11 PLASTIC COLLAPSE THEOREMS

In this chapter, we will present some classical *plastic collapse theorems* that have been used extensively during the years to evaluate the forces in metal forming processes and to determine the maximum load capacity of concrete structures and soil foundations. Moreover, these theorems often form the basis of various national building codes.

The body is assumed to consist of a material that may either be *stiff-ideal plastic* or *elastic-ideal plastic*, cf. Fig. 9.2. In that case the external load on the body cannot be increased indefinitely. This means that the structure possesses a certain maximum load capacity, i.e. a certain *limit load* exists. *Limit design* is concerned with the determination of this maximum load capacity - the *collapse load* - and the plastic collapse theorems take the forms of a *lower bound theorem* and an *upper bound theorem.*

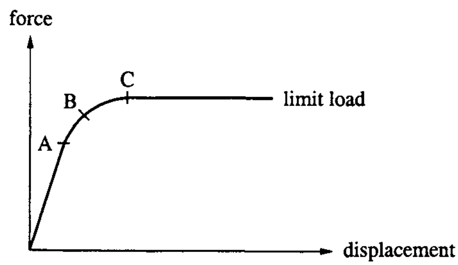

Figure 11.1: Illustration of limit load for a structure consisting of an elastic-ideal plastic material.

For an arbitrary structure consisting of an elastic-ideal plastic material, the concept of the limit load is shown in Fig. 11.1. At load stage A, plasticity is initiated and at load stage C, the stress distribution caused by plasticity is fully developed; load stage B is an intermediate load. For a beam loaded by a point force, these stages are illustrated in Fig. 11.2. From Fig. 11.1, it is apparent that, at the limit load, the external forces are constant and that the deformations can

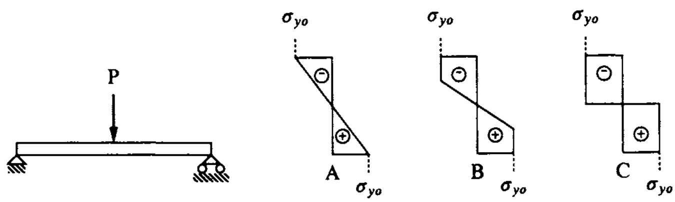

Figure 11.2: Elastic-ideal plastic beam and the stress distribution at the midsection at load stages A, B and C, cf. Fig. 11.1.

then take infinitely large values. Since small strains are assumed, we are only concerned with the initial stage of collapse.

Strict formal proofs of the lower and upper bound theorems were given by Gvozdev (1938) and they were later formulated independently by Drucker *et al.* (1952). However, the essential content of these theorems was known by intuition long before the establishment of the formal proofs. Indeed, the manner in which Coulomb (1776) investigated the collapse load of soil structures is essentially an application of the upper bound theorem. The formulation by Johansen (1943) of the so-called *yield line theory* applicable to concrete and steel plates also makes use of the upper bound theorem (an English translation is given by Johansen (1962)). On the other hand, determination of the collapse load of soil structures by means of a technique equivalent to the lower bound theorem was provided by Rankine in the mid-nineteenth century. The lower and upper bound theorems therefore belong to the classical topics in plasticity theory and we may refer to Koiter (1960), Lubliner (1990) and Nielsen (1984) for further historical remarks.

The upper and lower bound theorems hold for associated plasticity, but we will later touch upon the limit load for non-associated plasticity. Before we turn our interest to these theorems, we will first present some prerequisites.

Let us assume that the kinematic boundary conditions are given by

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) \quad along \quad S_u \tag{11.1}$$

cf. (3.34); since, in general, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ this implies that the displacements along S_u do not vary due to the loading, i.e. $\dot{\boldsymbol{u}} = \boldsymbol{0}$ along S_u, and this situation covers most cases of practical interest. Moreover, considering static conditions and choosing the arbitrary virtual displacement v_i as $v_i = \dot{u}_i$, the principle of virtual work (3.33) takes the form

$$\int_V \dot{\varepsilon}_{ij}\sigma_{ij}dV = \int_S \dot{u}_i t_i dS + \int_V \dot{u}_i b_i dV \tag{11.2}$$

Due to (11.1), $\dot{u}_i$=0 holds along S_u and we therefore obtain

$$\int_S \dot{u}_i t_i dS = \int_{S_u} \dot{u}_i t_i dS + \int_{S_t} \dot{u}_i t_i dS = \int_{S_t} \dot{u}_i t_i dS$$

i.e. the principle of virtual work (11.2) takes the form

$$\int_V \dot{\varepsilon}_{ij}\sigma_{ij} dV = \int_{S_t} \dot{u}_i t_i dS + \int_V \dot{u}_i b_i dV \tag{11.3}$$

By this approach, the contribution to the boundary term from the reaction forces acting along S_u has been eliminated. This means that the right-hand side of (11.3) in addition to the displacement rate $\dot{u}_i$ only involves the known external load.

The formulation (11.3) was obtained from the equilibrium equations $\sigma_{ij,j} + b_i = 0$. By differentiating these equations with respect to time we obtain the incremental equilibrium equations $\dot{\sigma}_{ij,j} + \dot{b}_i = 0$. Evidently, we can derive a weak form of these latter equations by completely analogous manipulations to those that resulted in (11.3). Therefore, we obtain

$$\int_V \dot{\varepsilon}_{ij}\dot{\sigma}_{ij} dV = \int_{S_t} \dot{u}_i \dot{t}_i dS + \int_V \dot{u}_i \dot{b}_i dV \tag{11.4}$$

At the limit load, the external forces are constant, i.e. at the limit load (11.4) leads to

$$\int_V \dot{\varepsilon}_{ij}\dot{\sigma}_{ij} dV = 0 \tag{11.5}$$

For associated plasticity, we have

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p = C_{ijkl}\dot{\sigma}_{kl} + \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

i.e.

$$\dot{\varepsilon}_{ij}\dot{\sigma}_{ij} = \dot{\sigma}_{ij}C_{ijkl}\dot{\sigma}_{kl} + \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} \tag{11.6}$$

For ideal plasticity where no hardening parameters exist, the consistency relation (10.11) gives $\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = 0$. Then (11.6) takes the form

$$\dot{\varepsilon}_{ij}\dot{\sigma}_{ji} = \dot{\sigma}_{ij}C_{ijkl}\dot{\sigma}_{kl} \tag{11.7}$$

and we observe that this expression holds for both a purely elastic behavior as well as for elastic-ideal plastic behavior when associated plasticity is assumed. Since the elastic flexibility tensor C_{ijkl} is positive definite, cf. (4.30), i.e. $\dot{\sigma}_{ij}C_{ijkl}\dot{\sigma}_{kl} > 0$ when $\dot{\sigma}_{ij} \neq 0$ it follows from (11.7) that (11.5) only can hold true if $\dot{\sigma}_{ij} = 0$ holds at the limit load. Since $\dot{\varepsilon}_{ij}^e = 0$ when $\dot{\sigma}_{ij} = 0$, we find

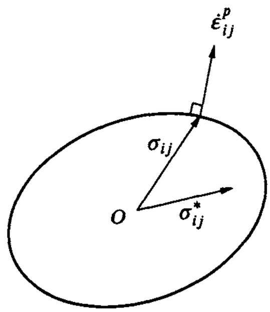

Figure 11.3: Quantities entering the postulate of maximum plastic dissipation.

that $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{p}$ applies at the limit load when the material is elastic-ideal plastic. Evidently, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{p}$ also holds for stiff-ideal plasticity, i.e. we have

For stiff-ideal plasticity and elastic-ideal plasticity

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{p} \tag{11.8}$$

holds at the limit point

Let us recall the postulate of maximum plastic dissipation given by (9.42) which reads

$$(\sigma_{ij} - \sigma_{ij}^{*})\dot{\varepsilon}_{ij}^{p} \geq 0 \tag{11.9}$$

Here σ_{ij} is the stress state on the yield surface related to the plastic strain rate $\dot{\varepsilon}_{ij}^{p}$ and σ_{ij}^{*} is any stress state within or on the yield surface; these features are illustrated in Fig. 11.3.

Since we are considering ideal plasticity, where no hardening parameters exist, the restriction on the stress state σ_{ij}^{*} can be expressed as

$$f(\sigma_{ij}^{*}) \leq 0$$

It is recalled that the postulate of maximum plastic dissipation leads to the associated flow rule. Moreover, since we are considering ideal plasticity where the convex yield surface certainly encloses the stress origin, the scalar product $\sigma_{ij}\dot{\varepsilon}_{ij}^{p}$ must be positive, cf. Fig. 11.3, i.e.

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} > 0 \tag{11.10}$$

11.1 Lower bound theorem

To establish the lower bound theorem, we first introduce the concept of an *allowable* and *statically admissible stress field* σ_{ij}^a. It is defined as

$$\boxed{\begin{array}{l} \textit{Allowable and statically admissible stress field } \sigma_{ij}^a: \\ \quad \sigma_{ij,j}^a + b_i^a = 0 \\ \quad f(\sigma_{ij}^a) \le 0 \\ \quad t_i^a = \mu^- t_i \quad \textit{is given along } S_t \\ \quad b_i^a = \mu^- b_i \quad \textit{in } V \end{array}} \tag{11.11}$$

The stress field σ_{ij}^a is created by the external forces t_i^a and b_i^a and is called statically admissible because it fulfills the differential equations of equilibrium and is called allowable because it does not violate the yield criterion. Moreover, this stress field σ_{ij}^a is created by external forces that are proportional to the real external forces and this proportionality is expressed by the factor μ^-. We will assume that by some means we are able to determine such a statically admissible and allowable stress field σ_{ij}^a.

Evidently, just as it is possible to make a weak formulation of the equilibrium equations for the real stress field σ_{ij}, it is possible to make a weak form of the equilibrium equations for the stress field σ_{ij}^a. Similar to (11.3), we then obtain

$$\int_V \dot{\varepsilon}_{ij}\sigma_{ij}^a dV = \int_{S_t} \dot{u}_i t_i^a dS + \int_V \dot{u}_i b_i^a dV$$

Due to (11.11), we then obtain

$$\int_V \dot{\varepsilon}_{ij}\sigma_{ij}^a dV = \mu^-[\int_{S_t} \dot{u}_i t_i dS + \int_V \dot{u}_i b_i dV] \tag{11.12}$$

Multiplication of (11.3) by μ^- and then subtraction of (11.12) give

$$\int_V (\mu^-\dot{\varepsilon}_{ij}\sigma_{ij} - \dot{\varepsilon}_{ij}\sigma_{ij}^a)dV = 0$$

As $\mu^- = (\mu^- - 1) + 1$, we obtain

$$\int_V [(\mu^- - 1)\dot{\varepsilon}_{ij}\sigma_{ij} + \dot{\varepsilon}_{ij}(\sigma_{ij} - \sigma_{ij}^a)]dV = 0 \tag{11.13}$$

Since σ_{ij}^a does not violate the yield criterion, we can as σ_{ij}^* take the stress σ_{ij}^a in Fig. 11.3 and the postulate of maximum plastic dissipation (11.9) in conjunction with (11.8) then shows that $\dot{\varepsilon}_{ij}(\sigma_{ij} - \sigma_{ij}^a) \ge 0$. Moreover, (11.10) and (11.8) show that $\dot{\varepsilon}_{ij}\sigma_{ij} > 0$. The only possibility for fulfillment of (11.13) is then $\mu^- \leq 1$. We are then led to the following important theorem

Lower bound theorem: *The external loading corresponding to an allowable and statically admissible stress field is less than or equal to the collapse load, i.e.* $\mu^- \leq 1$	(11.14)

Provided that we are able to identify an allowable and statically admissible stress field, the theorem provides a lower bound to the collapse load. The key question is then how such a stress field is identified. The simplest possibility is to perform an elastic analysis and then choose the factor μ^- such that the yield criterion is not violated. Clearly this would provide a very conservative μ^--factor and another possibility is to make some simplified analysis where the plastic effects are accounted for. As an example, consider a reinforced rectangular concrete plate. This plate is imagined to carry the load as two sets of beams at right angles to each other. It is then possible to satisfy the equilibrium differential equations for these two sets of beams when plastic effects are accounted for. If these stresses are within or on the yield surface and if they also satisfy the static boundary conditions, an allowable and statically admissible stress field has been established. This is the concept of the *strip method* proposed by Hillerborg (1974). There exists a number of approaches that often depend on the specific structure in question, and the reader may consult the specialized literature on this subject, see Chen and Han (1988), Johnson and Mellor (1983), Lubliner (1990) and Nielsen (1984). We will later give a simple example.

11.2 Upper bound theorem

Let us next introduce the concept of a *compatible displacement field* u_i^c; occasionally, such a displacement field is called an *admissible displacement field.* A compatible displacement field u_i^c is defined as one where $\dot{u}_i^c = 0$ holds along S_u and where the corresponding strain rate field is given by $\dot{\varepsilon}_{ij}^c = \frac{1}{2}(\dot{u}_{i,j}^c + \dot{u}_{j,i}^c)$. In accordance with (11.8), we will take $\dot{\varepsilon}_{ij}^c$ to be purely plastic. Moreover, the (plastic) strain rate $\dot{\varepsilon}_{ij}^c$ will be taken to be in accordance with the normality rule for associated plasticity. By this we mean that there exists some stress state σ_{ij}^c such that $\dot{\varepsilon}_{ij}^c = \dot{\lambda}(\frac{\partial f}{\partial \sigma_{ij}})_c$ where $(\frac{\partial f}{\partial \sigma_{ij}})_c$ indicates that $\frac{\partial f}{\partial \sigma_{ij}}$ is evaluated at some stress state σ_{ij}^c where $f(\sigma_{ij}^c) = 0$. We do not necessarily know this stress field σ_{ij}^c, but we do know that $\dot{\varepsilon}_{ij}^c = (\dot{\varepsilon}_{ij}^p)^c$ is in accordance with the normality rule. For von Mises ideal plasticity, for instance, where $\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sigma_{yo}}$, cf. (9.55), we can take $\dot{\varepsilon}_{ij}^c$ as any deviatoric strain state. With these remarks, we consider the following

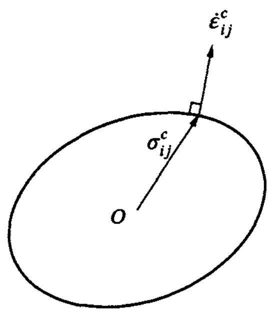

Figure 11.4: Correspondence between σ^c_{ij} and $\dot{\varepsilon}^c_{ij}$ according to the normality rule.

field

$$\boxed{\begin{array}{l}\textit{Compatible displacement field:}\\ \quad \dot{u}^c_i = 0 \text{ along } S_u\\ \quad \dot{\varepsilon}^c_{ij} = \frac{1}{2}(\dot{u}^c_{i,j} + \dot{u}^c_{j,i})\\ \textit{Moreover, } \dot{\varepsilon}^c_{ij} \textit{ fulfills the normality rule for some}\\ \textit{stress field } \sigma^c_{ij}\textit{, i.e.}\\ \quad \dot{\varepsilon}^c_{ij} = \dot{\lambda}(\frac{\partial f}{\partial \sigma_{ij}})_c\\ \textit{where}\\ \quad f(\sigma^c_{ij}) = 0\end{array}} \tag{11.15}$$

As the stress state σ^c_{ij} is located on the yield surface and as $\dot{\varepsilon}^c_{ij}$ is in accordance with the normality rule, we have the situation shown in Fig. 11.4. Since we are considering ideal plasticity, the convex yield surface certainly encloses the stress origin and we have

$$\sigma^c_{ij}\dot{\varepsilon}^c_{ij} > 0 \tag{11.16}$$

Moreover, as the real stress field σ_{ij} is located inside or on the convex yield surface, Fig. 11.5 illustrates the situation. It follows that the scalar product $\dot{\varepsilon}^c_{ij}(\sigma^c_{ij} - \sigma_{ij})$ is positive or zero, i.e.

$$\dot{\varepsilon}^c_{ij}(\sigma^c_{ij} - \sigma_{ij}) \geq 0 \tag{11.17}$$

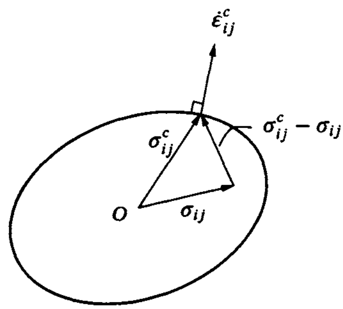

Figure 11.5: The real stress state σ_{ij} is located inside or on the yield surface.

As will be shown in a moment, we then obtain

Upper bound theorem:
Let $\dot{u}^c_i$ and $\dot{\varepsilon}^c_{ij}$ be defined by (11.15) and consider the loads
$t^c_i = \mu^+ t_i$ along S_t
$b^c_i = \mu^+ b_i$ in V
If μ^+ is calculated from
$$\mu^+ \underbrace{\left[\int_{S_t} \dot{u}^c_i t_i dS + \int_V \dot{u}^c_i b_i dV\right]}_{known} = \underbrace{\int_V \dot{\varepsilon}^c_{ij}\sigma^c_{ij} dV}_{known} \tag{11.18}$$
then
$\mu^+ \geq 1$

If μ^+ is calculated in this fashion, we have evidently determined an external load that is larger than or equal to the real collapse load that is, we have obtained an upper bound to the real collapse load. We mentioned previously that the stress field σ^c_{ij} does not necessary need to be known; however, as emphasized in (11.18) the quantity $\dot{\varepsilon}^c_{ij}\sigma^c_{ij}$ must be known up to, in principle, the unknown factor $\dot{\lambda}$ which then also enters the expression for $\dot{u}^c_i$. For von Mises ideal plasticity for instance, we have

$$\dot{\varepsilon}^c_{ij}\sigma^c_{ij} = \dot{\lambda}\left(\frac{\partial f}{\partial \sigma_{ij}}\right)_c \sigma^c_{ij} = \dot{\lambda}\frac{3s^c_{ij}}{2\sigma_{yo}}\sigma^c_{ij} = \dot{\lambda}\sigma_{yo} \tag{11.19}$$

To prove the upper bound theorem, we have

$$\begin{aligned}\int_V \dot{\varepsilon}^c_{ij}\sigma_{ij} dV = \int_V \dot{u}^c_{i,j}\sigma_{ij} dV &= \int_V [(\dot{u}^c_i\sigma_{ij})_{,j} - \dot{u}^c_i\sigma_{ij,j}] dV \\ &= \int_V (\dot{u}^c_i\sigma_{ij})_{,j} dV + \int_V \dot{u}^c_i b_i dV\end{aligned}$$

where the equilibrium equations $\sigma_{ij,j} + b_i = 0$ for the real stress field were used. With the divergence theorem and as $\sigma_{ij}n_j = t_i$, it then follows that

$$\int_V \dot{\varepsilon}_{ij}^c \sigma_{ij} dV = \int_S \dot{u}_i^c t_i dS + \int_V \dot{u}_i^c b_i dV$$

and since $\dot{u}_i^c = 0$ along S_u, we obtain

$$\int_{S_t} \dot{u}_i^c t_i dS + \int_V \dot{u}_i^c b_i dV = \int_V \dot{\varepsilon}_{ij}^c \sigma_{ij} dV$$

Multiplication by μ^+ and subtracting the result from (11.18) gives

$$\int_V (\dot{\varepsilon}_{ij}^c \sigma_{ij}^c - \mu^+ \dot{\varepsilon}_{ij}^c \sigma_{ij}) dV = 0$$

Since $1 = \mu^+ + (1 - \mu^+)$, we obtain

$$\int_V [\mu^+ \dot{\varepsilon}_{ij}^c (\sigma_{ij}^c - \sigma_{ij}) + (1 - \mu^+) \dot{\varepsilon}_{ij}^c \sigma_{ij}^c] dV = 0$$

The factor μ^+ is positive and according to (11.17) and (11.16) we have $\dot{\varepsilon}_{ij}^c(\sigma_{ij}^c - \sigma_{ij}) \geq 0$ and $\dot{\varepsilon}_{ij}^c \sigma_{ij}^c > 0$. We immediately conclude that $\mu^+ \geq 1$ in accordance with the result already stipulated in (11.18).

To use the upper bound theorem, use is made of the fact that $\dot{\varepsilon}_{ij}^c$ is a plastic strain. Then the body is often assumed to deform as rigid regions separated by narrow bands - so-called *slip bands* or *shear bands* - where the yield criterion is assumed to be fulfilled. Within these slip bands the displacement field is taken in accordance with the normality rule and such that $\dot{u}_i^c = 0$ along S_u. For plates, the similar approach is called *yield line* theory, cf. Johansen (1943, 1962). The reader may consult the specialized literature for further details, Chen and Han (1988), Hill (1950), Johnson and Mellor (1983), Lubliner (1990) and Nielsen (1984).

It is evident that the lower and upper bound theorems can be used to identify the bounds for the real limit load and if the lower bound solution is equal to the upper bound solution, then the true limit load has been determined.

11.3 Simple example

In order to illustrate the use of the lower and upper bound theorems in a simple fashion, the semi-infinite body loaded by the uniform pressure p shown in Fig. 11.6 is considered; von Mises ideal plasticity as well as plane strain are assumed.

It turns out to be convenient to work with the yield shear stress τ_{yo}, which according to (8.30) is related to the yield stress in tension σ_{yo} through

$$\tau_{yo} = \frac{\sigma_{yo}}{\sqrt{3}} \tag{11.20}$$

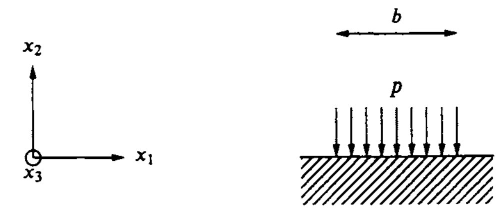

Figure 11.6: Semi-infinite body loaded by a pressure p.

The von Mises criterion (8.27) can then be written as

$$\sqrt{\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]} - \tau_{yo} = 0 \tag{11.21}$$

To use the lower bound theorem, an allowable and statically admissible stress field needs to be established. The stress field in Fig. 11.7 is characterized by

$$\begin{aligned} \text{Regions}: \ \mathrm{I} \quad & \sigma_1 = -2\tau_{yo}; \ \ \sigma_2 = 0; \ \ \sigma_3 = -\tau_{yo} \\ \mathrm{II} \quad & \sigma_1 = -2\tau_{yo}; \ \ \sigma_2 = -4\tau_{yo}; \ \ \sigma_3 = -3\tau_{yo} \end{aligned} \tag{11.22}$$

It appears that the out-of-plane stress σ_3 is taken as the mean value of σ_1 and

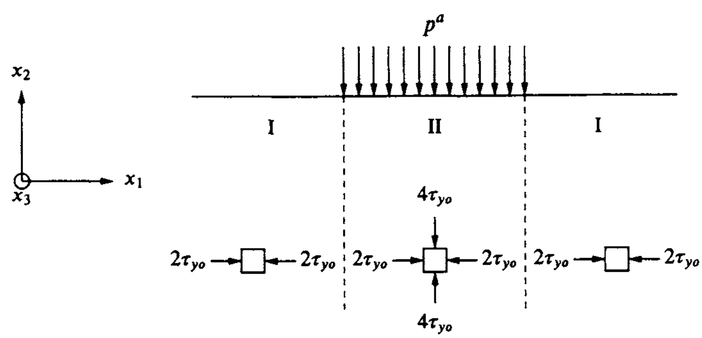

Figure 11.7: Allowable and statically admissible stress distribution.

σ_2. By inspection, these stress fields are seen to fulfill the yield criterion (11.21) so they are allowable stress fields. Since the stress fields are constant they also satisfy the equilibrium differential equations. The stress field in regions I fulfill the static boundary condition along the free surface ($t_i = 0$) and if we assume that the pressure p^a, see Fig. 11.7, creates the stress fields and that

$$p^a = 4\tau_{yo}$$

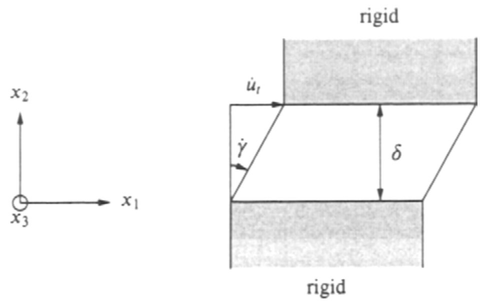

Figure 11.8: Shear band between two rigid blocks; $\dot{u}_t$ is the relative tangential velocity between these blocks.

then the static boundary condition along the pressurized surface is also fulfilled. This implies that the stress fields (11.22) in addition to being allowable are also statically admissible. The lower bound theorem (11.14) then provides

$$4\tau_{yo} \leq p_{\text{limit load}} \tag{11.23}$$

We may note that there is a discontinuity in the stress field along the border between region I and II. However, along this border the traction vector $t_i = \sigma_{ij}n_j$ for region I is opposite, but otherwise equal to the traction vector for region II. This implies that the equilibrium equations also are fulfilled along this border and the discontinuity in the stress field therefore creates no problems.

To be able to use the upper bound theorem, consider the shear band between two rigid blocks as shown in Fig. 11.8. We will take the displacements within the shear band as a compatible displacement field. From Fig. 11.8 it follows that

$$\dot{u}_1^c = \dot{\gamma}x_2 \; ; \qquad u_2^c = u_3^c = 0 \tag{11.24}$$

According to Fig. 11.8, $\dot{u}_t$ is the relative tangential velocity between the two rigid blocks, i.e. $\dot{\gamma}\delta = \dot{u}_t$ where δ is the thickness of the shear band. From (11.24), the corresponding strain rates then become

$$\dot{\varepsilon}_{12}^c = \frac{1}{2}\dot{\gamma} = \frac{\dot{u}_t}{2\delta} \; ; \qquad \dot{\varepsilon}_{ij} = 0 \quad \text{otherwise} \tag{11.25}$$

According to (11.15), the compatible displacement field is required to fulfill $\dot{u}_i^c = 0$ along S_u; however, since for the problem in question there is no S_u-surface, this requirement is fulfilled trivially.

The last requirement to the compatible displacement field is that $\dot{\varepsilon}_{ij}^c$ fulfills the normality rule, i.e. $\dot{\varepsilon}_{ij}^c = \dot{\lambda}(\frac{\partial f}{\partial \sigma_{ij}})_c$ where $f(\sigma_{ij}^c) = 0$, cf. (11.15). For von Mises ideal plasticity, we therefore require

$$\dot{\varepsilon}_{ij}^c = \dot{\lambda}\frac{3s_{ij}^c}{2\sigma_{yo}}$$

A comparison with (11.25) shows that we must have

$$\dot{\varepsilon}^c_{12} = \frac{\dot{u}_t}{2\delta} = \dot{\lambda}\frac{3s^c_{12}}{2\sigma_{yo}}; \qquad s^c_{ij} = 0 \quad \text{otherwise} \tag{11.26}$$

Finally, σ^c_{ij} must fulfill $f(\sigma^c_{ij}) = 0$ and as the stress state defined above corresponds $\sigma_1 = s^c_{12}$, $\sigma_2 = -s^c_{12}$ and $\sigma_3 = 0$, (11.21) provides

$$s^c_{12} = \tau_{yo} \tag{11.27}$$

as expected. According to (11.15), we have then demonstrated that u^c_i is a compatible displacement field. Indeed, we observe that the shearing displacement mode in Fig. 11.8 may be caused by yielding due to a shear stress alone and this motivates the terminology of the deformation mode of Fig. 11.8 being a shear band. Insertion of (11.20) and (11.27) into (11.26) then gives

$$\dot{\lambda} = \frac{1}{\sqrt{3}}\frac{\dot{u}_t}{\delta}$$

Use of this expression in (11.19) and taking advantage of (11.20) result in

$$\dot{\varepsilon}^c_{ij}\sigma^c_{ij} = \frac{\dot{u}_t}{\delta}\tau_{yo}$$

Let w be the width in the x_3-direction of the shear band. Then the plastic dissipation per unit length of the shear band becomes

$$\int_A \dot{\varepsilon}^c_{ij}\sigma^c_{ij}dA = \int_0^w \Big(\int_0^\delta \frac{\dot{u}_t}{\delta}\tau_{yo}dx_2\Big)dx_3 = \int_0^w \dot{u}_t\tau_{yo}dx_3 = w\dot{u}_t\tau_{yo}$$

We then obtain

$$\int_V \dot{\varepsilon}^c_{ij}\sigma^c_{ij}dV = \int_{\mathcal{L}} w\dot{u}_t\tau_{yo}d\mathcal{L} = w\dot{u}_t\tau_{yo}\mathcal{L} \tag{11.28}$$

where $\mathcal{L}$ is the length of the shear band.

Adopting the failure mechanism shown in Fig. 11.9, which corresponds to a compatible displacement field, we are now in a position to apply the upper bound theorem (11.18). Since there are no body forces and as $\mu^+ t_i = t^c_i$ we have

$$\int_{S_t} \dot{u}^c_i t^c_i dS = \int_V \dot{\varepsilon}^c_{ij}\sigma^c_{ij}dV$$

The left-hand side is the rate of external work. In view of Fig. 11.9 and (11.28) and recalling that w is the width in the x_3-direction, we obtain

$$p^c bw\frac{1}{2}\dot{u}_t = w\dot{u}_t\tau_{yo}\pi b$$

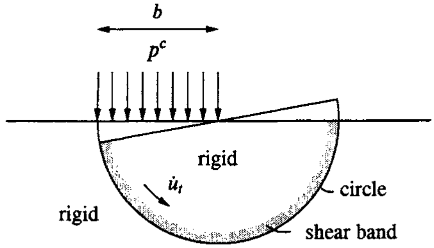

Figure 11.9: Compatible displacement field.

i.e.

$$p^c = 2\pi\tau_{yo}$$

and as this is an upper bound solution, we have

$$p_{\text{limit load}} \leq 2\pi\tau_{yo}$$

Combination with (11.23) gives

$$4\tau_{yo} \leq p_{\text{limit load}} \leq 2\pi\tau_{yo} \tag{11.29}$$

It is of interest that the mean of the lower and upper bound becomes $(2+\pi)\tau_{yo}$ and this can, in fact, be shown to be the exact solution cf. for instance Nielsen (1984). It is also shown there that the same result (11.29) is obtained if Tresca's yield criterion is used instead of the von Mises criterion. Indeed, the exact solution of the more general problem where the material follows Coulomb's yield criterion was already established by Prandtl (1920).

11.4 Nonassociated plasticity

To be able to apply the lower and upper bound theorem, the material must, in addition to being stiff-ideal plastic or elastic-ideal plastic, fulfill the normality rule, i.e. these theorems hold for associated plasticity. However, it turns out to be possible to obtain certain theorems for the limit load of nonassociated ideal plastic bodies; Lubliner (1990) gives a detailed discussion as well as relevant references. Here, we shall simply mention one important theorem which is due to Radenkovic (1961). It reads

Consider a body made of an ideal plastic material with a certain given yield criterion. *The limit load for a nonassociated flow rule* $\leq$ *the limit load for an associated flow rule*

(11.30)

The proof is straightforward. Observing that the yield criterion is assumed to be given, consider the body with a nonassociated flow rule at its limit load. The corresponding stress field can certainly be taken as allowable and admissible, cf. (11.11). Adopting this allowable and admissible stress field in the lower bound theorem (11.14), which holds for the body with an associated flow rule, it then follows that the stress field for nonassociated plasticity corresponds to an external loading that is less than or equal to the limit load for the body with an associated flow rule. This proves the correctness of statement (11.30).