10 GENERAL PLASTICITY THEORY

In the previous chapter, the ingredients of the plasticity theory were introduced. The objective was to familiarize the reader with various important topics, but the exposition was tuned more towards illustrative viewpoints rather than a systematic treatment. With this background, we will now present a systematic exposition of the general plasticity theory. Reference may also be made to the textbooks of Chen and Han (1988) and Chen (1994) as well as to Hill (1950), Khan and Huang (1995), Lubliner (1990), Mroz (1966) and Stouffer and Dame (1996).

10.1 Fundamental equations

Let us first list the fundamental equations that comprise the general plasticity theory. Later on, we shall manipulate these equations in order to obtain a final framework that is suitable also from a computational point of view.

The total strains consist of the sum of the elastic and plastic strains, i.e.

$$\boxed{\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}} \tag{10.1}$$

where a dot, as usual, denotes the time rate. The elastic strains are determined from Hooke's law, i.e.

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl}) \tag{10.2}$$

where D_{ijkl} is the elastic stiffness tensor. We shall allow general elasticity, i.e. D_{ijkl} may even refer to anisotropic elasticity. Due to the symmetry of σ_{ij} and ε_{ij}, D_{ijkl} possesses the usual symmetry properties $D_{ijkl} = D_{jikl}$ and $D_{ijkl} = D_{ijlk}$, cf. (4.21). Moreover, we assume the symmetry property given by (4.23), i.e. we have in total that

$$\boxed{D_{ijkl} = D_{jikl} \ ; \quad D_{ijkl} = D_{ijlk} \ ; \quad D_{ijkl} = D_{klij}} \tag{10.3}$$

which leads to symmetry of the elastic stiffness matrix $\boldsymbol{D}$, cf. (4.38). In accordance with (4.24a) and (4.25), D_{ijkl} is positive definite, i.e.

$$a_{ij}D_{ijkl}a_{kl} > 0 \qquad \text{for any } a_{ij} \neq 0 \tag{10.4}$$

holds for any symmetric second-order tensor a_{ij}. It follows that D_{ijkl} is non-singular, i.e. the homogeneous equation system

$$D_{ijkl}a_{kl} = 0 \qquad \Rightarrow \qquad a_{kl} = 0 \tag{10.5}$$

possesses only the trivial solution $a_{kl} = 0$.

Assuming the tensor D_{ijkl} to be constant with respect to the loading, we obtain from (10.2) that

$$\boxed{\dot{\sigma}_{ij} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl})} \tag{10.6}$$

We assume the existence of a yield function $f(\sigma_{ij}, K_\alpha)$ so that development of plastic strains requires that

$$\boxed{f(\sigma_{ij}, K_\alpha) = 0 \qquad \textit{for development of plasticity}} \tag{10.7}$$

In this expression K_α denotes the hardening parameters, which may be scalars or higher-order tensors. Since $\alpha = 1, 2 \ldots$ we may have one, two or more hardening parameters, cf. the discussion relating to (9.6) - (9.8). To confirm with the previous terminology, (10.7) denotes the current yield surface which for $K_\alpha = 0$ reduces to the initial yield surface $F(\sigma_{ij})$. However, since we have discussed these concepts in detail in the previous chapter, we shall for convenience merely refer to (10.7) as the *yield function*. We also recall that the manner in which we go from the initial to the current yield criterion is controlled by the hardening rule, which may be chosen in the form of an isotropic, kinematic or mixed hardening rule.

For fixed hardening parameters, (10.7) describes a surface in the stress space. The sign of the function $f(\sigma_{ij}, K_\alpha)$ is chosen such that

$$\boxed{f(\sigma_{ij}, K_\alpha) < 0 \qquad \Rightarrow \qquad \textit{elastic behavior}} \tag{10.8}$$

Therefore, in order that changes may occur in the hardening parameters K_α and the plastic strains ε^p_{ij}, it is necessary that (10.7) is fulfilled.

The state of the material is described by the internal variables, which may be scalars or higher-order tensors. In analogy with the notation above, we denote the internal variables by κ_α. Since the internal variables describe the condition, i.e. the state, of the material, they are often termed state variables in the literature. In principle, the only quantities that can be directly measured, i.e. observed, are the total strains ε_{ij} and the temperature and the internal variables κ_α are therefore non-observable variables; this is the reason why κ_α are occasionally called 'hidden' variables in the literature . The internal variables κ_α

memorize the plastic loading history of the material. As an example of an internal variable, we may take the effective plastic strain. As the internal variables characterize the elasto-plastic material, we have

$$K_\alpha = K_\alpha(\kappa_\beta) \tag{10.9}$$

and the number of hardening parameters equals the number of internal variables; otherwise, the relation between K_α and κ_α will not be unique.

Like the yield function $f(\sigma_{ij}, K_\alpha)$, we assume the existence of the potential function g defined by

$$\boxed{g = g(\sigma_{ij}, K_\alpha)}$$

i.e. the potential function depends on the same parameters as the yield function. In accordance with (9.43), the flow rule is written in the following general form

$$\boxed{\dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}\,; \qquad \dot{\lambda} \geq 0} \tag{10.10}$$

If $g = f$, we have associated plasticity and if $g \neq f$, nonassociated plasticity holds. The flow rule states that the direction of $\dot{\varepsilon}_{ij}^p$ is given by the gradient $\partial g/\partial \sigma_{ij}$ whereas the plastic multiplier $\dot{\lambda}$ determines the magnitude of $\dot{\varepsilon}_{ij}^p$. If we have $\dot{\lambda} = 0$, then no plastic strains develop; otherwise, $\dot{\lambda} > 0$ ensures that $\dot{\varepsilon}_{ij}^p$ and $\partial g/\partial \sigma_{ij}$ possess the same direction.

The consistency relation states that during development of plastic strains, the yield criterion (10.7) is fulfilled, i e.

$$\boxed{\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha}\dot{K}_\alpha = 0} \tag{10.11}$$

where the summation convention also holds for Greek letters. The consistency relation involves the time rate of the hardening parameters and from (10.9) we obtain the following evolution of K_α

$$\boxed{\dot{K}_\alpha = \frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta} \tag{10.12}$$

In turn, this expression involves the time rate of internal variables, i.e. how κ_α evolves with time. In accordance with (9.48), we assume the following evolution laws

$$\boxed{\dot{\kappa}_\alpha = \dot{\lambda}k_\alpha(\sigma_{ij}, K_\beta)} \tag{10.13}$$

where the evolution functions k_α, in general, are allowed to depend on the same variables as the yield function and the potential function. However, nothing is changed in the following exposition, if the evolution functions k_α are also allowed to depend on other variables than σ_{ij} and K_α. In accordance with our objective to formulate constitutive relations for time-independent materials, we observe that (10.13) is an expression that is homogeneous in time, i.e. the denominator dt present in both $\dot{\kappa}_\alpha = d\kappa_\alpha/dt$ and $\dot{\lambda} = d\lambda/dt$ may be canceled. This illustrates that the occurrence of time in (10.13), as well as in the other expressions, is purely artificial, enabling a convenient notation. Therefore, the evolution laws are written as (10.13) instead of the following more clumsy notation: $d\kappa_\alpha = d\lambda\, k_\alpha(\sigma_{ij}, K_\alpha)$. We also observe that (10.13) comprises a very general form of the evolution laws.

Equations (10.1) - (10.13) form the backbone of the general plasticity theory. No more assumptions or concepts are needed and the rest of this chapter describes the implications of these equations. It is emphasized that the relations above hold not only for isotropic materials, but also for anisotropic materials.

We finally notice that in order to obtain a specific plasticity model we must choose the yield function f, the potential function g, the hardening parameters K_α, the internal variables κ_α, the manner in which K_α and κ_α relate to each other (10.9) and the evolution functions k_α present in (10.13). We notice that the choice of hardening parameters K_α implies the choice of the hardening rule (isotropic, kinematic or mixed). For advanced plasticity models, all these choices are certainly not trivial and they must be based on our experimental and general knowledge of the material behavior. Often, a trial-and-error process is involved in making these choices whereby the predictions of the plasticity model are tuned to fit the pertinent experimental data.

It is of significant interest that the formulation of the general plasticity theory as presented above can also be derived entirely by means of the principles of *thermodynamics*. A detailed exposition of this thermodynamic approach will require the introduction of a number of concepts and in order to set the scene, we will postpone this fruitful, but complex treatment to Chapter 21. However, the fact that the general plasticity theory can be derived from thermodynamics means that it rests on the basic laws of nature and not just on our interpretation of experimental evidence and some reasonable assumptions. The key point in thermodynamics is the fulfillment of the *second law of thermodynamics*, i.e. the so-called *dissipation inequality*, which turns out to take the form $\sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha\dot{\kappa}_\alpha \geq 0$. Essentially this inequality excludes the existence of a *perpetual motion of the second kind*. If the yield function $f(\sigma_{ij}, K_\alpha)$ is a convex function, then it can be shown that the dissipation inequality is fulfilled for the following evolution laws: $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial f/\partial\sigma_{ij}$ and $\dot{\kappa}_\alpha = -\dot{\lambda}\partial f/\partial K_\alpha$. A comparison with (10.10) and (10.13) shows that we have recovered associated plasticity and that the evolution function in (10.13) is in the form $k_\alpha = -\partial f/\partial K_\alpha$. Even nonassociated plasticity can be derived from thermodynamics and for the potential function $g = g(\sigma_{ij}, K_\alpha)$ being convex and fulfilling some rather weak requirements, we obtain

$$\boxed{\begin{array}{l}\textit{Potential function approach gives}\\ \dot{\varepsilon}^p_{ij} = \dot{\lambda}\dfrac{\partial g}{\partial \sigma_{ij}}\\ \dot{\kappa}_\alpha = -\dot{\lambda}\dfrac{\partial g}{\partial K_\alpha}\end{array}} \tag{10.14}$$

which again is contained in the formulation given by (10.10) and (10.13).

Two important observations follow from this discussion. The first observation arises from the second law given by $\sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha\dot{\kappa}_\alpha \geq 0$. Here, the term $\sigma_{ij}\dot{\varepsilon}^p_{ij}$ has the dimension of energy rate per unit volume and consequently, the term $K_\alpha\dot{\kappa}_\alpha$ has the same dimension. One uses the phrase that σ_{ij} is *conjugated* to $\dot{\varepsilon}^p_{ij}$ since their product is an energy rate (per unit volume). In the same sense, K_α is conjugated to $\dot{\kappa}_\alpha$ (or, in view of the minus sign in the second law, it may be more reasonable to say that $-K_\alpha$ is conjugated to $\dot{\kappa}_\alpha$). Therefore, just as it is natural to work with the stresses σ_{ij} and the plastic strains ε^p_{ij}, it is natural to work with the hardening parameters K_α and the internal variables κ_α.

The second observation is that just as $g(\sigma_{ij}, K_\alpha)$ serves as a potential for $\dot{\varepsilon}^p_{ij}$ via $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial g/\partial\sigma_{ij}$ so does g serve as a potential for $\dot{\kappa}_\alpha$ via $\dot{\kappa}_\alpha = -\dot{\lambda}\partial g/\partial K_\alpha$. This underlines again the duality between σ_{ij} and ε^p_{ij} and between K_α and κ_α.

Here we have merely indicated some of the results that follow from thermodynamics. Since the use of thermodynamics requires the introduction of a number of abstract concepts, at the present stage we will continue to investigate plasticity from a purely mechanical point of view.

Another reason for this choice of presentation is that not all valid plasticity formulations fit into the format given by (10.14). This is not to say that they do not fulfill the second law of thermodynamics. To appreciate the difference, compare (10.13), i.e. $\dot{\kappa}_\alpha = \dot{\lambda}k_\alpha$ and the evolution law for $\dot{\kappa}_\alpha$ given by (10.14), i.e. $\dot{\kappa}_\alpha = -\dot{\lambda}\partial g/\partial K_\alpha$; it appears that the potential approach requires the evolution function k_α to be derived from the potential function $g(\sigma_{ij}, K_\alpha)$ i.e. $k_\alpha = -\partial g/\partial K_\alpha$. This is certainly a restriction and we conclude that our present approach (10.13) is somewhat more general that the potential function approach given by (10.14). Again this does not imply that models based on (10.13) do not fulfill the second law of thermodynamics; if they fulfill the inequality $\sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha\dot{\kappa}_\alpha \geq 0$ all formal requirements are fulfilled. The principal difference between (10.13) and (10.14) is that if the potential function g is convex and fulfills some rather weak requirements, then we know *a priori* that the formulation (10.14) fulfills the second law $\sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha\dot{\kappa}_\alpha \geq 0$ whereas with the formulation (10.13) we have, in principle, to check *a posteriori* that this inequality is fulfilled.

In Chapters 21 and 22 we will return to the thermodynamic approach and explore its ramifications in detail, not only for plasticity, but also for other nonlinear material behaviors.

10.2 Generalized plastic modulus - Relation between stress rates and total strain rates

The flow rule (10.10) determines the direction of the plastic strain rates, but the magnitude of $\dot{\varepsilon}_{ij}^{p}$ is unknown since the plastic multiplier $\dot{\lambda}$ is still unknown. To determine $\dot{\lambda}$, advantage is taken of the consistency relation and the evolution laws for $\dot{\kappa}_{\alpha}$.

Insert the evolution laws (10.13) into (10.12) to obtain

$$\boxed{\dot{K}_{\alpha} = \dot{\lambda}\frac{\partial K_{\alpha}}{\partial \kappa_{\beta}}k_{\beta}} \tag{10.15}$$

Insertion of (10.15) in the consistency relation (10.11) then provides

$$\boxed{\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0} \tag{10.16}$$

where the generalized plastic modulus H is defined by

$$\boxed{H = -\frac{\partial f}{\partial K_{\alpha}}\frac{\partial K_{\alpha}}{\partial \kappa_{\beta}}k_{\beta}} \tag{10.17}$$

We will see in a moment that once the yield function f, the potential function g and the generalized plastic modulus H are known then the plasticity formulation is complete. Referring to (10.17) and (10.15) it is allowable to directly specify the combined term $\partial K_{\alpha}/\partial \kappa_{\beta} k_{\beta}$ instead of specifying each of the separate factors $\partial K_{\alpha}/\partial \kappa_{\beta}$ and k_{β}. This implies that in some plasticity models the term $\partial K_{\alpha}/\partial \kappa_{\beta} k_{\beta}$ is directly postulated whereas in other models each of the factors $K_{\alpha}/\partial \kappa_{\beta}$ and k_{β} are postulated separately.

If $H \neq 0$, then (10.16) and the flow rule (10.10) give

$$\dot{\varepsilon}_{ij}^{p} = \frac{1}{H}\left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}\right)\frac{\partial g}{\partial \sigma_{ij}} \tag{10.18}$$

Moreover, (10.1) and (10.6) provide

$$\dot{\varepsilon}_{ij}^{e} = C_{ijkl}\,\dot{\sigma}_{kl} \tag{10.19}$$

where C_{ijkl} is the elastic flexibility tensor. Combination of (10.1), (10.18) and (10.19) gives

$$\dot{\varepsilon}_{ij} = C_{ijkl}^{ep}\dot{\sigma}_{kl} \tag{10.20}$$

where the *elasto-plastic flexibility tensor* C_{ijkl}^{ep} is given by

$$C_{ijkl}^{ep} = C_{ijkl} + \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}} \tag{10.21}$$

Therefore, if $H \neq 0$ and if the stress rate $\dot{\sigma}_{kl}$ is given, then (10.20) determines the response completely. This formulation comprises the *stress driven* format.

However, we want to be able to determine the response in the general case where we may have $H = 0$. This turns out to be possible if - instead of a prescribed stress rate $\dot{\sigma}_{ij}$ - the total strain rate $\dot{\varepsilon}_{ij}$ is given. Moreover, this general format also has the advantage of fitting directly into our numerical formulation given in terms of the nonlinear finite element method, as we shall see later.

To obtain this general format, we insert the flow rule (10.10) into Hooke's law (10.6), i.e.

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} - \dot{\lambda} D_{ijst}\frac{\partial g}{\partial \sigma_{st}} \tag{10.22}$$

Multiply this expression by $\partial f/\partial \sigma_{ij}$ and use (10.16) to obtain

$$\boxed{\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl}\ ; \qquad \dot{\lambda} \geq 0} \tag{10.23}$$

Here the parameter A is defined by

$$\boxed{A = H + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}}\ ; \qquad A > 0} \tag{10.24}$$

To be able to derive (10.23), we must require that $A \neq 0$. However, we will later show that A is also a positive parameter.

If (10.23) is inserted into the flow rule (10.10), we find that

$$\dot{\varepsilon}^p_{ij} = \frac{1}{A}\left(\frac{\partial f}{\partial \sigma_{kl}} D_{klmn}\dot{\varepsilon}_{mn}\right)\frac{\partial g}{\partial \sigma_{ij}} \tag{10.25}$$

which proves that for given total strain rates, the plastic strain rates are known. Of more importance, however, is that insertion of (10.23) into (10.22) yields

$$\boxed{\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\varepsilon}_{kl}} \tag{10.26}$$

where the *elasto-plastic stiffness tensor* is given by

$$\boxed{D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A} D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}} \tag{10.27}$$

This result comprises our objective, namely that for a given total strain rate $\dot{\varepsilon}_{ij}$, the stress rate $\dot{\sigma}_{ij}$ is determined from (10.26). This expression holds in general and we have in (10.24) emphasized that A is a positive parameter. This will be proved shortly. For evident reasons, (10.26) comprises the *strain driven* format. We also observe that D^{ep}_{ijkl} determines the current *tangential stiffness* of the material.

The establishment of the important relation (10.26) was first given by Hill (1950) for a von Mises material and later by Hill (1958) for general associated plasticity and by Mroz (1966) for general nonassociated plasticity. Often however, the establishment of the strain driven format (10.26) is attributed to Yamada *et al.* (1968) and Zienkiewicz *et al.* (1969).

Evidently, since D^{ep}_{ijkl} depends on the stress state σ_{ij}, the hardening parameters K_α and the plastic modulus H, (10.26) determines a nonlinear material response. However, we observe that D^{ep}_{ijkl} does not depend on either the stress rate $\dot{\sigma}_{ij}$ or the strain rate $\dot{\varepsilon}_{ij}$, i.e. relation (10.26) is incrementally linear.

It appears that the general format (10.26) looks very much as the incremental form of Hooke's law and it is therefore not surprising that (10.26) is of fundamental importance in nonlinear finite element calculations. Due to (10.3a,b) it follows from (10.27), that we always have

$$\boxed{D^{ep}_{ijkl} = D^{ep}_{jikl}\ ; \quad D^{ep}_{ijkl} = D^{ep}_{ijlk}}$$

It also follows from (10.27) and (10.3c) that

$$\boxed{\begin{array}{ll} D^{ep}_{ijkl} = D^{ep}_{klij} & \textit{for associated plasticity} \\ D^{ep}_{ijkl} \neq D^{ep}_{klij} & \textit{for nonassociated plasticity} \end{array}} \tag{10.28}$$

Similar to (4.37), we may write (10.26) in the following matrix format

$$\boxed{\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\,\dot{\boldsymbol{\varepsilon}}} \tag{10.29}$$

which is of importance in nonlinear finite element calculations.

It appears from (10.28) that associated plasticity $f = g$ implies symmetry, i.e. $\boldsymbol{D}^{ep} = \boldsymbol{D}^{ep^T}$ whereas nonassociated plasticity $f \neq g$ implies $\boldsymbol{D}^{ep} \neq \boldsymbol{D}^{ep^T}$. We conclude that nonassociated plasticity is more complicated, not only from a conceptual viewpoint, but also from a computational viewpoint. The advantage that is offered by nonassociated plasticity is the greater possibility to fit a specific plasticity model so that it complies well with the experimental evidence. In general, associated plasticity gives accurate predictions for metals and steel whereas nonassociated plasticity is often required when concrete, soil and rocks are considered.

10.3 Evaluation of plastic modulus H

When deriving (10.23), we implicitly assumed that the parameter $A \neq 0$. Let us now prove that A is a positive quantity as already stated by (10.24). The theory that we develop is supposed to be general, so it must also hold for an associated flow rule ($f = g$) even when the plastic modulus H is zero; according to (10.17), the plastic modulus H is zero for ideal plasticity, where the yield function does not involve any hardening parameters K_α, i.e. $\partial f/\partial K_\alpha = 0$. In that case, parameter A as given by (10.24) reduces to

$$A = \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f}{\partial \sigma_{kl}}$$

which is clearly a positive quantity, cf. (10.4). We are then led to the general requirement that

$$\boxed{A > 0}$$

always holds. Having proved this inequality, we will obtain an evaluation of the generalized plastic modulus H that, up to now, has just been defined by expression (10.17).

When using the general format (10.26), we may be interested in a situation where non-zero total strain rates $\dot{\varepsilon}_{ij} \neq 0$ imply $\dot{\sigma}_{ij} = 0$, i.e. we want to investigate possible non-trivial solutions $\dot{\varepsilon}_{kl}$ of the following homogeneous equation system

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\, \dot{\varepsilon}_{kl} = 0 \tag{10.30}$$

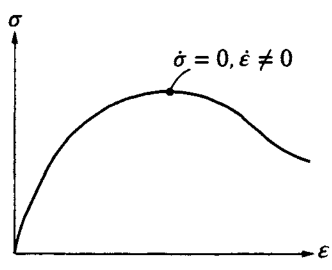

Figure 10.1: Limit point, i.e. peak stress where we have $\dot{\varepsilon} \neq 0$, but $\dot{\sigma} = 0$.

For uniaxial loading, this situation is illustrated in Fig. 10.1 and it is called a *limit point*. Since $\dot{\sigma}_{ij} = 0$, it follows from Hooke's law that $\dot{\varepsilon}^e_{ij} = 0$ i.e. $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^p_{ij}$. With the flow rule (10.10), a solution to (10.30) must therefore be of the form

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^p_{ij} = \dot{\lambda} \frac{\partial g}{\partial \sigma_{ij}} \tag{10.31}$$

To obtain a non-trivial solution we must require $\dot{\lambda} \neq 0$, i.e. insertion of (10.31) into (10.30) yields

$$D^{ep}_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} = 0$$

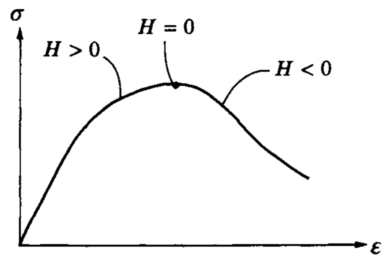

Figure 10.2: Interpretation of the plastic modulus H.

which with (10.27) and (10.24) takes the form

$$\frac{H}{A} D_{ijst} \frac{\partial g}{\partial \sigma_{st}} = 0 \tag{10.32}$$

Since D_{ijkl} is non-singular, cf. (10.5), (10.32) implies that

$$H = 0$$

We have then proved that (10.30) possesses the non-trivial solution (10.31) only when $H = 0$. Referring to the interpretation of (10.30) illustrated in Fig. 10.1, we are considering a situation where the stresses take their peak values and the tangent to the stress-strain curve is horizontal. This situation also holds for ideal plasticity and in that case the requirement $H = 0$, is indeed, not surprising, since the yield function for ideal plasticity, per definition, does not depend on any hardening parameters, cf. (9.15); since no hardening parameters K_α appear in the yield function, (10.17) shows that $H = 0$. More generally, however, the derivation above shows that (10.30) only has a non-trivial solution when $H = 0$ and that this solution is given by (10.31).

Considering (10.27) and recalling that $A > 0$, it appears that $D^{ep}_{ijkl} \to D_{ijkl}$ for $A \to \infty$, which requires $H \to \infty$. Therefore, the response of our elasto-plastic material approaches the linear elastic response when $H \to \infty$. We conclude that $H > 0$ corresponds to hardening plasticity and, consequently, $H < 0$ corresponds to softening plasticity.

The findings above are illustrated in Fig. 10.2, which may be summarized as

$$\boxed{\begin{array}{lcl} H > 0 & \Rightarrow & \textit{hardening plasticity} \\ H = 0 & \Rightarrow & \textit{ideal plasticity} \\ H < 0 & \Rightarrow & \textit{softening plasticity} \end{array}} \tag{10.33}$$

For isotropic hardening of a von Mises material, we observed in (9.67) that H denotes the slope $d\sigma/d\varepsilon^p$ for uniaxial loading. This finding is in accordance with (10.33), but we observe that $H = d\sigma/d\varepsilon^p$ is not a general interpretation. The general interpretation is given by (10.33) and Fig. 10.2.

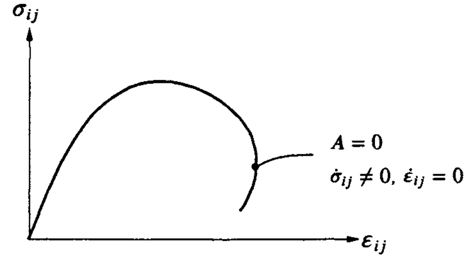

Figure 10.3: Illustration of the consequence of $A \to 0$. Note that all strain rate components $\dot{\varepsilon}_{ij} = 0$.

From (10.27) appears that formulation ceases to be valid if $A \to 0$ and we will next show that requirement $A > 0$ places restrictions on the amount of softening that can be modeled.

Against this background and as a complement to the problem posed by (10.30), it may be of interest to investigate whether it is possible to have the following situation

$$\dot{\sigma}_{ij} \neq 0 \ ; \quad \dot{\varepsilon}_{ij} = 0$$

This situation is illustrated in Fig. 10.3. According to (10.20) and (10.21) we have

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} = (C_{ijkl} + \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}})\dot{\sigma}_{kl} \tag{10.34}$$

Since $\dot{\varepsilon}_{ij} = 0$ and $\dot{\varepsilon}^p_{ij} = \dot{\lambda}\partial g/\partial \sigma_{ij}$ hold, (10.6) gives

$$\dot{\sigma}_{kl} = -\dot{\lambda} D_{klmn}\frac{\partial g}{\partial \sigma_{mn}} \tag{10.35}$$

Use of (10.35) in (10.34) and noting that $\dot{\varepsilon}_{ij} = 0$ and $\dot{\lambda} \neq 0$ imply

$$0 = \frac{\partial g}{\partial \sigma_{ij}} + \frac{1}{H}\frac{\partial g}{\partial \sigma_{ij}}(\frac{\partial f}{\partial \sigma_{kl}} D_{klmn}\frac{\partial g}{\partial \sigma_{mn}})$$

With the definition (10.24), we obtain

$$0 = \frac{A}{H}\frac{\partial g}{\partial \sigma_{ij}}$$

and it is required that $A = 0$. It follows that the limit $A \to 0$ corresponds to the situation illustrated in Fig. 10.3. For this situation to be possible, the stress rates must fulfill (10.35). It is also of importance that all strain rate components must be zero, i.e. $\dot{\varepsilon}_{ij} = 0$, in order that the solution (10.35) be possible. For an isotropic hardening von Mises material, (10.35) implies that the incremental stresses should be purely deviatoric (for instance pure shear) and uniaxial tension will therefore not give rise to the situation discussed above.

Let us finally specialize to associated hardening plasticity. In that case we obtain with (10.18)

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{p} = \frac{1}{H}\left(\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}\right)\left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}\right) > 0 \quad \text{assoc. hard. plasticity}$$

in accordance with the implication of Drucker's postulate, cf. (9.38).

10.4 General loading and unloading criteria

In the previous chapter, we have derived criteria that make it possible to determine whether we have plastic loading or elastic unloading, cf. (9.53). As mentioned there, these criteria are only applicable to hardening plasticity and we will therefore now derive loading and unloading criteria that hold in general.

If the stress state is located inside the yield surface, i.e. $f < 0$, then, in accordance with (10.8) we have incrementally an elastic behavior. Therefore, evolution of plastic strains requires that $f = 0$. Let us define the *elastic stress rate* $\dot{\sigma}_{ij}^{e}$ by

$$\boxed{\dot{\sigma}_{ij}^{e} = D_{ijkl}\dot{\varepsilon}_{kl}} \tag{10.36}$$

where the term 'elastic' refers to the fact that this is the stress rate that would result for a given total strain rate provided the material responded elastically. With this definition, (10.23) takes the form

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{e} \geq 0 \tag{10.37}$$

where $A > 0$. From (10.37) it follows that plastic strains develop if $\partial f/\partial\sigma_{i}\dot{\rho}_{ij}^{e} > 0$. If $\partial f/\partial\sigma_{ij}\ \dot{\sigma}_{ij}^{e} < 0$, (10.37) implies $\dot{\lambda} < 0$ which cannot occur, i.e. this case must correspond to elastic unloading. Finally, in the limiting case where $\partial f/\partial\sigma_{ij}\ \dot{\sigma}_{ij}^{e} = 0$ we have neutral loading, which may formally be treated as the development of plastic strains even though the incremental response is purely elastic. We may summarize these findings into

$$\boxed{\begin{array}{llll} f = 0 & \textit{and} & \dfrac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{e} > 0 \Rightarrow & \textit{plastic loading} \\ f = 0 & \textit{and} & \dfrac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{e} = 0 \Rightarrow & \textit{neutral loading} \\ f = 0 & \textit{and} & \dfrac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{e} < 0 \Rightarrow & \textit{elastic unloading} \end{array}} \tag{10.38}$$

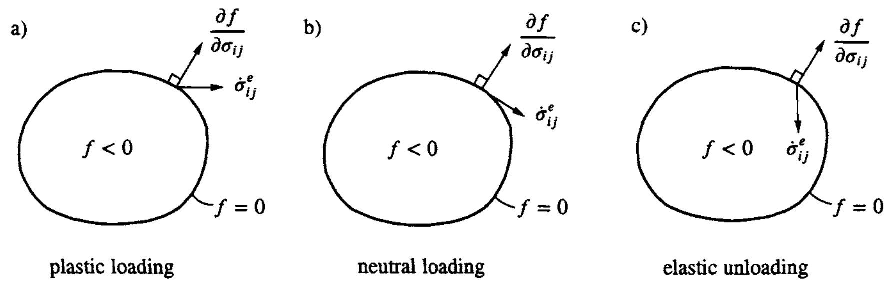

Figure 10.4: General loading/unloading criteria.

These general loading and unloading criteria are illustrated in Fig. 10.4.

We emphasize that the loading/unloading criteria (10.38) hold in general irrespective of whether we have hardening, ideal or softening plasticity. In these criteria, the elastic stress rate $\dot{\sigma}^e_{ij}$ given by (10.36) plays an important role. It is of significant interest that the term $\dot{\sigma}^e_{ij}$ also turns out to be of major importance in the numerical treatment of the constitutive equation, as will be discussed later on in Chapter 18. Moreover, it is emphasized that these loading/unloading criteria are not postulated, but they rather follow from the assumption that the plastic multiplier is non-negative, i.e. $\dot{\lambda} \geq 0$. Due to (10.36), the loading/unloading criteria (10.38) are strain driven.

Previously, we established the loading/unloading criteria given by (9.53) which formally have the same structure as the criteria (10.38). However, we claimed that (9.53) is not of general validity and we will now prove this point. From the consistency relation (10.16) we obtain with (10.37)

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = \frac{H}{A}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}^e_{ij} \tag{10.39}$$

With the definitions (10.33), evaluation of (10.39) shows that the general loading/unloading criteria (10.38) only coincide with the loading/unloading criteria (9.53) when $H > 0$, i.e. when we have hardening plasticity.

To further substantiate this conclusion, we may consider softening plasticity obtained via an isotropic 'hardening' concept where the yield surface shrinks. Figure 10.5 shows this situation and it is obvious that during the plastic deformation we have $\partial f/\partial \sigma_{ij}\, \dot{\sigma}_{ij} < 0$ in contradiction with the criteria (9.53).

If we have hardening plasticity, we may equally well use the loading criteria (9.53). However, even in this case, there are significant advantages relating to the general format (10.38) when nonlinear finite element calculations are considered. In a nonlinear finite element scheme, we know the nodal displacement increments from which the total strain increments $d\epsilon_{ij}$ can be determined di-

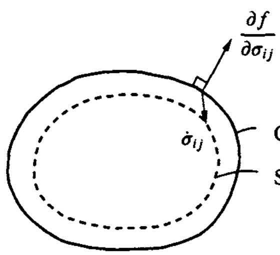

Figure 10.5: Softening plasticity modelled by an isotropic 'hardening' rule.

rectly, i.e. also the elastic stress increments $d\sigma_{ij}^e$ are known directly from (10.36) and the general criteria (10.38) can therefore be applied directly. If, however, the criteria (9.53) are used, we need to determine the stress increments $d\sigma_{ij}$ from (10.26) which require a (numerical) integration since the elasto-plastic stiffness tensor D_{ijkl}^{ep} varies with the loading. Therefore also for hardening plasticity, the general criteria (10.38) are preferable to the criteria (9.53) and we will return to this subject in Chapter 18.

The general criteria (10.38) were, in fact, introduced already by Hill (1958), but it took some time before the generality and advantages of this format were recognized in the literature.

The inability of the stress driven loading/unloading criteria (9.53) to include ideal and softening materials led Naghdi and Trapp (1975), see also Yoder and Iwan (1981), to propose a so-called *strain space plasticity*. This means that, for instance, the yield criterion is expressed as $f(\varepsilon_{ij}, K_\alpha)$ instead of $f(\sigma_{ij}, K_\alpha)$. However, due to (10.36) the loading/unloading criteria (10.38) are strain driven and since they apply to hardening, ideal and softening plasticity, there is no need to resort to the full strain space formulation mentioned above.

10.5 Plane strain

In practice, the two-dimensional problem of plane strain is of great importance and it is easy to derive the general plasticity equations also in that case. Focus will be directed to the strain driven format.

Let the plane of interest be described by the x_1x_2-coordinates, i.e. the x_3-direction denotes the out-of-plane direction. Per definition, plane strain is then characterized by

$$\dot{\varepsilon}_{13} = \dot{\varepsilon}_{23} = \dot{\varepsilon}_{33} = 0$$

Let Greek indices take the values 1, 2 whereas Latin indices, as usual, take the values 1, 2 and 3. From (10.26) and (10.27), the relation between the in-plane stress rate $\dot{\sigma}_{\alpha\beta}$ and the in-plane total strain rates $\dot{\epsilon}_{\alpha\beta}$ is then given by

$$\boxed{\dot{\sigma}_{\alpha\beta} = D^{ep}_{\alpha\beta\gamma\delta}\dot{\epsilon}_{\gamma\delta}} \tag{10.40}$$

where

$$\boxed{D^{ep}_{\alpha\beta\gamma\delta} = D_{\alpha\beta\gamma\delta} - \frac{1}{A}D_{\alpha\beta st}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}}D_{mn\gamma\delta}} \tag{10.41}$$

One may note the simultaneous occurrence of Greek and Latin indices in this latter expression. For convenience, we again list expression (10.24) for the parameter A, i.e.

$$\boxed{A = H + \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}}; \qquad A > 0} \tag{10.42}$$

For isotropic elasticity, we obtain from (4.110)

$$\boxed{D_{\alpha\beta\gamma\delta} = 2G[\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \tfrac{\nu}{1-2\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}]} \tag{10.43}$$

We also emphasize that the out-of-plane stress rates $\dot{\sigma}_{13}$, $\dot{\sigma}_{23}$ and $\dot{\sigma}_{33}$, in general, need to be determined from (10.26) since these stresses, in general, are non-zero and since they, in general, enter the expression for the yield function. Expression (10.26) leads to

$$\dot{\sigma}_{i3} = D^{ep}_{i3\gamma\delta}\dot{\epsilon}_{\gamma\delta}$$

In general, also the out-of-plane plastic components $\dot{\epsilon}^{p}_{13}$, $\dot{\epsilon}^{p}_{23}$ and $\dot{\epsilon}^{p}_{33}$ are of interest since they may enter the internal variables κ_α. From (10.25) and the flow rule follow that

$$\dot{\epsilon}^{p}_{i3} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{i3}} \quad \text{where} \quad \dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{kl}}D_{kl\gamma\delta}\dot{\epsilon}_{\gamma\delta}$$

10.6 Plane stress

The plane of interest is given by the x_1x_2-coordinates, i.e. the x_3-direction denotes the out-of-plane direction. Per definition, plane stress is then defined by

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \tag{10.44}$$

The aim is to establish the strain driven format. It will turn out that for plane stress conditions, this derivation is much more involved than the corresponding derivations for the plane strain case. The problem is that we do not know beforehand the out-of-plane strain components, implying that we cannot make direct use of the formulation (10.26). On the other hand, since the out-of-plane stress components are known, we start instead with

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij}$$

Hooke's law then states

$$\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij} = C_{ijkl}\dot{\sigma}_{kl}$$

The plane stress condition reduces this expression to

$$\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij} = C_{ij\gamma\delta}\dot{\sigma}_{\gamma\delta} \tag{10.45}$$

i.e., the in-plane components are given by

$$\dot{\varepsilon}_{\alpha\beta} - \dot{\varepsilon}^p_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\dot{\sigma}_{\gamma\delta}$$

or

$$\dot{\sigma}_{\alpha\beta} = D^*_{\alpha\beta\gamma\delta}(\dot{\varepsilon}_{\gamma\delta} - \dot{\varepsilon}^p_{\gamma\delta}) \tag{10.46}$$

Here $D^*_{\alpha\beta\gamma\delta}$ denotes the inverse to $C_{\alpha\beta\gamma\delta}$, i.e.

$$D^*_{\alpha\beta\theta\psi}C_{\theta\psi\gamma\delta} = \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$

For isotropic elasticity, $D^*_{\alpha\beta\gamma\delta}$ is given by (4.114), i.e.

$$\boxed{D^*_{\alpha\beta\gamma\delta} = 2G[\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \tfrac{\nu}{1-\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}]} \tag{10.47}$$

The difference between $D_{\alpha\beta\gamma\delta}$ given by (10.43) and $D^*_{\alpha\beta\gamma\delta}$ given by (10.47) should be observed.

From the general flow rule (10.10), we obtain

$$\dot{\varepsilon}^p_{\alpha\beta} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{\alpha\beta}} \tag{10.48}$$

The plane stress conditions (10.44) reduce the general consistency relation (10.16) to

$$\frac{\partial f}{\partial \sigma_{\alpha\beta}}\dot{\sigma}_{\alpha\beta} - \dot{\lambda}H = 0 \tag{10.49}$$

where the plastic modulus H is defined as before, i.e. (10.17).

Insertion of the flow rule (10.48) into Hooke's law (10.46) gives

$$\dot{\sigma}_{\alpha\beta} = D^*_{\alpha\beta\gamma\delta}\dot{\varepsilon}_{\gamma\delta} - \dot{\lambda}D^*_{\alpha\beta\pi\theta}\frac{\partial g}{\partial \sigma_{\pi\theta}} \tag{10.50}$$

Multiplication by $\partial f/\partial\sigma_{\alpha\beta}$ and use of (10.49) then yield

$$\dot{\lambda} = \frac{1}{A^*}\frac{\partial f}{\partial\sigma_{\alpha\beta}}D^*_{\alpha\beta\gamma\delta}\dot{\varepsilon}_{\gamma\delta} \tag{10.51}$$

where A^* is defined by

$$\boxed{A^* = H + \frac{\partial f}{\partial\sigma_{\alpha\beta}}D^*_{\alpha\beta\gamma\delta}\frac{\partial g}{\partial\sigma_{\gamma\delta}}\ ;\quad A^* > 0} \tag{10.52}$$

Finally, insertion of (10.51) into (10.50) results in

$$\boxed{\dot{\sigma}_{\alpha\beta} = D^{*ep}_{\alpha\beta\gamma\delta}\dot{\varepsilon}_{\gamma\delta}} \tag{10.53}$$

where

$$\boxed{D^{*ep}_{\alpha\beta\gamma\delta} = D^*_{\alpha\beta\gamma\delta} - \frac{1}{A^*}D^*_{\alpha\beta\pi\theta}\frac{\partial g}{\partial\sigma_{\pi\theta}}\frac{\partial f}{\partial\sigma_{\phi\psi}}D^*_{\phi\psi\gamma\delta}} \tag{10.54}$$

It is of interest to compare (10.52)-(10.54) with the corresponding formulation (10.40)-(10.42) applicable to plane strain. First of all, only Greek indices enter the plane stress formulation whereas a mixture of Greek and Latin indices enters the plane strain formulation. Second, whereas only $D^*_{\alpha\beta\gamma\delta}$ enters the plane stress case, both $D_{\alpha\beta\gamma\delta}$, $D_{\alpha\beta st}$ and D_{ijkl} enter the plane strain case. Third, the difference between A^* and A should be recognized.

For plane stress, it may be of importance to determine the out-of-plane plastic components $\dot{\varepsilon}^p_{i3}$, since they may enter the internal variables κ_α. From the flow rule (10.10) we have

$$\dot{\varepsilon}^p_{i3} = \dot{\lambda}\frac{\partial g}{\partial\sigma_{i3}} \tag{10.55}$$

where $\dot{\lambda}$ is given by (10.51). In general, we observe that $\partial g/\partial\sigma_{i3} \neq 0$ even though $\sigma_{i3} = 0$.

Moreover, the out-of-plane components $\dot{\varepsilon}_{i3}$ is determined from (10.45), i.e.

$$\dot{\varepsilon}_{i3} = \dot{\varepsilon}^p_{i3} + C_{i3\gamma\delta}\dot{\sigma}_{\gamma\delta}$$

where $\dot{\varepsilon}^p_{i3}$ is given by (10.55) and $\dot{\sigma}_{\gamma\delta}$ by (10.53).

The present derivation of the plane stress relations differs from most expositions. Here, we interpret $\partial f/\partial\sigma_{ij}$ and $\partial g/\partial\sigma_{ij}$ in the same manner as in the general three-dimensional case. This means that we use the general three-dimensional formulation of f and g, perform the differentiations $\partial f/\partial\sigma_{ij}$ and $\partial g/\partial\sigma_{ij}$ and then, finally, introduce the plane stress conditions into these expressions. In other expositions, the plane stress conditions are directly introduced into the expressions for f and g and then the differentiations are performed. In general, the two approaches differ and the advantage of the present approach is that all quantities have the same meaning, irrespective of whether the three-dimensional case, the plane strain case or the plane stress case is considered. In turn, this means that, for instance, the plastic multiplier $\dot{\lambda}$ and the plastic modulus H have the same interpretation in all these cases.

This discussion also illustrates an interesting observation, namely that the most general case, the three-dimensional case, preserves all symmetry properties of the problem whereas these properties may be less apparent when specialized conditions are considered. We are then led to the somewhat surprising conclusion that it is easier to derive general three-dimensional elasto-plasticity than plane stress elasto-plasticity.