3 STRESS TENSOR

Having obtained a description of the deformation of the body, the next important point is to be able to describe the loading on the body at an arbitrary point. It turns out that this description is provided by the so-called *stress tensor*. Just like the strain tensor, the stress tensor turns out to be a symmetric second-order tensor and we will therefore take advantage of a number of properties that were proven in Chapter 2 for the strain tensor. For further studies of the stress tensor, the reader may consult, for instance, Fung (1965), Malvern (1969), Sokolnikoff (1946) or Spencer (1980).

3.1 Introduction

The body is supposed to be continuous and two kinds of forces are assumed: *body forces* (i.e. force per unit volume) and *surface forces* (i.e. force per unit area).

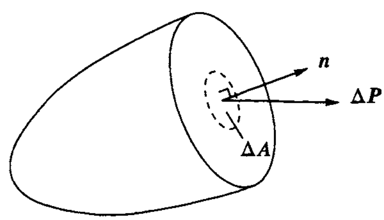

Figure 3.1: Force $\Delta \boldsymbol{P}$ on area ΔA with outer unit normal vector $\boldsymbol{n}$.

Consider a surface of the body as shown in Fig. 3.1. This surface can be an external surface or an internal surface obtained by a section of the body. The vector $\boldsymbol{n}$ is a unit vector normal to the surface and directed out of the body. The incremental force vector $\Delta \boldsymbol{P}$ acts on the incremental surface area ΔA. When ΔA approaches zero, it is assumed that the ratio $\Delta \boldsymbol{P}/\Delta A$ approaches a value $\boldsymbol{t}$, i.e.

$$\boxed{t = \left(\frac{\Delta \boldsymbol{P}}{\Delta A}\right)_{\Delta A \to 0} \quad ; \qquad t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}}$$

The vector $\boldsymbol{t}$, with components t_1, t_2 and t_3 in the x_1-, x_2- and x_3-directions respectively, is termed the *traction vector* and has the unit [N/m^2].

In principle, the surface forces around a point may also give rise to a moment about that point even when the ΔA-area approaches zero. Likewise, the body forces might result in a moment about a point even when the volume approaches zero. In these situations, so-called *couple stresses* will be present, but in classical continuum mechanics, these possible couple stresses are ignored; consideration of couple stresses leads to the so-called *polar continuum mechanics*, cf. Fung (1965), Malvern (1969), and Jaunzemis (1967) for introductory remarks and Cosserat and Cosserat (1909), Mindlin and Tiersten (1962), Mindlin (1964), Koiter (1964) and Eringen (1999) for more details.

The traction vector $\boldsymbol{t}$ defined above is related to a surface with the outer unit normal vector $\boldsymbol{n}$. It is obvious that the traction vector will, in general, be different when other sections through the same point are considered. What we are looking for is a quantity - the *stress tensor* - which for a particular point contains all the information necessary to determine the traction vector for arbitrary sections through that point.

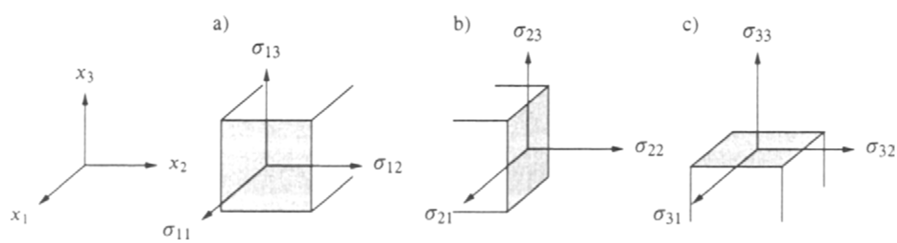

Figure 3.2: Illustration of stress components.

Let us first consider some special traction vectors, namely those obtained when sections perpendicular to the coordinate axes are considered. Assume that the outer normal vector $\boldsymbol{n}$ (see Fig. 3.1) is taken in the direction of the x_1-axis. The corresponding traction vector is denoted by $\boldsymbol{t}_1$ and we can resolve this vector into its components along the coordinate axes, i.e.

$$\boldsymbol{t}_1^T = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \end{bmatrix} \tag{3.1}$$

where σ_{11}, σ_{12} and σ_{13} denote the components of $\boldsymbol{t}_1$ in the x_1-, x_2- and x_3-directions respectively. These components are illustrated in Fig. 3.2a.

Likewise, if the outer normal unit vector $\boldsymbol{n}$ is taken in the direction of the x_2-axis, we denote the corresponding traction vector by $\boldsymbol{t}_2$, i.e.

$$\boldsymbol{t}_2^T = \begin{bmatrix} \sigma_{21} & \sigma_{22} & \sigma_{23} \end{bmatrix} \tag{3.2}$$

where σ_{21}, σ_{22} and σ_{23} denote the components of $\boldsymbol{t}_2$ in the x_1-, x_2- and x_3-directions respectively, cf. Fig. 3.2b. Finally, if the outer normal unit vector $\boldsymbol{n}$ is taken in the direction of the x_3-axis, we denote the corresponding traction vector by $\boldsymbol{t}_3$, i.e.

$$\boldsymbol{t}_3^T = \begin{bmatrix} \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.3}$$

where σ_{31}, σ_{32} and σ_{33} denote the components of $\boldsymbol{t}_3$ in the x_1-, x_2- and x_3-directions respectively, cf. Fig. 3.2c.

The components given by (3.1)-(3.3) are termed the *stress components* and σ_{11}, σ_{22}, σ_{33} are called *normal stresses*, whereas σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31} and σ_{32} are referred to as *shear stresses*. We observe the consistent notation of the stress components where, for instance, σ_{23} is the x_3-component of the traction vector for a surface with the outer unit vector in the x_2-direction. Likewise, σ_{12} is the x_2-component of the traction vector for a surface with the outer unit vector in the x_1-direction.

Using the special traction vectors considered above, we define the quantity σ_{ij} by

$$\boxed{[\sigma_{ij}] = \begin{bmatrix} \boldsymbol{t}_1^T \\ \boldsymbol{t}_2^T \\ \boldsymbol{t}_3^T \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}} \tag{3.4}$$

We shall later prove that σ_{ij} is a second-order tensor and σ_{ij} is therefore called the *stress tensor*.

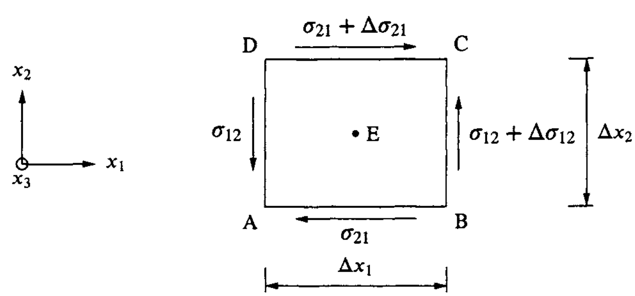

Figure 3.3: Moment about an axis through the center E and parallel to the x_3-axis.

Let us first prove that σ_{ij} is symmetric. From the body we cut a small parallelepiped with planes parallel to the coordinate planes. We then consider the moment equilibrium about an axis through the center E of this parallelepiped and parallel to the x_3-axis, cf. Fig. 3.3. It appears that body forces do not provide a moment about this axis. It is also obvious that only forces acting on planes parallel to the moment axis can contribute to the moment equilibrium. On these planes, only shear stresses normal to the moment axis can give rise to the moments; see Fig. 3.3.

Referring to this figure, the positive direction of the shear stresses along BC and DC is in accordance with the previous interpretation of the stress components, cf. Fig. 3.2. The positive direction of the shear stresses along AB and AD follows from the *law of action and reaction*. Taking moments as positive in the counter-clockwise direction, moment equilibrium about point E yields

$$(\sigma_{12} + \Delta\sigma_{12})\Delta x_2 \Delta x_3 \frac{1}{2}\Delta x_1 - (\sigma_{21} + \Delta\sigma_{21})\Delta x_1 \Delta x_3 \frac{1}{2}\Delta x_2$$
$$+ \sigma_{12}\Delta x_2 \Delta x_3 \frac{1}{2}\Delta x_1 - \sigma_{21}\Delta x_1 \Delta x_3 \frac{1}{2}\Delta x_2 = 0$$

i.e.

$$2\sigma_{12} - 2\sigma_{21} + \Delta\sigma_{12} - \Delta\sigma_{21} = 0$$

Letting Δx_1, Δx_2 and Δx_3 approach zero, both $\Delta\sigma_{12}$ and $\Delta\sigma_{21}$ also approach zero; that is, moment equilibrium requires that $\sigma_{12} = \sigma_{21}$. Likewise, considering moment equilibrium about axes parallel to the x_1- and x_2-axes implies that $\sigma_{23} = \sigma_{32}$ and $\sigma_{13} = \sigma_{31}$ respectively. In conclusion, we have proved that σ_{ij} is symmetric, i.e.

$$\boxed{\sigma_{ij} = \sigma_{ji} \qquad or \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T}$$

Our aim was to establish a quantity which contains all the information necessary to determine the traction vector $\boldsymbol{t}$ for arbitrary sections through the point in question. We shall now prove that the stress tensor σ_{ij} contains this information.

Consider the small tetrahedron shown in Fig. 3.4a). At the surface ABC with the outer unit normal vector $\boldsymbol{n}$, we have the traction vector $\boldsymbol{t}$. On the planes parallel to the coordinate planes, the traction vectors are $-\boldsymbol{t}_1$, $-\boldsymbol{t}_2$ and $-\boldsymbol{t}_3$, cf. (3.1)-(3.3); (minus signs appear because of the law of action and reaction and because the outer normal vectors are in the negative direction of the coordinate axes). The area ABC is denoted by ΔA, the area AOC by ΔA_1, the area AOB by ΔA_2 and the area BOC by ΔA_3. In Fig. 3.4b) the line CP is orthogonal to the line AB. As $\boldsymbol{n}$ is perpendicular to the surface ABC, it is also perpendicular to the lines CP and AB. The vector $\boldsymbol{n}$ is therefore located in the plane OCP. The components of the unit vector n_i are given by $n_i = (n_1, n_2, n_3)$ and by definition we have $n_2 = \cos\theta$ where θ is the angle shown in Fig. 3.4b). From Fig. 3.4b) follows that

$$\Delta A_2 = \frac{1}{2}|OP| \cdot |AB| \,; \qquad |OP| = |CP|\cos\theta = |CP|n_2 \qquad \text{i.e.}$$
$$\Delta A_2 = \frac{1}{2}|CP| \cdot |AB| n_2 = n_2 \Delta A$$

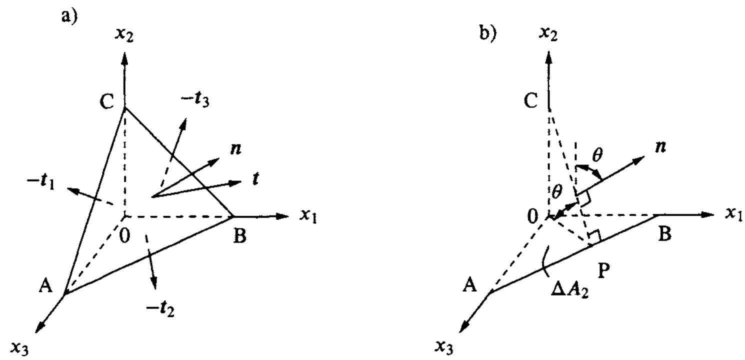

Figure 3.4: a) Traction vectors on a tetrahedron: $\boldsymbol{t}$ acts on ABC, -$\boldsymbol{t}_1$ on AOC, -$\boldsymbol{t}_2$ on AOB and -$\boldsymbol{t}_3$ on BOC; b) determination of ΔA by geometrical arguments. Vector $\boldsymbol{n}$ is located in the plane OCP.

By analogous arguments we find that

$$\Delta A_1 = n_1 \Delta A \; ; \qquad \Delta A_2 = n_2 \Delta A \; ; \qquad \Delta A_3 = n_3 \Delta A \tag{3.5}$$

The condition of force equilibrium of the tetrahedron of Fig. 3.4a) requires that

$$\boldsymbol{t}\Delta A - \boldsymbol{t}_1 \Delta A_1 - \boldsymbol{t}_2 \Delta A_2 - \boldsymbol{t}_3 \Delta A_3 + \boldsymbol{b}\Delta V = \boldsymbol{0} \tag{3.6}$$

where $\boldsymbol{b}$ is the body force per unit volume and ΔV is the volume of the small tetrahedron. The body force $\boldsymbol{b}$ has the components

$$\boxed{\boldsymbol{b}^T = [b_1 \;\; b_2 \;\; b_3]}$$

Use of (3.5) in (3.6) gives

$$\boldsymbol{t} - \boldsymbol{t}_1 n_1 - \boldsymbol{t}_2 n_2 - \boldsymbol{t}_3 n_3 + \boldsymbol{b}\frac{\Delta V}{\Delta A} = \boldsymbol{0}$$

Letting the size of the tetrahedron shrink towards zero, we have $\Delta V/\Delta A \to 0$ (volume has the unit m^3 and area has the unit m^2) and we then obtain

$$\boldsymbol{t} = \boldsymbol{t}_1 n_1 + \boldsymbol{t}_2 n_2 + \boldsymbol{t}_3 n_3$$

which may be written as

$$\boldsymbol{t} = [\boldsymbol{t}_1 \;\; \boldsymbol{t}_2 \;\; \boldsymbol{t}_3] \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \boldsymbol{\sigma}^T \boldsymbol{n}$$

where (3.4) was used. Due to the symmetry of $\boldsymbol{\sigma}$ we arrive at

$$\boxed{t_i = \sigma_{ij} n_j \qquad or \qquad \boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}} \tag{3.7}$$

This expression proves that knowledge of the stress tensor $\boldsymbol{\sigma}$ provides sufficient information for the traction vector $\boldsymbol{t}$ to be derived for any direction $\boldsymbol{n}$. It should be observed that on the exterior surface of the body, (3.7) represents a *boundary condition* expressing a relation between the forces acting on the external surface and the stress tensor. Equation (3.7) was derived by Cauchy in 1822 and it is therefore occasionally referred to as *Cauchy's formula*; the stress tensor is called the *Cauchy stress tensor*. When considering large deformations, it turns out that a number of different stress tensors exist, but for small strains and rotations they all reduce to the Cauchy stress tensor.

Moreover, since t_i and n_i are first-order tensors (vectors), it follows from the quotient theorem (cf. the argument that led to (1.41)) that σ_{ij} - indeed - is a second-order tensor.

3.2 Change of coordinate system

If we instead of the x_i-coordinate system change to a x'_i-coordinate system, we have as usual that

$$x'_i = A_{ij}(x_j - c_j) \qquad \text{or} \qquad \boldsymbol{x}' = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{c})$$

where A_{ij} is the transformation matrix and where $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, cf. (1.17) and (1.22).

Since σ_{ij} is known to be a second-order tensor, we can directly from (1.41) and (1.42) write the following relations between the components σ_{ij} in the x_i-system and the components σ'_{ij} in the x'_i-system.

$$\boxed{\sigma'_{ij} = A_{ik}\sigma_{kl}A_{jl} \qquad or \qquad \boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T} \tag{3.8}$$

and

$$\sigma_{ij} = A_{ki}\sigma'_{kl}A_{lj} \qquad \text{or} \qquad \boldsymbol{\sigma} = \boldsymbol{A}^T\boldsymbol{\sigma}'\boldsymbol{A} \tag{3.9}$$

3.3 Principal stresses and principal directions - Invariants

The traction vector $\boldsymbol{t}$ on a surface with the outer normal unit vector $\boldsymbol{n}$ is given by (3.7). The traction vector $\boldsymbol{t}$ can be resolved into a component parallel to $\boldsymbol{n}$ and a component perpendicular to $\boldsymbol{n}$. The component parallel to $\boldsymbol{n}$ is called the normal stress in direction $\boldsymbol{n}$ and denoted by σ_n. From (3.7) we obtain

$$\boxed{\sigma_n = n_i t_i = n_i\sigma_{ij}n_j \qquad or \qquad \sigma_n = \boldsymbol{n}^T\boldsymbol{t} = \boldsymbol{n}^T\boldsymbol{\sigma}\boldsymbol{n}} \tag{3.10}$$

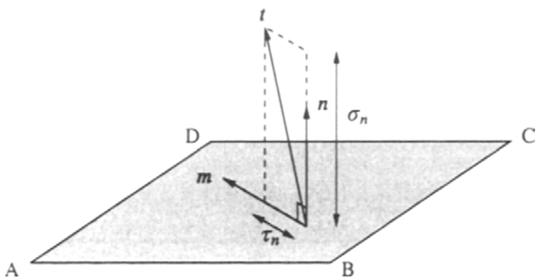

Figure 3.5: Illustration of normal stress σ_n and shear stress τ_n.

The component of $\boldsymbol{t}$ perpendicular to $\boldsymbol{n}$ is called the shear stress and is denoted by τ_n. Both σ_n and τ_n are illustrated in Fig. 3.5, where the unit vector $\boldsymbol{m}$ is perpendicular to $\boldsymbol{n}$ and located in the plane ABCD. It readily appears that

$$\boxed{\tau_n = m_i t_i = m_i \sigma_{ij} n_j \qquad or \qquad \tau_n = \boldsymbol{m}^T \boldsymbol{t} = \boldsymbol{m}^T \boldsymbol{\sigma} \boldsymbol{n}} \tag{3.11}$$

Alternatively, we may write

$$\tau_n^2 = t_i t_i - \sigma_n^2$$

With these preliminary results, we may obtain a physical interpretation of the important eigenvalue problem of the stress tensor and with the solution of the eigenvalue problem, we arrive at the stress invariants. Moreover, it turns out that for a special choice of coordinate system, the stress tensor takes a particularly simple form.

Returning to Fig. 3.5, we look for a situation where the traction vector $\boldsymbol{t}$ is collinear with the unit vector $\boldsymbol{n}$. From Fig. 3.5, the direction $\boldsymbol{n}$ should be chosen so that

$$t_i = \lambda n_i \tag{3.12}$$

where λ is some factor and (3.10) implies that $\lambda = \sigma_n$. Since n_i and m_i are orthogonal, (3.11) gives in the present situation that the shear stress $\tau_n = 0$. Insertion of (3.7) into (3.12) yields

$$\boxed{(\sigma_{ij} - \lambda \delta_{ij}) n_j = 0 \qquad or \qquad (\boldsymbol{\sigma} - \lambda \boldsymbol{I}) \boldsymbol{n} = \boldsymbol{0}} \tag{3.13}$$

This constitutes an eigenvalue problem and a comparison with (2.24) shows a complete analogy. Therefore all the conclusions that were derived for the strain tensor apply also for the stress tensor. That is, the characteristic equation

$$\det(\boldsymbol{\sigma} - \lambda \boldsymbol{I}) = 0$$

determines the three principal stresses $\sigma_1 = \lambda_1$, $\sigma_2 = \lambda_2$ and $\sigma_3 = \lambda_3$ and for each λ-value, (3.13) provides the corresponding principal direction $\boldsymbol{n}$. The principal stresses and directions are real; the principal stresses are invariants and the principal directions may always be taken to be orthogonal. If the coordinate system is taken collinear with the principal directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$, the stress tensor takes the following simple form

$$\boxed{\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad where \quad \boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{n}_1 \, \boldsymbol{n}_2 \, \boldsymbol{n}_3 \end{bmatrix}}$$

cf. (2.37)

Also the stress tensor satisfies the Cayley-Hamilton theorem. Moreover, the coefficients in the characteristic equation are the Cauchy-stress invariants, but of more importance are the following generic stress invariants

$$\boxed{I_1 = \sigma_{ii} \; ; \qquad I_2 = \tfrac{1}{2}\sigma_{ij}\sigma_{ji} \; ; \qquad I_3 = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}} \tag{3.14}$$

where the term 'generic' refers to the systematic definition of these invariants (we may refer to (2.51) for a comparison with the corresponding strain invariants).

3.4 Stress deviator tensor

Similarly to the exposition of the strain tensor, we define the *stress deviator tensor* by

$$\boxed{s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}} \tag{3.15}$$

where $\sigma_{kk}/3$ is called the *hydrostatic stress*. The σ_{ij}- and s_{ij}-tensors have identical off-diagonal elements and thus they have identical principal directions.

The generic invariants of the stress deviator tensor are given by

$$\boxed{J_1 = s_{ii} = 0 \; ; \qquad J_2 = \tfrac{1}{2}s_{ij}s_{ji} \; ; \qquad J_3 = \tfrac{1}{3}s_{ij}s_{jk}s_{ki}} \tag{3.16}$$

Similar to (2.53c), we have

$$J_3 = s_1 s_2 s_3 \tag{3.17}$$

Moreover, similar to (2.54) and (2.55) we find the following relations

$$J_2 = I_2 - \frac{1}{6}I_1^2 \; ; \qquad J_3 = I_3 - \frac{2}{3}I_1 I_2 + \frac{2}{27}I_1^3 \tag{3.18}$$

and

$$I_2 = J_2 + \frac{1}{6}I_1^2 \; ; \qquad I_3 = J_3 + \frac{2}{3}I_1 I_2 - \frac{2}{27}I_1^3 \tag{3.19}$$

Therefore, instead of using the set of invariants I_1, I_2 and I_3 we may equally well use the set I_1, J_2 and J_3.

Finally, and in analogy with (2.56), we have the *octahedral normal stress* σ_0 and *octahedral shear stress* τ_0 defined by

$$\boxed{\sigma_0 = \tfrac{1}{3}I_1 \; ; \qquad \tau_0 = \sqrt{\tfrac{2}{3}J_2}} \tag{3.20}$$

where σ_0 and τ_0 are the normal stress shear stress respectively, that act on an octahedral plane. Here, the normal to an octahedral plane makes equal angles to the principal stress directions; when comparing (3.20) with (2.56) note the difference between engineering shear strain and tensorial shear strain.

3.5 Change of coordinate system - Mohr's circle

Let us consider a coordinate system obtained by rotating the old coordinate system by an angle α in the counter-clockwise direction around the x_3-axis, cf. Fig. 3.6a).

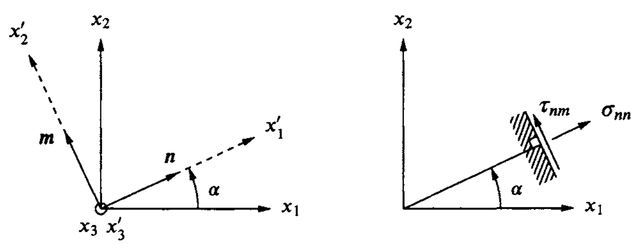

Figure 3.6: a) Rotation of coordinate system about the x_3-axis; b) normal stress σ_{nn} and shear stress τ_{nm} acting on section perpendicular to direction $\boldsymbol{n}$.

The directions of the x_1'- and x_2'-axes are given by the unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ respectively. For a section perpendicular to the $\boldsymbol{n}$-vector, the normal stress σ_{nn} acts in the direction of $\boldsymbol{n}$ and the shear stress τ_{nm} acts in the direction of $\boldsymbol{m}$, cf. Fig. 3.6b). According to (3.10) and (3.11) we have

$$\sigma_{nn} = n_i \sigma_{ij} n_j \; ; \qquad \tau_{nm} = m_i \sigma_{ij} n_j \tag{3.21}$$

We emphasize that σ_{nn} and τ_{nm} are measured positive in the direction of $\boldsymbol{n}$ and $\boldsymbol{m}$ respectively.

Comparing (3.21) with (2.11) and (2.17), we observe that the expressions for σ_{nn} and τ_{nm} are identical with the expressions for ε_{nn} and ε_{nm}. That is, all that has been derived for Mohr's circle of strain applies directly for Mohr's circle of

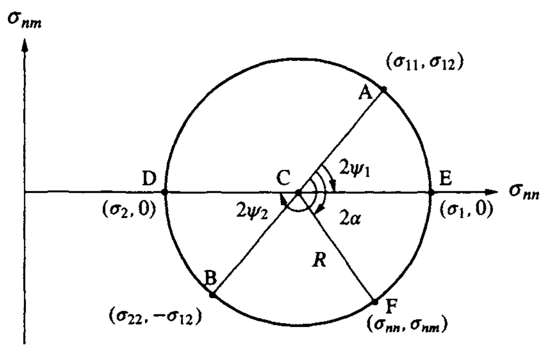

Figure 3.7: Mohr's circle of stress ($\sigma_1 \geq \sigma_2$).

stress, when ε_{ij} is replaced by σ_{ij}, ε_{nn} is replaced by σ_{nn} and ε_{nm} is replaced by τ_{nm}. In particular, we conclude from (2.57)-(2.59) that

$$\boxed{\begin{aligned} \sigma_{nn} &= \sigma_{11}\cos^2\alpha + \sigma_{22}\sin^2\alpha + \sigma_{12}\sin 2\alpha \\ \sigma_{mm} &= \sigma_{11}\sin^2\alpha + \sigma_{22}\cos^2\alpha - \sigma_{12}\sin 2\alpha \\ \tau_{nm} &= -\tfrac{1}{2}(\sigma_{11} - \sigma_{22})\sin 2\alpha + \sigma_{12}\cos 2\alpha \end{aligned}} \tag{3.22}$$

where σ_{mm} is the normal stress acting on a section perpendicular to the $\boldsymbol{m}$-direction. From (2.60) we obtain the inverse relations to (3.22), namely

$$\begin{aligned} \sigma_{11} &= \sigma_{nn}\cos^2\alpha + \sigma_{mm}\sin^2\alpha - \tau_{nm}\sin 2\alpha \\ \sigma_{22} &= \sigma_{nn}\sin^2\alpha + \sigma_{mm}\cos^2\alpha + \tau_{nm}\sin 2\alpha \\ \sigma_{12} &= \tfrac{1}{2}(\sigma_{nn} - \sigma_{mm})\sin 2\alpha + \tau_{nm}\cos 2\alpha \end{aligned} \tag{3.23}$$

In (3.22) and (3.23), it is sufficient to consider α in the range

$$\boxed{0 \leq \alpha < \pi}$$

From (3.22) appears that a particular α-value exists, $\alpha = \psi$, for which the shear stress τ_{nm} becomes zero. This angle ψ is determined by

$$\boxed{\tan 2\psi = \frac{2\tau_{nm}}{\sigma_{nn} - \sigma_{mm}}\,; \qquad 0 \leq \psi < \pi}$$

If the x_3'-direction, cf. Fig. 3.6, is a principal stress direction, it follows that the directions $\boldsymbol{n}$ and $\boldsymbol{m}$ defined by $\alpha = \psi$ are also principal directions. This situation will be assumed below.

Similar to (2.71), the principal stresses in the x_1x_2-plane are given by

$$\boxed{\left.\begin{matrix}\sigma_1 \\ \sigma_2\end{matrix}\right\} = \tfrac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \sqrt{[\tfrac{1}{2}(\sigma_{11} - \sigma_{22})]^2 + \sigma_{12}^2}\,; \qquad (\sigma_1 \geq \sigma_2)}$$

Moreover, *Mohr's circle of stress* (Mohr, 1882) takes the form shown in Fig. 3.7. Similar to the comments related to Fig. 2.11, we conclude that the value of the angle ψ_1, apparent from Fig. 3.7, is the angle α towards the principal direction having the largest principal stress.

Finally, from Fig. 3.7 we conclude that

$$\boxed{\tau_{nm,max} = \tfrac{1}{2}(\sigma_1 - \sigma_2)\ ; \qquad \tau_{nm,min} = -\tfrac{1}{2}(\sigma_1 - \sigma_2)} \tag{3.24}$$

where $\sigma_1 \geq \sigma_2$.

3.6 Special states of stress

Several special states of stress, which are often encountered in practice, will now be discussed.

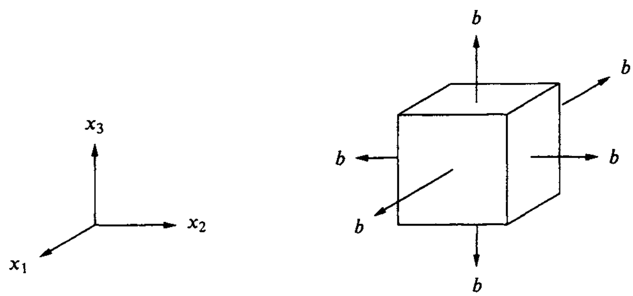

Figure 3.8: Hydrostatic stress state.

A state of *hydrostatic stress* exists, if the stress tensor is given by

$$\sigma_{ij} = b\,\delta_{ij}$$

where b is an arbitrary scalar. It appears that the deviatoric stress tensor s_{ij} is zero and that the loading consists of equal normal stresses having the amount b, cf. Fig. 3.8.

Uniaxial stress occurs if the stress tensor is given by

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Plane stress exists if the stress tensor is given by

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.25}$$

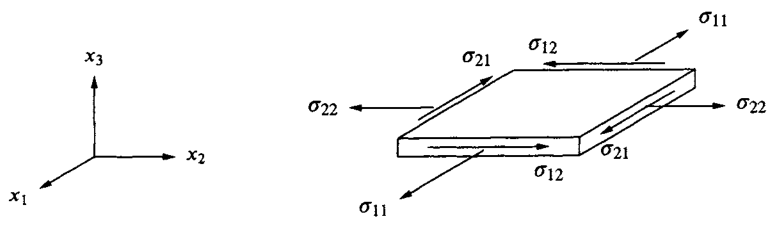

Figure 3.9: Plane stress.

and a disc loaded by *in-plane* stresses comprises an illustration of this type of loading, cf. Fig. 3.9.

Finally, a state of *pure shear* exists, if

$$[\sigma_{ij}] = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which holds for pure torsion of a cylindrical specimen. It is easily shown that the principal stresses become $\sigma_1 = \sigma_{12}$, $\sigma_2 = -\sigma_{12}$ and $\sigma_3 = 0$.

3.7 Equations of motion

We have previously used the equilibrium condition on an infinitesimal tetrahedron (Fig. 3.4) to obtain the connection between the traction vector and the stress tensor, (3.7), which can be considered as a static boundary condition. Let us now formulate the equations of motion for an arbitrary part of the body. The

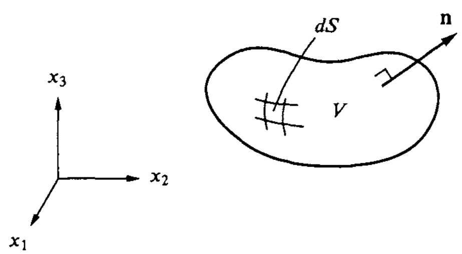

Figure 3.10: Volume V with surface boundary S and outer normal unit vector $\boldsymbol{n}$.

arbitrary part of the body has the volume V and the outer surface S with the outer normal unit vector $\boldsymbol{n}$, as shown in Fig. 3.10. The forces acting on this arbitrary body are given by the traction vector t_i along the boundary surface S

and the body force b_i per unit volume in the region V. The acceleration vector is denoted by $\ddot{u}_i$ where u_i is the displacement vector and a dot denotes the time derivative. *Newton's second law* states that

$$\int_S t_i dS + \int_V b_i dV = \int_V \rho \ddot{u}_i dV \tag{3.26}$$

where ρ is the mass density.

Before reformulating this equation, we recall the *divergence theorem of Gauss*, which states that for an arbitrary vector $\boldsymbol{q}$, the following relation holds

$$\int_V \operatorname{div}\boldsymbol{q}\, dV = \int_S \boldsymbol{q}^T \boldsymbol{n}\, dS$$

We have per definition that

$$\operatorname{div}\boldsymbol{q} = \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} = q_{i,i}\,; \qquad \boldsymbol{q}^T \boldsymbol{n} = q_i n_i$$

i.e. the divergence theorem can be written as

$$\boxed{\int_V q_{i,i} dV = \int_S q_i n_i dS} \tag{3.27}$$

If we choose q_i as the quantity c_{1i}, a relation analogous with (3.27) is obtained. Likewise, similar relations can be obtained by choosing q_i as c_{2i} and q_i as c_{3i}. Collecting all these results, we obtain for an arbitrary quantity c_{ij}, that (3.27) generalizes to

$$\boxed{\int_V c_{ij,j} dV = \int_S c_{ij} n_j dS} \tag{3.28}$$

With (3.7), we may then reformulate (3.26) by means of (3.28) to obtain

$$\int_V (\sigma_{ij,j} + b_i - \rho \ddot{u}_i) dV = 0$$

As this equation holds for arbitrary regions V of the body, we conclude that

$$\boxed{\sigma_{ij,j} + b_i = \rho \ddot{u}_i} \tag{3.29}$$

These equations express the *equations of motion* for the body and they are of course of fundamental importance in a number of applications; they were derived by Cauchy already in 1822.

3.8 Weak formulation - Principle of virtual work

From the equations of motion (3.29), i.e. the balance equations, we shall now derive one of the most powerful principles in mechanics namely the corresponding so-called *weak formulation.* The establishment of this weak form can be applied to any balance equation and within solid mechanics it is most often referred to as the celebrated *principle of virtual work.*

We multiply the equations of motion (3.29) by an arbitrary vector v_i - the *weight vector* - and integrate over the volume to obtain

$$\int_V v_i(\sigma_{ij,j} + b_i - \rho \ddot{u}_i)dV = 0$$

This equation may be written as

$$\int_V \left[(\sigma_{ij}v_i)_{,j} - \sigma_{ij}v_{i,j}\right] dV + \int_V (v_i b_i - \rho v_i \ddot{u}_i)dV = 0 \qquad (3.30)$$

From the divergence theorem (3.27) and (3.7) we have

$$\int_V (\sigma_{ij}v_i)_{,j}dV = \int_S \sigma_{ij}v_i n_j dS = \int_S v_i t_i dS$$

Use of this expression in (3.30) gives

$$\int_V \rho v_i \ddot{u}_i dV + \int_V v_{i,j}\sigma_{ij}dV = \int_S v_i t_i dS + \int_V v_i b_i dV \qquad (3.31)$$

This is the result aimed at, but we may occasionally use a slight reformulation. We recall that v_i is an arbitrary vector which, in general, has nothing to do with the displacement vector u_i. However, we may determine a quantity ε^v_{ij} defined by

$$\boxed{\varepsilon^v_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})} \qquad (3.32)$$

i.e. the tensor ε^v_{ij} is related to the arbitrary vector v_i in the same manner as the strain tensor ε_{ij} is related to the displacement vector u_i; therefore ε^v_{ij} is the 'strain' associated with v_i. Moreover, due to the symmetry of σ_{ij} we have

$$v_{i,j}\sigma_{ij} = \frac{1}{2}(v_{i,j}\sigma_{ij} + v_{j,i}\sigma_{ji}) = \frac{1}{2}(v_{i,j}\sigma_{ij} + v_{j,i}\sigma_{ij}) = \varepsilon^v_{ij}\sigma_{ij}$$

With this result, we may write (3.31) as

Weak form of equations of motion = principle of virtual work

$$\boxed{\int_V \rho v_i \ddot{u}_i dV + \int_V \varepsilon^v_{ij}\sigma_{ij}dV = \int_S v_i t_i dS + \int_V v_i b_i dV} \qquad (3.33)$$

This is the *weak form* of the equations of motion. However, we may think of the weight vector v_i as being some fictitious displacement - a so-called *virtual displacement*. In that case the expression on the right-hand side of (3.31) or (3.33) may be interpreted as the external work done during the 'virtual' displacement v_i suggesting the terminology of (3.31) and (3.33) being the *principle of virtual work*. We emphasize that v_i is an arbitrary vector that has nothing to do with the displacement vector u_i. Naturally, we may choose $v_i = u_i$, but in general this is not the case.

Moreover, we stress that (3.31) and (3.33) follow from the equations of motion alone and they hold therefore for any kind of material behavior. As the material response is described by certain relations between stresses and strains - so-called *constitutive relations* - we conclude that (3.31) and (3.33) hold for any kind of constitutive relation.

We have already used the terminology that (3.33) is the 'weak' form of the equations of motion. On the other hand, the differential equations of motion given by (3.29) is often called the *strong form* of the equations of motion. The reason for this terminology is whereas the strong form (3.29) contains derivatives of the stress tensor, the weak form does not and this is utilized in the numerical solution technique provided by the *finite element* formulation, cf. for instance Hughes (1987) and Ottosen and Petersson (1992). The weak form - i.e. the principle of virtual work - forms the basis not only for the finite element method, but also for other numerical solution techniques and it is also central for the establishment of a number of important theorems in solid mechanics. Thus, the weak form is one of the most important principles within solid mechanics.

In fact, the approach by which the weak form was derived above can be applied to arbitrary balance differential equations; therefore it provides a very powerful means for the establishment of various approximate solution techniques within different fields of physics.

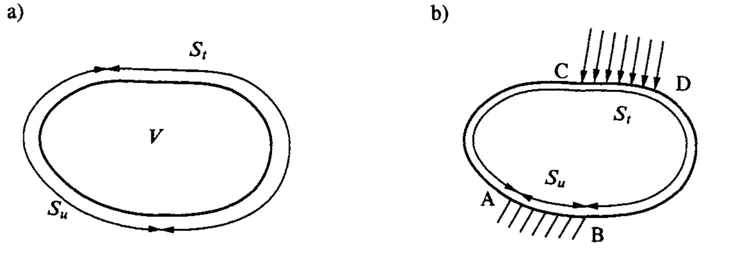

Figure 3.11: Boundary conditions; a) general situation, b) specific situation.

To solve the *field equations* within solid mechanics, i.e. the kinematic relations involving the strains and the displacements, the equations of motion and the constitutive relations relating stresses and strains, *boundary conditions* are required. They are given in the following form

$$\boxed{\begin{array}{ll} \boldsymbol{u} & \textit{is given along} \quad S_u \\ \boldsymbol{t} & \textit{is given along} \quad S_t \end{array}} \tag{3.34}$$

i.e. on the part S_u of the boundary, the displacements are known and prescribed whereas on the part S_t of the boundary, the traction vector is known and prescribed. The total boundary S consists of S_u and S_t. Figure 3.11a) illustrates the situation in general and in Fig. 3.11b) a specific example is shown. Here the support AB is the part S_u whereas ACDB is the part S_t (here the traction vector is zero along AC and DB and the traction vector is non-zero and known along CD).

The boundary condition where $\boldsymbol{u}$ is given along S_u is called a *kinematic boundary condition* and as the displacement vector $\boldsymbol{u}$ is the fundamental quantity to be determined, one also refers to this boundary condition as being an *essential boundary condition*. On the other hand, the boundary condition where $\boldsymbol{t}$ is given along S_t is called a *static boundary condition* and as the traction vector $\boldsymbol{t}$ emerges as a logical consequence in the boundary term in the weak form (3.33), it is also called a *natural boundary condition.*

We obtained the weak form of the equations of motion from their strong form (3.29), so the strong form implies the weak form. It is of significant interest, however, that if we accept the weak form and observe that the weight vector v_i is arbitrary, it then turns out that the weak form implies the strong form. The two forms are therefore equivalent expressions for the equations of motion, i.e.

$$\boxed{\begin{array}{c} \textit{Strong form of equations of motion} \\ \Longleftrightarrow \\ \textit{Weak form of equations of motion} \end{array}} \tag{3.35}$$

To prove this statement we assume that the weak form (3.33) holds. Choose the arbitrary weight vector v_i as an arbitrary constant vector; this implies that we choose the virtual displacements v_i as a virtual rigid-body translation. Since we then have $\varepsilon_{ij}^v = 0$ and as v_i does not depend on position, (3.33) takes the form

$$v_i[\int_V \rho \ddot{u}_i dV - \int_S t_i dS - \int_V b_i dV] = 0$$

and as this expression holds for arbitrary vectors v_i, it follows that

$$\int_V \rho \ddot{u}_i dV = \int_S t_i dS + \int_V b_i dV$$

But this expression is precisely Newton's second law, cf. (3.26), which leads to the strong form of the equations of motion given by (3.29). In the first place, we started out with the strong form and then derived the weak form. Now we have shown that if we start out with the weak form we can obtain the strong form. This proves the correctness of statement (3.35). For other implications of the weak form, i.e. the principle of virtual work, the reader is referred to the detailed discussion provided by Maugin (1980).

Finally, we mention that some authors present the principle of virtual work in a form where the virtual displacements must fulfill $v_i = 0$ along S_u, cf. (3.34), and with v_i otherwise being arbitrary. Certainly, this format is allowable, but it implies an unnecessary restriction on the general formulation of the principle of virtual work. Here, we therefore adopt the formulation of this important principle in its most general form.