Problems in Constitutive Modeling

September 3, 2021

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June 2020

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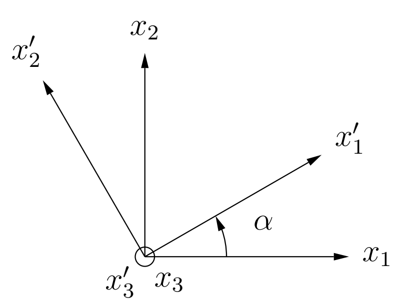
1. Notations and Cartesian tensors

Problem 1.1. The x_i-coordinate system is transformed to the $x_i^{'}$-coordinate system by the transformation matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \frac{1}{25}\begin{bmatrix} 12 & -9 & 20 \\ 15 & 20 & 0 \\ -16 & 12 & 15 \end{bmatrix}$$

Show that the point $(0, 1, -1)$ in the x_i-system coincides with the point $(-29/25$, $4/5$, $-3/25)$ in the $x_i^{'}$-system.

Problem 1.2. The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new $x_i^{'}$-coordinate system, see figure.



The relation between the x_i- and the $x_i^{'}$-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{x}^{'}$$

a) Determine for this special case $\boldsymbol{A}^T$ and $\boldsymbol{A}$

b) Show for this special case that

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}\ , \quad \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$$

i.e., that $\boldsymbol{A}$ is an orthogonal matrix (this is a general property of $\boldsymbol{A}$).

Problem 1.3. If Φ is a scalar, show the following:

a) $\Phi_{,i}$ is a first-order tensor.

b) $\Phi_{,ij}$ is a second-order tensor.

c) $\Phi_{,kk}$ is a zero-order tensor (scalar).

Problem 1.4. Using the transformation matrix $\boldsymbol{A}$ given in problem 1.1, show that the following two planes coincide:

$$2x_1 - \frac{1}{3}x_2 + x_3 = 1 \qquad \text{expressed in the } x_i\text{-system}$$

$$\frac{47}{25}x_1^{'} + \frac{14}{15}x_2^{'} - \frac{21}{25}x_3^{'} = 1 \qquad \text{expressed in the } x_i^{'}\text{-system.}$$

Problem 1.5. If $b_i = a_i/\sqrt{a_j a_j}$, show that b_i is a unit vector.

Problem 1.6. For Kronecker's delta δ_{rs} show that $\delta_{ij}\delta_{jk} = \delta_{ik}$.

Problem 1.7. Given the relations

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$J_2 = \frac{1}{2}s_{ij}s_{ji}$$

where σ_{ij} and s_{ij} are symmetric second-order tensors, show that:

a) $s_{ii} = 0$

b) $\partial J_2/\partial\sigma_{ij} = s_{ij}$

Problem 1.8. Prove that there is no pair of vectors a_i and b_i such that $a_i b_j = \delta_{ij}$.

Problem 1.9. For an arbitrary second-order tensor σ_{ij}, define s_{ij} by

$$s_{ij} = \sigma_{ij} - \alpha\delta_{ij}$$

Determine α such that $s_{ii} = 0$. The quantity s_{ij} is called the *deviatoric part* of σ_{ij}. Prove that if σ_{ij} is symmetric, also s_{ij} is symmetric.

Problem 1.10. Show that if all components of a tensor vanish in one coordinate system, then they vanish in all other coordinate systems.

Problem 1.11. Prove the theorem: The sum or difference of two tensors of the same type is again a tensor of the same type.

Problem 1.12. If a_{ij} is a tensor and the components $a_{ij} = a_{ji}$, then the tensor is called a *symmetric tensor*. If the components $a_{ij} = -a_{ji}$, then the tensor is said to be *anti-symmetric*. Show that these symmetry properties are conserved under coordinate transformations.

Problem 1.13. Hooke's law for a linear elastic material can be written

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl} \tag{1}$$

where D_{ijkl} is the elastic stiffness matrix. The expression in (1) is valid in the x_i-coordinate system. If we instead express (1) in a new $x_i^{'}$-coordinate system we get

$$\sigma_{ij}^{'} = D_{ijkl}^{'}\epsilon_{kl}^{'}$$

where the x_i- and $x_i^{'}$-coordinates are related to each other by

$$x_i^{'} = A_{ij}x_j - c_i$$

and A_{ij} is the coordinate transformation matrix. This matrix fulfils the conditions

$$A_{ki}A_{kj} = \delta_{ij}\ , \qquad A_{ik}A_{jk} = \delta_{ij}$$

Since D_{ijkl} is a fourth-order tensor, the components D_{ijkl} and $D_{ijkl}^{'}$ are related by

$$D_{ijkl}^{'} = A_{im}A_{jn}D_{mnpq}A_{kp}A_{lq} \tag{2}$$

If (1) expresses Hooke's law for isotropic materials, D_{ijkl} is given as

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \tag{3}$$

where G is the shear modulus and ν is Poisson's ratio. Calculate $D_{ijkl}^{'}$ using (2) and (3) and comment upon the result.

2. Strain tensor

Problem 2.1. Prove formally that the Cauchy strain invariants

$$\theta_1 = \epsilon_{ii} \qquad \theta_2 = \frac{1}{2}\theta_1^2 - \frac{1}{2}\epsilon_{ij}\epsilon_{ji} \qquad \theta_3 = \det(\epsilon_{ij})$$

are invariants.

<u>Hint:</u> Determine the invariants in a new coordinate system and use the transformation rules for a tensor. Moreover, since $\det \boldsymbol{A} = 1$, we have

$$\det(\boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T) = \det \boldsymbol{A} \cdot \det \boldsymbol{\epsilon} \cdot \det \boldsymbol{A} = (\det \boldsymbol{A})^2 \cdot \det \boldsymbol{\epsilon} = \det \boldsymbol{\epsilon}$$

Problem 2.2. Prove that $J_3^{'} = \frac{1}{3}e_{ij}e_{jk}e_{ki}$ can be expressed as

$$J_3^{'} = \frac{1}{3}(e_1^3 + e_2^3 + e_3^3)$$

Problem 2.3. The displacement field for a deformed body is given by

$$u_i = k(x_1^2,\ 2x_1x_2,\ 2x_1x_3)$$

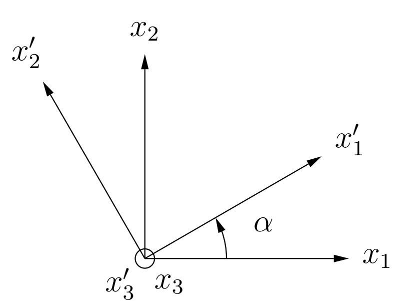
where k is a constant.

a) Determine the strain tensor ϵ_{ij}.

b) At the point (1, 2, 3) determine the tensorial shear strain and the engineering shear strain between the two directions defined by the orthogonal vectors (3, 4, 0) and $(\frac{1}{3},\ -\frac{1}{4},\ \frac{1}{4})$. Determine also the normal strain in the direction of (3, 4, 0).

c) At the point (1, 4, 0) determine the principal strains and the corresponding principal directions.

Problem 2.4. The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new $x_i^{'}$-coordinate system, see figure.



The relation between the x_i- and the $x_i^{'}$-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}^{'}$$

In this particular case, we have

$$\boldsymbol{A}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Since the strain tensor ϵ_{ij} is a second-order tensor, the components ϵ_{ij} in the x_i-coordinate system and the corresponding components $\epsilon^{'}_{ij}$ in the $x^{'}_i$-coordinate system are related to each other by

$$\epsilon^{'}_{kl} = A_{ki}\epsilon_{ij}A_{lj} \qquad \text{or} \qquad \boldsymbol{\epsilon}^{'} = \boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T$$

Suppose that ϵ_{ij} is given by

$$[\epsilon_{ij}] = \frac{10^{-4}}{2}\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Calculate the components of $\epsilon^{'}_{ij}$.

b) From the result in a), calculate the $\epsilon^{'}_{ij}$-components when the angle α is chosen as α=45^o. Comment upon the result.

Problem 2.5. If the strain tensor ϵ_{ij} is given as

$$[\epsilon_{ij}] = \frac{10^{-4}}{2}\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

a) Determine the principal strains and the corresponding principal directions.

b) Compare the result in a) with the result in problem 2.4 b) and comment upon the result.

3. Stress tensor

Problem 3.1. The deviatoric stress tensor s_{ij} is defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$$

Show that the principal directions for s_{ij} and σ_{ij} coincide.

Problem 3.2. Using tensor notation, Hooke's law for general isotropic elastic behaviour can be written as

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{1}$$

where E = Young's modulus and ν = Poisson's ratio.

a) Do the principal directions for ϵ_{ij} and σ_{ij} coincide or not? Prove your statement.

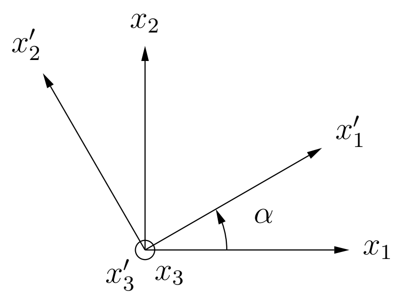
b) Using matrix notation, (1) can be written as $\boldsymbol{\epsilon} = \boldsymbol{C}\boldsymbol{\sigma}$ where

$$\boldsymbol{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23})$$

$$\boldsymbol{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$$

Determine the matrix $\boldsymbol{C}$.

Problem 3.3. The original cartesian x_i-coordinate system is rotated an angle α about the x_3-axis to get a new $x_i^{'}$-coordinate system, see figure.



The relation between the x_i- and the $x_i^{'}$-coordinate system can be written in matrix form as

$$\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{x}^{'}$$

In this particular case, we have

$$\boldsymbol{A}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Since the stress tensor σ_{ij} is a second-order tensor, the components σ_{ij} in the x_i-coordinate system and the corresponding components $\sigma_{ij}^{'}$ in the $x_i^{'}$-coordinate system are related to each other by

$$[\sigma_{kl}^{'}] = A_{ki}\sigma_{ij}A_{lj} \qquad \text{or} \qquad \boldsymbol{\sigma}^{'} = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$$

Suppose that σ_{ij} is given by

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

Calculate the components of σ'_{ij}.

b) From the result in a), calculate the σ'_{ij}-components when the angle α is chosen as α=45^o. Comment upon the result.

Problem 3.4. If σ_{ij} is given as

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

a) Determine the principal stresses and the corresponding principal directions.

b) Compare the result in a) with the result in problem 3.3 b) and comment upon the result.

Problem 3.5. Assume that the coordinate system is colinear with the principal directions of the stress tensor.

a) Determine the traction vector $\boldsymbol{t}$ on the surface, where the outer normal unit vector $\boldsymbol{n}$ is given by $\boldsymbol{n}^T = (1,\, 1,\, 1)/\sqrt{3}$.

b) The component of $\boldsymbol{t}$ along $\boldsymbol{n}$ is called the normal stress σ_n and the component of $\boldsymbol{t}$ along the surface is called the shear stress τ_n. For this particular choice of coordinate system and $\boldsymbol{n}$-vector the following notation is often employed:

$$\sigma_n = \sigma_o = \text{octahedral normal stress.}$$

$$\tau_n = \tau_o = \text{octahedral shear stress.}$$

Show that

$$\tau_n^2 = \tau_o^2 = \frac{2}{3}J_2 \quad \text{where} \quad J_2 = \frac{1}{2}tr(\boldsymbol{s}^2) = \frac{1}{2}s_{ij}s_{ji}$$

$$\sigma_n = \sigma_o = \frac{1}{3}I_1 \quad \text{where} \quad I_1 = tr\boldsymbol{\sigma} = \sigma_{kk}$$

Problem 3.6. A circular disk without any holes and made of an arbitrary material is loaded along the circular boundary by a uniform radial pressure $p_1 > 0$.

a) Prove that all boundary conditions and equilibrium conditions are fulfilled by the following stress state

$$\sigma_{11} = \sigma_{22} = -p_1 \qquad \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

where the x_1- and x_2-axes are located in the disk plane.

b) Prove that the stress state given above is not a proper stress state, if there also exists a circular hole in the center of the disk loaded by a uniform radial pressure $p_2 > 0$, unless $p_2 = p_1$.

Problem 3.7. At a point the following stress state is given

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = p \quad \text{and} \quad \sigma_{33} = 0$$

a) By hand-calculations, determine the principal stresses and directions and check that the principal directions are orthogonal.

b) Same as a), but now use CALFEM or another program; compare the results.

c) The original coordinate system is given by the x_i-coordinates. A new x_i'-coordinate system is chosen to be colinear with the principal directions. The relation between the new and old coordinates can be written as $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Determine $\boldsymbol{A}$ and show that for the particular $\boldsymbol{A}$-matrix in question, we have $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, i.e. $\boldsymbol{A}$ is an orthogonal matrix (this is a general property of $\boldsymbol{A}$).

d) Demonstrate by inspection that the $\boldsymbol{\sigma}$-matrix given above fulfils Cayley-Hamilton's theorem.

Problem 3.8. In the x_i-coordinate system an arbitrary stress matrix $\boldsymbol{\sigma}$ is given. A new x_i'-coordinate system is chosen, so that the unit vectors along the x_1'-, x_2'- and x_3'-axes are given by the components $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}/\sqrt{2}$, $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}/\sqrt{2}$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, respectively, in the old coordinate system. Determine the stress matrix $\boldsymbol{\sigma}'$ in the new coordinate system, which corresponds to $\boldsymbol{\sigma}$ in the old coordinate system.

Problem 3.9. For the unit vector $\mathbf{n}$ and the parameter k, we can define a stress tensor $\boldsymbol{\sigma}$ given by

$$\boldsymbol{\sigma} = k\boldsymbol{n}\boldsymbol{n}^T$$

Show that this stress tensor corresponds to pure tension in the direction $\mathbf{n}$.

Problem 3.10. A stress tensor is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Determine σ_{11} so that a section will exist on which the traction vector $\mathbf{t}$ is zero. Determine a unit vector normal to that section.

4. Hyper-elasticity

Problem 4.1. Prove the following relations

$$\frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} = \delta_{ij} \quad , \qquad \frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} = \epsilon_{ij} \quad , \qquad \frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}} = \epsilon_{ik}\epsilon_{kj}$$

Problem 4.2. Determine the results of the partial derivatives

$$\frac{\partial I_1}{\partial \sigma_{ij}} \quad , \qquad \frac{\partial J_2}{\partial \sigma_{ij}} \quad , \qquad \frac{\partial J_3}{\partial \sigma_{ij}}$$

Problem 4.3. Let D_{ijkl} be given by

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]$$

Show that $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$ then implies

$$\sigma_{ij} = 2G\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right]$$

Demonstrate also that D_{ijkl} fulfils the symmetry properties $D_{ijkl} = D_{jikl}$, $D_{ijkl} = D_{ijlk}$ and $D_{ijkl} = D_{klij}$.

Problem 4.4. For linear isotropic elasticity we have

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]$$

and

$$C_{ijkl} = \frac{1}{2G}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\nu}{1+\nu}\delta_{ij}\delta_{kl}\right]$$

Show that these expressions fulfil the equation

$$D_{ijmn}C_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Problem 4.5. We have $\epsilon_{ij} = C_{ijkl}\ \sigma_{kl}$. Multiply with D_{pqij} and show that we obtain $\sigma_{ij} = D_{ijkl}\ \epsilon_{kl}$.

Problem 4.6. We have $\sigma_{ij} = D_{ijkl}\ \epsilon_{kl}$. Write this equation in matrix form $\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$ and identify all terms. For isotropic elasticity show that $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

Problem 4.7. Use superposition of simple loading situations to show that the strain-stress relation for an isotropic material can be written as

$$\boldsymbol{\epsilon} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\boldsymbol{\sigma}$$

Hint, to obtain the relations between shear strains and stresses use can be made of Mohr's circles of stress and strain.

Problem 4.8. For a general isotropic hyper-elastic material the strain energy can be written in term of the strain invariants, i.e.

$$W = W(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$$

where

$$\tilde{I}_1 = \epsilon_{kk}, \quad \tilde{I}_2 = \frac{1}{2}\epsilon_{ij}\epsilon_{ji}, \quad \tilde{I}_3 = \frac{1}{3}\epsilon_{ik}\epsilon_{kj}\epsilon_{ji}$$

a) Determine the strain-stress relation based on

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

b) Based on a) and that

$$\frac{\partial W}{\partial \tilde{I}_1} = \lambda \epsilon_{kk}$$

determine that fourth-order tensor D^s_{ijkl} given by

$$\sigma_{ij} = D^s_{ijkl}\epsilon_{kl}$$

c) Derive the incremental relation

$$d\sigma_{ij} = D^t_{ijkl} d\epsilon_{kl} \qquad \text{where} \qquad D^t_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

and identify D^t_{ijkl}. Assume that

$$\frac{\partial W}{\partial \tilde{I}_2} \quad \text{and} \quad \frac{\partial W}{\partial \tilde{I}_3} \quad \text{are constants}$$

d) In a uniaxial loading situation, i.e. in a $\sigma - \epsilon$-graph, identify D^s and D^t (the corresponding uniaxial quantities). No calculations are necessary, a graphical illustration is sufficient.

Problem 4.9. The strain energy can be written as $W = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$. Show that it also can be written as $W = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\sigma}$.

Problem 4.10. A hyper-elastic material is assumed to possess the following complementary energy

$$C = aJ_2 + bI_1J_2 \tag{1}$$

where a and b are constants.

a) Derive the constitutive relation $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$.

b) For uniaxial tension, the stress-strain behaviour as experienced in the laboratory can be approximated by

$$\epsilon = 10^{-4}\sigma + 10^{-5}\sigma^2$$

where σ is measured in MPa. Determine the parameters a and b in (1).

c) For loading in pure shear where $\tau = 100$ MPa calculate the corresponding strains.

d) For the load defined in c), determine the volumetric strain ϵ_{kk} and compare with the volumetric strain for a linear elastic isotropic material.

Problem 4.11. Using two perpendicular symmetry planes show that the general (orthotropic) hyper-elastic stress-strain relation can be written as

$$\boldsymbol{\sigma} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \boldsymbol{\epsilon}$$

What will the use of a third symmetry plane yield?

Problem 4.12. Similar to problem (4.7), use simple loading and engineering definitions for the elastic moduli that the othotropic strain-stress relation can be written as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix} \boldsymbol{\sigma}$$

Give also a physical interpretation of the elastic constants, as well as, assuming hyper-elasticity comment upon the symmertry reqirements.

Problem 4.13. Derive the constitutive matrix $\boldsymbol{D}$ corresponding to plane stress for an orthotropic material.

5. Cauchy-elasticity

Problem 5.1. Consider isotropic hyper-elasticity. The most general format of the complementary energy C is then given by

$$C = C(I_1,\ J_2,\ J_3)$$

a) Derive the most general hyper-elasticity format $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$.

b) Write the most general non-linear Hooke formulation for hyper-elasticity expressed in terms of the non-linear parameters G and K. Can G and K depend on J_3?

c) Consider isotropic Cauchy-elasticity. What is the most general Cauchy-elasticity format $\epsilon_{ij} = \epsilon_{ij}(\sigma_{kl})$?

d) Write the most general non-linear Hooke formulation for Cauchy-elasticity expressed in terms of the nonlinear parameters G and K. Can G and K depend on J_3?

e) For which materials are the influence of J_3 important?

f) For nonlinear elasticity, what is the difference in response during loading and unloading? Which materials behave like nonlinear elasticity during unloading?

6. Representation theorems

Problem 6.1. Prove by inspection that

$$\boldsymbol{\epsilon} = \alpha_1 \boldsymbol{I} + \alpha_2 \boldsymbol{\sigma} + \alpha_3 \boldsymbol{\sigma}^2$$

satisfies the conditions of coordinate invariance and isotropy.

Problem 6.2. Write in index notation the tensor generators $\mathbf{G}_1 \ldots \mathbf{G}_8$ given by

$$\begin{array}{l} \boldsymbol{G}_1 = \boldsymbol{I} \ ; \ \boldsymbol{G}_2 = \boldsymbol{N} \ ; \ \boldsymbol{G}_3 = \boldsymbol{N}^2 \ ; \ \boldsymbol{G}_4 = \boldsymbol{P} \ ; \ \boldsymbol{G}_5 = \boldsymbol{P}^2 \\ \boldsymbol{G}_6 = \boldsymbol{N}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N} \ ; \ \boldsymbol{G}_7 = \boldsymbol{N}^2\boldsymbol{P} + \boldsymbol{P}\boldsymbol{N}^2 \ ; \\ \boldsymbol{G}_8 = \boldsymbol{N}\boldsymbol{P}^2 + \boldsymbol{P}^2\boldsymbol{N} \end{array}$$

Problem 6.3. Derive the constitutive relation for the Maxwell model and the Kevin model and determine the responses for a sudden applied load for the uniaxial loading situation.

7. Hypo-elasticity

Problem 7.1. Equation

$$\begin{aligned} \dot{\sigma}_{ij} \ = \ & \beta_1 \dot{\epsilon}_{kk} \delta_{ij} + \beta_2 \dot{\epsilon}_{ij} + \beta_3 \dot{\epsilon}_{kk} \sigma_{ij} + \beta_4 \sigma_{mn} \dot{\epsilon}_{mn} \delta_{ij} \\ & + \beta_5 (\sigma_{ik} \dot{\epsilon}_{kj} + \dot{\epsilon}_{ik} \sigma_{kj}) + \beta_6 \dot{\epsilon}_{mm} \sigma_{ik} \sigma_{kj} + \beta_7 \sigma_{mn} \dot{\epsilon}_{nm} \sigma_{ij} \\ & + \beta_8 \sigma_{lm} \sigma_{mn} \dot{\epsilon}_{nl} \delta_{ij} + \beta_9 (\sigma_{ik} \sigma_{kl} \dot{\epsilon}_{lj} + \dot{\epsilon}_{ik} \sigma_{kl} \sigma_{lj}) \\ & + \beta_{10} \sigma_{mn} \dot{\epsilon}_{nm} \sigma_{ik} \sigma_{kj} + \beta_{11} \sigma_{lm} \sigma_{mn} \dot{\epsilon}_{nl} \sigma_{ij} \\ & + \beta_{12} \sigma_{lm} \sigma_{mn} \dot{\epsilon}_{nl} \sigma_{ik} \sigma_{kj} \end{aligned}$$

may be written as $\dot{\sigma}_{ij} = D_{ijst}\dot{\epsilon}_{st}$. Determine the tensor D_{ijst}.

8. Failure and initial yield criteria

Problem 8.1. For the stress tensor σ_{ij} given by:

$$\begin{bmatrix} \frac{3}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{11}{4} & -\frac{5}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{5}{4} & \frac{11}{4} \end{bmatrix} \text{MPa}$$

a) Find the principal stresses and the corresponding principal directions.

b) Find the deviatoric stress tensor, s_{ij}, and the principal deviatoric stresses s_1, s_2 and s_3.

c) Determine the deviatoric stress invariants J_1, J_2 and J_3.

Problem 8.2. Answer the following questions and explain the answer.

a) If $s_1 > s_2 > s_3$, can s_3 be equal to zero?

b) Can J_2 be negative?

c) Can J_3 be positive?

Problem 8.3. Show that subtracting a hydrostatic stress from a given state of stress does not change the principal directions.

Problem 8.4. The stress state at a point is given by

$$[\sigma_{ij}] = \begin{bmatrix} 30 & 45 & 60 \\ 45 & 20 & 50 \\ 60 & 50 & 10 \end{bmatrix} \text{MPa}$$

Determine the stress invariants I_1, J_2, J_3 and the Lode angle θ.

Problem 8.5. A metal yields when the maximum shear stress, τ_{max}, reaches the value of 125 MPa. A material element of this metal is subjected to a biaxial state of stress:

$$\sigma_1 = \sigma\,; \quad \sigma_2 = \alpha\sigma\,; \quad \sigma_3 = 0$$

where α is a constant and σ is positive. For what values of (σ, α) will yielding occur?

Problem 8.6. For biaxial stress states and adopting the Tresca criterion, draw the yield curve in the $\sigma_1\sigma_2$-coordinate system. Note that in this coordinate system, the usual convention of $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is abandoned.

Problem 8.7. A metal yields at a state of plane stress with

$$\sigma_{11} = 80 \text{ MPa}, \quad \sigma_{22} = 40 \text{ MPa}, \quad \sigma_{12} = 80 \text{ MPa}$$

Assume isotropy, independence of hydrostatic pressure, and equality of properties for reversed loading (for instance, that tension and compression gives the same yield stress).

a) Derive other biaxial states of stress at yield in the (σ_1, σ_2)-space using the above informations.

b) Plot the result in part a) in the (σ_1, σ_2)-space and estimate the yield stress in axial tension and in simple shear, respectively, and give limits of possible error of your estimate, based on convexity.

c) Determine the yield stresses in b), based on the von Mises criterion and the Tresca criterion, respectively.

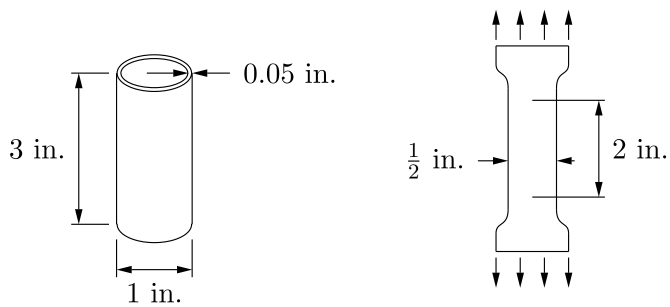
Problem 8.8. A long circular steel tube having a mean diameter of 254 mm and 3.2 mm wall thickness is subjected to an internal pressure of 4.83 MPa. The ends of the tube are closed. The yield stress of the steel is 227 MPa. Find the additional axial tensile load F which is needed to cause yielding of the tube, based on the von Mises criterion and the Tresca criterion, respectively.

Problem 8.9. The Coulomb failure criterion can be written as (assume $k > 1$)

$$k\sigma_1 - \sigma_3 = \sigma_c \tag{1}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses (tension is considered as positive) For plane stress conditions and considering only principal stresses show the failure curve corresponding to (1) in the $\sigma_1\sigma_2$-coordinate system (where the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$ now is abandoned). Express the uniaxial tensile strength in terms of σ_c and k.

Problem 8.10. A simple torsion test of certain material, using a hollow cylinder specimen as shown in the figure, shows that the load-deflection curve is linear for a shearing stress below 125 MPa and that at the stress 125 MPa yielding occurs. If the von Mises yield criterion is adopted, what is the expected value of the tensile stress at which yielding occur in a tension test specimen as shown in the figure?



Shear test

Tension test

Problem 8.11. A closed-ended thin-walled tube of thickness t and mean radius r is subjected to an axial tensile force F, which is less than the value F_o necessary to cause yielding. If a gradually increasing internal pressure p is now applied, show that the tube will yield according to the Tresca criterion when

$$\frac{pr}{t\sigma_{y0}} = \begin{cases} 1 & \text{when } \dfrac{F}{F_o} \leq \dfrac{1}{2} \\ 2(1 - \frac{F}{F_o}) & \text{when } \dfrac{F}{F_o} \geq \dfrac{1}{2} \end{cases}$$

and according to the von Mises criterion when

$$\frac{pr}{t\sigma_{y0}} = \frac{2}{\sqrt{3}}\sqrt{1 - (\frac{F}{F_o})^2}$$

Problem 8.12. Given the yield stresses σ_t and σ_c in uniaxial tension and compression, respectively, find the yield stress in shear resulting from the following yield criteria: a) Coulomb, b) Drucker-Prager, c) von Mises and d) Tresca.

Problem 8.13. Show that for a state of plane stress with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and $\sigma_{22} = 0$, both the Tresca and von Mises yield criterion can be expressed in the form

$$(\frac{\sigma}{\sigma_{yo}})^2 + (\frac{\tau}{\tau_{y0}})^2 = 1$$

How is σ_{y0} and τ_{y0} related for the Tresca and the von Mises criterion, respectively?

Problem 8.14. The initial yield criterion of Drucker-Prager is defined by

$$f = \sqrt{3J_2} + \alpha I_1 - \beta = 0$$

where α and β are parameters and

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \qquad s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \qquad I_1 = \sigma_{kk}$$

Consider the stress state

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) Assume that loading takes place such that $\tau = \sigma$. Calculate the value of σ for which yielding starts. Both $\sigma > 0$ and $\sigma < 0$ should be considered.

b) In the meridian plane, $\sqrt{3J_2} - I_1$, draw the shape of the Drucker-Prager yield criterion and the loading paths given by $\tau = \sigma$. Both $\sigma > 0$ and $\sigma < 0$ should be considered.

c) In the deviatoric plane illustrate the shape of the Drucker-Prager yield criterion and the loading path given by $\sigma = 0$ and $\tau \neq 0$. (one path is sufficient). Hint: the angle is given by

$$\cos(3\theta) = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \quad \text{where} \quad J_3 = \frac{1}{3} s_{ik} s_{kj} s_{ji}$$

Problem 8.15. von Mises isotropic criterion can be written as

$$s_{ij} P_{ijkl} s_{kl} - 1 = 0 \qquad \text{or} \qquad \boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s} - 1 = 0$$

where

$$P_{ijkl} = \frac{3}{4\sigma_{yo}^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Derive the matrix $\boldsymbol{P}$ corresponding to P_{ijkl}. Next, use that $s_{ii} = 0$ to derive an alternative format of the von Mises isotropic criterion, i.e.

$$\boldsymbol{s}^T \hat{\boldsymbol{P}} \boldsymbol{s} - 1 = 0 \qquad \text{where} \quad \hat{\boldsymbol{P}} \neq \boldsymbol{P}$$

Problem 8.16. Show that

$$\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} - 1 = 0$$

for the special choice

$$\boldsymbol{P} = \begin{bmatrix} F+G & -F & -G & 0 & 0 & 0 \\ -F & F+H & -H & 0 & 0 & 0 \\ -G & -H & G+H & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

can be written as

$$\boldsymbol{s}^T \boldsymbol{P} \boldsymbol{s} - 1 = 0$$

Problem 8.17. Derive the general format of $\boldsymbol{P}$ and $\boldsymbol{q}$ in

$$\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} - \boldsymbol{q}^T \boldsymbol{\sigma} - 1 = 0$$

if orthotropy and pressure independent response is assumed. How may independent parameters does the model have?

Problem 8.18. The von Mises criterion for orthotropy can be written as $\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma} - 1 = 0$ where

$$\boldsymbol{P} = \begin{bmatrix} A & -F & -G & 0 & 0 & 0 \\ -F & B & -H & 0 & 0 & 0 \\ -G & -H & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}$$

Establish (imagined) test methods such that all parameters can be determined.

9. Introduction to plasticity theory

Problem 9.1. Let the effective plastic strain rate $\dot{\epsilon}^p_{eff}$ and the effective stress σ_{eff} be defined by

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\, \dot{\epsilon}^p_{ij}\, \dot{\epsilon}^p_{ij})^{1/2} \; ; \qquad \sigma_{eff} = (\frac{3}{2}\, s_{ij}\, s_{ij})^{1/2}$$

Show that we then have the convenient properties that $\sigma_{eff} = \sigma$ holds for uniaxial tension and that $\dot{\epsilon}^p_{eff} = \dot{\epsilon}^p$ holds for uniaxial tension if plastic incompressibility and isotropy are assumed.

Problem 9.2. A von Mises material is considered. For isotropic hardening, we have

$$f = (\frac{3}{2}s_{ij}s_{ij})^{1/2} - \sigma_y = 0$$

whereas kinematic hardening is given by

$$f = [\frac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)]^{1/2} - \sigma_{y0} = 0$$

For associated plasticity, derive expressions for $\dot{\epsilon}_{ij}^p$.

Problem 9.3. Ideal plasticity according to the following criteria is considered:

a) the von Mises yield criterion: $\sqrt{3J_2} - \sigma_{yo} = 0$

b) the Tresca yield criterion: $\sigma_1 - \sigma_3 - \sigma_{yo} = 0$; $\quad \sigma_1 \geq \sigma_2 \geq \sigma_3$

c) the Drucker - Prager criterion: $\sqrt{3J_2} + \alpha I_1 - \beta = 0$

d) the Coulomb criterion: $k\sigma_1 - \sigma_3 - \sigma_{y0} = 0$;
$\sigma_1 \geq \sigma_2 \geq \sigma_3$

A material element is subjected to proportional loading. The principal stresses are given by (2σ, σ, 0) where σ is an increasing stress value. Find the magnitude of σ where the material begins to yield. Adopting the associated flow rule, find also the plastic strain rate $\dot{\epsilon}_{ij}^p$ at onset of yielding expressed in terms of the plastic multiplier $\dot{\lambda}$. If the effective plastic strain rate $\dot{\epsilon}_{eff}^p$ is defined as $\dot{\epsilon}_{eff}^p = (\frac{2}{3}\dot{\epsilon}_{ij}^p\dot{\epsilon}_{ij}^p)^{1/2}$, how is $\dot{\lambda}$ related to $\dot{\epsilon}_{eff}^p$?

Problem 9.4. The same problem as 9.3, but for the principal stresses (σ, σ, 0) and only considering the von Mises and the Drucker-Prager criteria. When considering the Tresca and the Coulomb criteria, what problem is encountered if you should determine $\dot{\epsilon}_{ij}^p$?

Problem 9.5. Isotropic hardening of a von Mises material is given by

$$f(\sigma_{ij}, K) = \sqrt{3J_2} - \sigma_{yo} - K(\kappa) = 0 \tag{1}$$

where

$$J_2 = \frac{1}{2}s_{kl}s_{kl} \quad ; \qquad s_{kl} = \sigma_{kl} - \frac{1}{3}\delta_{kl}\sigma_{pp}$$

and σ_{yo} is the initial yield stress in tension. The current yield stress σ_y is then given by

$$\sigma_y(\kappa) = \sigma_{yo} + K(\kappa)$$

i.e. (1) takes the form

$$f = \sqrt{3J_2} - \sigma_y(\kappa) = 0 \tag{2}$$

The associated flow rule provides

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} \quad ; \qquad \dot{\lambda} \geq 0 \tag{3}$$

The effective plastic strain rate is defined by

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij})^{1/2} \tag{4}$$

Strain hardening is assumed, i.e. the following evolution law for κ is assumed

$$\dot{\kappa} = \dot{\epsilon}^p_{eff}$$

a) Based on (2) and (3) determine the explicit form of the plastic strain rate, i.e. $\dot{\epsilon}^p_{ij}$.

b) From the definition (4) prove that we have

$$\dot{\epsilon}^p_{eff} = \dot{\lambda} \tag{5}$$

c) Consider uniaxial tension where

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

For this stress state prove that yielding requires that

$$\sigma = \sigma_y(\epsilon^p_{eff})$$

d) For the stress state defined by (6), identify the expression

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\begin{bmatrix} \\ ? \\ 3 \times 3 \end{bmatrix}$$

Denote $\dot{\epsilon}^p$ by $\dot{\epsilon}^p = \dot{\epsilon}^p_{11}$ and show that in the present case, we have

$$\dot{\epsilon}^p_{eff} = \dot{\epsilon}^p$$

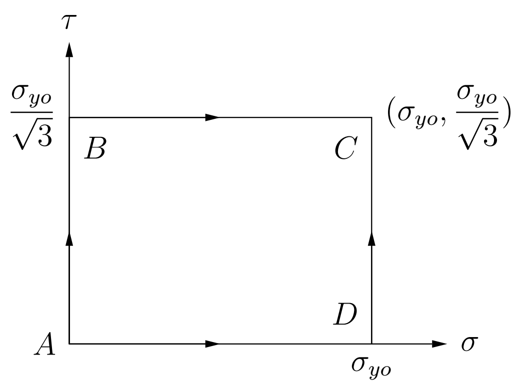
Problem 9.6. The initial yield stress is σ_{y0} and during uniaxial tension in the plastic regime, we have $d\sigma/d\epsilon_p = H$ where H is a constant (i.e. linear hardening). Investigate isotropic hardening of a von Mises material. Associated plasticity is adopted. Calculate the resulting elastic and plastic strains at point C for the load histories:

a) load path ABC

b) load path ADC

c) proportial loading, i.e. load path AC

d) calculate the curve $\sigma_{eff} = \sigma_{eff}(\epsilon^p_{eff})$ for the three load cases mentioned above and comment upon the result.



Note: $\int \frac{x^2\, dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\frac{1}{\sqrt{ab}} \arctan \frac{x\sqrt{ab}}{a}$

10. General plasticity theory

Problem 10.1. Write all necessary equations to establish $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$.

Problem 10.2. The interpretation of the plastic modulus in the uniaxial case is shown in the figure below.

Show that this is true for all plasticity models.

12. Common plasticity models

Problem 12.1. From $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\epsilon}_{kl}$ where

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{9G^2}{A}\frac{s_{ij}s_{kl}}{\sigma_y^2}$$

and

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]$$

derive the corresponding matrix format, i.e. $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\epsilon}}$.

Problem 12.2.

a) Derive the plane strain formula of $\dot{\boldsymbol{\sigma}} = \boldsymbol{D}^{ep}\dot{\boldsymbol{\epsilon}}$ in problem (12.1).

b) Determine also the expression for the out-of-plane stress rates $\dot{\sigma}_{13}$, $\dot{\sigma}_{23}$ and $\dot{\sigma}_{33}$.

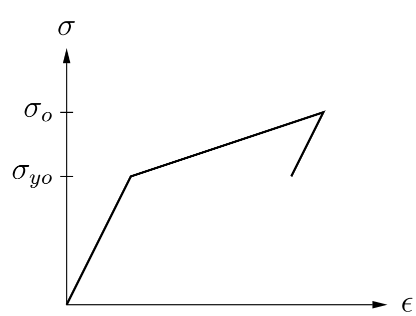
Problem 12.3. Using the von Mises model with the initial yield stress $\sigma_{y\circ}$, the following loading test is conducted:

$$(\sigma,\tau) = (0,0) \rightarrow (2\sigma_{y\circ},0) \rightarrow (0,2\sigma_{y\circ}) \rightarrow (2\sigma_{y\circ},2\sigma_{y\circ})$$

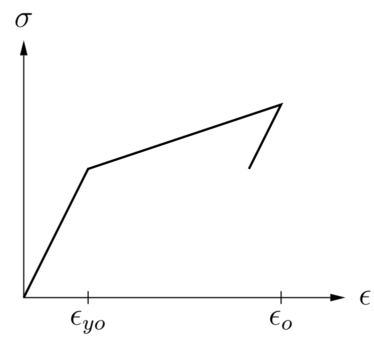
Assuming that the behaviour of this material follows the isotropic hardening rule, draw the initial yield surface and the subsequent yield surfaces in the $\sigma-\tau$ space at the ends of the loading paths. Note that in each loading step, the load is varied proportionally.

Problem 12.4. Show how i) the isotropic and ii) the kinematic von Mises bilinear model behaves in uniaxial loading when

a) the stress is cycled 4 times between σ_o and -σ_o.



b) the strain is cycled 4 times between ϵ_o and -ϵ_o



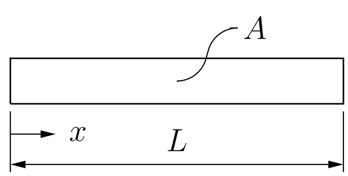
Problem 12.5. Using the Drucker-Prager criterion as the yield function $f(\sigma_{ij}, K)$, and the von Mises criterion as the plastic potential function $g(\sigma, K)$ in

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$$

derive the expression for the scalar $\dot{\lambda}$.

17. Solution of global equations

Problem 15.1. Euler forward scheme and Newton-Raphson equilibrium iterations. Consider a bar with constant area A and length L. No body forces act on the bar.



The equilibrium condition states that

$$\frac{d}{dx}(A\sigma) = 0 \tag{1}$$

a) Show that the weak form of (1) is given by

$$\int_0^L \frac{dv}{dx} A\sigma dx = [vA\sigma]_0^L \tag{2}$$

where v is an arbitrary weight function.

b) The axial displacement $u = u(x)$ (measured positive in the x-direction) is approximated by

$$u = \boldsymbol{N}\boldsymbol{a}$$

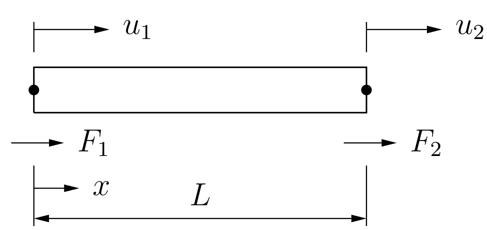
i.e. the axial strain $\epsilon = \epsilon(x)$ becomes

$$\epsilon = \boldsymbol{B}\boldsymbol{a} \qquad \text{where} \qquad \boldsymbol{B} = \frac{d\boldsymbol{N}}{dx}$$

Use the Galerkin method to express the equilibrium condition (2) as

$$\boldsymbol{\psi} = [\boldsymbol{N}^T A\sigma]_0^L - A\int_0^L \boldsymbol{B}^T \sigma dx\,; \qquad \boldsymbol{\psi} = \boldsymbol{0} \tag{3}$$

c) As indicated, the behaviour of the bar is approximated by one linear finite element. The figure shows the nodal displacements u_1 and u_2 as well as the external forces F_1 and F_2 acting on the nodal points.



Show that the equilibrium condition (3) then takes the form

$$\boldsymbol{\psi} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - \frac{A}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\int_0^L \sigma dx\,; \qquad \boldsymbol{\psi} = \boldsymbol{0} \tag{4}$$

d) The behaviour of the material is assumed to be given by the constitutive equation

$$\dot{\sigma} = E^{ep}\dot{\epsilon} \tag{5}$$

where

$$E^{ep} = E(1 - 2\alpha\epsilon) \tag{6}$$

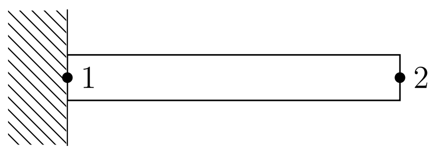
and E is Young's modulus whereas α is a dimensionless positive parameter. With this material model show that the incremental form of (4) takes the form

$$\frac{AE^{ep}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix} \tag{7}$$

(Hint: how does ϵ vary along the bar?; how does E^{ep} vary along the bar?)

e) The boundary conditions are given by

$$u_1 = 0\ ; \qquad F_2 \text{ prescribed} \tag{8}$$



Show that the part of the incremental formulation (7) that is of interest becomes

$$\frac{AE^{ep}}{L}\dot{u}_2 = \dot{F}_2 \tag{9}$$

Discuss how and (7) and (9) are related to (13.23). Likewise, show that the part of the equilibrium condition (4) that is of interest becomes

$$\psi = F_2 - A\sigma\,; \quad \psi = 0 \tag{10}$$

Discuss how (4) and (10) are related to (13.15) and (13.16).

f) With the definitions

$$K = \frac{AE^{ep}}{L}\ ; \qquad a = u_2\ ; \qquad f = F_2 \tag{11}$$

the incremental formulation (9) can be written

$$K\dot{a} = \dot{f} \tag{12}$$

whereas the equilibrium condition (10) takes the form

$$\psi = f - A\sigma\,; \quad \psi = 0 \tag{13}$$

We assume $\alpha = 10^2$.

g) Using the Euler forward scheme, determine the result for two load steps $f_1 = 10^{-3}AE$ and $f_2 = 2 \cdot 10^{-3}AE$.
(Result: $a_1 = 10^{-3}L$; $a_2 = 2.25 \cdot 10^{-3}L$).

h) Determine the correct response by integrating (5) exactly.
(Result: $a_1 = 1.127 \cdot 10^{-3}L$; $a_2 = 2.764 \cdot 10^{-3}L$).

i) Using the Newton-Raphson approach with 3 equilibrium iterations in each load step obtain the response (4 digits-calculations).
(Result: $a_1^1 = 10^{-3}L$, $a_1^2 = 1.125 \cdot 10^{-3}L$, $a_1^3 = \underline{1.127 \cdot 10^{-3}L}$
$a_2^1 = 2.418 \cdot 10^{-3}L$, $a_2^2 = 2.741 \cdot 10^{-3}L$, $a_2^3 = \underline{2.765 \cdot 10^{-3}L}$).

j) Show the results of g), h) and i) in a figure and comment upon the results.

18 Integration of elasto-plastic constitutive equations

Problem 18.1. For the plane stress case the von Mises yield surface can be expressed in terms of principal stresses as

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

The contact state γ_c, i.e. where the stress path intersect with the yield surface can be determined from the condition

$$f = \left[(\sigma_1^c)^2 + (\sigma_2^c) + \sigma_1^c\sigma_2^c\right]^{1/2} - \sigma_{y0} = 0$$

where

$$\sigma_{ij}^c = \sigma_{ij}^n + \gamma^c D_{ijkl}\Delta\epsilon_{kl}$$

Determine the contact state for the data below

$$\begin{aligned}
&\boldsymbol{\sigma}^{nT} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{21}] = [180 \quad -40 \quad 0 \quad 0] \\
&\boldsymbol{\Delta\epsilon}^T = [\Delta\epsilon_{11} \quad \Delta\epsilon_{22} \quad \Delta\epsilon_{33} \quad 2\Delta\epsilon_{21}] = [0.001 \quad 0.001 \quad 0 \quad 0] \\
&E = 210\ GPa \quad \nu = 0 \qquad \sigma_{y0} = 240MPa
\end{aligned}$$

Hint: When $\nu = 0$ the relation $\sigma_{ij} = D_{ijkl}\epsilon_{kl}$ reduces to $\sigma_{ij} = E\epsilon_{ij}$.

Problem 18.2. Consider the yield function for an von Mises elastic-ideal plastic material material under plane stress condition

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

a) Express $\dot{\lambda}$ in terms of the strain rate $\dot{\epsilon}_{ij}$ using the consistency condition $\dot{f} = 0$.

b) Integrate the the constitutive equations using one forward Euler step and one fourth-order RK step, for the scheme given in the text book, for the strain increment used in (18.1).

Problem 18.3. For the plane stress case the von Mises yield surface can be expressed in terms of principal stresses as

$$f = \sigma_e - \sigma_{y0} = \left[\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right]^{1/2} - \sigma_{y0}$$

Calculate the updated state for the data in task (18.1) using the backward Euler method

a) Calculate the trial stress, i.e. the updated state for elastic loading.

b) Express the the updated state in terms of $\Delta\lambda$.

c) Calculate $\Delta\lambda$ using the yield condition, $f = 0$, at the updated state.

d) Calculate the updated state, $\boldsymbol{\sigma}^{n+1}$, and compare the result with the result obtained from task (18.2).

Answers

Answer 1.1. By definition:

$$x'_i = A_{ij} x_j.$$

In matrix format we have,

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x},$$

inserting the given quantities we get

$$\frac{1}{25}\begin{bmatrix} 12 & -9 & 20 \\ 15 & 20 & 0 \\ -16 & 12 & 15 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -29/25 \\ 4/5 \\ -3/25 \end{bmatrix}.$$

Answer 1.2. a) From the picture we have

$$x_1 = \cos(\alpha)x'_1 - \sin(\alpha)x'_2$$

$$x_2 = \sin(\alpha)x'_1 + \cos(\alpha)x'_2$$

$$x_3 = x'_3$$

In matrix format this is

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{A}^T}\underbrace{\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}}_{\boldsymbol{x}'}$$

Therefore

$$\boldsymbol{A}^T = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) We perform the multiplication using $\cos^2(\alpha) + \sin^2(\alpha) = 1$

$$\boldsymbol{A}^T\boldsymbol{A} =$$

$$\begin{bmatrix} \cos^2(\alpha) + \sin^2(\alpha) & \cos(\alpha)\sin(\alpha) - \cos(\alpha)\sin(\alpha) & 0 \\ \cos(\alpha)\sin(\alpha) - \cos(\alpha)\sin(\alpha) & \cos^2(\alpha) + \sin^2(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

From this it follows directly from transposition that

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I} = \boldsymbol{I}^T = (\boldsymbol{A}^T\boldsymbol{A})^T = \boldsymbol{A}\boldsymbol{A}^T$$

Answer 1.3. a) By definition

$$\Phi_{,i} \triangleq \frac{\partial \Phi}{\partial x_i}$$

The derivative $\Phi_{,i}$ is a first order tensor if and only if it satisfies

$$\frac{\partial \Phi}{\partial x_i} = \frac{\partial \Phi}{\partial x'_j} A_{ji} = A_{ji} \frac{\partial \Phi}{\partial x'_j}$$

where

$$x'_i = A_{ij} x_j$$

We may deploy the chain rule as

$$\frac{\partial \Phi}{\partial x_i} = \frac{\partial \Phi}{\partial x'_j}\frac{\partial x'_j}{\partial x_i} = \frac{\partial \Phi}{\partial x'_j}\frac{\partial (A_{jk}x_k)}{\partial x_i} = \frac{\partial \Phi}{\partial x'_j} A_{jk}\delta_{ik} = \frac{\partial \Phi}{\partial x'_j} A_{ji}$$

b) By definition

$$\Phi_{,ij} \triangleq \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$$

The derivative $\Phi_{,ij}$ is a second order tensor if and only if it satisfies

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = A_{ki} \frac{\partial^2 \Phi}{\partial x'_k \partial x'_l} A_{lj}$$

Again we deploy the chain rule

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\Big(\frac{\partial \Phi}{\partial x'_k}\frac{\partial x'_k}{x_i}\Big) = \frac{\partial}{\partial x_j}\Big(\frac{\partial \Phi}{\partial x'_k}\frac{\partial (A_{kl}x_l)}{\partial x_i}\Big) =$$

$$\frac{\partial}{\partial x_j}\Big(\frac{\partial \Phi}{\partial x'_k} A_{kl}\delta_{il}\Big) = \frac{\partial}{\partial x_j}\Big(\frac{\partial \Phi}{\partial x'_k} A_{ki}\Big) = \frac{\partial}{\partial x'_k}\Big(\frac{\partial \Phi}{\partial x_j} A_{ki}\Big) =$$

$$\frac{\partial}{\partial x'_k}\Big(\frac{\partial \Phi}{\partial x'_l}\frac{\partial x'_l}{\partial x_j} A_{ki}\Big) = A_{lj}\frac{\partial^2 \Phi}{\partial x'_k \partial x'_l} A_{ki} = A_{ki}\frac{\partial^2 \Phi}{\partial x'_k \partial x'_l} A_{lj}$$

where it was used that the order of partial derivation is interchangeable.

c) By definition

$$\Phi_{,kk} \triangleq \frac{\partial^2 \Phi}{\partial x_k \partial x_k}$$

The derivative $\Phi_{,kk}$ is a zero order tensor if and only if it satisfies

$$\frac{\partial^2 \Phi}{\partial x_k \partial x_k} = \frac{\partial^2 \Phi}{\partial x'_k \partial x'_k}$$

Using the results of b) setting $i = j$ and using $A_{li} A_{ki} = \delta_{lk}$ we find directly

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_i} = A_{li} \frac{\partial^2 \Phi}{\partial x'_k \partial x'_l} A_{ki} = \frac{\partial^2 \Phi}{\partial x'_k \partial x'_l} \delta_{lk} = \frac{\partial^2 \Phi}{\partial x'_k \partial x'_k}$$

Answer 1.4. The plane normals are given as

$$\boldsymbol{n} = \begin{bmatrix} 2 \\ -1/3 \\ 1 \end{bmatrix}, \quad \boldsymbol{n}' = \begin{bmatrix} 47/25 \\ 14/15 \\ -21/25 \end{bmatrix}$$

The two planes coincide if and only if their normals coincide and they share a common point. We investigate the necessary normal relation

$$\boldsymbol{n} = \boldsymbol{A}^T \boldsymbol{n}'$$

Since $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$ we may equivalently write

$$\boldsymbol{n}' = \boldsymbol{A}\boldsymbol{n}$$

This is readily shown true by performing the explicit calculation

$$\frac{1}{25} \begin{bmatrix} 12 & -9 & 20 \\ 15 & 20 & 0 \\ -16 & 12 & 15 \end{bmatrix} \begin{bmatrix} 2 \\ -1/3 \\ 1 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 24 + 3 + 20 \\ 30 - (20/3) \\ -32 - 4 + 15 \end{bmatrix} = \begin{bmatrix} 47/25 \\ 14/15 \\ -21/25 \end{bmatrix}$$

Secondly we seek a common point of the two planes. Since either all points, or no points will now be shared between the planes, we may select any point included in one of the planes and perform a simple check. For simplicity we choose the point $\boldsymbol{p} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, which is included in the first plane. We find in the primed system that

$$\boldsymbol{p}' = \boldsymbol{A}\boldsymbol{p} = \frac{1}{25} \begin{bmatrix} 12 & -9 & 20 \\ 15 & 20 & 0 \\ -16 & 12 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 20 \\ 0 \\ 15 \end{bmatrix}.$$

Inserting this in the equation of the $'$-plane we find

$$\frac{47}{25}x_1' + \frac{14}{15}x_2' - \frac{21}{25}x_3' = \frac{47}{25}\frac{20}{25} - \frac{21}{25}\frac{15}{25} = 1.$$

Thus the planes must be one and the same.

Answer 1.5. By definition: b_i is a unit vector if $\sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{b_i b_i} = 1$. We find from the given relation that

$$\sqrt{b_i b_i} = \sqrt{\frac{a_i}{\sqrt{a_j a_j}}\frac{a_i}{\sqrt{a_k a_k}}} = \sqrt{\frac{a_i a_i}{a_l a_l}} = 1$$

and thus b_i is a unit vector.

Answer 1.6. We first expand the dummy index following the definition of Einstein summation

$$\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}$$

We now argue that the first term $\delta_{i1}\delta_{1k}$ is identical to 1 if and only if i=k=1, and if so is the case the other two terms, $\delta_{i2}\delta_{2k}$ and $\delta_{i3}\delta_{3k}$, become 0. The same argument can be applied to the second and third term, rendering the right hand side to be identical 1 if $i = k$ and 0 otherwise. This is the definition of the Kronecker's delta!

Answer 1.7. a) From the first relation we have

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Setting $j = i$ we get

$$s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} = \sigma_{ii} - \sigma_{kk} = \sigma_{ii} - \sigma_{ii} = 0,$$

where it was used that $\delta_{ii} = 3$.

b) Inserting the above expression for s_{ij} in the second given relation we find We note that

$$\frac{\partial s_{ij}}{\partial \sigma_{kl}} = \frac{\partial(\sigma_{ij} - \frac{1}{3}\sigma_{mm}\delta_{ij})}{\partial \sigma_{kl}} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

and that equivalently the transpose, s_{ji}, becomes

$$\frac{\partial s_{ji}}{\partial \sigma_{kl}} = \delta_{jk}\delta_{il} - \frac{1}{3}\delta_{ji}\delta_{kl}$$

The sought derivative is found via the chain rule as

$$\frac{\partial J_2}{\partial \sigma_{kl}} = \frac{1}{2}\frac{\partial}{\partial \sigma_{kl}}\Big(s_{ij}s_{ji}\Big) = \frac{1}{2}\Big(\frac{\partial s_{ij}}{\partial \sigma_{kl}}s_{ji} + s_{ij}\frac{\partial s_{ji}}{\partial \sigma_{kl}}\Big) =$$

$$\frac{1}{2}\Big(\big(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\big)s_{ji} + s_{ij}\big(\delta_{jk}\delta_{il} - \frac{1}{3}\delta_{ji}\delta_{kl}\big)\Big) =$$

$$\frac{1}{2}\Big(s_{lk} - \frac{1}{3}s_{ii}\delta_{kl} + s_{lk} - \frac{1}{3}s_{ii}\delta_{kl}\Big) = s_{lk} = s_{kl}$$

where it was used that $s_{ii} = 0$ and that $s_{ij} = s_{ji}$.

Answer 1.8. If $a_i b_j = \delta_{ij}$, then it must be so that

$$a_1b_1 = a_2b_2 = a_3b_3 = 1, \quad \text{and}$$

$$a_ib_j = 0 \quad \text{if} \quad i \neq j$$

The first condition implies that $a_i \neq 0$ and $b_i \neq 0$ for all i which is in contradiction with the second requirement, and thus $a_ib_j \neq \delta_{ij}$.

Answer 1.9. We set $i = j$ in the given expression and find α as

$$s_{ii} = \sigma_{ii} - \alpha\delta_{ii} = \sigma_{ii} - 3\alpha = 0 \quad \Rightarrow \quad \alpha = \frac{1}{3}\sigma_{ii}$$

Using that $\delta_{ij} = \delta_{ji}$ and that $\sigma_{ij} = \sigma_{ji}$ we find directly

$$s_{ij} = \sigma_{ij} - \alpha\delta_{ij} = \sigma_{ji} - \alpha\delta_{ji} = s_{ji}$$

Answer 1.10. We may first consider the zero order tensor $b = 0$, it follows directly that $b' = b = 0$. For a first order tensor $a_j = 0$ we have

$$a'_i = A_{ij}a_j = 0$$

For a second order tensor $b_{kl} = 0$ we also have

$$b'_{ij} = A_{lj}b_{kl}A_{ki} = 0$$

(Consider the tensor $T_{ijklmn...}$ of arbitrary order. The transformation of $T_{ijklmn...}$ unto a primed system is given as a sum of terms in which all include a multiplication with the component $T_{ijklmn...}$. Thus, if $T_{ijklmn...} = 0$ for all $ijklmn..$ the transformed tensor is also identical zero.)

Answer 1.11. Let a_i and b_i be two first order tensors, the transformations,

$$a'_i = A_{ij} a_j$$

$$b'_i = A_{ij} b_j,$$

then hold. We introduce the two quantities $c_i = a_i + b_i$ and $c'_i = a'_i + b'_i$ and find directly

$$c'_i = a'_i + b'_i = A_{ij} a_j + A_{ij} b_j = A_{ij}(a_j + b_j) = A_{ij} c_j$$

which proves that c_i is again a first order tensor. The same can be applied for higher order tensors.

Answer 1.12. From the definition of a second order tensor we find

$$a'_{ij} = A_{ik} a_{kl} A_{jl} = A_{ik} a_{lk} A_{jl} = A_{jl} a_{lk} A_{ik} = a'_{ji}$$

where $a_{kl} = a_{lk}$ was used. Similarly if $a_{kl} = -a_{lk}$ we find

$$a'_{ij} = A_{ik} a_{kl} A_{jl} = -A_{ik} a_{lk} A_{jl} = -A_{jl} a_{lk} A_{ik} = -a'_{ji}$$

Answer 1.13. We have given that

$$D'_{ijkl} = A_{im} A_{jn} D_{mnpq} A_{kp} A_{lq}$$

and

$$D_{mnpq} = 2G\big[\frac{1}{2}(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) + \frac{\nu}{1-2\nu}\delta_{mn}\delta_{pq}\big] \qquad (3)$$

putting the equations together we find

$$D'_{ijkl} = A_{im} A_{jn} 2G\big[\frac{1}{2}(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}) + \frac{\nu}{1-2\nu}\delta_{mn}\delta_{pq}\big] A_{kp} A_{lq} =$$

$$\frac{1}{2}(A_{im}A_{jn}\delta_{mp}\delta_{nq}A_{kp}A_{lq} + A_{im}A_{jn}\delta_{mq}\delta_{np}A_{kp}A_{lq}) + \frac{\nu}{1-2\nu}A_{im}A_{jn}\delta_{mn}\delta_{pq}A_{kp}A_{lq} =$$

$$\frac{1}{2}(A_{ip}A_{jq}A_{kp}A_{lq} + A_{iq}A_{jp}A_{kp}A_{lq}) + \frac{\nu}{1-2\nu}A_{in}A_{jn}A_{kq}A_{lq} =$$

$$\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl} = D_{ijkl}$$

And so it is found that the elastic stiffness matrix is an invariant in this case ($D'_{ijkl} = D_{ijkl}$). This is expected as the material is isotropic and thus the elastic behaviour is the same in all directions.

Answer 2.1. Starting with θ_1 we find from the definition of a second order tensor:

$$\epsilon_{ij} = A_{ki}\epsilon'_{kl}A_{lj} \Rightarrow \epsilon_{ii} = A_{ki}\epsilon'_{kl}A_{li} = A_{ki}A_{li}\epsilon'_{kl} = \delta_{kl}\epsilon'_{kl} = \epsilon'_{kk}$$

Since a sum of invariants is again invariant, for θ_2 it is enough to show that $\epsilon_{ij}\epsilon_{ji}/2$ is an invariant, using again the transformation rule of a second order tensor we find:

$$\frac{1}{2}\epsilon_{ij}\epsilon_{ji} = \frac{1}{2}A_{ki}\epsilon'_{kl}A_{lj}A_{mj}\epsilon'_{mn}A_{ni} = \frac{1}{2}\delta_{kn}\delta_{lm}\epsilon'_{kl}\epsilon'_{mn} = \frac{1}{2}\epsilon'_{nm}\epsilon'_{mn}$$

where $A_{lj}A_{mj} = \delta_{lm}$ and $A_{ki}A_{ni} = \delta_{kn}$ was used. For θ_3 we have

$$det(\epsilon_{ij}) = det(A_{ki}\epsilon'_{kl}A_{lj}) = det(A_{ki})det(\epsilon'_{kl})det(A_{lj}) = det(\epsilon'_{kl})$$

using that $det(A_{ij}) = 1$.

Answer 2.2. We recall the definition of the deviatoric strain

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij}$$

If the coordinate system is defined to be co-linear with the principal strains we have that

$$\epsilon_{kk} = \epsilon_k \quad \text{(no summation i } k\text{)}$$

where ϵ_k are the principal strains. We also have that

$$\epsilon_{ij} = 0 \quad \text{if} \quad i \neq j$$

Thus it follows that

$$e_{ij} = 0 \quad \text{if} \quad i \neq j$$

So we realise that only when $i = j = k$ do the product $e_{ij}e_{jk}e_{ki} \neq 0$. We note that a single contribution becomes

$$e_{kk} = e_k = \epsilon_k - \frac{1}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3) \quad \text{(no summation in } k\text{)}$$

and thus we conclude that in this particular coordinate system

$$J'_3 = \frac{1}{3}(e_{ij}e_{jk}e_{ki}) = \frac{1}{3}(e_{11}e_{11}e_{11} + e_{22}e_{22}e_{22} + e_{33}e_{33}e_{33}) = \frac{1}{3}(e_1^3 + e_2^3 + e_3^3)$$

If J_3' is an invariant this holds good for all coordinate systems. We check to see if this is the case

$$\frac{1}{3}(e'_{mn}e'_{np}e'_{pm}) = \frac{1}{3}A_{ms}e_{st}A_{nt}A_{nk}e_{kl}A_{pl}A_{pi}e_{ij}A_{mj} =$$

$$\frac{1}{3}\delta_{sj}\delta_{tk}\delta_{li}e_{st}e_{kl}e_{ij} = \frac{1}{3}e_{jk}e_{ki}e_{ij}$$

and thus the above proof is valid.

Answer 2.3. a) The small strain tensor, e_{ij}, is by definition

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

We produce the derivatives as

$$\epsilon_{11} = \frac{1}{2}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}) = 2kx_1$$

$$\epsilon_{12} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) = kx_2$$

$$\epsilon_{13} = \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) = kx_3$$

$$\epsilon_{22} = \frac{1}{2}(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}) = 2kx_1$$

$$\epsilon_{23} = \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) = 0$$

$$\epsilon_{33} = \frac{1}{2}(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}) = 2kx_1$$

Since the small strain tensor is symmetric we have now

$$[\epsilon_{ij}] = k\begin{bmatrix} 2x_1 & x_2 & x_3 \\ x_2 & 2x_1 & 0 \\ x_3 & 0 & 2x_1 \end{bmatrix}$$

b) at the point (1,2,3) we have

$$[\epsilon_{ij}] = k\begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

The tensorial shear strain is found as

$$m_i\epsilon_{ij}n_j \Rightarrow k\frac{1}{5\sqrt{34}}\begin{bmatrix}3\\4\\0\end{bmatrix}^T\begin{bmatrix}2&2&3\\2&2&0\\3&0&2\end{bmatrix}\begin{bmatrix}4\\-3\\3\end{bmatrix} = \frac{41k}{5\sqrt{34}}$$

The normal strain is found as

$$m_i\epsilon_{ij}m_j \Rightarrow k\frac{1}{25}\begin{bmatrix}3\\4\\0\end{bmatrix}^T\begin{bmatrix}2&2&3\\2&2&0\\3&0&2\end{bmatrix}\begin{bmatrix}3\\4\\0\end{bmatrix} = \frac{98}{25}k$$

and the engineering shear strain is defined as two times the tensorial shear strain, thus

$$\gamma_{nm} = 2\epsilon_{nm} = 2m_i\epsilon_{ij}n_j = \frac{82k}{5\sqrt{34}}$$

c) At the point (1,4,0) we have

$$[\epsilon_{ij}] = k\begin{bmatrix}2&4&0\\4&2&0\\0&0&2\end{bmatrix}$$

The characteristic equation becomes

$$k^3\Big((2-\lambda)^3 - (2-\lambda)16\Big) = 0 \Rightarrow (2-\lambda)\Big((2-\lambda)^2 - 16\Big) = 0$$

And the eigenvalues are

$$\lambda_{1,2} = 2k \pm 4k, \quad \lambda_3 = 2k$$

We seek now the eigenvectors that fulfill

$$\epsilon_{ij}v_j = \lambda v_j$$

Starting with $\lambda_1 = 6$ we have

$$k\begin{bmatrix}2&4&0\\4&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = 6k\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$

$$2a_1 + 4a_2 = 6a_1 \Rightarrow a_2 = a_1$$

$$2a_3 = 6a_3 \Rightarrow a_3 = 0$$

moving on to $\lambda_2 = -2$ we find

$$k \begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = -2k \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$2b_1 + 4b_2 = -2b_1 \Rightarrow b_2 = -b_1$$

$$2b_3 = -2b_3 \Rightarrow b_3 = 0$$

Finally for $\lambda_3 = 2$

$$k \begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 2k \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$2c_1 + 4c_2 = 2c_1 \Rightarrow c_2 = 0$$

$$4c_1 + 2c_2 = 2c_2 \Rightarrow c_1 = 0$$

$$2c_3 = 2c_3 \Rightarrow c_3 = c_3$$

We write the normalised principal directions as

$$\boldsymbol{a} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Answer 2.4. a) By expansion we find

$$\boldsymbol{\epsilon}' = \boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T$$

$$= \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{10^{-4}}{2} \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{10^{-4}}{2} \begin{bmatrix} 5\cos(\alpha) + 10\sin(\alpha) & 10\cos(\alpha) + 5\sin(\alpha) & 0 \\ -5\sin(\alpha) + 10\cos(\alpha) & -10\sin(\alpha) + 5\cos(\alpha) & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{10^{-4}}{2} \begin{bmatrix} 5 + 20\cos(\alpha)\sin(\alpha) & 10\cos^2(\alpha) - 10\sin^2(\alpha) & 0 \\ 10\cos^2(\alpha) - 10\sin^2(\alpha) & 5 - 20\cos(\alpha)\sin(\alpha) & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

b) Inserting $\alpha = \pi/4$ we find:

$$\boldsymbol{\epsilon}' = \frac{10^{-4}}{2}\begin{bmatrix} 15 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

This means that a rotation of 45^o will bring the Cartesian system to be co-linear with the principal directions.

Answer 2.5. a) We seek the eigenvectors, $v_j \neq 0$, that will result in the mapping

$$\epsilon_{ij} v_j = \lambda v_j$$

where λ is a scalar eigenvalue. We may alternatively write

$$(\epsilon_{ij} - \lambda\delta_{ij})v_j = 0$$

Non trivial solutions, where $v_j \neq 0$, exists if the inverse $[(\epsilon_{ij} - \lambda\delta_{ij})]^{-1}$ does not exists. Thus we require that the determinant be identical zero

$$det(\epsilon_{ij} - \lambda\delta_{ij}) = 0$$

The characteristic equation becomes

$$\frac{10^{-4}}{2}\Big((5-\lambda)^2(-6-\lambda) - (-6-\lambda)100\Big) = 0$$

Which can be written

$$(-6-\lambda)\Big((5-\lambda)^2 - 100\Big) = 0$$

The solutions become

$$\lambda_{1,2} = 5 \pm 10, \quad \lambda_3 = -6$$

The eigenvalues are the principal strains. We now seek the principal directions, v_j, these must solve the original problem

$$\epsilon_{ij} v_j = \lambda v_j$$

Since we seek only the directions of the eigenvectors, we may drop the scaling term $10^{-4}/2$. We use $[\epsilon_{ij}]$ and $\lambda_1 = 15$ to find three equations

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 15\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$5a_1 + 10a_2 = 15a_1 \Rightarrow a_1 = a_2$$

$$-6a_3 = 15a_3 \Rightarrow a_3 = 0$$

Then we use $\lambda_2 = -5$ in the same way

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = -5 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$5b_1 + 10b_2 = -5b_1 \Rightarrow b_1 = -b_2$$

$$-6b_3 = -5b_3 \Rightarrow b_3 = 0$$

And finally for $\lambda_3 = -6$ we find

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = -6 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\left. \begin{aligned} 5c_1 + 10c_2 = -6c_1 \Rightarrow c_2 = \frac{-11}{10}c_1 \\ 10c_1 + 5c_2 = -6c_2 \Rightarrow c_1 = \frac{-11}{10}c_2 \end{aligned} \right\} \Rightarrow c_1 = c_2 = 0$$

$$-6c_3 = -6c_3 \Rightarrow c_3 = c_3$$

We summarise by selecting the three eigenvectors to be normalised

$$\boldsymbol{a} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

these are the principal directions.

b) We conclude that the rotation of 45^o made in problem 2.4 b) results in the same strain tensor as a swap to the principal system. This means, in this particular case, that the principal directions could also be found by rotation of the unprimed coordinate system by 45^o.

Answer 3.1. Consider the eigenvalue problem

$$\sigma_{ij} v_j = \lambda_\sigma v_j \Rightarrow (\sigma_{ij} - \lambda_\sigma \delta_{ij}) v_j = 0, \quad v_j \neq 0$$

We may equivalently write

$$(s_{ij} - (\lambda_\sigma - \frac{1}{3}\sigma_{kk})\delta_{ij}) v_j = 0 \Rightarrow (s_{ij} - \lambda_s \delta_{ij}) v_j = 0$$

and note that the terms $\frac{1}{3}\sigma_{kk}\delta_{ij}$ cancels out from the definition of s_{ij}. This finishes the proof, v_j in the above equation is namely nothing else than the eigenvectors of the quantity s_{ij}. Thus the principal directions of σ_{ij} and s_{ij} coincide. (We note that the eigenvalues are shifted by a subtraction of $\frac{1}{3}\sigma_{kk}$.)

Answer 3.2. Consider the eigenvalue problem

$$\epsilon_{ij}v_j = \lambda_\epsilon v_j \Rightarrow (\epsilon_{ij} - \lambda_\epsilon\delta_{ij})v_j = 0, \quad v_j \neq 0$$

Inserting the given format of ϵ_{ij} we find

$$\begin{aligned}
&\Big(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} - \lambda_\epsilon\delta_{ij}\Big)v_j = 0 \\
&\Rightarrow \Big((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} + E\lambda_\epsilon\delta_{ij}\Big)v_j = 0 \\
&\Rightarrow \Big((1+\nu)\sigma_{ij} - (\nu\sigma_{kk} + E\lambda_\epsilon)\delta_{ij}\Big)v_j = 0 \\
&\Rightarrow \left(\sigma_{ij} - \left(\frac{\nu}{(1+\nu)}\sigma_{kk} + E\frac{\lambda_\epsilon}{(1+\nu)}\right)\delta_{ij}\right)v_j = 0 \\
&\Rightarrow (\sigma_{ij} - \lambda_\sigma\delta_{ij})v_j = 0
\end{aligned}$$

Thus the principal directions, v_j, of ϵ_{ij} and σ_{ij} coincide.

b) By direct insertion into the given formula we find for the 11 component

$$\epsilon_{11} = \frac{1}{E}((1+\nu)\sigma_{11} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})) = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33})$$

The component 22 and 33 are found in the same way. The component 12 is found as

$$2\epsilon_{12} = \frac{2}{E}(1+\nu)\sigma_{12}$$

and 13 and 23 are found in the same way. In total we have:

$$\boldsymbol{C} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

Answer 3.3. a) In analogy with problem 2.4 we find by expansion

$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T =$

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 5\cos(\alpha) + 10\sin(\alpha) & 10\cos(\alpha) + 5\sin(\alpha) & 0 \\ -5\sin(\alpha) + 10\cos(\alpha) & -10\sin(\alpha) + 5\cos(\alpha) & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 5 + 20\cos(\alpha)\sin(\alpha) & 10\cos^2(\alpha) - 10\sin^2(\alpha) & 0 \\ 10\cos^2(\alpha) - 10\sin^2(\alpha) & 5 - 20\cos(\alpha)\sin(\alpha) & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

b) Inserting $\alpha = \pi/4$ we find:

$$\boldsymbol{\sigma}' = \begin{bmatrix} 15 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

This means that a rotation of 45^o will bring the Cartesian system to be co-linear with the principal directions.

Answer 3.4. a) In direct analogy with problem 2.5 We seek the eigenvectors, $v_j \neq 0$, that will result in the mapping

$$\sigma_{ij} v_j = \lambda v_j$$

where λ is a scalar eigenvalue. We may alternatively write

$$(\sigma_{ij} - \lambda\delta_{ij})v_j = 0$$

Non trivial solutions, where $v_j \neq 0$, exists if the inverse $[(\sigma_{ij} - \lambda\delta_{ij})]^{-1}$ does not exists. Thus we require that the determinant be identical zero

$$det(\sigma_{ij} - \lambda\delta_{ij}) = 0$$

The characteristic equation becomes

$$\Big((5-\lambda)^2(-6-\lambda) - (-6-\lambda)100\Big) = 0$$

Which can be written

$$(-6-\lambda)\Big((5-\lambda)^2-100\Big)=0$$

The solutions become

$$\lambda_{1,2}=5\pm 10, \quad \lambda_3=-6$$

The eigenvalues are the principal stresses. We now seek the principal directions, v_j, these must solve the original problem

$$\sigma_{ij}v_j=\lambda v_j$$

We use $[\sigma_{ij}]$ and $\lambda_1=15$ to find three equations

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 15 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$5a_1+10a_2=15a_1 \Rightarrow a_1=a_2$$

$$-6a_3=15a_3 \Rightarrow a_3=0$$

Then we use $\lambda_2=-5$ in the same way

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = -5 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$5b_1+10b_2=-5b_1 \Rightarrow b_1=-b_2$$

$$-6b_3=-5b_3 \Rightarrow b_3=0$$

And finally for $\lambda_3=-6$ we find

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = -6 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\left.\begin{aligned} &5c_1+10c_2=-6c_1 \Rightarrow c_2=\frac{-11}{10}c_1 \\ &10c_1+5c_2=-6c_2 \Rightarrow c_1=\frac{-11}{10}c_2 \end{aligned}\right\} \Rightarrow c_1=c_2=0$$

$$-6c_3=-6c_3 \Rightarrow c_3=c_3$$

We summarise by selecting the three eigenvectors to be normalised

$$\boldsymbol{a} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

these are the principal directions.

b) We conclude that the rotation of 45^o made in problem 3.3 b) results in the same stress tensor as a swap to the principal system. This means, in this particular case, that the principal directions could also be found by rotation of the unprimed coordinate system by 45^o.

Answer 3.5. a) The stress tensor σ_{ij} is diagonal in the principal coordinate system:

$$[\sigma_{ij}] = \begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}$$

Given the normal, $\boldsymbol{n}^T = (1,1,1)/\sqrt{3}$, we find the traction vector ,$\boldsymbol{t}$, via Cauchy's formula

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}$$

b) We find the normal stress as

$$\sigma_n = \boldsymbol{t}\cdot\boldsymbol{n} = t_i n_i = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{\sigma_{kk}}{3}$$

The squared shear stress τ_n^2 becomes

$$\tau_n^2 = ||\boldsymbol{t} - \sigma_n\boldsymbol{n}||^2 =$$

$$\frac{1}{3}\Big((\sigma_1 - \frac{\sigma_{kk}}{3})^2 + (\sigma_2 - \frac{\sigma_{kk}}{3})^2 + (\sigma_3 - \frac{\sigma_{kk}}{3})^2\Big) =$$

$$\frac{1}{3}((\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{2}{3}\sigma_{kk}^2 + \frac{3}{9}\sigma_{kk}^2) =$$

$$\frac{1}{3}((\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{3}\sigma_{kk}^2)$$

We may rewrite $\dfrac{2}{3}J_2$ as

$$\frac{2}{3}J_2 = \frac{1}{3}(s_{ij}s_{ji}) = \frac{1}{3}\big((\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij})(\sigma_{ji} - \frac{1}{3}\sigma_{kk}\delta_{ji})\big) =$$

$$\frac{1}{3}\Big(\sigma_{ij}\sigma_{ji} - \frac{2}{3}\sigma_{kk}^2 + \frac{1}{3}\sigma_{kk}^2\Big) = \frac{1}{3}(\sigma_{ij}\sigma_{ji} - \frac{1}{3}\sigma_{kk}^2) =$$

$$\frac{1}{3}((\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{3}\sigma_{kk}^2)$$

We have now proven $\tau_n^2 = 2J_2/3$ as well as $\sigma_n = I_1/3 = \sigma_{kk}/3$.

Answer 3.7. a) The given stress tensor is of the form

$$[\sigma_{ij}] = \begin{bmatrix} p & p & p \\ p & p & p \\ p & p & 0 \end{bmatrix}$$

We seek to solve the eigenvalue problem

$$(\sigma_{ij} - \lambda\delta_{ij})v_j = 0, \quad v_j \neq 0$$

The characteristic equation becomes

$$-\lambda(p-\lambda)^2 + 2p^3 - 2(p-\lambda)p^2 + \lambda p^2 = 0$$

This can be expanded and rearranged as

$$\lambda^3 - 2p\lambda^2 - 2p^2\lambda = 0$$

Solving this polynomial equation gives

$$\lambda_{1,2} = p(1 \pm \sqrt{3}), \quad \lambda_3 = 0$$

and so we must now proceed to solve for the eigenvectors, starting with $\lambda_1 =$

$p(1+\sqrt{3})$

$$\begin{bmatrix} p & p & p \\ p & p & p \\ p & p & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = p(1+\sqrt{3}) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$pa_1 + pa_2 + pa_3 = p(1+\sqrt{3})a_1 \Rightarrow a_2 + a_3 = a_1\sqrt{3}$$

$$pa_1 + pa_2 + pa_3 = p(1+\sqrt{3})a_2 \Rightarrow a_1 + a_3 = a_2\sqrt{3}$$

$$pa_1 + pa_2 = p(1+\sqrt{3})a_3 \Rightarrow a_1 + a_2 = (1+\sqrt{3})a_3$$

$$\Rightarrow 2a_1 = (1+\sqrt{3})a_3$$

$$\Rightarrow a_2 = a_1$$

For $\lambda_2 = p(1-\sqrt{3})$ we have instead

$$\begin{bmatrix} p & p & p \\ p & p & p \\ p & p & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = p(1-\sqrt{3}) \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$pb_1 + pb_2 + pb_3 = p(1-\sqrt{3})b_1 \Rightarrow b_2 + b_3 = -b_1\sqrt{3}$$

$$pb_1 + pb_2 + pb_3 = p(1-\sqrt{3})b_2 \Rightarrow b_1 + b_3 = -b_2\sqrt{3}$$

$$pb_1 + pb_2 = p(1-\sqrt{3})b_3 \Rightarrow b_1 + b_2 = (1-\sqrt{3})b_3$$

$$\Rightarrow 2b_1 = (1-\sqrt{3})b_3$$

$$\Rightarrow b_2 = b_1$$

Setting $\lambda_3 = 0$ we find

$$\begin{bmatrix} p & p & p \\ p & p & p \\ p & p & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} &pc_1 + pc_2 + pc_3 = 0 \Rightarrow c_1 + c_2 + c_3 = 0 \\ &pc_1 + pc_2 = 0 \Rightarrow c_1 = -c_2 \end{aligned}\right\} \Rightarrow c_3 = 0$$

With normalisation the principal directions now become

$$\boldsymbol{a} = \frac{1}{\sqrt{6-\sqrt{12}}}\begin{bmatrix} 1 \\ 1 \\ 2/(1+\sqrt{3}) \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{\sqrt{6+\sqrt{12}}}\begin{bmatrix} 1 \\ 1 \\ 2/(1-\sqrt{3}) \end{bmatrix}, \quad \boldsymbol{c} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

We check to see that the eigenvectors are all orthogonal (dropping the scaling factors for simplicity):

$$\boldsymbol{a}^T\boldsymbol{b} = 1 + 1 + (1+\sqrt{3})(1-\sqrt{3}) = 3 - \sqrt{3} + \sqrt{3} - 3 = 0$$

$$\boldsymbol{a}^T\boldsymbol{c} = 1 - 1 + 0 = 0$$

$$\boldsymbol{b}^T\boldsymbol{c} = 1 - 1 + 0 = 0$$

b) Matlab calculation, note eigenvalues should be multiplied with p

```
>> S=[1 1 1;1 1 1;1 1, 0]

S =

     1     1     1
     1     1     1
     1     1     0

>> [V,D]=eig(S)

V =

    0.3251    0.7071    0.6280
    0.3251   -0.7071    0.6280
   -0.8881         0    0.4597


D =

   -0.7321         0         0
         0   -0.0000         0
         0         0    2.7321
```

where columns in V contains the eigenvectors and diagonal elements in D the eigenvalues.

c) From the relation $\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{x}'$ we have that the axis $\boldsymbol{x}'^T = [1\,0\;0]$ in the primed system is defined by the eigenvector $\boldsymbol{x} = \boldsymbol{a}$ in the unprimed system, i.e.

$$\boldsymbol{a} = \boldsymbol{A}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering all axis, we require the matrix $\boldsymbol{A}$ to execute the mapping

$$\begin{bmatrix} \boldsymbol{a}\; \boldsymbol{b}\; \boldsymbol{c} \end{bmatrix} = \boldsymbol{A}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the right matrix is the identity matrix we have that

$$\boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{a}\; \boldsymbol{b}\; \boldsymbol{c} \end{bmatrix}$$

Since we have already normalised the eigenvectors we find directly that

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{a}^T \\ \boldsymbol{b}^T \\ \boldsymbol{c}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^T\boldsymbol{a} & \boldsymbol{a}^T\boldsymbol{b} & \boldsymbol{a}^T\boldsymbol{c} \\ \boldsymbol{b}^T\boldsymbol{a} & \boldsymbol{b}^T\boldsymbol{b} & \boldsymbol{b}^T\boldsymbol{c} \\ \boldsymbol{c}^T\boldsymbol{a} & \boldsymbol{c}^T\boldsymbol{b} & \boldsymbol{c}^T\boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is required.

d) We seek to verify Cayley-Hamilton's theorem for this specific case. We insert $[\sigma_{ij}]$ in the characteristic equation

$$[\sigma_{ij}]^3 - 2p[\sigma_{ij}]^2 - 2p^2[\sigma_{ij}] = \mathbf{0}$$

It is useful to compute

$$[\sigma_{ij}]^2 = p^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = p^2 \begin{bmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

as well as

$$[\sigma_{ij}]^3 = [\sigma_{ij}]^2[\sigma_{ij}] = p^3 \begin{bmatrix} 8 & 8 & 6 \\ 8 & 8 & 6 \\ 6 & 6 & 4 \end{bmatrix}$$

inserting these results in the Cayley-Hamilton equation yields

$$p^3 \begin{bmatrix} 8 & 8 & 6 \\ 8 & 8 & 6 \\ 6 & 6 & 4 \end{bmatrix} - 2pp^2 \begin{bmatrix} 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix} - 2p^2p \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

$$p^3 \begin{bmatrix} 8 & 8 & 6 \\ 8 & 8 & 6 \\ 6 & 6 & 4 \end{bmatrix} - p^3 \begin{bmatrix} 6 & 6 & 4 \\ 6 & 6 & 4 \\ 4 & 4 & 4 \end{bmatrix} - p^3 \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} = \mathbf{0}$$

Answer 3.8. The transformation from $\boldsymbol{x}$ to $\boldsymbol{x}'$ is given as

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{I} \Rightarrow \boldsymbol{A} = \boldsymbol{x}'$$

hence the transformation matrix could be selected as

$$\boldsymbol{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

note that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$ since the two involved basis systems are orthonormal . From the definition of a second order tensor we now see that the stress in the primed system becomes

$$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$$

Answer 3.9. We have

$$\boldsymbol{\sigma} = k\boldsymbol{n}\boldsymbol{n}^T$$

We introduce the primed system $\boldsymbol{x}'$ as

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

and select the orthonormal basis

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{m}^T \\ \boldsymbol{q}^T \end{bmatrix}$$

where $\boldsymbol{m}$, $\boldsymbol{q}$ and $\boldsymbol{n}$ form an orthonormal basis. The stress in the primed system becomes

$$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{m}^T \\ \boldsymbol{q}^T \end{bmatrix} k\boldsymbol{n}\boldsymbol{n}^T \begin{bmatrix} \boldsymbol{n} & \boldsymbol{m} & \boldsymbol{q} \end{bmatrix} =$$

$$k \begin{bmatrix} \boldsymbol{n}^T\boldsymbol{n} \\ \boldsymbol{m}^T\boldsymbol{n} \\ \boldsymbol{q}^T\boldsymbol{n} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T\boldsymbol{n} & \boldsymbol{n}^T\boldsymbol{m} & \boldsymbol{n}^T\boldsymbol{q} \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thus we have pure tension in the $\boldsymbol{n}$ direction.

Answer 3.10. We have from Cauchy's formula

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The two bottom equations yield

$$2n_1 + 2n_3 = 0 \Rightarrow n_3 = -n_1$$

$$n_1 + 2n_2 = 0 \Rightarrow n_2 = -n_1/2$$

we select a normal unit vector, $\boldsymbol{n}$, fulfilling this

$$\boldsymbol{n} = \frac{2}{3}\begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix}$$

insertion into Cauchy's formula yields

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \frac{2}{3}\begin{bmatrix} 1 \\ -1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation reads

$$\frac{2}{3}(\sigma_{11} - 1 - 1) = 0 \Rightarrow \sigma_{11} = 2$$

Thus if $\sigma_{11} = 2$ there exist a section with unit normal $\boldsymbol{n} = 2/3\begin{bmatrix} 1 & -1/2 & -1 \end{bmatrix}^T$ on which the traction $\boldsymbol{t}$ is zero.

Answer 4.1. The invariants $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3$ are defined as

$$\tilde{I}_1 = \epsilon_{ii}$$

$$\tilde{I}_2 = \frac{1}{2}\epsilon_{ji}\epsilon_{ij}$$

$$\tilde{I}_3 = \frac{1}{3}\epsilon_{ij}\epsilon_{jk}\epsilon_{ki}$$

we find first

$$\frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} = \frac{\partial \epsilon_{kk}}{\partial \epsilon_{ij}} = \delta_{ik}\delta_{kj} = \delta_{ij}$$

secondly

$$\frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} = \frac{1}{2}\frac{\partial \epsilon_{kl}\epsilon_{lk}}{\partial \epsilon_{ij}} = \frac{1}{2}\Big(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{ij}}\epsilon_{lk} + \frac{\partial \epsilon_{lk}}{\partial \epsilon_{ij}}\epsilon_{kl}\Big)$$

$$= \frac{1}{2}\Big(\delta_{ik}\delta_{lj}\epsilon_{lk} + \delta_{li}\delta_{kj}\epsilon_{kl}\Big) = \frac{1}{2}\Big(\epsilon_{ji} + \epsilon_{ij}\Big) = \epsilon_{ij}$$

where it was used that $\epsilon_{ij} = \epsilon_{ji}$. Finally

$$\frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}} = \frac{1}{3}\frac{\partial \epsilon_{kl}\epsilon_{lp}\epsilon_{pk}}{\partial \epsilon_{ij}} = \frac{1}{3}\Big(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{ij}}\epsilon_{lp}\epsilon_{pk} + \epsilon_{kl}\frac{\partial \epsilon_{lp}\epsilon_{pk}}{\partial \epsilon_{ij}}\Big)$$

$$= \frac{1}{3}\Big(\delta_{ki}\delta_{lj}\epsilon_{lp}\epsilon_{pk} + \epsilon_{kl}\Big(\frac{\partial \epsilon_{lp}}{\partial \epsilon_{ij}}\epsilon_{pk} + \frac{\partial \epsilon_{pk}}{\partial \epsilon_{ij}}\epsilon_{lp}\Big)\Big)$$

$$= \frac{1}{3}\Big(\epsilon_{jp}\epsilon_{pi} + \epsilon_{kl}\Big(\delta_{li}\delta_{jp}\epsilon_{pk} + \delta_{pi}\delta_{jk}\epsilon_{lp}\Big)\Big)$$

$$= \frac{1}{3}\Big(\epsilon_{jp}\epsilon_{pi} + \epsilon_{ki}\epsilon_{jk} + \epsilon_{jl}\epsilon_{li}\Big) = \frac{1}{3}\Big(\epsilon_{ip}\epsilon_{pj} + \epsilon_{ik}\epsilon_{kj} + \epsilon_{il}\epsilon_{lj}\Big)$$

$$= \epsilon_{ik}\epsilon_{kj}$$

where it was used multiple times that $\epsilon_{ij} = \epsilon_{ji}$.

Answer 4.2. The invariants I_1, I_2, I_3 are defined as

$$I_1 = \sigma_{ii}$$

$$J_2 = \frac{1}{2}s_{ji}s_{ij}$$

$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

we first see that

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{\partial(\sigma_{kl} - (1/3)\delta_{kl}\sigma_{mm})}{\partial \sigma_{ij}} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{im}\delta_{jm}\delta_{kl} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \delta_{ik}\delta_{kj} = \delta_{ij}$$

secondly

$$\frac{\partial J_2}{\partial \sigma_{ij}} = \frac{1}{2}\frac{\partial s_{kl}s_{lk}}{\partial \sigma_{ij}} = \frac{1}{2}\Big(\frac{\partial s_{kl}}{\partial \sigma_{ij}}s_{lk} + \frac{\partial s_{lk}}{\partial \sigma_{ij}}s_{kl}\Big)$$

$$= \frac{1}{2}\Big((\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl})s_{lk} + (\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{lk})s_{kl}\Big)$$

$$= \frac{1}{2}\Big(s_{ij} - \frac{1}{3}\delta_{ij}s_{kk} + s_{ji} - \frac{1}{3}\delta_{ij}s_{kk}\Big) = s_{ij}$$

where it was used that $s_{ij} = s_{ji}$ and $s_{kk} = 0$. Finally by use of the chain rule we find

$$\frac{\partial J_3}{\partial \sigma_{ij}} = \frac{\partial J_3}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}$$

Where

$$\frac{\partial J_3}{\partial s_{kl}} = \frac{1}{3}\frac{\partial (s_{mn}s_{np}s_{pm})}{\partial s_{kl}}$$

$$= \frac{1}{3}\Big(s_{np}s_{pm}\frac{\partial s_{mn}}{\partial s_{kl}} + s_{mn}s_{pm}\frac{\partial s_{np}}{\partial s_{kl}} + s_{mn}s_{np}\frac{\partial s_{pm}}{\partial s_{kl}}\Big)$$

$$= \frac{1}{3}\Big(s_{np}s_{pm}\delta_{mk}\delta_{ln} + s_{mn}s_{pm}\delta_{kn}\delta_{lp} + s_{mn}s_{np}\delta_{kp}\delta_{lm}\Big)$$

$$= \frac{1}{3}\Big(s_{lp}s_{pk} + s_{mk}s_{lm} + s_{ln}s_{nk}\Big) = s_{km}s_{ml}$$

Thus we find

$$\frac{\partial J_3}{\partial \sigma_{ij}} = s_{km}s_{ml}(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl})$$

$$= s_{im}s_{mj} - \frac{1}{3}s_{km}s_{mk}\delta_{ij} = s_{im}s_{mj} - \frac{2}{3}J_2\delta_{ij}$$

Answer 4.3. We plug in the definition of D_{ijkl} in the expression find

$$\sigma_{ij} = D_{ijkl}\epsilon_{ij} = 2G\Big[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\Big]\epsilon_{kl}$$

$$= 2G\Big[\frac{1}{2}(\delta_{ik}\delta_{jl}\epsilon_{kl} + \delta_{il}\delta_{jk}\epsilon_{kl}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\epsilon_{kl}\Big]$$

$$= 2G\Big[\frac{1}{2}(\epsilon_{ij} + \epsilon_{ji}) + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\Big]$$

$$= 2G\Big[\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\Big]$$

where it was used that $\epsilon_{ij} = \epsilon_{ji}$. The symmetry is found by swapping the indices as desired and equating the output expressions for D.

Answer 4.4. We have

$$D_{ijmn} = 2G\Big[\frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{mn}\Big] = (a+b)$$

and

$$C_{mnkl} = \frac{1}{2G}\Big[\frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}) - \frac{\nu}{1+\nu}\delta_{mn}\delta_{kl}\Big] = (c+d)$$

We know that

$$D_{ijmn}C_{mnkl} = (ac + ad + bc + bd)$$

We start by finding ac:

$$(ac) = \frac{2G}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\frac{1}{4G}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}) =$$

$$\frac{1}{4}(\delta_{im}\delta_{jn}\delta_{mk}\delta_{nl} + \delta_{im}\delta_{jn}\delta_{ml}\delta_{nk} + \delta_{in}\delta_{jm}\delta_{mk}\delta_{nl} + \delta_{in}\delta_{jm}\delta_{ml}\delta_{nk}) =$$

$$\frac{1}{4}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Now we evaluate ad:

$$(ad) = \frac{2G}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\frac{1}{2G}(-\frac{\nu}{1+\nu}\delta_{mn}\delta_{kl}) =$$

$$\frac{-1}{2}\frac{\nu}{1+\nu}(\delta_{im}\delta_{jn}\delta_{mn}\delta_{kl} + \delta_{in}\delta_{jm}\delta_{mn}\delta_{kl}) =$$

$$\frac{-\nu}{2(1+\nu)}(\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl}) = \frac{-\nu}{(1+\nu)}\delta_{ij}\delta_{kl}$$

and bc:

$$(bc) = 2G\frac{\nu}{1-2\nu}\delta_{ij}\delta_{mn}\frac{1}{2G}\frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}) =$$

$$\frac{\nu}{2(1-2\nu)}(\delta_{mk}\delta_{nl}\delta_{ij}\delta_{mn} + \delta_{ml}\delta_{nk}\delta_{ij}\delta_{mn}) =$$

$$\frac{\nu}{2(1-2\nu)}(\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl}) = \frac{\nu}{(1-2\nu)}\delta_{ij}\delta_{kl}$$

and bd:

$$(bd) = 2G\frac{\nu}{1-2\nu}\delta_{ij}\delta_{mn}\frac{1}{2G}(-\frac{\nu}{1+\nu}\delta_{mn}\delta_{kl}) =$$

$$\frac{-3\nu^2}{(1-2\nu)(1+\nu)}\delta_{ij}\delta_{kl}$$

Thus we have that $ad + bc + bd$ is

$$\frac{-3\nu^2}{(1-2\nu)(1+\nu)}\delta_{ij}\delta_{kl} + \frac{\nu}{(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{-\nu}{(1+\nu)}\delta_{ij}\delta_{kl}$$

$$\Big(\frac{-3\nu^2 + \nu(1+\nu) - \nu(1-2\nu)}{(1-2\nu)(1+\nu)}\Big)\delta_{ij}\delta_{kl} =$$

$$\Big(\frac{-3\nu^2 + \nu + \nu^2 - \nu + 2\nu^2}{(1-2\nu)(1+\nu)}\Big)\delta_{ij}\delta_{kl} = 0$$

And hence we have

$$D_{ijmn}C_{mnkl} = (ac + ad + bc + bd = ac) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

which was to be shown.

Answer 4.5. We have

$$D_{pqij}\epsilon_{ij} = D_{pqij}C_{ijkl}\sigma_{kl}$$

from previous problem we found

$$D_{pqij}C_{jikl} = \frac{1}{2}(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk})$$

Hence

$$D_{pqij}\epsilon_{ij} = D_{pqij}C_{ijkl}\sigma_{kl} = \frac{1}{2}(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk})\sigma_{kl} =$$

$$= \frac{1}{2}(\sigma_{pq} + \sigma_{qp}) = \sigma_{pq}$$

Swapping the index p to i, q to j, j to l and i to k we get

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

which was to be proven.

Answer 4.6. For isotropic elasticity we have that

$$D_{ijkl} = 2G\Big[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\Big], \quad \textit{and} \quad \sigma_{ij} = D_{ijkl}\epsilon_{kl}$$

Considering that $\epsilon_{ij} = \epsilon_{ji}$ and that D_{ijkl} has both major and minor symmetry ,$D_{ijkl} = D_{klij}$,$D_{ijkl} = D_{ijlk}$, we can find that:

$$\sigma_{11} = D_{1111}\epsilon_{11} + D_{1122}\epsilon_{22} + D_{1133}\epsilon_{33} + 2D_{1112}\epsilon_{12} + 2D_{1113}\epsilon_{13} + 2D_{1123}\epsilon_{23}$$

$$\sigma_{22} = D_{2211}\epsilon_{11} + D_{2222}\epsilon_{22} + D_{2233}\epsilon_{33} + 2D_{2212}\epsilon_{12} + 2D_{2213}\epsilon_{13} + 2D_{2223}\epsilon_{23}$$

$$\vdots$$

It make sense to define the strain with the factor two included. We define stress and strain in matrix format as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

It now follows that $[D_{ijkl}]$ must be defined as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & D_{1112} & D_{1113} & D_{1123} \\ D_{2211} & D_{2222} & D_{2233} & D_{2212} & D_{2213} & D_{2223} \\ D_{3311} & D_{3322} & D_{3333} & D_{3312} & D_{3313} & D_{3323} \\ D_{1211} & D_{1222} & D_{1233} & D_{1212} & D_{1213} & D_{1223} \\ D_{1311} & D_{1322} & D_{1333} & D_{1312} & D_{1313} & D_{1323} \\ D_{2311} & D_{2322} & D_{2333} & D_{2312} & D_{2313} & D_{2323} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

We want to identify all instances of $\boldsymbol{D}$. We start with

$$D_{1111} = D_{2222} = D_{3333} = 2G\Big(1 + \frac{\nu}{1-2\nu}\Big) = \frac{E}{(1+\nu)(1-2\nu)}(1-\nu)$$

where it was used that $G = E/2(1+\nu)$. Secondly we find

$$D_{1122} = D_{2211} = D_{1133} = D_{3311} = D_{3322} = D_{2233} =$$

$$2G\frac{\nu}{1-2\nu} = \frac{E}{(1+\nu)(1-2\nu)}\nu$$

Thirdly we find

$$D_{1212} = D_{1313} = D_{2323} = 2G = \frac{E}{(1+\nu)(1-2\nu)}(1-2\nu)$$

The remaining terms will be identical to zero as realised by inspection. This concludes that

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Answer 4.7. We seek the strain stress relation:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

We start by considering pure tension. The stress is of the format:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and the corresponding strains are related as

$$\epsilon_{11} = C_{1111}\sigma_{11}$$

$$\epsilon_{22} = C_{2211}\sigma_{11}$$

$$\epsilon_{33} = C_{3311}\sigma_{11}$$

$$2\epsilon_{12} = C_{1211}\sigma_{11}$$

$$2\epsilon_{13} = C_{1311}\sigma_{11}$$

$$2\epsilon_{23} = C_{2311}\sigma_{11}$$

Clearly the shear strains, $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$ must be zero if we have a pure tension for an isotropic material. Thus

$$C_{1211} = C_{1311} = C_{2311} = 0$$

Using the "traditional" one dimensional Hooke's law we have

$$\sigma_{11} = E\epsilon_{11}, \quad \epsilon_{22} = -\nu\epsilon_{11}, \quad \epsilon_{33} = -\nu\epsilon_{11}$$

where ν is Poisson's ratio and E Young's modulus. The remaining relations become

$$C_{1111} = 1/E$$

$$C_{2211} = -\nu/E$$

$$C_{3311} = -\nu/E$$

Since the material considered is assumed to be isotropic we will find the same results by applying uniaxial tension in σ_{22} or σ_{33}.
We may also apply pure shear

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 & \sigma_{12} & 0 & 0 \end{bmatrix}^T$$

This gives the relations

$$\epsilon_{11} = C_{1112}\sigma_{12}$$

$$\epsilon_{22} = C_{2212}\sigma_{12}$$

$$\epsilon_{33} = C_{3312}\sigma_{12}$$

$$2\epsilon_{12} = C_{1212}\sigma_{12}$$

$$2\epsilon_{13} = C_{1312}\sigma_{12}$$

$$2\epsilon_{23} = C_{2312}\sigma_{12}$$

From classical theory and isotropic materials we have that

$$G = \frac{\sigma_{12}}{2\epsilon_{12}} = \frac{E}{2(1+\nu)} \Rightarrow 2\epsilon_{12} = \frac{2(1+\nu)}{E}\sigma_{12}$$

Hence

$$C_{1212} = \frac{2(1+\nu)}{E}$$

And the same applies for shear stress in 13 and 23 direction. The axial strain must be zero here so therefore those components are zero.
Alternatively:

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \sigma_0 & 0 \\ \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{\epsilon} = \begin{bmatrix} 0 & \epsilon_0 & 0 \\ \epsilon_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(implication due to isotropy pure shear stress gives pure shear strain) Transforming into the principal coordinate system we get the characteristic equation from the eigenvalue problem

$$-\lambda^3 + \lambda\sigma_0^2 = \lambda(\lambda^2 - \sigma_0^2) \Rightarrow \lambda_{1,2} = \pm\sigma_0, \quad \lambda_3 = 0$$

Thus we have that in the principal system pure shear is

$$\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & -\sigma_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_0 & 0 & 0 \\ 0 & -\epsilon_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This then yields the equations

$$\epsilon'_{11} = C_{1111}\sigma'_{11} + C_{1122}\sigma'_{22}$$

$$\epsilon'_{22} = C_{2211}\sigma'_{11} + C_{2222}\sigma'_{22}$$

$$\epsilon'_{33} = C_{3311}\sigma'_{11} + C_{3322}\sigma'_{22}$$

$$2\epsilon'_{12} = C_{1211}\sigma'_{11} + C_{1222}\sigma'_{22}$$

$$2\epsilon'_{13} = C_{1311}\sigma'_{11} + C_{1322}\sigma'_{22}$$

$$2\epsilon'_{23} = C_{2311}\sigma'_{11} + C_{2322}\sigma'_{22}$$

We then get

$$2\epsilon'_{11} = 2(C_{1111} - C_{1122})\sigma'_0 \Rightarrow 2\epsilon'_{11} = \frac{1}{E}2(1+\nu)\sigma_0$$

Since $\sigma_0 = \sigma_{12}$ in the other system, and $\epsilon'_{11} = \epsilon_0 = \epsilon_{12}$ in the other system we have found the relation and

$$C_{1212} = \frac{1}{E}2(1+\nu)$$

The same applies for other components.

Answer 4.8. a) The chain rule yields

$$\frac{\partial W}{\partial \epsilon_{ij}} = \frac{\partial W}{\partial \tilde{I}_1}\frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial \tilde{I}_2}\frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} + \frac{\partial W}{\partial \tilde{I}_3}\frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}}$$

We find first

$$\frac{\partial \tilde{I}_1}{\partial \epsilon_{ij}} = \frac{\partial \epsilon_{kk}}{\partial \epsilon_{ij}} = \delta_{ik}\delta_{jk} = \delta_{ij}$$

Then

$$\frac{\partial \tilde{I}_2}{\partial \epsilon_{ij}} = \frac{1}{2}\frac{\partial \epsilon_{kl}\epsilon_{lk}}{\partial \epsilon_{ij}} = ... = \epsilon_{ij}$$

and finally

$$\frac{\partial \tilde{I}_3}{\partial \epsilon_{ij}} = \frac{1}{3}\frac{\partial \epsilon_{kl}\epsilon_{lm}\epsilon_{mk}}{\partial \epsilon_{ij}} = ... = \epsilon_{il}\epsilon_{lj}$$

In conclusion

$$\sigma_{ij} = \frac{\partial W}{\partial \tilde{I}_1}\delta_{ij} + \frac{\partial W}{\partial \tilde{I}_2}\epsilon_{ij} + \frac{\partial W}{\partial \tilde{I}_3}\epsilon_{il}\epsilon_{lj}$$

b) We insert and get

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + \frac{\partial W}{\partial \tilde{I}_2}\epsilon_{ij} + \frac{\partial W}{\partial \tilde{I}_3}\epsilon_{il}\epsilon_{lj} = [\lambda\delta_{ij}\delta_{lk} + \frac{\partial W}{\partial \tilde{I}_2}\delta_{ki}\delta_{lj} + \frac{\partial W}{\partial \tilde{I}_3}\delta_{ki}\epsilon_{lj}]\epsilon_{kl}$$

hence

$$D^s_{ijkl} = [\lambda\delta_{ij}\delta_{lk} + \frac{\partial W}{\partial \tilde{I}_2}\delta_{ki}\delta_{lj} + \frac{\partial W}{\partial \tilde{I}_3}\delta_{ki}\epsilon_{lj}]$$

c) We know from a) that σ_{ij} is a function of ϵ_{ij} as

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

Thus an increment $d\sigma_{ij}$ corresponds to an increment in ϵ_{ij} multiplied with the function derivative as

$$d\sigma_{ij} = \frac{\partial^2 W}{\partial \epsilon_{kl}\epsilon_{ij}} d\epsilon_{kl}$$

We find that

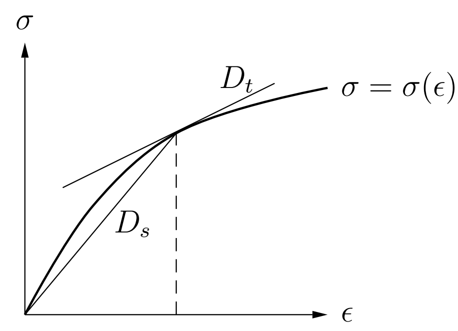
$$\frac{\partial^2 W}{\partial \epsilon_{kl}\epsilon_{ij}} = \frac{\partial}{\partial \epsilon_{kl}}\Big(\frac{\partial W}{\partial \epsilon_{ij}}\Big) = \frac{\partial}{\partial \epsilon_{kl}}\Big(\lambda\epsilon_{mm}\delta_{ij} + \frac{\partial W}{\partial \tilde{I}_2}\epsilon_{ij} + \frac{\partial W}{\partial \tilde{I}_3}\epsilon_{ip}\epsilon_{pj}\Big)$$

$$= \lambda\delta_{mk}\delta_{lm}\delta_{ij} + \frac{\partial W}{\partial \tilde{I}_2}\delta_{ik}\delta_{lj} + \frac{\partial W}{\partial \tilde{I}_3}(\delta_{ik}\delta_{pl}\epsilon_{pj} + \delta_{pk}\delta_{jl}\epsilon_{ip})$$

$$= \lambda\delta_{lk}\delta_{ij} + \frac{\partial W}{\partial \tilde{I}_2}\delta_{ik}\delta_{lj} + \frac{\partial W}{\partial \tilde{I}_3}(\delta_{ik}\epsilon_{lj} + \delta_{jl}\epsilon_{ik})$$

where it is assumed that the derivatives of W are constant.

d)



where D_t is the material tangent stiffness and D_s is the material secant stiffness.

Answer 4.9. Given

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

We have

$$\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\sigma} = \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} + 2\sigma_{12}\epsilon_{12} + 2\sigma_{13}\epsilon_{13} + 2\sigma_{23}\epsilon_{23}$$

We expand the given index notation expression as

$$W = \sigma_{ij}\epsilon_{ij} = \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} +$$

$$\sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13} + \sigma_{23}\epsilon_{23} + \sigma_{21}\epsilon_{21} + \sigma_{31}\epsilon_{31} + \sigma_{32}\epsilon_{32}$$

Since these tensors are symmetric we have

$$\sigma_{12}\epsilon_{12} = \sigma_{21}\epsilon_{21}$$

$$\sigma_{13}\epsilon_{13} = \sigma_{31}\epsilon_{31}$$

$$\sigma_{23}\epsilon_{23} = \sigma_{32}\epsilon_{32}$$

And thus

$$W = \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} +$$

$$2\sigma_{12}\epsilon_{12} + 2\sigma_{13}\epsilon_{13} + 2\sigma_{23}\epsilon_{23} = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon} = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\sigma}$$

Answer 4.10. a) By definition we have

$$C = \sigma_{ij}\epsilon_{ij} - W$$

where W is the strain energy. It can be shown that

$$\epsilon_{ij} = \frac{\partial C}{\partial \sigma_{ij}}$$

In our case we get

$$\frac{\partial C}{\partial \sigma_{ij}} = \frac{\partial(aJ_2 + bI_1J_2)}{\partial \sigma_{ij}} = a\frac{\partial J_2}{\partial \sigma_{ij}} + bI_1\frac{\partial J_2}{\partial \sigma_{ij}} + bJ_2\frac{\partial I_1}{\partial \sigma_{ij}}$$

The stress invariants are defined as

$$I_1 = \sigma_{kk}, \quad J_2 = \frac{1}{2} s_{kl} s_{lk}$$

and the derivatives are found from previous calculations

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij}, \quad \frac{\partial J_2}{\partial \sigma_{ij}} = s_{ij}$$

And we find

$$\frac{\partial C}{\partial \sigma_{ij}} = a s_{ij} + b I_1 s_{ij} + b J_2 \delta_{ij} = b J_2 \delta_{ij} + (a + b I_1) s_{ij}$$

b) We set $i = j = 1$ and find

$$\epsilon_{11} = 10^{-4} \sigma_{11} + 10^{-5} \sigma_{11}^2 = b J_2 + (a + b I_1) s_{11}$$

For uniaxial tension we have

$$[s_{ij}] = \frac{1}{3} \begin{bmatrix} 2\sigma_{11} & 0 & 0 \\ 0 & -\sigma_{11} & 0 \\ 0 & 0 & -\sigma_{11} \end{bmatrix}$$

And thus

$$J_2 = \frac{1}{3} \sigma_{11}^2$$

$$I_1 = \sigma_{kk} = \sigma_{11}$$

$$s_{11} = \frac{2}{3} \sigma_{11}$$

This yields

$$10^{-4} \sigma_{11} + 10^{-5} \sigma_{11}^2 = b \frac{1}{3} \sigma_{11}^2 + \frac{2}{3} (a + b\sigma_{11}) \sigma_{11} = \frac{2}{3} a \sigma_{11} + b \sigma_{11}^2$$

Thus it must hold that

$$a = \frac{3}{2} \times 10^{-4}$$

$$b = 10^{-5}$$

c) For pure shear we have

$$[s_{ij}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

note that $I_1 = \sigma_{kk} = 0$. We then have that

$$J_2 = \tau^2$$

Therefore

$$[\epsilon_{ij}] = b\tau^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\tau = 100$ MPa we get

$$[\epsilon_{ij}] = \begin{bmatrix} b100^2 & a100 & 0 \\ a100 & b100^2 & 0 \\ 0 & 0 & b100^2 \end{bmatrix}$$

d) The volumetric strain becomes

$$\epsilon_{kk} = 3b\tau^2$$

For a standard isotropic material and pure shear

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we get from the constitutive matrix relation in problem 4.7 that

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0$$

Thus pure shear gives $\epsilon_{kk} = 0$, i.e no volumetric change!

Answer 4.11. Assume that we define the given relation as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{bmatrix}$$

Let us then start by introducing the symmetry plane x_1x_2. The Coordinate transformation is per definition

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

and so we must have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The transformations for the stress and the strain is per definition

$$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T, \quad \boldsymbol{\epsilon}' = \boldsymbol{A}\boldsymbol{\epsilon}\boldsymbol{A}^T$$

Thus we have

$$\boldsymbol{\sigma}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{21} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix}$$

The same is true for the strain, this concludes

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{21} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon'_{11} & \epsilon'_{12} & \epsilon'_{13} \\ \epsilon'_{21} & \epsilon'_{22} & \epsilon'_{23} \\ \epsilon'_{31} & \epsilon'_{32} & \epsilon'_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & -\epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & -\epsilon_{23} \\ -\epsilon_{31} & -\epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

Now, since we have a symmetry plane, it is implied that the constitutive matrix $\boldsymbol{D}$ is identical in the two systems. In other words

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$$

$$\boldsymbol{\sigma}' = \boldsymbol{D}\boldsymbol{\epsilon}'$$

Selecting a component of stress, say σ_{11} and σ'_{11}, provide

$$\sigma_{11} = D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} + D_{14}\epsilon_{12} + D_{15}\epsilon_{13} + D_{16}\epsilon_{23}$$

$$\sigma'_{11} = D_{11}\epsilon'_{11} + D_{12}\epsilon'_{22} + D_{13}\epsilon'_{33} + D_{14}\epsilon'_{12} - D_{15}\epsilon'_{13} - D_{16}\epsilon'_{23}$$

and making use of the symmetry plane gives for the primed system

$$\sigma_{11} = D_{11}\epsilon_{11} + D_{12}\epsilon_{22} + D_{13}\epsilon_{33} + D_{14}\epsilon_{12} - D_{15}\epsilon_{13} - D_{16}\epsilon_{23}$$

As this should be the same as for the unprimed system, i.e. we should have the same stress response for all possible strain changes, requires that we must have

$$D_{15} = D_{16} = 0$$

Likewise it is found from the relations

$$\sigma_{22} = \sigma'_{22} \quad \Rightarrow \quad D_{25} = D_{26} = 0$$

$$\sigma_{33} = \sigma'_{33} \quad \Rightarrow \quad D_{35} = D_{36} = 0$$

$$\sigma_{12} = \sigma'_{12} \quad \Rightarrow \quad D_{45} = D_{46} = 0$$

$$\sigma_{13} = -\sigma'_{13} \quad \Rightarrow \quad D_{51} = D_{52} = D_{53} = D_{54} = 0$$

$$\sigma_{23} = -\sigma'_{23} \quad \Rightarrow \quad D_{61} = D_{62} = D_{63} = D_{64} = 0$$

We have now found that the relation for a single symmetry plane becomes

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} & 0 & 0 \\ D_{41} & D_{42} & D_{43} & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & D_{56} \\ 0 & 0 & 0 & 0 & D_{65} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{bmatrix}$$

We choose a perpendicular plane, let say the x_1x_3 plane, and repeat the above process. The transformation matrix becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & \sigma_{13} \\ -\sigma_{21} & \sigma_{22} & -\sigma_{23} \\ \sigma_{31} & -\sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon'_{11} & \epsilon'_{12} & \epsilon'_{13} \\ \epsilon'_{21} & \epsilon'_{22} & \epsilon'_{23} \\ \epsilon'_{31} & \epsilon'_{32} & \epsilon'_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & -\epsilon_{12} & \epsilon_{13} \\ -\epsilon_{21} & \epsilon_{22} & -\epsilon_{23} \\ \epsilon_{31} & -\epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

Since the constitutive matrix is the same in the two systems we can set the components equal again and find that

$$\sigma_{11} = \sigma'_{11} \quad \Rightarrow \quad D_{14} = 0$$

$$\sigma_{22} = \sigma'_{22} \quad \Rightarrow \quad D_{24} = 0$$

$$\sigma_{33} = \sigma'_{33} \quad \Rightarrow \quad D_{34} = 0$$

$$\sigma_{12} = -\sigma'_{12} \quad \Rightarrow \quad D_{41} = D_{42} = D_{43} = 0$$

$$\sigma_{13} = \sigma'_{13} \quad \Rightarrow \quad D_{56} = 0$$

$$\sigma_{23} = -\sigma'_{23} \quad \Rightarrow \quad D_{65} = 0$$

And so we have found that

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{bmatrix}$$

Using one additional symmetry plane will not reduce $\boldsymbol{D}$ further.

Answer 4.12. Applying uniaxial stress in xx we find form Hooke's law that

$$\epsilon_{xx} = \frac{1}{E_x}\sigma_{xx}$$

The Poisson's ratio gives in this case that

$$\epsilon_{yy} = \frac{-\nu_{xy}}{E_x}\sigma_{xx}$$

$$\epsilon_{zz} = \frac{-\nu_{xz}}{E_x}\sigma_{xx}$$

For the yy and zz direction we do the same and find from Hooke's law that

$$\epsilon_{yy} = \frac{1}{E_y}\sigma_{yy}$$

$$\epsilon_{zz} = \frac{1}{E_z}\sigma_{zz}$$

Poisson's ratio gives

$$\epsilon_{xx} = \frac{-\nu_{yx}}{E_y}\sigma_{yy}$$

$$\epsilon_{zz} = \frac{-\nu_{yz}}{E_y}\sigma_{yy}$$

$$\epsilon_{xx} = \frac{-\nu_{zx}}{E_z}\sigma_{zz}$$

$$\epsilon_{yy} = \frac{-\nu_{zy}}{E_z}\sigma_{zz}$$

Simple shear test

$$\gamma_{xy} = \frac{1}{G_{xy}}\sigma_{xy}, \quad 2\epsilon_{xy} = \gamma_{xy}$$

Since the material is hyper elastic the matrix $\boldsymbol{D}$ is symmetric and

$$\frac{-\nu_{xy}}{E_x} = \frac{-\nu_{yx}}{E_y}$$

Answer 7.1. Use $\dot{\epsilon}_{kk} = \delta_{st}\dot{\epsilon}_{st}$ and $\sigma_{mn}\dot{\epsilon}_{mn} = \sigma_{st}\dot{\epsilon}_{st}$. Moreover, use $\dot{\epsilon}_{ij} = \frac{1}{2}(\delta_{is}\delta_{jt} + \delta_{it}\delta_{js})\dot{\epsilon}_{st}$ (where also the right-hand side is symmetric in i and j). Some manipulations will show that $\dot{\sigma}_{ij} = D_{ijst}\dot{\epsilon}_{st}$ where

$$\begin{aligned}
D_{ijst} \;=\;& \beta_1\delta_{ij}\delta_{st} + \tfrac{1}{2}\beta_2(\delta_{is}\delta_{jt} + \delta_{it}\delta_{js}) + \beta_3\sigma_{ij}\delta_{st} + \beta_4\delta_{ij}\sigma_{st} \\
&+\tfrac{1}{2}\beta_5(\sigma_{is}\delta_{jt} + \sigma_{it}\delta_{js} + \sigma_{tj}\delta_{is} + \sigma_{sj}\delta_{it}) \\
&+\beta_6\sigma_{ik}\sigma_{kj}\delta_{st} + \beta_7\sigma_{ij}\sigma_{st} + \tfrac{1}{2}\beta_8\delta_{ij}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt}) \\
&+\tfrac{1}{2}\beta_9[\sigma_{ik}(\sigma_{ks}\delta_{jt} + \sigma_{kt}\delta_{js}) + \sigma_{lj}(\sigma_{tl}\delta_{is} + \sigma_{sl}\delta_{it})] \\
&+\beta_{10}\sigma_{ik}\sigma_{kj}\sigma_{st} + \tfrac{1}{2}\beta_{11}\sigma_{ij}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt}) \\
&+\tfrac{1}{2}\beta_{12}\sigma_{ik}\sigma_{kj}(\sigma_{tm}\sigma_{ms} + \sigma_{sm}\sigma_{mt})
\end{aligned}$$

Answer 8.1. a) The principal stresses are found as the eigenvalues λ

$$\sigma_{ij}v_j = \lambda v_j \quad \Rightarrow \quad (\sigma_{ij} - \lambda\delta_{ij})v_j = 0, \quad v_j \neq 0$$

This is done by setting the determinant to zero

$$\det(\sigma_{ij} - \lambda\delta_{ij}) = 0 \quad \text{or} \quad \det(\boldsymbol{\sigma} - \lambda\boldsymbol{I}) = \boldsymbol{0}$$

Thus we seek

$$\det\left(\begin{bmatrix} (\frac{3}{2}-\lambda) & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & (\frac{11}{4}-\lambda) & -\frac{5}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{5}{4} & (\frac{11}{4}-\lambda) \end{bmatrix}\right)$$

After simplification, we find the characteristic equation

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$$

The eigenvalues then becomes

$$\lambda_1 = \sigma_1 = 4\text{MPa}, \quad \lambda_2 = \sigma_2 = 2\text{MPa}, \quad \lambda_3 = \sigma_3 = 1\text{MPa}$$

We can now solve for the eigendirections using the equation

$$\boldsymbol{\sigma n} = \lambda \boldsymbol{n}$$

We find after some cumbersome calculations (see previous eigenproblem computations) that

$$\boldsymbol{n}^{(1)} = (1/\sqrt{2})\begin{bmatrix}0 & 1 & -1\end{bmatrix}^T$$

$$\boldsymbol{n}^{(2)} = (1/2)\begin{bmatrix}-\sqrt{2} & 1 & 1\end{bmatrix}^T$$

$$\boldsymbol{n}^{(3)} = (1/2)\begin{bmatrix}\sqrt{2} & 1 & 1\end{bmatrix}^T$$

b) The deviatoric stress is per definition

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Alternatively

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Considering the initial eigenvalue problem

$$(\sigma_{ij} - \lambda\delta_{ij})v_j = 0$$

We could insert the above expression for the stress as

$$(s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} - \lambda\delta_{ij})v_j = 0$$

To clarify our finding we write

$$(s_{ij} - (\lambda - \frac{1}{3}\sigma_{kk})\delta_{ij})v_j = 0 \quad \Rightarrow \quad (s_{ij} - \lambda^{(s)}\delta_{ij})v_j = 0$$

where

$$\lambda^{(s)} = (\lambda^{(\sigma)} - \frac{1}{3}\sigma_{kk})$$

Since the eigenvalues of σ_{ij}, $\lambda^{(\sigma)}$, are known form assignment (a), as well as σ_{kk}, the principal deviatoric stresses become via insertion

$$s_1 = 5/3 \text{ MPa}, \quad s_2 = -1/3 \text{ MPa}, \quad s_3 = -4/3 \text{ MPa}$$

c) The deviatoric stress invariants are defined as

$$J_1 = s_{ii}, \quad J_2 = \frac{1}{2}s_{ij}s_{ji}, \quad J_3 = \frac{1}{3}s_{ij}s_{kj}s_{ik}$$

Since they are all invariant, it is enough to determine them in the principal coordinate system. The principal deviatoric stress tensor is

$$[s_{ij}] = \begin{bmatrix} \frac{5}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix}$$

And so we find first:

$$J_1 = s_{ii} = \frac{5}{3} - \frac{1}{3} - \frac{4}{3} = 0$$

Secondly using that $s_{ij} = 0$ if $i \neq j$ we find

$$J_2 = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) = (\frac{1}{2})(\frac{5^2 + 1^2 + 4^2}{3^2}) = \frac{7}{3}$$

Finally, again using that $s_{ij} = 0$ if $i \neq j$ we find

$$J_3 = \frac{1}{3}s_{ij}s_{kj}s_{ik} = \frac{1}{3}(s_1^3 + s_2^3 + s_3^3) = (\frac{1}{3})(\frac{5^3 - 1^3 - 4^3}{3^3}) = \frac{20}{27}$$

Answer 8.2. a) By definition

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

We are given the relation

$$s_1 > s_2 > s_3$$

If then $s_3 = 0$ it follows directly that

$$s_1 > 0, \quad s_2 > 0$$

But the definition of s_{ij} requires

$$s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} = \sigma_{ii} - \sigma_{kk} = 0$$

And so clearly

$$s_{ii} = s_1 + s_2 + s_3 \neq 0$$

If $s_3 = 0, s_1 > 0, s_2 > 0$, and thus we must require $s_3 \neq 0$.

b) Clearly J_2 cannot be negative since

$$J_2 = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) > 0$$

c) We know from previous assignments that

$$J_3 = \frac{1}{3}(s_1^3 + s_2^3 + s_3^3)$$

In analogy with the proof on page number 38 in the book we can find that

$$J_3 = s_1 s_2 s_3$$

For J_3 to be positive it is then only required that two principal deviatoric strains components are negative and the third positive. Such a requirement can still satisfy $s_{ii} = 0$. Thus J_3 can be positive.

Answer 8.3. A hydro static stress is defined as

$$\sigma_{ij}^{(hydro)} = \frac{1}{3}\bar{\sigma}_{kk}\delta_{ij}$$

where $\bar{\sigma}_{kk}$ is an arbitrary stress. So if we subtract the hydrostatic stress from a given stress state we find

$$\sigma_{ij} - \sigma_{ij}^{(hydro)} = \sigma_{ij} - \frac{1}{3}\bar{\sigma}_{kk}\delta_{ij}$$

The eigenvalue problem for this stress state is given by

$$(\sigma_{ij} - \frac{1}{3}\bar{\sigma}_{kk}\delta_{ij} - \lambda\delta_{ij})v_j = 0$$

rewriting and we find the eigenvalue problem from the stress σ_{ij} as

$$(\sigma_{ij} - \bar{\lambda}\delta_{ij})v_j = 0$$

where $\bar{\lambda} = \lambda + \bar{\sigma}_{kk}/3$. This reveals that the principal directions coincide whereas the eigenvalues are different. Similar to what was showed previously, that the principal directions of s_{ij} and σ_{ij} coincide (cf. Problem 3.1).

Answer 8.4. By definition

$$I_1 = \sigma_{ii}, \quad J_2 = \frac{1}{2}s_{ij}s_{ji}, \quad J_3 = \frac{1}{3}s_{ij}s_{kj}s_{ik}$$

We find directly by insertion that

$$I_1 = \sigma_{ii} = 30 + 20 + 10 = 60 \text{ MPa}$$

The deviatoric stress is in this case

$$[s_{ij}] = [\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}] = \begin{bmatrix} 10 & 45 & 60 \\ 45 & 0 & 50 \\ 60 & 50 & -10 \end{bmatrix}$$

and so we find

$$\begin{aligned} J_2 &= \frac{1}{2}s_{ij}s_{ji} \\ &= \frac{1}{2}(s_{11}s_{11} + s_{21}s_{12} + s_{31}s_{13} + s_{12}s_{21} + s_{22}s_{22} + s_{32}s_{23} \\ &\quad + s_{13}s_{31} + s_{23}s_{32} + s_{33}s_{33}) \\ &= \frac{1}{2}(10^2 + 2 \times 45^2 + 2 \times 60^2 + 2 \times 50^2 + 10^2) = 8225 \text{ MPa} \end{aligned}$$

To find J_3 we consider that

$$J_3 = s_1 s_2 s_3$$

which is proven in analogy with the proof on page 38. in the course book. Computing the eigenvalues (use Matlab or similar) of the deviatoric stress tensor, s_{ij}, yields

$$\lambda_1 = -63.75..\text{MPa}, \quad \lambda_2 = -40.07..\text{MPa}, \quad \lambda_3 = 103.8..\text{MPa},$$

and so

$$J_3 = s_1 s_2 s_3 \approx 265250 \text{ M}^3\text{Pa}$$

The Lode angle can be calculated from relation (8.18) in the course book

$$\cos(3\theta) = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} = \frac{3\sqrt{3}}{2}\frac{265250 \times 10^{18}}{(8225 \times 10^{12})^{3/2}} = -0.923854$$

And so the Lode angle becomes

$$\theta \approx 0.13 \text{ rad}$$

Answer 8.5. For metals it is appropriate to choose a yield criteria that is independent of hydrostatic stress. Thus we can choose the von Mises yield criteria:

$$\sqrt{3J_2} - \sigma_{y0} = 0$$

or the Tresca yield criterion:

$$\sigma_1 - \sigma_3 - \sigma_{y0} = 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

Given the biaxial state

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \alpha\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we find the deviatoric stress as

$$\boldsymbol{s} = \begin{bmatrix} \sigma - \frac{\sigma}{3}(1+\alpha) & 0 & 0 \\ 0 & \alpha\sigma - \frac{\sigma}{3}(1+\alpha) & 0 \\ 0 & 0 & -\frac{\sigma}{3}(1+\alpha) \end{bmatrix}$$

Computing J_2 then gives

$$J_2 = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{3}\sigma^2(\alpha^2 - \alpha + 1)$$

and so we have that the material yields for von Mises criterion when

$$\sqrt{\frac{1}{3}\sigma^2(\alpha^2-\alpha+1)}-\sigma_{y0}=0$$

For Tresca's criterion whe have to consider three different cases. First, when $\alpha \geq 1$ we have

$$\alpha\sigma-\sigma_{y0}=0$$

When instead $\alpha \leq 0$ we get

$$(1-\alpha)\sigma-\sigma_{y0}=0$$

and finally when $0 \leq \alpha \leq 1$ we have

$$\sigma-\sigma_{y0}=0$$

Alternatively, one may replace the parameter σ_{y0} for $2\tau_{max}$. This can be revealing considering the relation (3.24) in the course book. Tresca's criterion is then interpreted as giving yielding when the maximum shear stress reaches a critical value.

Answer 8.6. The Tresca yield criterion is defined as

$$\sigma_1-\sigma_3-\sigma_{y0}=0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

Or, equivalently

$$\sigma_{max}-\sigma_{min}-\sigma_{y0}=0$$

We know that $\sigma_{33}=0$ and find now a series cases (dropping the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$)

$$(1)\ \sigma_1=\sigma_{y0}, \quad \text{if} \quad \sigma_1=\sigma_{max} \geq \sigma_2 \geq 0$$

$$(2)\ \sigma_2=\sigma_{y0}, \quad \text{if} \quad \sigma_2=\sigma_{max} \geq \sigma_1 \geq 0$$

$$(3)\ \sigma_1=-\sigma_{y0}, \quad \text{if} \quad \sigma_1=\sigma_{min} \leq \sigma_2 \leq 0$$

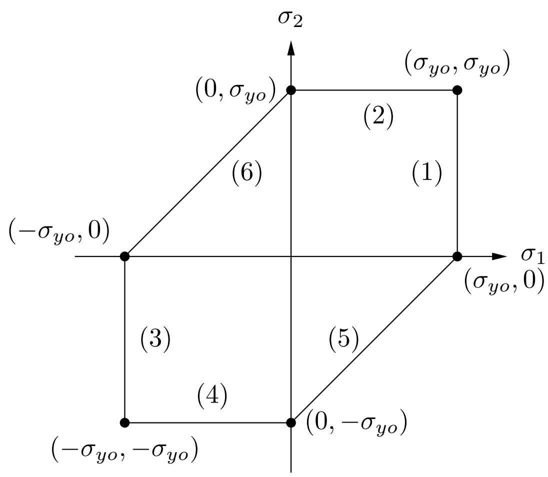
$$(4)\ \sigma_2=-\sigma_{y0}, \quad \text{if} \quad \sigma_2=\sigma_{min} \leq \sigma_1 \leq 0$$

In the case when $\sigma_{max}=\sigma_1$ and $\sigma_{min}=\sigma_2$ (and reverse) such that $\sigma_{33}=0$ becomes the intermediate value we get

$$(5)\ \sigma_1-\sigma_2=\sigma_{y0}, \quad \text{if} \quad \sigma_1=\sigma_{max} \geq 0 \geq \sigma_2$$

$$(6)\ \sigma_2-\sigma_1=\sigma_{y0}, \quad \text{if} \quad \sigma_2=\sigma_{max} \geq 0 \geq \sigma_1$$

With the 6 above relations we can now draw the yield curve



Answer 8.7. We have that

$$[\sigma_{ij}] = \begin{bmatrix} 80 & 80 & 0 \\ 80 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

Solving the eigenvalue problem (as per usual...) we get the principal stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} 142.46 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -22.46 \end{bmatrix} \text{ MPa}$$

We know now that yielding will initiate at the biaxial state $\sigma_1 = 142.46$ MPa and $\sigma_3 = -22.46$ MPa. Using reverse loading we find that the material will also yield at

$$[\sigma_{ij}] = \begin{bmatrix} -142.46 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 22.46 \end{bmatrix} \text{ MPa}$$

From isotropy we have yielding also at

$$[\sigma_{ij}] = \begin{bmatrix} -22.46 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 142.46 \end{bmatrix} \text{ MPa}$$

Applying reverse loading again gives yet another state

$$[\sigma_{ij}] = \begin{bmatrix} 22.46 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -142.46 \end{bmatrix} \text{ MPa}$$

Independence of hydrostatic pressure gives yet other yield states such as for instance found by subtracting 22.46 of the diagonal

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -22.46 & 0 \\ 0 & 0 & -164.92 \end{bmatrix} \text{ MPa}$$

Instead of subtracting 22.46 we could have added 142.46 instead, i.e.

$$[\sigma_{ij}] = \begin{bmatrix} 164.92 & 0 & 0 \\ 0 & 142.46 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

For both these states we should consider reversed loading and isotropy, in the same way as for the original stress state.
We will collect a series of coordinates that we know lie on the yield surface (The convention of $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is dropped!). We start by using the known yield state together with reverse loading and isotropy (like we did above). Note that σ_1 and σ_2 do not in general lie in some specific direction of space. Rather we are collecting states of plane stress that will give yielding, no mater how that plane may or may not be oriented. Whenever the stress tensor is altered, the eigenvalues will change, and so will the principal directions. We start by collecting the data above for reversed loading and isotropy, i.e.

σ_1	σ_2
142.46	-22.46
-142.46	22.46
-22.46	142.46
22.46	-142.46

Next we consider the case when a hydrostatic stress of 22.46 MPa is added and consider reversed loading and isotropy in the same way as above

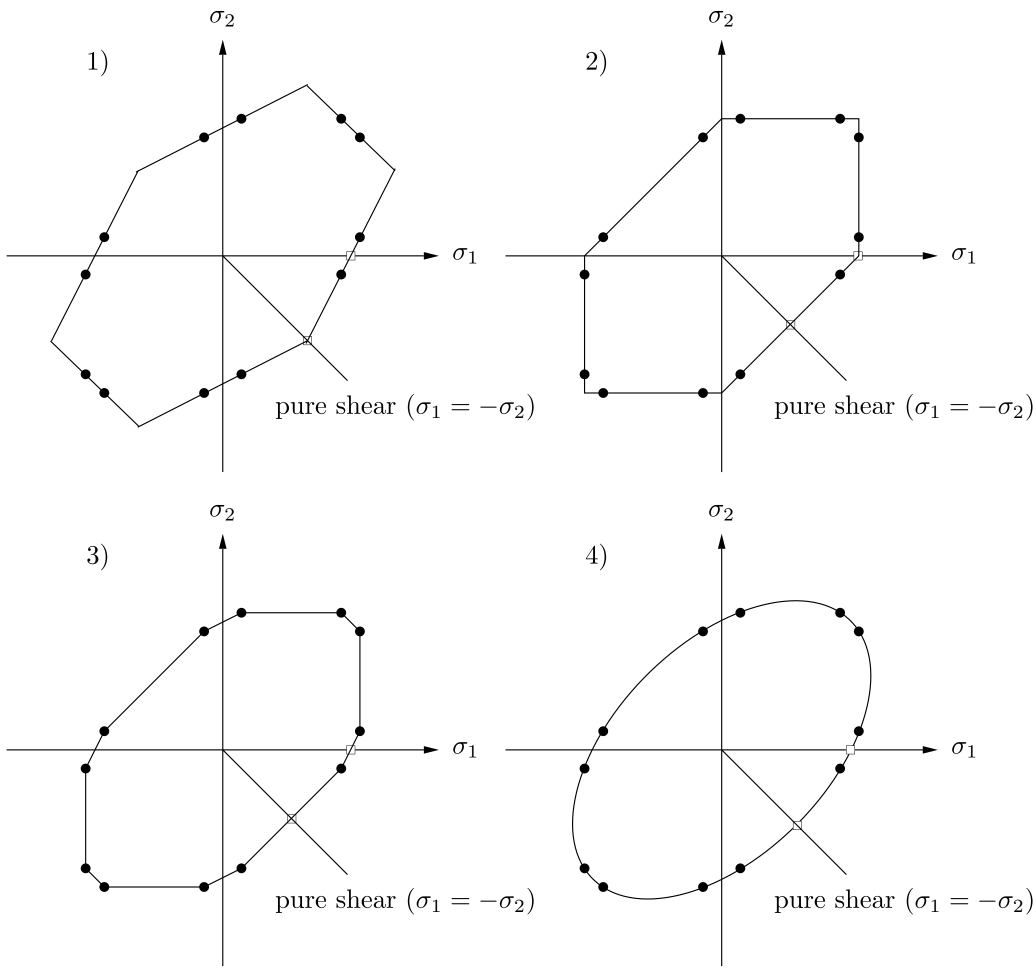
σ_1	σ_2
-22.46	-164.92
22.46	164.92
-164.92	-22.46
164.92	22.46

And finally the state when the hydrostatic 142.49 MPa is added, from reversed loading and isotropy we find

σ_1	σ_2
164.92	142.46
-164.92	-142.46
142.46	164.92
-142.46	-164.92

These are all stress state where yielding is initiated.

b) Let us consider different possibilities to connect all data points based on convexity



Evidently 2) is the Tresca Yield criterion and 4) is the von Mises yield criterion. The pure shear state is found when $\sigma_1 = -\sigma_2$ (a pure shear tensor gives this in

principal system). The values sought are found graphically as approximately, and we see that the then yield limit varies depending upon the selected convex surface. The smallest value is found from 2) and 3) providing

$$\sigma_1 = -\sigma_2 = 83 \text{ MPa}$$

and the largest from 1) providing

$$\sigma_1 = -\sigma_2 = 102 \text{ MPa}$$

For axial tension the largest value is found in 2) providing

$$\sigma_1 = 165 \text{ MPa}$$

and the smallest is found in 1) and 3) as

$$\sigma_1 = 154 \text{ MPa}$$

c) For Tresca found in figure 2) is in this case simple to calculate, here we find directly from the definition that

$$\sigma_{y0} = \sigma_1 - \sigma_3 = 22.46 + 142.46 \approx 165 \text{ MPa}$$

The yield stress in von Mises model found in figure 4) is given as

$$\sqrt{3J_2} - \sigma_{y0} = 0 \quad \Rightarrow \quad \sigma_{y0} = \sqrt{3J_2}$$

we know that in the principal system

$$J_2 = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2)$$

From the previous calculated principal stresses at yield $\sigma_1, \sigma_2, \sigma_3$ we find

$$s_1 = \sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = 102.5 \text{ MPa}$$

$$s_2 = \sigma_2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = -62.5 \text{ MPa}$$

$$s_3 = \sigma_3 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = -40 \text{ MPa}$$

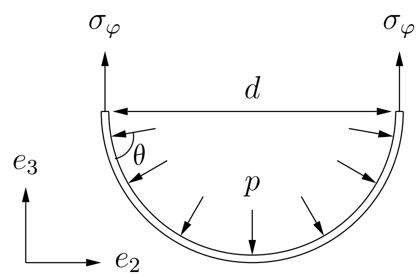
this gives

$$J_2 = \frac{1}{2}(102.5^2 + 62.5^2 + 40^2) = 8006.25 \text{ MPa}$$

and so

$$\sigma_{y0} = \sqrt{3 \times 8006.25} \approx 155 \text{ MPa}$$

Answer 8.8. Let us assume that the cylinder at hand can be considered to be thin-walled in the sense that the radial stress is approximately zero. A free body diagram of a cut through the cylinder gives:



At an incremental area element dA, with the pressure p, we have the downwards force

$$dF_\downarrow(\theta) = p \times dA \times \sin(\theta)$$

Such an area element is constantly equal to

$$dA = \frac{d}{2} \times d\theta \times dl$$

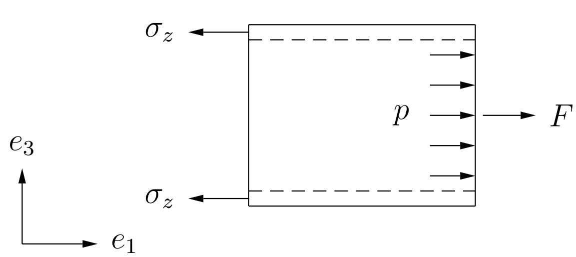
where dl is an increment in length of the tube and $d\theta$ is an increment in angle between the wall and pressure normal (here we used $(d-t)/2 \approx d/2$). To find the total downwards force, $F_\downarrow$ we must integrate

$$F_\downarrow = \int_0^L \int_0^\pi dF_\downarrow(\theta) = \int_0^L \int_0^\pi p\frac{d}{2}\sin(\theta) dl d\theta = pdL$$

The balance of force now reads

$$\sigma_\varphi 2tL - F_\downarrow = 0 \;\;\Rightarrow\;\; \sigma_\varphi = \frac{pd}{2t}$$

Let us now draw a free body diagram in the length direction



The stress in the length direction of the cylinder σ_z can be connected to the applied force F, and pressure as

$$-\sigma_z d\pi t + \frac{pd^2\pi}{4} + F = 0$$

where it was used that the tube is thin walled again. This now gives

$$\sigma_z = \frac{pd}{4t} + \frac{F}{\pi dt}$$

In conclusion we have the stress tensor

$$\begin{bmatrix} \sigma_z & 0 & 0 \\ 0 & \sigma_\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and now inquire what the value of F must be for the tube to yield. Von Mises criterion is

$$\sqrt{3J_2} - \sigma_{y0} = 0$$

or alternatively

$$\sigma_{y0}^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

We insert our principal stresses and find

$$\sigma_{y0}^2 = \left(\frac{pd}{4t} + \frac{F}{\pi dt}\right)^2 + \left(\frac{pd}{2t}\right)^2 - \left(\frac{pd}{4t} + \frac{F}{\pi dt}\right)\left(\frac{pd}{2t}\right)$$

inserting all known values and solving for $F > 0$ gives the solution

$$F \approx 395.3 \text{ kN}$$

The Tresca criterion instead reads

$$\sigma_{max} - \sigma_{min} = \sigma_{y0}$$

We know that if $F > 0$ we also have that $\sigma_z > 0$ at yielding. We also know that $\sigma_\varphi > 0$ due to the positive pressure in the tube, and so, we get two scenarios for $F > 0$, either $\sigma_\varphi = \sigma_{max}$, $\sigma_r = \sigma_{min} = 0$, or $\sigma_z = \sigma_{max}$, $\sigma_r = \sigma_{min} = 0$. The first corresponds to the pressure breaking the tube independent of force:

$$\sigma_\varphi - 0 = \sigma_{y0} \Rightarrow \sigma_{y0} = \frac{pd}{2t}$$

and since we know the tube can sustain the given pressure, at $F = 0$, clearly the second must be true:

$$\sigma_z - 0 = \sigma_{y0} \Rightarrow \sigma_{y0} = (\frac{pd}{4t} + \frac{F}{\pi dt}) \Rightarrow F \approx 334.9 \text{ kN}$$

(If F instead is allowed to be negative we find $\sigma_z = \sigma_{min} < 0$, but still $\sigma_\varphi > 0$ since $p > 0$ and thus

$$\sigma_\varphi - \sigma_z = \sigma_{y0} \Rightarrow \sigma_{y0} = \frac{pd}{2t} - \frac{pd}{4t} - \frac{F}{\pi dt} \Rightarrow F \approx -824.4 \text{ kN}$$

i.e it is a lot harder to collapse the tube inwards than to pop it open)

Answer 8.9. The Coulomb criterion reads in alternate form

$$k\sigma_{max} - \sigma_{min} = \sigma_c$$

where σ_{max} and σ_{min} are principal stresses. For positive tension and plane stress we have the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus the principal stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

where always one of the three principal stresses must be zero as there is no out of plane stress. For this purpose assume $\sigma_3 = 0$.
Let us consider the possibilities found by use permutation of the principal stresses

$$k\sigma_1 = \sigma_c, \quad 0 \leq \sigma_2 \leq \sigma_1$$

$$-\sigma_2 = \sigma_c, \quad \sigma_2 \leq \sigma_1 \leq 0$$

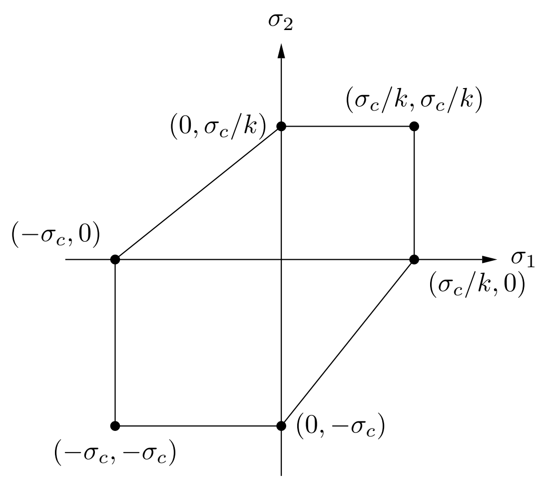
$$k\sigma_2 - \sigma_1 = \sigma_c, \quad \sigma_1 \leq 0 \leq \sigma_2$$

and

$$k\sigma_2 = \sigma_c, \quad 0 \leq \sigma_1 \leq \sigma_2$$

$$-\sigma_1 = \sigma_c, \quad \sigma_1 \leq \sigma_2 \leq 0$$

$$k\sigma_1 - \sigma_2 = \sigma_c, \quad \sigma_2 \leq 0 \leq \sigma_1$$



and so we find the tensile strength, $\sigma_1 = \sigma_t$:

$$\sigma_t = \frac{\sigma_c}{k}$$

Answer 8.10. Simple torsion is a state of pure shear

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives the principal stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}$$

following the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and assuming $\tau > 0$. Von Mises criterion reads

$$\sqrt{3J_2} - \sigma_{y0} = 0$$

inserting the definition of $J_2 = 0.5 s_{ij} s_{ji}$ we can arrive at

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} = \sigma_{y0}$$

Inserting the principal stresses we get

$$\frac{1}{\sqrt{2}}\sqrt{(\tau - 0)^2 + (0 + \tau)^2 + (\tau + \tau)^2} = \sigma_{y0} \;\;\Rightarrow\;\; \sigma_{y0} = \tau\sqrt{3}$$

With $\tau = 125$ MPa at yielding we find the yield stress $\sigma_{y0} \approx 216.5$ MPa.

Answer 8.11. From the figures and equations of problem 8.8 we immediately conclude that

$$\sigma_\varphi = \frac{pd}{2t}, \quad \sigma_z = \frac{pd}{4t} + \frac{F}{\pi dt}$$

Let us assume that $F > 0$ such that also $\sigma_z > 0$. Furthermore, let us assume that the pressure is positive, $p > 0$. The Tresca criterion reads

$$\sigma_{max} - \sigma_{min} = \sigma_{y0}$$

We find the force F_0 that will cause yielding from the zero pressure scenario $p = 0$, which gives $\sigma_z = \sigma_{max}$ and $\sigma_{min} = 0$

$$\frac{F_0}{\pi dt} = \sigma_{y0} \;\;\Rightarrow\;\; F_0 = \pi dt \sigma_{y0}$$

In the case of $\sigma_z = \sigma_{max}$ we have $\sigma_r = \sigma_{min} = 0$ since $\sigma_\varphi > 0$. We find

$$\frac{pd}{4t} + \frac{F}{\pi dt} = \sigma_{y0} \;\;\Rightarrow\;\; \frac{pd}{4t\sigma_{y0}} + \frac{F}{F_0} = 1 \;\;\Rightarrow\;\; \frac{pd}{2t\sigma_{y0}} = 2(1 - \frac{F}{F_0})$$

If instead $\sigma_\varphi = \sigma_{max}$ and $\sigma_r = \sigma_{min} = 0$ we get

$$\frac{pd}{2t} = \sigma_{y0} \;\;\Rightarrow\;\; \frac{pd}{2t\sigma_{y0}} = 1 \;\;\Rightarrow\;\; \frac{pr}{t\sigma_{y0}} = 1$$

Comparing σ_z with σ_φ we find that $\sigma_z \geq \sigma_\varphi$ is true if

$$\frac{pd}{4t} + \frac{F}{\pi dt} \geq \frac{pd}{2t} \;\;\Rightarrow\;\; F \geq \frac{\pi p d^2}{4}$$

and so

$$\frac{F}{F_0} \geq \frac{\pi p d^2}{4\pi dt\sigma_{y0}} = \frac{pd}{4t\sigma_{y0}} = \frac{1}{2}2(1 - \frac{F}{F_0}) \;\;\Rightarrow\;\; \frac{F}{F_0} \geq \frac{1}{2}$$

i.e

$$\frac{pr}{t\sigma_{y0}} = 2(1 - \frac{F}{F_0}) \;\;\Rightarrow\;\; \frac{F}{F_0} \geq \frac{1}{2}$$

Likewise we find for the opposite case that $F/F_0 \leq 1/2$. This concludes the Tresca case.

Von Mises criterion reads

$$\sqrt{3J_2} = \sigma_{y0}$$

or expressed in principal stress

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} = \sigma_{y0}$$

Inserting $\sigma_2 = \sigma_\varphi$ and $\sigma_3 = \sigma_z$ we find

$$\frac{1}{\sqrt{2}}\sqrt{(0-\sigma_\varphi)^2 + (\sigma_\varphi - \sigma_z)^2 + (0-\sigma_z)^2} = \sigma_{y0}$$

This gives

$$\sigma_\varphi^2 + \sigma_z^2 - \sigma_\varphi\sigma_z = \sigma_{y0}^2$$

and so, inserting the known format of the stresses gives

$$\sigma_{y0}^2 = \Big(\frac{pd}{2t}\Big)^2 + \Big(\frac{pd}{4t} + \frac{F}{\pi dt}\Big)^2 - \Big(\frac{pd}{2t}\Big)\Big(\frac{pd}{4t} + \frac{F}{\pi dt}\Big)$$

This is simplified to

$$\sigma_{y0}^2 = \frac{3}{4}\Big(\frac{pd}{2t}\Big)^2 + \Big(\frac{F}{d\pi t}\Big)^2$$

We know that for the zero pressure state, applying the force F_0 the stress is zero except for in the z-direction

$$\sigma_z = \frac{F_0}{d\pi t}, \quad \sigma_\varphi = 0, \quad \sigma_r = 0$$

Thus the yield stress is found from this state as

$$\sqrt{2J_2} = \sigma_{y0} \;\; \Rightarrow \;\; \sigma_{y0}^2 = \sigma_z^2$$

Rearranging previous expression and combining gives

$$1 = \frac{1}{\sigma_{y0}^2}\frac{3}{4}\Big(\frac{pd}{2t}\Big)^2 + \frac{1}{\sigma_{y0}^2}\Big(\frac{F}{d\pi t}\Big)^2 = \frac{1}{\sigma_{y0}^2}\frac{3}{4}\Big(\frac{pd}{2t}\Big)^2 + \Big(\frac{d\pi t}{F_0}\Big)^2\Big(\frac{F}{d\pi t}\Big)^2$$

This then simplifies to

$$\frac{pr}{t\sigma_{y0}} = \frac{2}{\sqrt{3}}\sqrt{2 - \Big(\frac{F}{F_0}\Big)^2}$$

which was to be proven.

Answer 8.12. We follow the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$. For pure tension we have

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\; \Rightarrow \;\; \sigma_1 = \sigma_t, \quad \sigma_2 = 0, \quad \sigma_3 = 0$$

for pure compression

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma_c \end{bmatrix} \Rightarrow \sigma_1 = 0, \quad \sigma_2 = 0, \quad \sigma_3 = -\sigma_c$$

for pure shear

$$[\sigma_{ij}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow [\sigma_{ij}] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix} \Rightarrow$$

$$\sigma_1 = \tau, \quad \sigma_2 = 0, \quad \sigma_3 = -\tau$$

Let us start with the Coulomb criterion

$$k\sigma_1 - \sigma_3 - \sigma_c = 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

From tension we find

$$k = \frac{\sigma_c}{\sigma_t}$$

Pure shear then gives

$$k\tau + \tau = \sigma_c \Rightarrow \tau_{y0} = \frac{\sigma_c}{k+1} = \frac{\sigma_c \sigma_t}{\sigma_c + \sigma_t}$$

We proceed with the Drucker-Prager criterion

$$\sqrt{3J_2} + \alpha I_1 - \beta = 0$$

we have that the invariant is

$$J_2 = \frac{1}{2} s_{ij} s_{ji} = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2)$$

This gives for pure tension

$$J_2 = \frac{\sigma_t^2}{3}$$

and for pure compression

$$J_2 = \frac{\sigma_c^2}{3}$$

and for pure shear

$$J_2 = \tau^2$$

Similarly

$$I_1 = \sigma_t \quad \text{for pure tension}$$

$$I_1 = -\sigma_c \quad \text{for pure compression}$$

$$I_1 = 0 \quad \text{for pure shear}$$

Pure tension gives the equation

$$\sigma_t + \alpha\sigma_t - \beta = 0$$

and compression gives

$$\sigma_c - \alpha\sigma_c - \beta = 0$$

Thus we find

$$\alpha = \frac{\sigma_c - \sigma_t}{\sigma_c + \sigma_t}$$

$$\beta = \sigma_t + \frac{\sigma_t(\sigma_c - \sigma_t)}{\sigma_c + \sigma_t}$$

For pure shear we have

$$\tau\sqrt{3} - \beta = 0 \quad \Rightarrow \quad \tau_{y0} = \frac{\beta}{\sqrt{3}}$$

with β given above.
Let us now consider von Mises criterion

$$\sqrt{3J_2} = \sigma_{y0}$$

Pure tension gives

$$\sigma_{y0} = \sigma_t$$

Pure shear then gives

$$\tau\sqrt{3} = \sigma_{y0} \quad \Rightarrow \quad \tau_{y0} = \frac{\sigma_t}{\sqrt{3}}$$

Finally we consider the Tresca criterion

$$\sigma_1 - \sigma_3 = \sigma_{y0}$$

This is nothing else than the Coulomb criterion for $k = 1$, thus it follows directly

$$\tau_{y0} = \frac{\sigma_c}{2}$$

Answer 8.13. Let us start by computing the principal stress tensor. Given the state

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we find the principal stresses

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2}$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} + \tau^2}$$

$$\sigma_3 = 0$$

The Tresca criterion is $\sigma_1 - \sigma_3 = \sigma_{y0}$, and gives (assuming $\sigma_1 > 0$ and $\sigma_2 < 0$)

$$\frac{\sigma}{2} + \sqrt{\frac{\sigma^2}{4} + \tau^2} - \Big(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} + \tau^2}\Big) = \sigma_{y0}$$

and so

$$4(\frac{\sigma^2}{4} + \tau^2) = \sigma_{y0}^2 \quad \Rightarrow \quad \Big(\frac{\sigma}{\sigma_{y0}}\Big)^2 + \Big(\frac{\tau}{\sigma_{y0}/2}\Big)^2 = 1$$

and so we must have that $\sigma_{y0} = 2\tau_{y0}$ for Tresca. This can be verified with a pure shear state as

$$[\sigma_{ij}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives in principal system

$$[\sigma_{ij}] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}$$

Tresca then reads

$$\tau - (-\tau) = \sigma_{y0} \quad \Rightarrow \quad \sigma_{y0} = 2\tau_{y0}$$

Von Mises criterion is instead

$$\sqrt{3J_2} = \sigma_{y0}, \quad J_2 = \frac{1}{2}s_{ij}s_{ji}$$

we have that

$$[s_{ij}] = \begin{bmatrix} \frac{2}{3}\sigma & \tau & 0 \\ \tau & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$$

and so we get

$$J_2 = \frac{1}{2}(2\tau^2 + \frac{6}{9}\sigma^2) = \tau^2 + \frac{1}{3}\sigma^2$$

and so von Mises gives

$$3\tau^2 + \sigma^2 = \sigma_{y0}^2 \quad \Rightarrow \quad (\frac{\sigma}{\sigma_{y0}})^2 + \Big(\frac{\tau}{(\sigma_{y0}/\sqrt{3})}\Big)^2 = 1$$

and therefore we must have that $\tau_{y0} = (\sigma_{y0}/\sqrt{3})$ for von Mises. This can again be verified with a pure shear state, which gives in principal system

$$[\sigma_{ij}] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}$$

in this case

$$J_2 = \tau^2$$

and so von Mises reads

$$3\tau^2 = \sigma_{y0}^2 \quad \Rightarrow \quad \tau_{y0} = \sigma_{y0}/\sqrt{3}$$

Answer 8.14. a) In previous problem (8.13) we considered the exact same stress state. We concluded then that

$$J_2 = \tau^2 + \frac{1}{3}\sigma^2$$

naturally also

$$I_1 = \sigma_{kk} = \sigma$$

If then $\sigma = \tau$ the Drucker-Prager criterion reads

$$\sqrt{4\sigma^2} + \alpha\sigma - \beta = 0$$

If $\sigma > 0$ we solve for sigma and find

$$\sigma = \frac{\beta}{(2 + \alpha)}$$

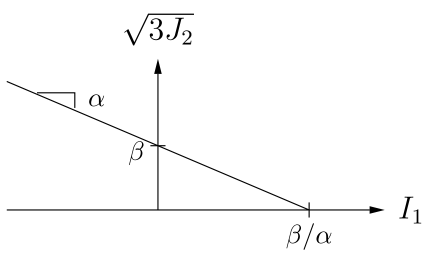
If instead $\sigma < 0$ we have $\sqrt{\sigma^2} = -\sigma$ which gives

$$\sigma = \frac{\beta}{(\alpha - 2)}$$

b) We consider $\sqrt{3J_2}$ as a function of I_1 and find the linear relation

$$\sqrt{3J_2} = -\alpha I_1 + \beta$$

This looks something like:



If then $\sigma = \tau$ we have the expression for I_1 and J_2 given above. We have

$$\sqrt{3J_2} = -\alpha\sigma + \beta$$

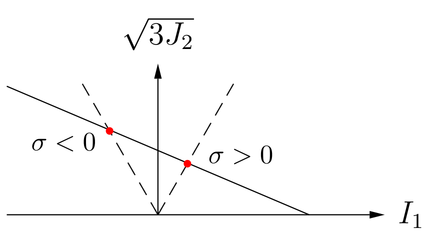
Given that we have also calculated the values of β and α expressed in σ we could rewrite for $\sigma = \tau$ and $\sigma > 0$:

$$\sqrt{3J_2} = 2\sigma$$

and for $\sigma < 0$:

$$\sqrt{3J_2} = -2\sigma$$

We draw the load paths and mark the yields points with red dots:



c) For pure shear we have the principal stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}$$

Since $\sigma_{kk} = 0$ we have that

$$s_{ij} = \sigma_{ij}$$

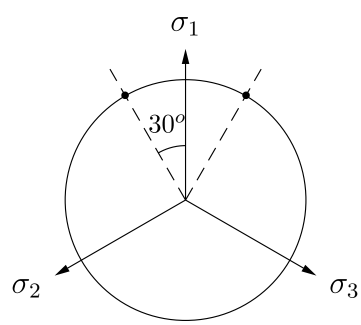
This gives J_3:

$$J_3 = \frac{1}{3}(s_1^3 + s_2^3 + s_3^2) = \frac{1}{3}(\tau^3 + (-\tau)^3) = 0$$

The angle θ can then be calculated from

$$\cos(3\theta) = 0 \ \Rightarrow \ \theta = \pm 30^o \pm 120^o$$

We draw the yield surface and pure shear load paths in the deviatoric plane, setting J_2 constant:



(Note that the surface is a 3D cone in principal stress space, and only reduces to a constant radius circle for a cut in the plane where I_1 =constant)

Answer 8.15. To find the matrix format define $s_{ij}v_{ij}$ where $v_{ij} = P_{ijkl}s_{kl}$ which are easy to evaluate. We find

$$s_{ij}v_{ij} = [s_{11}\ s_{22}\ s_{33}\ s_{12}\ s_{13}\ s_{23}] \begin{bmatrix} v_{11} \\ v_{22} \\ v_{33} \\ 2v_{12} \\ 2v_{13} \\ 2v_{23} \end{bmatrix} = \boldsymbol{s}^T \boldsymbol{v}$$

$$v_{ij} = [P_{ij11}\ P_{ij22}\ P_{ij33}\ 2P_{ij12}\ 2P_{ij13}\ 2P_{ij23}\ ]\,\boldsymbol{s}$$

Based on the result above we find that the matrix $\boldsymbol{P}$ is of the format

$$\boldsymbol{P} = \begin{bmatrix} P_{1111} & P_{1122} & P_{1133} & 2P_{1112} & 2P_{1113} & 2P_{1123} \\ P_{2211} & P_{2222} & P_{2233} & 2P_{2212} & 2P_{2213} & 2P_{2223} \\ P_{3311} & P_{3322} & P_{3333} & 2P_{3312} & 2P_{3313} & 2P_{3323} \\ 2P_{1211} & 2P_{1222} & 2P_{1233} & 4P_{1212} & 4P_{1213} & 4P_{1223} \\ 2P_{1311} & 2P_{1322} & 2P_{1333} & 4P_{1312} & 4P_{1313} & 4P_{1323} \\ 2P_{2311} & 2P_{2322} & 2P_{2333} & 4P_{2312} & 4P_{2313} & 4P_{2323} \end{bmatrix}$$

it can be found via direct insertion into the given expression for P_{ijkl}. Before we start, we note that

$$P_{ijkl} = P_{klij}$$

and that

$$P_{ijkl} = P_{jikl}$$

Next we insert and find

$$P_{1111} = P_{2222} = P_{3333} = \frac{3}{2\sigma_{y0}^2}$$

$$P_{1212} = P_{1313} = P_{2323} = \frac{3}{2\sigma_{y0}^2}$$

The of diagonal terms are all zero, this is realised as only if $i = k$ and $j = l$ or $i = l$ and $j = k$ is $P_{ijkl} \neq 0$. We obtain

$$\boldsymbol{P} = \frac{3}{4\sigma_{y0}^2} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

We have now the option to include any terms of the form $ks_{ii} = 0$ in the product $\boldsymbol{Ps}$, where k is a scalar. We can select

$$\hat{\boldsymbol{P}} = \frac{3}{4\sigma_{y0}^2} \begin{bmatrix} 2+k & k & k & 0 & 0 & 0 \\ k & 2+k & k & 0 & 0 & 0 \\ k & k & 2+k & & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

since this will result in

$$\hat{\boldsymbol{P}}\boldsymbol{s} = \frac{3}{4\sigma_{y0}^2}\begin{bmatrix}2s_{11}+ks_{ii} & 2s_{22}+ks_{ii} & 2s_{33}+ks_{ii} & 4s_{12} & 4s_{13} & 4s_{23}\end{bmatrix}^T$$

$$= \frac{3}{4\sigma_{y0}^2}\begin{bmatrix}2s_{11} & 2s_{22} & 2s_{33} & 4s_{12} & 4s_{13} & 4s_{23}\end{bmatrix}^T = \boldsymbol{P}\boldsymbol{s}$$

and so the quantity $\boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s}$ remains unchanged for this alternate form.

Answer 8.16. We note first that

$$\boldsymbol{P}\boldsymbol{a} = 0$$

for all

$$\boldsymbol{a} = \begin{bmatrix}k & k & k & 0 & 0 & 0\end{bmatrix}^T$$

where k is a scalar. In matrix format we have from the definition of deviatoric stress that

$$\boldsymbol{\sigma} = \boldsymbol{s} + \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\boldsymbol{I}$$

where $\boldsymbol{I}$ is the "unit vector"

$$\boldsymbol{I} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

Abbreviating define the second term as $\boldsymbol{b}$:

$$\boldsymbol{b} = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\boldsymbol{I}$$

The multiplication term now becomes

$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} = (\boldsymbol{s}+\boldsymbol{b})^T\boldsymbol{P}(\boldsymbol{s}+\boldsymbol{b})$$

we multiply together the parts and find

$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} = \boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} + \boldsymbol{b}^T\boldsymbol{P}\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{P}\boldsymbol{b} + \boldsymbol{b}^T\boldsymbol{P}\boldsymbol{b}$$

Note now that $\boldsymbol{b}$ is on the format of $\boldsymbol{a}$, and thus, $\boldsymbol{P}\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{P} = 0$, we are left with

$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} = \boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s}$$

And thus

$$\boldsymbol{\sigma}^T\boldsymbol{P}\boldsymbol{\sigma} - 1 = 0$$

can equally be written as

$$\boldsymbol{s}^T\boldsymbol{P}\boldsymbol{s} - 1 = 0$$

Answer 9.1. For uniaxial tension we have the stress tensor ($\sigma < 0$)

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives the devatoric stress tensor

$$s_{ij} = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$$

and so we find the effective stress

$$\sigma_{eff} = \sqrt{\frac{3}{2}(\frac{4}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2)} = \sigma$$

For uniaxial tension the plastic strain develop as

$$[\dot{\epsilon}^p_{ij}] = \begin{bmatrix} \dot{\epsilon}^p_{11} & 0 & 0 \\ 0 & \dot{\epsilon}^p_{22} & 0 \\ 0 & 0 & \dot{\epsilon}^p_{33} \end{bmatrix}$$

and so the effective plastic strain becomes

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\big((\dot{\epsilon}^p_{11})^2 + (\dot{\epsilon}^p_{22})^2 + (\dot{\epsilon}^p_{33})^2\big)}$$

Plastic incompressibility is defined as

$$\dot{\epsilon}^p_{ii} = 0$$

Thus we find that

$$\dot{\epsilon}^p_{33} = -\dot{\epsilon}^p_{11} - \dot{\epsilon}^p_{22}$$

and so

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\big((\dot{\epsilon}^p_{11})^2 + (\dot{\epsilon}^p_{22})^2 + (\dot{\epsilon}^p_{11} + \dot{\epsilon}^p_{22})^2\big)}$$

This gives

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\big(2(\dot{\epsilon}^p_{11})^2 + 2(\dot{\epsilon}^p_{22})^2 + 2\dot{\epsilon}^p_{11}\dot{\epsilon}^p_{22}\big)}$$

We must now use isotropy to get further

$$\dot{\epsilon}^p_{11} = \dot{\epsilon}^p, \quad \dot{\epsilon}^p_{22} = -q\dot{\epsilon}^p, \quad \dot{\epsilon}^p_{33} = -q\dot{\epsilon}^p$$

For plastic incompressibility it is then required that $q = 0.5$. This gives

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\big(2(\dot{\epsilon}^p)^2 + 2\nu^2(\dot{\epsilon}^p)^2 - 2\nu(\dot{\epsilon}^p)^2\big)} = \dot{\epsilon}^p\sqrt{\frac{2}{3}(2 + \frac{1}{2} - 1)} = \dot{\epsilon}^p$$

assuming that $\dot{\epsilon}^p > 0$

Answer 9.2. Associated plasticity is the assumption that

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

and so we seek the partial derivative of f with respect to σ_{ij}

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{3s_{kl}}{2\big((\frac{3}{2}s_{pq}s_{pq})^{1/2}\big)}(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}) = \frac{3s_{ij}}{2\sqrt{3J_2}} = \frac{3s_{ij}}{2\sigma_y}$$

where it was used that the derivative of $3J_2$ with respect to s_{kl} is $3s_{kl}$. It was also used that $s_{ii} = 0$.
For kinematic hardening we have instead

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial (s_{kl} - \alpha^d_{kl})}\frac{\partial (s_{kl} - \alpha^d_{kl})}{\partial \sigma_{ij}}$$

Let us compute the derivatives separately this time, we find

$$\frac{\partial f}{\partial (s_{kl} - \alpha^d_{kl})} = \frac{3(s_{kl} - \alpha^d_{kl})}{2[\frac{3}{2}(s_{ij} - \alpha^d_{ij})(s_{ij} - \alpha^d_{ij})]^{1/2}}$$

and

$$\frac{\partial (s_{kl} - \alpha^d_{kl})}{\partial \sigma_{ij}} = (\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl})$$

together we have

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3(s_{kl} - \alpha^d_{kl})}{2[\frac{3}{2}(s_{ij} - \alpha^d_{ij})(s_{ij} - \alpha^d_{ij})]^{1/2}}(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}) = \frac{3(s_{ij} - \alpha^d_{ij})}{2\sigma_{y0}}$$

using that $s_{ii} = 0$ and assuming that $\alpha_{ii} = 0$. In conclusion for associated plasticity

$$\dot{\epsilon}_{ij} = \dot{\lambda}\frac{\partial f}{\sigma_{ij}}$$

we find

$$\dot{\epsilon}_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_y} \quad \text{isotropic hardening}$$

$$\dot{\epsilon}_{ij} = \dot{\lambda}\frac{3(s_{ij} - \alpha^d_{ij})}{2\sigma_{y0}} \quad \text{kinematic hardening}$$

Answer 9.3. With the principal stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} 2\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we get the deviatoric stress as

$$[s_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma \end{bmatrix}$$

We calculate J_2 and I_1 as

$$J_2 = \sigma^2$$

$$I_1 = 3\sigma$$

a) For a von Mises material we have that

$$f = \sqrt{3J_2} - \sigma_{y0} = \sqrt{3\sigma^2} - \sigma_{y0} = 0$$

thus

$$\sigma = \frac{\sigma_{y0}}{\sqrt{3}}$$

will give yielding, note $\sigma > 0$ is assumed. The derivative with respect to stress is

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sqrt{3J_2}}$$

as calculated in previous problem. The flow rule becomes for associated plasticity

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}} = \dot{\lambda}\frac{\sqrt{3}}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The effective plastic strain becomes

$$\dot{\epsilon}^p_{eff} = \sqrt{\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}} = \sqrt{\frac{2}{3}\dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}}\dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}}} = \dot{\lambda}\sqrt{\frac{2}{3}\frac{9}{6}} = \dot{\lambda}$$

where it was used that $s_{ij}s_{ij} = 2J_2$.
b) Assuming σ is positive the Tresca criterion gives

$$f = \sigma_1 - \sigma_3 = 2\sigma - 0 = \sigma_{y0} \Rightarrow \sigma = \frac{\sigma_{y0}}{2}$$

We differentiate and find

$$[\frac{\partial f}{\partial \sigma_{ij}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \dot{\epsilon}^p_{ij} = \dot{\lambda}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The effective plastic strain the becomes

$$\dot{\epsilon}^p_{eff} = \Big(\frac{2}{3}\dot{\lambda}^2 \operatorname{tr}(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix})\Big)^{1/2} = \frac{2\dot{\lambda}}{\sqrt{3}}$$

c) Drucker-Prager gives

$$f = \sqrt{3J_2} + \alpha I_1 - \beta = \sigma\sqrt{3} + \alpha 3\sigma - \beta = 0 \Rightarrow \sigma = \frac{\beta}{\sqrt{3} + 3\alpha}$$

Using that we know the derivative of J_2 since previously the derivative becomes

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sqrt{3J_2}} + \alpha\delta_{ij}$$

The plastic strain rate is found as

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}(\frac{\sqrt{3}}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \alpha\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix})$$

and the plastic effective strain becomes

$$\dot{\epsilon}^p_{eff} = \dot{\lambda}\sqrt{\frac{2}{3}(\frac{3s_{ij}}{2\sqrt{3J_2}} + \alpha\delta_{ij})(\frac{3s_{ij}}{2\sqrt{3J_2}} + \alpha\delta_{ij})}$$

$$= \dot{\lambda}\sqrt{\frac{2}{3}(\frac{3s_{ij}}{2\sqrt{3J_2}}\frac{3s_{ij}}{2\sqrt{3J_2}} + 3\alpha^2)} = \dot{\lambda}\sqrt{1 + 2\alpha^2}$$

where, again, it was used that $s_{ij}s_{ij} = 2J_2$.
d) Finally for the Coulomb criterion we can use the result in (b), finding the derivative

$$[\frac{\partial f}{\partial \sigma_{ij}}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \dot{\epsilon}^p_{ij} = \dot{\lambda} \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and thus the effective plastic strain

$$\dot{\epsilon}^p_{eff} = \Big(\frac{2}{3} \dot{\lambda}^2 \operatorname{tr}(\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}) \Big)^{1/2} = \dot{\lambda} \sqrt{\frac{2}{3}(k^2 + 1)}$$

Answer 9.4. With the principal stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we get the deviatoric stress as

$$[s_{ij}] = \begin{bmatrix} \frac{1}{3}\sigma & 0 & 0 \\ 0 & \frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{2}{3}\sigma \end{bmatrix}$$

We calculate J_2 and I_1 as

$$J_2 = \frac{1}{3}\sigma^2$$

$$I_1 = 2\sigma$$

a) For a von Mises material we have that

$$f = \sqrt{3J_2} - \sigma_{y0} = \sqrt{\sigma^2} - \sigma_{y0} = 0 \Rightarrow \sigma = \sigma_{y0}$$

The flow rule and the effective plastic strain is the same as in 9.3 since no use was made of the format of the stress tensor to arrive at these conclusions

$$\dot{\epsilon}^p_{eff} = \dot{\lambda}, \qquad \dot{\epsilon}^p_{ij} = \dot{\lambda} \frac{3s_{ij}}{2\sqrt{3J_2}}$$

For Drucker-Prager we have instead that

$$f = \sqrt{3J_2} + \alpha I_1 - \beta = \sigma + \alpha 2\sigma - \beta = 0 \Rightarrow \sigma = \frac{\beta}{1 + 2\alpha}$$

The flow rule and effective plastic strain remains unchanged

$$\dot{\epsilon}^p_{eff} = \dot{\lambda}\sqrt{1+2\alpha}, \qquad \dot{\epsilon}^p_{ij} = \dot{\lambda}(\frac{3s_{ij}}{2\sqrt{3J_2}} + \alpha\delta_{ij})$$

For the criteria of Tresca and Coulomb, we obtain the situation shown in Fig.9.30. This is dealt with by Koiters flow rule, not treated in the course.

Answer 9.5. a) We know already the derivative

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sqrt{3J_2}}$$

and thus

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}}$$

b) We recall

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij})^{1/2} = (\frac{2}{3}\dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}}\dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}})^{1/2} = \dot{\lambda}$$

simply using that $J_2 = \frac{1}{2}s_{ij}s_{ij}$.
c) For the given stress state we get

$$[s_{ij}] = \begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & \frac{-1}{3}\sigma & 0 \\ 0 & 0 & \frac{-1}{3}\sigma \end{bmatrix}$$

and thus

$$J_2 = \frac{1}{3}\sigma^2$$

we find in the yield criteria that

$$f = \sqrt{3J_2} - \sigma_y(\kappa) = \sigma - \sigma_y(\kappa) = 0$$

Thus we must have

$$\sigma = \sigma_y(\kappa)$$

d) We insert in the explicit evolution law

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sqrt{3J_2}} = \dot{\lambda}\frac{3s_{ij}}{2\sigma}$$

in matrix format we get

$$[\dot{\epsilon}^p_{ij}] = \frac{3\dot{\lambda}}{2\sigma}\begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & \frac{-1}{3}\sigma & 0 \\ 0 & 0 & \frac{-1}{3}\sigma \end{bmatrix} = \frac{\dot{\lambda}}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus the effective plastic strain becomes

$$\dot{\epsilon}^p_{eff} = (\frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij})^{1/2} = (\frac{2}{3}\frac{\dot{\lambda}}{2}\frac{\dot{\lambda}}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} : \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix})^{1/2} = \dot{\lambda} = \dot{\epsilon}^p_{11}$$

Answer 9.6. Everywhere $\epsilon^p_{13} = \epsilon^p_{23} = 0$ and $\epsilon^p_{22} = \epsilon^p_{33} = -\epsilon^p_{11}/2$.
Irrespective of the load path, at point C

$$\begin{aligned} &\epsilon^e_{11} = \tfrac{\sigma_{y0}}{E}\,; \quad \epsilon^e_{12} = \tfrac{\sigma_{y0}}{2\sqrt{3}G} \\ &\epsilon^e_{22} = \epsilon^e_{33} = -\nu\epsilon^e_{11}\,; \quad \epsilon^e_{13} = \epsilon^e_{23} = 0 \end{aligned}$$

a) Load path AB is purely elastic; plasticity is intiated at point B.
Load path B $\rightarrow$ C

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}(1 - \frac{\pi}{4})\,; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{4}\ln 2$$

b) Load path AD is purely elastic; plasticity is initiated at point D.

Load path D $\rightarrow$ C

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}\frac{1}{2}\ln 2\,; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{2}(1 - \frac{\pi}{4})$$

c) Proportional loading

$$\sigma = k\sigma_{y0}\,; \qquad \tau = \frac{1}{\sqrt{3}}k\sigma_{y0}$$

where $0 \leq k \leq 1$. Yielding is initiated when $k = 1/\sqrt{2}$, i.e. plasticity occurs when $1/\sqrt{2} \leq k \leq 1$.

$$\epsilon^p_{11} = \frac{\sigma_{y0}}{H}(1 - \frac{1}{\sqrt{2}})\,; \quad \epsilon^p_{12} = \frac{\sigma_{y0}}{H}\frac{\sqrt{3}}{2}(1 - \frac{1}{\sqrt{2}})$$

d) The same curve $\sigma_{eff} = \sigma_{eff}(\epsilon^p_{eff})$ is obtained for all the load cases a), b) and c).

Answer 10.1. Following the pages 247-254 in the course book (beginning of Chapter 10) we reproduce the reasoning starting with the split of strain into elastic and plastic as

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij}$$

The time derivative becomes

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$$

Hookes law gives

$$\sigma_{ij} = D_{ijkl}\epsilon^e_{kl}$$

The time derivative becomes

$$\dot{\sigma}_{ij} = \dot{D}_{ijkl}\epsilon^e_{kl} + D_{ijkl}\dot{\epsilon}^e_{kl}$$

Assuming

$$\dot{D}_{ijkl} = 0$$

We can write

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^p_{kl})$$

For some potential g we use the flow rule

$$\dot{\epsilon}^p_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$$

and get

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - D_{ijkl}\dot{\lambda}\frac{\partial g}{\partial \sigma_{kl}} \tag{1}$$

Since we require for some yield function $f(K_\alpha, \kappa_\beta)$ that

$$\dot{f} = 0$$

we have by the chain rule, for some set of hardening variables, $K_\alpha = K_\alpha(\kappa_\beta)$, dependent on internal variables κ_β, that

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta = 0$$

This now gives us

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = -\frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta$$

multiplying through equation (1) by $\dfrac{\partial f}{\partial \sigma_{ij}}$ we find

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl} - \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\lambda}\frac{\partial g}{\partial \sigma_{kl}}$$

This then leads to

$$-\frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\kappa}_\beta = \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl} - \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\lambda}\frac{\partial g}{\partial \sigma_{kl}}$$

Let now $\dot{\kappa}_\beta$ be given by a a constitutive assumption which leads to a time independent formulation of plasticity:

$$\dot{\kappa}_\beta = \dot{\lambda}k_\beta(\sigma_{ij}, K_\alpha)$$

where $k_\beta(\sigma_{ij}, K_\alpha)$ is some function of stress and hardening parameters. We find then

$$-\frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}\dot{\lambda}k_\beta = \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl} - \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\lambda}\frac{\partial g}{\partial \sigma_{kl}}$$

solving for $\dot{\lambda}$ gives

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\epsilon}_{kl}$$

where

$$A = \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}} - \frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial \kappa_\beta}k_\beta$$

(it is possible to show $A > 0$). Inserting this in equation (1) gives

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - D_{ijpq}\frac{\partial g}{\partial \sigma_{pq}}\frac{1}{A}\frac{\partial f}{\partial \sigma_{rs}}D_{rskl}\dot{\epsilon}_{kl}$$

or alternatively

$$\dot{\sigma}_{ij} = \dot{D}^{ep}_{ijkl}\dot{\epsilon}_{kl}$$

where

$$\dot{D}^{ep}_{ijkl} = D_{ijkl} - D_{ijpq}\frac{\partial g}{\partial \sigma_{pq}}\frac{1}{A}\frac{\partial f}{\partial \sigma_{rs}}D_{rskl}$$

Answer 10.2. Since we have

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - H\dot{\lambda} = 0$$

from eq (10.16) (consistency relation + evolution laws). Since $\dot{\lambda} > 0$ during plastic loading we have

$$\text{sign}(H) = \text{sign}(\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij})$$

Clearly if $\dot{\sigma}_{ij} = 0$ also $H = 0$. Furthermore, if f is taken to be a convex function and we have that $f = 0$ during plastic loading, as well as that the gradient is the out pointing normal from the surface f, we have that for an expanding surface f (hardening) that

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0 \quad \Rightarrow \quad H > 0$$

For shrinking surface we have that

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0 \quad \Rightarrow \quad H < 0$$

since $\dot{\sigma}_{ij}$ is pointing into the elastic region.

Answer 12.1. We use the Voigt notation and define

$$\dot{\boldsymbol{\sigma}} = \begin{bmatrix} \dot{\sigma}_{11} & \dot{\sigma}_{22} & \dot{\sigma}_{33} & \dot{\sigma}_{12} & \dot{\sigma}_{13} & \dot{\sigma}_{23} \end{bmatrix}^T$$

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} \dot{\epsilon}_{11} & \dot{\epsilon}_{22} & \dot{\epsilon}_{33} & 2\dot{\epsilon}_{12} & 2\dot{\epsilon}_{13} & 2\dot{\epsilon}_{23} \end{bmatrix}^T$$

The matrix format of the elastoplastic tangent, D^{mat}_{1111}, is found via insertion into the given index expression, taking care to include a factor $\frac{1}{2}$ when necessary

$$D^{mat}_{1111} = D_{1111} - \frac{9G^2}{A} \frac{s_{11}s_{11}}{\sigma_y^2} = 2G[1 + \frac{\nu}{1-2\nu}] - \frac{9G^2}{A} \frac{s_{11}s_{11}}{\sigma_y^2}$$

$$D^{mat}_{1122} = D_{1122} - \frac{9G^2}{A} \frac{s_{11}s_{22}}{\sigma_y^2} = 2G[\frac{\nu}{1-2\nu}] - \frac{9G^2}{A} \frac{s_{11}s_{22}}{\sigma_y^2}$$

...

The shear strain terms is

$$D^{mat}_{1112} = (D_{1112} - \frac{9G^2}{A} \frac{s_{11}s_{12}}{\sigma_y^2}) = -\frac{9G^2}{A} \frac{s_{11}s_{12}}{\sigma_y^2}$$

Answer 12.2. For plane strain we have $\dot{\epsilon}_{23} = 0$, $\dot{\epsilon}_{13} = 0$ and $\dot{\epsilon}_{33} = 0$. The format in (12.1) reduces can be written in matrix forma

$$\begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{2211} & D_{2222} & D_{2212} \\ D_{1211} & D_{1222} & D_{1212} \end{bmatrix} \begin{bmatrix} \dot{\epsilon}_{11} \\ \dot{\epsilon}_{22} \\ \dot{\epsilon}_{12} \end{bmatrix}$$

considering that the remaining components of strain are zero. The explicit values of the above matrix is found from (12.1). The $\dot{\sigma}_{13}$ and $\dot{\sigma}_{23}$ components are by definition zero for plane strain. Furthermore

$$\dot{\sigma}_{33} = (D_{3311} - \frac{9G^2}{A}\frac{s_{33}s_{11}}{\sigma_y^2})\dot{\epsilon}_{11} + (D_{1122} - \frac{9G^2}{A}\frac{s_{33}s_{22}}{\sigma_y^2})\dot{\epsilon}_{22} + (D_{3312} - \frac{9G^2}{A}\frac{s_{33}s_{12}}{\sigma_y^2})\dot{\epsilon}_{12}$$

$$= 2G[\frac{\nu}{1-2\nu}]\dot{\epsilon}_{11} - \frac{9G^2}{A}\frac{s_{33}s_{11}}{\sigma_y^2}\dot{\epsilon}_{11} + 2G\frac{\nu}{1-2\nu}\dot{\epsilon}_{22} - \frac{9G^2}{A}\frac{s_{33}s_{22}}{\sigma_y^2}\dot{\epsilon}_{22} - \frac{9G^2}{A}\frac{s_{33}s_{12}}{\sigma_y^2}\dot{\epsilon}_{12}$$

Answer 12.4. For the isotropic case (expanding radius of yield surface) stress cycling gives a closed curve, and strain cycling is a limit value. For the kinematic (translating yield surface) case both strain and stress cycling gives closed curves. Illustrations are given below for the four cases. See page 287 and page 299 in the course book for more details.

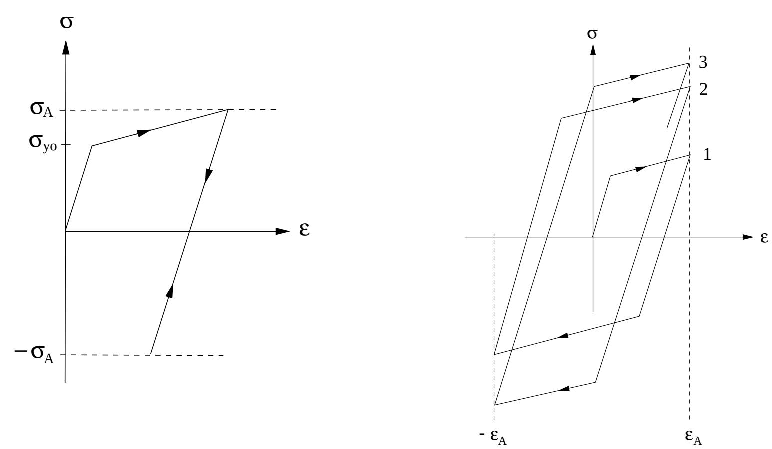


Figure 1: Stress and strain cycling for isotropic hardening von Mises plasticity model.

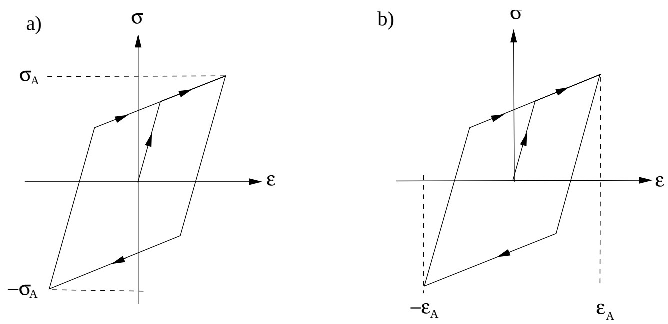


Figure 2: Strain and strain cycling for kinematic hardening von Mises plasticity model.

Answer 17.1. a) We multiply first by the arbitrary function v

$$v\frac{d}{dx}(A\sigma) = 0$$

Next we integrate over the body (V)

$$\int_V v\frac{d}{dx}(A\sigma)dV = 0$$

In one dimension this simplifies to

$$\int_0^L v\frac{d}{dx}(A\sigma)dL = 0$$

Integration by parts claim that for two functions f and g

$$\int fgdx = \int Fg'dx - \int Fg$$

Thus we can write

$$\int_0^L v\frac{d}{dx}(A\sigma)dL = \int_0^L (A\sigma)\frac{dv}{dx}dL - [vA\sigma]_0^L = 0$$

Thus

$$\int_0^L (A\sigma)\frac{dv}{dx}dL = [vA\sigma]_0^L$$

b) The Galerkin method states that we select $v = \boldsymbol{Nc}$, i.e the arbitrary function is selected in the same basis as the displacement field. Note that the function is still arbitrary in the sense that any bilinear representation is possible ($\boldsymbol{c}$ is arbitrary). Since v is scalar we have

$$v^T = \boldsymbol{c}^T\boldsymbol{N}^T = \boldsymbol{N}\boldsymbol{c} = v$$

Inserting this into our integral expression yields

$$\int_0^L (A\sigma)\boldsymbol{c}^T\boldsymbol{B}^T dL = [\boldsymbol{c}^T\boldsymbol{N}^T A\sigma]_0^L$$

This gives

$$\boldsymbol{c}^T(\int_0^L (A\sigma)\boldsymbol{B}^T dL - [\boldsymbol{N}^T A\sigma]_0^L) = 0$$

Thus

$$[\boldsymbol{N}^T A\sigma]_0^L - A\int_0^L \boldsymbol{B}^T\sigma dL = 0$$

as was to be proven

c) The shape functions become

$$N_1 = 1 - \frac{x}{L}$$

$$N_2 = \frac{x}{L}$$

Thus we have

$$\boldsymbol{N}^T = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

Since the bar is fixed in one end we must have that

$$A\sigma = -F_1 \quad \text{at} \quad x = 0$$

and

$$A\sigma = F_2 \quad \text{at} \quad x = L$$

Thus

$$[\boldsymbol{N}^T A\sigma]_0^L = \begin{bmatrix} 0 \\ F_2 \end{bmatrix} - \begin{bmatrix} -F_1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

The derivatives become

$$\boldsymbol{B}^T = \frac{d\boldsymbol{N}^T}{dx} = \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

inserting this into the integral expression gives

$$\boldsymbol{\Psi} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - \frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L \sigma dx; \quad \boldsymbol{\Psi} = \mathbf{0}$$

as was to be proven

d) We now take the time derivative

$$\dot{\boldsymbol{\Psi}} = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix} - \frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L \dot{\sigma} dx = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix} - \frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L E^{ep} \dot{\epsilon} dx = 0$$

Since we have

$$\epsilon = \boldsymbol{B u}$$

we also have

$$\dot{\epsilon} = \boldsymbol{B \dot{a}} = \begin{bmatrix} -1 & 1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}$$

Inserting this we find

$$\frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L E^{ep} \begin{bmatrix} -1 & 1 \end{bmatrix} \frac{1}{L} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} dx = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix}$$

Since neither E^{ep} nor the displacements vary along the bar the integral provides only L and we arrive at

$$\frac{AE^{ep}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix}$$

e) Simple insertion yields

$$\frac{AE^{ep}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{F}_1 \\ \dot{F}_2 \end{bmatrix}$$

the equations become

$$-\frac{AE^{ep}}{L} \dot{u}_2 = \dot{F}_1$$

and

$$-\frac{AE^{ep}}{L} \dot{u}_2 = \dot{F}_2$$

For the equilibrium condition we find

$$\boldsymbol{\Psi} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - \frac{A}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L \sigma dx = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - A\sigma \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$$

Since the strain is uniform so is the stress. Since F_2 is known, the second equation is of interest:

$$\Psi = F_2 - A\sigma = 0$$

g) Euler forward reads

$$(K_t)_n(a_{n+1} - a_n) = f_{n+1} - f_n$$

Thus for a step to $n+1 = 1$ from $n = 0$ we get

$$a_{n+1} = (K_t)_n^{-1}(f_{n+1} - f_n) + a_n$$

We compute

$$(K_t)_n = \frac{A(E^{ep})_n}{L} = \frac{AE(1 - 2\alpha\epsilon_n)}{L}$$

Thus

$$(K_t)_n^{-1} = \frac{L}{AE(1 - 2\alpha\epsilon_n)}$$

with $\epsilon_n = \epsilon_0 = 0$ we find

$$(K_t)_n^{-1} = \frac{L}{AE}$$

Inserting the forces and considering $a_n = a_0 = 0$ we get

$$a_{n+1} = a_1 = (K_t)_0^{-1}(f_1 - f_0) + a_0 = 10^{-3}L$$

For a load step from $n+1 = 2$ to $n = 1$ we have

$$(K_t)_n^{-1} = \frac{L}{AE(1 - 2\alpha\epsilon_n)} = \frac{L}{AE(1 - 2\alpha(u_2 - u_1)/L)} = \frac{10L}{AE8}$$

with $a_n = 10^{-3}L$ we now find

$$a_{n+1} = a_2 = (K_t)_1^{-1}(f_2 - f_1) + a_1 = 2.25 \cdot 10^{-3}L$$

h) The incremental form reads

$$d\sigma = E(1 - 2\alpha\frac{a}{L})d(\frac{a}{L})$$

Thus

$$\sigma = \int_0^{a/L} E(1 - 2\alpha\frac{a}{L})d(\frac{a}{L}) = E\frac{a}{L} - E\alpha(\frac{a}{L})^2 + C$$

where C is a constant. We insert in the equilibrium equation to find

$$f - EA\frac{a}{L} + EA\alpha(\frac{a}{L})^2 - AC = 0$$

Due to the initial condition that $a = 0$ for $f = 0$ we find $AC = 0$. We now solve for a and find

$$a = L\Big(\frac{1}{2\alpha} \pm \sqrt{\frac{1}{(2\alpha)^2} - \frac{f}{EA\alpha}}\Big)$$

Inserting the given constant α and the forces gives

$$a_1 = 1.127 \cdot 10^{-3} L$$

$$a_2 = 2.764 \cdot 10^{-3} L$$

(selecting the solutions closest to zero)
i) The Newton-Raphson scheme reads

$$\Psi^i = \Big(\frac{\partial \Psi}{\partial a}\Big)^{i-1}(a^i - a^{i-1}) + \Psi^{i-1} = 0$$

and iteration takes place so that $\Psi \to 0$. We compute

$$\frac{\partial \Psi}{\partial a} = \frac{\partial}{\partial a}(f - \sigma A) = -A\frac{\partial \sigma}{\partial a} = -K_t$$

Recall that

$$\Psi^{i-1} = (f - \sigma A)^{i-1} = f_{n+1} - A\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} d\epsilon$$

Here the force is fixed and the integration bounds changes throughout iteration (eq 17.21). Also, remember that

$$\Big(\frac{\partial \Psi}{\partial a}\Big)^{i-1} = -K_t^{i-1} = \frac{-AE(1 - 2\alpha a^{i-1}/L)}{L}$$

Inserting this in the iteration schemes gives.

$$\frac{-AE(1 - 2\alpha a^{i-1}/L)}{L}(a^i - a^{i-1}) + f_{n+1} - A\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} d\epsilon = 0$$

solving for a^i gives

$$a^i = \frac{L}{-AE(1 - 2\alpha a^{i-1}/L)}\Big(\frac{-AE(1 - 2\alpha a^{i-1}/L)}{L}a^{i-1} - f_{n+1} + (A\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} d\epsilon)\Big)$$

or more simple

$$a^i = \Big(a^{i-1} + \frac{L(f_{n+1} - A\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} d\epsilon)}{AE(1 - 2\alpha a^{i-1}/L)}\Big)$$

For step 1 the integral is zero and $a^{i-1} = 0$ so we get

$$a^1 = L \cdot 10^{-3}$$

For the second step we need to compute the integral in general as

$$\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} = [E(\epsilon - \alpha\epsilon^2)]_{\epsilon_n}^{\epsilon^{i-1}} = E(\epsilon^{i-1} - \alpha(\epsilon^{i-1})^2) = E(\frac{a^{i-1}}{L} - \alpha(\frac{a^{i-1}}{L})^2)$$

and so for the second step we find

$$\int_{\epsilon_n}^{\epsilon^{i-1}} E^{ep} = E(f_1 - \alpha f_1^2)$$

This gives

$$a^2 = \Big(L \cdot 10^{-3} + \frac{L(\cdot 10^{-3} AE - AE(\cdot 10^{-3} - \alpha \cdot 10^{-6}))}{AE(1 - 2\alpha \cdot 10^{-3})}\Big)$$

this simplifies to

$$a^2 = L\Big(10^{-3} + \frac{(\cdot 10^{-3} - (10^{-3} - 10^{-4}))}{0.8}\Big) = 1.125 \cdot 10^{-3} L$$

The final step follows the same procedure, as does the second load step.

Answer 18.2. a) The chain rule gives

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0$$

From Hookes law

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\epsilon}_{kl}^e = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p),$$

and the flow rule of associated plasticity

$$\dot{\epsilon}_{kl}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{kl}},$$

we then find

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\lambda} \frac{\partial f}{\partial \sigma_{kl}}) = 0$$

Solving for $\dot{\lambda}$ gives

$$\dot{\lambda} = \frac{1}{A} \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \dot{\epsilon}_{kl}$$

where the scalar A becomes

$$A = \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f}{\partial \sigma_{kl}}$$

b) To perform the integration we first evaluate

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3s_{ij}}{2\sqrt{3J_2}}$$

we then find

$$A = \frac{3s_{ij}}{2\sqrt{3J_2}} E \frac{3s_{ij}}{2\sqrt{3J_2}} = \frac{3}{2}E$$

Given that $\sigma_{ij} = D_{ijkl}\epsilon_{kl} = E\epsilon_{ij}$ (since $\nu = 0$ as stated in 18.1). Thus we have

$$\dot{\lambda} = \frac{2}{3E} \frac{3s_{ij}}{2\sqrt{3J_2}} E\dot{\epsilon}_{ij} = \frac{s_{ij}}{\sqrt{3J_2}}\dot{\epsilon}_{ij}$$

Since

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - D_{ijkl}\dot{\epsilon}^p_{kl} = D_{ijkl}\dot{\epsilon}_{kl} - \dot{\lambda} D_{ijkl} \frac{\partial f}{\partial \sigma_{kl}}$$

we get

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\epsilon}_{kl} - D_{ijkl} \frac{3s_{kl}}{2\sqrt{3J_2}} \frac{s_{pq}}{\sqrt{3J_2}} \dot{\epsilon}_{pq}$$

Thus

$$\dot{\sigma}_{ij} = E\dot{\epsilon}_{ij} - E \frac{3s_{ij}}{2\sqrt{3J_2}} \frac{s_{pq}}{\sqrt{3J_2}} \dot{\epsilon}_{pq}$$

$$\dot{\sigma}_{ij} = E(\delta_{ik}\delta_{lj} - \frac{3s_{ij}}{2\sqrt{3J_2}} \frac{s_{pq}}{\sqrt{3J_2}} \delta_{pk}\delta_{lq})\dot{\epsilon}_{kl}$$

which simplifies to

$$\dot{\sigma}_{ij} = E(\delta_{ik}\delta_{lj} - \frac{s_{ij}s_{kl}}{2J_2})\dot{\epsilon}_{kl}$$

Thus the elasto-plastic stiffness is identified as

$$D^{ep}_{ijkl} = E(\delta_{ik}\delta_{lj} - \frac{s_{ij}s_{kl}}{2J_2})$$

The forward Euler step is then taken as

$$\sigma^{(2)}_{ij} = \sigma^{(1)}_{ij} + D^{ep(1)}_{ijkl}\Delta\epsilon_{kl} = \sigma^{(1)}_{ij} + D^{ep(1)}_{ijkl}(\epsilon^{(2)}_{kl} - \epsilon^{(1)}_{kl})$$

The previous stress $\sigma^{(1)}_{ij}$, previous strain $\epsilon^{(1)}_{ij}$ and Youngs Modulus E is given (18.1). Insertion yields the sought answer.

Answer 18.3. a) By definition the trial stress is

$$\sigma_{ij}^{(t)} = \sigma_{ij}^{(1)} + \Delta\sigma_{ij}^{e}$$

where $\Delta\sigma_{ij}^{e}$ is given from Hook's linear elastic law

$$\Delta\sigma_{ij}^{e} = D_{ijkl}\Delta\epsilon_{kl}$$

and in this special case, with $\nu = 0$, we have

$$\sigma_{ij}^{(t)} = \sigma_{ij}^{(1)} + E\Delta\epsilon_{kl}$$

where $\Delta\epsilon_{kl} = \epsilon_{kl}^{(2)} - \epsilon_{kl}^{(1)}$, and superscript (1) and (2) denotes the current and updates step respectively. Inserting numerical values yields

$$\sigma_{ij}^{(t)} = \begin{bmatrix} 390 & 0 & 0 \\ 0 & 170 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

b) To return to the yield surface we have for our case that the new stress must equal

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{(t)} - E\Delta\epsilon_{ij}^{p}$$

where $\Delta\epsilon_{ij}^{p}$ is the plastic strain increment. Assuming associated plasticity, it follows from the flow rule that

$$\Delta\epsilon_{ij}^{p} = \int_{(c)}^{(2)} \frac{\partial f}{\partial\sigma_{ij}} d\lambda \approx \Big(\frac{\partial f}{\partial\sigma_{ij}}\Big)^{*}\Delta\lambda$$

where (c) stands for the contact point, and $\Big(\frac{\partial f}{\partial\sigma_{ij}}\Big)^{*}$ is $\frac{\partial f}{\partial\sigma_{ij}}$ evaluated at some point along the integration path between (1) and (2). For a backwards euler scheme we evaluate $\frac{\partial f}{\partial\sigma_{ij}}$ at (2) and get

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{(t)} - E\Big(\frac{\partial f}{\partial\sigma_{ij}}\Big)^{(2)}\Delta\lambda$$

The derivative is found, as has been shown several times previously for a von Mises yield surface, to be

$$\Big(\frac{\partial f}{\partial\sigma_{ij}}\Big)^{(2)} = \frac{3s_{ij}^{(2)}}{2\sqrt{3J_2^{(2)}}}$$

i.e we have

$$\sigma_{ij}^{(2)} = \sigma_{ij}^{(t)} - E\frac{3s_{ij}^{(2)}}{2\sqrt{3J_2^{(2)}}}\Delta\lambda$$

From this equation we conclude that

$$\sigma_{kk}^{(2)} = \sigma_{kk}^{(t)}$$

From this we see that

$$\sigma_{ij}^{(2)} - \sigma_{ij}^{(t)} = \sigma_{ij}^{(2)} + \frac{1}{3}\sigma_{kk}^{(2)}\delta_{ij} - \sigma_{ij}^{(t)} - \frac{1}{3}\sigma_{kk}^{(t)}\delta_{ij} = s_{ij}^{(2)} - s_{ij}^{(t)}$$

using this result we find from () that

$$s_{ij}^{(2)} - s_{ij}^{(t)} = -E\frac{3s_{ij}^{(2)}}{2\sqrt{3J_2^{(2)}}}\Delta\lambda$$

Which allows for identifying that

$$s_{ij}^{(2)} = \frac{s_{ij}^{(t)}}{\left(1 + \frac{3E}{2\sqrt{3J_2^{(2)}}}\Delta\lambda\right)}$$

by squaring the left and right hand side we get

$$2J_2^{(2)} = \frac{2J_2^{(t)}}{\left(1 + \frac{3E}{2\sqrt{3J_2^{(2)}}}\Delta\lambda\right)^2}$$

Multiplying through by a factor 3 and taking the square root gives

$$(1 + \frac{3E}{2\sqrt{3J_2^{(2)}}}\Delta\lambda)\sqrt{3J^{(2)}}_2 = \sqrt{3J_2^{(t)}}$$

from which we find

$$\sqrt{3J_2^{(2)}} = \sqrt{3J_2^{(t)}} - \frac{3E\Delta\lambda}{2}$$

using this we can now express the deviatoric stress at as state 2 as

$$s_{ij}^{(2)} = \frac{s_{ij}^{(t)}}{\left(1 + \frac{3E\Delta\lambda}{2\sqrt{3J_2^{(t)}} - 3E\Delta\lambda}\right)}$$

and so we have expressed all parts of the stress at the updated state (2).

c) We have from $f = 0$ and the previous results regarding $\sqrt{3J_2^{(2)}}$ that

$$f = \sqrt{3J_2^{(2)}} - \sigma_{y0} = \sqrt{3J_2^{(t)}} - \frac{3E\Delta\lambda}{2} - \sigma_{y0} = 0$$

and instantly we conclude that

$$\Delta\lambda = \frac{2}{3E}\left(\sqrt{3J_2^{(t)}} - \sigma_{y0}\right)$$

Inserting numerical values yields

$$s_{ij}^{(t)} = \begin{bmatrix} 203.3 & 0 & 0 \\ 0 & -16.67 & 0 \\ 0 & 0 & -186.7 \end{bmatrix} \text{ MPa}$$

$$\sqrt{3J_2^{(t)}} = 338.67 \text{ MPa}$$

$$\Delta\lambda = 3.13 \cdot 10^{-4}$$

d) The updated stress is found by combining the formulae for deviatoric and hydro static updated stresses as

$$\sigma_{ij}^{(2)} = \frac{s_{ij}^{(t)}}{\left(1 + \dfrac{3E\Delta\lambda}{2\sqrt{3J_2^{(t)}} - 3E\Delta\lambda}\right)} + \frac{1}{3}\delta_{ij}\sigma_{kk}^{(t)}$$

numerical values gives

$$\sigma_{ij}^{(2)} = \begin{bmatrix} 230.8 & 0 & 0 \\ 0 & 174.9 & 0 \\ 0 & 0 & 54.4 \end{bmatrix} \text{ MPa}$$