# 16 NONLINEAR FINITE ELEMENT METHOD

As engineers, we are interested in the response of structures and thereby in the solution of boundary value problems. Having established the equations that control elasto-plastic, viscoplastic and creep behavior, it is evident that these equations are so complex that exact analytical solutions of boundary value problems cannot, in general, be established. Instead we must look for approximative solution methods and, today, the most powerful numerical means turns out to be the *finite element method*.

For linear problems, use of the finite element (FE) method is straightforward. Elasto-plastic, viscoplastic and creep problems, however, are nonlinear and this gives rise to a number of questions that must be resolved before a reliable solution can be established. These new questions can be summarized into: formulation of the nonlinear finite element method, solution of the nonlinear global equations, and integration of the constitutive equations. Here, we will first present the formulation of the nonlinear finite element method whereas the next two chapters will deal with the solution of the nonlinear global equations for static problems and numerical integration of the constitutive equations.

Let us therefore first formulate the nonlinear finite element method for general nonlinear solid mechanics. The formulation of the nonlinear FE method is very similar to that of the linear FE method, i.e. it is based on the weak formulation of the equations of motion, i.e. on the principle of virtual work. Detailed discussions of the finite element method are given, for instance, by Bathe (1996), Belytschko *et al.* (2000), Hughes (1987), Ottosen and Petersson (1992) and Zienkiewicz and Taylor (1989).

## 16.1 Equations of motion

In the first place, we will express the equations of motion in a finite element format. The weak form of the equations of motion is given by (3.33), i.e.

$$\int_V \rho v_i \ddot{u}_i dV + \int_V \varepsilon^v_{ij} \sigma_{ij} dV = \int_S v_i\, t_i dS + \int_V v_i\, b_i dV \tag{16.1}$$

where it is recalled that $v_i = v_i(x_i)$ is an arbitrary weight vector, $t_i$ is the traction vector, $b_i$ the body force, i.e. force per unit volume and $\ddot{u}_i$ is the acceleration vector. Moreover, $V$ denotes the region, i.e. the volume, of the body whereas $S$ is the boundary surface of the body. Finally, according to (3.32) the quantity $\varepsilon^v_{ij}$ is defined by

$$\varepsilon^v_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{16.2}$$

It appears that $\varepsilon^v_{ij}$ is defined in a similar manner as the strain tensor $\varepsilon_{ij}$, but to emphasize that $\varepsilon^v_{ij}$ just is a quantity defined by (16.2), we have used the superscript $v$.

In finite element formulations, matrix notation turns out to be particularly convenient. With the notations, cf. (4.35)

$$\boldsymbol{\varepsilon}^v = \begin{bmatrix} \varepsilon^v_{11} \\ \varepsilon^v_{22} \\ \varepsilon^v_{33} \\ 2\varepsilon^v_{12} \\ 2\varepsilon^v_{13} \\ 2\varepsilon^v_{23} \end{bmatrix}; \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

$$\ddot{\boldsymbol{u}} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix}; \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}; \quad \boldsymbol{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}; \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we may write the weak form (16.1) of the equations of motion as

$$\int_V \rho \boldsymbol{v}^T \ddot{\boldsymbol{u}} dV + \int_V (\boldsymbol{\varepsilon}^v)^T \boldsymbol{\sigma} dV = \int_S \boldsymbol{v}^T \boldsymbol{t} dS + \int_V \boldsymbol{v}^T \boldsymbol{b} dV \tag{16.3}$$

The *boundary conditions* of the body can be expressed as

$$\boxed{\begin{array}{ll} \boldsymbol{u} = & \textit{is given along } S_u \\ \boldsymbol{t} = & \textit{is given along } S_t \end{array}}$$

that is, the displacement vector $\boldsymbol{u}$ is prescribed along the boundary surface $S_u$ and the traction vector $\boldsymbol{t}$ is prescribed along the boundary surface $S_t$. The sum $S_u$ and $S_t$ comprises the entire boundary $S$ as illustrated in Fig. 16.1.

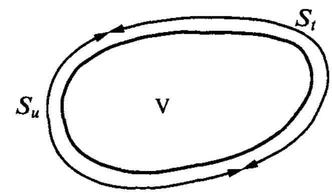


**Figure 16.1:** Boundary conditions

The finite element method is based on the concept that the displacement vector $\boldsymbol{u}$ throughout the body can be expressed in an approximate manner as

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a} \tag{16.4}$$

where $\boldsymbol{N}$ denotes the *global shape functions* and $\boldsymbol{a}$ is a column matrix that includes all the *nodal displacements* of the body. The displacement vector $\boldsymbol{u}$ depends on both position and time whereas the global shape functions only depend on position, i.e. we have

$$\boldsymbol{u} = \boldsymbol{u}(x_i, t)\ ; \qquad \boldsymbol{N} = \boldsymbol{N}(x_i)\ ; \qquad \boldsymbol{a} = \boldsymbol{a}(t) \tag{16.5}$$

It follows that

$$\ddot{\boldsymbol{u}} = \boldsymbol{N}\ddot{\boldsymbol{a}}$$

With the displacements given by (16.4) we can determine the corresponding strains, which, in a matrix form, may be expressed as

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}\ ; \qquad \boldsymbol{B} = \boldsymbol{B}(x_i) \tag{16.6}$$

where the matrix $\boldsymbol{B}$ is derived from the matrix $\boldsymbol{N}$.

The fundamental issue of the standard finite element method is that the arbitrary weight vector $\boldsymbol{v}$ is chosen in accordance with *Galerkin's method*, i.e. it is approximated in the same manner as the displacement $\boldsymbol{u}$. In analogy with (16.4) we therefore write

$$\boldsymbol{v} = \boldsymbol{N}\boldsymbol{c} \tag{16.7}$$

Since $\boldsymbol{v}$ is arbitrary and the global shape functions $\boldsymbol{N}$ are specified by us, the column matrix $\boldsymbol{c}$ is arbitrary. Like (16.5), we observe that $\boldsymbol{c}$ does not depend on position. From (16.7), we can determine the quantity $\boldsymbol{\varepsilon}^v$ similar to (16.6) i.e.

$$\boldsymbol{\varepsilon}^v = \boldsymbol{B}\boldsymbol{c} \tag{16.8}$$

Use of (16.7) and (16.8) in the weak form (16.3) of the equations of motion and noting that $\boldsymbol{c}$ is independent of position gives

$$\boldsymbol{c}^T \left[ \left( \int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV \right) \ddot{\boldsymbol{a}} + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \int_S \boldsymbol{N}^T \boldsymbol{t} dS - \int_V \boldsymbol{N}^T \boldsymbol{b} dV \right] = 0$$

As this equation holds for arbitrary $\boldsymbol{c}$-matrices, we conclude that

$$\boxed{\boldsymbol{M}\ddot{\boldsymbol{a}} + \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV = \boldsymbol{f}} \tag{16.9}$$

where the *mass matrix* $\boldsymbol{M}$ is defined by

$$\boldsymbol{M} = \int_V \rho \boldsymbol{N}^T \boldsymbol{N} dV$$

i.e. $\boldsymbol{M}$ is symmetric. Moreover, $\boldsymbol{f}$ defines the *external forces* according to

$$\boldsymbol{f} = \int_S \boldsymbol{N}^T \boldsymbol{t} dS + \int_V \boldsymbol{N}^T \boldsymbol{b} dV$$

We emphasize that expression (16.9) was derived entirely from the equations of motion without any information on the particular constitutive relation and (16.9) therefore holds for any constitutive relation.

## 16.2 Static conditions

Let us now specialize to static conditions, i.e. the nodal accelerations $\ddot{\boldsymbol{a}}$ are zero; the equations of motion (16.9) then reduce to the *equilibrium equations*

$$\boxed{\boldsymbol{\psi} = \boldsymbol{0}} \tag{16.10}$$

where

$$\boxed{\boldsymbol{\psi} = \int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV - \boldsymbol{f}} \tag{16.11}$$

It is recalled that $\boldsymbol{f}$ is an expression for the external loading of the body given by the traction vector $\boldsymbol{t}$ and the body force $\boldsymbol{b}$. Likewise, the term $\int_V \boldsymbol{B}^T \boldsymbol{\sigma} dV$ expresses the *internal forces* that the stresses $\boldsymbol{\sigma}$ give rise to and (16.10) and (16.11) therefore states that in order that the body be in equilibrium, the external forces must be equal to the internal forces.

Equations (16.10) and (16.11) express the equilibrium equations for the body and since no consideration was made of the constitutive relation, they hold for any body in equilibrium.

Evidently, the constitutive relation must be invoked in order to solve a specific boundary value problem. For linear elasticity, this step is straightforward. In that case, we have with (16.6)

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{a} \tag{16.12}$$

where, in general, the elastic stiffness $\boldsymbol{D}$ depends on position, i.e. $\boldsymbol{D} = \boldsymbol{D}(x_i)$, but not on the loading. Introduction of (16.12) in (16.11) and use of (16.10) provide

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f} \quad \text{where} \quad \boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} dV \tag{16.13}$$

where the elastic stiffness matrix $\boldsymbol{K}$ is constant. It appears that (16.13) is a linear equation system that - after consideration of the boundary conditions - can be solved directly of provide the current value of the nodal displacements $\boldsymbol{a}$.

For general nonlinear problems, the situation is quite different. Here we cannot express the current stresses $\boldsymbol{\sigma}$ directly in terms of the current strains $\boldsymbol{\varepsilon}$; instead, we simply know the incremental relation between the stress rate and the strain rate. For elasto-plastic problems, for instance, this relation is given by (10.29), i.e.

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\varepsilon}} \tag{16.14}$$

and the current stresses $\boldsymbol{\sigma}$ must be obtained by integration of (16.14) along the actual load history.

Since the constitutive relation (16.14) is nonlinear, it is no surprise that also the global equilibrium equations (16.10) and (16.11) become nonlinear. This is the first problem we encounter when solving, for instance, elasto-plastic boundary value problems. The next problem is that at each material point, we have to integrate the constitutive equations given by (16.14).

In order to further illustrate these problems, we observe that (16.14) is given in an incremental form. It is therefore tempting to differentiate (16.10) with respect to time to obtain

$$\int_V \boldsymbol{B}^T \dot{\boldsymbol{\sigma}} dV = \dot{\boldsymbol{f}} \tag{16.15}$$

where

$$\dot{\boldsymbol{f}} = \int_S \boldsymbol{N}^T \dot{\boldsymbol{t}} dS + \int_V \boldsymbol{N}^T \dot{\boldsymbol{b}} dV$$

and where it was used that the global shape functions $\boldsymbol{N}$ and the matrix $\boldsymbol{B}$ are independent of time, cf. (16.5) and (16.6). We observe that whereas (16.10) expresses the total equilibrium of the body, (16.15) expresses the incremental equilibrium condition of the body. For elasto-plasticity, it follows from (16.14) and (16.6) that

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{D}_t \boldsymbol{B}\dot{\boldsymbol{a}} \quad \text{where} \quad \boldsymbol{D}_t = \begin{cases} \boldsymbol{D} & \text{if the point behaves elastically} \\ \boldsymbol{D}^{ep} & \text{if the point behaves plastically} \end{cases} \tag{16.16}$$

Since the incremental nodal displacements $\dot{\boldsymbol{a}}$ are independent of position, use of (16.16) in (16.15) yields

$$\boldsymbol{K}_t \, \dot{\boldsymbol{a}} = \dot{\boldsymbol{f}} \tag{16.17}$$

where the *tangential stiffness matrix* $\boldsymbol{K}_t$ of the entire body is given by

$$\boldsymbol{K}_t = \int_V \boldsymbol{B}^T \boldsymbol{D}_t \, \boldsymbol{B} \, dV$$

Expression (16.17) comprises the global equation system that we need to solve in order to determine the incremental response of the body.

It is of importance that the tangential stiffness matrix $\boldsymbol{K}_t$ is not a constant matrix, i.e. (16.17) comprises a system of nonlinear equations. Expression (16.17) suggests that the external load $\boldsymbol{f}$ is increased in small steps and for each of these steps, the corresponding change of the nodal displacements $\boldsymbol{a}$ is determined by means of (16.17). Therefore, the solution of our boundary value problem has been reformulated into a *stepwise*, i.e. an *incremental solution procedure* that traces the response of the body with increased loading.

The first fundamental problem in the nonlinear finite element method is therefore to solve the global nonlinear equations (16.17). Moreover, this solution must have a form that ensures that the total equilibrium conditions (16.10) for the body are also satisfied. However, to use (16.10) – and considering for the moment elasto-plastic problems – we need to know the total stresses $\boldsymbol{\sigma}$ which, in turn, requires an integration of the constitutive equations (16.16) at each material point along the load path which is certainly not trivial and this comprises the second fundamental problem in the nonlinear FE method. The next chapter discusses different solution strategies for solution of the equilibrium equations whereas Chapter 18 is concerned with the integration of the constitutive equations for elasto-plasticity, viscoplasticity and creep.

In reality, a reliable solution scheme of the global equilibrium equations is based directly on (16.10) and not on (16.17). This will be discussed in detail in the next chapter and we emphasize that the format (16.17) was mainly introduced in order to provide a suitable background for a discussion of the principal problems.