4 HYPER-ELASTICITY

In the previous chapters we established the concepts of stresses and strains. No reference was made to the material as such, and we emphasize that within the assumption of small strains and small rotations, the results hold for any material which may be treated as a continuum. It is obvious, however, that stresses and strains must be related in some way or another and the specific manner of this relation is controlled by the specific material in question. The expression between stresses and strains is called the *constitutive relation* and a variety of such relations has been established. Examples are *elasticity, plasticity, viscoelasticity, viscoplasticity* and *creep*. In the present chapter, we will consider the simplest constitutive theory, namely *hyper-elasticity*.

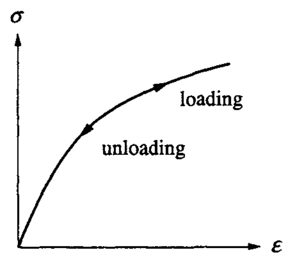

Figure 4.1: Nonlinear elasticity for uniaxial loading.

Let us first define what is meant by elasticity:

$\boxed{\textit{Elastic response is independent of the load history}}$

Alternatively, one says that the material response is *path independent* and it follows that the response for loading or unloading follows the same path as illustrated in Fig. 4.1. After removal of the loading, the material therefore returns to its original configuration. We emphasize that elastic response, in general, is nonlinear, as also illustrated in Fig. 4.1.

In accordance with these statements, the stresses are uniquely given by the strains, i.e. we have the constitutive relation

$$\boxed{\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl})} \tag{4.1}$$

If we say nothing more than that, we have the most general form of elasticity, namely so-called *Cauchy-elasticity* (in honor of Cauchy, 1789-1857, who in 1822 formulated the constitutive law for isotropic linear elastic materials). This type of elasticity will be treated in the next chapter. However, it turns out that a slightly restricted form of elasticity can be established by considerations to the strain energy. This restricted form of elasticity is called *hyper-elasticity* and it turns out to be very simple to derive the most general form of nonlinear hyper-elasticity for isotropic materials.

In general, we observe that if the constitutive relation depends on position, then the material is termed *inhomogeneous*; otherwise, it is termed *homogeneous*.

In addition to a treatment of hyper-elasticity, in the present chapter we will introduce and discuss a number of questions that are of general importance and applicable to a number of constitutive theories like, for instance, plasticity theory. Examples are matrix formulation of constitutive laws, discussion of symmetry properties of the constitutive matrix and topics within anisotropy. Thus, the present chapter is rather extensive, but in the following chapters we will take advantage of the concepts introduced here.

4.1 Strain energy and hyper-elasticity

Let us first introduce the concept of *strain energy* W per unit volume of the body, i.e. W has the unit [Nm/m^3]. For a uniaxial stress state, the incremental strain energy is defined by

$$dW = \sigma d\varepsilon \quad \text{i.e.} \quad W = \int_0^{\varepsilon} \sigma(\varepsilon^*) d\varepsilon^* \tag{4.2}$$

where advantage was taken of (4.1) and where ε^* is an integration variable whereas ε denotes the current strain. Equation (4.2) is illustrated in Fig. 4.2.

Adopting this approach to the general situation we obtain

$$\boxed{dW = \sigma_{ij} d\varepsilon_{ij} \quad i.e. \quad W = \int_0^{\varepsilon_{ij}} \sigma_{ij}(\varepsilon_{kl}^*) d\varepsilon_{ij}^*} \tag{4.3}$$

where ε_{ij} denote the current strains whereas ε_{ij}^* denotes the integration variables.

Even though the current stresses σ_{ij} only depend on the current strains ε_{ij}, cf. (4.1), we will in general have that the strain energy W as determined by

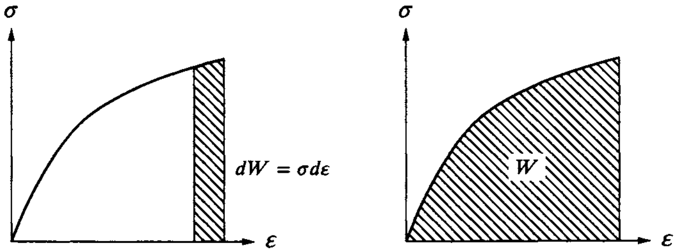

Figure 4.2: Incremental strain energy dW and strain energy W for uniaxial loading.

(4.3b) depends both on the current strains ε_{ij} as well as on the manner in which these strains were achieved, i.e.

$$W = W(\varepsilon_{ij}\text{ , load history}) \tag{4.4}$$

This is just to say that W as determined by (4.3b) depends not only on the current strains, but also on the integration path, where the integration path represents the load history.

We will now make the assumption that W is independent on the integration path and (4.4) then reduces to

$$\boxed{W = W(\varepsilon_{ij})} \tag{4.5}$$

From this expression follows that

$$dW = \frac{\partial W}{\partial \varepsilon_{ij}} d\varepsilon_{ij} \tag{4.6}$$

Subtraction of (4.6) from (4.3a) gives

$$(\sigma_{ij} - \frac{\partial W}{\partial \varepsilon_{ij}})d\varepsilon_{ij} = 0 \tag{4.7}$$

In general, the incremental strains $d\varepsilon_{ij}$ can be chosen arbitrarily and independently of each other and we therefore conclude from (4.7) that

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}} \tag{4.8}$$

There is one exception where the incremental strains $d\varepsilon_{ij}$ cannot be chosen arbitrarily, namely the case of incompressible response and we will return to this special situation in Section 4.13. Since σ_{ij} is obtained from W by a differentiation, one uses the phrase that W serves as a *potential function* for the stresses.

We observe that (4.5) and (4.8) imply (4.1) and a material that obeys the constitutive relation (4.5) and thereby (4.8) is called a *hyper-elastic material*;

'hyper' meaning 'to a higher degree'. Occasionally, the term *Green-elasticity* is used since this formulation was adopted by Green in 1839 and even today most work on elasticity is based on this format.

Another feature often related to elasticity is *reversibility* also from a thermodynamical point of view. Later, in Chapter 21 we will show that two thermodynamical reversible processes result in hyper-elasticity, namely, a reversible, adiabatic process and a reversible, isothermal process. Therefore, the term hyper - meaning 'to a higher degree' - is attributed to this kind of elasticity, as hyper-elasticity implies reversibility not only between stresses and strains, but also reversibility in the thermodynamical sense. In the next chapter we will encounter elasticity models (Cauchy-elasticity), which only implies reversibility between stresses and strains.

Let us return to (4.3b) and (4.8) and the issue of the strain energy W being independent of the integration path i.e. independent of the load history. As an illustration consider the quantity Q given by

$$Q = \int_A^B (L\,dx + M\,dy)$$

which means that Q is obtained as an integration along some curve in the xy-plane from point A to point B; moreover, $L = L(x, y)$ and $M = M(x, y)$. From standard mathematics, it is well known that Q only depends on the end points A and B and not on the path between A and B if $L\ dx + M\ dy$ is a *perfect differential*. The necessary and sufficient condition for $L\ dx + M\ dy$ being a perfect differential is

$$\frac{\partial L}{\partial y} = \frac{\partial M}{\partial x}$$

Generalizing these concepts to (4.3b), we see that W is independent on the integration path if

$$\boxed{\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}} \tag{4.9}$$

and use of (4.8) demonstrates this condition to be fulfilled – as expected.

Using the transformation rule for the second-order tensors σ_{ij} and $d\varepsilon_{ij}$, it is readily shown that $\sigma_{ij}d\varepsilon_{ij}$ is an invariant. It therefore follows from (4.3a) that dW is an invariant, i.e. we have

$$\boxed{\textit{Strain energy } W \textit{ is an invariant}} \tag{4.10}$$

4.2 Complementary energy and hyper-elasticity

Having established the strain energy W and the fundamental relation (4.8), we will now perform an interesting reformulation. Define the function C - the *complementary energy* per unit volume - by

$$\boxed{C = \sigma_{ij}\varepsilon_{ij} - W(\varepsilon_{ij})} \tag{4.11}$$

It is obvious that C only depends on the current state and not on the manner in which this state was established. By differention we obtain

$$dC = d\sigma_{ij}\varepsilon_{ij} + \sigma_{ij}d\varepsilon_{ij} - \frac{\partial W}{\partial \varepsilon_{ij}}d\varepsilon_{ij}$$

which together with (4.8) gives

$$dC = \varepsilon_{ij}d\sigma_{ij} \tag{4.12}$$

Instead of (4.1), we assume that the inverse relation exists i.e.

$$\boxed{\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl})} \tag{4.13}$$

and we obtain

$$\boxed{C(\sigma_{ij}) = \int_0^{\sigma_{ij}} \varepsilon_{kl}(\sigma^*_{mn})d\sigma^*_{kl}} \tag{4.14}$$

where σ_{ij} is the current stress state whereas σ^*_{kl} denotes the integration variable.

We mentioned that the complementary energy C only depends on the current state and not on the history. Moreover, we found from (4.14) that $C = C(\sigma_{ij})$. This may seem a little surprising since ε_{ij} enters (4.11). To convince ourselves that $C = C(\sigma_{ij})$, we assume that $C = C(\sigma_{ij}, \varepsilon_{ij})$ and obtain

$$dC = \frac{\partial C}{\partial \sigma_{ij}}d\sigma_{ij} + \frac{\partial C}{\partial \varepsilon_{ij}}d\varepsilon_{ij}$$

and a comparison with (4.12) indicates that $\partial C/\partial \varepsilon_{ij} = 0$ i.e. $C = C(\sigma_{ij})$. We therefore have

$$dC = \frac{\partial C}{\partial \sigma_{ij}}d\sigma_{ij} \tag{4.15}$$

It appears that by the format (4.11) we have shifted the old variable ε_{ij} in $W = W(\varepsilon_{ij})$ into a new variable σ_{ij} in $C = C(\sigma_{ij})$ without knowing the explicit relation between ε_{ij} and σ_{ij}. The format (4.11) is an example of the use of the *Legendre transformation* that is frequently used in mechanics and – in particular – in thermodynamics; in Chapter 21, we will encounter a number of applications of the Legendre transformation. Subtraction of (4.15) from (4.12) yields

$$(\varepsilon_{ij} - \frac{\partial C}{\partial \sigma_{ij}})d\sigma_{ij} = 0$$

Since this relation holds for arbitrary stress states, it follow that

$$\boxed{\varepsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}}; \qquad C = C(\sigma_{ij})} \tag{4.16}$$

i.e. the complementary energy C serves as a potential function for ε_{ij}. In the uniaxial case, an illustration of C given by (4.14) is shown in Fig. 4.3.

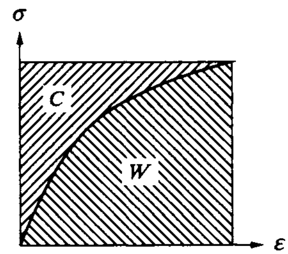

Figure 4.3: Complementary energy C and strain energy W for uniaxial loading.

By arguments similar to those adopted when we evaluated the strain energy W, cf. (4.10), it follows that

$$\boxed{\textit{Complementary energy } C \textit{ is an invariant}} \tag{4.17}$$

Moreover, from (4.16) appears that

$$\boxed{\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}} \tag{4.18}$$

4.3 Linear hyper-elasticity - Anisotropy

A material is *anisotropic*, if it behaves differently when loaded in the same manner in different directions. As an illustration, consider the piece of wood shown in Fig. 4.4. In Fig. 4.4a), we have uniaxial tension along the x_1−axis and we may express the relation between σ_{11} and ε_{11} as $\sigma_{11} = E_a\varepsilon_{11}$ where E_a is some experimentally determined stiffness parameter. Likewise, in Fig. 4.4b) we also have uniaxial tension along the x_1−axis and the relation between σ_{11} and ε_{11} can now be written $\sigma_{11} = E_b\varepsilon_{11}$ where E_b again is some experimentally determined stiffness parameter. Comparison of Figs. 4.4a) and 4.4b) clearly indicates that $E_a \neq E_b$. We are thereby led to the following general conclusion:

Material anisotropy means that the constitutive relation takes different forms depending on the Cartesian coordinate system we use

(4.19)

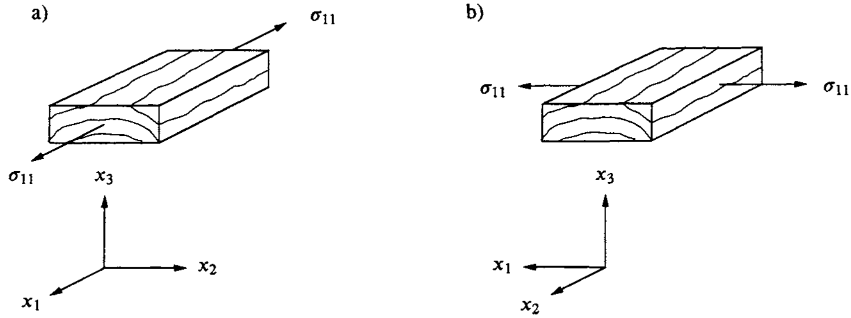

Figure 4.4: Example of anisotropy. Piece of wood loaded by the same uniaxial tension in different directions.

Let us assume that the constitutive law between stresses and strains is linear and let us investigate the general properties of this relation for a hyper-elastic material. The most general linear relation must be of the form

$$\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl}\ ; \qquad D_{ijkl} = D_{ijkl}(x_i)} \tag{4.20}$$

where D_{ijkl} is the *elastic stiffness tensor*. That D_{ijkl} indeed is a tensor of fourth order follows from the quotient theorem and the fact that σ_{ij} and ε_{ij} are second-order tensors, cf. the derivation of (1.41) from (1.37). The formulation (4.20) covers both anisotropic and isotropic elastic materials and in another coordinate system x'_i, we have $\sigma'_{ij} = D'_{ijkl}\varepsilon'_{kl}$ where D'_{ijkl} is related to D_{ijkl} via (1.43). For anisotropic materials, we have $D_{ijkl} \neq D'_{ijkl}$ in accordance with (4.19). Due to the linearity, D_{ijkl} is independent of the amount of loading. However, D_{ijkl} is allowed to depend on the position x_i in which case we have an inhomogeneous material; otherwise it is homogeneous. The constitutive relation (4.20) is referred to as *Hooke's law* or *Hooke's generalized law* since it generalizes the uniaxial form $\sigma = E\varepsilon$ suggested by Hooke in 1676.

Writing all terms in (4.20) explicitly and using the symmetry of ε_{ij}, we obtain

$$\begin{aligned}\sigma_{ij} &= D_{ij11}\varepsilon_{11} + D_{ij22}\varepsilon_{22} + D_{ij33}\varepsilon_{33} + \\ &\quad (D_{ij12} + D_{ij21})\varepsilon_{12} + (D_{ij13} + D_{ij31})\varepsilon_{13} + (D_{ij23} + D_{ij32})\varepsilon_{23}\end{aligned}$$

It appears that it is no restriction to rename the term $D_{ij12} + D_{ij21}$ by $2D_{ij12}$. If the new term D_{ij12} now has the symmetry property $D_{ij12} = D_{ij21}$ then we would again obtain the same result as (4.21). Likewise, we rename the term $D_{ij13} + D_{ij31}$ by $2D_{ij13}$ and the term $D_{ij23} + D_{ij32}$ by $2D_{ij23}$ where now $D_{ij13} = D_{ij31}$ as well as $D_{ij23} = D_{ij32}$. By this operation, we have achieved that $D_{ijkl} = D_{ijlk}$.

Moreover, since the stress tensor σ_{ij} is symmetric, it follows directly from (4.20) that $D_{ijkl} = D_{jikl}$. Thus, from the symmetries of σ_{ij} and ε_{ij} we generally have the so-called *minor symmetry* properties

$$\boxed{D_{ijkl} = D_{jikl} \quad ; \quad D_{ijkl} = D_{ijlk} \quad \textit{minor symmetry}} \tag{4.21}$$

With the property (4.21b), (4.21) may be written

$$\begin{aligned}\sigma_{ij} &= D_{ij11}\varepsilon_{11} + D_{ij22}\varepsilon_{22} + D_{ij33}\varepsilon_{33} \\ &\quad +2D_{ij12}\varepsilon_{12} + 2D_{ij13}\varepsilon_{13} + 2D_{ij23}\varepsilon_{23}\end{aligned} \tag{4.22}$$

Up until now, we have just based our considerations on (4.20) and not used that the material is assumed to be hyper-elastic. However, hyper-elasticity implies (4.9) and if (4.20) is inserted into this relation, we immediately conclude the following additional so-called *major symmetry* property

$$\boxed{D_{ijkl} = D_{klij} \qquad \textit{for hyper-elasticity; major symmetry}} \tag{4.23}$$

We saw previously that the strain energy W plays a central role in hyper-elasticity, so it is tempting to derive W for the present material model. For this purpose, we obtain with Hooke's law (4.20) and the symmetry property (4.23) that

$$\begin{aligned} d(\frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl}) &= \frac{1}{2}d\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} + \frac{1}{2}\varepsilon_{ij}D_{ijkl}d\varepsilon_{kl} \\ &= \frac{1}{2}d\varepsilon_{ij}\sigma_{ij} + \frac{1}{2}\sigma_{kl}d\varepsilon_{kl} \\ &= \sigma_{ij}d\varepsilon_{ij}\end{aligned}$$

From (4.3) then follows that

$$\boxed{W = \frac{1}{2}\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} \qquad or \qquad W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}} \tag{4.24}$$

Since ε_{ij} is a second-order tensor and D_{ijkl} is a fourth-order tensor, it follows that the quantity $\varepsilon_{ij}D_{ijkl}\varepsilon_{kl}$ is an invariant. It then follows from (4.24) that the strain energy W is an invariant and this conclusion is in accordance with (4.10). Moreover, adopting (4.24a) together with the general expression (4.8), we recover Hooke's generalized law (4.20).

For uniaxial loading where $\sigma = E\varepsilon$ holds, we have $W = \frac{1}{2}E\varepsilon^2$ which is positive since we certainly expect *Young's modulus E* to be positive. It seems natural also to expect that the strain energy is positive for the general case, i.e.

$$\boxed{W > 0} \tag{4.25}$$

Referring to (4.24a) and borrowing from the terminology of matrices, it appears that the linear elastic stiffness tensor D_{ijkl} is *positive definite* (i.e. for any $\varepsilon_{ij} \neq 0$, we have $\varepsilon_{ij}D_{ijkl}\varepsilon_{kl} > 0$, cf. (1.7)). This property turns out to be of importance in order to ensure that a linear elastic boundary value problem possesses a unique solution (see Chapter 24).

As the entire concept of linear elasticity relies on the concept of a one-to-one relation between stresses and strains, it is possible to invert the constitutive relation (4.20) to obtain

$$\boxed{\varepsilon_{ij} = C_{ijkl}\sigma_{kl}} \tag{4.26}$$

where C_{ijkl} is termed the *elastic flexibility tensor* and C_{ijkl} is independent of the amount of loading. In the next section, we shall provide a formal proof that this inversion is mathematically possible. Based on the symmetry of ε_{ij} and σ_{ij}, we obtain similarly to (4.21) that

$$\boxed{C_{ijkl} = C_{jikl} \quad ; \quad C_{ijkl} = C_{ijlk} \quad \textit{minor symmetry}} \tag{4.27}$$

We now require that the constitutive law (4.26), i.e. Hooke's generalized law, should comply with the concepts of hyper-elasticity. Insertion of (4.26) into (4.18) then reveals that

$$\boxed{C_{ijkl} = C_{klij} \qquad \textit{for hyper-elasticity; major symmetry}} \tag{4.28}$$

in accordance with (4.23).

The complementary energy C is defined by (4.11) which with (4.24b) proves that for linear hyper-elastic materials, we have

$$\boxed{C = W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} > 0} \tag{4.29}$$

Recalling the interpretations of C and W, cf. Fig. 4.3, and the assumed linearity between stresses and strains, this relation is certainly not surprising. Insertion of Hooke's law in the form of (4.26) into (4.29) gives

$$\boxed{C = \frac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl} > 0} \tag{4.30}$$

Since σ_{ij} is a second-order tensor and C_{ijkl} a fourth-order tensor, it follows that the quantity $\sigma_{ij}C_{ijkl}\sigma_{kl}$ is an invariant. It then follows from (4.30) that C is an invariant and this conclusion is in accordance with (4.17). We also conclude from (4.30) that C_{ijkl} is positive definite. Moreover, (4.14) shows that $dC = \varepsilon_{ij}d\sigma_{ij}$.

Insertion of the constitutive relation (4.26) into (4.20) yields

$$\sigma_{ij} = D_{ijmn}C_{mnkl}\sigma_{kl}$$

Direct inspection shows that the expression $\sigma_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\sigma_{kl}$ holds and we also observe that both the left-hand and right-hand side of this relation fulfill the symmetry in i and j. Therefore, we obtain

$$[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - D_{ijmn}C_{mnkl}]\sigma_{kl} = 0$$

and as this expression holds for arbitrary stresses σ_{kl}, we conclude that

$$\boxed{D_{ijmn}C_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})} \tag{4.31}$$

In fact, (4.31) shows that the stiffness tensor is the inverse of the flexibility tensor. If we instead insert the constitutive relation (4.20) into (4.26) we obtain in a similar manner

$$\boxed{C_{ijmn}D_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})} \tag{4.32}$$

4.4 Linear elasticity – Matrix formulation

As discussed in Chapter 1, tensor formulation is usually most convenient when deriving a theory whereas matrix notation is of advantage when it comes to numerical applications. As our objective is to be able to solve general boundary value problems, which in practice implies the use of numerical solution strategies in terms of the finite element method, we need to reformulate the tensor equations of linear elasticity into their corresponding matrix form.

Let us assume that Hooke's law is written in the form

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \tag{4.33}$$

With the summation convention and the symmetry property $D_{ijkl} = D_{ijlk}$, (4.33) can be written as

$$[\sigma_{ij}] = \begin{bmatrix} D_{ij11} & D_{ij22} & D_{ij33} & D_{ij12} & D_{ij13} & D_{ij23} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \tag{4.34}$$

in accordance with (4.22). Define the following matrices

$$\boxed{\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} \quad ; \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}} \tag{4.35}$$

and

$$
\boldsymbol{D} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & D_{1112} & D_{1113} & D_{1123} \\ D_{2211} & D_{2222} & D_{2233} & D_{2212} & D_{2213} & D_{2223} \\ D_{3311} & D_{3322} & D_{3333} & D_{3312} & D_{3313} & D_{3323} \\ D_{1211} & D_{1222} & D_{1233} & D_{1212} & D_{1213} & D_{1223} \\ D_{1311} & D_{1322} & D_{1333} & D_{1312} & D_{1313} & D_{1323} \\ D_{2311} & D_{2322} & D_{2333} & D_{2312} & D_{2313} & D_{2323} \end{bmatrix} \tag{4.36}
$$

then (4.34) can be written as

$$
\boxed{\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}} \tag{4.37}
$$

It may be slightly confusing that we use the notation $\boldsymbol{\sigma}$ both for the column matrix defined by (4.35) as well as for the stress tensor. Likewise, we use $\boldsymbol{\varepsilon}$ to denote the column matrix defined by (4.35) as well as the strain tensor. Unfortunately, such double designation cannot be avoided, but remembering that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ defined by (4.35) most often will appear in connection with the matrix **D** defined by (4.36), uncertainty of what is meant by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ will hardly emerge in practice.

It is of interest that the engineering shear strains $2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}$ occur in $\boldsymbol{\varepsilon}$ defined by (4.35). In principle, we could equally well have placed the digit 2 at appropriate locations in the **D**-matrix. By not doing so we obtain the significant advantage that owing to the major symmetry property $D_{ijkl} = D_{klij}$ that holds for hyper-elasticity, **D** as defined by (4.36) becomes symmetric, i.e.

$$
\boxed{\boldsymbol{D} = \boldsymbol{D}^T \qquad when\ D_{ijkl} = D_{klij}} \tag{4.38}
$$

A further advantage is obtained by the definition given in (4.35) since it appears that

$$
\boxed{\sigma_{ij}\varepsilon_{ji} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T\boldsymbol{\sigma}} \tag{4.39}
$$

where $\sigma_{ij}\varepsilon_{ji}$ clearly is an invariant.

From $\sigma_{ij}\varepsilon_{ji} = \boldsymbol{\varepsilon}^T\boldsymbol{\sigma}$ we conclude from (4.24b), (4.25) and (4.37) that the strain energy W is given by

$$
\boxed{W = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{D}\boldsymbol{\varepsilon} > 0} \tag{4.40}
$$

i.e. $\boldsymbol{D}$ is positive definite. It therefore possesses an inverse defined by

$$
\boxed{\boldsymbol{C} = \boldsymbol{D}^{-1}}
$$

i.e. (4.37) leads to

$$
\boxed{\boldsymbol{\varepsilon} = \boldsymbol{C}\boldsymbol{\sigma}} \tag{4.41}
$$

The matrix $\boldsymbol{D}$ is called the *linear elastic stiffness matrix* whereas $\boldsymbol{C}$ is called the *linear elastic flexibility matrix* . From (4.38), it is obvious that

$$\boxed{\boldsymbol{C} = \boldsymbol{C}^T \qquad \textit{for hyper-elasticity}}$$

It is evident that (4.41) in tensor notation takes the form stipulated by (4.26) and we have therefore proved formally that the inversion of (4.20) that leads to (4.26) is mathematically possible. Finally, we obtain from $\sigma_{ij}\varepsilon_{ji} = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}$, (4.29) and (4.41) that the complementary energy C is given by

$$\boxed{C = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{C}\boldsymbol{\sigma}}$$

Returning to the definition of the column matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ defined by (4.35), we mentioned previously that other definitions are possible. However, the present definitions are the classical ones and as mentioned they possess certain advantages; they were introduced by Voigt (1928). We may mention that if $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are defined as $\boldsymbol{\sigma}^T = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\sqrt{2}\sigma_{12}\ \sqrt{2}\sigma_{13}\sqrt{2}\sigma_{23}]$ and $\boldsymbol{\varepsilon}^T = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \sqrt{2}\varepsilon_{12}\ \sqrt{2}\varepsilon_{13}\ \sqrt{2}\varepsilon_{23}]$ then the corresponding stiffness matrix also becomes symmetric. Moreover, $\boldsymbol{\sigma}^T\boldsymbol{\sigma} = \sigma_{ij}\sigma_{ji}$ and $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \varepsilon_{ij}\varepsilon_{ji}$ as well as $\boldsymbol{\sigma}\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$. This approach was adopted by Argyris (1965a) and by Horgan (1973) and its further ramifications is discussed by Pedersen (1995) as well as Cowin and Mehrabadi (1995).

4.5 Change of coordinate system when using matrix format

Suppose that in the x_i–coordinate system, we have determined the stress tensor σ_{ij}, the strain tensor ε_{ij} and the constitutive tensor D_{ijkl}. In another coordinate system given by x'_i, the stress tensor σ'_{ij}, the strain tensor ε'_{ij} and the constitutive tensor D'_{ijkl} are given by the corresponding transformation formulas that are characteristic for tensors, cf. (3.8), (1.43) and (2.19). However, if we work in the matrix form $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$, cf. (4.37), then neither $\boldsymbol{\sigma}$ nor $\boldsymbol{\varepsilon}$ and nor $\boldsymbol{D}$ are tensors so we need to establish rules for how these quantities transform.

Let us first establish the transformation rules for the column matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$. From the tensor transformation rule (3.8) we have

$$\sigma'_{ij} = A_{ik}\sigma_{kl}A_{jl} = L_{ijkl}\sigma_{kl} \qquad \text{where} \qquad L_{ijkl} = A_{ik}A_{jl}$$

from which we can establish the following transformation rule in matrix format

$$\boxed{\boldsymbol{\sigma}' = \boldsymbol{L}\boldsymbol{\sigma}} \tag{4.42}$$

where $\boldsymbol{L}$ is a 6x6 transformation matrix with components that can be determined in a trivial, but tedious manner. We noted in (4.39) that $\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}$ is an invariant. Therefore, we have with (4.42) that

$$\boldsymbol{\sigma}^T\boldsymbol{\varepsilon} = \boldsymbol{\sigma}'^T\boldsymbol{\varepsilon}' \quad \text{i.e.} \quad \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon} - \boldsymbol{L}^T\boldsymbol{\varepsilon}') = 0$$

and as this expression holds for arbitrary values of $\boldsymbol{\sigma}^T$, we conclude that

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{L}^T\boldsymbol{\varepsilon}'} \tag{4.43}$$

It is important to observe that the transformation matrix $\boldsymbol{L}$ is not orthogonal, i.e. $\boldsymbol{L}^T \neq \boldsymbol{L}^{-1}$. To prove this, consider the quantity $\boldsymbol{\sigma}^T\boldsymbol{\sigma}$ which is certainly not an invariant (note that $\boldsymbol{\sigma}^T\boldsymbol{\sigma} \neq \sigma_{ij}\sigma_{ij}$). With (4.42), we obtain $\boldsymbol{\sigma}'^T\boldsymbol{\sigma}' = \boldsymbol{\sigma}^T\boldsymbol{L}^T\boldsymbol{L}\boldsymbol{\sigma}$, i.e. if $\boldsymbol{L}^T\boldsymbol{L} = \boldsymbol{I}$ then $\boldsymbol{\sigma}'^T\boldsymbol{\sigma}' = \boldsymbol{\sigma}^T\boldsymbol{\sigma}$ in contradiction with the fact that $\boldsymbol{\sigma}^T\boldsymbol{\sigma}$ is not an invariant.

Having established the transformation rules for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, we next turn our attention to the constitutive matrix $\boldsymbol{D}$. In the x_i–coordinate system, we have

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon} \tag{4.44}$$

and in the x_i'–system, the constitutive relation reads

$$\boldsymbol{\sigma}' = \boldsymbol{D}'\boldsymbol{\varepsilon}' \tag{4.45}$$

Insertion of (4.42) and (4.43) into (4.44) yields

$$\boldsymbol{L}^{-1}\boldsymbol{\sigma}' = \boldsymbol{D}\boldsymbol{L}^T\boldsymbol{\varepsilon}' \quad \text{i.e.} \quad \boldsymbol{\sigma}' = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T\boldsymbol{\varepsilon}'$$

and a comparison with (4.45) reveals that

$$\boxed{\boldsymbol{D}' = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T} \tag{4.46}$$

Evidently, this transformation rule is not restricted to linear hyper-elasticity and it holds for any constitutive relation that is given in a format similar to (4.44).

4.6 Anisotropy in linear hyper-elasticity

We mentioned previously, cf. (4.19), that anisotropy, in general, means that the constitutive relation takes different forms depending on the Cartesian coordinate system we use. Anisotropy appears when the material behaves differently when loaded in the same manner in different directions; a piece of wood may be taken as an example, cf. Fig. 4.4.

On the other hand, if the material behaves identically when loaded in the same manner in all directions, the material is said to be *isotropic*. We are then led to the following conclusion:

Material isotropy means that the constitutive relation remains the same irrespective of the Cartesian coordinate system we use	(4.47)

Isotropy means that the material has no properties that depend on the direction and a piece of steel may be taken as an example of an isotropic material. In the x_i-coordinate system, we have $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$ and in another coordinate system x'_i we have $\sigma'_{ij} = D'_{ijkl}\varepsilon'_{kl}$ where $D_{ijkl} = D'_{ijkl}$ according to (4.47) holds for isotropic materials. In agreement with the discussion in Section 1.6, the elastic stiffness tensor D_{ijkl} for isotropic materials is then an isotropic fourth-order tensor and we shall later derive its explicit format.

In (4.37), we expressed Hooke's law in matrix form as

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon} \tag{4.48}$$

where the assumption of hyper-elasticity implies that $\boldsymbol{D} = \boldsymbol{D}^T$. It is convenient to redefine the components of $\boldsymbol{D}$ given by (4.36) according to

$$\boxed{\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{63} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}} \tag{4.49}$$

It appears that for a completely anisotropic material, the stiffness matrix $\boldsymbol{D}$ is fully populated. Due to the symmetry of $\boldsymbol{D}$, however, it comprises 21 independent components. We will now evaluate different forms of $\boldsymbol{D}$ for different forms of anisotropy where some kind of material symmetry exists.

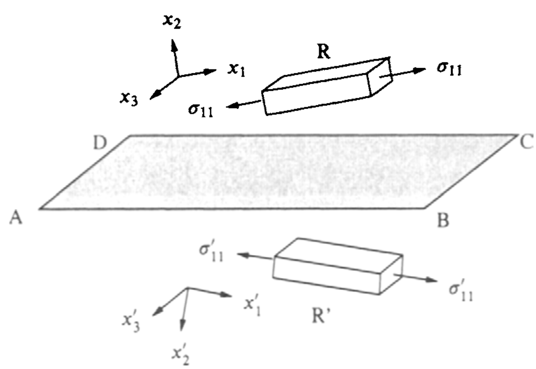

Figure 4.5: Region R' is a reflection of region R; coordinate system x'_i is a reflection of coordinate system x_i.

To illustrate such a symmetry property, consider two regions R and R' of a homogeneous material. According to Fig. 4.5, these regions are reflections of each other about the plane ABCD. Moreover, the coordinate system x'_i is a reflection (i.e. mirror image) of coordinate system x_i about this plane. Suppose that region R is subjected to a stress state where $\sigma_{11} \neq 0$ and all other stress components are equal to zero, cf. Fig. 4.5; via Hooke's law $\boldsymbol{\varepsilon} = \boldsymbol{C\sigma}$, this stress state results in a strain state ε_{ij}. Likewise, let region R' be subjected to the stress state $\sigma'_{11} \neq 0$ and all other stress components equal to zero; this gives rise to the strain state ε'_{ij}. Take the stress component $\sigma_{11} = \sigma'_{11}$; assume that this implies that all strain components $\varepsilon_{ij} = \varepsilon'_{ij}$. To generalize this situation, consider an arbitrary stress state and take the stress components $\sigma_{ij} = \sigma'_{ij}$. Assume that this implies that all strain components $\varepsilon_{ij} = \varepsilon'_{ij}$; in that case the material is said to possess a *plane of elastic symmetry* and, in the present case, this symmetry plane is plane ABCD.

From the discussion above, we arrive at the following definition

If the constitutive relation takes the same form for every pair of Cartesian coordinate systems that are mirror images (reflections) of each other in a certain plane, this plane is a plane of elastic symmetry	(4.50)

cf. Malvern (1969) and Love (1944). As an example, we may assume that the x_1x_2–plane is a plane of elastic symmetry, cf. Fig. 4.6a). In this coordinate system, Hooke's law is given by (4.48). Figure 4.6b) illustrates that the x'_i–coordinate system is a mirror image (reflection) of the x_i–system given in Fig. 4.6a), i.e. $x'_1 = x_1$, $x'_2 = x_2$ and $x'_3 = -x_3$. In the x'_i–coordinate system, Hooke's law reads

$$\boldsymbol{\sigma}' = \boldsymbol{D\varepsilon}' \tag{4.51}$$

where it was used that the constitutive matrix is the same in the two coordinate systems since the x_1x_2–plane is assumed to be a plane of elastic symmetry.

Having motivated and discussed the implications of elastic symmetry, we now consider the situation where the stress states $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the same, but

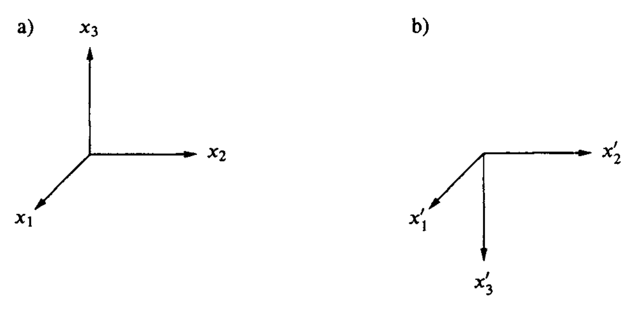

Figure 4.6: Coordinate change when x_1x_2–plane is a plane of elastic symmetry.

measured in different coordinate systems. This implies that also the strain states ε and ε' are the same, but measured in different coordinate systems. According to (1.27) and (1.28), the coordinate change in Fig. 4.6 implies

$$x = A^T x' \qquad \text{where} \qquad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From the transformation rules (3.8) and (2.19) we have $\sigma' = A\sigma A^T$ and $\varepsilon' = A\varepsilon A^T$ respectively, and this leads to

$$\begin{aligned} &\sigma'_{13} = -\sigma_{13} \;;\; \sigma'_{23} = -\sigma_{23} \qquad \text{otherwise} \qquad \sigma'_{ij} = \sigma_{ij} \\ &\varepsilon'_{13} = -\varepsilon_{13} \;;\; \varepsilon'_{23} = -\varepsilon_{23} \qquad \text{otherwise} \qquad \varepsilon'_{ij} = \varepsilon_{ij} \end{aligned} \tag{4.52}$$

It seems appropriate to point out that when we in Chapter 1 discussed a change of coordinate system, a translation and a rotation of the coordinate system was considered. This is the most general change of a coordinate system by which a right-handed (left-handed) coordinate system is preserved as a right-handed (left-handed) coordinate system. However, if a reflection of the coordinate system is involved, as shown in Fig. 4.6, the right-handed coordinate system x_i changes into the left-handed coordinate system x'_i. Whereas det$A$=1 holds for all changes of right-handed coordinate systems, cf. (1.26), we now have det$A$=-1, cf. the transformation matrix above. Apart from that, it is evident that we still have the transformation rule $x' = A(x - c)$, cf. (1.17), as well as the previously established tensor transformation rules that still hold even when reflections of the coordinate system are involved. In the literature, a transformation where det$A$=1 and $A^T A = I$ is called a *proper orthogonal transformation* whereas det$A$=-1 and $A^T A = I$ is called an *improper orthogonal transformation*.

The first equation of (4.48) provides

$$\sigma_{11} = D_{11}\varepsilon_{11} + D_{12}\varepsilon_{22} + D_{13}\varepsilon_{33} + 2D_{14}\varepsilon_{12} + 2D_{15}\varepsilon_{13} + 2D_{16}\varepsilon_{23}$$

Likewise, the first equation of (4.51) gives with (4.52) that

$$\sigma_{11} = D_{11}\varepsilon_{11} + D_{12}\varepsilon_{22} + D_{13}\varepsilon_{33} + 2D_{14}\varepsilon_{12} - 2D_{15}\varepsilon_{13} - 2D_{16}\varepsilon_{23}$$

A comparison of these two expressions reveals that

$$D_{15} = D_{16} = 0$$

Proceeding in the same manner for the remaining equations of (4.48) and (4.51) results in

$$D_{25} = D_{26} = D_{35} = D_{36} = D_{45} = D_{46} = 0$$

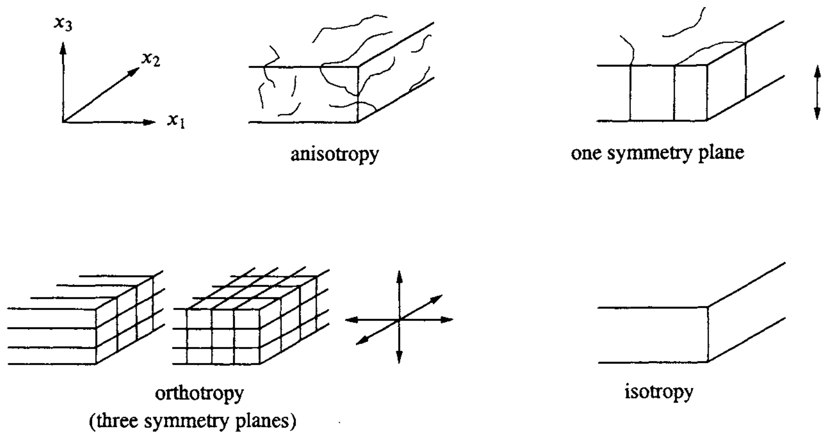

Figure 4.7: Illustration of increasing degree of symmetry.

We have therefore proved that if the x_1x_2–plane is a plane of elastic symmetry, then the constitutive matrix $\boldsymbol{D}$ takes the form

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} & 0 & 0 \\ D_{41} & D_{42} & D_{43} & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{65} & D_{66} \end{bmatrix} \quad \textit{one symmetry plane} \tag{4.53}$$

Therefore, recalling the symmetry of $\boldsymbol{D}$, it appears that one plane of symmetry reduces the 21 independent components of $\boldsymbol{D}$ to 13 independent components. The case of the x_1x_2–plane being a plane of elastic symmetry is illustrated in Fig. 4.7.

If not only the x_1x_2-plane is a symmetry plane, but also the x_1x_3-plane is a symmetry plane, then we obtain

$$\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}' \qquad \text{where} \qquad \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and with $\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$ and $\boldsymbol{\varepsilon}' = \boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T$, this implies that

$$\begin{aligned} \sigma'_{12} &= -\sigma_{12} \;;\; \sigma'_{32} = -\sigma_{32} \qquad \text{otherwise} \qquad \sigma'_{ij} = \sigma_{ij} \\ \varepsilon'_{12} &= -\varepsilon_{12} \;;\; \varepsilon'_{32} = -\varepsilon_{32} \qquad \text{otherwise} \qquad \varepsilon'_{ij} = \varepsilon_{ij} \end{aligned} \tag{4.54}$$

From $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$ and the first equation of (4.53) we obtain

$$\sigma_{11} = D_{11}\varepsilon_{11} + D_{12}\varepsilon_{22} + D_{13}\varepsilon_{33} + 2D_{14}\varepsilon_{12}$$

Likewise, the first equation of (4.51) gives with (4.54)

$$\sigma_{11} = D_{11}\varepsilon_{11} + D_{12}\varepsilon_{22} + D_{13}\varepsilon_{33} - 2D_{14}\varepsilon_{12}$$

A comparison of these two expressions reveals that

$$D_{14} = 0$$

Proceeding in the same manner for the remaining equations of (4.53) and (4.51) results in

$$D_{24} = D_{34} = D_{56} = 0$$

i.e. (4.53) reduces to

$$\boxed{\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \quad \textit{orthotropy}} \tag{4.55}$$

If finally, not only the x_1x_2- and x_1x_3-planes are planes of symmetry, but also the x_2x_3-plane is a plane of symmetry, we obtain

$$\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}' \qquad \text{where} \qquad \boldsymbol{A}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e.

$$\begin{aligned} \sigma'_{12} &= -\sigma_{12} \; ; \quad \sigma'_{13} = -\sigma_{13} \qquad \text{otherwise} \qquad \sigma'_{ij} = \sigma_{ij} \\ \varepsilon'_{12} &= -\varepsilon_{12} \; ; \quad \varepsilon'_{13} = -\varepsilon_{13} \qquad \text{otherwise} \qquad \varepsilon'_{ij} = \varepsilon_{ij} \end{aligned} \tag{4.56}$$

However, a comparison of (4.56) with (4.52) and (4.54) shows that no new information can be achieved and as a material possessing three orthogonal planes of elastic symmetry is called *orthotropic* we conclude

$$\boxed{\begin{aligned} \textit{Orthotropy} &= \textit{three orthogonal symmetry planes} \\ &= \textit{two orthogonal symmetry planes} \end{aligned}} \tag{4.57}$$

From (4.55), we then conclude that an orthotropic material possesses nine independent elastic parameters. Several important engineering materials are orthotropic and examples are wood, paper, rolled steel and rolled aluminum and an illustration is given in Fig. 4.7. It is emphasized that (4.55) holds when the coordinate axes are chosen as parallel to the *material axes of orthotropy*. If this is not the case, the coordinate system must be rotated accordingly and the stress and strain components in the new aligned coordinate system must be determined using (3.8) and (2.19) before use can be made of (4.55).

From a physical point of view, the form (4.55) of the orthotropic $\boldsymbol{D}$-matrix is certainly not surprising since it implies that normal strains only result in normal stresses and that a shear strain only affects the corresponding shear stress. In view of the orthotropic symmetry properties this is exactly the response we would expect and, in fact, (4.55) could be written down directly just utilizing these physical evident properties. The reason that we have here adopted a more formal route to derive (4.55) is that when it comes to yield criteria for orthotropic materials, the direct physical approach is difficult to apply and the benefits of the formal approach introduced already now are then considerable, cf. Section 8.13. In Section 6.5, we will return to orthotropic elasticity, but adopt another and very elegant approach.

Let us finally invert (4.55) and obtain Hooke's law in the format

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \\ \frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} \tag{4.58}$$

Since the flexibility matrix is symmetric, we have

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

The notation for Poisson's ratios needs an explanation. Suppose that we have uniaxial loading in the x_2-direction; all stress components except σ_{22} are zero. From (4.58) follows that $\varepsilon_{22} = \sigma_{22}/E_2$ whereas $\varepsilon_{11} = -\nu_{21}\varepsilon_{22}$ and $\varepsilon_{33} = -\nu_{23}\varepsilon_{22}$. Therefore, ν_{21} is Poisson's ratio associated with loading in the x_2-direction and strain in the x_1-direction and a similar interpretation holds for ν_{23}.

Apart from orthotropy, other symmetry properties like various crystal symmetries may occasionally be of importance and reference is given to, for instance, Love (1944), Green and Adkins (1960), Lekhnitskii (1981) and Cowin and Mehrabadi (1995) for further information.

We have already mentioned, cf. (4.47), that isotropy means that the constitutive relation is the same irrespective of the Cartesian coordinate system we use. This implies that every plane is a plane of elastic symmetry and, referring for instance to Sokolnikoff (1946) or Malvern (1969), the constitutive matrix then takes the form

$$
\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \tag{4.59}
$$

We now have only two independent material parameters and they are *Young's modulus* E and *Poisson's ratio* ν. We shall deal with isotropic linear elasticity in more details in Section 4.9 and we shall then derive (4.59) by another approach. For isotropic materials, it is straightforward to invert $\boldsymbol{D}$ to obtain

$$
\boldsymbol{C} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{4.60}
$$

We finally mention that whereas each stress component for general anisotropic elasticity couples with all strain components, cf. (4.49), isotropy implies that the normal stresses are given by the normal strains and each shear stress is determined entirely by the corresponding shear strain, cf. (4.59).

4.7 Initial strains - Thermoelasticity

Let us assume that there exists a given strain state ε_{ij}^o for which the strain energy is zero. In analogy with (4.24), this may be achieved by assuming that

$$
\boxed{W = \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ij}^o) D_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^o)} \tag{4.61}
$$

From (4.8) then follows that

$$
\boxed{\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^o)} \tag{4.62}
$$

It appears that by the choice (4.61), we have obtained an extension of Hooke's law, which implies that $\sigma_{ij} = 0$ when $\varepsilon_{ij} = \varepsilon_{ij}^o$. The strains ε_{ij}^o are called *initial strains*. In obvious matrix notation (4.62) reads

$$
\boldsymbol{\sigma} = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^o)
$$

An important example of initial strains is *thermal strains*, i.e. strains caused by thermal expansion of the material. When thermal strains are considered in Hooke's law (4.62), one speaks of *thermoelasticity*.

To determine the thermal strains, consider a specimen of isotropic and homogeneous material which is free to expand or contract as a result of a uniform change in temperature. As the material is free to expand or contract, $\sigma_{ij} = 0$ holds and the total strains ε_{ij} and the thermal strains ε_{ij}^o are equal. We have

$$\varepsilon_{ij}^o = \alpha \Delta T \delta_{ij} \tag{4.63}$$

where α is the *thermal expansion coefficient* and ΔT is the change of temperature from some reference temperature where no thermal strains exist. Note that in accordance with the assumption of isotropy, the thermal normal strains are equal and no thermal shear strains exist.

For a anisotropic material, (4.63) is replaced by

$$\varepsilon_{ij}^o = \alpha_{ij} \Delta T \tag{4.64}$$

where α_{ij} is some symmetric second-order tensor that characterizes the thermal expansion properties of the material. Consider orthotropy, for instance, we have

$$[\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}$$

Finally, we note that (4.62) may be written as

$$\boxed{\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^o}$$

where

$$\boxed{\varepsilon_{ij}^e = C_{ijkl}\sigma_{kl} \quad ; \qquad \varepsilon_{ij}^o = \alpha_{ij}\Delta T}$$

I.e. the *total strains* ε_{ij} consist of the sum of the *elastic strains* ε_{ij}^e determined by Hooke's law and the thermal strains ε_{ij}^o. The formulation (4.62) was introduced by Duhamel in 1837.

4.8 Most general isotropic hyper-elasticity

In the previous sections, we discussed various aspects of linear hyper-elasticity, its matrix formulation and different forms of anisotropy. We will now derive the most general form of hyper-elasticity for isotropic materials and we shall see that this opens for modeling of nonlinear elastic behavior.

The strain energy depends on the current strains ε_{ij} through $W = W(\varepsilon_{ij})$ and according to (4.10), the strain energy is an invariant. The strain tensor ε_{ij} can be expressed by the principal strains ε_1, ε_2 and ε_3 and the corresponding principal strain directions. Isotropy means that the material has no directional properties and this implies that we may write the strain energy as

$W = W(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. As the principal strains are given uniquely by the strain invariants, we may equally well write the strain energy W as

$$\boxed{W = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)} \tag{4.65}$$

where the generic strain invariants are given (2.51), i.e.

$$\tilde{I}_1 = \varepsilon_{kk} \; ; \quad \tilde{I}_2 = \frac{1}{2}\varepsilon_{kl}\varepsilon_{lk} \; ; \quad \tilde{I}_3 = \frac{1}{3}\varepsilon_{kl}\varepsilon_{lm}\varepsilon_{mk} \tag{4.66}$$

Formulation (4.65) is evidently in accordance with (4.10), stating that W is an invariant. The choice of the set of invariants given by (4.66) is particularly convenient, since we have the following neat relations

$$\frac{\partial \tilde{I}_1}{\partial \varepsilon_{ij}} = \delta_{ij} \; ; \quad \frac{\partial \tilde{I}_2}{\partial \varepsilon_{ij}} = \varepsilon_{ij} \; ; \quad \frac{\partial \tilde{I}_3}{\partial \varepsilon_{ij}} = \varepsilon_{ik}\varepsilon_{kj} \tag{4.67}$$

We are now in a position to derive the most general constitutive law for isotropic hyper-elastic materials. From (4.8) and (4.65) we obtain

$$\sigma_{ij} = \frac{\partial W}{\partial \tilde{I}_1}\frac{\partial \tilde{I}_1}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \tilde{I}_2}\frac{\partial \tilde{I}_2}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \tilde{I}_3}\frac{\partial \tilde{I}_3}{\partial \varepsilon_{ij}} \tag{4.68}$$

With the notation

$$\boxed{\phi_1 = \frac{\partial W}{\partial \tilde{I}_1} \; ; \quad \phi_2 = \frac{\partial W}{\partial \tilde{I}_2} \; ; \quad \phi_3 = \frac{\partial W}{\partial \tilde{I}_3}} \tag{4.69}$$

(4.68) reduces with (4.67) to

$$\boxed{\sigma_{ij} = \phi_1\delta_{ij} + \phi_2\varepsilon_{ij} + \phi_3\varepsilon_{ik}\varepsilon_{kj}} \tag{4.70}$$

Instead of the index notation, we may write $\sigma_{ij} = \boldsymbol{\sigma}$ and $\varepsilon_{ij} = \boldsymbol{\varepsilon}$, i.e. (4.70) can be written as

$$\boxed{\boldsymbol{\sigma} = \phi_1\boldsymbol{I} + \phi_2\boldsymbol{\varepsilon} + \phi_3\boldsymbol{\varepsilon}^2} \tag{4.71}$$

From the definition of the parameters ϕ_1, ϕ_2 and ϕ_3 it follows directly that they may depend on the strain invariants. However, these parameters are not independent, since (4.69) results in the following constraints

$$\boxed{\frac{\partial \phi_1}{\partial \tilde{I}_2} = \frac{\partial \phi_2}{\partial \tilde{I}_1} \; ; \quad \frac{\partial \phi_1}{\partial \tilde{I}_3} = \frac{\partial \phi_3}{\partial \tilde{I}_1} \; ; \quad \frac{\partial \phi_2}{\partial \tilde{I}_3} = \frac{\partial \phi_3}{\partial \tilde{I}_2}} \tag{4.72}$$

It is of interest that (4.70) or (4.71) shows that the stress tensor and the strain tensor have identical principal directions. To see this, assume that the coordinate system is chosen collinearly with the principal directions of the strain tensor, i.e. $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^2$ become diagonal matrices. It follows immediately from (4.71) that this implies that also $\boldsymbol{\sigma}$ becomes diagonal proving that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ have identical principal directions.

The most general form of isotropic hyper-elasticity is given by (4.70) and it appears that the presence of the quadratic term $\varepsilon_{ik}\varepsilon_{kj}$ implies a nonlinear relation between stresses and strains. It is of importance that even though we restrict ourselves to small strain theory, we cannot ignore the quadratic term $\varepsilon_{ik}\varepsilon_{kj}$ when compared to the linear term ε_{ij}, since this depends entirely on the magnitude of the parameters ϕ_2 and ϕ_3. Moreover, even if we choose $\phi_3 = 0$, i.e. the strain energy W does not depend on the third strain invariant $\tilde{I}_3$, cf. (4.69), we may obtain a nonlinear formulation if ϕ_1 and ϕ_2 are not constants. The implications of this approach is discussed in Section 4.10.

Another point of interest is that the presence of the three parameters ϕ_1, ϕ_2 and ϕ_3 means that the most general form of isotropic nonlinear hyper-elasticity involves three material parameters.

In the discussion above, we used the strain energy W as the vehicle for our derivation. However, we could equally well use the complementary energy C as the basis for our discussions and we shall now see the implications of this approach.

In general, the strain energy is given by $W = W(\varepsilon_{ij})$ and for isotropic materials this reduces to $W = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$, cf. (4.65). In general, the complementary energy is given by $C = C(\sigma_{ij})$ and similarly to the result above, we have for isotropic materials

$$\boxed{C = C(I_1, I_2, I_3)} \tag{4.73}$$

where the generic stress invariants are given by (3.14), i.e.

$$I_1 = \sigma_{kk}\ ; \quad I_2 = \frac{1}{2}\sigma_{kl}\sigma_{lk}\ ; \quad I_3 = \frac{1}{3}\sigma_{kl}\sigma_{lm}\sigma_{mk}$$

We have the following relations

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij}\ ; \quad \frac{\partial I_2}{\partial \sigma_{ij}} = \sigma_{ij}\ ; \quad \frac{\partial I_3}{\partial \sigma_{ij}} = \sigma_{ik}\sigma_{kj} \tag{4.74}$$

From (4.16) and (4.73) follow that

$$\varepsilon_{ij} = \frac{\partial C}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial C}{\partial I_2}\frac{\partial I_2}{\partial \sigma_{ij}} + \frac{\partial C}{\partial I_3}\frac{\partial I_3}{\partial \sigma_{ij}} \tag{4.75}$$

With the notation

$$\boxed{\psi_1 = \frac{\partial C}{\partial I_1}\ ; \quad \psi_2 = \frac{\partial C}{\partial I_2}\ ; \quad \psi_3 = \frac{\partial C}{\partial I_3}} \tag{4.76}$$

(4.75) reduces with (4.74) to

$$\boxed{\varepsilon_{ij} = \psi_1\delta_{ij} + \psi_2\sigma_{ij} + \psi_3\sigma_{ik}\sigma_{kj}} \tag{4.77}$$

which in matrix notation reads

$$\boxed{\boldsymbol{\varepsilon} = \psi_1 \boldsymbol{I} + \psi_2 \boldsymbol{\sigma} + \psi_3 \boldsymbol{\sigma}^2} \tag{4.78}$$

These results may be compared with the forms given by (4.70) and (4.71).

From the definition of the parameters ψ_1, ψ_2 and ψ_3 it appears that, in general, they depend on the stress invariants. However, these parameters are not independent, since (4.76) results in the following constraints

$$\boxed{\frac{\partial \psi_1}{\partial I_2} = \frac{\partial \psi_2}{\partial I_1}\,; \quad \frac{\partial \psi_1}{\partial I_3} = \frac{\partial \psi_3}{\partial I_1}\,; \quad \frac{\partial \psi_2}{\partial I_3} = \frac{\partial \psi_3}{\partial I_2}} \tag{4.79}$$

Formulation (4.70) or (4.77) corresponds to isotropic hyper-elasticity in its most general form and it evidently includes nonlinear elasticity. A detailed discussion of the various types of nonlinearity that can be modeled by this formulation is given by Evans and Pister (1966).

4.9 Isotropic linear elasticity

We have already touched upon linear isotropic hyper-elasticity, cf. (4.59), but we shall now treat it in a more consistent manner.

The general format of isotropic hyper-elasticity is given by (4.70) where ϕ_1, ϕ_2 and ϕ_3 are defined by (4.69). To obtain a linear relation between stresses and strains, we must have $\phi_3 = 0$. Moreover, let us assume that

$$\phi_1 = \lambda \tilde{I}_1\,; \quad \phi_2 = 2\mu\,; \quad \phi_3 = 0 \tag{4.80}$$

where λ and μ are constants – the so-called *Lamé parameters* introduced by Lamé in 1852. It appears that this choice fulfills the constraints given by (4.72). We also observe that $\phi_3 = 0$ implies that the strain energy W does not depend on $\tilde{I}_3$, cf. (4.69). I.e. (4.69) and (4.80) lead to the following expression for the strain energy

$$W = \frac{1}{2}\lambda \tilde{I}_1^2 + 2\mu \tilde{I}_2$$

With (4.70) and (4.80) and as $\tilde{I}_1 = \varepsilon_{kk}$ we find that

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4.81}$$

Since λ and μ are material constants, we observe that (4.81) expresses a linearity between stresses and strains, i.e. we have recovered Hooke's generalized law for isotropic materials. We mentioned previously in relation to (4.59) that linear isotropic elasticity is characterized by two material parameters and the present derivation provides a formal proof for that. These two parameters may be expressed in terms of the more familiar material parameters E = Young's modulus and ν = Poisson's ratio through

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}; \quad \mu = \frac{E}{2(1+\nu)} \tag{4.82}$$

Moreover, the *shear modulus* G, where $G = \mu$, and the *bulk modulus* K are defined by

$$\boxed{G = \frac{E}{2(1+\nu)}; \qquad K = \frac{E}{3(1-2\nu)}} \tag{4.83}$$

With (4.82) and (4.83), Hooke's law (4.81) takes the form

$$\boxed{\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right]} \tag{4.84}$$

An interesting reformulation of Hooke's law can be obtained as follows. Contraction of (4.84) yields

$$\boxed{\sigma_{kk} = 3K\varepsilon_{kk}} \tag{4.85}$$

From the definition of deviatoric stresses $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ we obtain from (4.84) and (4.85) that

$$s_{ij} = 2G\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right)$$

which, by means of the definition of the deviatoric strains e_{ij}, takes the form

$$\boxed{s_{ij} = 2G\, e_{ij}} \tag{4.86}$$

We recall that the off-diagonal terms of the deviatoric stress and strain tensors are equal to the shear stresses and shear strains respectively. From (4.85) and (4.86) follow that

Hooke's linear isotropic law implies decoupling between volumetric and deviatoric response	(4.87)

Let us now make a further reformulation of (4.84). We have that $\varepsilon_{kk} = \delta_{kl}\varepsilon_{kl}$. Moreover, direct inspection shows that $\varepsilon_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl}$ where both the left-hand and right-hand side of this expression are symmetric in i and j. With these two reformulations, (4.84) takes the form

$$\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl}} \tag{4.88}$$

where

$$\boxed{D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]} \tag{4.89}$$

and D_{ijkl} is the *elastic isotropic stiffness tensor*. We have then recovered the formulation (4.20) with an explicit expression for D_{ijkl} valid for an isotropic material. Expression (4.89) clearly fulfills the general symmetry properties given by (4.21) and (4.23). Writing all components in (4.84) - or (4.88) - and changing to a matrix formulation, we recover the stiffness matrix $\boldsymbol{D}$ given by (4.59).

Insertion of (4.89) into the transformation formula (1.43) provides

$$\boxed{D'_{ijkl} = D_{ijkl}} \tag{4.90}$$

Since the isotropic elastic stiffness tensor takes the same form in all coordinate systems, it is an *isotropic fourth-order tensor*. As a comparison, we have previously shown that Kronecker's delta δ_{ij} is an isotropic second-order tensor, cf. (1.45). That D_{ijkl} is an isotropic tensor is certainly not surprising since we are considering an isotropic material, which means that we expect Hooke's law to be the same irrespective of which Cartesian coordinate system we choose.

To further scrutinize different expressions of Hooke's law, we will now make an inversion so that we obtain the strains as function of the stresses. This inversion is easily accomplished when noting Hooke's law in the format given by (4.85) and (4.86). Finding ε_{kk} and e_{ij} from these expressions and using the definition $\varepsilon_{ij} = e_{ij} + \delta_{ij}\varepsilon_{kk}/3$, we arrive at

$$\boxed{\varepsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right]} \tag{4.91}$$

Since $\sigma_{ij} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\sigma_{kl}$ and $\sigma_{kk} = \delta_{kl}\sigma_{kl}$, we obtain

$$\boxed{\varepsilon_{ij} = C_{ijkl}\sigma_{kl}} \tag{4.92}$$

where

$$\boxed{C_{ijkl} = \frac{1}{2G}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{1+\nu}\delta_{ij}\delta_{kl}\right]} \tag{4.93}$$

and C_{ijkl} is termed the *elastic isotropic flexibility tensor*. We have then recovered the formulation (4.26) with an explicit expression for C_{ijkl} valid for an isotropic material. Expression (4.93) clearly fulfills the general symmetry properties given by (4.27) and (4.28).

Insertion of (4.93) into the transformation formula similar to (1.43) shows that

$$\boxed{C'_{ijkl} = C_{ijkl}} \tag{4.94}$$

i.e. the flexibility tensor is an isotropic tensor - as expected. It is straightforward to convince oneself that the explicit expressions given by (4.89) and (4.93) fulfill the general inversion properties stated by (4.31) and (4.32). Writing all components in (4.91) - or (4.92) - and changing to a matrix formulation, we recover the flexibility matrix $\boldsymbol{C}$ given by (4.60).

Due to (4.24) and (4.25), the stiffness tensor D_{ijkl} must be positive definite. Let us now see, what kind of constraints this requirement sets on the two material parameters E and ν. From (4.24) and (4.25) follow that

$$W = \frac{1}{2}\varepsilon_{ij}\sigma_{ij} > 0$$

Since $\varepsilon_{ij} = e_{ij} + \frac{1}{3}\delta_{ij}\varepsilon_{kk}$ and $\sigma_{ij} = s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$, we then find

$$W = \frac{1}{2}\left(e_{ij}s_{ij} + \frac{1}{3}\varepsilon_{kk}\sigma_{mm}\right) > 0$$

where advantage was taken of the fact that $e_{ii} = s_{ii} = 0$. Use of (4.85) and (4.86) then provide

$$W = \frac{1}{2}\left(2Ge_{ij}e_{ij} + K\varepsilon_{kk}\varepsilon_{mm}\right) > 0 \tag{4.95}$$

Both $e_{ij}e_{ij}$ and $\varepsilon_{kk}\varepsilon_{mm}$ are non-negative numbers. Moreover, e_{ij} and ε_{kk} can be chosen arbitrarily and independently of each other. Thus, we are led to the requirement $G > 0$ and $K > 0$. With reference to (4.83) and as we certainly must require that $E > 0$, we conclude that the elastic parameters must fulfill the constraints

$$\boxed{E > 0\,; \quad -1 < \nu < \frac{1}{2}} \tag{4.96}$$

in order to fulfill (4.95).

4.10 Nonlinear isotropic Hooke formulation

In order to provide a firm basis, we have given quite an extensive description of linear material behavior. However, our objective is to investigate nonlinear material behavior, so it is timely to change our focus of interest.

Our first objective is to derive a nonlinear elasticity theory that can be expressed in a form similar to Hooke's law

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \tag{4.97}$$

but where D_{ijkl} now depends on the amount of loading. To derive such a model, we shall restrict ourselves to isotropic elasticity.

The most general isotropic nonlinear elasticity may be derived from the complementary energy C. According to (4.73), C depends on the stress invariants I_1, I_2 and I_3, but we could equally well use the invariants given by I_1, J_2 and

J_3, cf. (3.18) and (3.19). For our present purpose, this latter set of invariants turns out out to give a more convenient description. We therefore have

$$C = C(I_1, J_2, J_3)$$

where

$$I_1 = \sigma_{kk} \quad ; \quad J_2 = \frac{1}{2} s_{kl} s_{lk} \quad ; \quad J_3 = \frac{1}{3} s_{kl} s_{lm} s_{mk}$$

From (4.16) we then obtain with the chain rule

$$\varepsilon_{ij} = \frac{\partial C}{\partial I_1}\delta_{ij} + \frac{\partial C}{\partial J_2} s_{ij} + \frac{\partial C}{\partial J_3}(s_{ik}s_{kj} - \frac{2}{3}J_2\delta_{ij}) \tag{4.98}$$

Since our objective is to derive a nonlinear elasticity theory that can be expressed in the Hooke format (4.97), we must in (4.98) require that the term containing the quadratic quantity $s_{ik}s_{kj}$ must disappear, i.e. $\partial C/\partial J_3 = 0$. This leads to

$$C = C(I_1, J_2)$$

With this restriction, (4.98) reduces to

$$\varepsilon_{ij} = \frac{\partial C}{\partial I_1}\delta_{ij} + \frac{\partial C}{\partial J_2} s_{ij}$$

Contraction of this equation and use of the definition of the deviatoric strains e_{ij} lead to

$$\varepsilon_{kk} = 3\frac{\partial C}{\partial I_1} \quad ; \quad e_{ij} = \frac{\partial C}{\partial J_2} s_{ij} \tag{4.99}$$

Let us make the following choices

$$\frac{\partial C}{\partial I_1} = \frac{\sigma_{kk}}{9K} \quad ; \quad \frac{\partial C}{\partial J_2} = \frac{1}{2G} \tag{4.100}$$

where the material functions K and G, in general, depend on the stress invariants I_1 and J_2, i.e.

$$\boxed{K = K(I_1, J_2) \quad ; \quad G = G(I_1, J_2)} \tag{4.101}$$

With (4.100), (4.99) reduces to

$$\boxed{\sigma_{kk} = 3K\varepsilon_{kk} \quad ; \quad s_{ij} = 2Ge_{ij}} \tag{4.102}$$

and we have therefore obtained a formulation that is completely similar to linear elasticity, cf. (4.85) and (4.86). In analogy with linear elasticity, (4.102) may be written in the format of (4.97) where D_{ijkl} is given by (4.89) and where G and ν (or E and ν) now depend on I_1 and J_2, cf. (4.101) and (4.83).

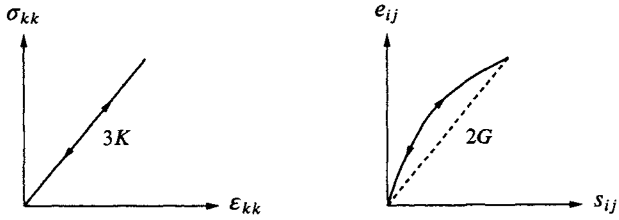

Figure 4.8: Illustration of response predicted by (4.102) and (4.105) - metals and steel.

By the above approach, we have created a nonlinear Hooke formulation with quite remarkable properties, as will be illustrated shortly.

The bulk modulus K and shear modulus G given by (4.101) cannot take arbitrary expressions of I_1 and J_2. Since $\partial(\partial C/\partial I_1)/\partial J_2 = \partial(\partial C/\partial J_2)/\partial I_1$, we obtain from (4.100) the following constraint

$$\boxed{\frac{\sigma_{kk}}{3}\frac{\partial}{\partial J_2}(\frac{1}{3K}) = \frac{\partial}{\partial I_1}(\frac{1}{2G})} \tag{4.103}$$

Let us now investigate whether the theory derived can be used to model the nonlinear behavior of important engineering materials. The general experimental evidence for nonlinear time-independent behavior of metals and steel can be summarized as follows:

For metals and steel, the volumetric response is linear elastic and all nonlinearity is related to the deviatoric response

(4.104)

This implies that the volumetric and deviatoric responses are uncoupled. These features can be captured in a very simple manner by the nonlinear Hooke formulation presented above. For this purpose and referring to (4.101), we make the following choice

$$K = \text{constant} \quad ; \quad G = G(J_2) \tag{4.105}$$

which evidently fulfills the constraint (4.103). Relations (4.102) in combination with (4.105) fulfill precisely the general requirements given by (4.104) and we have then obtained the so-called *deformation plasticity theory* proposed by Hencky (1924) and used extensively in older literature. The principal response predicted by (4.102) and (4.105) is illustrated in Fig. 4.8. It appears that the volumetric and deviatoric response is uncoupled just like in linear elasticity theory, cf. (4.87). The deformation plasticity theory states a relation between the current stresses and the current strains independently of the load history. Therefore, in contrast to experimental evidence where unloading implies the occurrence of permanent (plastic) strains, the present model predicts that loading and unloading follow the same path and this is the major disadvantage of deformation plasticity. In 'real' plasticity theory, the constitutive relations are given in an incremental form that allows for different behaviors in loading and unloading. However, in Section 9.7 we shall show that the 'real' so-called von Mises plasticity theory and the present deformation plasticity theory are identical for proportional increasing loading.

The nonlinear time-independent response of soil, rocks and concrete is more complicated and the general experimental evidence may be summarized as follows

For concrete, soil and rocks, the volumetric and deviatoric response is coupled and both the volumetric and deviatoric response are nonlinear

(4.106)

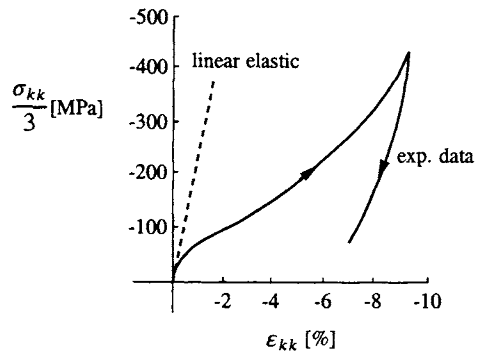

Figure 4.9: Hydrostatic compression of concrete ($\sigma_1 = \sigma_2 = \sigma_3 < 0$). Experimental data of Green and Swanson (1973); uniaxial compressive strength=48.5 MPa.

As examples of such behavior, Fig. 4.9 illustrated the nonlinear response of concrete for purely hydrostatic compression and Fig. 4.10 illustrates the nonlinear response of concrete for uniaxial compressive loading. Especially the volumetric response in Fig. 4.10b) is of interest, as it shows that the volume starts to increase when the failure state, i.e. the peak stress, is approached.

An example of the coupled response of concrete, soil and rocks is shown in Fig. 4.10. For constant hydrostatic stress $I_1 = \sigma_{kk}$ = constant, increase of the deviatoric stresses not only leads to changes of the deviatoric strains, as given by (4.102b), but also to a change of the volumetric strain ε_{ii}, cf. Fig. 4.11. For

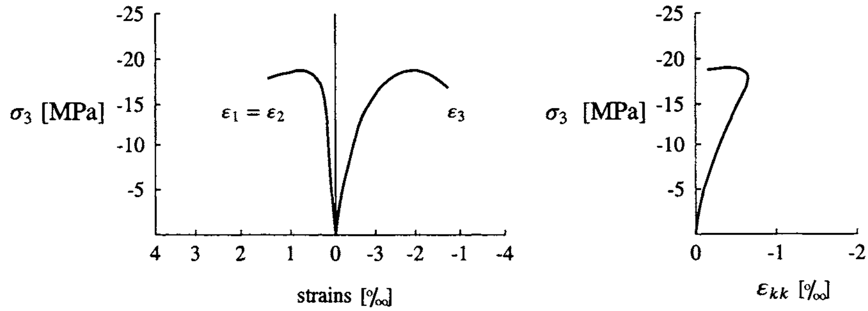

Figure 4.10: Uniaxial compression of concrete ($\sigma_1 = \sigma_2 = 0, \sigma_3 < 0$). Experimental data of Kupfer (1973); uniaxial compressive strength=18.7 MPa. a) stress-strain curves; b) development of volumetric strain ε_{ii}: first the volume decreases and then it increases.

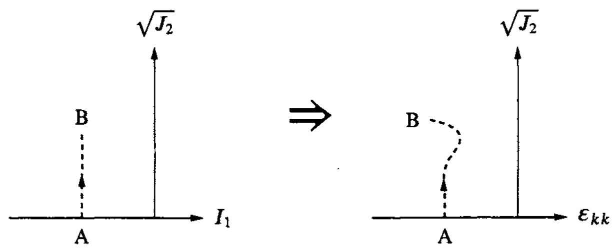

Figure 4.11: Coupled deviatoric and volumetric response characteristic for concrete, soil and rocks. Constant hydrostatic stress and increasing deviatoric stresses result in a change of volumetric strain.

σ_{kk} = constant, we obtain with (4.102a) and (4.101) that

$$\underbrace{\sigma_{kk}}_{\text{const.}} = 3K(\underbrace{I_1}_{\text{const.}}, \overbrace{J_2}^{\text{varies}})\, \varepsilon_{kk}$$

It is evident that fulfillment of this relation requires a change of ε_{kk} even though σ_{kk} is constant. This means that our nonlinear elasticity model allows for the coupled response characteristic for concrete, soil and rocks.

Another example of coupled response is illustrated in Fig. 4.12. For constant deviatoric stresses (s_{ij} = constant i.e. J_2 = constant), change of the hydrostatic stress I_1 not only results in a change of the volumetric strain ε_{ii} according to (4.102a), but also in a change of the deviatoric strains. For s_{ij} = constant,

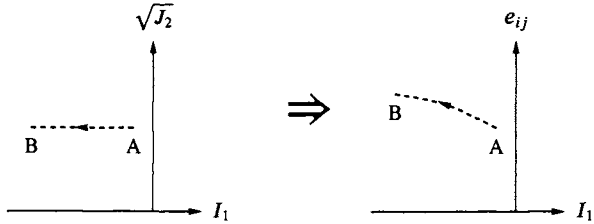

Figure 4.12: Coupled deviatoric and volumetric response characteristic for concrete, soil and rocks. Constant deviatoric stresses and change of hydrostatic stress result in a change of deviatoric strains.

(4.101) and (4.102b) yield

$$\underbrace{s_{ij}}_{\text{const.}} = 2G(\overbrace{I_1}^{\text{varies}}, \underbrace{J_2}_{\text{const.}})\, e_{ij}$$

and fulfillment of this expression requires a change of e_{ij} even though s_{ij} is constant, i.e. this expression allows the complicated response just described.

In conclusion, we obtain

A simple nonlinear Hooke formulation allows coupling effects between volumetric and deviatoric responses

and this coupling effect is characteristic for soils, rock and concrete.

It is therefore no surprise that a number of specific models have been proposed in the literature, where the moduli K and G have been calibrated to experimental data for concrete, soil and rocks. For obvious reasons, these models are often called *variable moduli models* and for a more detailed discussion, we may refer to Chen and Saleeb (1982) and Desai and Siriwardane (1984). We shall return to such models in Section 5.4, where it turns out that within the concept of Cauchy-elasticity, it is allowable to obtain a model similar to (4.102). However, in contrast to (4.101), Cauchy-elasticity allows that K and G even may depend on the third stress invariant J_3 and this makes for a much more accurate calibration of K and G to experimental data for concrete, soil and rocks. Finally, we recall the major drawback of nonlinear elasticity, namely that unloading follows the same path as loading. Remedies to remove this drawback will also be discussed in Section 5.4.

4.11 Plane strain

In plane strain, the only non-zero strains are ε_{11}, ε_{22} and ε_{12}, cf. (2.72). Using these conditions in (4.37) and assuming isotropy, we obtain with (4.59) that

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}} \tag{4.107}$$

where

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} ; \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} \tag{4.108}$$

and

$$\boxed{\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}} \tag{4.109}$$

With these definitions, (4.107) describes the *in-plane* conditions. Moreover, from (4.37) and (4.59) the *out-of-plane* stresses are given by

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22}) ; \quad \sigma_{13} = \sigma_{23} = 0$$

Alternatively, we may use (4.88) and (4.89) to obtain the following in-plane description

$$\boxed{\sigma_{\alpha\beta} = D_{\alpha\beta\gamma\lambda}\varepsilon_{\gamma\lambda}}$$

where Greek subscripts take the range 1, 2 and where

$$\boxed{D_{\alpha\beta\gamma\lambda} = 2G[\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\gamma}) + \frac{\nu}{1-2\nu}\delta_{\alpha\beta}\delta_{\gamma\lambda}]} \tag{4.110}$$

4.12 Plane stress

In plane stress, the only non-zero stresses are σ_{11}, σ_{22} and σ_{12}, cf. (3.25). Using these conditions in (4.41) and assuming isotropy, we obtain with (4.60) the following in-plane description

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{4.111}$$

as well as

$$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) ; \quad \varepsilon_{13} = \varepsilon_{23} = 0$$

Inversion of (4.111) gives

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{D}^* \boldsymbol{\varepsilon}}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are defined by (4.108) and $\boldsymbol{D}^*$ is given by

$$\boxed{\boldsymbol{D}^* = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}} \tag{4.112}$$

Note that $\boldsymbol{D}^*$ differs from $\boldsymbol{D}$ given by (4.109).

Using tensor notation, (4.92) and (4.93) gives for the in-plane behavior

$$\varepsilon_{\alpha\beta} = C_{\alpha\beta\gamma\lambda}\sigma_{\gamma\lambda} \tag{4.113}$$

where

$$C_{\alpha\beta\gamma\lambda} = \frac{1}{2G}[\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\gamma}) - \frac{\nu}{1+\nu}\delta_{\alpha\beta}\delta\gamma\lambda]$$

Define the tensor $D^*_{\alpha\beta\gamma\lambda}$ by

$$\boxed{D^*_{\alpha\beta\gamma\lambda} = 2G[\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\gamma}) + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\delta_{\gamma\lambda}]} \tag{4.114}$$

It appears that

$$D^*_{\alpha\beta\gamma\lambda}C_{\gamma\lambda\theta\kappa} = \frac{1}{2}(\delta_{\alpha\theta}\delta_{\beta\kappa} + \delta_{\alpha\kappa}\delta_{\beta\theta}) \tag{4.115}$$

Note the difference in the term containing Poisson's ratio when comparing (4.110) and (4.114). With (4.115), multiplication of (4.113) by $D^*_{\kappa\theta\alpha\beta}$ provides

$$\boxed{\sigma_{\alpha\beta} = D^*_{\alpha\beta\gamma\lambda}\varepsilon_{\gamma\lambda}}$$

4.13 Incompressible linear hyper-elasticity

When deriving the fundamental relation (4.8), i.e. $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$, we argued that $d\varepsilon_{ij}$ could be chosen arbitrarily in (4.7). However, this argument fails if some restrictions exist on $d\varepsilon_{ij}$ and this is the case if the material behaves *incompressibly*, i.e. $\varepsilon_{ii} = 0$.

To circumvent this obstacle, we may instead use the formulation (4.16), i.e.

$$\varepsilon_{ij} = \frac{\partial C}{\partial\sigma_{ij}} \tag{4.116}$$

When deriving this relation, we used that the incremental stresses $d\sigma_{ij}$ can be chosen arbitrarily, i.e. (4.116) holds in general. In the following let us restrict ourselves to isotropic elasticity, i.e. the complementary energy C depends only on the invariants of the stress tensor. It turns out to be convenient to choose the following invariants

$$C = C(I_1, J_2, J_3)$$

From (4.116) we then obtain in accordance with (4.98)

$$\varepsilon_{ij} = \frac{\partial C}{\partial I_1}\delta_{ij} + \frac{\partial C}{\partial J_2}s_{ij} + \frac{\partial C}{\partial J_3}(s_{ik}s_{kj} - \frac{2}{3}J_2\delta_{ij}) \tag{4.117}$$

Let us furthermore restrict ourselves to linear elasticity. Then (4.117) shows that we must have $\partial C/\partial J_3 = 0$, i.e.

$$C = C(I_1, J_2)$$

i.e.

$$\varepsilon_{ij} = \frac{\partial C}{\partial I_1}\delta_{ij} + \frac{\partial C}{\partial J_2}s_{ij} \tag{4.118}$$

The incompressibility condition $\varepsilon_{ii} = 0$ then implies that $\partial C/\partial I_1 = 0$, i.e. $C = C(J_2)$. Moreover, since (4.118) should provide a linear relation between stresses and strains, we choose

$$C = \frac{J_2}{2G}$$

where G is a constant material parameter. With this expression, (4.118) becomes

$$\boxed{\varepsilon_{ij} = e_{ij} = \frac{s_{ij}}{2G}\ ; \quad \varepsilon_{ii} = 0} \tag{4.119}$$

and we immediately identify the material parameter G to be the shear modulus, cf. (4.86).

Expression (4.119) states the constitutive law for an isotropic, incompressible and linear elastic solid. It is of major importance that, for any given strain state, it is only possible to identify the deviatoric stresses s_{ij} and not the full stress tensor σ_{ij} since the hydrostatic stress σ_{kk} is not determined by the constitutive relation. On the other hand, a change of the hydrostatic stress does not change the strain state.

As an example, we may imagine an incompressible material subject to a given stress state and a known strain state. On top of this stress state, we superpose a large hydrostatic stress by submerging the material onto the bottom of the ocean. However, this will not change the strain state. We therefore conclude that

> *The constitutive law for a linear elastic, isotropic and incompressible material provides no information of the hydrostatic stress*

Since the hydrostatic stress is not determined by the constitutive relation, it is entirely determined by the static conditions of the boundary value problem in question.

Finally, a comparison of (4.119) with (4.85) and (4.86) shows that incompressibility occurs when the bulk modulus $K \to \infty$, i.e. $\nu \to 1/2$, and rubber provides an example of an incompressible material.