22 PLASTICITY, VISCOPLASTICITY AND VISCOELASTICITY

Having established the general framework provided by thermodynamics, we will now illustrate its use on plasticity, viscoplasticity and viscoelasticity. The reader may consult the textbooks of Lemaitre and Chaboche (1990), Maugin (1992), Simo and Hughes (1998) and Nguyen (2000) for related viewpoints. We will first present the general plasticity equations and then give examples of the establishment of some specific and typical plasticity models.

22.1 Fundamental equations of plasticity

For plasticity theory where the viscous stress σ_{ij}^{v} disappears, the results (21.37) read

$$\boxed{\begin{aligned} &\psi = \psi(\theta, \varepsilon_{ij} - \varepsilon_{ij}^{p}, \kappa_\alpha) \\ &\textit{and} \\ &\gamma_{mech} \equiv \sigma_{ij}\dot{\varepsilon}_{ij}^{p} - K_\alpha \dot{\kappa}_\alpha \geq 0 \\ &\textit{where} \\ &\sigma_{ij} = \rho\frac{\partial \psi}{\partial \varepsilon_{ij}}\,; \qquad K_\alpha \equiv \rho\frac{\partial \psi}{\partial \kappa_\alpha} \end{aligned}} \tag{22.1}$$

We certainly know that at least the stress σ_{ij} must enter the yield function and the question is what other variables are needed to monitor hardening/softening behavior. According to the mechanical dissipation inequality, the 'forces' σ_{ij} and K_α are conjugated to the 'fluxes' $\dot{\varepsilon}_{ij}^{p}$ and $-\dot{\kappa}_\alpha$ respectively. It therefore seems natural to take these forces as the variables in the yield function f and we then have

$$\boxed{f = f(\sigma_{ij}, K_\alpha) \leq 0}$$

A comparison with Chapter 10 shows that the conjugated thermodynamic forces K_α for plasticity theory take the form of being the *hardening parameters*.

Once Helmholtz' free energy ψ has been chosen, both the stresses σ_{ij} and the forces K_α are given in accordance with (22.1). Disregarding for the moment thermo-plasticity, which we will deal with in the next chapter, and considering the expression for $\rho\psi$ given by (21.17) that is valid for thermo-elasticity, a natural choice for plasticity is

$$\boxed{\rho\psi(\theta, \varepsilon_{ij}-\varepsilon^p_{ij}, \kappa_\alpha) = \rho h(\theta) + \frac{1}{2}(\varepsilon_{ij}-\varepsilon^p_{ij})D_{ijkl}(\varepsilon_{kl}-\varepsilon^p_{kl}) + \rho\psi^p(\kappa_\alpha)} \tag{22.2}$$

Here, $h(\theta)$ is an arbitrary function of the temperature and $\psi^p(\kappa_\alpha)$ is an arbitrary function of the internal variables; moreover, D_{ijkl} is the elastic stiffness tensor which is considered to be a constant. It appears that the free energy function ψ is split into three separate parts and that no coupling effects are present between the three sets of variables: θ, $\varepsilon_{ij} - \varepsilon^p_{ij}$ and κ_α, cf. Lubliner (1972). From (22.2) and (22.1) follow that

$$\boxed{\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})\,; \qquad K_\alpha = \rho\frac{\partial\psi^p(\kappa_\beta)}{\partial\kappa_\alpha}} \tag{22.3}$$

where Hooke's law is recovered by (22.3a).

Having determined the 'forces' σ_{ij} and K_α, the next topic is to establish evolution laws for the 'fluxes' $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$. The only formal requirement is that these evolution laws must be such that the dissipation inequality (22.1) is fulfilled. A natural route is to require that γ_{mech} is as large as possible; we then evidently ensure that $\gamma_{mech} \geq 0$. This leads to the postulate of maximum dissipation discussed in Section 21.3. As the mathematical literature on extremum properties is by tradition concerned with minimization problems, we change our requirement of γ_{mech} being maximal to $-\gamma_{mech}$ being minimal. Moreover, we will treat the fluxes $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$ as given - although arbitrary - and consider the forces σ_{ij} and K_α as the variables. In agreement with (21.55) we then have

Postulate of maximum dissipation: *For given fluxes* $\dot{\varepsilon}^p_{ij}$ *and* $\dot{\kappa}_\alpha$, *find those stresses* σ_{ij} *and conjugated forces* K_α *that minimize* $-\gamma_{mech}$ *under the constraint that* $f(\sigma_{ij}, K_\alpha) \leq 0$. *The yield function* f *is assumed to be a convex function in* σ_{ij} *and* K_α	(22.4)

In accordance with the Appendix, we are faced with a minimization problem with a constraint in terms of an inequality. Following (A.16), we define the Lagrange function $\mathcal{L}(\sigma_{ij}, K_\alpha, \dot{\lambda})$ by

$$\mathcal{L}(\sigma_{ij}, K_\alpha, \dot{\lambda}) = -\gamma_{mech} + \dot{\lambda}f = -\sigma_{ij}\dot{\varepsilon}^p_{ij} + K_\alpha\dot{\kappa}_\alpha + \dot{\lambda}f(\sigma_{ij}, K_\alpha)$$

where $\dot{\lambda}$ is a Lagrange multiplier. In conformity with (A.17) and (A.15), it follows that $\partial\mathcal{L}/\partial\sigma_{ij} = -\dot{\varepsilon}^p_{ij}+\dot{\lambda}\partial f/\partial\sigma_{ij} = 0$ and $\partial\mathcal{L}/\partial K_\alpha = \dot{\kappa}_\alpha+\dot{\lambda}\partial f/\partial K_\alpha = 0$.

From these expressions and (A.15), the following *Kuhn-Tucker relations* are then obtained

Postulate of maximum dissipation implies:
The associated evolution equations

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial f}{\partial K_\alpha}$$

as well as the loading/unloading criteria

$$\dot{\lambda} \geq 0$$

$$\dot{\lambda} = 0 \quad \textit{for elastic behavior} \tag{22.5}$$

and

$$\dot{\lambda} f = 0$$

It appears that the postulate of maximum dissipation leads to the associated evolution laws and that the quantity $\dot{\lambda}$, which was originally introduced as a Lagrange multiplier, turns out to be the *plastic multiplier*. Moreover, the Kuhn-Tucker relations imply that $\dot{\lambda} \geq 0$ when $f = 0$ and $\dot{\lambda} = 0$ when $f < 0$. Previously, in Chapter 10 we had to assume these properties for $\dot{\lambda}$, but now they are mathematical consequences of the postulate of maximum dissipation. We also recall from Chapter 10 that these properties for $\dot{\lambda}$ lead to the general loading/unloading criteria given by (10.38).

To obtain the evolution equations for $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$, another fruitful route is to use the results given by (21.51) where the dissipation function $\phi = \phi(A_\Theta, Z^p)$ was introduced and where $A_\Theta = \{\sigma_{ij}, K_\alpha\}$, i.e. the set A_Θ comprises all σ_{ij} and K_α components and Z^p are simply some variables, cf. (21.39) and (21.46). Then the evolution equations are given by $\dot{a}_\Theta = \dot{\lambda}\partial\phi/\partial A_\Theta$ where $\dot{a}_\Theta = \{\dot{\varepsilon}^p_{ij}, -\dot{\kappa}_\alpha\}$, i.e. the set $\dot{a}_\Theta$ comprises all $\dot{\varepsilon}^p_{ij}$ and $-\dot{\kappa}_\alpha$ components, cf. (21.39). Since $\phi = \phi(A_\Theta, Z^p)$ and $A_\Theta = \{\sigma_{ij}, K_\alpha\}$ we may choose the dissipation function ϕ as the *potential function* $g(\sigma_{ij}, K_\alpha)$. The evolution equations $\dot{a}_\Theta = \dot{\lambda}\partial\phi/\partial A_\Theta$ then become $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial g/\partial\sigma_{ij}$ and $-\dot{\kappa}_\alpha = \dot{\lambda}\partial g/\partial K_\alpha$. With (21.51), we then obtain

If the potential function $g = g(\sigma_{ij}, K_\alpha)$ is a convex function in σ_{ij} and K_α and if

$$g(\sigma_{ij}, K_\alpha) - g(0, 0) \geq 0$$

then the nonassociated evolution equations (22.6)

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial g}{\partial K_\alpha}; \quad \dot{\lambda} \geq 0$$

fulfill the dissipation inequality

Referring to the discussion of (21.51), we observe that the requirement $g(\sigma_{ij}, K_\alpha) - g(0, 0) \geq 0$ is implied by the requirements $(\partial g/\partial\sigma_{ij})_{\sigma_{ij}=0, K_\alpha=0} = 0$ and

$(\partial g/\partial K_\alpha)_{\sigma_{ij}=0,K_\alpha=0} = 0$ (as well as convexity of g, that is, g is minimum at $g(0,0)$). It appears that we have retrieved nonassociated plasticity and that associated plasticity emerges if we choose $g = f$. However, in contrast to Chapter 10 we now know exactly the requirements that must be posed on the potential function g in order that the nonassociated evolution laws are physically meaningful (i.e. the dissipation inequality is fulfilled). This route of establishing the evolution equations only leave us with the information that $\dot{\lambda} \geq 0$; however, it is natural to impose the same further requirements on $\dot{\lambda}$ as in associated plasticity, i.e. $\dot{\lambda} \geq 0$ when $f = 0$ and $\dot{\lambda} = 0$ when $f < 0$.

It is recalled that (22.5) and (22.6) are convenient mathematical tools by which we can identify the precise requirements that ensure *a priori* that the dissipation inequality is fulfilled. However, in principle, it is possible to relax these requirements and simply write $\dot{\varepsilon}^p_{ij} = \dot{\lambda} h_{ij}$ and $\dot{\kappa}_\alpha = -\dot{\lambda} b_\alpha$ ($\dot{\lambda} \geq 0$) where the functions h_{ij} and b_α then should fulfill $\sigma_{ij}h_{ij} + K_\alpha b_\alpha \geq 0$. The drawback of this more general approach is that it is not possible to identify, *a priori*, the properties that h_{ij} and b_α should possess in order to fulfill this inequality. Therefore, for each specific model, i.e. each specific choice of h_{ij} and b_α, one must, *a posteriori*, check that the inequality is fulfilled. With these remarks, in the following we will restrict ourselves to the formulations given by (22.5) and (22.6).

Both for associated and nonassociated plasticity we have in accordance with the previous discussion $\dot{\lambda} f = 0$, cf. (22.5). Differentiation gives $\ddot{\lambda} f + \dot{\lambda}\dot{f} = 0$. For development of plasticity to occur, we must have $f = 0$ which implies $\dot{\lambda}\dot{f} = 0$ and since development of plastic strains (and of the internal variables) requires $\dot{\lambda} > 0$ we are left with $\dot{f} = 0$. According to Chapter 10, we have then recovered the *consistency relation* in an elegant fashion. With $f = f(\sigma_{ij}, K_\alpha)$, the consistency relation becomes

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha}\dot{K}_\alpha = 0 \tag{22.7}$$

From (22.1) and (22.6), we have

$$\dot{K}_\alpha = \rho \frac{\partial^2 \psi^p(\kappa^\gamma)}{\partial \kappa_\alpha \partial \kappa_\beta}\dot{\kappa}_\beta = -\dot{\lambda}\rho \frac{\partial^2 \psi^p(\kappa^\gamma)}{\partial \kappa_\alpha \partial \kappa_\beta}\frac{\partial g}{\partial K_\beta}$$

Insertion of this expression in (22.7) leads to

$$\boxed{\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0} \tag{22.8}$$

where the *generalized plastic modulus* H is defined by

$$\boxed{H \equiv \frac{\partial f}{\partial K_\alpha}\rho \frac{\partial^2 \psi^p(\kappa^\gamma)}{\partial \kappa_\alpha \partial \kappa_\beta}\frac{\partial g}{\partial K_\beta}} \tag{22.9}$$

in complete analogy with Section 10.2.

Assuming the elastic stiffness D_{ijkl} to be constant, (22.3a) gives

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) = D_{ijkl}\dot{\varepsilon}_{kl} - \dot{\lambda} D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}}$$

and insertion into the consistency relation (22.8) provides

$$\boxed{\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl} \quad \textit{where} \quad A = H + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}} \geq 0} \tag{22.10}$$

in accordance with (10.23) and where, as usual, it is required that $A > 0$.

It appears that all fundamental equations of general plasticity previously discussed in Chapter 10 have now been recovered in a fashion that fulfills all formal thermodynamical requirements. All the subsequent manipulations of these fundamental equations are therefore completely identical to those discussed in Chapter 10; thus, they will not be repeated here.

The only remaining topic is the choice of the function $\psi^p(\kappa_\alpha)$, present in (22.3), and thereby also the choice of the internal variables κ_α. Some typical examples will be presented in Section 22.3, but before that specific discussion it is worthwhile to return to a discussion of the postulate of maximum dissipation.

22.2 Further discussion of the postulate of maximum dissipation

The postulate of maximum dissipation was used to derive associated plasticity and it was emphasized in Section 21.3 that this postulate is not a law of nature. Indeed it may simply be viewed as a convenient mathematical means to ensure that the mechanical dissipation inequality is fulfilled. On the other hand, it may be possible to appeal to some kind of physics behind this postulate. Let us consider a body where no mechanical work input and no heat input are supplied (i.e. $\delta W = \delta Q = 0$). Following (20.63) it was shown that if $\delta W = \delta Q = 0$ and if the entropy has a maximum value then thermodynamical equilibrium exists. Suppose that we consider a process where the body is carried through such stages of maximum entropy and thermodynamic equilibrium. Then it seems reasonable to assume that a process of maximum entropy production occurs and, consequently, this process is one of maximum dissipation. However, this kind of process is an assumption and not a strict result; otherwise nonassociated plasticity would not exist.

Before we enter a further discussion of the postulate of maximum dissipation, a few results about the concepts of a *convex function* and a *convex set of points* need to be recalled.

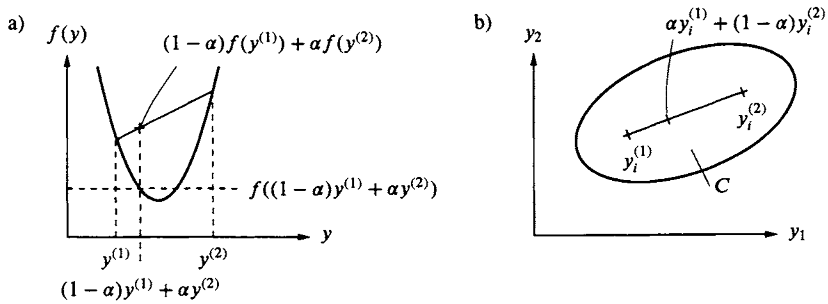

Figure 22.1: a) One-dimensional convex function, b) two-dimensional convex set of points.

A function $f(y_i)$ of the N variables y_i, i.e. $i = 1, 2, \cdots, N$ is *convex* if and only if

Convex function:

$$\boxed{f((1-\alpha)y_i^{(1)} + \alpha y_i^{(2)}) \leq (1-\alpha) f(y_i^{(1)}) + \alpha f(y_i^{(2)})}$$

where $0 \leq \alpha \leq 1$ and where $y_i^{(1)}$ and $y_i^{(2)}$ are two arbitrary points, cf. Appendix (A.1). Consider next a *set of points* C where both $y_i^{(1)}$ and $y_i^{(2)}$ belong to C, i.e. $y_i^{(1)}$ and $y_i^{(2)} \in C$. A set of points C is *convex* if and only if

Convex set of points C :

$$\boxed{\textit{If } y_i^{(1)} \textit{ and } y_i^{(2)} \in C \textit{ then } \alpha y_i^{(1)} + (1-\alpha) y_i^{(2)} \in C} \qquad (22.11)$$

where $0 \leq \alpha \leq 1$. A one-dimensional example of a convex function is shown in Fig. 22.1a) and a two-dimensional example of a convex set of points is illustrated in Fig. 22.1b).

Consider an arbitrary function $f(y_i)$ and define the set of points C by those points y_i that fulfill $f(y_i) \leq K = constant$, i.e. the set of points is bounded by a contour curve of the function f. It then turns out that

Let $f(y_i)$ be a convex function.
The set of points C defined by

$$f(y_i) \leq K = constant \qquad (22.12)$$

is then a convex set of points

To prove this, we first use that f is a convex function, i.e.

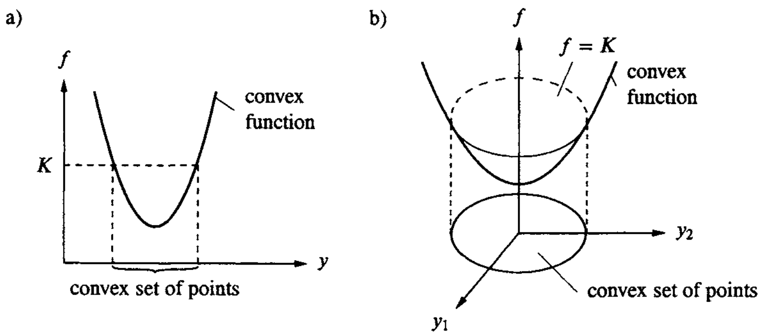

Figure 22.2: Convexity of the function f implies convexity of the set of points C defined by $f(y_i) \leq K$ = *constant*; a) one-dimensional case, b) two-dimensional case.

$$f(\alpha y_i^{(1)} + (1-\alpha) y_i^{(2)}) \leq \alpha f(y_i^{(1)}) + (1-\alpha) f(y_i^{(2)})$$

From the definition of the set C, we have $f(y_i^{(1)}) \leq K$ and $f(y_i^{(2)}) \leq K$ and insertion in the expression above gives

$$f(\alpha y_i^{(1)} + (1-\alpha) y_i^{(2)}) \leq \alpha f(y_i^{(1)}) + (1-\alpha) f(y_i^{(2)}) \leq \alpha K + (1-\alpha) K = K$$

Consequently, the point $\alpha y_i^{(1)} + (1-\alpha) y_i^{(2)}$ also belongs to the set C. Following (22.11) we then conclude that the set C is convex.

The result (22.12) is illustrated in Figs. 22.2a) and 23.2b). While convexity of a function f implies convexity of the set of points C defined by $f(y_i) \leq K$ = *constant*, the opposite is not true; simple examples are given in Fig. 22.3.

With these preliminaries, let us next prove the following

If the yield function $f(\sigma_{ij}, K_\alpha)$ is a convex function in σ_{ij} and K_α, and if

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = -\dot{\lambda} \frac{\partial f}{\partial K_\alpha}; \quad \dot{\lambda} \geq 0$$

then

$$(\sigma_{ij} - \sigma_{ij}^*) \dot{\varepsilon}_{ij}^p - (K_\alpha - K_\alpha^*) \dot{\kappa}_\alpha \geq 0 \tag{22.13}$$

where

$$f(\sigma_{ij}, K_\alpha) = 0 \quad \textit{and} \quad f(\sigma_{ij}^*, K_\alpha^*) \leq 0$$

where the evolution laws are evaluated at the state (σ_{ij}, K_α), i.e. $\dot{\varepsilon}_{ij}^p$ and $\dot{\kappa}_\alpha$ belong to the state (σ_{ij}, K_α). It appears from (22.13) that the point (σ_{ij}, K_α) is

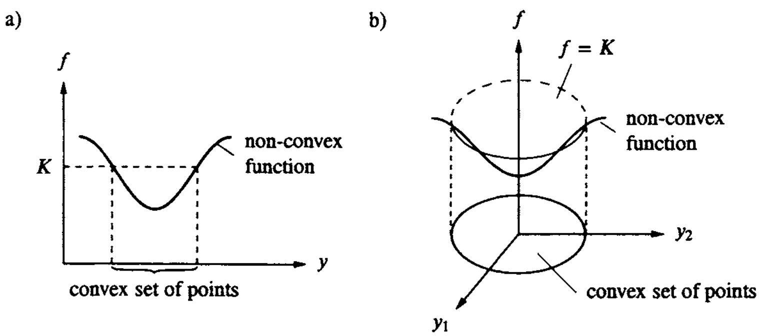

Figure 22.3: Convexity of the set of points C defined by $f(y_i) \leq K = constant$ does not imply convexity of the function f.

located on the yield surface whereas the point $(\sigma_{ij}^*, K_\alpha^*)$ is located on or inside the yield surface. Rewrite (22.13) as $\sigma_{ij}\dot{\varepsilon}_{ij}^p - K_\alpha\dot{\kappa}_\alpha \geq \sigma_{ij}^*\dot{\varepsilon}_{ij}^p - K_\alpha^*\dot{\kappa}_\alpha$ which, with (22.1) and evident notation, can be written as $\gamma_{mech} \geq \gamma_{mech}^*$. Then (22.13) can be interpreted as: the dissipation related to the real state (σ_{ij}, K_α) is larger than or equal to the dissipation related to any other state $(\sigma_{ij}^*, K_\alpha^*)$ within or on the yield surface. Indeed, this is just another way of expressing the postulate of maximum dissipation.

To prove (22.13), we accept that f is a convex function in σ_{ij} and K_α. From Appendix (A.5) we then have

$$f(\sigma_{ij}^*, K_\alpha^*) - f(\sigma_{ij}, K_\alpha) \geq \frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij}^* - \sigma_{ij}) + \frac{\partial f}{\partial K_\alpha}(K_\alpha^* - K_\alpha)$$

where $\partial f / \partial \sigma_{ij}$ and $\partial f / \partial K_\alpha$ are evaluated at the point (σ_{ij}, K_α). Multiplication by $\dot{\lambda}\ (\geq 0)$, use of the evaluation equations as well as $f(\sigma_{ij}, K_\alpha) = 0$ give

$$\dot{\lambda} f(\sigma_{ij}^*, K_\alpha^*) \geq \dot{\varepsilon}_{ij}^p(\sigma_{ij}^* - \sigma_{ij}) - \dot{\kappa}_\alpha(K_\alpha^* - K_\alpha)$$

Therefore

$$(\sigma_{ij} - \sigma_{ij}^*)\dot{\varepsilon}_{ij}^p - (K_\alpha - K_\alpha^*)\dot{\kappa}_\alpha \geq -\dot{\lambda} f(\sigma_{ij}^*, K_\alpha^*) \geq 0$$

since $\dot{\lambda} \geq 0$ and $f(\sigma_{ij}^*, K_\alpha^*) \leq 0$; this is exactly the result stated in (22.13).

Let us next prove

$$\boxed{\begin{array}{l} \textit{Let } \dot{\varepsilon}^p_{ij} \textit{ and } \dot{\kappa}_\alpha \textit{ be given and let} \\ \quad (\sigma_{ij} - \sigma^*_{ij})\dot{\varepsilon}^p_{ij} - (K_\alpha - K^*_\alpha)\dot{\kappa}_\alpha \geq 0 \\ \textit{and } \; f(\sigma_{ij}, K_\alpha) = 0; \quad f(\sigma^*_{ij}, K^*_\alpha) \leq 0 \quad \textit{then} \\ \quad \dot{\varepsilon}^p_{ij} = \dot{\lambda}\dfrac{\partial f}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = -\dot{\lambda}\dfrac{\partial f}{\partial K_\alpha}; \quad \dot{\lambda} \geq 0 \\ \textit{and the set defined by} \\ \quad f(\sigma_{ij}, K_\alpha) \leq 0 \textit{ is a convex set of points} \end{array}} \tag{22.14}$$

where $\partial f/\partial\sigma_{ij}$ and $\partial f/\partial K_\alpha$ are evaluated at the point (σ_{ij}, K_α).

Let us first prove the associated evolution laws. Without losing generality, we can write

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} + p_{ij}(\dot{\lambda}, \sigma_{kl}, K_\beta); \qquad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial f}{\partial K_\alpha} + q_\alpha(\dot{\lambda}, \sigma_{kl}, K_\beta) \tag{22.15}$$

where p_{ij} and q_α are some general functions and since $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$ are given, p_{ij} and q_α do not depend on $(\sigma^*_{ij}, K^*_\alpha)$. Then the expression for maximum dissipation given by (22.14) becomes

$$\begin{aligned} &\dot{\lambda}\Big[\frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \sigma^*_{ij}) + \frac{\partial f}{\partial K_\alpha}(K_\alpha - K^*_\alpha)\Big] \\ &\qquad + (\sigma_{ij} - \sigma^*_{ij})p_{ij} - (K_\alpha - K^*_\alpha)q_\alpha \geq 0 \end{aligned} \tag{22.16}$$

From $f = f(\sigma_{ij}, K_\alpha)$ we obtain

$$df = \frac{\partial f}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial f}{\partial K_\alpha}dK_\alpha \tag{22.17}$$

The arbitrary state $(\sigma^*_{ij}, K^*_\alpha)$ is located within or on the yield surface. Thus, it is always possible to choose $(\sigma^*_{ij}, K^*_\alpha)$ such that

$$\sigma_{ij} - \sigma^*_{ij} = -d\sigma_{ij}; \qquad K_\alpha - K^*_\alpha = -dK_\alpha \tag{22.18}$$

Suppose that $(\sigma^*_{ij}, K^*_\alpha)$ is chosen on the yield surface; an illustration of σ_{ij}, σ^*_{ij} and $d\sigma_{ij}$ is shown below in Fig. 22.4a). Since we are moving tangentially to the yield surface we have $df = 0$ and (22.17) and (22.18) then provide

$$\frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \sigma^*_{ij}) + \frac{\partial f}{\partial K_\alpha}(K_\alpha - K^*_\alpha) = 0$$

Insertion of this expression and (22.18) into (22.16) gives

$$-d\sigma_{ij}p_{ij} + dK_\alpha q_\alpha \geq 0 \tag{22.19}$$

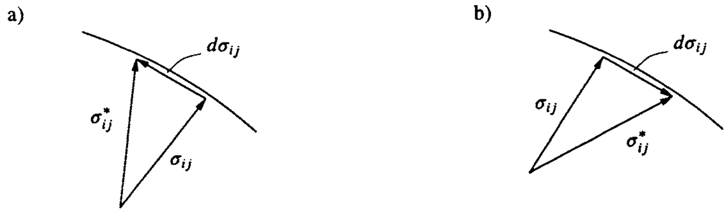

Figure 22.4: The same σ_{ij}-value applies, but σ^*_{ij} is chosen such that $(d\sigma_{ij})^b = -(d\sigma_{ij})^a$.

Since f is assumed to be a smooth surface, it is always possible to choose σ^*_{ij} and K^*_α such that $d\sigma_{ij}$ and dK_α given by (22.18) take the opposite values to those they had previously; an illustration of σ_{ij} and σ^*_{ij} and the new $d\sigma_{ij}$ is illustrated in Fig. 22.4b). In that case (22.19) takes the form

$$-(-d\sigma_{ij}p_{ij} + dK_\alpha q_\alpha) \geq 0 \tag{22.20}$$

and a comparison of (22.19) and (22.20) shows that $p_{ij} = 0$ and $q_\alpha = 0$; i.e. (22.15) reduces to the associated evolution laws $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial f/\partial\sigma_{ij}$ and $\dot{\kappa}_\alpha = -\dot{\lambda}\partial f/\partial K_\alpha$. Moreover, (22.16) reduces to

$$\dot{\lambda}[\frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \sigma^*_{ij}) + \frac{\partial f}{\partial K_\alpha}(K_\alpha - K^*_\alpha)] \geq 0 \tag{22.21}$$

We next have to prove that $\dot{\lambda} \geq 0$. For this purpose we choose the point $(\sigma^*_{ij}, K^*_\alpha)$ to be inside the yield surface. With (22.18) and since $(\sigma^*_{ij}, K^*_\alpha)$ is located inside the yield surface we have $df < 0$, i.e. (22.17) and (22.18) imply

$$-\frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij} - \sigma^*_{ij}) - \frac{\partial f}{\partial K_\alpha}(K_\alpha - K^*_\alpha) < 0$$

From this inequality and (22.21), we conclude that $\dot{\lambda} \geq 0$.

Finally, let us prove that the postulate of maximum dissipation given by (22.14) implies that the set of points C defined by $f(\sigma_{ij}, K_\alpha) \leq 0$ is a convex set of points. For the particular points (σ_{ij}, K_α) and $(\sigma^*_{ij}, K^*_\alpha)$ present in the postulate of maximum dissipation given by (22.14) we have $f(\sigma_{ij}, K_\alpha) = 0$ and $f(\sigma^*_{ij}, K^*_\alpha) \leq 0$; both these points therefore belong to the set of points C defined by $f(\sigma_{ij}, K_\alpha) \leq 0$. Assume that this set of points is not convex. Following (22.11) we can then have a situation where

$$f(\alpha\sigma^*_{ij} + (1-\alpha)\sigma_{ij},\ \alpha K^*_\beta + (1-\alpha)K_\beta) > 0$$

where $0 \leq \alpha \leq 1$. From this expression and since $f(\sigma_{ij}, K_\beta) = 0$ we obtain for $\alpha \neq 0$

$$\frac{f(\sigma_{ij} + \alpha(\sigma_{ij}^* - \sigma_{ij}),\ K_\beta + \alpha(K_\alpha^* - K_\beta)) - f(\sigma_{ij}, K_\beta)}{\alpha} > 0$$

Letting $\alpha \to 0$, the left-hand side of this expression is the so-called *directional derivative* of f when moving in the direction of $\sigma_{ij}^* - \sigma_{ij}$, $K_\beta^* - K_\beta$, see the discussion of (A.3). From (A.3), we then obtain

$$\frac{\partial f}{\partial \sigma_{ij}}(\sigma_{ij}^* - \sigma_{ij}) + \frac{\partial f}{\partial K_\beta}(K_\beta^* - K_\beta) > 0$$

where $\partial f/\partial \sigma_{ij}$ and $\partial f/\partial K_\beta$ are evaluated at the state (σ_{ij}, K_β). Multiplication by $-\dot{\lambda}$ (≤ 0) and use of the evolution laws give

$$\dot{\varepsilon}_{ij}^p(\sigma_{ij} - \sigma_{ij}^*) - \dot{\kappa}_\beta(K_\beta - K_\beta^*) < 0$$

However, this expression is in contradiction with the postulate of maximum dissipation given by (22.14); thus we have proved that the set of points defined by $f(\sigma_{ij}, K_\alpha) \leq 0$ must be convex. This concludes the proof of the results given in (22.14).

The assumptions and results in (22.14) are very close to the previous formulation of the postulate of maximum dissipation given by (22.4) and the resulting consequences given by the Kuhn-Tucker relations (22.5); however, there are subtle differences. In both cases, we obtain the associated evolution laws. In (22.4), however, the yield function is *a priori* assumed to be a convex function and as a result of the Kuhn-Tucker relations, we obtain $\dot{\lambda} \geq 0$ if $f = 0$ and $\dot{\lambda} = 0$ if $f < 0$. In (22.14), on the other hand, we *a priori* presuppose $f = 0$ to hold for plasticity to develop, but we can now instead prove that the set of points defined by $f(\sigma_{ij}, K_\alpha) \leq 0$ is convex.

While (22.4) and the resulting Kuhn-Tucker relations (22.5) are now used extensively in modern more advanced texts, the classical approach is that provided by (22.14). Indeed, we have seen some reference to (22.14) in relation to (9.39) - which emerged as a consequence of Drucker's postulate - where this expression was viewed as an example of the postulate of maximum dissipation. The expression in question reads

$$(\sigma_{ij} - \sigma_{ij}^*)\dot{\varepsilon}_{ij}^p \geq 0 \tag{22.22}$$

A comparison with (22.14) shows that (22.22) lacks the influence of the hardening parameters K_α and of the fluxes $\dot{\kappa}_\alpha$. However, it was proved in Chapter 9 that (22.22) leads to the associated flow rule $\dot{\varepsilon}_{ij}^p = \dot{\lambda}\partial f/\partial \sigma_{ij}$ whereas no information can be derived from (22.22) regarding the evolution laws for $\dot{\kappa}_\alpha$. However, one may state that if the postulate of maximum dissipation is accepted then the expression given by (22.14) makes physical sense whereas (22.22) cannot be justified (unless of course for ideal plasticity where no hardening parameters K_α and thereby no internal variables κ_α exist).

For further viewpoints and discussions, the reader is referred to the works of Moreau (1970) and Eve *et al.* (1990).

22.3 Examples of typical plasticity models

With the choice of Helmholtz' free energy ψ given by (22.2), the stresses σ_{ij} and hardening parameters K_α are obtained by (22.3). Moreover, the evolution laws are either given by (22.5) for associated plasticity or by (22.6) for nonassociated plasticity. The only remaining topic is the choice of the function $\rho\psi^p(\kappa_\alpha)$ present in (22.2) and thereby also the choice of the internal variables κ_α.

In this section, various choices of $\rho\psi^p(\kappa_\alpha)$ will be provided. Only some typical and illustrative cases will be presented, but they should enable the reader to easily make suitable generalizations; a number of specific models is also presented by Lemaitre and Chaboche (1990) and by Maugin (1992).

For ideal plasticity, no hardening parameters K_α and consequently no internal parameters exist, i.e.

$$\boxed{\psi^p(\kappa_\alpha) = 0 \quad \Rightarrow \quad \textit{ideal plasticity}}$$

Isotropic hardening of von Mises material

For associated plasticity, consider next isotropic hardening of a von Mises material. According to (12.4) the yield function is then given by

$$\begin{aligned} &f = \sigma_{eff} - \sigma_{yo} - K \\ &\text{where} \quad \sigma_{eff} = (\tfrac{3}{2}s_{kl}s_{kl})^{1/2} \quad \text{and} \quad \sigma_y = \sigma_{yo} + K \end{aligned} \tag{22.23}$$

where s_{ij} is the deviatoric stress tensor, σ_{y0} is the initial yield stress and σ_y is the current yield stress. It follows that

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}} \quad \text{i.e.} \quad \dot{\lambda} = \dot{\varepsilon}^p_{eff} = (\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij})^{1/2} \tag{22.24}$$

Since only one hardening parameter K exists, we have only one internal variable κ, cf. the evolution law $\dot{\kappa}_\alpha = -\dot{\lambda}\partial f/\partial K_\alpha$, which becomes

$$\dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda} = \dot{\varepsilon}^p_{eff} \tag{22.25}$$

It appears that the effective plastic strain ε^p_{eff} emerges as a natural internal variable for isotropic von Mises plasticity. This neat result is a consequence of the thermodynamic formulation whereas in Section 9.6 we had to posit various intuitive, but reasonable arguments for this choice. From (22.23) and (22.9) the plastic modulus H becomes

$$H = \rho \frac{d^2 \psi^p(\kappa)}{d\kappa^2}$$

Since $K \equiv \rho d\psi^p / d\kappa$ we obtain with $\kappa = \varepsilon^p_{eff}$

$$H = \frac{dK}{d\varepsilon^p_{eff}} = \frac{d(\sigma_{yo} + K)}{d\varepsilon^p_{eff}} = \frac{d\sigma_y}{d\varepsilon^p_{eff}}$$

This result is in agreement with (12.12).

If linear hardening is assumed then

$$\sigma_y = \sigma_{yo} + H\varepsilon^p_{eff}$$

where H is constant. Since $\sigma_y = \sigma_{yo} + K$, it follows that

$$K = \rho \frac{d\psi^p}{d\varepsilon^p_{eff}} = H\varepsilon^p_{eff}$$

i.e.

$$\rho\psi^p(\varepsilon^p_{eff}) = \frac{1}{2} H (\varepsilon^p_{eff})^2; \quad \varepsilon^p_{eff} = \kappa \tag{22.26}$$

If a Taylor expansion of $\psi^p(\varepsilon^p_{eff})$ is made about the point $\varepsilon^p_{eff} = 0$, we obtain $\rho\psi^p = A + B\varepsilon^p_{eff} + \frac{1}{2}H(\varepsilon^p_{eff})^2$ where higher order terms are ignored. Then $K = \rho d\psi^p / d\varepsilon^p_{eff} = B + H\varepsilon^p_{eff}$ and as $\varepsilon^p_{eff} = \kappa = 0$ must imply $K = 0$ - otherwise the yield condition for $\varepsilon^p_{eff} = 0$ does not coincide with the initial yield condition - $B = 0$ holds; moreover, the constant term A is of no influence. It then appears that if nothing is known *a priori*, the most simple and straightforward approach is to make a Taylor expansion of ψ^p and this leads to linear hardening.

If power law hardening is assumed, we have

$$\sigma_y = \sigma_{yo} + k\sigma_{yo}(\varepsilon^p_{eff})^n; \quad 0 < n \leq 1$$

cf. (9.4). Since $\sigma_y = \sigma_{yo} + K$, it follows that

$$K = \rho \frac{d\psi^p(\varepsilon^p_{eff})}{d\varepsilon^p_{eff}} = k\sigma_{yo}(\varepsilon^p_{eff})^n$$

i.e.

$$\rho\psi^p(\varepsilon^p_{eff}) = \frac{k\sigma_{yo}}{n+1}(\varepsilon^p_{eff})^{n+1}; \quad \varepsilon^p_{eff} = \kappa$$

Consider finally exponential hardening where

$$\sigma_y = \sigma_{yo} + K_\infty(1 - e^{-\frac{\varepsilon^p_{eff}}{\varepsilon_o}}) \tag{22.27}$$

cf. (9.3). Consequently

$$K = \rho\frac{d\psi^p(\varepsilon^p_{eff})}{d\varepsilon^p_{eff}} = K_\infty(1 - e^{-\frac{\varepsilon^p_{eff}}{\varepsilon_o}})$$

i.e.

$$\rho\psi^p(\varepsilon^p_{eff}) = K_\infty(\varepsilon^p_{eff} + \varepsilon_o e^{-\frac{\varepsilon^p_{eff}}{\varepsilon_o}}); \qquad \varepsilon^p_{eff} = \kappa$$

The associated evolution laws follow from the postulate of maximum dissipation, provided that the yield function $f = f(\sigma_{ij}, K_\alpha)$ is a convex function in σ_{ij} and K_α, cf. (22.4). For isotropic von Mises hardening, let us therefore prove formally that f is a convex function. According to the Appendix (A.8) the function $f = \sigma_{eff} - \sigma_{yo} - K$ is convex in σ_{ij} and K if and only if its *Hessian* is positive semi-definite. This implies that an arbitrary quadratic form I of the Hessian must be non-negative. With $(\bar{\sigma}_{ij}, \bar{K}_\alpha)$ being an arbitrary point we therefore require

$$\begin{aligned} I &= \begin{bmatrix} \bar{\sigma}_{ij} & \bar{K} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial\sigma_{ij}\partial\sigma_{kl}} & \frac{\partial^2 f}{\partial\sigma_{ij}\partial K} \\ \frac{\partial^2 f}{\partial K\partial\sigma_{ij}} & \frac{\partial^2 f}{\partial K^2} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{kl} \\ \bar{K} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\sigma}_{ij} & \bar{K} \end{bmatrix} \begin{bmatrix} A_{ijkl} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{kl} \\ \bar{K} \end{bmatrix} \geq 0 \end{aligned}$$

where

$$A_{ijkl} = \frac{3}{2\sigma^2_{eff}}\left\{\sigma_{eff}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] - s_{ij}\frac{3s_{kl}}{2\sigma_{eff}}\right\}$$

It follows that $I = \bar{\sigma}_{ij}A_{ijkl}\bar{\sigma}_{kl}$, i.e.

$$I = \frac{3}{2\sigma^2_{eff}}\left[\sigma_{eff}\bar{s}_{kl}\bar{s}_{kl} - \bar{s}_{ij}s_{ij}\frac{3s_{kl}\bar{s}_{kl}}{2\sigma_{eff}}\right] \geq 0 \tag{22.28}$$

It is always possible to write $\bar{s}_{ij}$ as $\bar{s}_{ij} = as_{ij} + bs^\perp_{ij}$ where the quantity $s^\perp_{ij}$ fulfills $s_{ij}s^\perp_{ij} = 0$ and a and b are constants. Then $\bar{s}_{ij}\bar{s}_{ij} = a^2 s_{ij}s_{ij} + b^2 s^\perp_{ij}s^\perp_{ij}$ and $\bar{s}_{ij}s_{ij} = as_{ij}s_{ij}$; with these expressions and $\sigma^2_{eff} = \frac{3}{2}s_{ij}s_{ij}$, (22.28) reduces to

$$I = \frac{3}{2\sigma_{eff}}b^2 s^\perp_{ij}s^\perp_{ij} \geq 0$$

which certainly is fulfilled. It follows that f is a convex function in σ_{ij} and K, i.e. the prerequisites for the Kuhn-Tucker relations (and thereby the associated evolution laws) are fulfilled; accordingly the postulate of maximum dissipation (22.4) ensures that $\gamma_{mech} \geq 0$.

However, in the present case of isotropic von Mises plasticity it is much easier to simply check *a posteriori* that $\gamma_{mech} \geq 0$ holds. With (22.24) and (22.25), we obtain

$$\gamma_{mech} \equiv \sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha \dot{\kappa}_\alpha = \dot{\lambda}(\sigma_{ij}\frac{3s_{ij}}{2\sigma_{eff}} - K) = \dot{\lambda}(\sigma_{eff} - K) \geq 0$$

which with the yield condition $f = 0$ becomes

$$\gamma_{mech} = \dot{\lambda}\sigma_{yo} \geq 0$$

that certainly is fulfilled. It is of interest that this expression for γ_{mech} holds irrespective of the particular hardening that is chosen.

A summary of these results reads

Isotropic von Mises hardening:

$$f = \sigma_{eff} - \sigma_{yo} - K$$

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}}$$

$$\dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda} = \dot{\varepsilon}^p_{eff}$$

$$\gamma_{mech} = \dot{\lambda}\sigma_{yo} \geq 0\,; \qquad H = \rho\frac{d^2\psi^p}{d\varepsilon^{p^2}_{eff}} \tag{22.29}$$

Linear hardening $\rho\psi^p = \frac{1}{2}(\varepsilon^p_{eff})^2 H$

Power law hardening $\rho\psi^p = \frac{k\sigma_{yo}}{n+1}(\varepsilon^p_{eff})^{n+1}$

Exponential hardening $\rho\psi^p = K_\infty(\varepsilon^p_{eff} + \varepsilon_o e^{-\frac{\varepsilon^p_{eff}}{\varepsilon_o}})$

Isotropic hardening of von Mises material – Nonassociated formulation

In the presentation above, it was more or less implied that there exists only one manner in which isotropic hardening of a von Mises material with a given stress-strain response can be achieved. This is not the case. Indeed there exists a number of formulations that lead to an identical stress-strain response; however, the mechanical dissipation connected with these formulations differs and this can be utilized when heat generation is modeled. The key point is that a nonassociated formulation is adopted.

Take the plastic part of Helmholtz' free energy according to

$$\rho\psi^p = \frac{1}{2}c\kappa^2 \tag{22.30}$$

where c is a constant. It follows that

$$K = \rho\frac{\partial\psi}{\partial\kappa} = c\kappa \tag{22.31}$$

As before the yield function is given by

$$f = \sigma_{eff} - \sigma_{yo} - K\,; \qquad \sigma_y = \sigma_{yo} + K$$

Nonassociated plasticity is adopted according to

$$g = f(\sigma_{ij}, K) + g^*(K) \tag{22.32}$$

The evolution equations then become

$$\begin{aligned} \dot{\varepsilon}^p_{ij} &= \dot{\lambda}\frac{\partial g}{\partial\sigma_{ij}} = \dot{\lambda}\frac{\partial f}{\partial\sigma_{ij}} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}} \\ \dot{\kappa} &= -\dot{\lambda}\frac{\partial g}{\partial K} = -\dot{\lambda}(-1 + \frac{\partial g^*}{\partial K}) \end{aligned} \tag{22.33}$$

It appears that the potential function has been chosen in such a way that the flow rule corresponds exactly to associated plasticity; it is only the evolution law for the internal variable that is influenced by the formulation being nonassociated.

As a specific example, suppose that we want to establish a model that exhibits exponential hardening. According to (22.27), we then require

$$K = K_\infty(1 - e^{-\frac{\varepsilon^p_{eff}}{\varepsilon_0}})$$

Let us assume that

$$g^* = \frac{K^2}{2K_\infty} \tag{22.34}$$

Taking advantage of (22.33b) leads to

$$\dot{\kappa} = \dot{\lambda}(1 - \frac{K}{K_\infty}) \tag{22.35}$$

From (22.33a) follows as usual that $\dot{\varepsilon}^p_{eff} = \dot{\lambda}$ and insertion of the expression above into (22.31) provides

$$\dot{K} = c\,\dot{\varepsilon}^p_{eff}(1 - \frac{K}{K_\infty})$$

Using $K(0) = 0$ an integration results in

$$K = K_\infty(1 - e^{-\frac{c\varepsilon^p_{eff}}{K_\infty}})$$

Thus by choosing $c = K_\infty/\varepsilon_o$ the nonassociated formulation given by (22.30), (22.32) and (22.34) results in the same stress-strain response as the associated formulation given by (22.29); even so, the mechanical dissipation relating to these two models differs.

By way of demonstration, let us determine the mechanical dissipation for the nonassociated formulation. From (22.33) and (22.35) we obtain

$$\gamma_{mech,nonas} = \sigma_{ij}\dot{\varepsilon}^p_{ij} - K\dot{\kappa} = \dot{\lambda}(\sigma_{eff} - K + \frac{K^2}{K_\infty})$$

Since $\sigma_{eff} = \sigma_{yo} + K$, it follows that

$$\gamma_{mech,nonas} = \dot{\lambda}(\sigma_{yo} + \frac{K^2}{K_\infty})$$

A comparison with (22.29) shows that $\gamma_{mech,nonas} \geq \gamma_{mech,ass}$. Since the mechanical dissipation enters the heat equation, we have now obtained a situation by which we can adjust the model so that for a given stress-strain response it exhibits different heat generation properties.

Linear kinematic hardening of von Mises material

Consider next linear kinematic hardening von Mises plasticity and, again, our purpose is to identify the quantity $\rho\psi^p$ present in (22.2). According to (12.37), we have

$$f(\sigma_{ij}, K_{ij}) = [\frac{3}{2}(s_{kl} - K^d_{kl})(s_{kl} - K^d_{kl})]^{1/2} - \sigma_{yo}$$

where K^d_{ij} is the deviatoric part of K_{ij} and $K^d_{ij} = \alpha^d_{ij}$ is the deviatoric *back-stress tensor*. Associated plasticity is assumed, i.e.

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} \quad \text{i.e.} \quad \dot{\lambda} = \dot{\varepsilon}^p_{eff}$$

and the evolution law for the internal variable becomes

$$\dot{\kappa}_{ij} = -\dot{\lambda}\frac{\partial f}{\partial K_{ij}} = -\dot{\lambda}\frac{\partial f}{\partial K^d_{mn}}\frac{\partial K^d_{mn}}{\partial K_{ij}} = \dot{\lambda}\frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} \tag{22.36}$$

It appears that κ_{ij} is a purely deviatoric quantity and that

$$\kappa_{ij} = \varepsilon^p_{ij}$$

From (22.1) the hardening parameters K_{ij} are obtained as

$$K_{ij} = \rho \frac{\partial \psi^p(\kappa_{kl})}{\partial \kappa_{ij}}$$

i.e.

$$\dot{K}_{ij} = \rho \frac{\partial^2 \psi^p(\kappa_{mn})}{\partial \kappa_{ij} \partial \kappa_{kl}} \dot{\kappa}_{kl} = \rho \frac{\partial^2 \psi^p(\kappa_{mn})}{\partial \kappa_{ij} \partial \kappa_{kl}} \dot{\varepsilon}^p_{kl} \tag{22.37}$$

To obtain linear *Melan-Prager kinematic hardening*, cf. (12.46) and (12.47), we require

$$\dot{K}_{ij} = \dot{K}^d_{ij} = \dot{\alpha}^d_{ij} = c\dot{\varepsilon}^p_{ij}$$

where c is a constant. A comparison with (22.37) shows that

$$\rho \frac{\partial^2 \psi^p(\kappa_{mn})}{\partial \kappa_{ij} \partial \kappa_{kl}} = c\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

As $\kappa_{ij} = \varepsilon^p_{ij} = 0$ must imply $K_{ij} = \rho \partial \psi^p / \partial \kappa_{ij} = 0$ - otherwise the initial yield condition makes no sense - we obtain

$$\rho \psi^p = \frac{1}{2} c \kappa_{ij} \kappa_{ij} \tag{22.38}$$

Moreover, from (22.9) and as associated plasticity is adopted, we obtain with (22.36) and (22.38)

$$\begin{aligned} H &= \frac{\partial f}{\partial K_{ij}} \rho \frac{\partial^2 \psi^p}{\partial \kappa_{ij} \partial \kappa_{kl}} \frac{\partial f}{\partial K_{kl}} \\ &= \frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} c\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{3(s_{kl} - K^d_{kl})}{2\sigma_{yo}} \end{aligned}$$

i.e.

$$H = \frac{3}{2} c$$

in accordance with (12.49). Just like linear isotropic hardening can be obtained by making a Taylor expansion of $\psi^p(\varepsilon^p_{eff})$, cf. the discussion following (22.26), it is of interest that simply by making a Taylor expansion of $\psi^p(\kappa_{ij})$ we are led to Melan-Prager's kinematic hardening rule, cf. (22.38).

Linear mixed von Mises hardening

Consider next linear mixed von Mises hardening. From (12.54) we then have

$$f(\sigma_{ij}, K_{ij}, K) = [\frac{3}{2}(s_{kl} - K^d_{kl})(s_{kl} - K^d_{kl})]^{1/2} - \sigma_{yo} - K$$

where $K^d_{ij} = \alpha^d_{ij}$. It follows that

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij} - K^d_{ij})}{2\bar{\sigma}_{yo}} \quad \text{i.e.} \quad \dot{\lambda} = \dot{\varepsilon}^p_{eff}$$

where

$$\bar{\sigma}_{yo} = \sigma_{yo} + K$$

Moreover, with the internal variables κ_{ij} conjugated to K_{ij} and the internal variable κ conjugated to K, the associated evolution laws give

$$\dot{\kappa}_{ij} = -\dot{\lambda}\frac{\partial f}{\partial K_{ij}} = \dot{\lambda}\frac{(s_{ij} - K^d_{ij})}{2\bar{\sigma}_{yo}} = \dot{\varepsilon}^p_{ij}; \quad \dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda} = \dot{\varepsilon}^p_{eff} \tag{22.39}$$

From (22.1) and as $\rho\psi^p(\kappa_{ij}, \kappa)$

$$K_{ij} = \rho\frac{\partial \psi^p(\kappa_{mn}, \kappa)}{\partial \kappa_{ij}}; \quad K = \rho\frac{\partial \psi^p(\kappa_{ij}, \kappa)}{\partial \kappa}$$

These expressions imply that

$$\begin{aligned} \dot{K}_{ij} &= \rho\frac{\partial^2 \psi^p}{\partial \kappa_{ij}\partial \kappa_{kl}}\dot{\kappa}_{kl} + \rho\frac{\partial^2 \psi^p}{\partial \kappa_{ij}\partial \kappa}\dot{\kappa} \\ \dot{K} &= \rho\frac{\partial^2 \psi^p}{\partial \kappa\partial \kappa_{ij}}\dot{\kappa}_{ij} + \rho\frac{\partial^2 \psi^p}{\partial \kappa^2}\dot{\kappa} \end{aligned} \tag{22.40}$$

Since mixed linear hardening is considered, we require with (12.65), (12.67) and (12.75) that

$$\begin{aligned} \dot{K}_{ij} &= \dot{K}^d_{ij} = \dot{\alpha}^d_{ij} = (1-m)c\dot{\varepsilon}^p_{ij} = (1-m)c\dot{\kappa}_{ij} \\ \dot{K} &= m\frac{3}{2}c\dot{\lambda} = m\frac{3}{2}c\dot{\kappa} \end{aligned} \tag{22.41}$$

where the *mixed hardening parameter* m is in the range $0 \le m \le 1$; for $m = 0$ we obtain a purely linear kinematic hardening whereas $m = 1$ implies a purely linear isotropic hardening. A comparison of (22.40) and (22.41) gives

$$\rho\psi^p = \frac{1}{2}(1-m)c\kappa_{ij}\kappa_{ij} + \frac{3}{4}mc\kappa^2$$

Indeed this expression is not surprising since it can also be obtained by writing $\rho\psi^p(\kappa_{ij}, \kappa)$ as $\rho\psi^p(\kappa_{ij}, \kappa) = (1-m)p(\kappa_{ij}) + mq(\kappa)$ and then making a Taylor expansion of the functions $p(\kappa_{ij})$ and $q(\kappa)$.

For the plastic modulus H, we obtain with (22.9)

$$H = \frac{\partial f}{\partial K_{ij}} \rho \frac{\partial^2 \psi^p}{\partial \kappa_{ij} \partial \kappa_{kl}} \frac{\partial f}{\partial K_{kl}} + \frac{\partial f}{\partial K} \rho \frac{\partial^2 \psi^p}{\partial \kappa^2} \frac{\partial f}{\partial K} = \frac{3}{2} c$$

in accordance with (12.76).

Let us finally calculate the mechanical dissipation. From (22.1) and (22.39) and the yield condition $f = 0$ follow that

$$\begin{aligned} \gamma_{mech} &= \sigma_{ij}\dot{\varepsilon}^p_{ij} - K_{ij}\dot{\kappa}_{ij} - K\dot{\kappa} \\ &= \dot{\lambda}[\sigma_{ij}\frac{3(s_{ij} - K^d_{ij})}{2\bar{\sigma}_{yo}} - K_{ij}\frac{3(s_{ij} - K^d_{ij})}{2\bar{\sigma}_{yo}} - K] \\ &= \dot{\lambda}(\bar{\sigma}_{yo} - K) = \dot{\lambda}\sigma_{yo} \geq 0 \end{aligned} \tag{22.42}$$

and this inequality is certainly fulfilled. It is of interest that γ_{mech} is independent of the mixed hardening parameter m and that (22.42) coincides with expression (22.29) valid for arbitrary isotropic von Mises hardening.

In summary, we have

Linear mixed von Mises hardening:

$$f = [\frac{3}{2}(s_{ij} - K^d_{ij})(s_{ij} - K^d_{ij})]^{1/2} - \bar{\sigma}_{yo}\,; \qquad \bar{\sigma}_{yo} = \sigma_{yo} + K$$

$$\rho\psi^p = \frac{1}{2}(1-m)c\kappa_{ij}\kappa_{ij} + \frac{3}{4}mc\kappa^2\,; \qquad 0 \leq m \leq 1$$

This implies

$$\begin{aligned} &\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij} - K^d_{ij})}{2\bar{\sigma}_{yo}}\,; \qquad \dot{\lambda} = \dot{\varepsilon}^p_{eff} \\ &\dot{\kappa}_{ij} = -\dot{\lambda}\frac{\partial f}{\partial K_{ij}} = \dot{\varepsilon}^p_{ij} \\ &\dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda} \end{aligned} \tag{22.43}$$

Moreover

$$\gamma_{mech} = \dot{\lambda}\sigma_{yo} \geq 0$$

$$\dot{K}_{ij} = \dot{K}^d_{ij} = (1-m)c\dot{\varepsilon}^p_{ij}\,; \qquad \dot{K} = \dot{\lambda}\frac{3}{2}mc$$

$$H = \frac{3}{2}c$$

Armstrong-Frederick kinematic hardening

Let us next turn to Armstrong-Frederick kinematic hardening, which, within a kinematic hardening concept, allows nonlinear hardening to be modeled in an elegant fashion. The yield function is again given by

$$f = [\frac{3}{2}(s_{kl} - K^d_{kl})(s_{kl} - K^d_{kl})]^{1/2} - \sigma_{yo} \tag{22.44}$$

Following Armstrong and Frederick (1966) and in accordance with (13.70), it is required that

$$\dot{K}_{ij} = \dot{K}^d_{ij} = \dot{\alpha}^d_{ij} = h(\frac{2}{3}\dot{\varepsilon}^p_{ij} - \frac{K^d_{ij}}{\alpha_\infty}\dot{\varepsilon}^p_{eff}) \tag{22.45}$$

where h and α_∞ are constant parameters and $\alpha_\infty \geq 0$; it appears that the Armstrong-Frederick formulation degenerates to Melan-Prager hardening for $\alpha_\infty \to \infty$ and $h = 3c/2$. According to (22.43), we have seen that an associated format leads to Melan-Prager hardening and to obtain (22.45) a nonassociated format must therefore be adopted. However, as the plastic flow rule is still to be given in the form $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial f/\partial\sigma_{ij}$, we are led to the conclusion that the evolution law for $\dot{\varepsilon}^p_{ij}$ is to follow an associated format whereas the evolution law for $\dot{\kappa}_{ij}$ must follow a nonassociated format. The result is a potential function of the form

$$g(\sigma_{ij}, K_{ij}) = f(\sigma_{ij}, K_{ij}) + g^*(K_{ij}) \tag{22.46}$$

With the general nonassociated formulation given by (22.6), use of (22.44) and (22.46) give

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} \quad \text{i.e.} \quad \dot{\lambda} = \dot{\varepsilon}^p_{eff} \tag{22.47}$$

Moreover

$$\begin{aligned} \dot{\kappa}_{ij} &= -\dot{\lambda}\frac{\partial g}{\partial K_{ij}} = -\dot{\lambda}\frac{\partial f}{\partial K_{ij}} - \dot{\lambda}\frac{\partial g^*}{\partial K_{ij}} \\ &= \dot{\varepsilon}^p_{ij} - \dot{\lambda}\frac{\partial g^*}{\partial K_{ij}} \end{aligned} \tag{22.48}$$

and (22.1) gives

$$K_{ij} = \rho\frac{\partial \psi^p(\kappa_{mn})}{\partial \kappa_{ij}}$$

i.e.

$$\dot{K}_{ij} = \rho\frac{\partial^2 \psi^p}{\partial \kappa_{ij}\partial \kappa_{kl}}\dot{\kappa}_{kl}$$

Insertion of (22.48) and use of (22.47b) provide

$$\dot{K}_{ij} = \rho \frac{\partial^2 \psi^p}{\partial \kappa_{ij} \partial \kappa_{kl}} \dot{\varepsilon}^p_{kl} - \rho \frac{\partial^2 \psi^p}{\partial \kappa_{ij} \partial \kappa_{kl}} \frac{\partial g^*}{\partial K_{kl}} \dot{\varepsilon}^p_{eff} \tag{22.49}$$

It appears that if we choose

$$\rho \psi^p = \frac{1}{3} h \kappa_{ij} \kappa_{ij}$$

and

$$g^* = \frac{3}{2\alpha_\infty} K^d_{ij} K^d_{ij} \tag{22.50}$$

then (22.49) reduces to (22.45). It may be of interest to observe that the most simple expression for $g^*(K_{ij})$ is obtained by a straightforward Taylor expansion which, in fact, is the one given by (22.50).

The plastic modulus H defined by (22.9) becomes

$$\begin{aligned} H &= \frac{\partial f}{\partial K_{ij}} \rho \frac{\partial^2 \psi^p}{\partial \kappa_{ij} \partial \kappa_{kl}} \frac{\partial g}{\partial K_{kl}} \\ &= -\frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} \frac{h}{3} \frac{1}{2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) [-\frac{3(s_{kl} - K^d_{kl})}{2\sigma_{yo}} + \frac{3}{\alpha_\infty} K^d_{kl}] \end{aligned}$$

i.e.

$$H = h[1 - \frac{3(s_{ij} - K^d_{ij}) K^d_{ij}}{2\alpha_\infty \sigma_{yo}}]$$

in accordance with (13.78).

Let us finally determine the mechanical dissipation. From (22.47), (22.48) and (22.50), we obtain

$$\begin{aligned} \gamma_{mech} &= \sigma_{ij} \dot{\varepsilon}^p_{ij} - K^d_{ij} \dot{\kappa}_{ij} \\ &= \dot{\lambda} [\sigma_{ij} \frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} - K^d_{ij} \frac{3(s_{ij} - K^d_{ij})}{2\sigma_{yo}} + \frac{3}{\alpha_\infty} K^d_{ij} K^d_{ij}] \geq 0 \end{aligned}$$

which with the yield condition $f = 0$ gives

$$\gamma_{mech} = \dot{\lambda} (\sigma_{yo} + \frac{3}{\alpha_\infty} K^d_{ij} K^d_{ij}) \geq 0$$

which certainly is fulfilled.

In conclusion

Armstrong-Frederick kinematic hardening:

$$f = [\frac{3}{2}(s_{ij}-K_{ij}^d)(s_{ij}-K_{ij}^d)]^{1/2}-\sigma_{yo}$$

$$g = f + \frac{3}{2\alpha_\infty}K_{ij}^d K_{ij}^d$$

$$\rho\psi^p = \frac{h}{3}\kappa_{ij}\kappa_{ij}$$

This implies

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3(s_{ij}-K_{ij}^d)}{2\sigma_{yo}}; \qquad \dot{\lambda} = \dot{\varepsilon}_{eff}^p$$

$$\dot{\kappa}_{ij} = -\dot{\lambda}\frac{\partial g}{\partial K_{ij}} = \dot{\varepsilon}_{ij}^p - \dot{\lambda}\frac{3}{\alpha_\infty}K_{ij}^d; \qquad \kappa_{ij} = \kappa_{ij}^d$$

Moreover

$$\gamma_{mech} = \dot{\lambda}(\sigma_{yo} + \frac{3}{\alpha_\infty}K_{ij}^d K_{ij}^d) \geq 0$$

$$K_{ij} = \rho\frac{\partial\psi^p}{\partial\kappa_{ij}} = \frac{2h}{3}\kappa_{ij}$$

$$i.e.\ \dot{K}_{ij} = h(\frac{2}{3}\dot{\varepsilon}_{ij}^p - \frac{1}{\alpha_\infty}K_{ij}^d\dot{\lambda}) \quad and \quad K_{ij} = K_{ij}^d$$

$$H = h[1 - \frac{3(s_{ij}-K_{ij}^d)K_{ij}^d}{2\alpha_\infty\sigma_{yo}}]$$

We emphasize that this formulation of Armstrong-Frederick hardening is an interesting example of a format that is intrinsically nonassociated. However, the potential function is such that the plastic flow rule turns out to be associated; thus, only the evolution law for the internal variables is nonassociated.

Isotropic Drucker-Prager hardening

As a simplistic prototype for soil and concrete mechanics consider finally isotropic Drucker-Prager hardening. From (12.114), we have

$$f = \sigma_{eff} + \alpha I_1 - \beta - K$$

where - for practical purposes - both α and β are non-negative parameters. Nonassociated plasticity is adopted and the potential function is taken as

$$g = \sigma_{eff} + \alpha^* I_1 - \beta^* - K$$

where α^* is a non-negative parameter and β^* is a parameter. It appears that associated plasticity is obtained if $\alpha^* = \alpha$ and $\beta^* = \beta$. The plastic flow rule becomes

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}(\frac{3s_{ij}}{2\sigma_{eff}} + \alpha^*\delta_{ij}) \tag{22.51}$$

i.e. $\dot{\varepsilon}^p_{ii} = 3\dot{\lambda}\alpha^*$ and

$$\dot{\varepsilon}^p_{eff} = (\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij})^{1/2} = \dot{\lambda}\sqrt{1 + 2(\alpha^*)^2}$$

The evolution law for the internal variable κ becomes

$$\dot{\kappa} = -\dot{\lambda}\frac{\partial g}{\partial K} = \dot{\lambda} \tag{22.52}$$

and it is observed that the internal variable is proportional to the effective plastic strain.

Using (22.51) and (22.52), the mechanical dissipation takes the form

$$\begin{aligned} \gamma_{mech} &= \sigma_{ij}\dot{\varepsilon}^p_{ij} - K\dot{\kappa} = \dot{\lambda}[\sigma_{ij}(\frac{3s_{ij}}{2\sigma_{eff}} + \alpha^*\delta_{ij}) - K] \\ &= \dot{\lambda}(\sigma_{eff} + \alpha^* I_1 - K) \geq 0 \end{aligned}$$

Use of the yield condition $f = 0$ implies

$$\gamma_{mech} = \dot{\lambda}[\beta - I_1(\alpha - \alpha^*)] \geq 0 \tag{22.53}$$

Since the model primarily intends to work for compressive stresses, (22.53) must be fulfilled for arbitrary negative values of I_1 and we therefore require

$$\alpha \geq \alpha^* \tag{22.54}$$

This restriction is shown in Fig. 22.5

Leaving aside the specific Drucker-Prager material in question and considering frictional materials in general, it is found experimentally, see for instance Ko and Scott (1968) and Lade and Duncan (1973), that the frictional angle (α) is larger than the dilatancy angle (α^*). The theoretical result (22.54) is in neat agreement with this experimental evidence and it underlines the significance of nonassociated plasticity for frictional materials.

Let us next confine ourselves to associated plasticity. In that case $\alpha^* = \alpha$ and (22.53) reduces to

$$\gamma_{mech} = \dot{\lambda}\beta \geq 0$$

For $\beta > 0$ this expression is certainly fulfilled. However, if a cohesionless material like sand is considered, $\beta = 0$ holds and we always obtain $\gamma_{mech} = 0$. Since

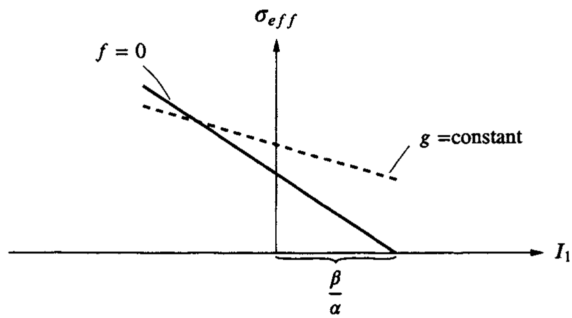

Figure 22.5: Illustration of the restriction $\alpha \geq \alpha^*$.

$\gamma_{mech} > 0$ implies an irreversible material behavior whereas $\gamma_{mech} = 0$ implies a reversible material behavior, the model in question will imply the development of plastic strains and even so it will be reversible! This evident contradiction cannot be accepted and it is concluded that associated isotropic hardening of a cohesionless Drucker-Prager material violates the physical implication of the second law of thermodynamics. If we furthermore restrict ourselves to ideal plasticity, the rejection of such a model has long been known in the literature without, however, referring to strict thermodynamical arguments. Instead, it was observed, see for instance Vermeer and de Borst (1984) that the rate of plastic work for associated ideal plasticity of a cohesionless Drucker-Prager material becomes $\dot{W}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij} = \dot{\lambda}\sigma_{ij}[3s_{ij}/(2\sigma_{eff}) + \alpha\delta_{ij}] = \dot{\lambda}(\sigma_{eff} + \alpha I_1)$ and as the yield condition reads $f = \sigma_{eff} + \alpha I_1 = 0$, we obtain $\dot{W}^p = 0$. A graphical illustration of this result is shown in Fig. 22.6 where the observation that $\dot{\varepsilon}^p_{ij}$ is orthogonal to the yield surface immediately implies that the scalar product $\sigma_{ij}\dot{\varepsilon}^p_{ij} = 0$. Here we can extend this rejection also to include the rejection of isotropic hardening/softening of a cohesionless associated Drucker-Prager material; we additionally note that these rejections are intimately connected with the existence of straight meridians in the Drucker-Prager criterion, cf. Fig. 22.6.

For a cohesionless nonassociated formulation, (22.53) reduces to $\gamma_{mech} = -\dot{\lambda}I_1(\alpha - \alpha^*) \geq 0$, which is only fulfilled for $I_1 < 0$, i.e. we must exclude hardening and only ideal and softening plasticity are allowed.

In general, from (22.53) it is concluded that $\gamma_{mech} \geq 0$ requires

$$I_1 \leq \frac{\beta}{\alpha - \alpha^*}$$

From (80) and $f = 0$ the maximum value of I_1 is given by $I_{1,max} = (\beta + K)/\alpha$ and if the model is to be thermodynamically acceptable this $I_{1,max}$-value must

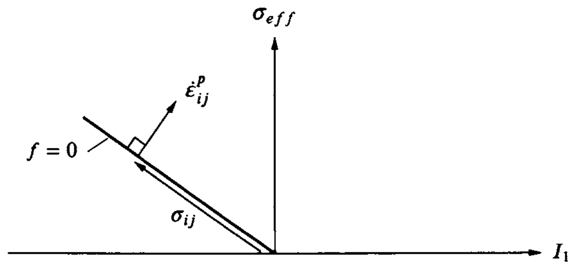

Figure 22.6: Illustration that $\dot{W}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij} = 0$ for associated ideal plasticity of a cohesionless Drucker-Prager material.

be less than or equal to the limit given above. This leads to the following restriction

$$K(\alpha - \alpha^*) \leq \beta\alpha^* \tag{22.55}$$

Considering in the following a Drucker-Prager material with cohesion (i.e. $\beta > 0$) and as $\alpha^* \geq 0$ then (22.55) is certainly fulfilled for softening plasticity (i.e. $K < 0$) and for ideal plasticity (i.e. $K = 0$); however, hardening nonassociated plasticity places a restriction on the amount of hardening.

With these observations, (22.9) gives the following expression for the generalized plastic modulus

$$H = \frac{\partial f}{\partial K}\rho\frac{d^2\psi^p(\kappa)}{d\kappa^2}\frac{\partial g}{\partial K} = \rho\frac{d^2\psi^p}{d\kappa^2}$$

Let us in particular assume linear hardening where H is constant. Then the expression above gives

$$\rho\psi^p = \frac{1}{2}H\kappa^2$$

Moreover

$$K = \rho\frac{d\psi^p}{d\kappa} = H\kappa$$

Since the hardening parameter K for linear hardening can take arbitrarily large positive values, (22.55) will, for nonassociated plasticity, be violated at some state; this formulation must thus be rejected.

In conclusion

Drucker-Prager isotropic hardening:

$$f = \sigma_{eff} + \alpha I_1 - \beta - K$$
$$g = \sigma_{eff} + \alpha^* I_1 - \beta^* - K$$

where $\alpha \geq 0$, $\beta \geq 0$ *and* $\alpha^* \geq 0$

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}(\frac{3s_{ij}}{2\sigma_{eff}} + \alpha^*\delta_{ij}); \qquad \dot{\varepsilon}^p_{eff} = \dot{\lambda}\sqrt{1 + 2(\alpha^*)^2}$$
$$\dot{\kappa} = -\dot{\lambda}\frac{\partial g}{\partial K} = \dot{\lambda}$$
$$\gamma_{mech} = \dot{\lambda}[\beta - I_1(\alpha - \alpha^*)] \geq 0$$

It is required that $\alpha \geq \alpha^*$

Material without cohesion: associated plasticity is not allowed; nonassociated plasticity valid only for ideal and softening plasticity

Material with cohesion: Associated plasticity is always allowed; nonassociated plasticity is valid for ideal and softening plasticity, but restrictions apply for hardening plasticity and linear hardening is not allowed

$$H = \rho\frac{d^2\psi(\kappa)}{d\kappa^2}; \qquad K = \rho\frac{d\psi^p(\kappa)}{d\kappa}$$

Linear hardening: $\rho\psi^p = \frac{1}{2}H\kappa^2$

22.4 Use of plastic work as internal variable

We have seen that when an internal variable takes the form of a scalar, this internal variable often turns out to be the effective plastic strain or a quantity proportional to the effective plastic strain; examples are isotropic and mixed von Mises hardening and isotropic Drucker-Prager hardening. Therefore, the natural manifestation of a scalar internal variable seems to be that of effective plastic strain.

In Section 9.6, isotropic von Mises hardening was introduced and two choices for the internal variable were discussed: *strain hardening* where κ was chosen as the effective plastic strain ε^p_{eff} and *work hardening* where κ was chosen as the plastic work W^p. It was shown that the two assumptions lead to the same model, since there exists a one-to-one relation between ε^p_{eff} and W^p. However, it was emphasized that whereas this conclusion holds for isotropic von Mises plasticity, it is not a general conclusion that holds for arbitrary plasticity models.

With the format of the specific plasticity models discussed above, we observed that the natural interpretation of the scalar internal variable κ is that of ε^p_{eff} (or proportional to ε^p_{eff}). However, in many specific plasticity models used in the literature - and especially for those plasticity models that are derived in the traditional way without making use of thermodynamics - the plastic work W^p is often used as an internal variable. Thus, let us establish a framework which provides W^p as the natural interpretation of the scalar internal variable κ. To do so, we will take advantage of the results presented by Ristinmaa (1999).

In soil and concrete constitutive modeling, the yield function and potential function are often written as

$$f(\sigma_{ij}, K) = p(K)F(\sigma_{ij}) - c_1 ; \quad g(\sigma_{ij}, K) = q(K)G(\sigma_{ij}) - c_2 \qquad (22.56)$$

i.e. only one scalar hardening parameter K and therefore only one scalar internal variable κ is involved; moreover, c_1 and c_2 are constants. Formulation (22.56) applies, for instance, to the soil and concrete model of Lade and Kim (1995). The characteristic feature of (22.56) is that the influence of σ_{ij} and K appears in a factorized form. Adopting nonassociated plasticity, it follows from (22.6) that

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda}q(K)\frac{\partial G(\sigma_{kl})}{\partial \sigma_{ij}} ; \quad \dot{\kappa} = -\dot{\lambda}\frac{\partial g}{\partial K} = -\dot{\lambda}\frac{dq(K)}{dK}G(\sigma_{ij}) \qquad (22.57)$$

Assume that the function $G(\sigma_{ij})$ is homogeneous in σ_{ij} of degree n. From Euler's theorem (21.47), we then have

$$\sigma_{ij}\frac{\partial G}{\partial \sigma_{ij}} = nG \qquad (22.58)$$

The rate of plastic work then becomes

$$\dot{W}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij} = \dot{\lambda}q\sigma_{ij}\frac{\partial G}{\partial \sigma_{ij}} = \dot{\lambda}qnG \qquad (22.59)$$

Since we want

$$\dot{\kappa} = \dot{W}^p$$

insertion of (22.57b) and (22.59) gives

$$-\frac{dq(K)}{dK} = nq$$

with the solution

$$q = e^{-nK} \qquad (22.60)$$

With (22.56), (22.58) and (22.60), we conclude that the internal variable becomes $\kappa = W^p$. Indeed, this formulation was adopted by Lade and Kim (1995) using traditional plasticity theory whereas we have here followed Ristinmaa (1999) for its thermodynamic formulation. As another implication of thermodynamics, the dissipation inequality becomes

$$\gamma_{mech} = \sigma_{ij}\dot{\varepsilon}^p_{ij} - K\dot{\kappa} = \dot{\lambda}(qnG + K\frac{dq}{dK}G) \geq 0$$

by using (22.59) and (22.57b). With (22.60), we arrive at

$$\gamma_{mech} = \dot{\lambda}Gne^{-nK}(1 - K) \geq 0$$

and it is required that the hardening parameter K fulfills $K < 1$. It turns out that in the formulation of Lade and Kim (1995), this restriction is fulfilled.

22.5 Corner plasticity

Many yield and potential surfaces contain *corners* and examples are given by the criteria of Tresca and Coulomb. Considering for simplicity associated plasticity we have used the format $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial f/\partial\sigma_{ij}$, but this certainly presumes that the yield surface is smooth. Considering ideal plasticity and adopting the postulate of maximum plastic dissipation, we have $(\sigma_{ij} - \sigma^*_{ij})\dot{\varepsilon}^p_{ij} \geq 0$ where σ_{ij} is the current stress state located on the yield surface, σ^*_{ij} is an arbitrary point on or inside the yield surface and the plastic strain rate $\dot{\varepsilon}^p_{ij}$ is related to the current stress state σ_{ij}. Regarding Fig. 22.7 this implies that $\dot{\varepsilon}^p_{ij}$ is located somewhere within the 'cone' defined by the two normals at the corner; as a result, $\dot{\varepsilon}^p_{ij}$ can be written as a linear combination of these two normals.

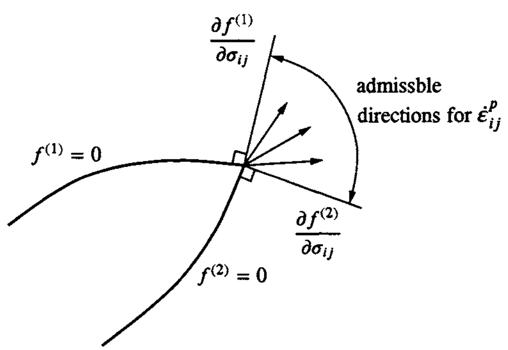

Figure 22.7: Stress space with two yield functions. Admissible directions for $\dot{\varepsilon}^p_{ij}$ at a corner.

Generalizing these ideas and assuming that F_{max} is the number of yield surfaces that meet at a corner, we are led to

Koiter's flow rule

$$\dot{\varepsilon}_{ij}^p = \sum_{I=1}^{F_{max}} \dot{\lambda}^I \frac{\partial f^I}{\partial \sigma_{ij}}$$

as suggested by Koiter (1953, 1960); here f^I, $I = 1, 2, \ldots, F_{max}$ denote the yield functions and $\dot{\lambda}^I$ the corresponding plastic multipliers.

Following Ottosen and Ristinmaa (1996) we will now generalize Koiter's flow rule both for associated and nonassociated plasticity. Each of the yield functions depends on the stresses and the hardening parameters

$$f^I = f^I(\sigma_{ij}, K_\alpha)\,; \qquad I = 1, 2, \ldots, F_{max}$$

As usual, the mechanical dissipation is given by $\gamma_{mech} = \sigma_{ij}\dot{\varepsilon}_{ij}^p - K_\alpha \dot{\kappa}_\alpha$. Adopting the postulate of maximum dissipation we are faced with the following problem: for given $\dot{\varepsilon}_{ij}^p$ and $\dot{\kappa}_\alpha$, minimize the quantity $-\gamma_{mech}$ subject to the constraints $f^I \leq 0$. From the Kuhn-Tucker relations given by Appendix (A.15), we obtain

Associated plasticity

$$\dot{\varepsilon}_{ij}^p = \sum_{I=1}^{F_{max}} \dot{\lambda}^I \frac{\partial f^I}{\partial \sigma_{ij}}\,; \qquad \dot{\kappa}_\alpha = -\sum_{I=1}^{F_{max}} \dot{\lambda}^I \frac{\partial f^I}{\partial K_\alpha}$$

where

$$\dot{\lambda}^I \geq 0 \quad \textit{and} \quad \dot{\lambda}^I f^I = 0$$

Following Appendix (A.13), a prerequisite for this result is that the point is a *regular point* meaning that

$$\begin{aligned} &\frac{\partial f^I}{\partial \sigma_{ij}} \quad \text{are linearly independent} \\ &\frac{\partial f^I}{\partial K_\alpha} \quad \text{are linearly independent} \end{aligned} \tag{22.61}$$

Moreover, according to Appendix (A.18) it is required that f^I are convex functions.

For nonassociated plasticity, the potential functions are given by

$$g^\Phi = g^\Phi(\sigma_{ij}, K_\alpha)\,; \qquad \Phi = 1, 2, \ldots, G_{max}$$

where G_{max} is the number of potential functions meeting at the corner; in general, we will allow the number of yield surfaces F_{max} to differ from the number of potential surfaces G_{max} and later we will return to this issue. It is required that the potential functions are convex and that $g^{\Phi}(\sigma_{ij}, K_\alpha) - g^{\Phi}(0,0) \geq 0$. Following (21.51) it is then evident that the following evolution laws fulfill the dissipation inequality

Nonassociated plasticity

$$\dot{\varepsilon}^p_{ij} = \sum_{\Phi=1}^{G_{max}} \dot{\lambda}^{\Phi} \frac{\partial g^{\Phi}}{\partial \sigma_{ij}}; \qquad \dot{\kappa}_\alpha = -\sum_{\Phi=1}^{G_{max}} \dot{\lambda}^{\phi} \frac{\partial g^{\phi}}{\partial K_\alpha} \tag{22.62}$$

where

$$\dot{\lambda}^{\phi} \geq 0$$

In accordance with (22.2) we choose Helmholtz' free energy according to

$$\rho\psi(\theta, \varepsilon_{ij} - \varepsilon^p_{ij}, \kappa_\alpha) = \rho h(\theta) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon^p_{ij}) D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl}) + \rho\psi^p(\kappa_\alpha)$$

This leads to

$$\sigma_{ij} = \rho\frac{\partial \psi}{\partial \varepsilon_{ij}} = D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})$$

$$K_\alpha = \rho\frac{\partial \psi}{\partial \kappa_\alpha} = \rho\frac{\partial \psi^p}{\partial \kappa_\alpha}$$

and it follows that

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}); \qquad \dot{K}_\alpha = d_{\alpha\beta}\dot{\kappa}_\beta \tag{22.63}$$

where

$$\boxed{D_{ijkl} = \rho\frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}; \qquad d_{\alpha\beta} = \rho\frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta}}$$

With these fundamental equations, we will now derive the incremental relation $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\varepsilon}_{kl}$. Differentiation of the yield function f^I gives

$$\dot{f}^I = \frac{\partial f^I}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f^I}{\partial K_\alpha}\dot{K}_\alpha$$

Insertion of (22.63b) gives

$$\dot{f}^I = \frac{\partial f^I}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f^I}{\partial K_\alpha} d_{\alpha\beta}\dot{\kappa}_\beta$$

and the evaluation law (22.62b) then provides

$$\boxed{\begin{array}{l} \dot{f}^I = \dfrac{\partial f^I}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - \displaystyle\sum_{\Phi=1}^{G_{max}} H^{I\Phi}\dot{\lambda}^{\Phi} \\ \textit{where} \\ H^{I\Phi} = \dfrac{\partial f^I}{\partial K_\alpha} d_{\alpha\beta} \dfrac{\partial g^{\Phi}}{\partial \kappa_\beta} \\ \textit{is the matrix of plastic moduli} \end{array}} \tag{22.64}$$

Insertion of the flow rule (22.62a) into Hooke's law (22.63a) gives

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} - D_{ijst} \sum_{\Phi=1}^{G_{max}} \frac{\partial g^{\Phi}}{\partial \sigma_{st}} \dot{\lambda}^{\Phi} \tag{22.65}$$

and with this expression (22.64) becomes

$$\dot{f}^I = \dot{a}^I - \sum_{\Phi=1}^{G_{max}} A^{I\Phi}\dot{\lambda}^{\Phi} \tag{22.66}$$

where

$$\dot{a}^I = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl}$$

and

$$\boxed{A^{I\Phi} = H^{I\Phi} + \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g^{\Phi}}{\partial \sigma_{kl}}} \tag{22.67}$$

When plastic loading occurs, the consistency relation states that $\dot{f}^I = 0$ which with (22.66) gives

$$\sum_{\Phi=1}^{G_{max}} A^{I\Phi}\dot{\lambda}^{\Phi} = \dot{a}^I \quad \text{or} \quad \boldsymbol{A}\dot{\boldsymbol{\lambda}} = \dot{\boldsymbol{a}} \tag{22.68}$$

To be able to derive a strain driven format, this equation must provide a unique $\dot{\lambda}^{\Phi}$-solution. The augmented matrix $\boldsymbol{T}$ is defined by $\boldsymbol{T} = [\boldsymbol{A}, \dot{\boldsymbol{a}}]$ and it is of dimension $F_{max} \times (G_{max} + 1)$. From Ayres (1962), for instance, a unique $\dot{\lambda}^{\Phi}$-solution requires that $\operatorname{rank}\boldsymbol{T} = \operatorname{rank}\boldsymbol{A} = G_{max}$; thus, it follows that $\dot{a}^I$ can be expressed by linear combinations of the columns in $A^{I\Phi}$. We therefore require

$$\operatorname{rank}\boldsymbol{A} = G_{max} \tag{22.69}$$

This requirement can never be achieved if $F_{max} < G_{max}$ and we conclude

$$\boxed{F_{max} \geq G_{max}} \tag{22.70}$$

That is, the number of yield surfaces must be larger than or equal to the number of potential surfaces. If these requirements above are fulfilled then (22.68) has a unique $\dot{\lambda}^{\Phi}$-solution.

Due to (22.69) the $F_{max} \times G_{max}$ matrix $\boldsymbol{A}$ has a *left inverse* $\underline{\boldsymbol{A}}$ such that

$$\underline{\boldsymbol{A}}\boldsymbol{A} = \boldsymbol{I} \quad \text{or} \quad \sum_{I=1}^{F_{max}} \underline{A}^{\Theta I} A^{I\Phi} = \delta^{\Theta\Phi}$$

cf. Ayres (1962). As a result, (22.68) provides the solution

$$\dot{\boldsymbol{\lambda}} = \underline{\boldsymbol{A}}\dot{\boldsymbol{a}} \quad \text{or} \quad \dot{\lambda}^{\Phi} = \sum_{I=1}^{F_{max}} \underline{A}^{\Phi I} \dot{a}^{I}$$

It is of interest that even though the left inverse $\underline{\boldsymbol{A}}$, is not, in general, uniquely determined, the $\dot{\lambda}^{\Phi}$-solution given above is a unique solution, cf. Eves (1980). Insertion of the result above into Hooke's law (22.65) gives the result sought for

$$\boxed{\begin{aligned} &\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl} \\ &\textit{where} \\ &D^{ep}_{ijkl} = D_{ijkl} - D_{ijst} \sum_{\Phi=1}^{G_{max}} \frac{\partial g^{\Phi}}{\partial \sigma_{st}} \sum_{I=1}^{F_{max}} \underline{A}^{\Phi I} \frac{\partial f^{I}}{\partial \sigma_{mn}} D_{mnkl} \end{aligned}}$$

which for $F_{max} = G_{max} = 1$ evidently reduces to expression (10.27) valid for smooth yield and potential functions. It also appears that $\boldsymbol{D}^{ep}$ for associated plasticity becomes a symmetric matrix.

In addition to requirement (22.69), let us now investigate the matrix $A^{I\Phi}$ in more detail. Observing (22.70), it is then evident that it is always possible to choose the numbering of the yield functions such that the $A^{I\Phi}$-matrix can be written as

$$A^{\Phi I} = \begin{bmatrix} P^{\Theta\Phi} \\ Q^{J\Phi} \end{bmatrix}; \quad \text{rank}\, P^{\Theta\Phi} = G_{max} \tag{22.71}$$

where $P^{\Theta\Phi}$ is of dimension $G_{max} \times G_{max}$ whereas $Q^{J\Phi}$ is of dimension $(F_{max} - G_{max}) \times G_{max}$. With (22.69) we immediately observe that det $\boldsymbol{P} \neq 0$ and we will now show that the eigenvalues of $\boldsymbol{P}$ are positive.

Consider as a special case ideal plasticity where no hardening parameters exist. In that case the matrix of plastic moduli $\boldsymbol{H}$ is zero and if we furthermore specialize to associated plasticity, $\boldsymbol{A}$ given by (22.67) reduces to the following symmetric matrix

$$A^{IJ} = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f^J}{\partial \sigma_{kl}}$$

Since D_{ijkl} is positive definite and as $\partial f^I / \partial \sigma_{ij}$ are linearly independent, cf. (22.61), it follows that also A^{IJ} is positive definite, i.e. it possesses only positive eigenvalues. As the theory we want to establish is to be general, we therefore assume - in recognition of (22.71) - that

$$\boxed{\textit{The eigenvalues of } P^{\Theta\Phi} \textit{ must be positive}} \qquad (22.72)$$

In the special case of associated plasticity, Sewell (1973) and Simo *et al.* (1988) argued that A^{IJ} should be positive definite and under these conditions, (22.72) reduces to that requirement. We may also note when $F_{max} = G_{max} = 1$, the requirement above reduces to the result $A > 0$ valid for conventional plasticity, cf. (10.24).

Having determined the properties of the matrix $A^{I\Phi}$, it also turns out to be important to evaluate the properties of the plastic moduli matrix. For this purpose, let us return to (22.64) and investigate the possibilities for a stress driven format. The consistency relation gives

$$\sum_{\Phi=1}^{G_{max}} H^{I\Phi} \dot{\lambda}^{\Phi} = \frac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \qquad (22.73)$$

A stress driven format requires that this equation provides a unique $\dot{\lambda}^{\Phi}$-solution and it is concluded that

$$\text{Stress driven format requires rank } H^{I\Phi} = G_{max}$$

Equivalent to (22.71) it is always possible to write

$$H^{I\Phi} = \begin{bmatrix} R^{\Theta\Phi} \\ S^{J\Phi} \end{bmatrix}; \qquad \begin{array}{l} \text{stress driven format requires} \\ \text{rank } R^{\Theta\Phi} = G_{max} \end{array} \qquad (22.74)$$

where $R^{\Theta\Phi}$ is of dimension $G_{max} \times G_{max}$ and $S^{J\Phi}$ is of dimension $(F_{max} - G_{max}) \times G_{max}$.

A *limit point* is defined according to

$$\dot{\sigma}_{ij} = 0 \quad \text{and} \quad \dot{\varepsilon}_{ij} \neq 0 \ \Rightarrow \ \text{limit point}$$

Since $\dot{\sigma}_{ij} = 0$, (22.73) gives $\sum_{\Phi=1}^{G_{max}} H^{I\Phi} \dot{\lambda}^{\Phi} = 0$ and it is concluded that

$$\text{limit point} \ \Leftrightarrow \ \text{rank } H^{\Theta\Phi} < G_{max} \qquad (22.75)$$

Consider now the special case of ideal plasticity. In that case no hardening parameters exist and we have

$$H^{\Theta\Phi} = 0 \;\Leftrightarrow\; \text{ideal plasticity}$$

That is, all components of $H^{I\Phi}$ are zero. We now observe a difference between corner plasticity and conventional plasticity. For conventional plasticity, ideal plasticity ($H = 0$) and the existence of a limit are identical statements, cf. Section 10.3. For corner plasticity, a glance at the two last equations reveals a fundamental difference between ideal plasticity and the existence of a limit point.

Considering now associated hardening plasticity, then during plastic loading we expect all the components of the quantity $\partial f^I/\partial\sigma_{ij}\dot{\sigma}_{ij}$ to be positive. Since the plastic multipliers $\dot{\lambda}^I$ are also positive during plastic loading, we obtain $\sum_{I=1}^{F_{max}} \dot{\lambda}^I \partial f^I/\partial\sigma_{ij}\dot{\sigma}_{ij} > 0$. From (22.73) it then follows that

$$\sum_{I=1}^{F_{max}} \sum_{J=1}^{F_{max}} \dot{\lambda}^I H^{IJ} \dot{\lambda}^J > 0 \quad \text{for associated hardening plasticity}$$

and the matrix of plastic moduli is therefore positive definite. Generalizing this result and observing (22.74) we arrive at the following definitions

$$\boxed{\textit{Hardening plasticity} \;\Leftrightarrow\; R^{\Theta\Phi} \textit{ has only positive eigenvalues}}$$

With reference to (22.75), the existence of a limit point is reformulated as

$$\boxed{\textit{Limit point} \;\Leftrightarrow\; R^{\Theta\Phi} \textit{ has at least one zero eigenvalue}}$$

We are then left with the following conclusion

$$\boxed{\textit{Softening plasticity} \;\Leftrightarrow\; \begin{array}{l} R^{\Theta\Phi} \textit{ has at least one negative eigen-} \\ \textit{value and no eigenvalues are zero} \end{array}}$$

It is recalled that ideal plasticity exists if all the components of $\boldsymbol{H}$ are zero. With these definitions, it follows that hardening plasticity allows a stress driven format, i.e. (22.73) can be solved for $\dot{\lambda}^\Phi$.

Suppose that two yield surfaces meet at a corner and that hardening of one yield surface is independent of what happens with the other yield surface, then *independent hardening* occurs and Fig. 22.8 shows an example. On the other hand, if hardening of one yield surface depends on what happens with the other yield surface, then *dependent hardening* occurs as illustrated in Fig. 22.9.

For associated plasticity where two yield surfaces meet at a corner, two classic formats of the plastic moduli matrix are given by

$$\boldsymbol{H} = \begin{bmatrix} k & k \\ k & k \end{bmatrix} \quad \text{Taylor hardening} \tag{22.76}$$

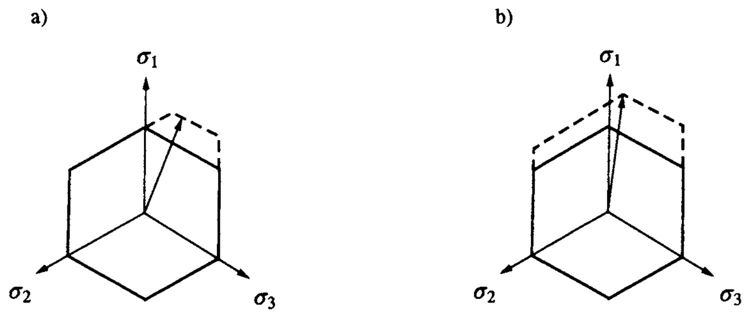

Figure 22.8: Tresca criterion with independent hardening; a) loading along a smooth yield surface, b) corner loading.

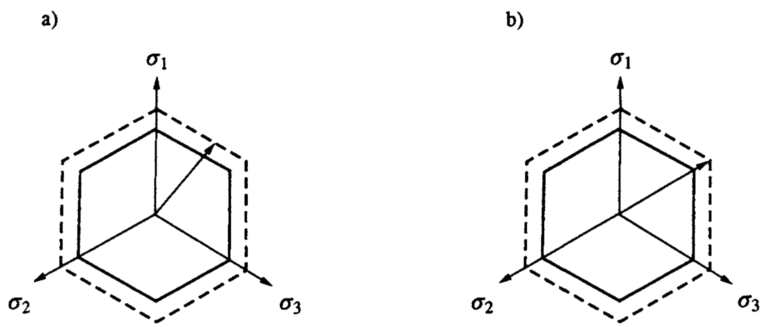

Figure 22.9: Tresca criterion with dependent hardening; a) loading along a smooth yield surface, b) corner loading.

and

$$\boldsymbol{H} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \quad \text{Koiter hardening}$$

where (22.76) is due to Taylor (1938). Koiter hardening implies independent hardening whereas Taylor hardening may be involved for dependent hardening. It is of interest that Taylor hardening does not allow a stress driven format, cf. (22.74); for further discussions, we refer to Mandel (1965), Hill (1966) and Ottosen and Ristinmaa (1996).

With the discussion above, we have established a firm basis for general nonassociated corner plasticity. It turns out, however, that in the most general case the loading/unloading criteria become rather complex and a discussion is provided by Ottosen and Ristinmaa (1996). The issue of numerical integration is treated by Simo *et al.* (1988), Pramono and Willam (1989) and Simo and Hughes (1998) for the associated case.

22.6 Viscoplasticity

As discussed in Chapter 15 there exist two major concepts when formulating viscoplasticity: the Perzyna and the Duvaut-Lions format. We will now illustrate how these formulations can be given a thermodynamic interpretation.

Perzyna viscoplasticity

The general nonassociated format is easily obtained. Let the potential function $g(\sigma_{ij}, K_\alpha)$ possess the property $g(\sigma_{ij}, K_\alpha) - g(0, 0) \geq 0$; then, according to (21.51) and (21.54) the dissipation inequality is fulfilled for the following evolution laws

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda \frac{\partial g}{\partial \sigma_{ij}}; \qquad \dot{\kappa}_\alpha = -\Lambda \frac{\partial g}{\partial K_\alpha} \tag{22.77}$$

where Λ is a non-negative function. Choosing this quantity as $\Lambda = \Phi(f)/\eta$ we have recovered the format (15.36).

For Perzyna viscoplasticity, the postulate of maximum dissipation was originally restricted to rely on subtle regularization and penalty techniques that only hold in the limit when viscoplasticity degenerates to inviscid plasticity, cf. Simo and Honein (1990). However, by using the concept of the dynamic yield surface, Ristinmaa and Ottosen (2000) demonstrated that this allows the postulate of maximum dissipation to be adopted in a straightforward manner as will be shown next.

According to (15.43) the dynamic yield function is given by

$$f^d(\sigma_{ij}, K_\alpha, \dot{\varepsilon}_{eff}^{vp}) = f(\sigma_{ij}, K_\alpha) - \varphi(\eta \dot{\varepsilon}_{eff}^{vp}) \tag{22.78}$$

where $f^d < 0$ implies elastic behavior, $f^d = 0$ results in development of viscoplasticity and $f^d > 0$ cannot occur; moreover, $f(\sigma_{ij}, K_\alpha)$ denotes the static yield function and the effective viscoplastic strain rate is defined by $\dot{\varepsilon}_{eff}^{vp} = (2\dot{\varepsilon}_{ij}^{vp}\dot{\varepsilon}_{ij}^{vp}/3)^{1/2}$.

The mechanical dissipation inequality is given by $\gamma_{mech} = \sigma_{ij}\dot{\varepsilon}_{ij}^{vp} - K_\alpha \dot{\kappa}_\alpha \geq 0$ and the postulate of maximum dissipation can accordingly be formulated as: For given fluxes $\dot{\varepsilon}_{ij}^{vp}$ and $\dot{\kappa}_\alpha$ find those stresses σ_{ij} and conjugated forces K_α that minimize $-\gamma_{mech}$ under the constraint that $f^d(\sigma_{ij}, K_\alpha, \dot{\varepsilon}_{eff}^{vp}) \leq 0$. Assuming the dynamic yield function to be convex, Appendix (A.15) and (A.18) provide the Kuhn-Tucker relations

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda \frac{\partial f^d}{\partial \sigma_{ij}}; \qquad \dot{\kappa}_\alpha = -\Lambda \frac{\partial f^d}{\partial K_\alpha}$$

where $\Lambda \geq 0$, $f^d \leq 0$ and $\Lambda f^d = 0$. With the dynamic yield function defined by (22.78), we obtain

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda \frac{\partial f}{\partial \sigma_{ij}}; \qquad \dot{\kappa}_\alpha = -\Lambda \frac{\partial f}{\partial K_\alpha} \tag{22.79}$$

From (22.79a) and the definition of the effective viscoplastic strain rate, we find

$$\dot{\varepsilon}_{eff}^{vp} = \Lambda \left(\frac{2}{3} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2} \tag{22.80}$$

When viscoplasticity develops $f^d = 0$ holds and (22.78) then provides

$$f(\sigma_{ij}, K_\alpha) = \varphi(\eta \dot{\varepsilon}_{eff}^{vp}) \quad \Rightarrow \quad \Phi(f) = \eta \dot{\varepsilon}_{eff}^{vp} \tag{22.81}$$

where Φ is the inverse function of φ. A comparison of (22.80) and (22.81b) shows that the non-negative function Λ is given by

$$\Lambda = \frac{\Phi(f)}{\eta \left(\frac{2}{3} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2}} \tag{22.82}$$

In most cases, for instance, when the static yield function is chosen in terms of the von Mises, Drucker-Prager or Coulomb criterion, the quantitity $\partial f / \partial \sigma_{ij}$ $\partial f / \partial \sigma_{ij}$ becomes a constant and (22.79) and (22.82) then coincide exacly with associated Perzyna viscoplasticity.

Duvaut-Lions viscoplasticity

To establish a thermodynamic framework for Duvaut-Lions viscoplasticity is somewhat more complex; in fact, it turns out that Duvaut-Lions viscoplasticity apparently can only be formulated as nonassociated viscoplasticity. We will follow the proposal by Ristinmaa and Ottosen (1998), which rests on the concept of an *additive split* of the conjugated forces.

The mechanical dissipation inequality is given by

$$\gamma_{mech} = \sigma_{ij} \dot{\varepsilon}_{ij}^{vp} - K_\alpha \dot{\kappa}_\alpha \geq 0 \tag{22.83}$$

With $f = f(\sigma_{ij}, K_\alpha)$ being the static yield function, it was shown in Section 15.4.2 that the closest-point-projection $\bar{\sigma}_{ij}, \bar{K}_\alpha$ on the static yield surface was given by (15.57), i.e.

$$\begin{aligned}
&-C_{ijkl}(\sigma_{kl} - \bar{\sigma}_{kl}) + \mu \frac{\partial f}{\partial \bar{\sigma}_{ij}} = 0 \\
&-c_{\alpha\beta}(K_\beta - \bar{K}_\beta) + \mu \frac{\partial f}{\partial \bar{K}_\alpha} = 0 \\
&f(\bar{\sigma}_{ij}, \bar{K}_\alpha) = 0
\end{aligned} \tag{22.84}$$

Define the quantities σ_{ij}^*, K_α^* according to

$$\sigma_{ij} = \sigma_{ij}^* + \bar{\sigma}_{ij}\,; \qquad K_\alpha = K_\alpha^* + \bar{K}_\alpha \tag{22.85}$$

It appears that we have made an additive split of the conjugated forces. Insertion into the mechanical dissipation inequality (22.83) gives

$$\gamma_{mech} = \sigma_{ij}^* \dot{\varepsilon}_{ij}^{vp} - K_\alpha^* \dot{\kappa}_\alpha + \bar{\sigma}_{ij} \dot{\varepsilon}_{ij}^{vp} - \bar{K}_\alpha \dot{\kappa}_\alpha \geq 0$$

This inequality is certainly fulfilled provided

$$\gamma_{mech,1} = \sigma_{ij}^* \dot{\varepsilon}_{ij}^{vp} - K_\alpha^* \dot{\kappa}_\alpha \geq 0\,; \qquad \gamma_{mech,2} = \bar{\sigma}_{ij} \dot{\varepsilon}_{ij}^{vp} - \bar{K}_\alpha \dot{\kappa}_\alpha \geq 0 \tag{22.86}$$

Each of these inequalities will now be fulfilled separately using the potential function approach. The static yield fuction is convex and it must fulfill $f(\bar{\sigma}_{ij}, \bar{K}_\alpha) - f(0,0) \geq 0$. According to (21.51) and (21.54) we then comply with the inequality $\gamma_{mech,2} \geq 0$ for

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda_2 \frac{\partial f}{\partial \bar{\sigma}_{ij}}\,; \qquad \dot{\kappa}_\alpha = -\Lambda_2 \frac{\partial f}{\partial \bar{K}_\alpha} \tag{22.87}$$

where Λ_2 is a non-negative quantity. Considering the other inequality, we adopt as potential function the same function as that which defines the distance between the current state and the closest-point-projection. According to (15.56), we then choose the potential function $G(\sigma_{ij}^*, K_\alpha^*)$ as

$$G(\sigma_{ij}^*, K_\alpha^*) = \frac{1}{2} \sigma_{ij}^* C_{ijkl} \sigma_{kl}^* + \frac{1}{2} K_\alpha^* c_{\alpha\beta} K_\beta^* \tag{22.88}$$

The elastic flexibility tensor is certainly positive definite and $c_{\alpha\beta}$ is also positive definite for hardening viscoplasticity, as shown in Ristinmaa and Ottosen (1998). There it is proved that the potential function introduced fulfills our requirements even when softening viscoplasticity is involved. Therefore, we obtain

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda_1 \frac{\partial G}{\partial \sigma_{ij}^*} = \Lambda_1 C_{ijkl} \sigma_{kl}^*\,; \qquad \dot{\kappa}_\alpha = -\Lambda_1 \frac{\partial G}{\partial K_\alpha^*} = -\Lambda_1 c_{\alpha\beta} K_\beta^* \tag{22.89}$$

The two sets of evolution equations given by (22.87) and (22.89) must be identical. Indeed, if Λ_2 is chosen as $\Lambda_2 = \mu\Lambda_1$ then, in view of (22.84), it appears that this equality is achieved. Thus, from (22.89) and making the definition $\Lambda = \Lambda_1$ we obtain

$$\dot{\varepsilon}_{ij}^{vp} = \Lambda C_{ijkl} (\sigma_{kl} - \bar{\sigma}_{kl})\,; \qquad \dot{\kappa}_\alpha = -\Lambda c_{\alpha\beta} (K_\beta - \bar{K}_\beta) \tag{22.90}$$

which is exactly the Duvaut-Lions formulation, cf. (15.58).

22.7 Viscoelasticity

Having illustrated how plasticity and viscoplasticity are placed within a thermodynamic framework, it is of interest to consider viscoelasticity; information is also provided by the textbooks of Lemaitre and Chaboche (1990), Maugin (1992) and Simo and Hughes (1998). An introduction to viscoelasticity was given in Chapter 14 and we will here investigate Maxwell and Kelvin viscoelasticity in their generalized forms. It turns out to be useful to return to the mechanical dissipation inequality (21.5), which reads

$$\gamma_{mech} = -\rho(\dot{\psi} + s\dot{\theta}) + \sigma_{ij}\dot{\varepsilon}_{ij} \geq 0 \tag{22.91}$$

In the framework discussed so far, it was emphasized in relation to (21.37) that viscoelasticity may involve the occurrence of a viscous stress. However, before we pursue this observation, it will first be shown that Maxwell viscoelasticity and generalized Maxwell viscoelasticity fit into the thermodynamic framework already discussed.

Maxwell models

A very simple form of viscoelasticity is given by Maxwell viscoelasticity for which we assume

$$\psi = \psi(\theta, \varepsilon_{ij} - \varepsilon_{ij}^{v}) \tag{22.92}$$

where ε_{ij}^{v} denotes the *viscous strains*. From (21.37) it then follows that

$$\sigma_{ij} = \rho\frac{\partial\psi}{\partial\varepsilon_{ij}} \tag{22.93}$$

and the dissipation inequality becomes

$$\boxed{\gamma_{mech} = \sigma_{ij}\dot{\varepsilon}_{ij}^{v} \geq 0} \tag{22.94}$$

In order to fulfill this inequality and as linear viscoelasticity is aimed at, we take the potential function Φ according to

$$\Phi = \frac{1}{2}\sigma_{ij}B_{ijkl}\sigma_{kl}$$

where the constant tensor B_{ijkl} is assumed to be positive definite. From this expression, the evolution equation becomes

$$\dot{\varepsilon}_{ij}^{v} = \frac{\partial\Phi}{\partial\sigma_{ij}} = B_{ijkl}\sigma_{kl} \tag{22.95}$$

Insertion into (22.94) immediately reveals that the dissipation inequality is fulfilled; we may note that the evolution law (22.95) can also be viewed as being an Onsager relation, cf. (21.42).

To be specific, take Helmholtz' free energy according to

$$\rho\psi(\theta, \varepsilon_{ij} - \varepsilon_{ij}^{v}) = A(\theta) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ij}^{v})D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{v})$$

The stresses are given by (22.93) and a time differentiation and use of (22.95) result in

$$C_{ijkl}\dot{\sigma}_{kl} + B_{klmn}\sigma_{mn} = \dot{\varepsilon}_{ij}$$

where C_{ijkl} is the flexibility tensor; this result coincides with the findings given by (14.20) and (14.22). In conclusion

$$\boxed{\begin{array}{l} \textit{Maxwell viscoelasticity} \\ \psi = \psi(\theta, \varepsilon_{ij} - \varepsilon_{ij}^{v}) \\ \sigma_{ij} = \rho\dfrac{\partial\psi}{\partial\varepsilon_{ij}}; \qquad \dot{\varepsilon}_{ij}^{v} = B_{ijkl}\sigma_{kl} \\ \textit{where } B_{ijkl} \textit{ is positive definite} \end{array}} \tag{22.96}$$

Since Maxwell viscoelasticity is a very simplistic model for viscoelastic behavior, it is of interest to consider generalized Maxwell viscoelasticity, cf. (14.14) and the discussion following that expression. With the result (22.96), this generalization can be achieved in a straightforward manner. Take the free energy according to

$$\psi = \sum_{k=1}^{n} \psi^{(k)} \quad \text{where} \quad \psi^{(k)} = \psi^{(k)}(\theta, \varepsilon_{ij} - \varepsilon_{ij}^{v(k)}) \tag{22.97}$$

Define $\sigma_{ij}^{(k)}$ by

$$\sigma_{ij}^{(k)} = -\rho\frac{\partial\psi^{(k)}}{\partial\varepsilon_{ij}^{v(k)}} \tag{22.98}$$

The stress tensor is still given by (22.93) and it follows that

$$\sigma_{ij} = \rho\frac{\partial\psi}{\partial\varepsilon_{ij}} = \sum_{k=1}^{n} \sigma_{ij}^{(k)} \tag{22.99}$$

Insertion of (22.97) into the dissipation inequality (22.91) and observing that $\partial\psi/\partial\theta = -s$ still holds, result in

$$\rho\sum_{k=1}^{n} \frac{\partial\psi^{(k)}}{\partial\varepsilon_{ij}^{v(k)}}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^{v(k)}) + \sigma_{ij}\dot{\varepsilon}_{ij} \geq 0$$

Due to (22.98) and (22.99), this expression takes the form

$$\sum_{k=1}^{n} \sigma_{ij}^{(k)} \dot{\varepsilon}_{ij}^{v(k)} \geq 0 \tag{22.100}$$

Take the potential function Φ according to

$$\Phi = \sum_{k=1}^{n} \Phi^{(k)} \quad \text{where} \quad \Phi^{(k)} = \frac{1}{2} \sigma_{ij}^{(k)} B_{ijkl}^{(k)} \sigma_{kl}^{(k)}$$

where the constant tensors $B_{ijkl}^{(k)}$ are assumed to be positive definite. From (21.51), the evolution equations then become

$$\dot{\varepsilon}_{ij}^{v(k)} = \frac{\partial \Phi^{(k)}}{\partial \sigma_{ij}^{(k)}} = B_{ijkl}^{(k)} \sigma_{kl}^{(k)} \tag{22.101}$$

which evidently fulfill the dissipation inequality (22.100).

To be specific, take Helmholtz' free energy according to

$$\rho \psi = A(\theta) + \sum_{k=1}^{n} \frac{1}{2} (\varepsilon_{ij} - \varepsilon_{ij}^{v(k)}) D_{ijkl}^{(k)} (\varepsilon_{kl} - \varepsilon_{kl}^{v(k)})$$

From (22.99) follows that

$$\sigma_{ij} = \sum_{k=1}^{n} D_{ijkl}^{(k)} (\varepsilon_{kl} - \varepsilon_{kl}^{v(k)}) \tag{22.102}$$

and (22.98) provides

$$\sigma_{ij}^{(k)} = D_{ijkl}^{(k)} (\varepsilon_{kl} - \varepsilon_{kl}^{v(k)}) \tag{22.103}$$

A comparison of (22.101)-(22.103) with the expressions following (14.14) shows that we have achieved the three-dimensional generalized Maxwell model. It is also noted that the three-dimensional formulation of the standard model illustrated in Fig. 14.21 is obtained by keeping only two terms in the summation (i.e. $n = 2$).

Kelvin models

The considerations above demonstrate that Maxwell and generalized Maxwell viscoelasticity fit into the usual thermodynamic framework without any *viscous stresses*. However, when Kelvin viscoelasticity is addressed, then the viscous stresses mentioned in (21.37) enter the formulation. Indeed, formulation (14.9) for the one-dimensional Kelvin model already suggests that the total stresses consist of the elastic stresses and some viscous stresses. We have also seen in (14.5) that these viscous stresses depend on some strain rates.

Let us begin with the simplest possible Kelvin formulation and take Helmholtz' free energy in the form

$$\psi = \psi(\theta, \varepsilon_{ij}) \tag{22.104}$$

According to (21.37) the stresses are now given by

$$\sigma_{ij} = \rho\frac{\partial\psi}{\partial\varepsilon_{ij}} + \sigma^{v}_{ij}(\dot{\varepsilon}_{kl}) \tag{22.105}$$

where the viscous stresses depend on the strain rate. Again we have $s = -\partial\psi/\partial\theta$ and insertion of (22.104) and (22.105) into the dissipation inequality (22.91) result in

$$\boxed{\gamma_{mech} = \sigma^{v}_{ij}\dot{\varepsilon}_{ij} \geq 0} \tag{22.106}$$

Take the dissipation function (pseudo-dissipation function) as

$$\Phi = \frac{1}{2}\dot{\varepsilon}_{ij}A_{ijkl}\dot{\varepsilon}_{kl} \tag{22.107}$$

where the constant tensor A_{ijkl} is assumed to be positive definite. With the evolution law

$$\sigma^{v}_{ij} = \frac{\partial\Phi}{\partial\dot{\varepsilon}_{ij}} = A_{ijkl}\dot{\varepsilon}_{kl} \tag{22.108}$$

it appears immediately that the dissipation inequality is fulfilled.

To be specific, choose the free energy as

$$\rho\psi(\theta, \varepsilon_{ij}) = A(\theta) + \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} \tag{22.109}$$

From (22.105) and (22.108), we then obtain

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} + A_{ijkl}\dot{\varepsilon}_{kl} \tag{22.110}$$

This result coincides with the findings given by (14.24)-(14.26). In conclusion

$$\boxed{\begin{array}{l} \textit{Kelvin viscoelasticity} \\ \psi = \psi(\theta, \varepsilon_{ij}) \\ \sigma_{ij} = \rho\dfrac{\partial\psi}{\partial\varepsilon_{ij}} + \sigma^{v}_{ij}; \quad \sigma^{v}_{ij} = A_{ijkl}\dot{\varepsilon}_{kl} \\ \textit{where } A_{ijkl} \textit{ is positive definite} \end{array}} \tag{22.111}$$

In general, this Kelvin model is all too simple to be able to simulate viscoelastic materials in an accurate fashion. However, with the result (22.111) it is easy to derive the formulation of generalized Kelvin viscoelasticity introduced in relation to (14.15). Let the free energy be given by

$$\psi = \sum_{k=0}^{n} \psi^{(k)} \quad \text{where} \quad \psi^{(k)} = \psi^{(k)}(\theta, \varepsilon_{ij}^{(k)}) \tag{22.112}$$

Insertion into the dissipation inequality (22.91) and noting that $s = -\partial\psi/\partial\theta$, we obtain

$$-\rho \sum_{k=0}^{n} \frac{\partial \psi^{(k)}}{\partial \varepsilon_{ij}^{(k)}} \dot{\varepsilon}_{ij}^{(k)} + \sigma_{ij}\dot{\varepsilon}_{ij} \geq 0 \tag{22.113}$$

Let the quantities $\varepsilon_{ij}^{(k)}$ fulfill

$$\varepsilon_{ij} = \sum_{k=0}^{n} \varepsilon_{ij}^{(k)} \tag{22.114}$$

Inserting this expression in (22.113), and take

$$\sigma_{ij} = \rho \frac{\partial \psi^{(k)}}{\partial \varepsilon_{ij}^{(k)}} + \sigma_{ij}^{v(k)} \quad \text{where} \quad \sigma_{ij}^{v(0)} = 0 \tag{22.115}$$

Then the dissipation inequality (22.113) becomes

$$\sum_{k=1}^{n} \sigma_{ij}^{v(k)} \dot{\varepsilon}_{ij}^{(k)} \geq 0 \tag{22.116}$$

Take the dissipation function as

$$\Phi = \sum_{k=1}^{n} \Phi^{(k)} \quad \text{where} \quad \Phi^{(k)} = \frac{1}{2} \dot{\varepsilon}_{ij}^{(k)} A_{ijkl}^{(k)} \dot{\varepsilon}_{kl}^{(k)} \tag{22.117}$$

where all the constant tensors $A_{ijkl}^{(k)}$ are assumed to be positive definite. With the following evolution laws

$$\sigma_{ij}^{v(k)} = \frac{\partial \Phi^{(k)}}{\partial \dot{\varepsilon}_{ij}^{(k)}} = A_{ijkl}^{(k)} \dot{\varepsilon}_{kl}^{(k)} \quad k = 1, 2, \ldots, n \tag{22.118}$$

the dissipation inequality becomes fulfilled.

To be specific, Helmholtz' free energy is now taken as

$$\rho\psi(\theta, \varepsilon_{ij}^{(k)}) = A(\theta) + \sum_{k=0}^{n} \frac{1}{2} \varepsilon_{ij}^{(k)} D_{ijkl}^{(k)} \varepsilon_{kl}^{(k)} \tag{22.119}$$

and (22.115) then provides

$$\sigma_{ij} = D^{(k)}_{ijkl}\varepsilon^{(k)}_{kl} + \sigma^{v(k)}_{ij} \quad \text{where} \quad \sigma^{v(0)}_{ij} = 0 \tag{22.120}$$

Otherwise, the viscous stresses are given by (22.118). A comparison of (22.114), (22.118) and (22.120) with the expressions following (14.15) shows that we have now achieved the three-dimensional generalized Kelvin model.