5 CAUCHY-ELASTICITY

We will now deal with a type of elasticity that is more general than hyper-elasticity. It is recalled that the essential property of elasticity is that there exists a one-to-one relation between strains and stresses, i.e.

$$\varepsilon_{ij} = \varepsilon_{ij}(\sigma_{kl}) \tag{5.1}$$

This expression is completely similar to (4.13), but now we do not assume that the strain energy or complementary energy is independent of the load history. As mentioned previously, the corresponding type of elasticity is called *Cauchy-elasticity* and we shall here derive the most general format of isotropic Cauchy-elasticity. Before that can be done, some introductory concepts have to be presented.

5.1 Response function, principle of coordinate invariance and isotropic tensor function

It turns out to be more instructive to write (5.1) as

$$\boxed{\varepsilon_{ij} = g_{ij}(\sigma_{kl})} \tag{5.2}$$

Evidently, $\varepsilon_{ij} = g_{ij}$, but the formulation (5.2) emphasizes that g_{ij} is a function that depends on the stress tensor σ_{kl}. The tensor function g_{ij} is called the *response function* and it determines the operations that are to be performed on the stress tensor σ_{kl} in order to determine the strain tensor ε_{ij}.

In another coordinate system x'_i, the constitutive law (5.2) states that

$$\varepsilon'_{ij} = g^*_{ij}(\sigma'_{kl}) \tag{5.3}$$

where g^* is the response function in the x'_i-coordinate system. In general, the operations that the response function g^*_{ij} performs on σ'_{kl} are different from the operations that the response function g_{ij} performs on σ_{kl}.

As an illustration, assume that the constitutive law (5.2) is linear, i.e.

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{5.4}$$

This is just Hooke's law where C_{ijkl} is the flexibility tensor, cf. (4.26). We may write (5.4) as

$$\varepsilon_{ij} = g_{ij}(\sigma_{kl}) \qquad \text{where} \qquad g_{ij}(\sigma_{kl}) = C_{ijkl}\sigma_{kl} \tag{5.5}$$

In another coordinate system, we have

$$\varepsilon'_{ij} = C'_{ijkl}\sigma'_{kl}$$

or

$$\varepsilon'_{ij} = g^*_{ij}(\sigma'_{kl}) \qquad \text{where} \qquad g^*_{ij}(\sigma'_{kl}) = C'_{ijkl}\sigma'_{kl} \tag{5.6}$$

Referring to the discussion in Sections 4.3 and 4.6, we have for anisotropic materials that $C'_{ijkl} \neq C_{ijkl}$. In fact, the two flexibility tensors are related through the transformation rule similar to (1.43), i.e.

$$C'_{ijkl} = A_{im}A_{jn}C_{mnpq}A_{kp}A_{lq} \tag{5.7}$$

and it is only for isotropic materials that $C'_{ijkl} = C_{ijkl}$, cf. (4.94). To further illustrate the concept of a response function, we observe that for anisotropic materials where $C'_{ijkl} \neq C_{ijkl}$ holds, the response function g^*_{ij} given by (5.6) operates on σ'_{kl} in a different manner than the response function g_{ij} given by (5.5) operates on σ_{kl}. It seems tempting to use the notation $\varepsilon'_{ij} = g'_{ij}(\sigma_{kl})$ in the x'_i-coordinate system whereas we prefer to write $\varepsilon'_{ij} = g^*_{ij}(\sigma_{kl})$. The reason is that the quantity g_{ij} is a tensor function, i.e. it defines the operations to be performed on the argument σ_{kl}. The notation g^*_{ij} therefore underlines that in the x'_i-coordinate system, the response function takes the form g^*_{ij}.

Consider a material point at which in the x_i-coordinate system we have the stress components σ_{ij} and the strain components ε_{ij}. In another x'_i-coordinate system, the same stress state and the same strain state are now given by the stress components σ'_{ij} and the strain components ε'_{ij}. Since σ_{ij} and ε_{ij} correspond to σ'_{ij} and ε'_{ij} respectively, and the only difference is that they are measured in different coordinate systems, we have according to (2.19) and (3.8) that

$$\varepsilon'_{ij} = A_{ik}\varepsilon_{kl}A_{jl}\ ; \qquad \sigma'_{kl} = A_{kp}\sigma_{pq}A_{lq}$$

Insertion of these expressions into (5.3) gives

$$A_{ik}\varepsilon_{kl}A_{jl} = g^*_{ij}(A_{kp}\sigma_{pq}A_{lq})$$

and use of (5.2) on the left-hand side results in

$$A_{ik}g_{kl}(\sigma_{mn})A_{jl} = g^*_{ij}(A_{kp}\sigma_{pq}A_{lq}) \qquad \text{coordinate invariance} \tag{5.8}$$

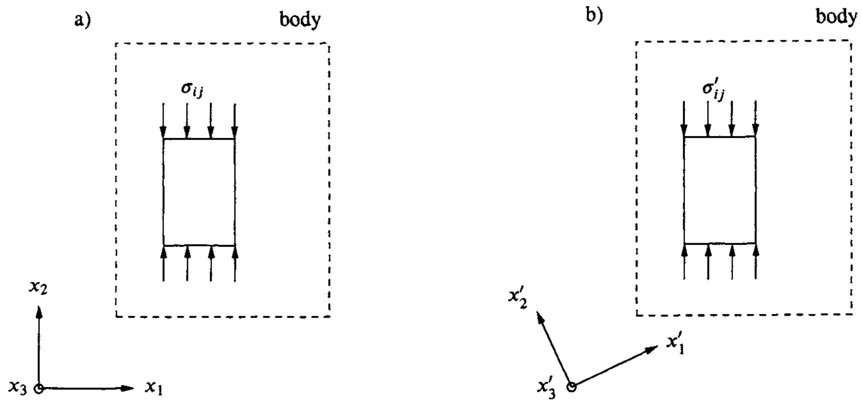

Figure 5.1: Body from which a homogeneously loaded piece of material is considered in the: a) x_i-coordinate system, b) x'_i-coordinate system. The components σ_{ij} and σ'_{ij} are different, but they refer to the same loading measured in different coordinate systems.

This important result is often referred to as being a result of the so-called *coordinate invariance* principle. Using matrix notation, (5.8) evidently reads

$$\boxed{\boldsymbol{A}\boldsymbol{g}(\boldsymbol{\sigma})\boldsymbol{A}^T = \boldsymbol{g}^*(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T) \qquad \textit{coordinate invariance}} \tag{5.9}$$

The principle of coordinate invariance can be stated as

Coordinate invariance principle: *The material response is independent* *of the coordinate system we choose*

(5.10)

cf. Truesdell and Toupin (1960).

The principle of coordinate invariance is trivially fulfilled when the constitutive law is written as a tensor relation and the principle of coordinate invariance as expressed by (5.9) follows trivially from tensor algebra. Indeed, as discussed in Chapter 1 the advantage of writing a constitutive relation in tensor format is that it ensures that if this relation holds in one coordinate system, it holds in all coordinate systems. However, the advantage of the format (5.9) is that it involves the, as yet, unknown response function and we shall see later that this format allow us to identify the response function for isotropic materials.

Leaving tensor algebra aside, it may be instructive to illustrate the principle of coordinate invariance in a simple fashion. For that purpose, we consider a piece of a material and assume that this piece is loaded in a homogeneous manner. In Fig. 5.1a) the loading is given by the stress components σ_{ij} measured in the x_i-coordinate system whereas in Fig. 5.1b) the loading is given by the stress components σ'_{ij} measured in the x'_i-coordinate system. We assume that the stress components σ_{ij} and σ'_{ij} refer to the same loading, but just measured in different coordinate systems. In Fig. 5.1a), the stresses σ_{ij} gives rise to the strains determined by ε_{ij} whereas in Fig. 5.1b), the stresses σ'_{ij} results in the strains given by ε'_{ij}. Since σ_{ij} and σ'_{ij} refer to the same loading, but measured in different coordinate systems, it is reasonable to assume that the strains ε_{ij} and ε'_{ij} must refer to the same deformation, but measured in different coordinate systems. This is the physical content of the principle of coordinate invariance.

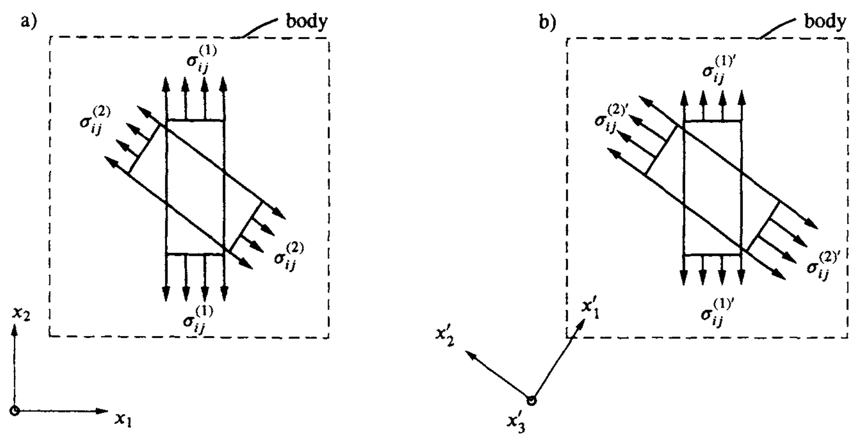

Figure 5.2: Two different stress states measured in the: a) x_i-system and b) x'_i-system.

Let us now try to illustrate the implications of material isotropy. For that purpose, consider Fig. 5.2. In Fig. 5.2a), two different stress states $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ are shown; these stress components are measured in the x_i-coordinate system. In Fig. 5.2b), the same two stress states are shown, but now these stress states are given by $\sigma_{ij}^{(1)'}$ and $\sigma_{ij}^{(2)'}$ measured in the x'_i-coordinate system. Therefore, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(1)'}$ refer to the same stress state, but merely measured in different coordinate systems. Likewise, $\sigma_{ij}^{(2)}$ and $\sigma_{ij}^{(2)'}$ refer to the same stress state, but merely measured in different coordinate systems. It follows that

$$\boldsymbol{\sigma}^{(1)'} = \boldsymbol{A}\boldsymbol{\sigma}^{(1)}\boldsymbol{A}^T \; ; \qquad \boldsymbol{\sigma}^{(2)'} = \boldsymbol{A}\boldsymbol{\sigma}^{(2)}\boldsymbol{A}^T$$

According to (5.2), we have

$$\varepsilon_{ij}^{(1)} = g_{ij}(\sigma_{ij}^{(1)}) \; ; \qquad \varepsilon_{ij}^{(2)'} = g_{ij}^*(\sigma_{ij}^{(2)'}) \tag{5.11}$$

It is always possible to choose the two stress states such that

$$\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)'}$$

i.e. the stress components $\sigma_{ij}^{(1)}$ are equal to the stress components $\sigma_{ij}^{(2)'}$; this situation is illustrated in Fig. 5.2. Insertion into (5.11) gives

$$\varepsilon_{ij}^{(1)} = g_{ij}(\sigma_{kl}^{(1)}) \; ; \qquad \varepsilon_{ij}^{(2)'} = g_{ij}^{*}(\sigma_{kl}^{(1)}) \tag{5.12}$$

We now assume that the material is isotropic. Since the material therefore has no directional properties and as $\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)'}$, we certainly expect that

$$\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(2)'}$$

and a comparison with (5.12) shows that $g_{ij}(\sigma_{kl}^{(1)}) = g_{ij}^{*}(\sigma_{kl}^{(1)})$ holds for isotropic materials, i.e. for isotropy, g_{ij} and g_{ij}^{*} express the same function. We have previously defined what is meant by anisotropy and isotropy, cf. (4.19) and (4.47). With the discussion above, we may express these definitions in the following more precise form

For isotropic materials, the response function is the same in all Cartesian coordinate systems, i.e. $g_{ij}(\sigma_{kl}) = g_{ij}^{*}(\sigma_{kl})$ *For anisotropic materials, the response function depends on the Cartesian coordinate system, i.e.* $g_{ij}(\sigma_{kl}) \neq g_{ij}^{*}(\sigma_{kl})$	(5.13)

In these comparisons of the response functions, it is important that the same argument enters in the functions g_{ij} and g_{ij}^{*}. Thus, isotropy could equally well be expressed as $g_{ij}(\sigma'_{kl}) = g_{ij}^{*}(\sigma'_{kl})$; the important point is that for isotropy, g_{ij} and g_{ij}^{*} define the same function.

To illustrate (5.13), we consider linear elasticity given by (5.5) and (5.6). For isotropic materials where $C_{ijkl} = C'_{ijkl}$ holds, the response function g_{ij} given by (5.5) operates on σ_{kl} in the same manner as the response function g_{ij}^{*} given by (5.6) operates on σ'_{kl}. Therefore, g_{ij} and g_{ij}^{*} define the same function and we have $g_{ij}(\sigma_{kl}) = g_{ij}^{*}(\sigma_{kl})$ in accordance with (5.13). On the other hand, for anisotropic materials where $C_{ijkl} \neq C'_{ijkl}$ holds, the response function g_{ij} given by (5.5) operates on σ_{kl} in a different manner than the response function g_{ij}^{*} given by (5.6) operates on σ'_{kl}. For anisotropic materials, g_{ij} and g_{ij}^{*} are therefore different functions in accordance with (5.13).

The response function g_{ij} refers to the x_i-coordinate system whereas the response function g_{ij}^{*} refers to the x'_i-coordinate system and for isotropic materials, these response functions define the same function. This may be expressed by saying that g_{ij} and g_{ij}^{*} possess the same form. In the literature, one therefore often uses the phrase that the response function for isotropic materials is *form-invariant* for arbitrary coordinate transformations, cf. Eringen (1975a) p.139. It is also referred to as the *principle of material invariance*, since the response function is invariant, i.e. has the same form, for arbitrary coordinate transformations, cf. Eringen (1975a).

With this discussion, let us now combine the coordinate invariance principle with the assumption of isotropic material behavior. Following (5.13), the assumption of isotropic material behavior implies that g_{ij} and g^*_{ij} expresses the same function, i.e.

$$g^*(A\sigma A^T) = g(A\sigma A^T)$$

Use of this expression in (5.9) provides

$$\boxed{Ag(\sigma)A^T = g(A\sigma A^T) \qquad \textit{coordinate invariance + isotropy}} \tag{5.14}$$

It is evident that an arbitrary response function g_{ij} cannot be expected to fulfill this relation; indeed, we will in a moment identify the most general response function that fulfills (5.14)

The response function g_{ij} is a tensor function and in case of isotropic materials, we have $g_{ij}(\sigma_{kl}) = g^*_{ij}(\sigma_{kl})$. In that case, we say that g_{ij} is an *isotropic tensor function* (of second-order).

An isotropic tensor function is not the same as an isotropic tensor. According to Section 1.6, the components of an isotropic tensor are the same in all Cartesian coordinate systems; examples are Kronecker's delta δ_{ij}, cf. (1.45), the isotropic stiffness tensor D_{ijkl} which obeys the transformation rule $D'_{ijkl} = D_{ijkl}$, cf. (4.90), and the isotropic flexibility tensor C_{ijkl} which obeys the transformation rule $C'_{ijkl} = C_{ijkl}$, cf. (4.94). On the other hand, for an isotropic tensor function $g_{ij}(\sigma_{kl})$ we have

$$\boxed{\begin{array}{c}\textit{The tensor function } g_{ij} \textit{ is isotropic if}\\ g_{ij}(\sigma_{kl}) = g^*_{ij}(\sigma_{kl})\end{array}} \tag{5.15}$$

The important point in (5.15) is that the same argument enters g_{ij} and g^*_{ij}. Since the argument is the same, we could equally well have defined an isotropic tensor function as $g_{ij}(\sigma'_{kl}) = g^*_{kl}(\sigma'_{kl})$; the essential issue is that g_{ij} and g^*_{ij} define the same function.

While a tensor is isotropic if its components are unchanged by a coordinate transformation, a tensor function is isotropic if it expresses the same function in different coordinate systems. For isotropic materials, the response functions are isotropic.

5.2 Most general isotropic Cauchy-elasticity

After these general remarks, we are now in a position to derive the most general form of isotropic Cauchy-elasticity.

Our starting point is (5.14), which follows from the coordinate invariance principle and the assumption of material isotropy, and the one-to-one relation between strains and stresses given by (5.2). It turns out that this information alone allows us to conclude that the most general form of isotropic Cauchy-elasticity is given by

$$\boxed{\boldsymbol{\varepsilon} = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{\sigma} + \alpha_3 \boldsymbol{\sigma}^2} \tag{5.16}$$

where the functions α_1,α_2 and α_3, in general, depend on the stress invariants, i.e.

$$\boxed{\alpha_1 = \alpha_1(I_1, I_2, I_3) \;\; ; \quad \alpha_2 = \alpha_2(I_1, I_2, I_3) \;\; ; \quad \alpha_3 = \alpha_3(I_1, I_2, I_3)} \tag{5.17}$$

The proof of this remarkable result is given in the next section.

It is of considerable interest, that (5.16) is exactly of the same form as that derived for hyper-elasticity, cf. (4.78). However, there is one significant difference, namely that whereas the coefficients α_1,α_2 and α_3 of (5.16) can take any form, the corresponding coefficients ψ_1,ψ_2 and ψ_3 of (4.78) are related through the constraints (4.79). We have already in Section 4.10 discussed that these constraints have significant consequences when modeling different nonlinear material behaviors and we therefore conclude that Cauchy-elasticity offers significant advantages as compared with hyper-elasticity. We will return to this topic in Section 5.4.

Since Cauchy-elasticity is a more general form of elasticity than hyper-elasticity, it may be of interest to investigate under what circumstances Cauchy-elasticity reduces to hyper-elasticity.

Let us first rewrite (5.16) as

$$\boxed{\varepsilon_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \sigma_{ij} + \alpha_3 \sigma_{ik}\sigma_{kj}} \tag{5.18}$$

Define the quantity dF by

$$dF = \varepsilon_{ij} d\sigma_{ij} \tag{5.19}$$

where ε_{ij} is given by (5.18). If we integrate up until the current stress state, F will depend not only on the current stress state, but also on the integration path, i.e. the load history. Let us investigate the conditions for which F is independent of the integration path. This requires that dF must be a perfect differential and thereby

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}} \tag{5.20}$$

cf. the discussion leading to (4.9). We observe that α_1,α_2 and α_3 depend on the stress invariants and we recall that

$$\frac{\partial I_1}{\partial \sigma_{kl}} = \delta_{kl} \; ; \qquad \frac{\partial I_2}{\partial \sigma_{kl}} = \sigma_{kl} \; ; \qquad \frac{\partial I_3}{\partial \sigma_{kl}} = \sigma_{km}\sigma_{ml}$$

A little algebra will then show that (5.20) is fulfilled if and only if

$$\frac{\partial \alpha_1}{\partial I_2} = \frac{\partial \alpha_2}{\partial I_1} \; ; \qquad \frac{\partial \alpha_1}{\partial I_3} = \frac{\partial \alpha_3}{\partial I_1} \; ; \qquad \frac{\partial \alpha_2}{\partial I_3} = \frac{\partial \alpha_3}{\partial I_2}$$

i.e. α_1,α_2 and α_3 must satisfy the same constraints as the functions ψ_1,ψ_2 and ψ_3 for a hyper-elastic material, cf. (4.79). When (5.20) is fulfilled, we immediately conclude that F as given by (5.19) becomes the complementary energy C, cf. (4.14) and (4.18) and we have then proved formally that Cauchy-elasticity contains hyper-elasticity as a special case.

Returning to (5.16), it is evident that this expression is in accordance with (2). Moreover, it is easy to see that it also fulfills (5.14). To show this, we write (5.16) as

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}) \qquad \text{where} \qquad \boldsymbol{g}(\boldsymbol{\sigma}) = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{\sigma} + \alpha_3 \boldsymbol{\sigma}\boldsymbol{\sigma} \tag{5.21}$$

Replacing $\boldsymbol{\sigma}$ by $\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$ and noting that α_1,α_2 and α_3 only depend on the stress invariants and therefore are unaffected by this operation, we obtain

$$\boldsymbol{g}(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T) = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T + \alpha_3 \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$$

Since $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$, it follows that

$$\boldsymbol{g}(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T) = \boldsymbol{A}(\alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{\sigma} + \alpha_3 \boldsymbol{\sigma}\boldsymbol{\sigma})\boldsymbol{A}^T$$

and use of (5.21) yields

$$\boldsymbol{g}(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T) = \boldsymbol{A}\boldsymbol{g}(\boldsymbol{\sigma})\boldsymbol{A}^T$$

in accordance with (5.14). We have then shown that (5.16) is an allowable form and in the next section we will prove that (5.16) is, in fact, the most general form.

Instead of the constitutive law (5.2), we may assume that

$$\sigma_{ij} = g_{ij}(\varepsilon_{kl})$$

In complete analogy with the discussion leading to (5.16), it follows that the most general isotropic Cauchy-elasticity may also be expressed as

$$\boxed{\boldsymbol{\sigma} = \beta_1 \boldsymbol{I} + \beta_2 \boldsymbol{\varepsilon} + \beta_3 \boldsymbol{\varepsilon}^2}$$

where β_1,β_2 and β_3 depend on the strain invariants; otherwise they are arbitrary.

5.3 Proof of most general form of isotropic Cauchy-elasticity

We claimed that (5.16) presents the most general form of isotropic Cauchy-elasticity. Moreover, we saw that it was easy to show that (5.16) fulfills the coordinate invariance principle and assumption of isotropy as given by (5.14). To prove that (5.16), in fact, is the most general form is much more complex and we will next present proof of this statement.

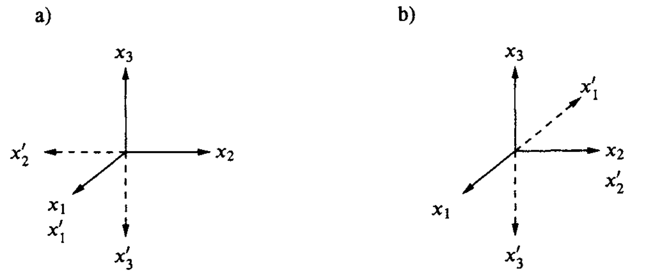

Figure 5.3: Different coordinate changes; a) 180°-rotation about the x_1-axis, b) 180°-rotation about the x_2-axis.

We will first prove that the assumption of isotropy implies that ε_{ij} and σ_{ij} have identical principal directions. Indeed this result is not surprising, since it says that an isotropic material loaded only by normal stresses will deform without appearance of shear strains. However, we will now provide formal proof of this property.

From (5.2), (5.3) and the assumption of isotropy (5.13) follow that

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}) \quad ; \qquad \boldsymbol{\varepsilon}' = \boldsymbol{g}(\boldsymbol{\sigma}') \tag{5.22}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ correspond to $\boldsymbol{\sigma}'$ and $\boldsymbol{\varepsilon}'$ respectively, but measured in different coordinate systems. Moreover, from (2.19) and (3.8) we have

$$\boldsymbol{\varepsilon}' = \boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T \quad ; \qquad \boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T \tag{5.23}$$

Now, choose the x_i-coordinate system collinear with the principal directions of $\boldsymbol{\sigma}$, i.e. $\boldsymbol{\sigma}$ becomes diagonal whereas nothing is known about $\boldsymbol{\varepsilon}$. We therefore have

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} ; \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{5.24}$$

Let us assume that the x_i'-coordinate system is obtained as a 180°- rotation about the x_1-axis, cf. Fig. 5.3a). Referring to (1.28), the transformation matrix is then given by

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \boldsymbol{A} \tag{5.25}$$

Insertion of (5.24) and (5.25) in (5.23) yields

$$\boldsymbol{\varepsilon}' = \begin{bmatrix} \varepsilon_{11} & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ -\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} ; \qquad \boldsymbol{\sigma}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{5.26}$$

Consequently, we have $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ and it then follows from (5.22) that we must have $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}$. A comparison of (5.24) and (5.26) then reveals that $\varepsilon_{12} = \varepsilon_{13} = 0$.

Assume next that the x_i'-coordinate system is obtained as a 180^o-rotation about the x_2-axis, cf. Fig. 5.3b). The transformation matrix is then given by

$$\boldsymbol{A}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \boldsymbol{A}$$

and using the same argumentation as above, it follows that $\varepsilon_{23} = 0$

Consequently, it has been shown that if $\boldsymbol{\sigma}$ is diagonal, so is $\boldsymbol{\varepsilon}$. I.e. the assumption of isotropy implies that the principal directions for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ coincide.

Therefore, if we choose the coordinate system collinearly with the principal directions of $\boldsymbol{\sigma}$, only principal strains are activated and (5.22a) implies

$$\varepsilon_1 = h_1(\sigma_1, \sigma_2, \sigma_3) \;\; ; \quad \varepsilon_2 = h_2(\sigma_1, \sigma_2, \sigma_3) \;\; ; \quad \varepsilon_3 = h_3(\sigma_1, \sigma_2, \sigma_3) \qquad (5.27)$$

where h_1,h_2 and h_3 are some functions. To summarize, we then have $\varepsilon_i = h_i(\sigma_k)$.

Consider now two different states: $\varepsilon_i^o = h_i(\sigma_k^o)$ which is assumed to be known and $\varepsilon_i = h_i(\sigma_k)$ which will be evaluated. Since h_i is a smooth function, a Taylor expansion of ε_i about the state σ_i^o gives

$$\begin{aligned} \varepsilon_i = h_i^o &+ \left(\frac{\partial h_i}{\partial \sigma_k}\right)^o (\sigma_k - \sigma_k^o) + \frac{1}{2}\left(\frac{\partial^2 h_i}{\partial \sigma_k \partial \sigma_l}\right)^o (\sigma_k - \sigma_k^o)(\sigma_l - \sigma_l^o) \\ &+ \frac{1}{6}\left(\frac{\partial^3 h_i}{\partial \sigma_k \partial \sigma_l \partial \sigma_p}\right)^o (\sigma_k - \sigma_k^o)(\sigma_l - \sigma_l^o)(\sigma_p - \sigma_p^o) + \ldots \end{aligned}$$

The superscript o is used to indicate that the quantities are evaluated at state σ_k^o. To simplify the following derivation let us now consider $i = 1$. Collecting terms, it then follows that the above expression can be written as

$$\begin{aligned} \varepsilon_1 =& [h_1^o + \left(\frac{\partial h_1}{\partial \sigma_\alpha}\right)^o (\sigma_\alpha - \sigma_\alpha^o) - \left(\frac{\partial h_1}{\partial \sigma_1}\right)^o \sigma_1^o + \ldots] \\ &+ [\left(\frac{\partial h_1}{\partial \sigma_1}\right)^o + \left(\frac{\partial^2 h_1}{\partial \sigma_\alpha \partial \sigma_1}\right)^o (\sigma_\alpha - \sigma_\alpha^o) + \ldots]\sigma_1 \\ &+ [\frac{1}{2}\left(\frac{\partial^2 h_1}{\partial \sigma_1 \partial \sigma_1}\right)^o + \frac{1}{2}\left(\frac{\partial^3 h_i}{\partial \sigma_\alpha \partial \sigma_1 \partial \sigma_1}\right)^o (\sigma_\alpha - \sigma_\alpha^o) + \ldots]\sigma_1^2 \\ &+ [\frac{1}{6}\left(\frac{\partial^3 h_i}{\partial \sigma_1 \partial \sigma_1}\right)^o + \frac{1}{6}\left(\frac{\partial^4 h_i}{\partial \sigma_\alpha \partial \sigma_1 \partial \sigma_1 \partial \sigma_1}\right)^o (\sigma_\alpha - \sigma_\alpha^o) + \ldots]\sigma_1^3 + \ldots \end{aligned}$$

where α takes the values 2 and 3. Introducing a more compact notation we find that

$$\varepsilon_1 = b_1 + b_2\sigma_1 + b_3\sigma_1^2 + b_4\sigma_1^3 + \cdots \qquad (5.28)$$

where b_1, b_2,... depend on σ_2 and σ_3.

The eigenvalue problem for the stress tensor is similar to the eigenvalue problem for the strain tensor. In complete conformity with (2.27), we therefore have for the principal stress σ_1 that

$$-\sigma_1^3 + \theta_1\sigma_1^2 - \theta_2\sigma_1 + \theta_3 = 0 \tag{5.29}$$

where θ_1,θ_2 and θ_3 are the Cauchy-stress invariants defined similar to (2.28). From (5.29) follows that

$$\sigma_1^{3+\alpha} = \theta_1\sigma_1^{2+\alpha} - \theta_2\sigma_1^{1+\alpha} + \theta_3\sigma_1^{\alpha} \tag{5.30}$$

where α is any non-negative integer. Repeated use of (5.30) in (5.28) means that all higher-order terms in (5.28) can be eliminated and we are then left with

$$\varepsilon_1 = p_1 + q_1\sigma_1 + r_1\sigma_1^2 \tag{5.31}$$

where p_1, q_1 and r_1 are some quantities that may depend on the stress invariants. Likewise, making a Taylor-expansion of the function h_2 and h_3 in (5.27) about σ_k^o, we obtain

$$\begin{aligned} \varepsilon_2 &= p_2 + q_2\sigma_2 + r_2\sigma_2^2 \\ \varepsilon_3 &= p_3 + q_3\sigma_3 + r_3\sigma_3^2 \end{aligned} \tag{5.32}$$

where p_2, q_2, r_2 as well as p_3, q_3 and r_3 are some quantities that may depend on the stress invariants.

From (5.31) and (5.32), we may write $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \boldsymbol{P} + \boldsymbol{Q}\boldsymbol{\sigma} + \boldsymbol{R}\boldsymbol{\sigma}^2 \tag{5.33}$$

where

$$\boldsymbol{P} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix};\ \boldsymbol{Q} = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix};\ \boldsymbol{R} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}$$

The only thing we know about $\boldsymbol{P}$,$\boldsymbol{Q}$ and $\boldsymbol{R}$ is that they are diagonal matrices which may depend on the stress invariants. We will now prove that they are, in fact, second-order tensors; indeed, it will turn out that they are isotropic second-order tensors

To show this, we may write (5.33) as

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}) \qquad \text{where} \qquad \boldsymbol{g}(\boldsymbol{\sigma}) = \boldsymbol{P} + \boldsymbol{Q}\boldsymbol{\sigma} + \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{\sigma} \tag{5.34}$$

The coordinate invariance principle and the assumption of isotropy led to (5.14), which may be written as

$$\boldsymbol{g}(\boldsymbol{\sigma}) = \boldsymbol{A}^T\boldsymbol{g}(\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T)\boldsymbol{A} \tag{5.35}$$

Since $\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$ and as $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ only depend on the stress invariants, these matrices take the same value irrespective of whether they are evaluated for the stress $\boldsymbol{\sigma}$ or for the stress $\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$. With $\boldsymbol{g}(\boldsymbol{\sigma})$ given by (5.34) we therefore obtain from (5.35)

$$\boldsymbol{P} + \boldsymbol{Q}\boldsymbol{\sigma} + \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{\sigma} = \boldsymbol{A}^T(\boldsymbol{P} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T + \boldsymbol{R}\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T)\boldsymbol{A}$$

Since $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, this can be written as

$$\overline{\boldsymbol{P}} + \overline{\boldsymbol{Q}}\boldsymbol{\sigma} + \overline{\boldsymbol{R}}\boldsymbol{\sigma}^2 = \boldsymbol{0} \tag{5.36}$$

where

$$\overline{\boldsymbol{P}} = \boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} \ ; \quad \overline{\boldsymbol{Q}} = \boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} \ ; \quad \overline{\boldsymbol{R}} = \boldsymbol{R} - \boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A} \tag{5.37}$$

The coordinate system was taken to be collinear with the principal stress directions implying that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^2$ are diagonal matrices in (5.36). For a given principal stress state σ_1,σ_2 and σ_3, the matrices $\boldsymbol{\sigma}$,$\boldsymbol{\sigma}^2$,$\boldsymbol{P}$,$\boldsymbol{Q}$ and $\boldsymbol{R}$ are then given and fixed quantities. However, we may in (5.37) choose the transformation matrix $\boldsymbol{A}$ in an arbitrary fashion. To fulfill (5.36) under these circumstances, we are led to the requirement

$$\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} \ ; \quad \boldsymbol{Q} = \boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} \ ; \quad \boldsymbol{R} = \boldsymbol{A}^T\boldsymbol{R}\boldsymbol{A}$$

We conclude that not only are $\boldsymbol{P}$,$\boldsymbol{Q}$ and $\boldsymbol{R}$ second-order tensors, but they are also isotropic second-order tensors, cf. (1.44). According to (1.45), the only isotropic second-order tensor is $k\delta_{ij}$ where the factor k may depend on some invariants. We are then led to

$$\boldsymbol{P} = \alpha_1\boldsymbol{I} \ ; \quad \boldsymbol{Q} = \alpha_2\boldsymbol{I} \ ; \quad \boldsymbol{R} = \alpha_3\boldsymbol{I}$$

where α_1,α_2 and α_3 may depend on the stress invariants. With these results, (5.33) takes the form

$$\boldsymbol{\varepsilon} = \alpha_1\boldsymbol{I} + \alpha_2\boldsymbol{\sigma} + \alpha_3\boldsymbol{\sigma}^2 \tag{5.38}$$

This result was derived on condition that the coordinate system is collinear with the principal stress directions. However, as $\boldsymbol{\varepsilon}$, $\boldsymbol{I}$ and $\boldsymbol{\sigma}$ are second-order tensors, (5.38) holds for arbitrary coordinate systems and we have then retrieved the format already stipulated in (5.16).

The result (5.38) derived from (5.2), the coordinate invariance principle and the assumption of isotropy is an example of a so-called *representation theorem* and we shall return to this topic in the next chapter; the result (5.38) was first derived by Prager (1945) (for incompressible materials) and Reiner (1945).

5.4 Nonlinear isotropic Hooke formulation

Let us return to the general format given by (5.16) i.e.

$$\varepsilon_{ij} = \alpha_1\delta_{ij} + \alpha_2\sigma_{ij} + \alpha_3\sigma_{ik}\sigma_{kj} \tag{5.39}$$

where α_1,α_2 and α_3 may depend on the stress invariants. Instead of the stress invariants indicated in (5.17), we may just as well choose

$$\alpha_1 = \alpha_1(I_1, J_2, J_3) \;;\quad \alpha_2 = \alpha_2(I_1, J_2, J_3) \;;\quad \alpha_3 = \alpha_3(I_1, J_2, J_3)$$

Similar to Section 4.10, our objective is to derive a nonlinear Hooke formulation, which implies that the quadratic term $\sigma_{ik}\sigma_{kj}$ must disappear. We therefore choose

$$\alpha_3 = 0$$

With this result, (5.39) may be written

$$\varepsilon_{kk} = 3\alpha_1 + \alpha_2\sigma_{kk}; \qquad e_{ij} = \alpha_2 s_{ij} \tag{5.40}$$

Let us choose

$$\alpha_1 = (\frac{1}{9K} - \frac{1}{6G})\sigma_{kk}; \qquad \alpha_2 = \frac{1}{2G}$$

then (5.40) becomes

$$\boxed{\sigma_{kk} = 3K\varepsilon_{kk}; \qquad s_{ij} = 2Ge_{ij}} \tag{5.41}$$

and it is evident that it is allowable to let K and G be arbitrary functions of the stress invariants, i.e.

$$\boxed{K = K(I_1, J_2, J_3); \qquad G = G(I_1, J_2, J_3)} \tag{5.42}$$

A comparison of (5.41) with (4.85) and (4.86) shows that K and G are the bulk and shear modulus respectively.

It may be of interest to compare (5.41) and (5.42) with the corresponding nonlinear Hooke formulation for hyper-elasticity. In the latter case, this formulation is given by (4.101)-(4.103). It is of considerable importance that the Cauchy-formulation is free from the constraint given by (4.103). Not only that, but K and G are now allowed also to depend on the third stress invariant J_3 and as we will see in Chapter 8 this is of major importance when modeling the behavior of materials like concrete, soil and rocks.

Indeed, (5.41) and (5.42) have been successfully applied to model such materials and, as examples, we may refer to Ottosen (1979) and Kotsovos (1980) for concrete and to DiMaggio and Sandler (1971) for soil (using hyper-elasticity); moreover, Chen and Saleeb (1982) present a comprehensive review

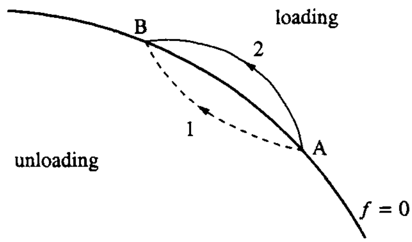

Figure 5.4: Stress space with surface $f = 0$ that divides the stress space into a space where unloading occurs and a space where loading occurs.

of such concrete and soil models. Recalling relation (4.91), we may write (5.41) in the alternative manner

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} \tag{5.43}$$

where Young's modulus E and Poisson's ratio ν now may depend on all the stress invariants, i.e.

$$E = E(I_1, J_2, J_3) \quad ; \qquad \nu = \nu(I_1, J_2, J_3)$$

Whereas a nonlinear Hooke formulation may provide close predictions to a variety of materials during loading, the major drawback is that unloading follows the same path as loading. This is certainly not a realistic prediction and various attempts have therefore been proposed in order to improve the response during unloading. However, as shown below there is only one manner in which the predictions during unloading may be improved in a consistent manner, Ottosen (1980)

For this purpose, assume that we introduce a criterion that tells us whether loading or unloading occurs. In Fig. 5.4 that illustrates the stress space, this criterion is expressed by the surface $f = 0$. We shall assume that Hooke's law also applies during unloading and that Young's modulus and Poisson's ratio are fixed material parameters during unloading.

Following Handelman *et al.* (1947), the essential property that we want to satisfy is that an infinitely small change of the applied loading only will result in an infinitely small change of the response. This requirement may be called the *continuity requirement.*

Consider path 1 in the unloading space of Fig. 5.4; the elastic parameters are fixed during unloading. Let path 1 approach the surface $f = 0$ so that path 1 in the limit follows the surface $f = 0$. For continuity reasons, we conclude that the elastic parameters are unchanged also when the path follows the surface $f = 0$.

Consider again path 1 in the unloading space and let E_u and ν_u be Young's modulus and Poisson's ratio respectively, during unloading. When moving from Point A to B along path 1, cf. Fig. 5.4, and recalling that E_u and ν_u are fixed quantities during unloading, we have according to (5.43) that the strain change during unloading $\Delta\varepsilon_{ij}^u$ is given by

$$\Delta\varepsilon_{ij}^u = -\frac{\nu_u}{E_u}\Delta\sigma_{kk}\delta_{ij} + \frac{1+\nu_u}{E_u}\Delta\sigma_{ij} \tag{5.44}$$

Let E_l and ν_l be Young's modulus and Poisson's ratio respectively, during loading. Moreover, let path 2 in the loading space, cf. Fig. 5.4, be infinitely close to the surface $f = 0$. Since the elastic parameters are unchanged when moving along $f = 0$, also E_l and ν_l are unchanged for this type of loading. According to (5.43), when moving from point A to B along path 2, the strain change during loading $\Delta\varepsilon_{ij}^l$ is then given by

$$\Delta\varepsilon_{ij}^l = -\frac{\nu_l}{E_l}\Delta\sigma_{kk}\delta_{ij} + \frac{1+\nu_l}{E_l}\Delta\sigma_{ij} \tag{5.45}$$

We have assumed that both path 1 and 2 are infinitely close to the surface $f = 0$. For continuity reasons, we must require that $\Delta\varepsilon_{ij}^u = \Delta\varepsilon_{ij}^l$, i.e. (5.44) and (5.45) provide

$$(-\frac{\nu_u}{E_u} + \frac{\nu_l}{E_l})\Delta\sigma_{kk}\delta_{ij} + (\frac{1+\nu_u}{E_u} - \frac{1+\nu_l}{E_l})\Delta\sigma_{ij} = 0$$

and this relation can only be satisfied if

$$\boxed{E_u = E_l \quad ; \qquad \nu_u = \nu_l} \tag{5.46}$$

That is, if unloading occurs, E_u and ν_u take those values of E_l and ν_l respectively, that were relevant immediately before unloading. We recall that during unloading, E_u and ν_u are fixed quantities.

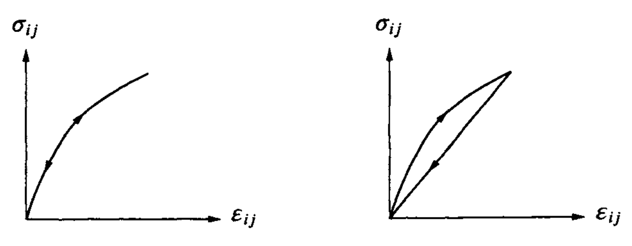

Figure 5.5: a) Nonlinear elastic response; b) elastic-fracturing response, secant approach.

If no loading/unloading criterion is introduced, our nonlinear elastic model of (5.43) predicts the same response in unloading as in loading, cf. Fig. 5.5a).

However, if a loading/unloading criterion is introduced, the only manner in which a consistent approach can be obtained that fulfills the continuity requirement, is to adopt the procedure indicated in (5.46) and illustrated in Fig. 5.5b). The material response shown in Fig. 5.5b) is often referred to as an *elastic fracturing material* after Dougill (1976). The response shown in this figure may also be termed as a *secant approach* since the linear unloading response is given by the secant values of E and ν just before unloading.

Whereas the unloading response in Fig. 5.5b) is much more realistic than that in Fig. 5.5a), it is still a crude approximation to the real unloading response of most materials. We shall later see that the plasticity theory makes for realistic predictions not only of the loading response, but also of the unloading response.