19 SOLUTION OF DYNAMIC FINITE ELEMENT EQUATIONS

We will now turn our interest to dynamic problems involving an elasto-plastic or viscoplastic response. Within the framework already established, it turns out that it is easy to include dynamics caused by the inertia effects. The only new issue that we have to introduce is that of different *time integration schemes*. These schemes are the same as those encountered in linear dynamic problems and, therefore, we will not discuss these time integration schemes in detail, but simply present some information that motivates these schemes to a sufficient degree. The interested reader may consult Bathe (1996), Hughes (1983, 1987) as well as Argyris and Mlejnek (1991), Belytschko *et al.* (2000) and Zienkiewicz and Taylor (1991) for a detailed discussion of time integration schemes.

Moreover, for convenience, damping effects will be ignored. The extension to consider damping is straightforward and as we are here interested in the fundamental numerical procedures, damping is ignored.

19.1 Introduction

According to (16.9), the FE discretization of the equations of motion is given by

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{\psi}(\boldsymbol{a}) = \boldsymbol{0} \tag{19.1}$$

where the mass matrix $\boldsymbol{M}$ is defined by

$$\boldsymbol{M} = \int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV$$

and the out-of-balance forces $\boldsymbol{\psi}$ are defined by

$$\boldsymbol{\psi}(\boldsymbol{a}) = \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \boldsymbol{f} \tag{19.2}$$

For a given time, the external forces are denoted by $\boldsymbol{f}$ and since the stresses $\boldsymbol{\sigma}$ for a given time depend on the nodal displacements $\boldsymbol{a}$, we have $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{a})$.

Relation (19.1) comprises a set of nonlinear differential equations. First we will transform them into a set of nonlinear algebraic equations; thereafter, these nonlinear algebraic equations will be solved in an iterative manner similar to that discussed in Chapter 17.

To transform the nonlinear differential equations (19.1) into a set of nonlinear algebraic equations, the time integration scheme proposed by Newmark (1959) is adopted. We assume that all quantities have been determined at time t_n and we want to proceed to time t_{n+1} and determine all relevant quantities. With evident notation, the *Newmark time integration scheme* then comprises the following approximations

Newmark time integration scheme

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t\dot{\boldsymbol{a}}_n + \tfrac{\Delta t^2}{2}[(1-2\beta)\ddot{\boldsymbol{a}}_n + 2\beta\ddot{\boldsymbol{a}}_{n+1}]$$

$$\dot{\boldsymbol{a}}_{n+1} = \dot{\boldsymbol{a}}_n + \Delta t[(1-\gamma)\ddot{\boldsymbol{a}}_n + \gamma\ddot{\boldsymbol{a}}_{n+1}] \tag{19.3}$$

where β and γ are certain parameters and Δt is the time step, i.e. $\Delta t = t_{n+1} - t_n$. Depending on particular choice of β and γ, we recover several special strategies like the central difference method, the trapezoidal rule etc. For a detailed discussion of the entire subclass of methods, we may refer to Bathe (1996) and Hughes (1983, 1987).

To substantiate the Newmark scheme (19.3), we adopt the *trapezoidal rule*, which provides the following approximations for $\boldsymbol{a}_{n+1}$ and $\dot{\boldsymbol{a}}_{n+1}$

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \frac{\Delta t}{2}(\dot{\boldsymbol{a}}_n + \dot{\boldsymbol{a}}_{n+1}) \;; \quad \dot{\boldsymbol{a}}_{n+1} = \dot{\boldsymbol{a}}_n + \frac{\Delta t}{2}(\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1}) \tag{19.4}$$

Insertion of (19.4b) into (19.4a) yields

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \Delta t\dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{2}\left(\frac{1}{2}\ddot{\boldsymbol{a}}_n + \frac{1}{2}\ddot{\boldsymbol{a}}_{n+1}\right) \tag{19.5}$$

It appears that (19.5) and (19.4b) correspond to (19.3a) and (19.3b) respectively, for $\beta = 1/4$ and $\gamma = 1/2$. Therefore, the Newmark scheme (19.3) can be viewed as a generalization of the trapezoidal rule.

19.2 Explicit scheme

Let us assume that

$$\beta = 0 \;; \quad \gamma = \frac{1}{2} \tag{19.6}$$

In that case (19.3) reduces to

$$\begin{aligned} \boldsymbol{a}_{n+1} &= \boldsymbol{a}_n + \Delta t \dot{\boldsymbol{a}}_n + \frac{\Delta t^2}{2} \ddot{\boldsymbol{a}}_n \\ \dot{\boldsymbol{a}}_{n+1} &= \dot{\boldsymbol{a}}_n + \frac{\Delta t}{2}(\ddot{\boldsymbol{a}}_n + \ddot{\boldsymbol{a}}_{n+1}) \end{aligned} \tag{19.7}$$

Let us write the equations of motion at the current time t_n, i.e. (19.1) becomes

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_n + \boldsymbol{\psi}(\boldsymbol{a}_n) = \boldsymbol{0} \tag{19.8}$$

where

$$\boldsymbol{\psi}(\boldsymbol{a}_n) = \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_n dV - \boldsymbol{f}_n$$

Let us now rewrite the time integration scheme such that the acceleration $\ddot{\boldsymbol{a}}$ is expressed in terms of displacements $\boldsymbol{a}$. Determination of $\ddot{\boldsymbol{a}}_n$ from (19.7a) yields

$$\ddot{\boldsymbol{a}}_n = \frac{2}{\Delta t^2}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n) - \frac{2}{\Delta t}\dot{\boldsymbol{a}}_n \tag{19.9}$$

i.e.

$$\ddot{\boldsymbol{a}}_{n+1} = \frac{2}{\Delta t^2}(\boldsymbol{a}_{n+2} - \boldsymbol{a}_{n+1}) - \frac{2}{\Delta t^2}\dot{\boldsymbol{a}}_{n+1} \tag{19.10}$$

Insertion of (19.9) and (19.10) into (19.7b) results in

$$\dot{\boldsymbol{a}}_{n+1} = \frac{1}{2\Delta t}(\boldsymbol{a}_{n+2} - \boldsymbol{a}_n)$$

and thereby

$$\dot{\boldsymbol{a}}_n = \frac{1}{2\Delta t}(\boldsymbol{a}_{n+1} - \boldsymbol{a}_{n-1}) \tag{19.11}$$

Then (19.9) and (19.11) imply

$$\boxed{\ddot{\boldsymbol{a}}_n = \frac{1}{\Delta t^2}(\boldsymbol{a}_{n+1} - 2\boldsymbol{a}_n + \boldsymbol{a}_{n-1})} \tag{19.12}$$

With the parameter choice (19.6), we obtain (19.12) which is, in fact, the *central difference approximation* to $\ddot{\boldsymbol{a}}_n$.

Insertion of (19.12) into (19.8) then provides the following scheme

$$\boxed{\boldsymbol{M}\boldsymbol{a}_{n+1} = \boldsymbol{M}(2\boldsymbol{a}_n - \boldsymbol{a}_{n-1}) + \Delta t^2(\boldsymbol{f}_n - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_n dV)} \tag{19.13}$$

The the nodal values $\boldsymbol{a}_n$ and $\boldsymbol{a}_{n-1}$ are known and, by the methods discussed in Chapter 18, we can determine the stress state $\boldsymbol{\sigma}_n$. It follows that all terms are

known at the right-hand side of (19.13) and this expression then provides the solution $\boldsymbol{a}_{n+1}$ at time t_{n+1}.

It is of considerable interest that even though we are facing a dynamic (elasto-plastic or viscoplastic) problem that certainly is a nonlinear problem, (19.13) provides the solution $\boldsymbol{a}_{n+1}$ directly without any iterations. This very fortunate situation is enhanced further if the mass matrix $\boldsymbol{M}$ is *lumped* i.e. diagonal. In that case the inversion of $\boldsymbol{M}$ is trivial and (19.13) gives the solution directly without in reality solving any equation system.

Unfortunately, we have to pay a price for these appealing solution properties and our price is that of *conditional stability*. Instability means that any small error introduced by truncation of numbers in the computer will jeopardize the solution in the sense that errors will accumulate with time and very quickly make the solution meaningless. On the other hand, conditional stability means that in order to maintain stability, the time step Δt must be within a certain limit. In our case, it turns out that we must have

$$\boxed{\Delta t \leq \frac{T_s}{\pi} \quad \Rightarrow \quad \textit{stability}} \tag{19.14}$$

where T_s is the shortest period of the finite element assemblage that comprises the discretized body, cf. for instance Bathe (1996) and Hughes (1983, 1987).

As a result, the scheme (19.13) is very efficient for situations with a short load duration; the effects of impacts and explosions as well as wave propagation are problems that are eminently well suited for the solution scheme (19.13). For longer load durations, like for instance earthquakes, the scheme (19.13) is not advantageous since the time step limitation (19.14) implies an excessive number of time steps to be considered. We will return to this aspect in the next section.

An interesting application of (19.13) is even static problems. However, in order to fulfill the restriction (19.14) and maintain a reasonable number of time steps, it is often necessary to modify the mass matrix in an appropriate manner, i.e. to increase the mass density ρ artificially and thereby increasing the smallest period T_s.

In accordance with the notation in Chapter 18, the solution scheme (19.13) is an *explicit scheme* since from the known state at t_n we extrapolate to obtain the response at t_{n+1}. This means that we enforce the equations of motion at time t_n, cf. (19.8). In addition, (19.13) provides an explicit solution where no iterations are involved, but, in principle, that has nothing to do with the method being explicit. In general, one may have an explicit scheme that only provides a solution that is implicitly given. We have previously encountered the opposite situation in Chapter 18 where the implicit return scheme (i.e. $\theta = 1$) for the isotropic von Mises material results in an explicit solution in terms of the radial return method, cf. Section 18.1.4.

19.3 Implicit schemes

We will now assume the parameter $\beta \neq 0$. From (19.3a) we then obtain

$$\ddot{\boldsymbol{a}}_{n+1} = c_1(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n) - c_2\dot{\boldsymbol{a}}_n - c_3\ddot{\boldsymbol{a}}_n \tag{19.15}$$

where

$$c_1 = \frac{1}{\beta\Delta t^2}\,; \quad c_2 = \frac{1}{\beta\Delta t}\,; \quad c_3 = \frac{1-2\beta}{2\beta} \tag{19.16}$$

All quantities are assumed to be known at the current time t_n. We will now derive *implicit schemes* and we therefore write the equations of motion (19.1) at time t_{n+1}, i.e.

$$\boldsymbol{M}\ddot{\boldsymbol{a}}_{n+1} + \boldsymbol{\psi}(\boldsymbol{a}_{n+1}) = \boldsymbol{0} \tag{19.17}$$

Insertion of (19.15) yields

$$\boldsymbol{v}(\boldsymbol{a}_{n+1}) = \boldsymbol{0} \tag{19.18}$$

where the column matrix $\boldsymbol{v}$ is given by

$$\boldsymbol{v}(\boldsymbol{a}_{n+1}) = \boldsymbol{M}[c_1(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n) - c_2\dot{\boldsymbol{a}}_n - c_3\ddot{\boldsymbol{a}}_n] + \boldsymbol{\psi}(\boldsymbol{a}_{n+1}) \tag{19.19}$$

Since the quantities $\boldsymbol{a}_n$, $\dot{\boldsymbol{a}}_n$ and $\ddot{\boldsymbol{a}}_n$ are known and fixed quantities and since we are considering a given fixed time t_{n+1}, it follows that $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{a}_{n+1})$.

By making use of the Newmark time integration scheme, we have transformed the nonlinear differential equations (19.1) into the nonlinear algebraic equations (20). This implies that we can take advantage of the methods described in detail in Chapter 17. In analogy with (17.14) and (17.15), we can therefore write (19.18) as

$$-(\boldsymbol{A}(\boldsymbol{a}_{n+1}))^{-1}\boldsymbol{v}(\boldsymbol{a}_{n+1}) = \boldsymbol{0} \tag{19.20}$$

where the iteration matrix $\boldsymbol{A}^{-1}$ is nonsingular, i.e. $\det \boldsymbol{A}^{-1} \neq 0$ and the components of $\boldsymbol{A}^{-1}$ are finite quantities; otherwise the square matrix $(\boldsymbol{A}(\boldsymbol{a}_{n+1}))^{-1}$ is arbitrary. We may also write (19.20) in the form

$$\boldsymbol{a}_{n+1} = \boldsymbol{F}(\boldsymbol{a}_{n+1}) \tag{19.21}$$

where

$$\boldsymbol{F}(\boldsymbol{a}_{n+1}) = \boldsymbol{a}_{n+1} - (\boldsymbol{A}(\boldsymbol{a}_{n+1}))^{-1}\boldsymbol{v}(\boldsymbol{a}_{n+1})$$

For convenience, let us drop the subscript $n+1$, that is, the equation above becomes

$$\boldsymbol{F}(\boldsymbol{a}) = \boldsymbol{a} - (\boldsymbol{A}(\boldsymbol{a}))^{-1}\boldsymbol{v}(\boldsymbol{a}) \tag{19.22}$$

As usual, cf. (17.13) the general iteration scheme for (19.21) is $\boldsymbol{a}^i = \boldsymbol{F}(\boldsymbol{a}^{i-1})$ for $i = 1, 2, \cdots$. Together with (19.21), (19.22) and (19.19), it then follows that our iteration scheme for dynamic nonlinear problems becomes

$$\boxed{\begin{array}{l} \boldsymbol{a}^i = \boldsymbol{a}^{i-1} \\ -(\boldsymbol{A}(\boldsymbol{a}^{i-1})^{-1})\{\boldsymbol{M}[c_1(\boldsymbol{a}^{i-1} - \boldsymbol{a}_n) - c_2\dot{\boldsymbol{a}}_n - c_3\ddot{\boldsymbol{a}}_n] + \boldsymbol{\psi}(\boldsymbol{a}^{i-1})\} \\ i = 1, 2 \cdots \end{array}} \tag{19.23}$$

where the out-of-balance forces $\boldsymbol{\psi}$ according to (19.2) are given by

$$\boldsymbol{\psi}(\boldsymbol{a}^{i-1}) = \int_V \boldsymbol{B}^T \boldsymbol{\sigma}^{i-1} dV - \boldsymbol{f}$$

The starting values for $i = 1$ are as usual taken as the last accepted values, i.e.

$$\boldsymbol{a}_{n+1}^0 = \boldsymbol{a}_n \; ; \qquad \boldsymbol{\sigma}_{n+1}^0 = \boldsymbol{\sigma}_n$$

Moreover, the stresses $\boldsymbol{\sigma}^{i-1}$ are obtained by a numerical integration of the constitutive equations; this was discussed in the previous chapter.

It appears that all terms on the right-hand side of (19.23) are known quantities and (19.23) can then be solved to provide the new values for the nodal displacements, i.e. $\boldsymbol{a}^i$.

Convergence is obtained for $\boldsymbol{a}^{i-1} \to \boldsymbol{a}^i$ and in that case (19.23) reduces to $\boldsymbol{v}(\boldsymbol{a}^i) \to 0$ where $\boldsymbol{v}$ is given by (19.19). In reality, $\boldsymbol{v}(\boldsymbol{a}^i) = \boldsymbol{0}$ is an expression for fulfillment of the equations of motion given by (19.17). Therefore, a criterion for convergence may not only involve $\boldsymbol{a}^{i-1} \to \boldsymbol{a}^i$, but it may also express the fact that $\boldsymbol{v}^i$ is close to zero. We therefore obtain the following possible convergence criteria:

$$(\boldsymbol{a}^i - \boldsymbol{a}^{i-1})^T(\boldsymbol{a}^i - \boldsymbol{a}^{i-1}) < \alpha_1 \boldsymbol{a}_n^T \boldsymbol{a}_n \tag{19.24}$$

and/or

$$\boldsymbol{v}^T(\boldsymbol{a}^i)\boldsymbol{v}(\boldsymbol{a}^i) < \alpha_2 NORM \tag{19.25}$$

and/or

$$\begin{array}{l} (\boldsymbol{a}^i - \boldsymbol{a}^{i-1})^T(\boldsymbol{M}\ddot{\boldsymbol{a}}^i + \boldsymbol{f}_{n+1} - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}^i dV) \\ < \alpha_3(\boldsymbol{a}^1 - \boldsymbol{a}_n)^T(\boldsymbol{M}\ddot{\boldsymbol{a}}_n + \boldsymbol{f}_{n+1} - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_n dV) \end{array} \tag{19.26}$$

where α_1, α_2 and α_3 are certain *tolerances* specified by us and *NORM* may be $\|\boldsymbol{f}\|$ or $\|\boldsymbol{M}\ddot{\boldsymbol{a}}\|$.

The convergence criteria (19.24) and (19.25) possess the drawback that quantities having one dimension may be added to quantities having another dimension; for instance, the components of $\boldsymbol{a}$ may involve both displacements and

rotations and, likewise, the components of $\boldsymbol{v}$ may involve forces as well as moments. In that sense, (19.26) is more appealing since it is an energy criterion where each force component is multiplied by the corresponding displacement component and each moment component is multiplied by the corresponding rotation component; accordingly, (19.26) is a consistent criterion.

Typical values for the tolerances α_1,α_2 and α_3 have been discussed by Bathe (1996) as well as Belytschko *et al.* (2000). It appears that the tolerances are of the following order: $\alpha_1 \simeq 10^{-3}$, $\alpha_2 \simeq 10^{-3}$ and $\alpha_3 \simeq 10^{-7}$.

As previously discussed, the scheme (19.23) is implicit and it appears that it is stable, irrespective of the time step, if the parameter values in the Newmark approximation fulfill the following bounds

$$\boxed{\gamma \geq \frac{1}{2} \; ; \quad \beta \geq \frac{1}{4}(\gamma + \frac{1}{2})^2 \quad \Rightarrow \quad \textit{unconditional stability}} \tag{19.27}$$

cf. Bathe (1996) and Hughes (1983, 1987). The parameter values in the most often applied implicit Newmark scheme are taken as $\beta = 1/4$ and $\gamma = 1/2$, which according to (19.27) implies unconditional stability. Moreover following (19.4) and (19.5), this scheme is equivalent to the trapezoidal rule.

Different expressions for the iteration matrix $\boldsymbol{A}$ present in the scheme (19.23) may be chosen, but in order to obtain some kind of feeling for an appropriate choice, it is instructive to derive the Newton-Raphson approach. In that case, a Taylor series expansion is made of the quantity $\boldsymbol{v}$ entering the nonlinear equation (19.18). Making the Taylor series expansion about the state $\boldsymbol{a}^{i-1}$, we obtain

$$\boldsymbol{v}(\boldsymbol{a}^i) = \boldsymbol{v}(\boldsymbol{a}^{i-1}) + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}})^{i-1}(\boldsymbol{a}^i - \boldsymbol{a}^{i-1}) \cdots$$

Ignoring higher order terms and assuming that $\boldsymbol{a}^i$ is close to the correct value, i.e. $\boldsymbol{v}(\boldsymbol{a}^i) = 0$, imply

$$\boldsymbol{a}^i = \boldsymbol{a}^{i-1} - [(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}})^{i-1}]^{-1}\boldsymbol{v}(\boldsymbol{a}^{i-1}) \tag{19.28}$$

From (19.19) we have

$$\boldsymbol{v}(\boldsymbol{a}) = \boldsymbol{M}[c_1(\boldsymbol{a} - \boldsymbol{a}_n) - c_2\dot{\boldsymbol{a}}_n - c_3\ddot{\boldsymbol{a}}_n] + \boldsymbol{\psi}(\boldsymbol{a}) \tag{19.29}$$

i.e.

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}} = c_1\boldsymbol{M} + \int_V \boldsymbol{B}^T \frac{d\boldsymbol{\sigma}}{d\boldsymbol{a}} dV$$

Since $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_t\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}_t\boldsymbol{B}\dot{\boldsymbol{a}}$, we find that

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}} = c_1\boldsymbol{M} + \boldsymbol{K}_t \tag{19.30}$$

where

$$\boldsymbol{K}_t = \int_V \boldsymbol{B}^T \boldsymbol{D}_t \boldsymbol{B} dV$$

Insertion of (19.30) into (19.28) and use of (19.29) then yield

$$\begin{aligned} \boldsymbol{a}^i = & \boldsymbol{a}^{i-1} \\ & - [c_1 \boldsymbol{M} + \boldsymbol{K}_t^{i-1}]^{-1} \{ \boldsymbol{M}[c_1(\boldsymbol{a}^{i-1} - \boldsymbol{a}_n) - c_2 \dot{\boldsymbol{a}}_n - c_3 \ddot{\boldsymbol{a}}_n] \\ & + \boldsymbol{\psi}(\boldsymbol{a}^{i-1}) \} \end{aligned}$$

A comparison with (19.23) shows that the following choice of the iteration matrix $\boldsymbol{A}$

$$\boxed{\boldsymbol{A}^{i-1} = c_1 \boldsymbol{M} + \boldsymbol{K}_t^{i-1} \quad \Rightarrow \quad \textit{Newton-Raphson}}$$

provides the Newton-Raphson approach. However, in accordance with Chapter 17 we may adopt other choices for the iteration matrix $\boldsymbol{A}$. Since the mass matrix $\boldsymbol{M}$ is constant, we may adopt the modified Newton-Raphson method where the stiffness matrix entering $\boldsymbol{A}$ is only updated once in every time step, the initial stiffness method where the stiffness matrix entering $\boldsymbol{A}$ is taken as the elastic stiffness $\boldsymbol{K}$ or a quasi-Newton method like the BFGS-approach.

If the parameters β and γ in the Newmark scheme fulfill (19.27), the time integration is always stable. However, considering a given mode in the response, the *accuracy* of the time integration scheme decreases as the ratio $\Delta t/T$ increases; here T is the period of the actual mode considered. This means that higher order modes (short period T) are integrated with a lower accuracy than higher order modes (long period T).

As discussed in the previous section, the implicit method (19.23) is especially suited to problems with long load durations, i.e. structural dynamics problems like modeling the effect of earthquakes. In that case, the contribution of higher order modes to the response is often of minor importance. On the other hand, due to the time restriction (19.14) the explicit scheme is geared to events of short duration as, for instance, wave propagation and impact, where higher order modes are often of importance. Moreover, the explicit scheme is much simpler to program than the implicit scheme, and evaluation of new concepts is therefore conveniently performed in an explicit environment.