2 STRAIN TENSOR

As we are concerned with the behavior of deformable bodies, it is essential to establish a quantity that only describes the deformation of the body, i.e. it should not be influenced by any rigid-body motions. Such a quantity, the *strain tensor*, will be derived in the present chapter.

We will present a detailed derivation of a number of properties of the strain tensor not only because of the importance of these properties, but also because it turns out that many of the properties can be transferred directly to the stress tensor that is treated in the next chapter. For further studies of these topics, we may refer to Fung (1965), Malvern (1969), Sokolnikoff (1946) and Spencer (1980).

2.1 Introduction

In the *reference configuration* before any deformation, a material point, i.e. a *particle*, has the position vector x_i in the fixed coordinate system. After deformation and in the same coordinate system, this material point has the position vector x_i^* given by

$$\boxed{x_i^* = x_i + u_i} \tag{2.1}$$

where $u_i = u_i(x_i, t)$ is the *displacement vector* and t denotes the time.

Referring to Fig. 2.1, we consider in the reference configuration the material points P and Q which are located infinitely close to each other; due to the deformation these positions change to the positions P* and Q* respectively. According to (2.1), point P given by the x_i-vector moves to the point P* given by the vector $x_i^* = x_i + u_i(x_i, t)$. Likewise, point Q given by the vector $x_i + dx_i$ moves to the point Q* given by the vector $x_i^* + dx_i^* = x_i + dx_i + u_i(x_i + dx_i, t)$. These geometric issues are illustrated in Fig. 2.1. From (2.1) it follows directly that

$$dx_i^* = dx_i + du_i = dx_i + u_{i,j}dx_j\,; \qquad du_i = u_{i,j}dx_j \tag{2.2}$$

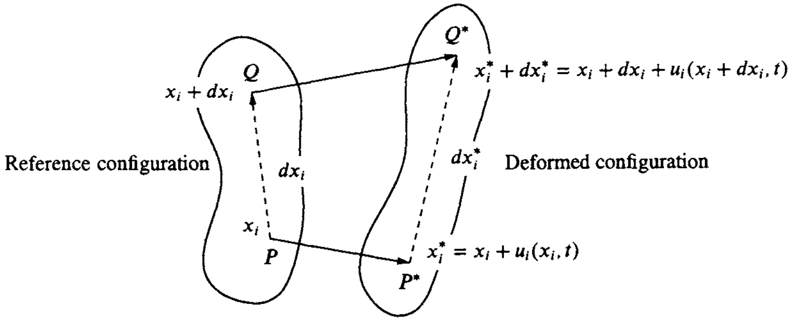

Figure 2.1: Displacements of neighboring material points P and Q

where the *displacement gradient* $u_{i,j}$ is defined by

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} \tag{2.3}$$

Equation (2.2) can also be written as

$$dx_i^* = (\delta_{ij} + u_{i,j})dx_j \tag{2.4}$$

Thus, due to the deformation the vector dx_i changes to the vector dx_i^* according to (2.4) and illustrated in Fig. 2.1.

Referring to this figure, let ds denote the length of the vector $\overline{PQ}$ and ds^* the length of the vector $\overline{P^*Q^*}$, i.e. $ds = |\overline{PQ}|$ and $ds^* = |\overline{P^*Q^*}|$. We then obtain

$$ds^2 = dx_j dx_j \; ; \qquad ds^{*^2} = dx_k^* dx_k^*$$

Using (2.4), we get

$$ds^{*^2} = (\delta_{kj} + u_{k,j})(\delta_{ki} + u_{k,i})dx_j dx_i$$

and as $dx_j dx_j = \delta_{ij} dx_i dx_j$, it appears that

$$ds^{*^2} - ds^2 = (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dx_j dx_i$$

This equation can be written as

$$\boxed{ds^{*^2} - ds^2 = 2dx_i E_{ij} dx_j} \tag{2.5}$$

where the *strain tensor* E_{ij} is defined by

$$\boxed{E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})} \tag{2.6}$$

It appears that E_{ij} is symmetric, i.e. $E_{ij} = E_{ji}$. This strain tensor was introduced by Green and St. Venant and it is often called *Green's strain tensor*.

Here we have described the displacement vector u_i as function of its position x_i before any deformations, i.e. $u_i = u_i(x_i, t)$ and such an approach is called a *Lagrangian description*. For that reason E_{ij} is often called *Lagrange's strain tensor* (occasionally, in the literature it is called the *Green-Lagrange strain tensor*; in fact, it was introduced by Green in 1841 and by St.-Venant in 1844). The alternative approach is the *Eulerian description*, often employed in fluid mechanics, where the displacement vector u_i is given as function of the current coordinates x_i^* i.e. $u_i = u_i(x_i^*, t)$.

That E_{ij} is, indeed, a second-order tensor follows from the fact that u_i is a vector and $u_{i,j}$ therefore is a second-order tensor, cf. (1.47).

2.2 Small strain tensor

In the following, we will only consider situations where the displacement gradients $u_{i,j}$ are small, i.e. each component is small when compared to unity

$$\boxed{|u_{i,j}| \ll 1}$$

In that case, the quadratic term in (2.6) can be ignored and the Lagrange strain E_{ij} can be approximated by the *small strain tensor* ε_{ij} defined by

$$\boxed{\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})} \tag{2.7}$$

which is also symmetric, i.e.

$$\boxed{\varepsilon_{ij} = \varepsilon_{ji}}$$

It is obvious that ε_{ij} is a second-order tensor.

2.3 Rigid-body motions

Our aim was to establish a quantity, the strain tensor, that is independent of rigid-body motions. Let us now prove that ε_{ij} possesses this property.

Any rigid-body motion is characterized by the fact that during motion, the vector $\overline{PQ}$ of two neighboring material points, cf. Fig. 2.1, changes into the vector $\overline{P^*Q^*}$ in such a way that its length remains constant. As $ds = |\overline{PQ}|$ and $ds^* = |\overline{P^*Q^*}|$, we can then write that during any rigid-body motion, we have

$$ds^{*2} - ds^2 = 0$$

Making use of (2.5) and (2.6) and noting that dx_i is arbitrary, we conclude that

$$2E_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} = 0$$

We observe that Green-Lagrange's strain tensor is unaffected by rigid-body motions and within our approximation of small displacement gradients we have $E_{ij} = \varepsilon_{ij}$, i.e. rigid-body motions do not influence the small strain tensor, which proves the desired property of this strain tensor.

2.4 Physical significance of the strain tensor

We shall now evaluate the physical significance of the strain tensor ε_{ij} and its components. Within our assumption of small displacement gradients, we have $E_{ij} = \varepsilon_{ij}$, i.e. (2.5) reads

$$\frac{ds^{*2} - ds^2}{ds^2} = 2\frac{dx_i}{ds}\varepsilon_{ij}\frac{dx_j}{ds} \tag{2.8}$$

where it is recalled from Fig. 2.1 that ds is the length of the vector dx_i between the two neighboring particles P and Q before any deformation takes place and ds^* is the distance between these two particles after the deformation. Therefore

$$n_i = \frac{dx_i}{ds}$$

is a unit vector in the direction of dx_i. From this expression and (2.8) follow that

$$\frac{ds^{*2} - ds^2}{2ds^2} = n_i\varepsilon_{ij}n_j \tag{2.9}$$

As the displacement gradients are small the components of ε_{ij} are also small and this implies that the left-hand side of (2.9) is small. Consequently ds^* is close to ds and we then obtain

$$\frac{ds^{*2} - ds^2}{2ds^2} = \frac{(ds^* + ds)(ds^* - ds)}{2ds^2} \approx \frac{2ds(ds^* - ds)}{2ds^2} = \frac{ds^* - ds}{ds}$$

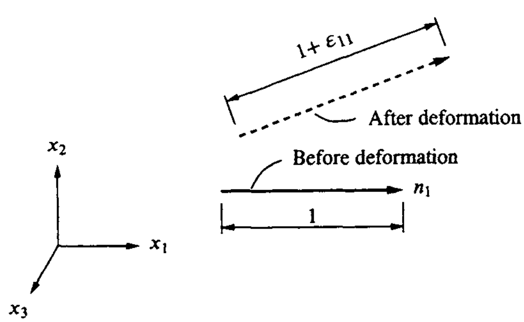

Figure 2.2: Illustration of strain component ε_{11}

We define the *relative elongation* or the *normal strain* ε of the vector $\overline{PQ}$ deforming into the $\overline{P^*Q^*}$-vector, cf. Fig. 2.1, by

$$\boxed{\varepsilon = \frac{ds^* - ds}{ds}} \tag{2.10}$$

in accordance with the elementary definition of normal strain. A combination of (2.9) - (2.10) yields

$$\boxed{\varepsilon = n_i \varepsilon_{ij} n_j \qquad or \qquad \varepsilon = \boldsymbol{n}^T \boldsymbol{\varepsilon}\, \boldsymbol{n}} \tag{2.11}$$

As an example, choose the direction n_i so that $n_i = (1, 0, 0)$, then we obtain $\varepsilon = \varepsilon_{11}$ as illustrated in Fig. 2.2. Likewise, for $n_i = (0, 1, 0)$, we obtain $\varepsilon = \varepsilon_{22}$ whereas $n_i = (0, 0, 1)$ yields $\varepsilon = \varepsilon_{33}$. Therefore, we have achieved a physical interpretation of all the diagonal terms of the strain tensor and it appears from (2.11) that the normal strain, i.e. the relative elongation, in an arbitrary direction given by the unit vector n_i, is known once the strain tensor is known.

To obtain a physical interpretation of the off-diagonal terms in the strain tensor, consider two directions $dx_i^{(1)}$ and $dx_i^{(2)}$ in the reference configuration before any deformations take place. These two directions are taken to be orthogonal, i.e.

$$dx_i^{(1)}\, dx_i^{(2)} = 0 \tag{2.12}$$

In accordance with Fig. 2.3, the lengths of $dx_i^{(1)}$ and $dx_i^{(2)}$ are given by $ds^{(1)}$

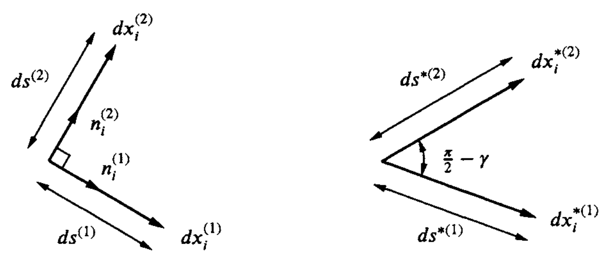

Figure 2.3: Change of orthogonal angle in reference configuration due to the deformation.

and $ds^{(2)}$ respectively, i.e. we have the following two orthogonal unit vectors

$$n_i^{(1)} = \frac{dx_i^{(1)}}{ds^{(1)}} \quad ; \qquad n_i^{(2)} = \frac{dx_i^{(2)}}{ds^{(2)}} \tag{2.13}$$

Due to the deformation, the vector $dx_i^{(1)}$ changes to $dx_i^{*(1)}$ with length $ds^{*(1)}$ whereas the vector $dx_i^{(2)}$ changes to $dx_i^{*(2)}$ with length $ds^{*(2)}$, cf. Fig. 2.3. The angle $90^o - \gamma$ between $dx^{*(1)}$ and $dx^{*(2)}$ is then given by

$$\cos(90^o - \gamma) = \frac{dx_i^{*(1)}}{ds^{*(1)}} \frac{dx_i^{*(2)}}{ds^{*(2)}} \tag{2.14}$$

From (2.4), we have

$$dx_i^{*(1)} = (\delta_{ij} + u_{i,j})dx_j^{(1)}\, ; \qquad dx_i^{*(2)} = (\delta_{ik} + u_{i,k})dx_k^{(2)}$$

Insertion into (2.14) yields

$$\sin\gamma = (\delta_{jk} + u_{k,j} + u_{j,k} + u_{i,j}u_{i,k})\frac{dx_j^{(1)}}{ds^{*(1)}}\frac{dx_k^{(2)}}{ds^{*(2)}} \tag{2.15}$$

Due to the small strain approximation, we can ignore the quadratic term and set $ds^{*(1)} \approx ds^{(1)}$ and $ds^{*(2)} \approx ds^{(2)}$. Consequently, (2.15) reduces with (2.7) to

$$\sin\gamma = \frac{dx_k^{(1)}}{ds^{(1)}}\frac{dx_k^{(2)}}{ds^{(2)}} + 2\varepsilon_{jk}\frac{dx_j^{(1)}}{ds^{(1)}}\frac{dx_k^{(2)}}{ds^{(2)}}$$

As we assume small strains we have $\sin\gamma \approx \gamma$. With (2.12) and (2.13) we then obtain

$$\gamma = 2\varepsilon_{jk}\, n_j^{(1)}\, n_k^{(2)}$$

To emphasize that the vectors $n_i^{(1)}$ and $n_i^{(2)}$ are orthogonal, we write $m_i = n_i^{(1)}$ and $n_i = n_i^{(2)}$ and the expression above takes the more convenient form

$$\gamma = 2m_i\varepsilon_{ij}n_j \tag{2.16}$$

Hence, due to the deformation the right angle between the unit vectors n_i and m_i in the reference configuration decreases by the amount γ given by (2.16).

As an example, choose $n_i = (1, 0, 0)$ then $\varepsilon_{ij}n_j$ becomes $\varepsilon_{ij}n_j = \varepsilon_{i1}$. If we then choose $m_i = (0, 1, 0)$, we obtain $\gamma = 2\varepsilon_{21}$ and if we choose $m_i = (0, 0, 1)$, we obtain $\gamma = 2\varepsilon_{31}$. I.e. $2\varepsilon_{21}$ is the decrease of the angle between the x_2- and x_1-axes due to deformation, whereas $2\varepsilon_{31}$ is the decrease of the angle between the x_3- and x_1-axes. A similar evaluation holds for $2\varepsilon_{23}$. These off-diagonal terms of the strain tensor are called *shear strains* as they describe the shearing, i.e. the distortion of the material. With obvious notation we can then write

$$\boxed{\gamma_{nm} = 2\varepsilon_{nm}}$$

where

$$\boxed{\varepsilon_{nm} = n_i\varepsilon_{ij}m_j = m_i\varepsilon_{ij}n_j \quad or \quad \varepsilon_{nm} = \boldsymbol{n}^T\boldsymbol{\varepsilon}\,\boldsymbol{m} = \boldsymbol{m}^T\boldsymbol{\varepsilon}\,\boldsymbol{n}} \tag{2.17}$$

In this expression n_i and m_i are arbitrary unit vectors, which are orthogonal in the reference configuration. The angle decrease γ_{nm} between n_i and m_i caused by the deformation is termed the *engineering shear strain* to be distinguished from the tensorial shear strain ε_{nm}. The shearing between two directions parallel with the x_1- and x_2-axes is illustrated in Fig. 2.4.

It appears that the strain tensor contains information by which relative elongation in arbitrary directions and angle changes between arbitrary orthogonal directions can be determined. Consequently, the strain tensor describes the deformation completely and, in addition, we have achieved a direct physical interpretation of all the components of this tensor. These results were already obtained by Cauchy in 1822.

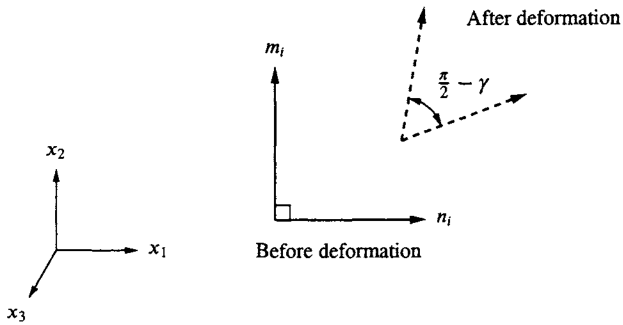

Figure 2.4: Illustration of shear component $\varepsilon_{12} = \gamma_{12}/2$

2.5 Change of coordinate system

The implications of coordinate system changes are important in many connections and we have already discussed this aspect in detail in Chapter 1.

Let us consider the change from the old x_i-coordinate system to the new x'_i-coordinate system. From (1.17) we have

$$x'_i = A_{ij}(x_j - c_i) \qquad \text{or} \qquad \boldsymbol{x}' = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{c}) \tag{2.18}$$

where A_{ij} is the transformation matrix. Suppose that we know the components of ε_{ij} in the x_i-system and suppose that we want to determine the components of ε'_{ij} in the x'_i-system. We have already proved that ε_{ij} is a second-order tensor, i.e. it follows directly from (1.41) that

$$\boxed{\varepsilon'_{ij} = A_{ik}\varepsilon_{kl}A_{jl} \qquad or \qquad \boldsymbol{\varepsilon}' = \boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T} \tag{2.19}$$

The inverse relations follow from (1.42), i.e.

$$\varepsilon_{ij} = A_{ki}\varepsilon'_{kl}A_{lj} \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \boldsymbol{A}^T\boldsymbol{\varepsilon}'\boldsymbol{A} \tag{2.20}$$

2.6 Principal strains and principal directions - Invariants

We have previously obtained a physical interpretation of the strain tensor components. However, it turns out that for a special choice of coordinate system, the strain tensor takes a particularly simple form.

For this purpose, consider a direction in the reference configuration given by the unit vector $\boldsymbol{n}$. We then define the vector $\boldsymbol{q}$ by

$$\boldsymbol{q} = \boldsymbol{\varepsilon}\boldsymbol{n} \tag{2.21}$$

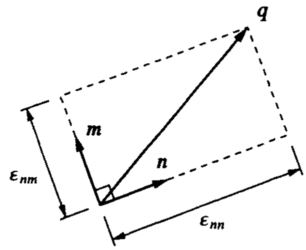

Figure 2.5: The vector $\boldsymbol{q} = \boldsymbol{\varepsilon}\boldsymbol{n}$ and its components after direction $\boldsymbol{n}$ and $\boldsymbol{m}$

Referring to Fig. 2.5, the unit vector $\boldsymbol{m}$ is orthogonal to $\boldsymbol{n}$. Following Fig. 2.5 and in accordance with (2.11) and (2.21), the component of $\boldsymbol{q}$ in the direction of $\boldsymbol{n}$ is given by

$$\varepsilon_{nn} = \boldsymbol{n}^T\boldsymbol{q} \tag{2.22}$$

where ε_{nn} is the normal strain in the direction $\boldsymbol{n}$. Likewise from (2.17) and (2.21), the component of $\boldsymbol{q}$ in the direction of $\boldsymbol{m}$ is given by

$$\varepsilon_{nm} = \boldsymbol{m}^T\boldsymbol{q}$$

where ε_{nm} is the shear strain between the directions $\boldsymbol{n}$ and $\boldsymbol{m}$.

We now look for the situation where the direction $\boldsymbol{n}$ is chosen so that $\boldsymbol{q}$ is collinear with $\boldsymbol{n}$, i.e. the shear strain $\varepsilon_{nm} = 0$. To achieve this situation, we must have

$$\boldsymbol{q} = \lambda\boldsymbol{n} \tag{2.23}$$

where λ is an unknown parameter and from (2.22) we conclude that $\varepsilon_{nn} = \lambda$. Use of (2.21) in (2.23) yields the following requirement

$$\boxed{(\boldsymbol{\varepsilon} - \lambda\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0} \qquad or \qquad (\varepsilon_{ij} - \lambda\delta_{ij})n_j = 0} \tag{2.24}$$

where $\boldsymbol{0}$ is defined as $\boldsymbol{0}^T = [0\ 0\ 0]$.

Expression (2.24) is an example of the well-known *eigenvalue problem.* It consists of a quadratic set of homogeneous equations and if a nontrivial solution $\boldsymbol{n}$ is to exist, we must require

$$\det(\boldsymbol{\varepsilon} - \lambda\boldsymbol{I}) = 0 \tag{2.25}$$

As $\boldsymbol{\varepsilon} - \lambda\boldsymbol{I}$ is a 3 x 3 matrix, the expression above provides a cubic equation for the determination of λ - the so-called *characteristic equation.* That is, (2.25) is fulfilled by three values of λ - the *eigenvalues* λ_1, λ_2 and λ_3. When λ_1, λ_2 and λ_3 have been determined, then substitution of λ_1 in (2.24) provides the solution $\boldsymbol{n}_1$, substitution of λ_2 provides the solution $\boldsymbol{n}_2$ and substitution of λ_3 yields the solution $\boldsymbol{n}_3$. The solutions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are the *eigenvectors*. In accordance with the theory of homogeneous equations the lengths of the eigenvectors will be undetermined whereas the direction will be known. Accordingly, it is always possible to choose a solution so that $\boldsymbol{n}$ becomes a unit vector and this situation will be assumed in the following. In the present context, the λ-values are most frequently called the *principal strains*, whereas the $\boldsymbol{n}$-vectors are called the *principal strain directions*.

The importance of the λ-values comes from the fact that they are *invariants*, i.e. they take the same values irrespective of the coordinate system. From a physical point of view, this is rather obvious as the magnitude of a principal strain λ was found above to be given by the relative elongation ε_{nn} in the fixed direction $\boldsymbol{n}$ and this relative elongation must be independent of the coordinate system chosen. To prove this formally, assume that we change the coordinate system from the old x_i-system to the new x_i'-system in accordance with (2.18). Following (2.24), the principal directions and principal strains in the new coordinate system are determined by

$$\boldsymbol{\varepsilon}'\boldsymbol{n}' = \lambda'\boldsymbol{n}' \tag{2.26}$$

where λ' denotes the principal strain in the new coordinate system. Since $\boldsymbol{n}$ is a vector, we have from (1.31) that

$$\boldsymbol{n}' = \boldsymbol{A}\boldsymbol{n}$$

Use of this expression and (2.19) in (2.26) yields

$$\boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{n} = \lambda'\boldsymbol{A}\boldsymbol{n}$$

Premultiplication by $\boldsymbol{A}^T$ and using that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, we find

$$\boldsymbol{\varepsilon}\boldsymbol{n} = \lambda'\boldsymbol{n}$$

and a comparison with (2.24) proves that $\lambda = \lambda'$ implying that the λ-values are invariants, i.e. independent of the coordinate system. However, since the components of the eigenvector $\boldsymbol{n}'$ are now measured in the new x_i'-coordinate system, these components differ from the components of the eigenvector $\boldsymbol{n}$.

Evaluation of the cubic equation (2.25) – the characteristic equation – gives after some algebra

$$\boxed{-\lambda^3 + \theta_1\lambda^2 - \theta_2\lambda + \theta_3 = 0} \tag{2.27}$$

where θ_1, θ_2 and θ_3 are defined by

$$\begin{aligned}
\theta_1 &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii} \\
\theta_2 &= \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{33} - \varepsilon_{23}^2 - \varepsilon_{12}^2 - \varepsilon_{13}^2 = \tfrac{1}{2}\theta_1^2 - \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ji} \\
\theta_3 &= \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - \varepsilon_{11}\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{13}^2 - \varepsilon_{33}\varepsilon_{12}^2 + 2\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} = \det(\varepsilon_{ij})
\end{aligned} \tag{2.28}$$

As the λ-values are invariants determined by the values of θ_1, θ_2 and θ_3 it is obvious that also the θ_1-, θ_2- and θ_3-values are invariants. They are called the *Cauchy-strain invariants* and any combination of these invariants is also an invariant.

An important issue is that the eigenvectors are orthogonal and that the eigenvalues are real; this is a consequence of the matrix $\boldsymbol{\varepsilon}$ being real and symmetric and it is a well-known result in mathematics. However, we will take the opportunity to prove it here.

To prove that the eigenvectors are orthogonal, assume that we have determined the two eigenvalues λ_1 and λ_2 and the corresponding two eigenvectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. We then have

$$\boldsymbol{\varepsilon n}_1 = \lambda_1 \boldsymbol{n}_1$$

$$\boldsymbol{\varepsilon n}_2 = \lambda_2 \boldsymbol{n}_2 \tag{2.29}$$

Transpose the first equation, utilize that $\boldsymbol{\varepsilon}$ is symmetric and postmultiply it by $\boldsymbol{n}_2$ to obtain

$$\boldsymbol{n}_1^T \boldsymbol{\varepsilon n}_2 = \lambda_1 \boldsymbol{n}_1^T \boldsymbol{n}_2 \tag{2.30}$$

Premultiply (2.29) by $\boldsymbol{n}_1^T$ to obtain

$$\boldsymbol{n}_1^T \boldsymbol{\varepsilon n}_2 = \lambda_2 \boldsymbol{n}_1^T \boldsymbol{n}_2 \tag{2.31}$$

Subtraction of (2.30) and (2.31) yields

$$(\lambda_1 - \lambda_2)\boldsymbol{n}_1^T \boldsymbol{n}_2 = 0$$

If we assume that $\lambda_1 \neq \lambda_2$ then it follows that $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ must be orthogonal. Similar arguments hold between $\boldsymbol{n}_1$ and $\boldsymbol{n}_3$ and between $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$, i.e. we obtain the following fundamental property

$$\boxed{\boldsymbol{n}_1^T \boldsymbol{n}_2 = \boldsymbol{n}_1^T \boldsymbol{n}_3 = \boldsymbol{n}_2^T \boldsymbol{n}_3 = 0 \quad \textit{orthogonality of eigenvectors}} \tag{2.32}$$

When proving this orthogonality, it was assumed that the principal strains were unequal. What happens if some of them are equal? Suppose that in a certain coordinate system, we have the following strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = a\boldsymbol{I}$$

It is obvious that in this coordinate system the principal strains are all equal and given by the quantity a. Suppose now that the coordinate system is changed from the present x_i-system to the new x_i'-system in accordance with (2.18). In this new x_i'-system, the strain tensor transforms into the one given by (2.19), i.e.

$$\boldsymbol{\varepsilon}' = \boldsymbol{A\varepsilon A}^T = a\boldsymbol{AIA}^T = a\boldsymbol{AA}^T = a\boldsymbol{I} = \boldsymbol{\varepsilon}$$

Consequently, we have proved that if all three principal strains are equal, then any coordinate system corresponds to the principal directions.

Suppose now that in a certain coordinate system, we have the following strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} = b\boldsymbol{I} + \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. two of the principal strains are equal. Suppose furthermore that we rotate the coordinate system according to (2.18). However, we will make the special choice that this rotation consists of a rotation about the x_1-axis. This implies that $\boldsymbol{e}_1'^T = [1\ 0\ 0]$, cf. Fig. 1.3. According to (2.19) and (1.28), the strain tensor in the new x_i'-system becomes

$$\begin{aligned} \boldsymbol{\varepsilon}' &= b\boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^T + \boldsymbol{A} \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{A}^T \\ &= b\boldsymbol{I} + \begin{bmatrix} \boldsymbol{e}_1'^T \\ \boldsymbol{e}_2'^T \\ \boldsymbol{e}_3'^T \end{bmatrix} \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\, \boldsymbol{e}_1'\ \boldsymbol{e}_2'\ \boldsymbol{e}_3' \,\right] \end{aligned}$$

As we only consider a rotation about the x_1-axis, i.e. $\boldsymbol{e}_1'^T = [1\ 0\ 0]$, we obtain

$$\boldsymbol{\varepsilon}' = b\boldsymbol{I} + \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\, \boldsymbol{e}_1'\ \boldsymbol{e}_2'\ \boldsymbol{e}_3' \,\right]$$

i.e.

$$\boldsymbol{\varepsilon}' = b\boldsymbol{I} + \begin{bmatrix} a-b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{\varepsilon}$$

Consequently, we have proved that if two of the principal strains are equal, then any coordinate system obtained by rotation about that axis, which corresponds to the principal strain different from the other principal strains, corresponds to the principal directions.

In conclusion, we find that it is always allowable to take the principal directions as orthogonal directions in accordance with (2.32).

Remembering the physical interpretation of λ, cf. the discussion of (2.23), it is evident that the λ-values must be real. However, a formal proof is readily achieved. For the eigenvalue λ and the corresponding eigenvector $\boldsymbol{n}$, $\boldsymbol{\varepsilon n} = \lambda \boldsymbol{n}$ holds. Take the complex conjugate of this equation to obtain

$$\boldsymbol{\varepsilon n}^* = \lambda^* \boldsymbol{n}^* \tag{2.33}$$

where an asterisk * for the time being denotes the complex conjugate and where it has been used that ε is real and that $(\lambda \boldsymbol{n})^* = \lambda^* \boldsymbol{n}^*$. Premultiplying $\boldsymbol{\varepsilon n} = \lambda \boldsymbol{n}$ by $\boldsymbol{n}^{*T}$ gives

$$\boldsymbol{n}^{*T} \boldsymbol{\varepsilon n} = \lambda \boldsymbol{n}^{*T} \boldsymbol{n} \tag{2.34}$$

whereas transposing (2.33), utilizing the symmetry of $\boldsymbol{\varepsilon}$ and postmultiplying by $\boldsymbol{n}$ provides

$$\boldsymbol{n}^{*T} \boldsymbol{\varepsilon n} = \lambda^* \boldsymbol{n}^{*T} \boldsymbol{n} \tag{2.35}$$

Then, finally, subtraction of (2.34) and (2.35) yields

$$(\lambda - \lambda^*) \boldsymbol{n}^{*T} \boldsymbol{n} = 0$$

However, $\boldsymbol{n}^{*T} \boldsymbol{n}$ is certainly different from zero implying that $\lambda = \lambda^*$ and it has then been proved that the eigenvalues are real. It follows immediately that also the eigenvectors are real, i.e.

$\boxed{\textit{The eigenvalues and eigenvectors are real}}$

We are now in a position to illustrate a significant feature related to the eigenvalues and eigenvectors. As $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are orthogonal, we can change our coordinate system from the x_i-system to a x'_i-system collinear with the $\boldsymbol{n}_1$-, $\boldsymbol{n}_2$– and $\boldsymbol{n}_3$- directions. Following (2.18) and (1.28), we then have

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{c} \qquad \text{where} \qquad \boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix}$$

In this new x'_i-system the strain tensor becomes, cf. (2.19)

$$\boldsymbol{\varepsilon}' = \begin{bmatrix} \boldsymbol{n}_1^T \\ \boldsymbol{n}_2^T \\ \boldsymbol{n}_3^T \end{bmatrix} \boldsymbol{\varepsilon} \begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{n}_1^T \\ \boldsymbol{n}_2^T \\ \boldsymbol{n}_3^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon n}_1 & \boldsymbol{\varepsilon n}_2 & \boldsymbol{\varepsilon n}_3 \end{bmatrix}$$

Using that $\boldsymbol{\varepsilon n}_1 = \lambda_1 \boldsymbol{n}_1$ and the similar relations, cf. (2.24), we obtain

$$\boldsymbol{\varepsilon}' = \begin{bmatrix} \boldsymbol{n}_1^T \\ \boldsymbol{n}_2^T \\ \boldsymbol{n}_3^T \end{bmatrix} \begin{bmatrix} \lambda_1 \boldsymbol{n}_1 & \lambda_2 \boldsymbol{n}_2 & \lambda_3 \boldsymbol{n}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \boldsymbol{n}_1^T \boldsymbol{n}_1 & \lambda_2 \boldsymbol{n}_1^T \boldsymbol{n}_2 & \lambda_3 \boldsymbol{n}_1^T \boldsymbol{n}_3 \\ \lambda_1 \boldsymbol{n}_2^T \boldsymbol{n}_1 & \lambda_2 \boldsymbol{n}_2^T \boldsymbol{n}_2 & \lambda_3 \boldsymbol{n}_2^T \boldsymbol{n}_3 \\ \lambda_1 \boldsymbol{n}_3^T \boldsymbol{n}_1 & \lambda_2 \boldsymbol{n}_3^T \boldsymbol{n}_2 & \lambda_3 \boldsymbol{n}_3^T \boldsymbol{n}_3 \end{bmatrix}$$

However, as the $\boldsymbol{n}$-vectors are unit vectors orthogonal to each other we finally obtain

$$\varepsilon' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.36}$$

Accordingly, we have obtained the important result that if the coordinate system is chosen collinearly with the principal directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$, then the strain tensor becomes diagonal and the normal strains become equal to λ_1, λ_2 and λ_3. This result is in accordance with the physical conditions, which were specified in the beginning when the eigenvalue problem was formulated. This important result also illustrates why the eigenvalues are called the principal strains and the eigenvectors the principal directions. The principal strains are often denoted by ε_1, ε_2 and ε_3, i.e. $\varepsilon_1 = \lambda_1$, $\varepsilon_2 = \lambda_2$ and $\varepsilon_3 = \lambda_3$.

The above result can be summarized by stating that if the coordinate system is collinear with the principal directions we have in accordance with (2.19) and (2.36) that

$$\boxed{\varepsilon' = \boldsymbol{A}\varepsilon\boldsymbol{A}^T = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \quad \textit{for} \quad \boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix}} \tag{2.37}$$

2.7 Extremum values of the normal strain

The normal strain ε in any direction n_i is determined by (2.11), i.e.

$$\varepsilon = n_i \varepsilon_{ij} n_j$$

For different directions of n_i, different ε-values are achieved. It will now be proved that the normal strain ε takes stationary values, i.e. maximum or minimum values, when the direction n_i is in the direction of one of the principal axes.

To find the stationary values of ε, the n_i-vector is varied. However, the n_i-components cannot be varied arbitrarily, as we have the constraint

$$n_i n_i - 1 = 0$$

Accordingly, we employ the *method of Lagrange* and find stationary values of the function

$$\psi = n_i \varepsilon_{ij} n_j - \alpha(n_i n_i - 1) \tag{2.38}$$

where now the n_i-components and α are independent quantities, α being a Lagrangian multiplier. From (2.38), where $\psi = \psi(n_i, \alpha)$ we obtain

$$\frac{\partial \psi}{\partial n_k} = \varepsilon_{kj} n_j + \varepsilon_{ki} n_i - \alpha(n_k + n_k) = 0 \tag{2.39}$$

and

$$\frac{\partial \psi}{\partial \alpha} = n_i n_i - 1 = 0 \tag{2.40}$$

Equation (2.39) can be written as

$$\varepsilon_{kj} n_j - \alpha n_k = 0 \qquad \text{or} \qquad (\varepsilon_{ij} - \alpha \delta_{ij}) n_j = 0 \tag{2.41}$$

Therefore, stationary values for the normal strain ε are obtained by solution of the homogeneous equation system (2.41) subject to the condition (2.40). We immediately observe that this is exactly the same eigenvalue problem as stated by (2.24) proving that stationary values, i.e. maximum and minimum values, of the normal strain ε occur in the principal directions.

2.8 Cayley-Hamilton's theorem

We will now prove an interesting relation for the strain tensor (occasionally also called the strain matrix).

Considering the eigenvalue problem (2.24), we premultiply this equation by $\boldsymbol{\varepsilon}$, i.e.

$$\boldsymbol{\varepsilon}^2 \boldsymbol{n} = \lambda \boldsymbol{\varepsilon} \boldsymbol{n} = \lambda^2 \boldsymbol{n}$$

where the notation

$$\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon} \tag{2.42}$$

has been used. Proceeding, we obtain the general result

$$\boldsymbol{\varepsilon}^\alpha \boldsymbol{n} = \lambda^\alpha \boldsymbol{n}\,; \qquad \alpha = 0, \pm 1, \pm 2 \ldots \tag{2.43}$$

where α is any integer (positive, negative or zero). If α is negative, say α = -2 then, in accordance with (2.42), we define

$$\boldsymbol{\varepsilon}^{-2} = \boldsymbol{\varepsilon}^{-1} \boldsymbol{\varepsilon}^{-1}$$

Hence, (2.43) holds even for negative values of the integer α provided that $\boldsymbol{\varepsilon}^{-1}$ exists i.e. provided that det $\boldsymbol{\varepsilon} \neq 0$. Moreover, in accordance with the usual definition that $x^0 = 1$ we make the following definition

$$\boldsymbol{\varepsilon}^0 = \boldsymbol{I}$$

From this definition follows that (2.43) holds even when $\alpha = 0$.

Equation (2.43) shows that if $\boldsymbol{\varepsilon}$ has the eigenvalue λ and eigenvector $\boldsymbol{n}$, then $\boldsymbol{\varepsilon}^\alpha$ will have the same eigenvector and the eigenvalue λ^α. Now, multiply the characteristic equation for λ, as given by (2.27), by $\boldsymbol{n}$ to obtain

$$-\lambda^3 \boldsymbol{n} + \theta_1 \lambda^2 \boldsymbol{n} - \theta_2 \lambda \boldsymbol{n} + \theta_3 \boldsymbol{n} = \boldsymbol{0} \tag{2.44}$$

where $\mathbf{0}$ is given by $\mathbf{0}^T = [0\ 0\ 0]$. Use of (2.43) in (2.44) gives

$$(-\boldsymbol{\varepsilon}^3 + \theta_1 \boldsymbol{\varepsilon}^2 - \theta_2 \boldsymbol{\varepsilon} + \theta_3 \boldsymbol{I})\boldsymbol{n} = \mathbf{0}$$

We know that this equation is fulfilled for $\boldsymbol{n}$ given by any of the three eigenvectors, i.e. these three matrix equations can be combined into the following format

$$(-\boldsymbol{\varepsilon}^3 + \theta_1 \boldsymbol{\varepsilon}^2 - \theta_2 \boldsymbol{\varepsilon} + \theta_3 \boldsymbol{I}) \begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix} = \mathbf{0} \tag{2.45}$$

where $\mathbf{0}$ now denotes the 3 x 3 null matrix. As the unit vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are orthogonal, we have according to (1.28) that

$$\begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix} = \boldsymbol{A}^T$$

where $\boldsymbol{A}$ is some transformation matrix. Expression (2.45) therefore takes the form

$$(-\boldsymbol{\varepsilon}^3 + \theta_1 \boldsymbol{\varepsilon}^2 - \theta_2 \boldsymbol{\varepsilon} + \theta_3 \boldsymbol{I})\boldsymbol{A}^T = \mathbf{0}$$

Postmultiplication by $\boldsymbol{A}$ and noting that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$ give

$$\boxed{-\boldsymbol{\varepsilon}^3 + \theta_1 \boldsymbol{\varepsilon}^2 - \theta_2 \boldsymbol{\varepsilon} + \theta_3 \boldsymbol{I} = \mathbf{0}} \tag{2.46}$$

This equation is similar to the characteristic equation for λ, cf. (2.27) and the result is thus often stated by saying that

The strain matrix satisfies its own characteristic equation

This important result is the *Cayley-Hamilton theorem.* Note that (2.46) is a matrix equation.

A significant implication of (2.46) is that an expression involving the term $\boldsymbol{\varepsilon}^3$ can always be simplified so that it only involves terms of $\boldsymbol{\varepsilon}^2$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{I}$. More generally, if we multiply (2.46) by $\boldsymbol{\varepsilon}^\alpha$, where α is any integer (positive, negative or zero), we obtain

$$\boxed{\boldsymbol{\varepsilon}^{3+\alpha} = \theta_1 \boldsymbol{\varepsilon}^{2+\alpha} - \theta_2 \boldsymbol{\varepsilon}^{1+\alpha} + \theta_3 \boldsymbol{\varepsilon}^\alpha}$$

If $\alpha \geq 0$ this means that any $\boldsymbol{\varepsilon}^{3+\alpha}$-term can be replaced by lower order powers of $\boldsymbol{\varepsilon}$. If $\alpha \leq 0$ (which presumes that $\boldsymbol{\varepsilon}^{-1}$ exists), then any $\boldsymbol{\varepsilon}^\alpha$-term can be replaced by higher order powers of $\boldsymbol{\varepsilon}$. Such manipulations are of importance in the so-called *representation theorems*, to be discussed later in Chapter 6.

2.9 Deviatoric strains

Instead of the full strain tensor, it is often convenient to operate with the so-called *deviatoric strain tensor* e_{ij} defined by

$$\boxed{e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}} \tag{2.47}$$

where $\frac{1}{3}\ \varepsilon_{kk}\ \delta_{ij}$ is the *volumetric* or *spherical strain tensor*, which only involves diagonal terms. As both ε_{ij} and δ_{ij} are second-order tensors, it follows directly that so is e_{ij}. Therefore, by analogy with (2.19) and (2.20) we have

$$\boxed{e'_{ij} = A_{ik}e_{kl}A_{jl} \qquad or \qquad e' = \boldsymbol{A}\boldsymbol{e}\boldsymbol{A}^T}$$

and

$$e_{ij} = A_{ki}e'_{kl}A_{lj} \qquad \text{or} \qquad \boldsymbol{e} = \boldsymbol{A}^T\boldsymbol{e}'\boldsymbol{A}$$

Moreover, we observe from definition (2.47) that

$$\boxed{e_{ii} = 0} \tag{2.48}$$

In a principal coordinate system the principal strains become ε_1, ε_2 and ε_3. Referring to (2.10), this means that the relative volume change due to the deformation becomes

$$\frac{dV^* - dV}{dV} = \frac{(1+\varepsilon_1)dx_1(1+\varepsilon_2)dx_2(1+\varepsilon_3)dx_3 - dx_1dx_2dx_3}{dx_1dx_2dx_3}$$

where dV is the infinitesimal volume before deformation, which owing to the deformation changes to dV^*. In accordance with our assumption of small strains, we ignore higher order strain terms and the expression above becomes

$$\boxed{\frac{dV^* - dV}{dV} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_{kk}} \tag{2.49}$$

We conclude that ε_{kk} is equal to the relative volume change, i.e. an incompressible material is characterized by ε_{kk}=0. Moreover, it may be recalled that ε_{kk} is an invariant.

Referring to (2.47), it appears that the off-diagonal terms of e_{ij} and ε_{ij} are identical. Consequently, it can be concluded that the volumetric strain tensor only influences the volumetric changes whereas the deviatoric strain tensor only influences the shearing (distortion) of the material.

Returning to the eigenvalue problem (2.24), we may eliminate ε_{ij} by means of (2.47) to obtain.

$$\left[\boldsymbol{e} - (\lambda - \frac{\varepsilon_{kk}}{3})\boldsymbol{I}\right]\boldsymbol{n} = \boldsymbol{0}$$

It is concluded that the eigenvalues of e_{ij} are given by λ - $\varepsilon_{kk}/3$ whereas the eigenvectors, i.e. the principal directions, are identical for the deviatoric strain tensor and the strain tensor. The fact that the principal directions of e_{ij} and ε_{ij} are identical follows also directly from the observation that they have identical off-diagonal terms, i.e. when ε_{ij} is diagonal, so is e_{ij}.

2.10 Important strain invariants

We have seen quite a number of different invariants and it might be convenient to summarize these invariants and make use of the opportunity to introduce additional invariants which later turn out to be of importance.

The Cauchy invariants are given by (2.28)

$$\boxed{\theta_1 = \varepsilon_{ii} \quad ; \quad \theta_2 = \frac{1}{2}\theta_1^2 - \frac{1}{2}\varepsilon_{ij}\varepsilon_{ji} \quad ; \quad \theta_3 = \det(\varepsilon_{ij}) = \varepsilon_1\varepsilon_2\varepsilon_3} \qquad (2.50)$$

In general, to prove that a quantity is an invariant, we must demonstrate that it takes the same value in all coordinate systems. As a prototype of such an evaluation we consider

$$\varepsilon'_{ij}\varepsilon'_{ij} = A_{ik}\varepsilon_{kl}A_{jl}A_{is}\varepsilon_{st}A_{jt} = \delta_{ks}\varepsilon_{kl}\varepsilon_{st}\delta_{lt} = \varepsilon_{sl}\varepsilon_{sl}$$

where advantage is taken of the transformation rule (2.19) as well as of (1.21). This demonstrates that the quantity $\varepsilon_{ij}\varepsilon_{ij}$ is an invariant. Likewise, it is easily shown that ε_{ii} and $\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$ are invariants. We can therefore list the following so-called *generic invariants* , where the term 'generic' reflects the systematic manner of their definition

$$\boxed{\begin{aligned} \tilde{I}_1 &= \varepsilon_{ii} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \tilde{I}_2 &= \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ji} = \tfrac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) \\ \tilde{I}_3 &= \tfrac{1}{3}\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} = \tfrac{1}{3}(\varepsilon_1^3 + \varepsilon_2^3 + \varepsilon_3^3) \end{aligned}} \qquad (2.51)$$

Occasionally, it is convenient to express these invariants in matrix notation and for that purpose, we define the *trace* of a 3 x 3 square matrix $\boldsymbol{B}$ by

$$tr\ \boldsymbol{B} = B_{ii}$$

i.e.

$$\tilde{I}_1 = tr\ \boldsymbol{\varepsilon}$$

Define the quantity B_{ij} by

$$B_{ij} = \varepsilon_{ik}\varepsilon_{kj} \qquad \text{or} \qquad \boldsymbol{B} = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^2$$

i.e. $tr\ \boldsymbol{B} = B_{ii} = \varepsilon_{ik}\ \varepsilon_{ki}$ and we therefore obtain

$$\tilde{I}_2 = \frac{1}{2}\ tr\ (\boldsymbol{\varepsilon}^2) \qquad \text{and likewise} \qquad \tilde{I}_3 = \frac{1}{3}\ tr\ (\boldsymbol{\varepsilon}^3)$$

It turns out that it is possible to obtain a unique relation between the Cauchy-invariants θ_1, θ_2 and θ_3 and the generic invariants $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$. We have

$$\tilde{I}_1 = \theta_1 \quad ; \quad \tilde{I}_2 = \frac{1}{2}\theta_1^2 - \theta_2 \quad ; \quad \tilde{I}_3 = \frac{1}{3}\theta_1^3 - \theta_1\theta_2 + \theta_3 \qquad (2.52)$$

The first two of these solutions follows directly from (2.50a), (2.51a) and (2.50b), (2.51b) respectively. Equation (2.52c) is easily established, for instance by taking the trace of matrix expression (2.46) given by Cayley-Hamilton's theorem and then using (2.52a) and (2.52b).

The inverse relations of (2.52) provide the following expressions

$$\theta_1 = \tilde{I}_1 \quad ; \quad \theta_2 = \frac{1}{2}\tilde{I}_1^2 - \tilde{I}_2 \quad ; \quad \theta_3 = \tilde{I}_3 + \frac{1}{6}\tilde{I}_1^3 - \tilde{I}_1\tilde{I}_2$$

It appears that a unique relation exists between θ_1, θ_2, θ_3 and $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_3$.

Now, let us turn to the *generic invariants* of the deviatoric strain tensor defined by analogy with (2.51). We have

$$\boxed{\begin{aligned} \tilde{J}_1 &= e_{ii} = tr\ e = e_1 + e_2 + e_3 = 0 \\ \tilde{J}_2 &= \tfrac{1}{2}e_{ij}e_{ji} = \tfrac{1}{2}tr\ (e^2) = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2) \\ \tilde{J}_3 &= \tfrac{1}{3}e_{ij}e_{jk}e_{ki} = \tfrac{1}{3}tr\ (e^3) = \tfrac{1}{3}(e_1^3 + e_2^3 + e_3^3) = e_1\ e_2\ e_3 \end{aligned}} \tag{2.53}$$

To prove the last relation that $\tilde{J}_3 = e_1e_2e_3$, we first observe that

$$(e_2 + e_3)^3 = e_2^3 + e_3^3 + 3e_2e_3(e_2 + e_3)$$

and since $e_2 + e_3 = -e_1$, we obtain

$$-e_1^3 = e_2^3 + e_3^3 - 3e_1e_2e_3$$

From the definition of $\tilde{J}_3 = \frac{1}{3}(e_1^3 + e_2^3 + e_3^3)$, it then follows that

$$\tilde{J}_3 = e_1e_2e_3$$

which was to be proved.

Moreover, using the definition of the deviatoric strain tensor as given by (2.47) in (2.53), we obtain

$$\tilde{J}_2 = \tilde{I}_2 - \frac{1}{6}\tilde{I}_1^2 \quad ; \quad \tilde{J}_3 = \tilde{I}_3 - \frac{2}{3}\tilde{I}_1\tilde{I}_2 + \frac{2}{27}\tilde{I}_1^3 \tag{2.54}$$

and the inverse relations become

$$\tilde{I}_2 = \tilde{J}_2 + \frac{1}{6}\tilde{I}_1^2 \quad ; \quad \tilde{I}_3 = \tilde{J}_3 + \frac{2}{3}\tilde{I}_1\tilde{J}_2 + \frac{1}{27}\tilde{I}_1^3 \tag{2.55}$$

Therefore, instead of using the set of invariants $\tilde{I}_1$, $\tilde{I}_2$, $\tilde{I}_3$ we may equally well use the set $\tilde{I}_1$, $\tilde{J}_2$, $\tilde{J}_3$.

An *octahedral plane* is defined as a plane where the normal to that plane makes equal angles to the three principal strain directions. Eight such planes exist and one example is shown in Fig. 2.6 where the axes 1,2 and 3 refer to

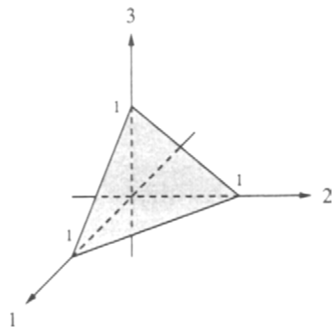

Figure 2.6: Example of octahedral plane.

the principal strain directions. For the normal to the octahedral plane shown in Fig. 2.6, we have

$$n = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In the coordinate system collinear with the principal strain directions, the strain tensor takes the form

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

The vector q is defined by $q = \varepsilon n$ cf. (2.21). It then follows from Fig. 2.5 that the normal strain ε_0 and tensorial shear strain $\gamma_0/2$ on the octahedral plane are given by

$$\varepsilon_o = n^T q \,; \qquad \frac{\gamma_0}{2} = \sqrt{q^T q - \varepsilon_0{}^2}$$

where ε_0 is called the *octahedral normal strain* and γ_0 is called the *octahedral shear strain*. It follows that

$$\varepsilon_o = \frac{1}{3}\tilde{I}_1 \,; \qquad \frac{\gamma_0}{2} = \sqrt{\frac{1}{3}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) - \frac{1}{9}\tilde{I}_1^2}$$

According to (2.47), we have

$$\varepsilon_1 = e_1 + \frac{1}{3}\tilde{I}_1 \,; \qquad \varepsilon_2 = e_2 + \frac{1}{3}\tilde{I}_1 \,; \qquad \varepsilon_3 = e_3 + \frac{1}{3}\tilde{I}_1$$

i.e.

$$\frac{\gamma_0}{2} = \sqrt{\frac{1}{3}\left[e_1^2 + e_2^2 + e_3^2 + \frac{1}{3}\tilde{I}_1^2 + \frac{2}{3}(e_1 + e_2 + e_3)\tilde{I}_1\right] - \frac{1}{9}\tilde{I}_1^2}$$

Due to (2.48) and (2.53b), we conclude that

$$\boxed{\varepsilon_0 = \frac{1}{3}\tilde{I}_1 \; ; \qquad \gamma_0 = 2\sqrt{\frac{2}{3}\tilde{J}_2}} \tag{2.56}$$

It is easily shown that these relations hold not only for the octahedral plane shown in Fig. 2.6, but also for all the other octahedral planes. Finally, it is emphasized that γ_0 is the engineering shear strain as already suggested by the notation.

2.11 Change of coordinate system - Mohr's circle

We have previously discussed the consequences of choosing a different coordinate system. Let us now consider the special case where the coordinate system is rotated about the x_3-axis as shown in Fig. 2.7.

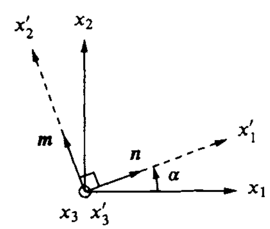

Figure 2.7: Rotation of coordinate system about the x_3-axis.

Thus, we change the coordinate system from the x_i-system to the x_i'-system, where the x_3- and x_3'-axes are identical and we shall investigate the strain tensor in this new coordinate-system. In the old x_i-coordinate system the unit vector $\boldsymbol{n}$ along the x_1'-axis and the unit vector $\boldsymbol{m}$ along the x_2'-axis have the components

$$[n_i] = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \; ; \qquad [m_i] = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}$$

where α is positive, when going in the counter-clockwise direction of the x_1x_2-plane, cf. Fig. 2.7. We then obtain

$$[\varepsilon_{ij}][n_j] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11}\cos\alpha + \varepsilon_{12}\sin\alpha \\ \varepsilon_{12}\cos\alpha + \varepsilon_{22}\sin\alpha \\ \varepsilon_{13}\cos\alpha + \varepsilon_{23}\sin\alpha \end{bmatrix}$$

With (2.11), the normal strain ε_{nn} in the direction of $\boldsymbol{n}$ becomes

$$\varepsilon_{nn} = n_i\varepsilon_{ij}n_j = \varepsilon_{11}\cos^2\alpha + \varepsilon_{22}\sin^2\alpha + \varepsilon_{12}\sin 2\alpha \tag{2.57}$$

Similarly, using (2.17), the shear strain ε_{nm} between the orthogonal axes $\boldsymbol{n}$ and $\boldsymbol{m}$ becomes

$$\varepsilon_{nm} = m_i \varepsilon_{ij} n_j = -\frac{1}{2}(\varepsilon_{11} - \varepsilon_{22}) \sin 2\alpha + \varepsilon_{12} \cos 2\alpha \tag{2.58}$$

It is obvious that when replacing α by $\alpha + \pi/2$ in (2.57), we obtain an expression for the normal strain ε_{mm} in the $\boldsymbol{m}$-direction, as this replacement corresponds to a rotation of 90° of the nm-coordinate system, i.e.

$$\varepsilon_{mm} = \varepsilon_{11} \sin^2\alpha + \varepsilon_{22} \cos^2\alpha - \varepsilon_{12} \sin 2\alpha \tag{2.59}$$

Thus, instead of the components ε_{11}, ε_{22}, ε_{12} in the old coordinate system, we have determined the components ε_{nn}, ε_{mm}, ε_{nm} in the new coordinate system by means of (2.57) - (2.59).

Occasionally, it is of interest to be able to determine ε_{11}, ε_{22}, ε_{12} provided that ε_{nn}, ε_{mm}, ε_{nm} are known. Let us assume that the nmx_3-axes comprise the original coordinate system and let us rotate this coordinate system the angle $-\alpha$ about the x_3-axis. We can then obtain the required result directly from (2.57) - (2.59) by replacing α with $-\alpha$ and ε_{11}, ε_{22}, ε_{12} with ε_{nn}, ε_{mm}, ε_{nm} respectively. This leads to

$$\boxed{\begin{aligned} \varepsilon_{11} &= \varepsilon_{nn} \cos^2\alpha + \varepsilon_{mm} \sin^2\alpha - \varepsilon_{nm} \sin 2\alpha \\ \varepsilon_{22} &= \varepsilon_{nn} \sin^2\alpha + \varepsilon_{mm} \cos^2\alpha + \varepsilon_{nm} \sin 2\alpha \\ \varepsilon_{12} &= \tfrac{1}{2}(\varepsilon_{nn} - \varepsilon_{mm}) \sin 2\alpha + \varepsilon_{nm} \cos 2\alpha \end{aligned}} \tag{2.60}$$

where the angle α still is measured positive in the counter-clockwise direction of the x_1x_2-plane, cf. Fig. 2.7.

It appears from (2.57) - (2.59) that $\varepsilon_{nn}(\alpha+\pi) = \varepsilon_{nn}(\alpha)$, $\varepsilon_{mm}(\alpha+\pi) = \varepsilon_{mm}(\alpha)$ and $\varepsilon_{nm}(\alpha + \pi) = \varepsilon_{nm}(\alpha)$, i.e. the strain components vary with a period of 180°. This property complies with the physical interpretation of the normal and the shear strain. Thus it appears that it is sufficiently general only to consider α-values in the range

$$\boxed{0 \leq \alpha < \pi} \tag{2.61}$$

It is of interest that (2.58) implies that a particular α-value exists, $\alpha = \psi$, for which the shear strain ε_{nm} becomes zero. This angle ψ is determined by

$$\boxed{\tan 2\psi = \frac{2\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{22}}} \tag{2.62}$$

which even applies when $\varepsilon_{11} = \varepsilon_{22}$, since (2.58) for $\varepsilon_{11} = \varepsilon_{22}$ provides $\cos 2\psi = 0$, i.e. the same solution as (2.62).

Therefore, when $\alpha = \psi$, we find that $\varepsilon_{nm} = 0$; however the shear strains ε_{nx_3} and ε_{mx_3} will, in general, be different from zero. Consequently, only if the x_3-

direction is a principal direction, will the directions $\boldsymbol{n}$ and $\boldsymbol{m}$ defined by $\alpha = \psi$ also be principal directions. This situation will be assumed in the following.

Consequently, when $\alpha = \psi$, where ψ is determined by (2.62), we have determined the position of the new coordinate system, so that it corresponds to the principal directions. In accordance with our previous discussion of principal strains and directions, the principal directions determined by ψ satisfying (2.62) are perpendicular to each other. In the present situation this implies that if the new coordinate system, collinear with the principal directions, is rotated by $\pm 90^o$, $\pm 180^o$ etc., then these new positions of the coordinate system will also be principal directions. Therefore, when we have determined the angle ψ, then any angle $\psi \pm n\pi/2$, where $n = 1, 2, 3$ .... will also be a valid ψ-value for which the shear strain $\varepsilon_{nm} = 0$.

With this discussion in mind, it is advantageous to accept some convention, when selecting a ψ-value which satisfies (2.62). In accordance with the convention (2.61) and as ψ is just a special value of α, for which no shear strains exist, it is advantageous to have the same range for ψ as for α. Therefore, we accept the following range for ψ

$$\boxed{0 \leq \psi < \pi} \tag{2.63}$$

However, as the tan-function is periodic with a period of π, this implies that when solving (2.62) subjected to (2.63), two ψ-solutions will result. Both ψ-solutions are acceptable and we will later – after having introduced Mohr's circle of strain – present a procedure, by which it is possible to select one uniquely defined ψ-value.

It turns out that an elegant geometrical interpretation can be made of (2.57) and (2.58). Define first the quantities a and b by

$$a = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}), \qquad b = \frac{1}{2}(\varepsilon_{11} - \varepsilon_{22}) \tag{2.64}$$

Observing that $\cos^2\alpha = (1+\cos 2\alpha)/2$ and $\sin^2\alpha = (1-\cos 2\alpha)/2$, (2.57) and (2.58) can be written as

$$\begin{aligned} \varepsilon_{nn} - a &= b\cos 2\alpha + \varepsilon_{12}\sin 2\alpha \\ \varepsilon_{nm} &= -b\sin 2\alpha + \varepsilon_{12}\cos 2\alpha \end{aligned} \tag{2.65}$$

By squaring each of these equations and adding, we obtain

$$(\varepsilon_{nn} - a)^2 + \varepsilon_{nm}^2 = R^2 \tag{2.66}$$

where the quantity R is defined by

$$R = \sqrt{b^2 + \varepsilon_{12}^2} = \sqrt{\left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2} \tag{2.67}$$

Since we are considering a given strain state, the quantity R is constant. Returning to expression (2.66), we observe that it is exactly the equation for a circle in

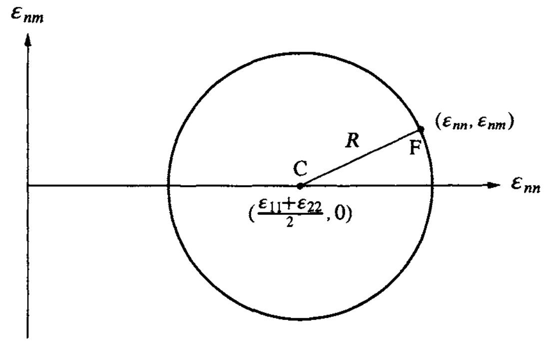

Figure 2.8: Mohr's circle of strain.

a ε_{nn}, ε_{nm}-coordinate system with the center of the circle at $(a, 0)$ and a radius R. Since the point $(\varepsilon_{nn}, \varepsilon_{nm})$ is located on this circle, we have obtained *Mohr's circle of strain* (Mohr, 1882) as shown in Fig. 2.8.

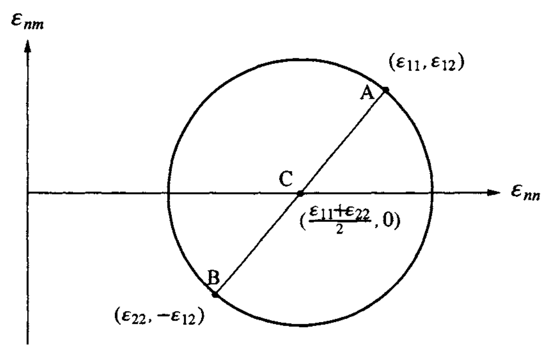

Figure 2.9: Graphical construction of Mohr's circle of strain.

The construction above of Mohr's circle involves the analytical determination of the center and of the radius. For practical purposes, however, a direct graphical construction is to be preferred. To this end, we note from (2.57) and (2.58) that

$$\varepsilon_{nn}(\alpha = 0) = \varepsilon_{11}\ ; \qquad \varepsilon_{nm}(\alpha = 0) = \varepsilon_{12} \qquad \text{point A}$$

and

$$\varepsilon_{nn}(\alpha = \frac{\pi}{2}) = \varepsilon_{22}\ ; \qquad \varepsilon_{nm}(\alpha = \frac{\pi}{2}) = -\varepsilon_{12} \qquad \text{point B}$$

These $(\varepsilon_{nn}, \varepsilon_{nm})$-values also correspond to points on Mohr's circle and they are referred to as point A and B respectively, in Fig. 2.9. If a line is drawn between point A and B, the midpoint of this line has the coordinates $(\frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}), 0)$, i.e. this midpoint is precisely the center of Mohr's circle. Having identified this center, Mohr's circle can be drawn directly.

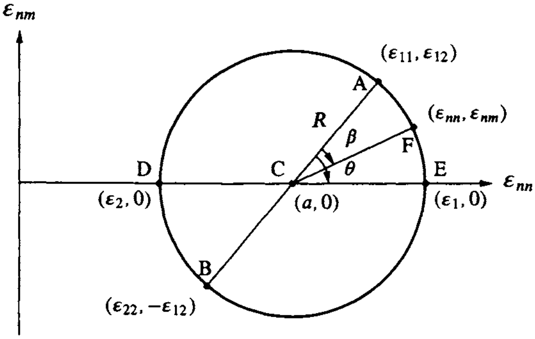

Figure 2.10: Identification of different angles $(\varepsilon_1 \geq \varepsilon_2)$.

The strain point $(\varepsilon_{nn}(\alpha), \varepsilon_{nm}(\alpha))$ is located somewhere on the circle and according to Fig. 2.10, its position F can be characterized by the angle β that is measured clockwise from the radius CA; also the angle θ is shown in this figure. From Fig. 2.10, point F can be identified as

$$\begin{aligned} \varepsilon_{nn} &= a + R\cos(\theta - \beta) \\ \varepsilon_{nm} &= R\sin(\theta - \beta) \end{aligned}$$

These expressions can be written as

$$\begin{aligned} \varepsilon_{nn} &= a + R\cos\theta\cos\beta + R\sin\theta\sin\beta \\ \varepsilon_{nm} &= R\sin\theta\cos\beta - R\cos\theta\sin\beta \end{aligned} \tag{2.68}$$

From Fig. 2.10, it also follows that $\frac{1}{2}(\varepsilon_{11} - \varepsilon_{22}) = R\cos\theta$ which with (2.64) results in $b = R\cos\theta$; we also have $\varepsilon_{12} = R\sin\theta$. Then (2.68) becomes

$$\begin{aligned} \varepsilon_{nn} &= a + b\cos\beta + \varepsilon_{12}\sin\beta \\ \varepsilon_{nm} &= \varepsilon_{12}\cos\beta - b\sin\beta \end{aligned}$$

and a comparison with (2.65) proves that $\beta = 2\alpha$.

We then obtain *Mohr's circle of strain* in the following fashion. For given strain components $\varepsilon_{11}, \varepsilon_{22}$ and ε_{12} plot the points A $(\varepsilon_{11}, \varepsilon_{12})$ and B $(\varepsilon_{22}, -\varepsilon_{12})$ in the $\varepsilon_{nn}, \varepsilon_{nm}$-coordinate system shown in Fig. 2.11. Draw a line between these two points. The intersection of this line with the ε_{nn}-axis defines the center C

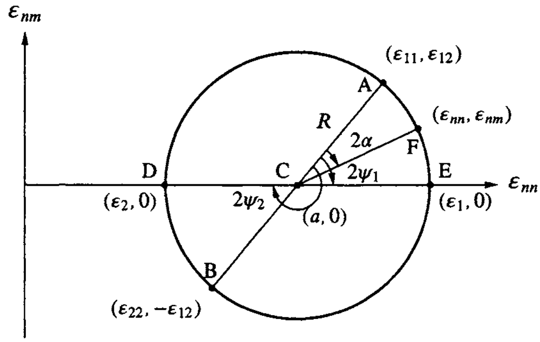

Figure 2.11: Identification of different angles ($\varepsilon_1 \geq \varepsilon_2$).

of a circle that contains the points A and B. Draw this circle. For an arbitrary angle α, cf. Fig. 2.7, the corresponding values of ε_{nn} and ε_{nm} are then given by point F in Fig. 2.11. Note that

The angle 2α in Mohr's circle in Fig. 2.11 is measured positive in the clockwise direction

and compare with Fig. 2.7 where α is measured positive in the counter-clockwise direction.

Note also that the graphical interpretation given by Mohr's circle only holds when both the $xy-$ and $\varepsilon_{nn}, \varepsilon_{nm}$-coordinate systems are right-handed coordinate systems; otherwise Mohr's circle of strain has to be modified.

The two points where the circle intersects the ε_{nn}-axis correspond to α-values where the shear strain $\varepsilon_{nm} = 0$; these two points therefore correspond to the principal strains and if we adopt the convention that

$$\boxed{\varepsilon_1 > \varepsilon_2} \tag{2.69}$$

the principal strains are given by the points E and D in Fig. 2.11. If we furthermore make the convention that the α-value corresponding to ε_1 is called ψ_1 and the α-value corresponding to ε_2 is called ψ_2, we then have the situation illustrated in Fig. 2.11. Due to (2.69) it appears for Fig. 2.11 that

$$\boxed{0 \leq \psi_1 < \pi} \tag{2.70}$$

To identify this angle, (2.62) could be used, but owing to the periodicity of the function $\tan 2\psi$, it is more convenient to observe from Fig. 2.11 that

$$\boxed{\begin{aligned} \cos 2\psi_1 &= \frac{\varepsilon_{11}-\varepsilon_{22}}{2R} \\ \sin 2\psi_1 &= \frac{\varepsilon_{12}}{R} \\ \textit{where} \quad R &= \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2} \end{aligned}}$$

Inserting the values for ε_{11}, ε_{22} and ε_{12} we can identify the signs for $\sin 2\psi_1$ and $\cos 2\psi_1$ and thereby the quadrant that the angle $2\psi_1$ is located in and thereby easily determine ψ_1 so that (2.70) is fulfilled.

Note the convention that elongation is considered positive and that $\varepsilon_1 \geq \varepsilon_2$. It is then emphasised that the angle $2\psi_1$ is the angle to the largest principal strain ε_1.

It is also concluded from Fig. 2.11 that the principal strains are given by

$$\boxed{\left.\begin{matrix} \varepsilon_1 \\ \varepsilon_2 \end{matrix}\right\} = \frac{1}{2}(\varepsilon_{11} - \varepsilon_{22}) \pm R} \tag{2.71}$$

We finally observe from Fig. 2.11 that the extremum values for the shear strain are determined by the radius R, i.e. (2.67) gives with the convention $\varepsilon_1 \geq \varepsilon_2$

$$\boxed{\varepsilon_{nm,max} = \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\ ; \qquad \varepsilon_{nm,min} = -\frac{1}{2}(\varepsilon_1 - \varepsilon_2)}$$

2.12 Special states of strain

Several special states of strain, which are often encountered in practice, will now be discussed.

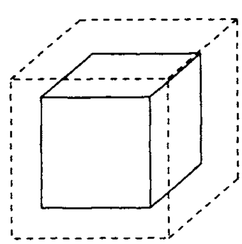

Figure 2.12: Uniform dilatation.

A state of *uniform dilatation* occurs, if the strain tensor is given by

$$\varepsilon_{ij} = b\,\delta_{ij}$$

where b is an arbitrary scalar. It appears from (2.47) that the deviatoric strain tensor e_{ij} becomes $e_{ij} = 0$ and according to the discussion of (2.49), the strain state corresponds to a uniform dilatation, i.e. a volume change, where the extension - or contraction - in any direction is the same and equal to the parameter b, cf. Fig. 2.12.

Uniaxial strain occurs if the displacement vector u_i is given by

$$[u_i] = \begin{bmatrix} u_1(x_1, t) \\ 0 \\ 0 \end{bmatrix}$$

which implies that $\varepsilon_{11} = \partial u_1/\partial x_1$ and all other strain components being zero, cf. Fig.2.13.

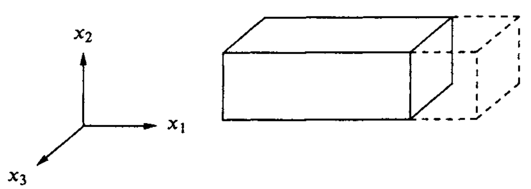

Figure 2.13: Uniaxial strain.

Plane strain or *plane deformation* occurs if the displacement vector u_i is given by

$$[u_i] = \begin{bmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ 0 \end{bmatrix}$$

which implies

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.72}$$

This strain state occurs often in practice when a long prismatic or cylindrical body is loaded by forces which are perpendicular to the longitudinal elements and which do not vary along the length. In this case, it can be assumed that all cross sections are in the same state and if, moreover, the body is restricted from moving in the length direction, a state of plane strain exists. An example is an internally pressurized tube with end sections confined between smooth and rigid walls, Fig. 2.14.

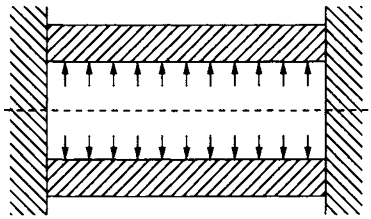

Figure 2.14: Example of plane strain. Pressurized tube with end sections confined between smooth and rigid walls.

So-called *generalized plane strain* or *generalized plane deformation* occurs if

$$[u_i] = \begin{bmatrix} u_1(x_1, x_2, t) \\ u_2(x_1, x_2, t) \\ u_3(x_1, x_2, t) \end{bmatrix}$$

which leads to

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & 0 \end{bmatrix}$$

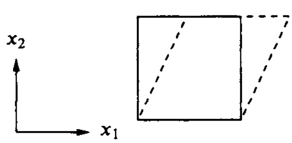

Figure 2.15: Simple shear.

Finally, a state of *simple shear* exists if

$$[\varepsilon_{ij}] = \begin{bmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

corresponding to $u_1 = u_1(x_2, t)$ and $u_2 = u_3 = 0$, as illustrated in Fig. 2.15. It appears that for simple shear, we have $\varepsilon_{ii} = 0$, i.e. no volume change and it is easily shown that the principal strains become $\varepsilon_1 = \varepsilon_{12}$, $\varepsilon_2 = -\varepsilon_{12}$ and $\varepsilon_3 = 0$.