscous strain is incompressible.

Let us next obtain a generalization of the uniaxial Kelvin model (6.32). Here we start by a generalization of (6.31) which then reads

$$\sigma_{ij} = \sigma^e_{ij} + \sigma^v_{ij}\ ; \qquad \varepsilon_{ij} = \varepsilon^e_{ij} = \varepsilon^v_{ij} \tag{6.39}$$

i.e. the stress tensor is split into an elastic part and a viscous part. In accordance with (6.26), we take

$$\sigma^e_{ij} = \sigma^e_{ij}(\varepsilon_{kl})\ ; \qquad \sigma^v_{ij} = \sigma^v_{ij}(\dot{\varepsilon}_{kl})$$

Starting with σ^e_{ij}, we choose in (6.18) $M_{ij} = \sigma^e_{ij}$, $N_{ij} = \varepsilon_{ij}$ and $P_{ij} = H_\alpha = 0$, then the representation theorem (6.24) gives

$$\sigma^e_{ij} = \alpha_1\delta_{ij} + \alpha_2\varepsilon_{ij} + \alpha_3\varepsilon_{ik}\varepsilon_{kj} \tag{6.40}$$

where α_1, α_2 and α_3 may depend on the strain invariants. Led by (4.84), we choose

$$\alpha_1 = 2G\frac{\nu}{1-2\nu}\varepsilon_{kk} \; ; \qquad \alpha_2 = 2G \; ; \qquad \alpha_3 = 0$$

i.e. we obtain

$$\sigma^e_{ij} = 2G\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \tag{6.41}$$

Turning to the viscous part of the stress tensor $\sigma^v_{ij} = \sigma^v_{ij}(\dot{\varepsilon}_{kl})$ similarly to (6.40), we obtain

$$\sigma^v_{ij} = \beta_1\delta_{ij} + \beta_2\dot{\varepsilon}_{ij} + \beta_3\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} \tag{6.42}$$

where β_1, β_2 and β_3 may depend on the invariants of the strain rate. To obtain a linear theory $\beta_3 = 0$ is chosen. Moreover, we choose

$$\beta_1 = -\eta_1\dot{\varepsilon}_{kk} \; ; \qquad \beta_2 = \eta_2 \; ; \qquad \beta_3 = 0$$

where η_1 and η_2 are some viscosity coefficients, i.e. (6.42) reduces to

$$\sigma^v_{ij} = -\eta_1\dot{\varepsilon}_{kk}\delta_{ij} + \eta_2\dot{\varepsilon}_{ij} \tag{6.43}$$

Often it is observed experimentally that only the deviatoric part of σ^v_{ij} is of importance and this can be achieved by choosing $\eta_1 = \frac{1}{3}\eta_2$. Combination of (6.39a), (6.41) and (6.43) gives

$$\boxed{\sigma_{ij} = 2G(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}) - \eta_1\dot{\varepsilon}_{kk}\delta_{ij} + \eta_2\dot{\varepsilon}_{ij} \qquad \textit{Kelvin}}$$

which is then the generalization of the Kelvin model (6.32). If $\eta_1 = \frac{1}{3}\eta_2$ the two last terms reduce to $\eta_2\dot{e}_{ij}$, i.e. only the deviatoric part displays a time-dependent behavior.

In the derivations above, the formulation was based on (6.33) for the Maxwell model and (6.39) for the Kelvin model and the advantages of this approach will become more evident when we focus on more complex viscoelastic models in Chapter 14. At the present stage, an alternative and perhaps more direct manner to derive the Maxwell model would be to choose $M_{ij} = \dot{\varepsilon}_{ij}$, $N_{ij} = \sigma_{ij}$, $P_{ij} = \dot{\sigma}_{ij}$ and $H_\alpha = 0$ in (6.18). Moreover, by choosing $\alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ and

$$\alpha_1 = -\frac{\sigma_{kk}}{\eta_1} - \frac{\nu}{2G(1+\nu)}\dot{\sigma}_{kk} \; ; \qquad \alpha_2 = \frac{1}{\eta_2} \; ; \qquad \alpha_3 = \frac{1}{2G}$$

in (6.24), we will retrieve the generalized Maxwell model, cf. (6.38). A similar approach could be adopted for establishment of the Kelvin model. However, while these approaches are prosperous within the present context they are not applicable within the general framework of viscoelasticity discussed in Chapter 14.

The models of Maxwell and Kelvin represent the two simplest models of viscoelasticity and more elaborate models can be obtained by various combinations of springs and dashpots, e.g. by connecting a Maxwell and a Kelvin model in series and thereby obtaining the so-called *Burgers model*. These more advanced models of viscoelasticity can also be derived using representation theorems for several variables. In Chapter 14, we will present a more detailed discussion of viscoelasticity.

6.5 Orthotropic linear elasticity

In Section 4.6, we derived the stiffness matrix $\boldsymbol{D}$ present in Hooke's law $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$ when the material is orthotropic; the result was given by (4.55). It is recalled that orthotropy means that the material possesses three symmetry planes and the result (4.55) holds when the coordinate planes are chosen to coincide with these symmetry planes. If another coordinate system is used, the stiffness matrix $\boldsymbol{D}$ has to be transformed in accordance with (4.46).

We will now derive linear elasticity for orthotropic materials by a completely different method, which makes use of representation theorems and which makes for an entirely different approach to deal with various kinds of anisotropy.

According to (4.50), the constitutive relation takes the same form in coordinate systems that are mirror images (reflections) of each other in a symmetry plane. One may also say that, for reflections about a symmetry plane, the constitutive relation is isotropic. For these reflections, the relation between the old coordinate system $\boldsymbol{x}$ and the new coordinate system $\boldsymbol{x}'$ is given by

$$\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}'$$

where the transformation matrix $\boldsymbol{A}$ is orthogonal. Choosing the coordinate planes to coincide with the three symmetry planes we found in Section 4.6 that the reflections are determined by

Symmetry group S:

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad \boldsymbol{A}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.44}$$

for reflections about the x_1x_2-plane, x_2x_3-plane and x_3x_1-plane respectively. For these reflections, the constitutive relation is isotropic. The reflections, i.e. the symmetry group defined by (6.44) is called the symmetry group S and it characterizes the symmetries related to orthotropy.

For orthotropy, assume that we can identify three second-order tensors $M_{ij}^{(1)}$, $M_{ij}^{(2)}$ and $M_{ij}^{(3)}$ that characterize the three symmetry planes; these $\boldsymbol{M}^{(\alpha)}$-tensors are called *structural tensors*. Since they are second-order tensors, the components $\boldsymbol{M}^{(\alpha)'}$ in a new coordinate system are related to the components $\boldsymbol{M}^{(\alpha)}$ in the old coordinate system through

$$\boldsymbol{M}^{(\alpha)'} = \boldsymbol{A}\boldsymbol{M}^{(\alpha)}\boldsymbol{A}^T$$

The structural tensors are chosen such that they are unchanged for the symmetry in question. In the present case, the symmetry group S is defined in (6.44), i.e. we have

$$\boldsymbol{M}^{(\alpha)} = \boldsymbol{A}\boldsymbol{M}^{(\alpha)}\boldsymbol{A}^T \quad \text{for} \quad \boldsymbol{A} \in S \tag{6.45}$$

Consider the strain energy W and assume that W depends on the strains and on the three structural tensors, i.e.

$$W = W(\boldsymbol{\varepsilon}, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}) \tag{6.46}$$

Assume that the strain energy is an isotropic scalar tensor function. According to (6.10) we then have

$$\begin{aligned} &W(\boldsymbol{\varepsilon}, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}) \\ &= W(\boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{M}^{(1)}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{M}^{(2)}\boldsymbol{A}^T, \boldsymbol{A}\boldsymbol{M}^{(3)}\boldsymbol{A}^T) \end{aligned} \tag{6.47}$$

for arbitrary $\boldsymbol{A}$ matrices. However, when the transformation matrix $\boldsymbol{A}$ is chosen in accordance with the symmetry group S given by (6.44), (6.45) holds and (6.47) then reduces to

$$\begin{aligned} &W(\boldsymbol{\varepsilon}, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}) \\ &= W(\boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}) \end{aligned} \quad \text{for} \quad \boldsymbol{A} \in S \tag{6.48}$$

Suppose that the material is isotropic. Then the strain energy is given by $W = W(\boldsymbol{\varepsilon})$ and for an arbitrary transformation matrix $\boldsymbol{A}$ we have $W(\boldsymbol{\varepsilon}) = W(\boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T)$. Therefore, expression (6.48) shows that for the reflections defined by (6.44), the relation for the strain energy is as if the material were isotropic and this is exactly what is meant by orthotropy. Therefore, with the format (6.46) and the structural tensors fulfilling property (6.45) we have achieved an intriguing new route to treat orthotropy as well as other kinds of material symmetries. Since (6.47) holds, we can use the representation theorem (6.11) on the format (6.46). Before this is performed, we have to identify the specific structural $\boldsymbol{M}^{(\alpha)}$-tensors.

Orthotropy means that three orthogonal symmetry planes exist. We could equally well speak of three orthogonal directions defined by the orthogonal unit vectors $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$ and $\boldsymbol{v}^{(3)}$, i.e.

$$\begin{aligned} &\boldsymbol{v}^{(1)T}\boldsymbol{v}^{(1)} = 1\,; \quad &&\boldsymbol{v}^{(2)T}\boldsymbol{v}^{(2)} = 1\,; \quad &&\boldsymbol{v}^{(3)T}\boldsymbol{v}^{(3)} = 1 \\ &\boldsymbol{v}^{(1)T}\boldsymbol{v}^{(2)} = 0\,; \quad &&\boldsymbol{v}^{(2)T}\boldsymbol{v}^{(3)} = 0\,; \quad &&\boldsymbol{v}^{(3)T}\boldsymbol{v}^{(1)} = 0 \end{aligned}$$

where the $\boldsymbol{v}^{(\alpha)}$-vectors are termed *material directions*. Suppose that we have identified a material direction, then its opposite direction also qualifies as a material direction. If the material directions $\boldsymbol{v}^{(\alpha)}$ are to describe the material, they must therefore appear in the form of equal powers of $\boldsymbol{v}^{(\alpha)}$. This is achieved by defining the structural tensors according to

$$\boxed{\boldsymbol{M}^{(1)} = \boldsymbol{v}^{(1)}\boldsymbol{v}^{(1)T}\,; \quad \boldsymbol{M}^{(2)} = \boldsymbol{v}^{(2)}\boldsymbol{v}^{(2)T}\,; \quad \boldsymbol{M}^{(3)} = \boldsymbol{v}^{(3)}\boldsymbol{v}^{(3)T}} \tag{6.49}$$

It follows directly that

$$\begin{aligned} &\boldsymbol{M}^{(1)}\boldsymbol{M}^{(1)}=\boldsymbol{M}^{(1)}\,; \quad &&\boldsymbol{M}^{(2)}\boldsymbol{M}^{(2)}=\boldsymbol{M}^{(2)}\,; \quad &&\boldsymbol{M}^{(3)}\boldsymbol{M}^{(3)}=\boldsymbol{M}^{(3)} \\ &\boldsymbol{M}^{(1)}\boldsymbol{M}^{(2)}=0\,; &&\boldsymbol{M}^{(2)}\boldsymbol{M}^{(3)}=0\,; &&\boldsymbol{M}^{(3)}\boldsymbol{M}^{(1)}=0 \end{aligned} \tag{6.50}$$

Suppose that the coordinate system is chosen collinearly with the material directions. Then we have $\boldsymbol{v}^{(1)T} = [1\,0\,0]$, $\boldsymbol{v}^{(2)T} = [0\,1\,0]$ and $\boldsymbol{v}^{(3)T} = [0\,0\,1]$ and (6.49) becomes

$$\boldsymbol{M}^{(1)}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}; \quad \boldsymbol{M}^{(2)}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}; \quad \boldsymbol{M}^{(3)}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \tag{6.51}$$

and it is trivial to check that these structural tensors fulfill requirement (6.45).

For the specific choice of coordinate system which leads to (6.51), we obtain

$$\boldsymbol{I} = \boldsymbol{M}^{(1)} + \boldsymbol{M}^{(2)} + \boldsymbol{M}^{(3)} \tag{6.52}$$

However, since the $\boldsymbol{M}^{(\alpha)}$-quantities are second-order tensors, cf. (6.49), it follows that (6.52) holds for arbitrary coordinate systems. Relation (6.52) shows that the three structural tensors depend on each other; if two structural tensors are known, the last can be derived from (6.52). This is another way of expressing that orthotropy can be defined as the existence of three orthogonal symmetry planes or as the existence of two orthogonal symmetry planes, cf. (4.57).

Similar to (6.52), we will now derive another useful relation. For the strain tensor $\boldsymbol{\varepsilon}$ we evidently have $\boldsymbol{\varepsilon}\boldsymbol{I} = \boldsymbol{\varepsilon}$ which together with (6.52) becomes

$$\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)} + \boldsymbol{\varepsilon}\boldsymbol{M}^{(2)} + \boldsymbol{\varepsilon}\boldsymbol{M}^{(3)} = \boldsymbol{\varepsilon}$$

Likewise, we have that $\boldsymbol{I}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$, i.e.

$$\boldsymbol{M}^{(1)}\boldsymbol{\varepsilon} + \boldsymbol{M}^{(2)}\boldsymbol{\varepsilon} + \boldsymbol{M}^{(3)}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$$

Adding these two expressions gives

$$\boldsymbol{\varepsilon}=\frac{1}{2}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)}+\boldsymbol{M}^{(1)}\boldsymbol{\varepsilon})+\frac{1}{2}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(2)}+\boldsymbol{M}^{(2)}\boldsymbol{\varepsilon})+\frac{1}{2}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(3)}+\boldsymbol{M}^{(3)}\boldsymbol{\varepsilon}) \tag{6.53}$$

Here, we have considered the strain tensor $\boldsymbol{\varepsilon}$, but it is evident that a relation similar to (6.53) holds for an arbitrary symmetric second-order tensor.

With this discussion, we will now use a representation theorem to determine the format of the strain energy. Since the three structural tensors depend on each other through (6.52), we choose to work with $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$, only. The strain energy is then taken as

$$W = W(\boldsymbol{\varepsilon}, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)})$$

Following (6.11), the strain energy is taken as function of the invariants defined by (6.12)-(6.15) where $\boldsymbol{N} = \boldsymbol{\varepsilon}$, $\boldsymbol{P} = \boldsymbol{M}^{(1)}$, $\boldsymbol{S} = \boldsymbol{M}^{(2)}$ and $\boldsymbol{H}_\alpha = 0$. Due to (6.50) and (6.49), we have, for instance, $I_{1M^{(1)}} = 1$, $I_{2M^{(1)}} = 1/2$, $I_{3M^{(1)}} = 1/3$, $J_4^0 = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(1)2}) = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(1)}) = J_3^0$ and $J_9^0 = tr(\boldsymbol{M}^{(1)}\boldsymbol{M}^{(2)}) = 0$. We therefore obtain

$$\begin{aligned} W =& W(tr\boldsymbol{\varepsilon}, \frac{1}{2}tr(\boldsymbol{\varepsilon}^2), \frac{1}{3}tr(\boldsymbol{\varepsilon}^3), \\ & tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)}), tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(1)}), tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(2)}), tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(2)})) \end{aligned} \tag{6.54}$$

To obtain an expression, where all the three structural tensors enter in a symmetric fashion, we make use of (6.53) to obtain

$$tr\boldsymbol{\varepsilon} = tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)}) + tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(2)}) + tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(3)}) \tag{6.55}$$

Replacing $\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon}^2$, we obtain in a similar way

$$tr(\boldsymbol{\varepsilon}^2) = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(1)}) + tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(2)}) + tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(3)})$$

Insertion of this expression and (6.55) into (6.54) provides

$$W = W(I_1, I_2, I_3, I_4, I_5, I_6, I_7) \tag{6.56}$$

where

$$\begin{aligned} & I_1 = tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)}); \quad I_2 = tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(2)}); \quad I_3 = tr(\boldsymbol{\varepsilon}\boldsymbol{M}^{(3)}) \\ & I_4 = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(1)}); \quad I_5 = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(2)}); \quad I_6 = tr(\boldsymbol{\varepsilon}^2\boldsymbol{M}^{(3)}); \quad I_7 = \tfrac{1}{3}tr(\boldsymbol{\varepsilon}^3) \end{aligned} \tag{6.57}$$

In hyper-elasticity where a strain energy exists, we have $\sigma_{ij} = \partial W/\partial \varepsilon_{ij}$, which with (6.56) and (6.57) lead to

$$\begin{aligned} \boldsymbol{\sigma} =& \frac{\partial W}{\partial I_1}\boldsymbol{M}^{(1)} + \frac{\partial W}{\partial I_2}\boldsymbol{M}^{(2)} + \frac{\partial W}{\partial I_3}\boldsymbol{M}^{(3)} + \frac{\partial W}{\partial I_4}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)}\boldsymbol{\varepsilon}) \\ & + \frac{\partial W}{\partial I_5}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(2)} + \boldsymbol{M}^{(2)}\boldsymbol{\varepsilon}) + \frac{\partial W}{\partial I_6}(\boldsymbol{\varepsilon}\boldsymbol{M}^{(3)} + \boldsymbol{M}^{(3)}\boldsymbol{\varepsilon}) + \frac{\partial W}{\partial I_7}\boldsymbol{\varepsilon}^2 \end{aligned} \tag{6.58}$$

in accordance with Boehler (1979). Since we want to determine the constitutive relation for linear elasticity, we choose

$$
\begin{aligned}
&\frac{\partial W}{\partial I_1} = \alpha_1 I_1 + \beta_1 I_2 + \beta_2 I_3 \\
&\frac{\partial W}{\partial I_2} = \alpha_2 I_1 + \alpha_3 I_2 + \beta_3 I_3 \\
&\frac{\partial W}{\partial I_3} = \alpha_4 I_1 + \alpha_5 I_2 + \alpha_6 I_3 \\
&\frac{\partial W}{\partial I_4} = \alpha_7 ; \qquad \frac{\partial W}{\partial I_5} = \alpha_8 ; \qquad \frac{\partial W}{\partial I_6} = \alpha_9 ; \qquad \frac{\partial W}{\partial I_7} = 0
\end{aligned}
\tag{6.59}
$$

where $\alpha_1 \cdots \alpha_9$ and $\beta_1, \beta_2, \beta_3$ are constants and where the reason for the specific notation will become evident in a moment. We have

$$
\frac{\partial}{\partial I_2}\left(\frac{\partial W}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W}{\partial I_2}\right); \quad \frac{\partial}{\partial I_3}\left(\frac{\partial W}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W}{\partial I_3}\right); \quad \frac{\partial}{\partial I_3}\left(\frac{\partial W}{\partial I_2}\right) = \frac{\partial}{\partial I_2}\left(\frac{\partial W}{\partial I_3}\right)
$$

which leads to

$$
\beta_1 = \alpha_2 ; \qquad \beta_2 = \alpha_4 ; \qquad \beta_3 = \alpha_5
$$

Insertion of these expressions and (6.59) into (6.58) then provides the result sought

$$
\boxed{
\begin{aligned}
\boldsymbol{\sigma} = {} & [\alpha_1 tr(\boldsymbol{\varepsilon M}^{(1)}) + \alpha_2 tr(\boldsymbol{\varepsilon M}^{(2)}) + \alpha_4 tr(\boldsymbol{\varepsilon M}^{(3)})]\boldsymbol{M}^{(1)} \\
& + [\alpha_2 tr(\boldsymbol{\varepsilon M}^{(1)}) + \alpha_3 tr(\boldsymbol{\varepsilon M}^{(2)}) + \alpha_5 tr(\boldsymbol{\varepsilon M}^{(3)})]\boldsymbol{M}^{(2)} \\
& + [\alpha_4 tr(\boldsymbol{\varepsilon M}^{(1)}) + \alpha_5 tr(\boldsymbol{\varepsilon M}^{(2)}) + \alpha_6 tr(\boldsymbol{\varepsilon M}^{(3)})]\boldsymbol{M}^{(3)} \\
& + \alpha_7(\boldsymbol{\varepsilon M}^{(1)} + \boldsymbol{M}^{(1)}\boldsymbol{\varepsilon}) + \alpha_8(\boldsymbol{\varepsilon M}^{(2)} + \boldsymbol{M}^{(2)}\boldsymbol{\varepsilon}) \\
& + \alpha_9(\boldsymbol{\varepsilon M}^{(3)} + \boldsymbol{M}^{(3)}\boldsymbol{\varepsilon})
\end{aligned}
}
\tag{6.60}
$$

in accordance with Boehler (1979). It appears that this expression contains nine material parameters ($\alpha_1 \cdots \alpha_9$) in agreement with (4.55). Indeed, if the coordinate system is chosen collinearly with the material directions then (6.51) holds and it is trivial from (6.60) to recover exactly the format given by (4.55).

One advantage of the format (6.60) is that it holds in all coordinate systems whereas (4.55) requires the coordinate system to be collinear with the material directions. Another advantage is that (6.60) is a tensorial expression and that, by use of the concept of structural tensors we have opened for a treatment of anisotropy in a very elegant fashion.

Smith and Rivlin (1957) introduced the concept of an *anisotropic tensor*, which has the property that it is unchanged by coordinate changes belonging to the symmetry group characterizing the material in question; an example is given by (6.45). This idea was further developed by Ericksen (1960)

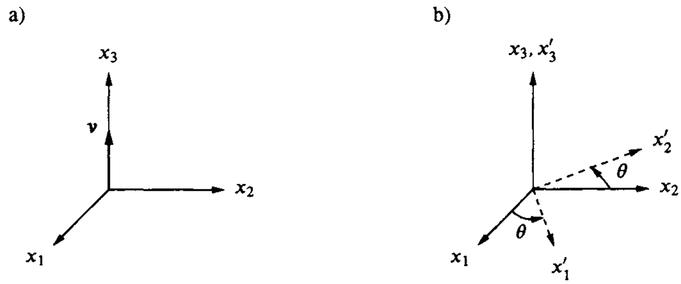

Figure 6.4: Transverse isotropy; a) material direction v in the x_3-direction, b) rotation of the coordinate system about v.

for transverse isotropic fluids where the quantity $M = vv^T$ was introduced and where v is the privileged direction of the medium; transversely isotropic solids were treated by Boehler and Sawczuk (1976). A systematic approach was established by Boehler (1978, 1979) and a comprehensive review is given by Boehler (1987a,b,c) where also the terminology of M being a *structural tensor* is adopted.

6.6 Transverse isotropic linear elasticity

We will now adopt the concept of structural tensors to establish linear elasticity for *transverse isotropy*. In a transversely isotropic material there exists a material direction defined by the unit vector v such that the constitutive relation is unchanged for arbitrary rotations of the coordinate system about that axis; transverse isotropy is occasionally termed as a material possessing an *axis of elastic symmetry*.

Choose the coordinate system such that the x_3-axis is in the v-direction, cf. Fig. 6.4. Following (1.28) we obtain

$$\text{Symmetry group S:} \qquad A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.61}$$

The structural tensor M is taken as

$$\boxed{M = vv^T}$$

and for the coordinate system shown in Fig. 6.4a) we get $v^T = [0\,0\,1]$, i.e.

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.62}$$

It is trivial to check that for the coordinate rotations shown in Fig. 6.4b) we obtain

$$M = AMA^T \quad \text{for} \quad A \in S$$

The strain energy is taken as

$$W = W(\varepsilon, M) \tag{6.63}$$

and assuming that W is an isotropic scalar tensor function, we have in accordance with (6.10)

$$W(\varepsilon, M) = W(A\varepsilon A^T, AMA^T)$$

In the particular, when the transformation matrix A belongs to the symmetry group S defined by (6.61), we obtain

$$W(\varepsilon, M) = W(A\varepsilon A^T, M) \quad \text{for} \quad A \in S$$

i.e. the expression for the strain energy changes as if the material is isotropic; this is precisely what transverse isotropy means.

For expression (6.63), we now use the representation theorem (6.11) with $N = \varepsilon$, $P = M$ and $S = H_\alpha = 0$. Observing that, for instance, $M = M^2$ we obtain

$$W = W(I_1, I_2, I_3, I_4, I_5)$$

where

$$I_1 = tr\varepsilon; \quad I_2 = \tfrac{1}{2}tr(\varepsilon^2); \quad I_3 = \tfrac{1}{3}tr(\varepsilon^3)$$
$$I_4 = tr(\varepsilon M); \quad I_5 = tr(\varepsilon^2 M)$$

For hyper-elasticity, we have $\sigma_{ij} = \partial W/\partial \varepsilon_{ij}$ and we then get

$$\sigma = \frac{\partial W}{\partial I_1}I + \frac{\partial W}{\partial I_2}\varepsilon + \frac{\partial W}{\partial I_3}\varepsilon^2 + \frac{\partial W}{\partial I_4}M + \frac{\partial W}{\partial I_5}(\varepsilon M + M\varepsilon) \tag{6.64}$$

in accordance with Boehler (1987b). Since linear elasticity is sought, we choose

$$\begin{aligned}
\frac{\partial W}{\partial I_1} &= \alpha_1 I_1 + \beta I_4 \\
\frac{\partial W}{\partial I_2} &= \alpha_2 \\
\frac{\partial W}{\partial I_3} &= 0 \\
\frac{\partial W}{\partial I_4} &= \alpha_3 I_1 + \alpha_4 I_4 \\
\frac{\partial W}{\partial I_5} &= \alpha_5
\end{aligned} \tag{6.65}$$

where $\alpha_1 \cdots \alpha_5$ and β are constants and where the reason for the specific notation will become evident in a moment. We have

$$\frac{\partial}{\partial I_4}\left(\frac{\partial W}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W}{\partial I_4}\right) \quad \Rightarrow \quad \beta = \alpha_3$$

Insertion of this result as well as (6.65) in (6.64) yields

$$\boxed{\begin{aligned}\boldsymbol{\sigma} = {} & [\alpha_1 tr\boldsymbol{\varepsilon} + \alpha_3 tr(\boldsymbol{\varepsilon M})]\, \boldsymbol{I} + \alpha_2 \boldsymbol{\varepsilon} \\ & + [\alpha_3 tr\boldsymbol{\varepsilon} + \alpha_4 tr(\boldsymbol{\varepsilon M})]\, \boldsymbol{M} + \alpha_5(\boldsymbol{\varepsilon M} + \boldsymbol{M \varepsilon})\end{aligned}} \tag{6.66}$$

in accordance with Boehler (1975). It appears that transverse isotropy involves five independent material parameters; moreover, (6.66) holds in an arbitrary coordinate system.

If the coordinate system is chosen such that the material direction $\boldsymbol{v}$ is in the direction of the x_3-axis, cf. Fig. 6.4, the structural tensor is given by (6.62). Then it is easily shown that (6.66) takes the format

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \alpha_1+\alpha_2 & \alpha_1 & \alpha_1+\alpha_3 & 0 & 0 & 0 \\ \alpha_1 & \alpha_1+\alpha_2 & \alpha_1+\alpha_3 & 0 & 0 & 0 \\ \alpha_1+\alpha_3 & \alpha_1+\alpha_3 & \begin{matrix}\alpha_1+\alpha_2+\alpha_4 \\ +2(\alpha_3+\alpha_5)\end{matrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(\alpha_2+\alpha_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\alpha_2\alpha_5) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

By comparison with (4.59) and Fig. 6.4, it appears that the material is isotropic in the $x_1 x_2$-plane.