9 INTRODUCTION TO PLASTICITY THEORY

The present chapter serves as a prelude to the next chapter where the general *plasticity theory* will be presented in a rather abstract and formal manner. Before we get to that point, it is necessary to become familiar with the various ingredients in the plasticity theory and this is the subject of the present chapter.

Plasticity theory is concerned with time-independent behavior that is nonlinear and where strains exist when the material is unloaded; these residual strains are the plastic strains. This is in contrast to nonlinear elasticity, where the body recovers its original configuration when unloaded. In Chapter 8, we discussed various initial yield criteria, i.e. conditions for which plastic effects are initiated. When the stress state exceeds the initial yield criterion, plastic strains will develop and this topic is one of the major issues that we will now address.

The basic behavior of an elasto-plastic material is summarized in Fig. 9.1. The behavior is linear elastic with stiffness E until the *initial yield stress* σ_{yo} is reached; after that *plastic strains* develop. Unloading from point A, see Fig. 9.1, occurs elastically with the stiffness E so that at complete unloading to point B, the residual strain amounts to the plastic strain ε^p developed at point A. There-

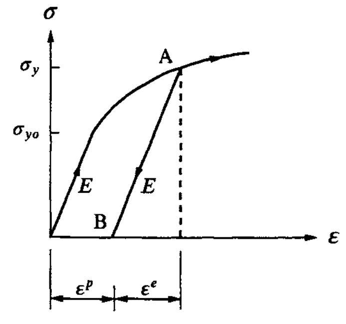

Figure 9.1: Basic response of elasto-plastic material.

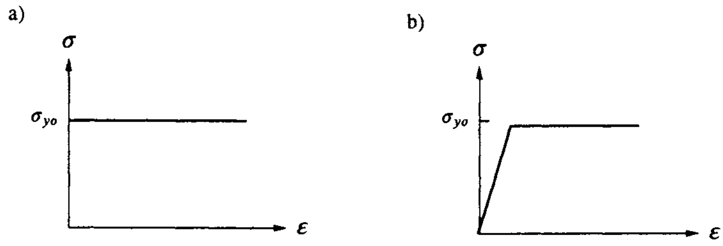

Figure 9.2: a) Stiff-ideal plastic behavior; b) elastic-ideal plastic behavior.

fore, at point A, the *total strain* ε consists of the sum of the *elastic* and *plastic* strains, i.e.

$$\boxed{\varepsilon = \varepsilon^e + \varepsilon^p} \tag{9.1}$$

If we reload again from point B, cf. Fig. 9.1, the material responds elastically until the stress reaches the value σ_y at point A. The value σ_y is therefore the *current yield stress* which, in general, differs from the initial yield stress σ_{yo}. On loading beyond point A the material behaves as if the previous unloading from point A had never occurred. Moreover, the response shown in Fig. 9.1 is assumed to be independent of time; this implies that we obtain the same response irrespective of the loading rate.

The behavior sketched in Fig. 9.1 is our model for the real material behavior, but it turns out that this model behavior closely agrees with the real behavior of elasto-plastic materials.

To characterize plastic behavior, a number of idealized responses have been identified. For the simplest response shown in Fig. 9.2a), the behavior is termed *stiff-ideal plastic* since no deformation occurs before the yield point has been reached and since the yield stress is unaffected by the amount of plastic strains. With obvious notation, the behavior shown in Fig. 9.2b) is termed *elastic-ideal plastic* behavior. Instead of ideal plasticity, the phrase *perfect plasticity* is often used.

In Fig. 9.3a) *hardening plasticity* is displayed; formally, the phrase elastic-hardening plasticity should be used, but the word elastic is ignored since it is evident that we will consider the elastic response. The hardening response shown in Fig. 9.3a) means that the current yield stress σ_y increases with increasing plastic strain, cf. Fig. 9.1, and this behavior is characteristic for alloyed steel and aluminum; moreover, aluminum lacks a sharply defined initial yield stress. In Fig. 9.3b), combined ideal and hardening plasticity is shown and this behavior is characteristic for mild steel.

Finally, Fig. 9.4 shows the development of hardening plasticity followed by *softening plasticity*; this response is typical for concrete, soil and rocks loaded

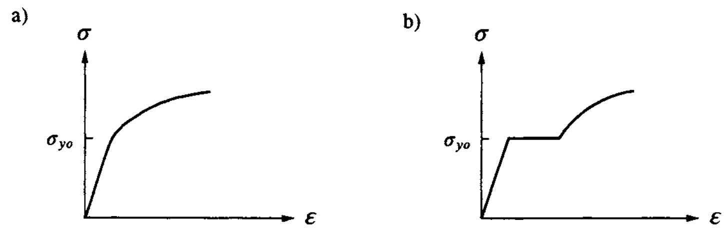

Figure 9.3: a) Hardening plasticity characteristic for alloyed steel and aluminum; b) combined ideal and hardening plasticity characteristic for mild steel.

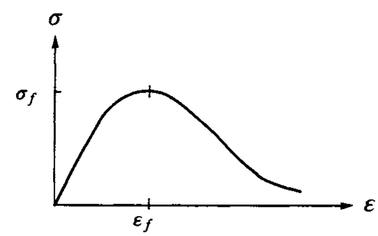

Figure 9.4: Hardening plasticity followed by softening plasticity; characteristic for rocks and concrete in compression.

in compression. Naturally, to achieve the softening branch it is required that the testing machine be operated by means of prescribed displacements and we shall return to this subject in relation to Fig. 9.27. In Fig. 9.4, σ_f denotes the failure stress and ε_f the corresponding strain.

For uniaxial loading, it is straightforward to establish various plasticity models. In essence, it merely consists of adjusting a mathematical curve so that it fits the experimental stress-strain data during loading and adopting a linear elastic response during unloading. We will first mention some well-known proposals that can be used to fit the experimental stress-strain data during uniaxial loading. However, for three-dimensional stress conditions, it is not possible to adopt this simple curve fitting technique; there are simply too many variables and too many different load cases. Thus, in that case, there exists a need for establishment of a general framework for plasticity formulations which will be discussed in detail in this and the following chapter. Before that we will consider some simple curve fitting techniques for uniaxial loading.

When no sharply defined initial yield stress exists, the uniaxial hardening stress-strain curve may be approximated by the Ramberg and Osgood (1943)

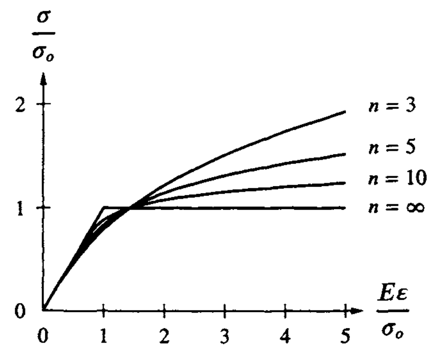

Figure 9.5: Ramberg-Osgood curve for $\alpha = 3/7$.

formula

$$\boxed{\varepsilon = \frac{\sigma}{E}[1 + \alpha(\frac{\sigma}{\sigma_o})^{n-1}] \; ; \qquad n > 1} \tag{9.2}$$

where α and n are dimensionless parameters whereas σ_o is a parameter with the dimension of stress. Expression (9.2) implies that plastic strains develop right from the onset of the loading. It appears that if $\sigma = \sigma_o$ then (9.2) predicts $\varepsilon = \sigma_o(1 + \alpha)/E$, i.e. the parameter σ_o may be interpreted as the stress value at which the curve given by (9.2) intersects the straight line given by $\sigma = \frac{E}{1+\alpha}\varepsilon$. The value $E/(1 + \alpha)$ may therefore be viewed as the secant modulus when the stress is σ_o. Most often the parameter α is chosen as $\alpha = 3/7$ implying that σ_o becomes the stress at which the secant modulus $E/(1 + \alpha)$ is $7E/10$. For this α-value, the appearance of (9.2) is shown in Fig. 9.5. From this figure, it appears that (9.2) for $n \to \infty$ corresponds to ideal plasticity.

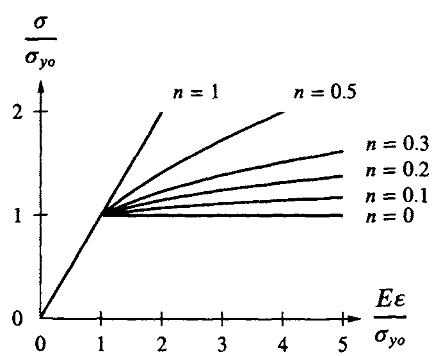

Figure 9.6: Ludwik curve.

If a sharply defined initial yield stress is required, the hardening stress-strain curve may be approximated by the following expression proposed by Ludwik (1909)

$$\boxed{\sigma = \begin{cases} E\varepsilon & when \;\; \varepsilon \leq \frac{\sigma_{yo}}{E} \\ \sigma_{yo}(\frac{E\varepsilon}{\sigma_{yo}})^n & when \;\; \varepsilon \geq \frac{\sigma_{yo}}{E} \quad (0 \leq n \leq 1) \end{cases}}$$

This expression is shown in Fig. 9.6 and it appears that the slope $d\sigma/d\varepsilon$ changes discontinuously at $\sigma = \sigma_{yo}$ (except when $n = 1$); moreover, ideal plasticity is recovered for $n = 0$.

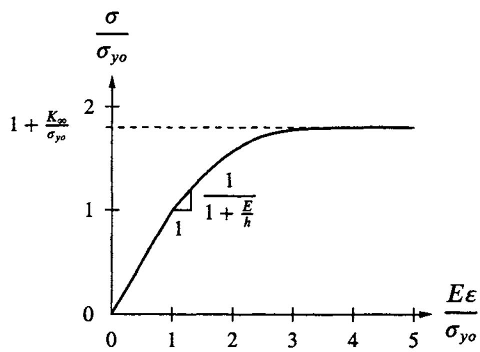

Figure 9.7: Exponential law curve when $E/\sigma_{yo} = 500$, $K_\infty/\sigma_{yo} = 0.8$ and $E/h = 1$.

Another expression that exhibits a sharply defined initial yield stress is given by the following *exponential law*

$$\boxed{\sigma = \begin{cases} E\varepsilon & when \;\; \varepsilon \leq \frac{\sigma_{yo}}{E} \\ \sigma_{yo} + K_\infty(1 - e^{-\frac{h}{K_\infty}\varepsilon^p}) & when \;\; \varepsilon \geq \frac{\sigma_{yo}}{E} \end{cases}} \tag{9.3}$$

where K_∞ and h are parameters with the dimension of stress. Using (9.1), we obtain $\varepsilon^p = \varepsilon - \frac{\sigma}{E}$ which means that (9.3) alternatively may be written as

$$\frac{\sigma}{\sigma_{yo}} = 1 + \frac{K_\infty}{\sigma_{yo}}[1 - e^{-\frac{\sigma_{yo}}{K_\infty}\frac{h}{E}(\frac{E\varepsilon}{\sigma_{yo}} - \frac{\sigma}{\sigma_{yo}})}]$$

Using E/σ_{yo}=500, which is typical for steel and choosing K_∞/σ_{yo}=0.8 and E/h=1, we obtain the curve shown in Fig. 9.7; the interpretation of the parameters K_∞ and h also appears from this figure.

To achieve an approximation for a hardening stress-strain curve that exhibits a sharply defined initial yield stress as well as a continuous variation of the slope

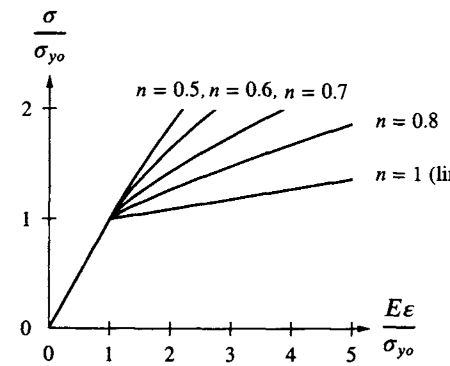

Figure 9.8: Power law curve when $E/\sigma_{yo} = 500$ and $k = 50$.

$d\sigma/d\varepsilon$ also at $\sigma = \sigma_{yo}$, an often used expression is the following *power law*

$$\boxed{\sigma = \begin{cases} E\varepsilon & when \quad \varepsilon \leq \frac{\sigma_{yo}}{E} \\ \sigma_{yo} + k\sigma_{yo}(\varepsilon^p)^n & when \quad \varepsilon \geq \frac{\sigma_{yo}}{E} \quad (0 < n \leq 1) \end{cases}} \tag{9.4}$$

where k is a dimensionless parameter. For $k = 0$, ideal plasticity emerges whereas $n = 1$ implies so-called *linear hardening* (in that case, the slope $d\sigma/d\varepsilon$ varies discontinuously at $\sigma = \sigma_{yo}$). Since $\varepsilon^p = \varepsilon - \sigma/E$, (9.4) may alternatively be written as

$$\frac{E\varepsilon}{\sigma_{yo}} = \frac{\sigma}{\sigma_{yo}} + \frac{E}{\sigma_{yo}} \frac{1}{k^{1/n}} (\frac{\sigma}{\sigma_{yo}} - 1)^{1/n} \quad \text{when} \quad \frac{E\varepsilon}{\sigma_{yo}} \geq 1$$

Using again $E/\sigma_{yo} = 500$ and choosing the parameter $k = 50$, we obtain the curves shown in Fig. 9.8.

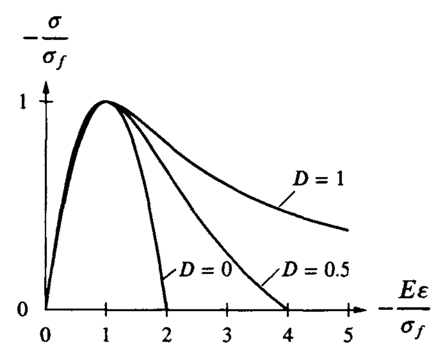

Figure 9.9: Sargin curve for $A = 2$.

To approximate the hardening/softening curve of Fig. 9.4, an expression proposed by Sargin (1971) is conveniently used. Since Fig. 9.4 is typical for concrete and rocks in compression, we shall write

$$\boxed{-\frac{\sigma}{\sigma_f} = \frac{-A\frac{\varepsilon}{\varepsilon_f} + (D-1)(\frac{\varepsilon}{\varepsilon_f})^2}{1-(A-2)\frac{\varepsilon}{\varepsilon_f} + D(\frac{\varepsilon}{\varepsilon_f})^2}} \tag{9.5}$$

Here σ_f and ε_f are the failure values of the stress and strain respectively, cf. Fig. 9.4; as expected, (9.5) then provides $\sigma = -\sigma_f$ when $\varepsilon = -\varepsilon_f$. Moreover, the parameter A is defined by $A = E/E_f$ in which $E_f = \sigma_f/\varepsilon_f$, i.e. E_f is the secant modulus at failure. Finally, the dimensionless parameter D is a parameter that mainly influences the descending curve in the post-failure region. To achieve that (9.5) reflects: 1) an increasing function without inflexion points before failure; 2) a decreasing function with at most one inflexion point after failure; and 3) a residual strength equal to zero at sufficiently large strain, it turns out that we must require

$$\begin{aligned} &A > \tfrac{4}{3} \\ &(1-\tfrac{A}{2})^2 \le D \le 1 + A(A-2) \quad \text{when} \quad A \le 2 \\ &0 \le D \le 1 \qquad\qquad\qquad\qquad \text{when} \quad A \ge 2 \end{aligned}$$

Sargin's expression is illustrated in Fig. 9.9 for $A = 2$. It appears that different softening behaviors can be simulated by means of the parameter D and that this only insignificantly affects the behavior before failure. Finally, we refer to Popovicz (1970) for other uniaxial stress-strain expressions relevant for concrete and rocks.

Whereas it is straightforward to posit different proposals for the uniaxial stress-strain curve, the elasto-plastic response for general three-dimensional loading is much more complex. In order to address this problem, we will first discuss various issues related to the yield surface.

9.1 Change of yield surface due to loading - Hardening rules

Since the yield stress most often varies with the plastic deformation and since the yield surface is the generalization of the yield stress to general stress states, it is evident that the yield surface will change with the plastic loading. This change of yield surface is called the *hardening rule*, i.e.

Hardening rule = rule for how the yield surface changes with the plastic loading

Since the yield surface is fundamental to the plasticity theory, we will first discuss this issue.

In general, we describe the *initial yield surface* by

$$\boxed{F(\sigma_{ij}) = 0\,; \quad \textit{initial yield surface}}$$

which for isotropic materials can also be written in the form given by (8.19). We know from Sections 6.5, 6.6 and 8.13 that if anisotropic materials are considered, say orthotropy, then structural tensors should also be included in the expression for the yield criterion. However, for the following discussion this aspect is not of importance; thus, we will not include structural tensors in the yield criterion even when anisotropy is present.

Since the yield surface in general varies with the development of plastic strains, we may express the *current yield surface* by

$$f(\sigma_{ij}, K_1, K_2, \ldots) = 0 \tag{9.6}$$

where we have introduced the so-called *hardening parameters* K_1, K_2,... that characterize the manner in which the current yield surface changes its size, shape and position with plastic loading. Before any plasticity is initiated, we know per definition that $K_\alpha = 0$. As yet, the number of hardening parameters is unknown, and, as indicated, we may have one, two or more hardening parameters. Moreover, at this point we do not know the type of the hardening parameters, which may be scalars or higher-order tensors. Therefore, we may collect all these hardening parameters into the notation K_α and use the following definition

$$\boxed{\begin{array}{ll} K_\alpha = \textit{hardening parameters}\ (\alpha = 1, 2, \ldots) & \\ K_\alpha = 0 & \textit{initially} \end{array}} \tag{9.7}$$

i.e. (9.6) can be written as

$$\boxed{f(\sigma_{ij}, K_\alpha) = 0\,; \quad \textit{current yield surface}} \tag{9.8}$$

Since $K_\alpha = 0$ holds initially, it follows that

$$f(\sigma_{ij}, 0) = F(\sigma_{ij}) \tag{9.9}$$

i.e. when the hardening parameters are zero, the current yield surface coincides with the initial yield surface. Through the hardening parameters, (9.8) describes how the size, shape and position of the current yield surface vary with plastic loading and the explicit manner in which this occurs is given by the hardening rule, i.e.

$$\boxed{\textit{Choice of hardening parameters} = \textit{choice of hardening rule}}$$

The hardening parameters K_α vary with the plastic loading. To model this, we assume that there exist some *internal variables* that characterize the condition, i.e. the state of the elasto-plastic material. As internal variables we may, for instance, use the plastic strains ε_{ij}^{p} or some combinations of this tensor; we will revert to this choice of the internal variables later and discuss it in detail, see for instance Section 9.6. The important point is that the internal variables are used to memorize the plastic loading history. As the internal variables characterize the state of the material, they are often termed *state variables*. In principle, the only variables that we can directly observe and measure are the total strains and the temperature. As we cannot directly observe or directly measure the internal variables, they are occasionally termed *hidden* variables in the literature. We shall follow the trend in recent literature and exclusively use the word 'internal variables' since the terminology of 'hidden' variables may act as a psychological block to acceptance. Summarizing these introductory remarks, we have

$$\boxed{\textit{internal variables} = \textit{state variables}} \tag{9.10}$$

In analogy with the notation above, we shall let κ_α denote the internal variables, i.e.

$$\boxed{\begin{array}{ll} \kappa_\alpha = \textit{internal variables}\ (\alpha = 1, 2 \ldots) \\ \kappa_\alpha = 0 \qquad \textit{initially} \end{array}} \tag{9.11}$$

Since the internal variables memorize the plastic loading history, they are, per definition, zero before any plasticity is initiated. As before, the above notation means that we may have one, two or more internal variables and at the present time we do not know whether κ_α are scalars or higher-order tensors. Since the internal variables κ_α characterize the elasto-plastic material, we can assume that

$$\boxed{K_\alpha = K_\alpha(\kappa_\beta)} \tag{9.12}$$

i.e. the hardening parameters K_α depend on the internal variables κ_α. It seems natural to assume that the number of hardening parameters equals the number of internal variables; otherwise (9.12) will not provide a unique relation between the set of hardening parameters and the set of internal variables, From (9.12) follows that

$$\dot{K}_\alpha = \frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta \tag{9.13}$$

where, as previously, a dot denotes the rate, i.e. the change. Here, the summation convention is also adopted for Greek letters. As the internal variables κ_α characterize the state of the elasto-plastic material, they can only change during plastic loading, i.e. $\dot{\kappa}_\alpha$ must be zero for an elastic response and, in view of (9.13), we conclude

$$\boxed{\dot{\kappa}_\alpha = \dot{K}_\alpha = 0\,; \quad \textit{for elastic behavior}} \tag{9.14}$$

With this general discussion, let us return to the hardening rule. Starting with the simplest case of ideal plasticity, as illustrated in Fig. 9.2b), the yield

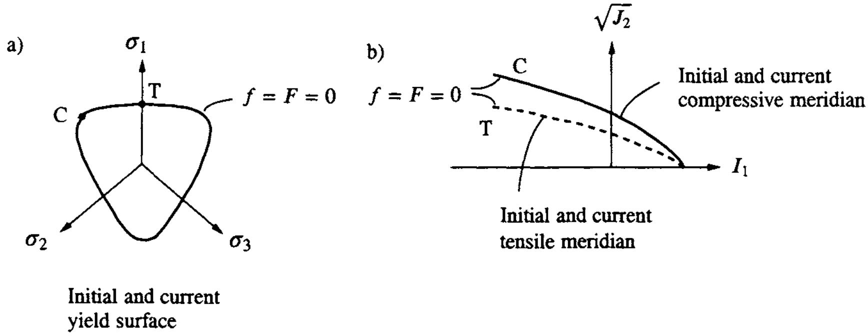

Figure 9.10: Ideal plasticity where the yield surface remains fixed; a) deviatoric plane; b) meridian plane.

surface is unaffected by the plastic deformations, i.e. it remains fixed in the stress space. This situation is illustrated in Fig. 9.10, where C refers to the compression meridian and T to the tensile meridian, cf. Fig. 8.6. In that case, no hardening parameters exist, i.e. (9.8) reduces with (9.9) to

$$\boxed{f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij}) = 0\,; \quad \textit{ideal plasticity}} \tag{9.15}$$

i.e. the current yield surface coincides with the initial yield surface. We conclude that

For ideal plasticity, the yield surface remains fixed in the stress space

Let us next assume that the shape and position of the yield surface remain fixed whereas the size of the yield surface changes. This situation is called *isotropic hardening* and is usually attributed to Hill (1950). As an example, we may consider the von Mises criterion where the initial yield surface is given by

$$F(\sigma_{ij}) = \sqrt{3J_2} - \sigma_{yo} = 0 \tag{9.16}$$

cf. (8.26). We may accomplish isotropic hardening by writing

$$f(\sigma_{ij}, K_\alpha) = \sqrt{3J_2} - \sigma_{yo} - K = 0 \tag{9.17}$$

where, for convenience, we have assumed that only one hardening parameter, K, controls the change of size of the yield surface. In turn, this implies the existence of only one internal parameter κ. The function $K(\kappa)$ describes how the size of the yield surface changes with the development of plastic strains and in accordance with (9.7) and (9.11) we have $K(0) = 0$ so that (9.17) reduces

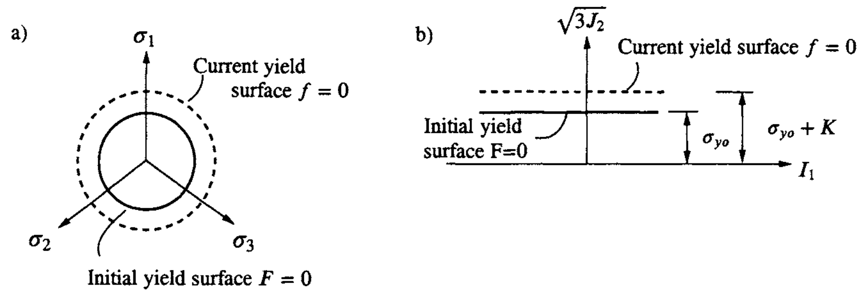

Figure 9.11: Isotropic hardening of the von Mises criterion; a) deviatoric plane; b) meridian plane.

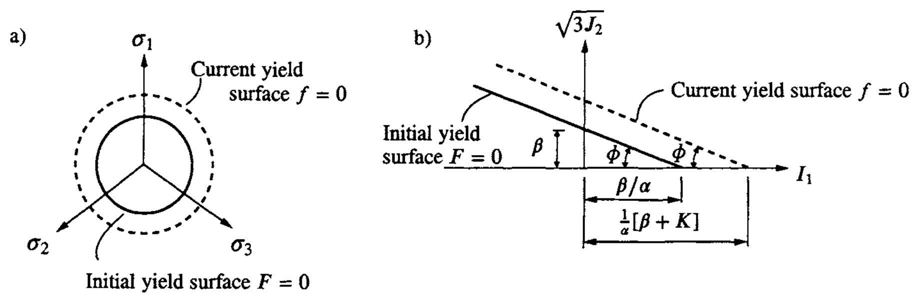

Figure 9.12: Isotropic hardening of the Drucker-Prager criterion; a) deviatoric plane; b) meridian plane.

to (9.16) before the development of plastic strains. Instead of the formulation (9.17), we may write

$$f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij}) - K = 0 \tag{9.18}$$

Isotropic hardening of the von Mises criterion is shown in Fig. 9.11. In this figure, the yield surface expands with increasing plastic deformation and this increase of the current yield stress evidently corresponds to the case of hardening plasticity illustrated in Figs. 9.1 and 9.3a). Mathematically, this is obtained by letting the function $K(\kappa)$ in (9.18) increase with increasing plastic deformation. It is of interest that if, at some stage, we let the function $K(\kappa)$ decrease with increasing plastic deformation then the von Mises surface shrinks in size and this decrease of the current yield stress corresponds to softening plasticity as illustrated in Fig. 9.4. By tradition, the terminology here is somewhat vague since we have achieved softening plasticity by means of the isotropic 'hardening' concept.

As the next example of isotropic hardening, consider the Drucker-Prager criterion. Referring to (8.33), the initial yield surface is here given by

$$F(\sigma_{ij}) = \sqrt{3J_2} + \alpha I_1 - \beta = 0$$

where α and β are parameters and α is dimensionless. We observe that if $\alpha = 0$ then the Drucker-Prager criterion reduces to the von Mises criterion of (9.16). The interpretation of the parameters α and β is illustrated in Fig. 9.12b). To obtain an isotropic hardening concept for the Drucker-Prager criterion, we recall that isotropic hardening is characterized by the shape and position of the yield surface being fixed while the size of the yield surface changes. Referring to the interpretation of the parameters α and β in Fig. 9.12b) we therefore obtain isotropic hardening by the formulation

$$f(\sigma_{ij}, K_\alpha) = \sqrt{3J_2} + \alpha I_1 - \beta - K = 0 \tag{9.19}$$

where the single hardening parameter, K, only depends on one internal variable κ, i.e. $K = K(\kappa)$. This isotropic hardening formulation is illustrated in Fig. 9.12 and we observe that it is possible to write (9.19) as

$$f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij}) - K = 0$$

i.e. a format identical to that achieved for isotropic von Mises hardening, cf. (9.18).

With this discussion, we may generally formulate isotropic hardening for an arbitrary yield function as

$$\boxed{f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij}) - K = 0\,; \qquad \textit{isotropic hardening}} \tag{9.20}$$

which may be expressed as

For isotropic hardening, the position and shape of the yield surface remain fixed whereas the size of the yield surface changes with plastic deformation

Returning to isotropic hardening of the von Mises criterion, it is obvious that we may write (9.17) as

$$\sqrt{3J_2} - \sigma_y(\kappa) = 0\,; \qquad \sigma_y(\kappa) = \sigma_{yo} + K(\kappa) \tag{9.21}$$

where σ_y is the current yield stress. For uniaxial loading, (9.21) reduces to $|\sigma| = \sigma_y$. As illustrated in Fig. 9.13a), this implies that if we reverse the loading from point A where $\sigma = \sigma_y$, the isotropic hardening model will predict elastic unloading until we reach point B where $\sigma = -\sigma_y$. As a result, even after plastic strains have developed, the isotropic hardening model of von Mises predicts the same yield stress in tension and in compression.

This prediction does not agree well with experimental results for metals and steel. Referring to Fig. 9.13b), experimental results show that point B, where plastic effects are again encountered, occurs much earlier than that predicted by the isotropic hardening model. This phenomenon was first observed by Bauschinger (1886) and is therefore called the *Bauschinger effect*. Let us now see how we can approximate this effect.

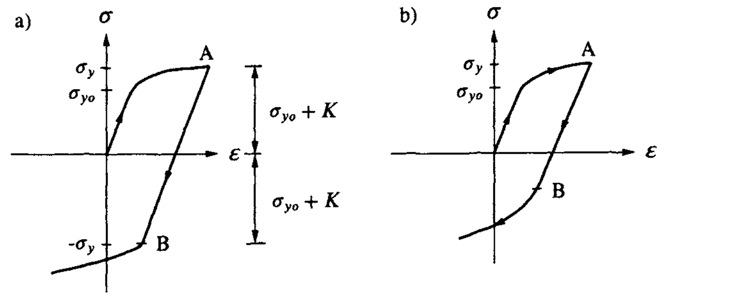

Figure 9.13: a) isotropic hardening; b) Bauschinger effect.

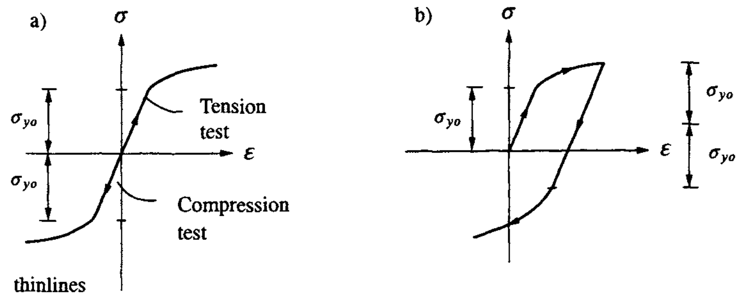

Figure 9.14: Metals and steel; a) tension and compression test; b) kinematic hardening model for uniaxial loading.

For metals and steel and in accordance with the von Mises model, the initial yield stress is the same whether we load in tension or in compression, cf. Fig. 9.14a). That means, the difference between these two yield points is $2\sigma_{yo}$. In an effort to approximate the Bauschinger effect, we assume that the difference between the two yield points is maintained at the value of $2\sigma_{yo}$ even after plastic deformations have occurred. This assumption is illustrated in Fig. 9.14b) and it appears to be a reasonable approximation of the real material behavior shown in Fig. 9.13b). This approximation to hardening is called *kinematic hardening* and it was introduced by Melan (1938) and later by Prager (1955).

Let us see how we can formulate the kinematic hardening assumption within a von Mises concept. Let us first rewrite the initial von Mises criterion (9.16) by using the definition (3.16) of the invariant J_2. This provides

$$F(\sigma_{ij}) = (\frac{3}{2} s_{ij} s_{ij})^{1/2} - \sigma_{yo} = 0 \tag{9.22}$$

In accordance with Fig. 9.14, we can assume that the size and form of the yield surface are unchanged during plastic loading. We are therefore left with the

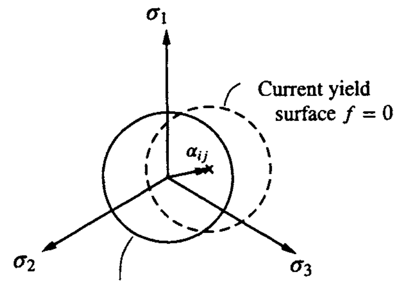

Figure 9.15: Deviatoric plane; kinematic hardening of the von Mises criterion.

possibility that the position of the yield surface changes as a result of the plastic loading. Let the center of the von Mises surface in the deviatoric plane be described by the tensor α_{ij}, where the initial value of α_{ij} is zero. Then we can accomplish our objective by writing the current yield surface as

$$f(\sigma_{ij}, K_\alpha) = \left[\frac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})\right]^{1/2} - \sigma_{yo} = 0 \tag{9.23}$$

which clearly reduces to (9.22) when $\alpha_{ij} = 0$. We see that there is only one hardening parameter and that it takes the form of the tensor α_{ij}. Often, the parameter α_{ij} is called the *back-stress*. This terminology refers to the deviatoric stresses being referred 'back' to the center α_{ij}. The kinematic hardening model of a von Mises material is illustrated in Fig. 9.15 which shows that the only thing that happens to the initial yield surface is that it moves as a rigid body in the stress space due to the plastic deformations.

Generalizing these ideas to arbitrary yield functions, we see that kinematic hardening is modelled by

$$\boxed{f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij} - \alpha_{ij}) = 0 \; ; \quad \textit{kinematic hardening}} \tag{9.24}$$

where we have one hardening parameter in terms of the tensor α_{ij} which describes the position of the current yield surface. Equation (9.24) may be expressed as

For kinematic hardening, the size and shape of the yield surface remain fixed whereas the position of yield surface changes with plastic deformation

Most materials consist of different constituents; concrete is a mixture of aggregate in a matrix of cement paste whereas metals and steel consist of polycrystals. When loading such a material into the plastic regime, the differences in stiffness and yield properties of the constituents imply that they experience

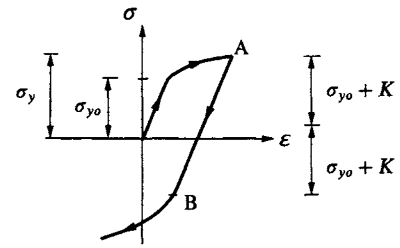

Figure 9.16: Uniaxial loading; mixed hardening of a von Mises material.

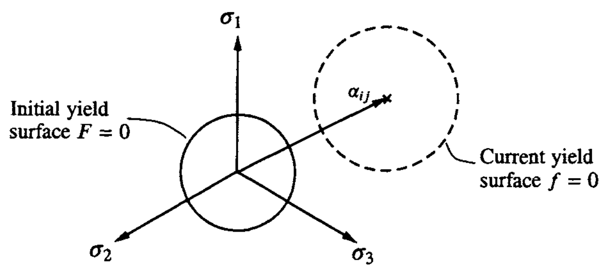

Figure 9.17: Deviatoric plane; mixed hardening of a von Mises material.

different plastic straining. When reversing the loading, this 'mismatch' of the constituents means that some constituents again enter the plastic regime before others and this manifests itself on the macrolevel as the Bauschinger effect and thereby kinematic hardening.

The two classic hardening rules: isotropic and kinematic hardening, may be combined into what is called *mixed hardening*. This concept was introduced by Hodge (1957). For the von Mises criterion, combination of the isotropic hardening given by (9.17) and the kinematic hardening expressed by (9.23) lead to

$$f(\sigma_{ij}, K_\alpha) = \left[\frac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})\right]^{1/2} - \sigma_{yo} - K = 0 \tag{9.25}$$

where the set of hardening parameters K_α consists of α_{ij} and K, i.e. $K_\alpha = \{\alpha_{ij}, K\}$. It appears that (9.25) for $K = 0$ reduces to kinematic hardening given by (9.23) whereas (9.25) for $\alpha_{ij} = 0$ reduces to isotropic hardening given by (9.17).

With reference to Figs. 9.13a) and 9.14b) the response predicted by mixed hardening in the case of uniaxial loading is illustrated in Fig. 9.16. Moreover, Fig. 9.17 shows the evolution of the yield surface in the deviatoric plane. Generalizing the concept of mixed hardening to an arbitrary yield function, we obtain

$$\boxed{f(\sigma_{ij}, K_\alpha) = F(\sigma_{ij} - \alpha_{ij}) - K = 0\,; \quad \textit{mixed hardening}} \tag{9.26}$$

where the set of hardening parameters K_α both consists of the back-stress α_{ij} and the hardening parameter K, i.e.

$$K_\alpha = \{\alpha_{ij}, K\}$$

Equation (9.26) may be expressed as

$$\boxed{\begin{array}{l}\textit{For mixed hardening, i.e. a combination of isotropic}\\ \textit{and kinematic hardening, the shape of the yield}\\ \textit{surface remains fixed whereas the size and position}\\ \textit{of the yield surface change with plastic deformation}\end{array}} \qquad (9.27)$$

In principle, it is possible to allow for a change not only in the size and position of the yield surface, but even of its shape. In that case, one speaks of *distortional hardening* - occasionally called *anisotropic hardening* - but for simplicity this more advanced mixed hardening concept shall not be treated here. The interested reader may, for instance, consult Baltov and Sawczuk (1965) and Axelsson (1979).

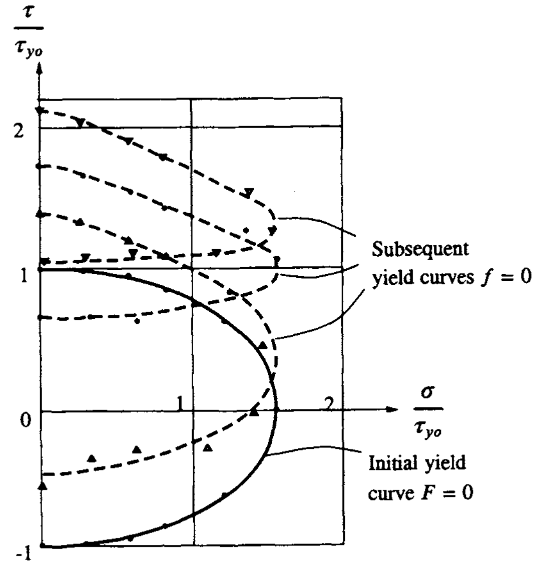

Figure 9.18: Aluminum alloy tested in combined torsion and tension by Ivey (1961); τ_{yo} is the initial yield stress in pure shear.

With this discussion it may be of interest to see what experimental results for metals and steel indicate. In Fig. 9.18, an aluminum alloy was tested in combined torsion and tension by Ivey (1961). In accordance with the von Mises criterion for such stress states, cf. (8.29), the initial yield curve can be described by an ellipse. The three subsequent yield curves show a pronounced

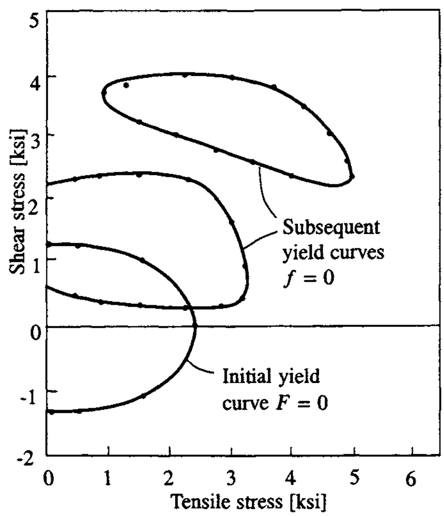

Figure 9.19: Pure aluminum tested in combined torsion and tension by Phillips and Tang (1972).

kinematic hardening as well as effects of isotropic hardening (or rather isotropic 'softening'). However, even an distortion of the form of the yield curve can be observed.

Figure 9.19 shows the results of Phillips and Tang (1972) for pure aluminum also tested in combined torsion and tension, but for a different load path than that used in Fig. 9.18. Again the kinematic hardening effect is pronounced and some isotropic hardening (softening) as well as distortion of the yield curve may be observed. For further experimental evidence and a historical account, the reader may consult Michno and Findley (1976).

In practice, it is seldom that models are used that go beyond combinations of isotropic and kinematic hardening. With reference to Figs. 9.18 and 9.19, this may seem to be a rather crude approximation. However, one should be aware of the fact that experimentally determined yield curves are highly sensitive to how the yield limit is defined. A detailed discussion of this aspect is given by Axelsson (1979). The yield limit is usually defined as some kind of deviation from linear response, but the threshold value used to identify this deviation influences the yield curve obtained most significantly. This means that for the same observed stress-strain curves, the position of the yield curve depends significantly on the experimental procedure. In practice, one is more interested in an accurate prediction of the stress-strain curves than in the determination of the yield curve as such. This implies that combined isotropic and kinematic hardening in most cases allows a prediction that is of sufficient engineering accuracy and, in practice, either isotropic or kinematic hardening is even adopted

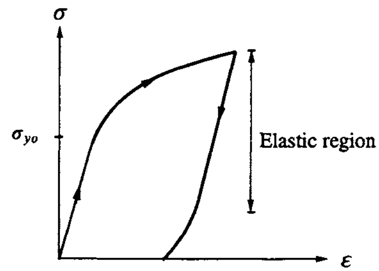

Figure 9.20: Due to the effect of kinematic hardening, plastic strains may develop during the unloading phase

and considered to be sufficiently accurate for not too complicated load paths. In general, isotropic hardening is adopted if the loading mainly increases whereas kinematic hardening is used when reversed loadings, i.e. cyclic loadings, are of interest.

Another point of interest is that both Figs. 9.18 and 9.19 show that the origin of the stress space may be located outside the current yield surface. When the material is completely unloaded, this implies that plastic strains develop during the unloading phase. This behavior, which at first sight may seem surprising, is illustrated in Fig. 9.20.

9.2 Development of plastic strains - Introductory remarks

Previously, we have discussed the yield surface and its change during plastic deformation, i.e. the hardening rule, in great detail. Let us now turn to the important issue of determining the development of the plastic strains.

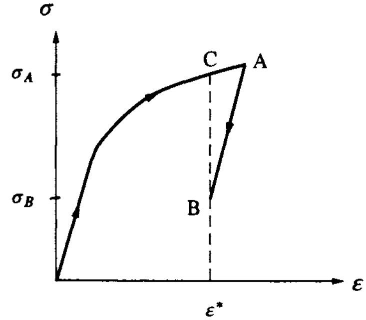

Figure 9.21: For a given strain ε^*, the corresponding stress is unknown unless we know the load history.

Consider the uniaxial loading in Fig. 9.21 where we unload to point B where the strain is ε^*. It is obvious that if only the strain value ε^* is known, we do not know whether the corresponding stress is σ_B or σ_C. We conclude that in plasticity, no unique relation exists between the stress state σ_{ij} and the strain state ε_{ij}. Therefore, the constitutive relation for elasto-plasticity must be of an incremental nature. This means that for a given strain state the corresponding stress state is obtained by an integration of the incremental constitutive relations and the result of this integration will depend on the integration path, i.e. the load history. This load history dependence is illustrated in Fig. 9.21.

Referring to (9.1), the total strains ε_{ij} are assumed to consist of the elastic and plastic strains, i.e.

$$\boxed{\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p} \tag{9.28}$$

The elastic strains are determined by Hooke's law (4.20) or (4.26) i.e.

$$\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl}^e \quad \text{or} \quad \varepsilon_{ij}^e = C_{ijkl}\sigma_{kl}}$$

Therefore, the incremental constitutive relation must be related to the increment $\dot{\varepsilon}_{ij}^p$ of the plastic strains. We recall that a dot denotes the time derivative, i.e. $\dot{\varepsilon}_{ij}^p = d\varepsilon_{ij}^p/dt$ and since we want to establish a plasticity theory that is independent of the loading rate, i.e. time does not influence the response, we merely use the format $\dot{\varepsilon}_{ij}^p$ instead of $d\varepsilon_{ij}^p$ to simplify the notation. This discussion is similar to the one given in Chapter 7.

In order to motivate the constitutive relationship for the plastic strains it turns out to be instructive to follow the historical development of the plasticity theory adopting a broad conceptual viewpoint.

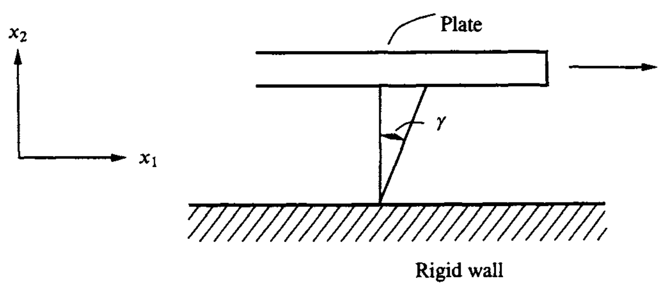

Figure 9.22: Movement of a plate in a Newton fluid.

In the 19th century, the scientific community was concerned with the simplest possible plasticity model, namely that of rigid–ideal plasticity, cf. Fig. 9.2a). With the word 'yielding' of materials, it seems tempting to associate this behavior with the response of viscous fluids. In Fig. 9.22, a plate is moved in a viscous fluid close to a rigid wall. Assuming the viscous fluid to be a linear Newton fluid, the shear stress τ in the fluid is then given by

$$\dot{\gamma} = \frac{\tau}{\mu}$$

where μ is the viscosity of the fluid, i.e.μ has the dimension $[Pa \cdot s]$ and γ is the engineering shear strain. We may also refer to the behavior of the dashpot shown in Fig. 6.1b) which obeys a similar constitutive relation, cf. (6.26). Referring to Fig. 9.22, we have $\gamma = 2\varepsilon_{12}$ and $\tau = s_{12}$ i.e. the relationship above may be written as $\dot{\varepsilon}_{12} = s_{12}/(2\mu)$. Generalizing this relation we obtain

$$\dot{\varepsilon}_{ij} = \frac{s_{ij}}{2\mu} \tag{9.29}$$

With this discussion, and considering rigid-ideal plastic materials where the elastic strains are zero, it is tempting to assume that the constitutive relation for such materials is given by

$$\dot{\varepsilon}_{ij} = \dot{\beta} s_{ij}; \qquad \dot{\beta} \geq 0 \tag{9.30}$$

Whereas (9.29) depends on time, we have by introduction of the quantity $\dot{\beta} = d\beta/dt$ ensured that (9.30) is independent of time, i.e. (9.30) is an expression that is homogeneous in dt. Since μ in (9.29) is positive, we introduce the constraint $\dot{\beta} \geq 0$ in (9.30) where $\dot{\beta} = 0$ implies that no strains develop; apart from that, the quantity $\dot{\beta}$ is at this stage unknown. It is of interest that the formulation (9.30) is precisely the formulation proposed by Saint-Venant (1870) for plane strain and by Lévy (1870) for general conditions. Later, von Mises (1913) also arrived at the same constitutive law which often is called the *Lévy-von Mises equations.*

Later on, interest shifted towards the behavior of elastic-ideal plastic materials and with (9.28) and (9.30) it seems natural to assume that

$$\dot{\varepsilon}_{ij}^{p} = \dot{\beta} s_{ij}; \qquad \dot{\beta} \geq 0 \tag{9.31}$$

and this format was suggested by Prandtl (1924) for two dimensions and by Reuss (1930) for three dimensions; expression (9.31) is therefore called the *Prandtl-Reuss equations.* We observe that $\dot{\beta} = 0$ implies that plastic strains do not develop. In general, a constitutive relation for $\dot{\varepsilon}_{ij}^{p}$ is called a *flow rule* and (9.31) is an example of a flow rule. It is of considerable interest that (9.31) implies that the plastic volumetric strain is $\dot{\varepsilon}_{ii}^{p} = 0$, i.e. plastic incompressibility and this property is in very close agreement with the behavior of metals and steels. For such materials, the experimental observation that the plastic response only depends on the deviatoric part of the stresses is also reflected by the flow rule (9.31).

In solid mechanics, as well as in other branches of mechanics, many problems may be formulated by means of a potential function. This means that one quantity is obtained by differentiation of a scalar function, the *potential function.*

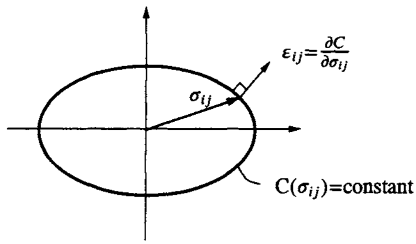

Figure 9.23: Linear elasticity. Normality of strain tensor ε_{ij} to the surface in the stress space described by $C(\sigma_{ij})$ = constant.

Well known examples are Airy's stress function for two-dimensional elasticity problems and Prandtl's stress function for torsion of non-circular elastic shafts.

In Chapter 4, we also encountered such a potential function, namely the complementary energy $C(\sigma_{ij})$ from which according to (4.16) we obtain the strains by a differentiation, i.e.

$$\varepsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}} \tag{9.32}$$

and this relation is characteristic for hyper-elasticity. For linear elasticity, the complementary energy C is given by (4.30) i.e.

$$C = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl} > 0 \tag{9.33}$$

which proves that the flexibility tensor C_{ijkl} is positive definite. From (9.32) and (9.33) we conclude that

$$\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\frac{\partial C}{\partial \sigma_{ij}} > 0 \tag{9.34}$$

If we consider the expression $C(\sigma_{ij})$ = constant, then this expression describes a surface in the stress space as illustrated in Fig. 9.23. According to (9.32), the strain tensor is orthogonal to this surface and following (9.34) the scalar product $\sigma_{ij}\varepsilon_{ij}$ is positive, i.e. ε_{ij} is directed outwards, as shown in Fig. 9.23. We have observed that the strain tensor ε_{ij} is normal to the surface $C(\sigma_{ij})$ = constant. Let us next prove that C is *convex*. For a one-dimensional function $g(x)$, convexity requires that $d^2g/dx^2 > 0$, cf. Fig. 9.24. For the multidimensional function $C(\sigma_{ij})$, the requirement of convexity is that the quantity $\partial^2 C/\partial\sigma_{ij}\partial\sigma_{kl}$ is positive definite, cf. for instance the Appendix. From Hooke's law $\varepsilon_{ij} = C_{ijkl}\sigma_{kl}$ and (9.32) we obtain

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = C_{ijkl} = \frac{\partial^2 C}{\partial \sigma_{kl}\partial \sigma_{ij}}$$

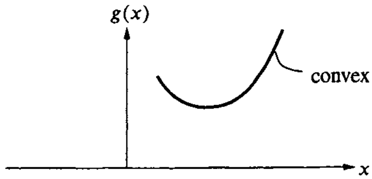

Figure 9.24: Convex function in one dimension.

Since C_{ijkl} is positive definite so is $\partial^2 C/\partial\sigma_{ij}\partial\sigma_{kl}$, i.e. C is convex.

We have discussed hyper-elasticity in some detail and we have shown that the concepts of a potential function, normality and convexity apply for this group of materials. We shall now see that similar concepts hold for the plasticity theory.

Previously, we established the Prandtl-Reuss equations given by (9.31) and we have argued for the natural wish to try to formulate problems in terms of potential functions. Around 1930, the von Mises criterion was well established and since this criterion provides a scalar function in terms of the yield function, it seems natural to investigate whether it can be used as a potential function. According to (9.21) we have

$$f(\sigma_{ij}, K) = \sqrt{3J_2} - \sigma_y(\kappa)\ ; \qquad f(\sigma_{ij}, K) = 0$$

where $\sigma_y(\kappa) = \sigma_{yo} + K(\kappa)$. Differentiation gives

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sqrt{3J_2}} = \frac{3}{2}\frac{s_{ij}}{\sigma_y}$$

It is of interest that this expression and (9.31) may be combined into

$$\boxed{\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}; \qquad \dot{\lambda} \geq 0} \tag{9.35}$$

where $\dot{\lambda} = 2\sigma_y\dot{\beta}/3$, i.e. the incremental plastic strains can be derived by using the von Mises function as a potential function.

For stress states inside the yield surface, we have $f < 0$, i.e. $\partial f/\partial\sigma_{ij}$ is normal to the yield surface $f = 0$ and directed outwards. Referring to Fig. 9.25, we have then established the important property of $\dot{\varepsilon}^p_{ij}$ being normal to the yield surface. Moreover, the von Mises surface is convex and as discussed in Chapter 8, experimental evidence shows that yield surfaces are, in general, convex.

The above discussion has led us to the important concept of *normality* expressed by the flow rule given by (9.35) in which the yield surface serves as a *potential function* for the determination of the incremental plastic strains. This flow rule is therefore called the *associated* flow rule since the yield criterion is

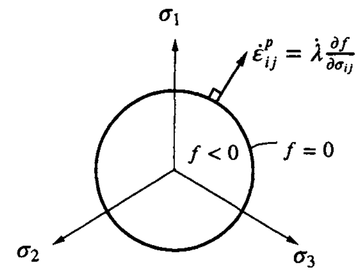

Figure 9.25: Normality of the incremental plastic strains to the von Mises surface in the deviatoric plane.

associated with - i.e. taken as - the potential function. Moreover, the *convexity* of the yield surface has also been discussed.

The flow rule (9.35) is simple and elegant and it was derived using the von Mises criterion as potential function. In principle however, it can be used in combination with any yield function and it was, in fact, proposed by von Mises (1928). We observe that only the direction of the incremental plastic strains is given by (9.35) and also to determine their magnitude we need to determine the so-called *plastic multiplier* $\dot{\lambda}$. This topic will be discussed later on.

Before turning to this point, it may be argued that the background for the establishment of associated flow rule (9.35) seems rather vague. However, since the use of this rule is elegant and provides close agreement with experimental results for metals and steel there have – over the years – been many attempts to strengthen the background for this flow rule. One of the most important attempts was proposed by Drucker (1951) and for that reason we will scrutinize his suggestion.

9.3 Drucker's postulate and its consequences

We have already touched upon what is meant by hardening and softening plasticity, and for uniaxial loading these phenomena are illustrated in Figs. 9.3 and 9.4. To obtain definitions applicable to general stress states, we will adopt the proposal of Drucker (1951, 1964) and it will turn out that this postulate leads to the associated flow rule as well as to the convexity of the yield surface.

Drucker's postulate makes use of a *stress cycle* and to illustrate this concept, we consider uniaxial loading of a hardening material, cf. Fig. 9.26a). First the material has been loaded to point B and then unloaded elastically to point A. The state indicated by point A with the stress σ^* is now considered as the existing state of the material. We now imagine that an additional load is first applied to the material; this brings us to point B with the stress σ. The additional load is now increased by the infinitesimal amount $d\sigma$ and this brings us to point C with

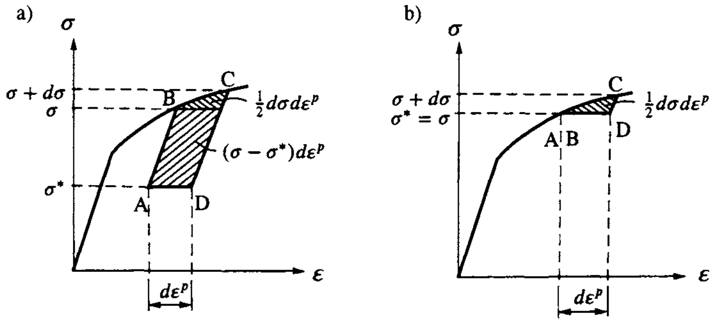

Figure 9.26: Hardening plasticity in uniaxial tension; a) stress cycle when the starting point A is below the current yield stress; b) stress cycle when the starting point A coincides with the current yield stress.

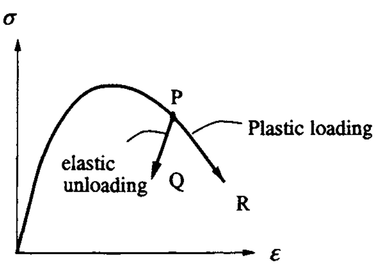

Figure 9.27: Softening material in uniaxial tension.Illustration of elastic unloading and plastic loading.

the stress $\sigma + d\sigma$. Then the entire additional load is removed and the material therefore unloads elastically to point D with the stress σ^* equal to the stress at point A. It appears that the additional load has carried the material through a complete *stress cycle*. This additional load which carries the material through a complete stress cycle is occasionally called an *external agency*.

As a special example of a stress cycle, we may imagine that points A and B coincide, cf. Fig. 9.26b). Also in this case is it possible for the material to go through a complete stress cycle by first applying a small additional load and then removing this load.

We have seen that it is always possible to let a hardening material go through a complete stress cycle and it is of interest to investigate whether the same is possible for a softening material. Before doing so we consider a softening material that has been loaded to point P in Fig. 9.27. It is of importance that a stress larger than the stress at point P can never be achieved. At point P, the material can only respond in two manners: either it unloads elastically towards point Q by decreasing the stress or strain or it responds plastically from point

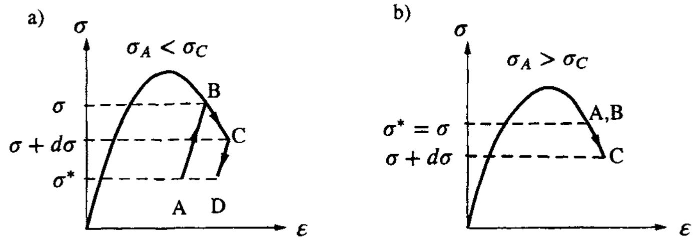

Figure 9.28: Softening material in uniaxial tension; a) stress cycle is possible; b) stress cycle is not possible.

P towards point R. However, such plastic loading can only occur if we force the total strain to increase. In particular, we cannot reach point R from point P either by prescribing an increasing or decreasing stress; increasing the stress is impossible and prescribing a decreasing stress would imply elastic unloading along the path PQ. We therefore conclude that

$$\boxed{\text{\textit{Elasto-plastic loading of a softening material can only be accomplished by strain control}}} \qquad (9.36)$$

This is in direct contrast to a hardening material where plastic loading can be accomplished by either prescribing an increasing stress or an increasing strain and we shall later see that this difference has important consequences for the establishment of general loading and unloading criteria, cf. the discussion relating to (10.38). Moreover, conclusion (9.36) has implications when testing softening materials in a testing machine. The only possibility for an experimental determination of the softening branch of the stress-strain curve is by prescribing an increasing strain, i.e. prescribing an increasing displacement on the specimen. This means that the testing machine must be operated in a *displacement controlled* manner.

With this discussion in mind, we consider a softening material and the situation shown in Fig. 9.28a) where $d\sigma$ is negative. Moreover, $\sigma_A < \sigma_C$ and by strain controlling the loading, it is possible for the additional loading - the external agency - to carry the material through a complete stress cycle ABCD. However, if point A and B coincide, cf. Fig. 9.28b), then by increasing the strain we reach point C where $\sigma_C < \sigma_A = \sigma_B$ and we have lost the possibility of reaching the stress given by $\sigma_A = \sigma_B$ irrespective of what load path we imagine. Consequently, a softening material in the state given by Fig. 9.28b) is not able to go through a stress cycle.

We have seen that the possibility for the material to go through a complete stress cycle is a property that distinguishes hardening plasticity from softening plasticity. Let us therefore return to the stress cycle for a hardening material shown in Fig. 9.26 and evaluate the work that is required by the additional load, i.e. by the external agency, to force the material to go through a stress cycle. Letting $W_{ex.ag.}$ denote the work per unit volume performed by the external agency during a complete stress cycle, we have

$$W_{ex.ag.} = \int_{ABCD} (\sigma - \sigma^*) d\varepsilon$$

The strain increment $d\varepsilon$ consists of its elastic and plastic components, i.e. $d\varepsilon = d\varepsilon^e + d\varepsilon^p$ and as plastic strains only develop during load path BC, we obtain

$$W_{ex.ag.} = \int_{ABCD} (\sigma - \sigma^*) d\varepsilon^e + \int_{BC} (\sigma - \sigma^*) d\varepsilon^p$$

As $d\varepsilon^e = d\sigma / E$, where Young's modulus E is assumed to be constant, and as σ^* is a constant quantity, we obtain

$$W_{ex.ag.} = \frac{1}{E} \int_{ABCD} \sigma d\sigma - \frac{\sigma^*}{E} \int_{ABCD} d\sigma + \int_{BC} (\sigma - \sigma^*) d\varepsilon^p$$

The path ABCD corresponds to a complete stress cycle and the contribution from the first two terms is therefore equal to zero. Denoting the plastic strain increment from B to C by $d\varepsilon^p$ ($d\varepsilon^p > 0$) and using the trapezoidal rule to the last term we obtain

$$W_{ex.ag.} = \frac{1}{2} \left[(\sigma - \sigma^*) + (\sigma + d\sigma - \sigma^*) \right] d\varepsilon^p = (\sigma - \sigma^*) d\varepsilon^p + \frac{1}{2} d\sigma d\varepsilon^p$$

The quantities appearing in this expression are illustrated in Fig. 9.26, where $\sigma > \sigma^*$ corresponds to Fig. 9.26a) whereas $\sigma = \sigma^*$ corresponds to Fig. 9.26b). It is obvious that we have $W_{ex.ag.} > 0$, i.e. the external agency must perform a positive work in order to force the hardening material to go through a stress cycle.

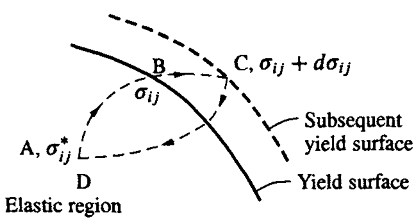

Figure 9.29: Stress cycle for hardening material produced by external agency.

To extend these observations to general stress states, we adopt the postulate of Drucker (1951, 1964). Again the external agency slowly applies an additional set of stresses to the already stressed material, thereby creating plastic strains and then slowly removing the added set. For hardening plasticity, the work done by the external agency during this stress cycle is postulated to be positive. The situation is sketched in Fig. 9.29, where σ_{ij}^* is the existing stress state at point A located on or inside the current yield surface. Additional stresses are then applied bringing the stress state to the yield surface at point B having the stress state σ_{ij}. Only changes in the elastic strains have occurred so far. Now, due to the external agency, a small stress increment $d\sigma_{ij}$ takes the stress state to point C on the subsequent yield surface thereby creating incremental plastic strains as well as incremental elastic strains. The external agency then releases the applied additional stresses thereby bringing the stress state back to the original stress state σ_{ij}^* along some stress path CD not necessarily coinciding with path ABC. According to the postulate of Drucker, the work done by the external agency during the stress cycle defined must be positive for hardening plasticity, i.e.

$$W_{ex-ag.} = \int_{ABCD} (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij} > 0$$

where $d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p$. Observing that plastic strains only occur during load path BC, and as σ_{ij}^* is a constant tensor, we obtain

$$W_{ex.ag.} = \int_{ABCD} \sigma_{ij} d\varepsilon_{ij}^e - \sigma_{ij}^* \int_{ABCD} d\varepsilon_{ij}^e + \int_{BC} (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p > 0$$

The first integral expresses the change of the elastic strain energy over the stress cycle considered and therefore this integral evidently becomes equal to zero. Likewise, as the elastic strains are determined entirely by Hooke's law $\varepsilon_{ij}^e = C_{ijkl}\sigma_{kl}$, the elastic strains before and after the stress cycle are equal; this implies that the second term also becomes equal to zero (provided that C_{ijkl} is constant). Denoting the plastic strain increments developing along path BC by $d\varepsilon_{ij}^p$ and letting $d\sigma_{ij}$ denote the corresponding stress increments whereas σ_{ij} denotes the stress state at point B, application of the trapezoidal rule to the last term implies that

$$W_{ex.ag.} = \frac{1}{2}\left[(\sigma_{ij} - \sigma_{ij}^*) + (\sigma_{ij} + d\sigma_{ij} - \sigma_{ij}^*)\right] d\varepsilon_{ij}^p > 0$$

i.e.

$$W_{ex.ag.} = (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p + \frac{1}{2} d\sigma_{ij} d\varepsilon_{ij}^p > 0 \tag{9.37}$$

If we choose $\sigma_{ij}^* = \sigma_{ij}$ then (9.37) implies

$$\boxed{d\sigma_{ij} d\varepsilon_{ij}^p > 0 \quad \textit{for associated hardening plasticity}} \tag{9.38}$$

cf. also Fig. 9.26b) As already indicated, we shall see later that (9.38) holds only for associated plasticity of a hardening material.

Let us return to (9.37) where we now know that the term $\frac{1}{2}d\sigma_{ij}d\varepsilon_{ij}^p$ is positive. However, if we choose σ_{ij}^* to be sufficiently different from σ_{ij} then the term $\frac{1}{2}d\sigma_{ij}d\varepsilon_{ij}^p$ becomes of second order providing

$$\boxed{(\sigma_{ij} - \sigma_{ij}^*)d\varepsilon_{ij}^p \geq 0} \tag{9.39}$$

cf. also Fig. 9.26a) We shall now see that (9.39) has remarkable implications. Consider the nine-dimensional coordinate system defined by all the stress com-

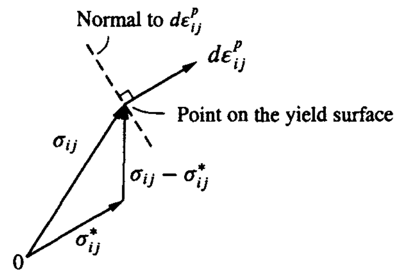

Figure 9.30: Convexity of yield surface in the stress space.

ponents and let the plastic strain increments be depicted in this coordinate system, then (9.39) states that the scalar product of the vectors $\sigma_{ij} - \sigma_{ij}^*$ and $d\varepsilon_{ij}^p$ is never obtuse. As σ_{ij} is located on the yield surface and as σ_{ij}^* is an arbitrary stress state on or within the yield surface, this implies that all points σ_{ij}^*, on or within the yield surface, must be located in the space opposite to the vector $d\varepsilon_{ij}^p$ and limited by the plane normal to $d\varepsilon_{ij}^p$, cf. Fig. 9.30. This implies that the yield surface must be *convex* in the stress space. Moreover, if the yield surface is assumed to be smooth, then the convexity of the yield surface and (9.39) imply that $d\varepsilon_{ij}^p$ must be normal to the yield surface, i.e.

$$\boxed{d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \; ; \quad d\lambda \geq 0} \tag{9.40}$$

which is exactly the associated flow rule established previously in (9.35). Therefore, Drucker's postulate for hardening plasticity implies the following two very important points: *convexity* of the yield surface as well as the *normality principle* stated by (9.40). In addition, we shall later see that fulfillment of Drucker's postulate also ensures the *uniqueness* of the elasto-plastic boundary value problem, cf. Chapter 24. We may also note that if the yield surface has a sharp corner - as, for instance, is the case for Tresca's and Coulomb's yield criteria - then (9.39) implies that $d\varepsilon_{ij}^p$ is located somewhere within the 'cone' defined by the two normals at the corner, see Fig. 9.31. In Section 22.5, this property will be utilized to establish the *flow rule of Koiter*. We might finally mention that the conclusion of the yield surface being convex is in close agreement with experimental evidence, cf. Chapter 8.

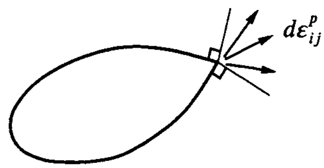

Figure 9.31: Possible directions of $d\varepsilon_{ij}^p$ when the yield surface has a corner.

The fundamental expression (9.39) was derived by Drucker's postulate for hardening plasticity and it implied convexity of the yield surface and the normality principle, i.e. the associated flow rule. It is of interest, however, that (9.39) holds even for ideal and softening plasticity, i.e. it holds even when Drucker's postulate does not hold.

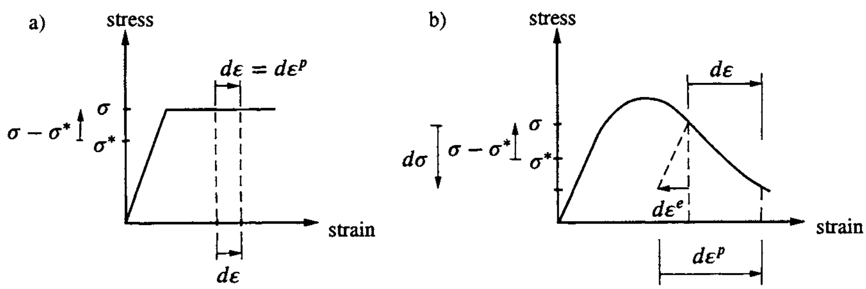

Figure 9.32: Expression (9.39) holds even for ideal and softening plasticity; The strain increment $d\varepsilon$ is imposed on the stress state given by σ.

To see this, consider Fig. 9.32 and recall that the stress σ^* is located inside or on the yield surface. Moreover, we do not consider a stress cycle, but we simply evaluate (9.39) when a strain increment $d\varepsilon > 0$ is imposed on the state with the stress σ. For ideal plasticity, cf. Fig. 9.32a), we have $\sigma - \sigma^* \geq 0$ and $d\varepsilon^p$ is positive, i.e. it follows that (9.39) holds. Considering the softening plasticity case shown in Fig. 9.32b), we have

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p = \frac{d\sigma}{E} + d\varepsilon^p$$

Since $d\varepsilon^e$ is a negative quantity, $d\varepsilon^p$ is certainly a positive quantity and as $\sigma - \sigma^* \geq 0$, (9.39) holds again. Consequently, even though (9.39) was derived for hardening materials using Drucker's postulate, it holds even for ideal and softening plasticity and we are then led to the normality rule and the convexity of the yield surface also for these materials.

It is evident, however, that the manner in which one may argue for the establishment of (9.39) is open to discussion. Already von Mises (1928) observed that if the associated flow rule (9.40) is accepted, then (9.39) follows. On the other hand, (9.39) was directly postulated by Taylor (1947) and Hill (1948b) and considering crystal plasticity it was derived by Bishop and Hill (1951).

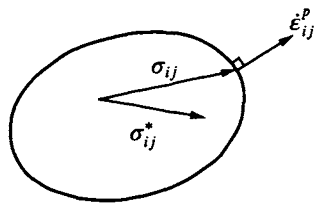

Figure 9.33: Quantities entering the postulate of maximum plastic dissipation.

Expression (9.39) has an interesting interpretation. The *rate of plastic work* related to the stress state σ_{ij} or σ^*_{ij} is defined by

$$\boxed{\dot{W}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij}\ ; \qquad \dot{W}^{p*} = \sigma^*_{ij}\dot{\varepsilon}^p_{ij}} \tag{9.41}$$

respectively. The rate of plastic work is also called the *plastic dissipation*. It appears that (9.39) may be written as

$$\boxed{\dot{W}^p \geq \dot{W}^{p*}} \tag{9.42}$$

We observe that the stress state σ_{ij} is the stress state on the yield surface which in reality is related to the incremental plastic strains $\dot{\varepsilon}^p_{ij}$, cf. Fig. 9.33. Accepting the value for $\dot{\varepsilon}^p_{ij}$, (9.42) shows that the plastic dissipation relating to the real stress (σ_{ij}) is larger than or equal to the plastic dissipation relating to any other stress state (σ^*_{ij}) within or on the yield surface. Therefore, when (9.39) is expressed in terms of (9.42), it is called the *postulate of maximum plastic dissipation*; later, in Chapter 22, we will return to this important postulate and present a more stringent discussion.

We have seen that there are much stronger reasons for accepting the associated flow rule (9.35) than we originally suggested. In fact, for many workers within the field, Drucker's postulate and the postulate of maximum plastic dissipation were accepted over a span of years as being more or less laws of nature. We have shown that there are good arguments for the acceptance of the associated flow rule, but this flow rule is not a necessity. To generalize the flow rule (9.35) we may therefore write

$$\boxed{\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}\ ; \qquad \dot{\lambda} \geq 0} \tag{9.43}$$

where g is a *potential function* that may or may not be equal to the yield function f. In general, we expect g to depend on the same quantities as f, i.e. in accordance with (9.8) we have

$$\boxed{g = g(\sigma_{ij}, K_\alpha)}$$

If $g = f$, we have an *associated flow rule* and if $g \neq f$ we have a *nonassociated flow rule*. The nonassociated flow was suggested by Melan (1938) and later by Prager (1949). Clearly, an associated flow rule simplifies the theory and it should be used if the predictions are in agreement with the experimental findings. This is the case for metals and steel. For frictional materials like concrete, rocks and soils, use of a nonassociated flow rule is most often required in order to obtain realistic predictions

9.4 Consistency relation and evolution laws

Based on experimental evidence, suppose that we have determined the potential function g, which for associated plasticity is taken as the yield function. Then use of the flow rule (9.43) only determines the direction of the incremental plastic strains $\dot{\varepsilon}_{ij}^p$. However, the magnitude of $\dot{\varepsilon}_{ij}^p$ is still unknown since the plastic multiplier $\dot{\lambda}$ is as yet unknown. The next task is therefore to determine this quantity.

It is a fundamental property of plasticity theory that during plastic development, the current stress state is always located on the current yield surface. The current yield surface changes in general during plastic loading, as we have previously discussed, but, by definition, the current stress state is always located on the current yield surface during this evolution.

Having chosen the hardening rule, i.e. the hardening parameters, the current yield function is given in its general form by (9.8), i.e.

$$f(\sigma_{ij}, K_\alpha) = 0 \tag{9.44}$$

where K_α are our, as yet, unspecified hardening parameters. Since $f = 0$ during plastic loading, we can express the so-called *consistency relation* by

$$\boxed{\dot{f} = 0}$$

which with (9.44) and the chain rule lead to

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha}\dot{K}_\alpha = 0 \tag{9.45}$$

where the summation convention is also used for the Greek letter α. The consistency relation was introduced by Prager (1949) and (9.45) tells us that during plastic loading where the stress state varies, also the hardening parameters K_α

vary in such a manner that the stress state always remains on the yield surface. According to (9.12) the hardening parameters K_α depend on the internal variables κ_α, i.e.

$$K_\alpha = K_\alpha(\kappa_\beta) \tag{9.46}$$

The choice of appropriate internal variables κ_α is given by us and also the relation between K_α and κ_α must be prescribed by us according to our experimental and other knowledge. Use of (9.46) in (9.45) provides

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta = 0 \tag{9.47}$$

In (9.47) the increments of the internal variables enter the consistency relation, and the next topic is therefore to establish some laws that tell us how κ_α evolutes with the plastic deformation. These laws are called the *evolution laws* and they must be specified by us based on our experimental knowledge. However, it turns out to be possible to establish the general format of these evolution equations.

A consequence of (9.46) is that $\dot{\kappa}_\alpha$ depends on the hardening parameters K_α. Moreover, since the incremental plastic strains $\dot{\varepsilon}^p_{ij}$ are the primary reason for the change of the internal variables, i.e. the change of the material, we may write

$$\dot{\kappa}_\alpha = a_\alpha(\dot{\varepsilon}^p_{ij}, K_\beta)$$

where a_α denote some functions. With (9.43) and since g depends on σ_{ij} and K_α we obtain

$$\dot{\kappa}_\alpha = a_\alpha(\dot{\lambda},\ \sigma_{ij},\ K_\beta)$$

Since our plasticity theory must be independent of time, we can accomplish this by writing $\dot{\kappa}_\alpha$ as an expression that is a homogeneous function of time. This is obtained by writing

$$\boxed{\dot{\kappa}_\alpha = \dot{\lambda}\ k_\alpha(\sigma_{ij},\ K_\beta)} \tag{9.48}$$

where k_α denote some functions, the *evolution functions*, that must be chosen by us based on experimental or other evidence. Equation (9.48) constitutes our *evolution laws*. Following the flow rule (9.43), no plastic strains develop when $\dot{\lambda} = 0$. In accordance with expectations, (9.48) shows that the internal variables also remain unchanged when $\dot{\lambda} = 0$, cf. also (9.14).

Two sets of evolution equations have now been established: the flow rule $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial g/\partial\sigma_{ij}$ and $\dot{\kappa}_\alpha = \dot{\lambda}k_\alpha$. It seems tempting to expect that the function $g(\sigma_{ij}, K_\alpha)$ not only serves as a potential function for $\dot{\varepsilon}^p_{ij}$, but also for $\dot{\kappa}_\alpha$. In Chapter 22 where the plasticity theory is formulated from the basic principles of thermodynamics, we will, in fact, show that these expectations become fulfilled as it turns out that

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \qquad \text{and} \qquad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial g}{\partial K_\alpha}$$

which means that $k_\alpha = -\partial g/\partial K_\alpha$; for associated plasticity these relations become

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} \qquad \text{and} \qquad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial f}{\partial K_\alpha}$$

However, at the present stage we shall use the more general format given by (9.48).

Insertion of (9.48) into (9.47) leads to

$$\boxed{\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\,\dot{\lambda} = 0} \tag{9.49}$$

where the quantity H is defined by

$$\boxed{H = -\frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}k_\beta} \tag{9.50}$$

This quantity is termed the *generalized plastic modulus* and in the next chapter we will evaluate it in more detail.

If $H \neq 0$ then (9.49) provides

$$\boxed{\dot{\lambda} = \frac{1}{H}\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} \geq 0} \tag{9.51}$$

i.e. the flow rule (9.43) takes the form

$$\boxed{\dot{\varepsilon}_{ij}^p = \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}} \tag{9.52}$$

Based on our experimental and other knowledge we choose the yield function f, the potential function g, relation (9.46) and the evolution functions k_α entering the evolution law (9.48). This implies that the plastic modulus H is also known. Consequently, all quantities on the right-hand side of (9.52) are known, i.e. the incremental plastic strain $\dot{\varepsilon}_{ij}^p$ is known once the incremental stresses $\dot{\sigma}_{ij}$ are given; for evident reasons, formulation (9.52) is called a *stress driven format.* This means that we have, at last, obtained a constitutive relation for the incremental plastic strain. It is of interest that we may even reverse the procedure outlined above by specifying the generalized plastic modulus H as well as all evolution functions except for one quantity. Expression (9.50) then serves as the vehicle to determine the remaining unknown quantity in the evolution law.

In the next chapter, we will derive an expression for $\dot{\varepsilon}_{ij}^p$ that also holds when $H = 0$ and which also expresses $\dot{\varepsilon}_{ij}^p$ in terms of the incremental total strains $\dot{\varepsilon}_{ij}$. This more general formulation is called a *strain driven format.*

9.5 Preliminary loading and unloading criteria

With reference to Fig. 9.21, it is obvious that we must establish criteria which make it possible to decide whether we have plastic loading or elastic unloading. In the next chapter, we will derive such criteria which hold in general, but here we will establish some preliminary rules that hold for hardening plasticity.

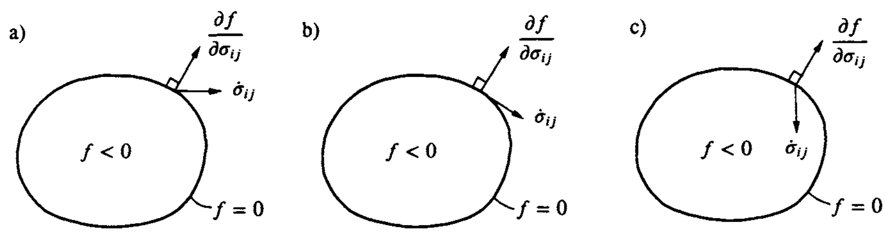

Figure 9.34: a) plastic loading; b) elastic unloading; c) neutral loading.

If the stress state is located inside the yield surface, i.e. $f < 0$, then we can only have elastic behavior. Therefore, evolution of plastic strains requires that $f = 0$. With reference to Fig. 9.34, we can conclude

$$\begin{aligned}
f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} > 0 \quad &\Rightarrow \quad \text{plastic loading} \\
f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} < 0 \quad &\Rightarrow \quad \text{elastic unloading} \\
f = 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = 0 \quad &\Rightarrow \quad \text{neutral loading}
\end{aligned} \tag{9.53}$$

For neutral loading, it follows from (9.51) and (9.53) that $\dot{\lambda} = 0$ and thereby $\dot{\varepsilon}_{ij}^{p} = 0$, i.e. neutral loading results in a purely elastic response even though it may formally be treated as the occurrence of plasticity.

Consider the following two situations: the first where $\dot{\sigma}_{ij}$ is directed into the plastic regime, but infinitely close to neutral loading and the second where $\dot{\sigma}_{ij}$ is directed into the elastic regime, but infinitely close to neutral loading. Since $\dot{\varepsilon}_{ij}^{p} = 0$ holds for neutral loading, the two cases therefore result in the same response. As a result, the plasticity theory fulfills the so-called *continuity requirement* introduced by Handelman *et al.* (1947) and previously discussed in relation to nonlinear elastic models, cf. Section 5.4.

9.6 Isotropic hardening of a von Mises material

We will illustrate some of the findings above by considering isotropic hardening of a von Mises material. Associated plasticity is adopted and (9.21) provides

$$f = \sqrt{3J_2} - \sigma_y(\kappa) \; ; \quad f = 0 \quad \text{where} \quad \sigma_y(\kappa) = \sigma_{yo} + K(\kappa) \tag{9.54}$$

i.e.

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2}\frac{s_{ij}}{\sigma_y} \tag{9.55}$$

We write the general evolution law for the internal variable κ in accordance with (9.48), i.e.

$$\dot{\kappa} = \dot{\lambda}\, k(\sigma_{ij}, K) \tag{9.56}$$

The plastic modulus H is defined by (9.50). Since $dK/d\kappa = d\sigma_y/d\kappa$, we obtain with (9.54) and (9.56)

$$\boxed{H = \frac{d\sigma_y}{d\kappa}k} \tag{9.57}$$

The flow rule (9.43) gives with $g = f$ and (9.55)

$$\boxed{\dot{\varepsilon}_{ij}^p = \dot{\lambda}\,\frac{3}{2}\frac{s_{ij}}{\sigma_y}} \tag{9.58}$$

from which it is concluded that $\dot{\varepsilon}_{ii}^p = 0$, i.e. we have the plastic incompressibility that is very characteristic for metals and steel; as $\dot{\varepsilon}_{ii}^p = 0$, it follows that the plastic strains are purely deviatoric. To evaluate the plastic multiplier $\dot{\lambda}$, we multiply each side of (9.58) by itself to obtain

$$\dot{\varepsilon}_{ij}^p\,\dot{\varepsilon}_{ij}^p = \dot{\lambda}^2\,\frac{3}{2}\,\frac{3 s_{ij}s_{ij}}{2\sigma_y^2} \tag{9.59}$$

Define the *effective plastic strain rate* $\dot{\varepsilon}_{eff}^p$ by

$$\boxed{\dot{\varepsilon}_{eff}^p = (\frac{2}{3}\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p)^{1/2}} \tag{9.60}$$

and it is evident that $\dot{\varepsilon}_{eff}^p$ is an invariant and that it is always non-negative. With (9.60) and (9.54), (9.59) takes the form

$$\boxed{\dot{\lambda} = \dot{\varepsilon}_{eff}^p} \tag{9.61}$$

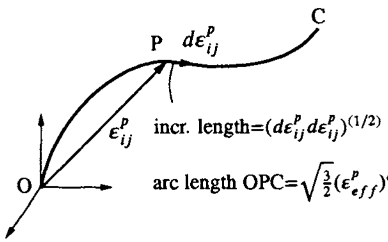

Figure 9.35: History of plastic strains ε^p_{ij} in plastic strain space.

Moreover, define the *effective stress* σ_{eff} by

$$\boxed{\sigma_{eff} = \sqrt{3J_2}} \qquad \text{i.e.} \qquad \dot{\sigma}_{eff} = \frac{3s_{ij}\dot{s}_{ij}}{2\sigma_{eff}} \tag{9.62}$$

where σ_{eff} is clearly an invariant. It follows from (9.54) that the yield criterion may then be expressed as

$$f = \sigma_{eff} - \sigma_y(\kappa)\ ; \qquad f = 0 \tag{9.63}$$

It turns out that the quantities σ_{eff} and $\dot{\varepsilon}^p_{eff}$ have particularly simple interpretations for uniaxial tension. For uniaxial tension we have $\sigma_{11} = \sigma$, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ which leads to $\sigma_{eff} = \sigma$. Moreover, we have $\dot{\varepsilon}^p_{11} = \dot{\varepsilon}^p$ and the flow rule (9.58) implies that $\dot{\varepsilon}^p_{22} = \dot{\varepsilon}^p_{33} = -\dot{\varepsilon}^p/2$ as well as $\dot{\varepsilon}^p_{12} = \dot{\varepsilon}^p_{13} = \dot{\varepsilon}^p_{23} = 0$ and (9.60) therefore leads to $\dot{\varepsilon}^p_{eff} = \dot{\varepsilon}^p$. In conclusion

$$\boxed{\sigma_{eff} = \sigma\ ; \quad \dot{\varepsilon}^p_{eff} = \dot{\varepsilon}^p \quad \textit{for uniaxial tension}} \tag{9.64}$$

The consistency relation (9.49) states that

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0 \tag{9.65}$$

From (9.63) follows that

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = \dot{\sigma}_{eff} \tag{9.66}$$

Insertion of (9.66) in (9.65) and noting that $\sigma_{eff} = \sigma_y$ during plastic loading, cf. (9.63), we find with (9.61) that

$$\boxed{H = \frac{d\sigma_y(\kappa)}{d\varepsilon^p_{eff}}} \tag{9.67}$$

Up until now, the internal variable κ has not been specified and it is of interest that (9.67) holds irrespective of our choice of κ. However, in view of (9.67) it seems tempting to choose the internal variable κ as the effective plastic strain ε^p_{eff}, i.e. $\sigma_y = \sigma_y(\varepsilon^p_{eff})$, which also leads to $K(\kappa) = K(\varepsilon^p_{eff})$. To substantiate this choice, we obtain from definition (9.60) that

$$\boxed{\varepsilon^p_{eff} = \int_0^t \dot{\varepsilon}^p_{eff} dt \quad \text{i.e.} \quad \varepsilon^p_{eff} = \int_0^C d\varepsilon^p_{eff}\ ; \quad C{=}current\ state} \tag{9.68}$$

where

$$d\varepsilon^p_{eff} = (\frac{2}{3} d\varepsilon^p_{ij} d\varepsilon^p_{ij})^{1/2}$$

The plastic strain space is shown in Fig. 9.35 and at some state during the load history, we have the plastic strains ε^p_{ij} indicated by point P. The current plastic strain state corresponds to point C and it appears that path OPC describes the evolution of the plastic strains up until the current state. At an arbitrary point P on this path, the incremental arc length is given by $(d\varepsilon^p_{ij} d\varepsilon^p_{ij})^{1/2} = \sqrt{\frac{3}{2}} d\varepsilon^p_{eff}$, i.e. the incremental arc length is proportional to $d\varepsilon^p_{eff}$. Moreover, it appears that the arc length OPC $= \sqrt{\frac{3}{2}}(\varepsilon^p_{eff})^c$ where $(\varepsilon^p_{eff})^c$ is the effective plastic strain at point C. We emphasize that in accordance with (9.68), $(\varepsilon^p_{eff})^c$ is obtained by an integration along the entire path OPC; this means that arc length OPC differs, in general, from the distance of the straight line between point O and point C. In general, we therefore have $(\varepsilon^p_{eff})^c \neq \sqrt{\frac{2}{3}(\varepsilon^p_{ij})^c (\varepsilon^p_{ij})^c}$.

On the basis of this description, it seems tempting to assume that the current value of ε^p_{eff} is an expression of the plastic history. We may therefore assume that the internal variable κ equals ε^p_{eff}, i.e. the evolution law becomes

$$\boxed{\dot{\kappa} = \dot{\varepsilon}^p_{eff} \quad \text{i.e.} \quad \dot{\kappa} = \dot{\lambda} \qquad strain\ hardening} \tag{9.69}$$

It is evident that κ is a non-decreasing quantity. A comparison with (9.56) shows that the evolution function $k = 1$ and (9.57) yields

$$\boxed{H = \frac{d\sigma_y}{d\varepsilon^p_{eff}}\ ; \qquad \sigma_y = \sigma_y(\varepsilon^p_{eff})} \tag{9.70}$$

in accordance with (9.67). The choice of internal variable given by (9.69) is often referred to as *strain hardening* and it was suggested by Odqvist (1933).

We will now see how (9.70) can be used to calibrate the plasticity model to experimental data. Suppose that we perform a uniaxial tension test into the plastic region. From these experimental data, we can plot the results in a stress

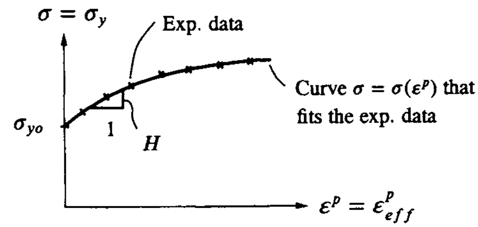

Figure 9.36: Uniaxial tension; experimental determination of the relation $\sigma_y = \sigma_y(\varepsilon^p_{eff})$.

(σ) – plastic strain (ε^p) diagram as shown in Fig. 9.36. Moreover, in accordance with Fig. 9.36 we may identify a curve given by some mathematical expression $\sigma = \sigma(\varepsilon^p)$ that fits these experimental data. For uniaxial tension, (9.63) and (9.64) show that $\sigma = \sigma_{eff} = \sigma_y$ and $\varepsilon^p = \varepsilon^p_{eff}$, i.e. the relation $\sigma = \sigma(\varepsilon^p)$ may equally well be written as $\sigma_y = \sigma_y(\varepsilon^p_{eff})$. Once this relation has been identified we may determine the plastic modulus H in accordance with (9.70) and illustrated in Fig. 9.36.

The essential point in this derivation is that we have determined the relation $\sigma_y(\varepsilon^p_{eff})$ and thereby $H = d\sigma_y/d\varepsilon^p_{eff}$ from a uniaxial experiment. However, since the expression $\sigma_y(\varepsilon^p_{eff})$ only involves invariants, it holds for any load history and we can now use this expression in the constitutive relation for any loading. In turn, this also implies that any load path can be used to determine the relation $\sigma_y(\varepsilon^p_{eff})$, but uniaxial loading is evidently very simple to accomplish and evaluate.

If the plastic modulus $H \neq 0$ (in Section 10.2 we will return to the general situation where H may even be zero), we obtain from the consistency relation (9.65) and (9.55) that

$$\boxed{\dot{\lambda} = \frac{1}{H}\frac{3s_{kl}\dot{\sigma}_{kl}}{2\sigma_y}} \tag{9.71}$$

At any state during the plastic loading, the current yield stress σ_y and the current plastic modulus H are known quantities. For any stress rate $\dot{\sigma}_{kl}$, the plastic multiplier $\dot{\lambda}$ is then determined by (9.71) and the flow rule (9.58) then provides us with the corresponding plastic strain rate $\dot{\varepsilon}^p_{ij}$. As $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}$ and as the elastic strain rates $\dot{\varepsilon}^e_{ij}$ are determined from Hooke's law, i.e.

$$\dot{\varepsilon}^e_{ij} = C_{ijkl}\dot{\sigma}_{kl}$$

where C_{ijkl} is the elastic flexibility tensor, we conclude that the entire response of the material can be identified for any stress increments $\dot{\sigma}_{kl}$.

Here we have identified the function $\sigma_y = \sigma_y(\varepsilon^p_{eff})$ and thereby the plastic modulus H, cf. (9.70), by means of a uniaxial tension test. However, as $\sigma_y =$

$\sigma_y(\varepsilon^p_{eff})$ is postulated to be a universal relation irrespective of the loading, other load paths may equally well be used even though the uniaxial tension test is especially convenient for this purpose. To illustrate this aspect, we imagine that we have performed a pure torsion test into the plastic region. Pure torsion is characterized by $\sigma_{12} = \tau$; $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ and the effective stress σ_{eff} defined by (9.62) then becomes $\sigma_{eff} = \sqrt{3}\,\tau$. Moreover, for pure torsion we have $\dot{\varepsilon}^p_{12} \neq 0$ and the flow rule (9.58) provides $\dot{\varepsilon}^p_{11} = \dot{\varepsilon}^p_{22} = \dot{\varepsilon}^p_{33} = \dot{\varepsilon}^p_{13} = \dot{\varepsilon}^p_{23} = 0$ and the effective plastic strain rate (9.60) then becomes $\dot{\varepsilon}^p_{eff} = \dot{\gamma}^p/\sqrt{3}$ where $\dot{\gamma}^p = 2\dot{\varepsilon}^p_{12}$. Therefore

$$\sigma_{eff} = \sigma_y = \sqrt{3}\tau\,; \qquad \dot{\varepsilon}^p_{eff} = \frac{\dot{\gamma}^p}{\sqrt{3}} \qquad \text{for pure torsion} \tag{9.72}$$

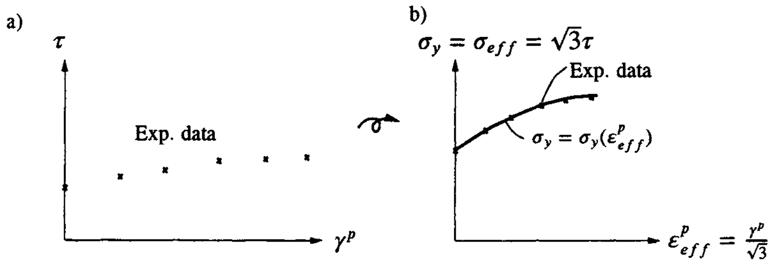

Figure 9.37: Pure torsion; experimental determination of the relation $\sigma_y = \sigma_y(\varepsilon^p_{eff})$.

From the torsion test, the experimental data illustrated in Fig. 9.37a) are obtained. Using (9.72), these data are converted into the experimental data shown in Fig. 9.37b). As shown, these data are again fitted by the curve $\sigma_y = \sigma_y(\varepsilon^p_{eff})$ which is now a known expression and we then use (9.70) to determine the plastic modulus H. It appears that any response measured in the laboratory may be used to identify the function $\sigma_y = \sigma_y(\varepsilon^p_{eff})$, but we also observe that a uniaxial tension test is particularly convenient for this purpose.

The remaining issue that needs to be addressed is the loading/unloading criteria which inform us whether the incremental response is elastic or elasto-plastic. If the current stress state is such that when inserted into the yield function f, we obtain $f < 0$ then we know that the incremental response is elastic. Therefore, for the incremental response to be elasto-plastic, it is necessary that $f = 0$ and the sign of the quantity $\dot{\sigma}_{ij}\partial f/\partial\sigma_{ij}$ then settles the question, cf. (9.53). In the present case, we obtain from (9.66) that

$$\text{if } f = 0 \quad \text{and} \quad \dot{\sigma}_{eff} = \begin{cases} > 0 & \text{plastic loading} \\ = 0 & \text{neutral loading} \\ < 0 & \text{elastic unloading} \end{cases}$$

where it is recalled that neutral loading produces no plastic strains.

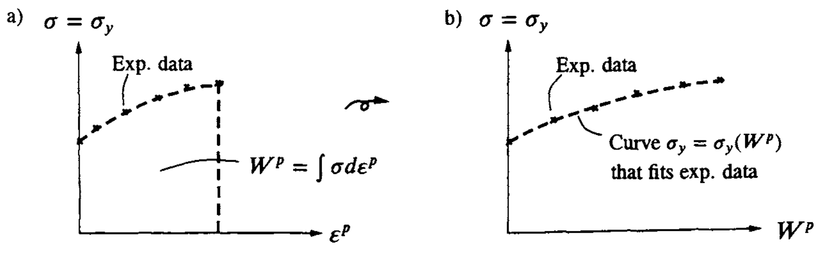

Figure 9.38: Uniaxial tension; experimental determination of the relation $\sigma_y = \sigma_y(W^p)$.

Previously, we adopted the strain hardening assumption that the internal variable κ is equal to the effective plastic strain ε^p_{eff}, cf. (9.69). Another common assumption is the *plastic work hardening* assumption usually attributed to Hill (1950) p.26, where the evolution law is assumed to be given by the format

$$\boxed{\dot{\kappa} = \dot{W}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij} \qquad \textit{plastic work hardening}} \tag{9.73}$$

where $\dot{W}^p$ denotes the *rate of plastic work* or *plastic dissipation*, cf. (9.41) and where $\dot{W}^p$ is clearly an invariant. According to (9.54) and (9.73) it is postulated that $\sigma_y = \sigma_y(W^p)$, i.e. we use W^p to express the plastic history. In order to write (9.73) in the general format (9.56), we introduce the flow rule (9.58) into (9.73) to obtain

$$\dot{\kappa} = \dot{W}^p = \dot{\lambda}k \qquad \text{where} \qquad k = \frac{3s_{ij}\sigma_{ij}}{2\sigma_y} = \frac{3s_{ij}s_{ij}}{2\sigma_y} = \sigma_y \tag{9.74}$$

and where advantage was taken of (9.54). It is evident that κ is an invariant and that it is a non-decreasing quantity. Concerning plastic work hardening where (9.74) holds, the plastic modulus H given by (9.57) becomes

$$\boxed{H = \frac{d\sigma_y}{dW^p}\sigma_y \; ; \qquad \sigma_y = \sigma_y(W^p)} \tag{9.75}$$

Indeed, the same result is obtained from (9.67), which with $\kappa = W^p$ gives

$$H = \frac{d\sigma_y(W^p)}{d\varepsilon^p_{eff}} = \frac{d\sigma_y(W^p)}{dW^p}\frac{dW^p}{d\varepsilon^p_{eff}} = \frac{d\sigma_y(W^p)}{dW^p}\frac{dW^p}{d\lambda} = \frac{d\sigma_y(W^p)}{dW^p}\sigma_y$$

where it also was used that $d\varepsilon^p_{eff} = d\lambda$ and that $dW^p/d\lambda = k = \sigma_y$, cf. (9.74).

To calibrate the isotropic von Mises model using the evolution law given by the work hardening assumption (9.73), we may consider uniaxial tension. In this case (9.73) degenerates to

$$\dot{\kappa} = \dot{W}^p = \sigma\dot{\varepsilon}^p \qquad \text{for uniaxial tension}$$

From the uniaxial test, we may plot the results in the $\sigma\varepsilon^p$ - diagram shown in Fig. 9.38a) and obtain values for the invariant W^p corresponding to each of the data points. Since $\sigma = \sigma_y$ we may then plot σ_y as a function of the experimentally determined W^p-values as illustrated in Fig. 9.38b); a curve given by the expression $\sigma_y = \sigma_y(W^p)$ is then fitted to these data points and the plastic modulus H is subsequently determined by means of (9.75).

It is of considerable interest to compare the assumptions of strain hardening and work hardening. Since $\dot{\lambda} = \dot{\varepsilon}^p_{eff}$, the plastic dissipation (9.74) may be written as

$$\dot{W}^p = \sigma_y \, \dot{\varepsilon}^p_{eff} \tag{9.76}$$

Assume first that we adopt the strain hardening assumption, i.e. $\sigma_y = \sigma_y(\varepsilon^p_{eff})$. It then follows from (9.76) that

$$W^p = \int_0^{\varepsilon^p_{eff}} \sigma_y(\varepsilon^p_{eff}) d\varepsilon^p_{eff} \quad \text{i.e.} \quad W^p = W^p(\varepsilon^p_{eff}) \quad \text{or} \quad \varepsilon^p_{eff} = \varepsilon^p_{eff}(W^p)$$

This means that the strain hardening assumption $\sigma_y = \sigma_y(\varepsilon^p_{eff})$ may equally well be written in the form $\sigma_y = \sigma_y(W^p)$ which is exactly the work hardening assumption. Therefore, in this case, the two assumptions are identical. Assume next that we adopt the work hardening assumption. From $\sigma_y = \sigma_y(W^p)$ and (9.76) follow that

$$\varepsilon^p_{eff} = \int_0^{W^p} \frac{dW^p}{\sigma_y(W^p)} \quad \text{i.e.} \quad \varepsilon^p_{eff} = \varepsilon^p_{eff}(W^p) \quad \text{or} \quad W^p = W^p(\varepsilon^p_{eff})$$

and we infer that the assumption $\sigma_y = \sigma_y(W^p)$ implies the relation $\sigma_y = \sigma_y(\varepsilon^p_{eff})$ i.e. the strain hardening assumption.

We conclude that, in reality, it is immaterial whether we assume strain hardening or work hardening. However, it is emphasized that whereas this coincidence holds for isotropic hardening of a von Mises material, it is not a general conclusion that holds for arbitrary plasticity models. From the discussion above, we observe that the coincidence of strain hardening and work hardening hinges on the property that the plastic dissipation $\dot{W}^p$ defined in general by (9.73), for the isotropic von Mises model takes the particular form given by (9.76). A further discussion is given by Bland (1957).

To deform a material plastically, external work is required. The development of plastic deformations is accompanied by irreversible phenomena that require dissipation of some of the external work within the material. These irreversible phenomena may manifest themselves in terms of a reorganization of the microstructure of the material and in terms of *heat generation.* It is of considerable interest that Taylor and Quinney (1934) found from experiments with metals and steel that as much as 90-95% of the incremental plastic work $dW^p = \sigma_{ij} d\varepsilon^p_{ij}$ is transformed into heat. This supports the everyday experience that plastic deformations may result in a considerable temperature increase and a strict derivation of such heat generation will be given in Chapter 21.

Finally, we mention that isotropic hardening of a von Mises material is treated more systematically in Section 12.2.

9.7 Proportional loading of isotropic hardening von Mises material

It turns out to be of interest to investigate the response of an isotropic hardening von Mises material during increasing *proportional loading*. Proportional loading is occasionally called *radial loading*.

The stress history for proportional loading is defined as

$$\sigma_{ij}(t) = \beta(t)\sigma^*_{ij} \ ; \qquad \sigma^*_{ij} = \text{constant}$$

where σ^*_{ij} is an arbitrary constant stress state and $\beta(t)$ is a scalar function that increases with time t; moreover, $\beta(t=0)=0$. It follows that

$$\sigma_{kk} = \beta\sigma^*_{kk} \ ; \qquad s_{ij} = \beta s^*_{ij} \ ; \qquad \sigma_{eff} = \beta\sigma^*_{eff}$$

During plastic loading, we have $\sigma_y = \sigma_{eff}$, i.e. the flow rule (9.58) becomes

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{3s^*_{ij}}{2\sigma^*_{eff}}$$

where $\dot{\lambda} = \dot{\varepsilon}^p_{eff}$. The expression above may be integrated to obtain

$$\varepsilon^p_{ij} = \lambda\frac{3s^*_{ij}}{2\sigma^*_{eff}} \tag{9.77}$$

Multiplication of each side of (9.77) by itself then gives

$$\lambda = \varepsilon^p_{eff} = (\frac{2}{3}\varepsilon^p_{ij}\varepsilon^p_{ij})^{1/2} \tag{9.78}$$

Let us assume isotropic elasticity. In that case, Hooke's law is given by (4.85) and (4.86), i.e.

$$\varepsilon^e_{kk} = \frac{\sigma_{kk}}{3K} \ ; \qquad e^e_{ij} = \frac{s_{ij}}{2G^e}$$

where the notation G^e is used to emphasize that this shear modulus refers to the linear elastic behavior. From (9.77) and (9.78), we obtain

$$\varepsilon^p_{kk} = 0 \ ; \qquad e^p_{ij} = \varepsilon^p_{ij} = \varepsilon^p_{eff}\frac{3s^*_{ij}}{2\sigma^*_{eff}} = \varepsilon^p_{eff}\frac{3s_{ij}}{2\sigma_{eff}}$$

It then follows that

$$\varepsilon_{kk} = \varepsilon^e_{kk} + \varepsilon^p_{kk} = \frac{\sigma_{kk}}{3K} \tag{9.79}$$

and

$$e_{ij} = e^e_{ij} + e^p_{ij} = (\frac{1}{2G^e} + \frac{3\varepsilon^p_{eff}}{2\sigma_{eff}})s_{ij} \tag{9.80}$$

Let us assume that we have strain hardening, i.e. $\sigma_y = \sigma_y(\varepsilon^p_{eff})$ and thereby $\sigma_y = \sigma_{eff} = \sigma_{eff}(\varepsilon^p_{eff})$. This may be written as $\varepsilon^p_{eff} = \varepsilon^p_{eff}(\sigma_{eff})$ and as $\sigma_{eff} = \sqrt{3J_2}$, it appears that (9.79) and (9.80) can be written as

$$\boxed{\sigma_{kk} = 3K\varepsilon_{kk} \ ; \quad s_{ij} = 2Ge_{ij}} \tag{9.81}$$

where

$$\boxed{K = constant \ ; \quad G = G(J_2) = \frac{G^e}{1 + G^e \dfrac{3\varepsilon^p_{eff}(\sigma_{eff})}{\sigma_{eff}}}} \tag{9.82}$$

These constitutive relations are referred to as the *deformational plasticity theory* of Hencky (1924) and we have seen that they may be derived from the incremental formulation of an isotropic hardening von Mises material during increasing proportional loading. Moreover, a comparison of (9.81) and (9.82) with (4.102) and (4.105) shows that we have recovered a nonlinear isotropic Hooke formulation. As discussed previously in Sections 4.10 and 5.4, the major drawback of such nonlinear elasticity formulations is their unrealistic predictions during unloading.

9.8 Conditions for plastic incompressibility

We have previously indicated in (4.104) that the nonlinearity of metals and steel is entirely related to the deviatoric response whereas the volumetric response is linear. Likewise, we indicated in (4.106) that for materials like concrete, rocks and soil, the nonlinear response is related both to the deviatoric and volumetric response. With these observations, it is of interest to investigate the conditions for which a plasticity model will imply plastic incompressibility, i.e. a linear elastic volumetric response.

The general flow rule (9.43) states that

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \tag{9.83}$$

where the potential function is given by $g = g(\sigma_{ij}, K_\alpha)$. It follows that plastic incompressibility $\dot{\varepsilon}^p_{kk} = 0$ is obtained for

$$\boxed{\frac{\partial g}{\partial \sigma_{ii}} = 0 \quad \Rightarrow \quad \textit{plastic incompressibility}}$$

In general, we observe that

$$\frac{\partial g}{\partial \sigma_{ii}} = \frac{\partial g}{\partial \sigma_{11}} + \frac{\partial g}{\partial \sigma_{22}} + \frac{\partial g}{\partial \sigma_{33}} \neq \frac{\partial g}{\partial I_1}$$

However, for isotropic materials the potential function can be written as

$$g = g(I_1, J_2, J_3, K_\alpha)$$

This implies

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial g}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial g}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial g}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}}$$

i.e.

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial g}{\partial I_1}\delta_{ij} + \frac{\partial g}{\partial J_2}s_{ij} + \frac{\partial g}{\partial J_3}(s_{ik}s_{kj} - \frac{2}{3}J_2\delta_{ij}) \tag{9.84}$$

It then follows that

$$\boxed{\frac{\partial g}{\partial \sigma_{ii}} = 3\frac{\partial g}{\partial I_1}} \tag{9.85}$$

For metals and steel, it is concluded that we can assume that $g = g(J_2, J_3, K_\alpha)$ whereas concrete, rocks and soil require the inclusion of the first stress invariant I_1, i.e. $g = g(I_1, J_2, J_3, K_\alpha)$.