23 THERMO-PLASTICITY

If the material is loaded so that not only plastic strains develop, but also the temperature is changed, then *thermo-plasticity* is encountered. This phenomenon was first studied in detail by Prager (1958) for rigid hardening plasticity and later extended to elasto-plasticity by Boley and Weiner (1960) and Naghdi (1960). Attempts to formulate thermo-plasticity within a thermodynamic framework were initiated by Dillon (1963), in the review paper by Perzyna and Sawczuk (1973) and by Raniecki and Sawczuk (1975); later efforts are reported, for instance, by Lehmann (1984), Gnoneim (1990), Allen (1991), Simo and Miehe (1992), Wu and Glockner (1996) and by Celentano *et al.* (1996). The latter papers use an approach somewhat similar to the one we will adopt.

The most evident implication of a change of temperature is that the total strains now consist of the elastic, plastic and thermal strains, i.e.

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^o$$

where the elastic strains, per definition, are defined by Hooke's law, i.e. $\varepsilon_{ij}^e = C_{ijkl}\sigma_{kl}$. However, change of temperature also influences the material parameters and it is worthwhile to evaluate this effect in some detail.

We also emphasize that plastic strains – or more generally, dissipative mechanisms – will result in a heat generation and thereby thermal strains; this phenomenon will also be considered here.

23.1   Change of material parameters with temperature

It is of interest to obtain knowledge about the variation of different material parameters when the temperature changes. The intention of this review is not to provide a detailed catalogue of the temperature dependence of different material parameters for various materials, but simply to provide typical illustrations of the effect of temperature. The reader is referred, for instance, to review contributions by Browne and Blundell (1972), Eibl *et al.* (1974), Schneider *et al.* (1981), Schneider (1988) and Vecchi (1998).
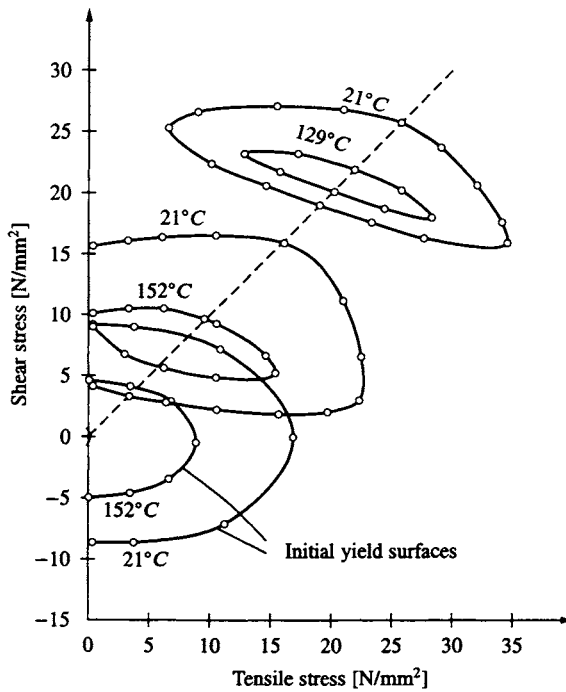
Figure 23.1: Effect of temperature on the yield surfaces; pure aluminum tested in combined torsion and tension by Phillips and Tang (1972).

The most characteristic feature of the temperature influence is that the yield stress changes with temperature; in practice it decreases with increasing temperature. This effect is evident from, for instance, the experimental results on pure aluminum of Phillips and Tang (1972) shown in Fig. 23.1 where the contraction of the yield surface with increasing temperature is apparent; further investigations are reported by Phillips (1974). For uniaxial stress conditions, the softening effect due to the temperature change shown in Fig. 23.1 is illustrated in Fig. 23.2 for aluminum 2024-T4 and carbon steel.

The elastic parameters are also influenced by the temperature; Fig. 23.3a) shows the decrease of the E-modulus with increasing temperature for carbon steel and pure aluminum and Fig. 23.3b) for quartzitic concrete, where the data are normalized by E_0, i.e. the E-modulus at $21^\circ C$.
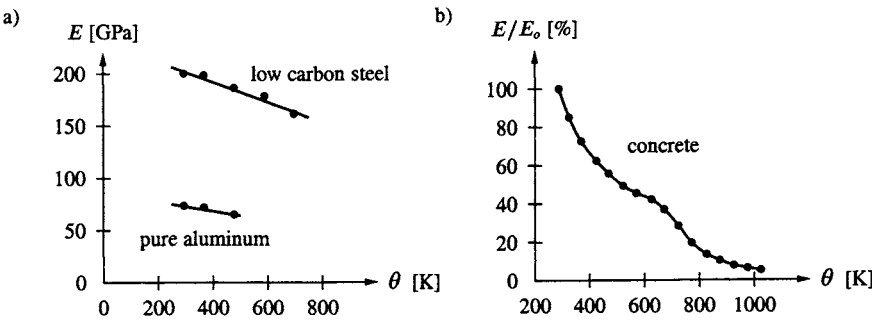
In Fig. 23.3a) the experimental data are interpolated linearly by

$$E(\theta) = E_0 + (\theta - \theta_0)E_1 \tag{23.1}$$

where $\theta_0 = 294K = 21^\circ C$ and E_0 is Young's modulus at this reference temper-
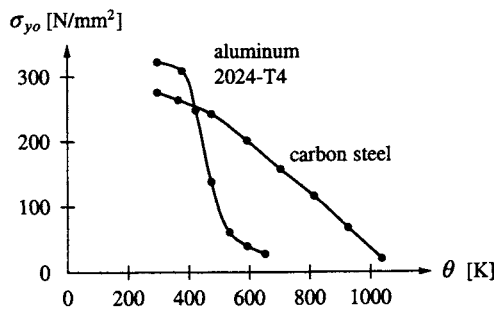
Figure 23.2: Effect of temperature on initial yield stress; experimental data of Eshbach and Souders (1975).

Figure 23.3: Effect of temperature on E-modulus; a) linear approximations and experimental data of Eshbach and Souders (1975), b) for quartzitic concrete, Schneider *et al.* (1981).

ature; the coefficients E_0 and E_1 are given in Table 23.1.

|	E_0[GPa]	E_1[GPa/K]
carbon steel	202	-0.096
pure aluminum	73	-0.043

Table 23.1: Coefficients for E-modulus in (23.1).

The fact that mechanical parameters like σ_{yo} and E in general decrease with increasing temperature is often referred to in the literature as the phenomenon of *thermal softening*.

The influence of temperature on Poisson's ratio v is shown in Fig. 23.4 and
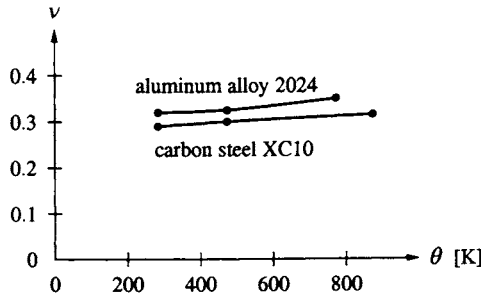
Figure 23.4: Effect of temperature on Poisson's ratio v; experimental data reported by Metals Handbook (1978)
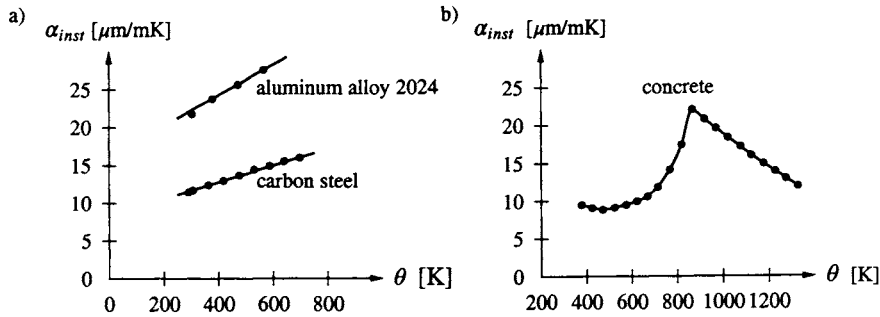
Figure 23.5: Effect of temperature on the instantaneous thermal expansion coefficient α_{inst}; a) linear approximations and experimental data for aluminum alloy and carbon steel by Metals Handbook (1978), b) for quartzitic concrete; Schneider *et al.* (1981).

it appears that the effect is very modest. Noting that the exact magnitude of v is only of secondary importance for the response, it is concluded that in most applications v can be taken as constant.

The temperature effect on the instantaneous thermal expansion coefficient α_{inst} (defined as $\dot{\varepsilon}^0 = \alpha_{inst}\dot{\theta}$) is shown in Fig. 23.5a) for carbon steel and aluminum alloy 2024. It appears that the experimental data are in close agreement with the linear approximation given by

$$\alpha_{inst}(\theta) = \alpha_0 + (\theta - \theta_0)2\alpha_1 \qquad (23.2)$$

where $\theta_0 = 294K = 21°C$ and the factor 2 ahead of α_1 is introduced for convenience, see (23.3). The coefficients α_0 and α_1 are given in Table 23.2.

	$\alpha_0 \, [10^{-6}/K]$	$\alpha_1 \, [10^{-6}/K^2]$
carbon steel	11.6	0.006
2024 al.alloy	22.1	0.010

Table 23.2: Coefficients for thermal expansion coefficient in (23.2) and (23.3).

The usual thermal expansion coefficient α is defined by $\varepsilon^0 = (\theta - \theta_0)\alpha$, i.e. $\dot\varepsilon^0 = \frac{d}{d\theta}[(\theta - \theta_0)\alpha]\dot\theta$ and it appears that $\alpha_{inst} = \frac{d}{d\theta}[(\theta - \theta_0)\alpha]$; from (23.2) it then follows that

$$\alpha(\theta) = \alpha_0 + (\theta - \theta_0)\alpha_1 \tag{23.3}$$

For quartzitic concrete, the variation of the thermal expansion coefficient is illustrated in Fig. 23.5b).
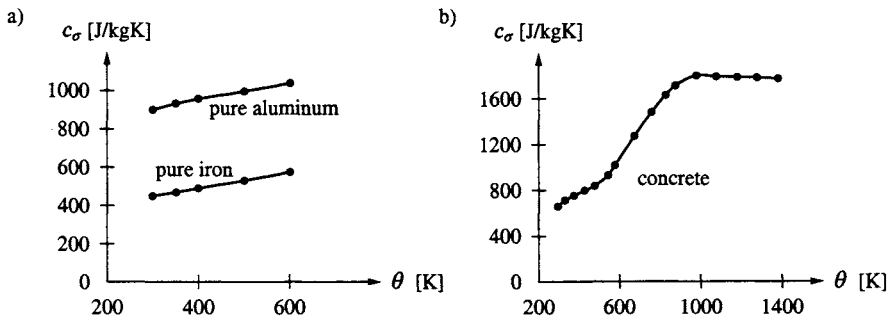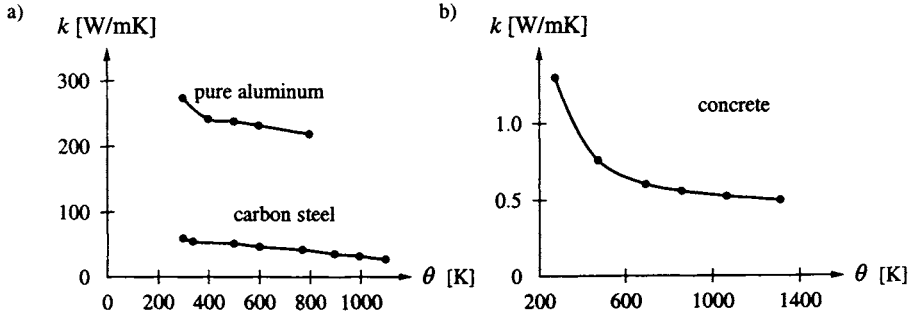

Figure 23.6: Effect of temperature on specific heat capacity; a) aluminum and iron, CRC (1995), b) quartzitic concrete, Schneider *et al.* (1981).

For the specific heat capacity c_σ, the temperature dependence for pure iron and pure aluminum is shown in Fig. 23.6a) (recall from (21.83) that $c_\varepsilon \approx c_\sigma$ holds with close accuracy) and for quartzitic concrete in Fig. 23.6b).

The variation of the thermal conductivity k with temperature is illustrated in Fig. 23.7a) for carbon steel and pure aluminum and in Fig. 23.7b) for granitic concrete.

Finally, for the mass density ρ, Perry's Chemical Engineering Handbook (1984) indicates that for Al, Cu and Fe the mass density only decreases by about 4% when the temperature is raised from room temperature to melting point. The mass density can therefore be considered in most applications as constant.

It appears that of all material parameters, it is the initial yield stress that exhibits the greatest temperature dependence and that the effect of temperature on Poisson's ratio can be ignored for most cases of practical interest.

Figure 23.7: Effect of temperature on thermal conductivity k; a) carbon steel, Metals Handbook (1978) and pure aluminum, Perry's Chemical Engineering Handbook (1984), b) granitic concrete, Betonghandbok (1982).

23.2 Equations of thermo-plasticity

With the discussion above of the influence of temperature on the various material parameters, it is timely to establish a form of Helmholtz' free energy ψ that results in thermo-plasticity. If all material parameters are taken as constants, a natural generalization of (22.2) valid for isothermal plasticity as well as (21.84) valid for thermo-elasticity becomes

$$\rho\psi(\theta, \varepsilon_{ij} - \varepsilon^p_{ij}, \kappa_\alpha) = \rho c_\varepsilon(\theta - \theta \ln\frac{\theta}{\theta^*}) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon^p_{ij})D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})$$

$$- \beta_{ij}(\varepsilon_{ij} - \varepsilon^p_{ij})(\theta - \theta_0) + \rho\psi^p(\kappa_\alpha) \qquad (23.4)$$

where θ^* is an arbitrary constant temperature and θ_0 is the reference temperature at which no thermal strains exist. In the expression above, the tensor β_{ij} is related to the thermal expansion tensor α_{ij} by (21.19)

$$\beta_{ij} = D_{ijkl}\alpha_{kl}$$

It is easily verified that $\theta\partial^2\psi/\partial\theta^2 = -c_\varepsilon$ in accordance with definition (21.78); moreover, from (21.37) and using the expression above we obtain

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl} - \varepsilon^o_{kl}) \quad \text{where} \quad \varepsilon^o_{ij} = \alpha_{ij}(\theta - \theta_0)$$

that is Hooke's law. Finally, the conjugated thermodynamic forces become

$$K_\alpha = \rho\frac{\partial\psi}{\partial\kappa_\alpha} = \rho\frac{\partial\psi^p(\kappa_\beta)}{\partial\kappa_\alpha}$$

These results were based on the assumption that D_{ijkl}, β_{ij} (and thereby α_{ij}) and c_ε are constants; moreover, the forces K_α depend only on the internal variables κ_α. In view of the previous discussion which showed that these quantities

in general depend on the temperature, it is natural to seek a proper generalization of (23.4).

Let us instead of (23.4) take Helmholtz' free energy in the format

$$\rho\psi(\theta, \varepsilon_{ij} - \varepsilon_{ij}^p, \kappa_\alpha) = \rho h(\theta) + \frac{1}{2}(\varepsilon_{ij} - \varepsilon_{ij}^p)D_{ijkl}(\theta)(\varepsilon_{kl} - \varepsilon_{kl}^p)$$
$$- \beta_{ij}(\theta)(\varepsilon_{ij} - \varepsilon_{ij}^p)(\theta - \theta_0) + \rho\psi^p(\theta, \kappa_\alpha) \qquad (23.5)$$

where $h(\theta)$, $D_{ijkl}(\theta)$, $\beta_{ij}(\theta)$ and $\psi^p(\theta, \kappa_\alpha)$ depend on the temperature. We then obtain Hooke's law

$$\sigma_{ij} = \rho\frac{\partial\psi}{\partial\varepsilon_{ij}} = D_{ijkl}(\theta)(\varepsilon_{kl} - \varepsilon_{kl}^p - \varepsilon_{kl}^o) \quad \text{where} \quad \varepsilon_{ij}^o = \alpha_{ij}(\theta)(\theta - \theta_0) \qquad (23.6)$$

and

$$K_\alpha = \rho\frac{\partial\psi^p(\theta, \kappa_\beta)}{\partial\kappa_\alpha} \qquad (23.7)$$

Moreover, assuming the mass density to be independent of the temperature, we find

$$\boxed{c_\varepsilon = -\theta\frac{\partial^2\psi}{\partial\theta^2} = -\theta\frac{d^2h(\theta)}{d\theta^2} + \tilde{c}_\varepsilon} \qquad (23.8)$$

where

$$\tilde{c}_\varepsilon = -\frac{\theta}{2\rho}(\varepsilon_{ij} - \varepsilon_{ij}^p)\frac{d^2D_{ijkl}}{d\theta^2}(\varepsilon_{kl} - \varepsilon_{kl}^p)$$
$$+ \frac{\theta}{\rho}(\varepsilon_{ij} - \varepsilon_{ij}^p)\frac{d^2}{d\theta^2}[\beta_{ij}(\theta - \theta_0)] - \theta\frac{\partial^2\psi^p}{\partial\theta^2} \qquad (23.9)$$

We will later show that the contribution of $\tilde{c}_\varepsilon$ is slight, i.e. $\tilde{c}_\varepsilon/c_\varepsilon \approx 0$.

Let us next consider the dissipation inequality and the evolution laws. Irrespective of whether we have isothermal plasticity or thermo-plasticity, the mechanical dissipation inequality is given by (21.37)

$$\gamma_{mech} \equiv \sigma_{ij}\dot{\varepsilon}_{ij}^p - K_\alpha\dot{\kappa}_\alpha \geq 0 \qquad (23.10)$$

If we accept the postulate of maximum dissipation as a means to fulfill this inequality then σ_{ij} and K_α are the natural variables in the yield function, i.e. $f = f(\sigma_{ij}, K_\alpha)$. However, we know that the temperature influences both the conjugated forces, see (23.7), and the initial yield stress σ_{yo}, cf. Figs. 23.1 and 23.2. This is accomplished by writing

$$\boxed{f = f(\sigma_{ij}, K_\alpha, \theta) \leq 0} \qquad (23.11)$$

For an isotropic hardening von Mises model - for instance - this expression reads

$$f = \sigma_{eff} - \sigma_{yo}(\theta) - K \leq 0$$

where we may have $K = K(\theta, \kappa)$ but since $f = f(\sigma_{ij}, K, \theta)$ it follows that $\partial f / \partial \theta = -d\sigma_{yo}/d\theta$.

To fulfill the dissipation inequality (23.10) we may adopt the nonassociated format and introduce a potential function $g = g(\sigma_{ij}, K_\alpha, \theta)$ and with (22.6) we obtain

$$\dot{\varepsilon}_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}} ; \quad \dot{\kappa}_\alpha = -\lambda \frac{\partial g}{\partial K_\alpha} \qquad (23.12)$$

with the restrictions $\lambda \geq 0$ $f \leq 0$ and $\lambda f = 0$; moreover, associated thermo-plasticity is retrieved if $g = f$.

As for isothermal plasticity, differentiation of $\lambda f = 0$ leads to $\dot{\lambda} f + \lambda \dot{f} = 0$. Development of plasticity requires $f = 0$ and $\lambda > 0$ and we are then led to the consistency relation $\dot{f} = 0$ for development of plasticity.

With (23.11), we have

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial K_\alpha} \dot{K}_\alpha + \frac{\partial f}{\partial \theta} \dot{\theta} \qquad (23.13)$$

From $K_\alpha = \rho \partial \psi^p(\theta, \kappa_\beta)/\partial \kappa_\alpha$ it follows that

$$\dot{K}_\alpha = \rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \dot{\kappa}_\beta + \rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \theta} \dot{\theta}$$

which with (23.12b) becomes

$$\dot{K}_\alpha = -\rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \frac{\partial g}{\partial K_\beta} \lambda + \frac{\partial K_\alpha}{\partial \theta} \dot{\theta}$$

Insertion of this expression into (23.13) provides

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - H\lambda + S\dot{\theta} \qquad (23.14)$$

where the generalized plastic modulus H is defined in the usual way as

$$H = \frac{\partial f}{\partial K_\alpha} \rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \frac{\partial g}{\partial K_\beta}$$

cf. (22.9); as previously $H > 0$, $H = 0$, $H < 0$ corresponds to harden-ing, ideal and softening thermo-plasticity respectively. Moreover, in (23.14) the quantity S is defined as

$$S = \frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial K_\alpha} \frac{\partial K_\alpha}{\partial \theta}$$

and it is observed that $S = df/d\theta$. It appears that S depends on how much the yield function parameters and the forces K_α change with temperature. Therefore, the *sensitivity parameter S* expresses how the yield function changes with temperature. To evaluate what the sign of S means, suppose that the state $(\sigma_{ij}, K_\alpha, \theta)$ initially corresponds to a point located inside the yield surface, i.e. $f < 0$. Consider then a situation where $\dot{\sigma}_{ij} = 0$ and as also $\dot{\lambda} = 0$ hold, (23.14) gives $\dot{f} = S\dot{\theta}$. If $S > 0$ then $\dot{f} > 0$ for $\dot{\theta} > 0$ and for a sufficiently elevated temperature we will be in a situation where $f = 0$ is achieved. It is concluded that $S > 0$ corresponds to the situation where the yield surface contracts with increasing temperature. Evidently $S = 0$ implies that the yield function is independent of the temperature and, finally, $S < 0$ must then be the situation where the yield surface expands with increasing temperature. With reference to Fig. 23.1, the normal situation is that the yield surface contracts with increasing temperature, i.e. $S > 0$ holds in most cases. In conclusion

$$
\begin{array}{l}
\textit{For change of temperature} \\[4pt]
S \left\{
\begin{array}{lll}
> 0 & \Rightarrow & \textit{yield surface contracts} \\
= 0 & \Rightarrow & \textit{yield surface remains constant} \\
< 0 & \Rightarrow & \textit{yield surface expands}
\end{array}
\right. \\[12pt]
\textit{where} \\[4pt]
S = \dfrac{\partial f}{\partial \theta} + \dfrac{\partial f}{\partial K_\alpha} \dfrac{\partial K_\alpha}{\partial \theta}
\end{array}
\tag{23.15}
$$

Let us now derive the incremental response for a thermo-plastic material as well as the loading/unloading criteria. Differentiation of Hooke's law (23.6) provides

$$
\dot{\sigma}_{ij} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^p) - P_{ij}\dot{\theta}
\tag{23.16}
$$

where

$$
P_{ij} = \beta_{ij} + \frac{d\beta_{ij}}{d\theta}(\theta - \theta_0) - \frac{dD_{ijkl}}{d\theta}(\varepsilon_{kl} - \varepsilon_{kl}^p)
$$

Since both β_{ij} and D_{ijkl} are symmetric in i and j, it follows that P_{ij} is a symmetric tensor. Use of (23.12a) in (23.16) implies

$$
\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} - D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}}\lambda - P_{ij}\dot{\theta}
\tag{23.17}
$$

Insertion of this expression into (23.14) and noting that $\dot{f} = 0$ holds then provide

$$
\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\dot{\varepsilon}_{kl} - A\dot{\lambda} + (S - \frac{\partial f}{\partial \sigma_{ij}}P_{ij})\dot{\theta} = 0
\tag{23.18}
$$

where the quantity A is defined in the usual way as

$$A = H + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} > 0$$

cf. (22.10) and where, again, it is required that $A > 0$. From (23.18) then follows

$$\dot{\lambda} = \frac{1}{A} \left[\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \dot{\varepsilon}_{kl} + (S - \frac{\partial f}{\partial \sigma_{ij}} P_{ij}) \dot{\theta} \right] \tag{23.19}$$

To obtain the incremental response, the expression above is inserted into (23.17) to obtain

$$
\boxed{
\begin{aligned}
&\textit{Incremental constitutive relation} \\[2mm]
&\quad \dot{\sigma}_{ij} = D^{ep}_{ijkl} \dot{\varepsilon}_{kl} - \beta^{ep}_{ij} \dot{\theta} \\[2mm]
&\textit{where} \\[2mm]
&\quad D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A} D_{ijst} \frac{\partial g}{\partial \sigma_{st}} \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} \\[2mm]
&\quad \beta^{ep}_{ij} = P_{ij} + \frac{1}{A} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} (S - \frac{\partial f}{\partial \sigma_{mn}} P_{mn}) \\[2mm]
&\quad P_{ij} = \beta_{ij} + \frac{d\beta_{ij}}{d\theta}(\theta - \theta_0) - \frac{d D_{ijkl}}{d\theta}(\varepsilon_{kl} - \varepsilon^p_{kl}) \\[2mm]
&\quad A = H + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} > 0 \\[2mm]
&\quad H = \frac{\partial f}{\partial K_\alpha} \rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \frac{\partial g}{\partial K_\beta}
\end{aligned}
}
\tag{23.20}
$$

The similarity in structure between (23.20) and (10.26) is evident. Moreover, it appears that D^{ep}_{ijkl} is the usual elasto-plastic stiffness tensor and that β^{ep}_{ij} is a symmetric tensor.

It is of interest to consider (23.17) for purely thermo-elastic behavior where $\dot{\lambda} = 0$; we then obtain $\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} - P_{ij} \dot{\theta}$. In analogy with (10.36), we then define the *thermo-elastic stress rate* $\dot{\sigma}^{te}_{ij}$ as

$$\boxed{\dot{\sigma}^{te}_{ij} \equiv D_{ijkl} \dot{\varepsilon}_{kl} - P_{ij} \dot{\theta}} \tag{23.21}$$

where the term 'thermo-elastic' refers to the fact that this is the stress rate that would result for a given total strain rate and a given temperature rate provided that the material responds thermo-elastically. With (23.21), (23.19) takes the form

$$\dot{\lambda} = \frac{1}{A} \left(\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}^{te}_{ij} + S\dot{\theta} \right) \tag{23.22}$$
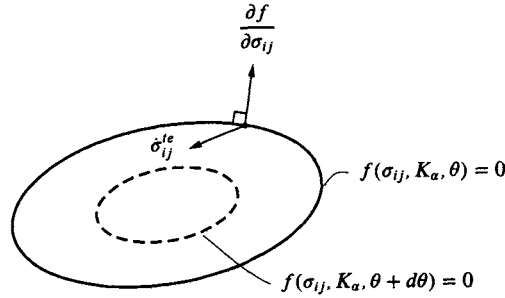
Figure 23.8: Illustration with $S > 0$ (i.e. the yield surface contracts with increasing temperature) where even $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} < 0$ may result in thermo-plastic loading for increasing temperature.

To obtain the general loading/unloading criteria, it is first observed that $f < 0$ implying that a thermo-elastic response occurs. Thermo-plasticity requires that $f = 0$ and $\dot\lambda \geq 0$. If $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} + S\dot\theta > 0$, then (23.22) implies $\dot\lambda > 0$ and we then have thermo-plastic loading. If $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} + S\dot\theta < 0$, (23.22) implying $\dot\lambda < 0$, which is impossible, and this case must therefore correspond to thermo-elastic unloading. Finally, in the limiting case where $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} + S\dot\theta = 0$, we have neutral thermo-plasticity loading, which is formally treated as thermo-plasticity even though the incremental response is purely thermo-elastic. These findings are summarized as

$$
\begin{aligned}
f = 0 \quad and \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{te} + S\dot\theta > 0 \quad &\Rightarrow \quad \textit{thermo-plastic loading} \\[1em]
f = 0 \quad and \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{te} + S\dot\theta = 0 \quad &\Rightarrow \quad \textit{neutral loading} \\[1em]
f = 0 \quad and \quad \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{te} + S\dot\theta < 0 \quad &\Rightarrow \quad \textit{thermo-elastic unloading}
\end{aligned}
$$

The similarity with the general loading/unloading criteria for isothermal plasticity given by (10.38) is striking. If $S = 0$, i.e. the yield function is unaffected by temperature changes, then $\dot\lambda = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{te}$ whereas for isothermal plasticity we have $\dot\lambda = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}^{e}$ where the elastic stress rate $\dot{\sigma}_{ij}^{e}$ is defined by $\dot{\sigma}_{ij}^{e} \equiv D_{ijkl}\dot{\epsilon}_{kl}$. If $S > 0$, which is normally the case, the yield surface contracts with increasing temperature and even if $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} < 0$ we may still have a situation where $\partial f / \partial \sigma_{ij} \; \dot{\sigma}_{ij}^{te} + S\dot\theta > 0$ for increasing temperature and thermo-plastic loading then occurs; this situation is illustrated in Fig. 23.8.

As a further illustration, consider a situation where thermo-plastic loading

$\dot{f} = 0$ holds. Suppose that the stresses are held constant, i.e. $\dot{\sigma}_{ij} = 0$; then (23.14) gives $-H\lambda + S\dot{\theta} = 0$. Assuming hardening behavior, $H > 0$, we then obtain $\lambda = \frac{S}{H}\dot{\theta}$. Again it appears that $S > 0$ (contraction of the yield surface with increasing temperature) implies $\lambda > 0$ and thereby development of plastic strains if $\dot{\theta} > 0$.

Let us now turn our interest to the energy equation. From (21.86) and $K_\alpha = \rho \partial \psi / \partial \kappa_\alpha$ and with $\rho \psi$ given by (23.5) we obtain

$$
\boxed{
\begin{aligned}
&\textit{Heat equation} \\[4pt]
&\quad \rho c_\varepsilon \dot{\theta} = r - q_{i,i} + Q \\[4pt]
&\textit{where} \\[4pt]
&\quad Q = \gamma_{mech} - \theta P_{ij}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij}) + \theta \frac{\partial K_\alpha}{\partial \theta}\dot{\kappa}_\alpha \\[4pt]
&\quad \gamma_{mech} = \sigma_{ij}\dot{\varepsilon}^p_{ij} - K_\alpha \dot{\kappa}_\alpha
\end{aligned}
}
\tag{23.23}
$$

where γ_{mech} is the mechanical dissipation, P_{ij} is defined in (23.20) and the heat flux vector q_i is related to the temperature gradient $\theta_{,i}$ by the constitutive equation in terms of Fourier's law, i.e.

$$
\boxed{q_i = -k_{ij}\theta_{,j}}
\tag{23.24}
$$

Insertion of the evolution equations for $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$ given by (23.12) and (23.19) into (23.23) leads, after a little algebra, to

$$
\boxed{
\begin{aligned}
&\textit{Heat equation for thermo-plasticity} \\[4pt]
&\quad Q^m_{kl}\dot{\varepsilon}_{kl} + [\rho c_\varepsilon + L(\frac{\partial f}{\partial \sigma_{mn}}P_{mn} - S)]\dot{\theta} = r - q_{i,i} \\[4pt]
&\textit{where} \\[4pt]
&\quad Q^m_{kl} = \theta P_{kl} - L\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl} \\[4pt]
&\quad L = \frac{1}{A}[(\theta P_{ij} + \sigma_{ij})\frac{\partial g}{\partial \sigma_{ij}} + (K_\alpha - \theta\frac{\partial K_\alpha}{\partial \theta})\frac{\partial g}{\partial K_\alpha}]
\end{aligned}
}
\tag{23.25}
$$

It appears that the symmetric tensor Q^m_{kl} expresses the mechanical coupling term that produces heat due to strain rates.

23.2.1 Specific heat

It appears that due to the contribution $\tilde{c}_\varepsilon$ in (23.8), c_ε now not only depends on the temperature, but $c_\varepsilon = c_\varepsilon(\theta, \varepsilon_{ij} - \varepsilon^p_{ij}, \kappa_\alpha)$. Intuitively, one may expect that the contribution $\tilde{c}_\varepsilon$ is slight and, accordingly, it is natural to seek the most general

circumstances for which $\tilde{c}_\varepsilon = 0$. From (23.9) follows that $\tilde{c}_\varepsilon = 0$, if and only if

$$D_{ijkl}(\theta) = D^0_{ijkl} + (\theta - \theta_0)D^1_{ijkl}$$

$$\beta_{ij} = D_{ijkl}\alpha_{kl} = \text{constant} \qquad (23.26)$$

$$\psi^p(\theta, \kappa_\alpha) = \psi^p_o(\kappa_\alpha) + (\theta - \theta_0)\tilde{\psi}^p(\kappa_\alpha)$$

where D^0_{ijkl} and D^1_{ijkl} are constant quantities. In that case, it is allowed that c_ε is an arbitrary function of the temperature alone. For an isotropic material in which Poisson's ratio is taken as constant whereas the E-modulus may vary with the temperature, we can, in view of (4.89), write D_{ijkl} as $b(\theta)D^0_{ijkl}$ where D^0_{ijkl} is constant and where $b(\theta) = E(\theta)/E_0$. According to (23.26a), we then have

$$D_{ijkl}(\theta) = b(\theta)D^0_{ijkl} = [1 + (\theta - \theta_0)b_1]D^0_{ijkl} \qquad (23.27)$$

where b_1 is a constant, which is in agreement with (23.1). With the result above, (23.26b) gives

$$\alpha_{ij} = \frac{1}{b(\theta)}C^0_{ijkl}\beta_{kl} = \frac{\alpha^0_{ij}}{b(\theta)} \qquad (23.28)$$

Since isotropic materials are considered (where $\alpha_{ij} = \alpha\delta_{ij}$), the expression above implies $E\alpha/(E_0\alpha_0) = 1$. This relation exhibits the right tendency: as E decreases, α increases, cf. Figs. 23.3a) and 23.5a). For carbon iron, (23.1) and (23.3) as well as Tables 23.1 and 23.2 imply that $E\alpha/(E_0\alpha_0)$ varies from 1 to 0.98 in the range $\theta = 294 - 700K$ and for aluminum $E\alpha/(E_0\alpha_0)$ varies from 1 to 0.93 in the range $\theta = 294 - 600K$. However, in general, the E-modulus must be zero when the material is in its liquid phase and $E\alpha/(E_0\alpha_0) = 1$ would then imply that the thermal expansion coefficient should be infinitely large. This absurdity emphasizes that (23.27) and (23.28) are far from being general.

In general, it is therefore concluded that the contribution $\tilde{c}_\varepsilon$ present in (23.8) is not zero and it is of interest to evaluate its magnitude; isotropy is assumed. For simplicity take ψ^p as a linear function of the temperature so that its contribution to $\tilde{c}_\varepsilon$ disappears. Moreover, based on Figs. 23.3a) and 23.4 it is a fair approximation to assume the variation of E to be linear and v to be constant. Since we then have $d^2D_{ijkl}/d\theta^2 = 0$, (23.9) reduces to

$$\tilde{c}_\varepsilon \approx \frac{\theta}{\rho}(\varepsilon_{ij} - \varepsilon^p_{ij})\frac{d^2}{d\theta^2}[D_{ijkl}\alpha_{kl}(\theta - \theta_0)] = \frac{\theta}{\rho}\frac{\varepsilon_{ii} - \varepsilon^p_{ii}}{1 - 2v}\frac{d^2}{d\theta^2}[E\alpha(\theta - \theta_0)]$$

For uniaxial stress conditions, $\varepsilon_{ii} - \varepsilon^p_{ii} = \varepsilon_{ii} = (1 - 2v)\varepsilon^e_{11}$ holds. Finally, with not only E, but also α taken as a linear function of the temperature, cf. Fig. 23.5a), the expression above reduces to

$$\tilde{c}_\varepsilon \approx 2\frac{\theta}{\rho}[\frac{dE}{d\theta}\frac{d\alpha}{d\theta}(\theta - \theta_0) + \frac{dE}{d\theta}\alpha + E\frac{d\alpha}{d\theta}]\varepsilon^e_{11}$$

Consider aluminum, for instance, and take the linear variations of E and α given by (23.1) and (23.3) and Tables 23.1 and 23.2; moreover $\rho \approx 2700$ kg/m^3 and $\varepsilon_{11}^e \approx 10^{-3}$. Then for the temperature in the range $\theta = 294 - 600K$, we obtain at most $\tilde{c}_\varepsilon \approx -0.3J/kgK$ and as c_ε for aluminum is about 900 J/kg$K$, cf. Fig. 23.6a), this gives $\tilde{c}_\varepsilon / c_\varepsilon \approx -3 \cdot 10^{-4}$ and $\tilde{c}_\varepsilon$ indeed turns out to be very small.

23.3 Isotropic hardening von Mises plasticity

As an illustration of thermo-plasticity, we will consider isotropic hardening von Mises plasticity and derive the incremental constitutive relation as well as the corresponding heat equation. For simplicity, only the initial yield stress σ_{yo} is assumed to be temperature dependent and Helmholtz' free energy is given by (23.4). The yield function is then given by

$$f = f(\sigma_{ij}, K, \theta) = (\frac{3}{2}s_{ij}s_{ij})^{1/2} - \sigma_y(\theta, K)$$

where $\sigma_y(\theta, K) = \sigma_{yo}(\theta) + K(\kappa)$. Isotropy is assumed, i.e.

$$D_{ijkl} = 2G[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}]; \quad \alpha_{ij} = \alpha\delta_{ij}$$

The evolution laws for associated plasticity provide

$$\dot{\varepsilon}_{ij}^p = \lambda\frac{\partial f}{\partial \sigma_{ij}} = \lambda\frac{3s_{ij}}{2\sigma_y}; \quad \dot{\kappa} = -\lambda\frac{\partial f}{\partial K} = \lambda$$

and it follows that $\dot{\varepsilon}_{eff}^p = (\frac{2}{3}\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p)^{1/2} = \lambda$. It also follows from (23.20) that

$$A = H + \frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\frac{\partial f}{\partial \sigma_{kl}} = H + 3G$$

where $H = \rho\partial^2\psi^p(\kappa)/\partial\kappa^2$ is temperature independent. Moreover

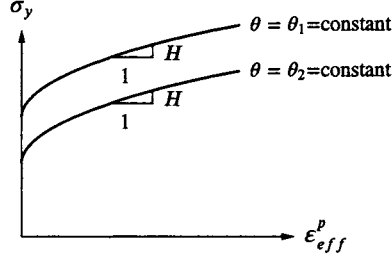$$D_{ijkl}^{ep} = D_{ijkl} - \frac{9G^2}{A}\frac{s_{ij}s_{kl}}{\sigma_y^2} \tag{23.29}$$

holds as usual, cf. (12.16).

The consistency relation $\dot{f} = 0$ is given by (23.14) and we obtain

$$\dot{\sigma}_{eff} - H\lambda + S\dot{\theta} = 0$$

At yield, $\sigma_{eff} = \sigma_y$ holds and for isothermal conditions, the usual condition

$$\frac{\partial \sigma_y}{\partial \varepsilon_{eff}^p} = H$$

Figure 23.9: Relation between σ_y and ε^p_{eff}; only the initial yield stress σ_{yo} is influenced by the temperature.

holds, cf. (12.12). Since H is temperature independent, the response shown in Fig. 23.9 is obtained. As the initial yield stress σ_{yo} only depends on temperature, the two curves in Fig. 23.9 are identical apart from a shift along the σ_y-axis.

To determine β^{ep}_{ij} present in the incremental relation $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\varepsilon}_{kl} - \beta^{ep}_{ij}\dot{\theta}$, we first calculate β_{ij} according to

$$\beta_{ij} = D_{ijkl}\alpha_{kl} = \frac{E}{1-2v}\alpha\delta_{ij} = 3K\alpha\delta_{ij}$$

Since only the initial yield stress σ_{yo} is assumed to depend on the temperature, (23.20) provides

$$P_{ij} = \beta_{ij} + \frac{d\beta_{ij}}{d\theta}(\theta - \theta_0) - \frac{dD_{ijkl}}{d\theta}(\varepsilon_{kl} - \varepsilon^p_{kl}) = \beta_{ij} = 3K\alpha\delta_{ij}$$

It then follows that

$$\frac{\partial f}{\partial\sigma_{mn}}P_{mn} = 0$$

Moreover

$$S = \frac{\partial f}{\partial\theta} + \frac{\partial f}{\partial K_\alpha}\frac{\partial K_\alpha}{\partial\theta} = \frac{\partial f}{\partial\theta} = -\frac{d\sigma_{y0}}{d\theta}$$

We then obtain from (23.20)

$$\beta^{ep}_{ij} = P_{ij} + \frac{1}{A}D_{ijkl}\frac{\partial f}{\partial\sigma_{kl}}(S - \frac{\partial f}{\partial\sigma_{mn}}P_{mn}) = 3K\alpha\delta_{ij} - \frac{3G}{A}\frac{s_{ij}}{\sigma_y}\frac{d\sigma_{yo}}{d\theta}$$

With this expression, as well as (23.29), the incremental constitutive relation becomes

$$\boxed{\dot{\sigma}_{ij} = (D_{ijkl} - \frac{9G^2}{A}\frac{s_{ij}s_{kl}}{\sigma_y^2})\dot{\varepsilon}_{kl} - (3K\alpha\delta_{ij} - \frac{3G}{A}\frac{s_{ij}}{\sigma_y}\frac{d\sigma_{yo}}{d\theta})\dot{\theta}}$$

in accordance with Ray and Utko (1989).

With these results, it is trivial to identify the heat equation and from (23.25) we obtain

$$(\theta 3K\alpha\delta_{ij} - \frac{3G}{H+3G}\frac{\sigma_{yo}}{\sigma_y}s_{ij})\dot{\varepsilon}_{ij} + (\rho c_\varepsilon + \frac{\sigma_{yo}}{H+3G}\frac{d\sigma_{yo}}{d\theta})\dot{\theta} = r - q_{i,i}$$

For adiabatic loading where $r - q_{i,i} = 0$ this expression may be compared with the Kelvin expression (21.92) for thermo-elasticity. It appears that plastic effects create a significant heat generation.

23.4   Field equations - Finite element formulation

Having evaluated in detail the mechanical constitutive equation and the heat equation, we will now discuss the field equations of thermo-plasticity as well as their finite element formulation.

Considering first the mechanical part of the problem, the equations of motion - i.e. the balance equations - read

$$\boxed{\sigma_{ij,j} + b_i = \rho \ddot{u}_i} \tag{23.30}$$

where b_i is the body force per unit volume and u_i is the displacement vector. The kinematic relation is

$$\boxed{\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})} \tag{23.31}$$

The second balance equation that needs to be considered is given by the energy equation. From (23.23) we have

$$\boxed{\rho c_\varepsilon \dot{\theta} = r - q_{i,i} + Q} \tag{23.32}$$

where the heat flux vector q_i is related to the temperature gradient $\theta_{,i}$ by the constitutive equation in terms of Fourier's law, i.e.

$$\boxed{q_i = -k_{ij}\theta_{,j}} \tag{23.33}$$

The mechanical field equations consist of (23.30) and (23.31) and the thermal field equations are given by (23.32) and (23.33). It appears that, in general, these field equations are coupled: as the stresses depend on the temperature, we cannot solve the mechanical field equations without knowledge of the temperature and - on the other hand - the thermal field equations cannot be solved without knowledge of the strains and stresses. Instead of (23.32) we could use (23.25) which directly links strain rates and temperature changes. However, (23.25) applies only to thermo-plasticity whereas (23.32) also holds for

viscoplasticity and creep if $\dot{\varepsilon}_{ij}^{p}$ is replaced by $\dot{\varepsilon}_{ij}^{vp}$ and $\dot{\varepsilon}_{ij}^{c}$ respectively. Consequently, by adopting (23.32) we obtain a very general approach which not only applies to plasticity, but also to viscoplasticity and creep.

To derive the finite element formulation of the field equations, we begin with the equations of motion given by (23.30); first the corresponding weak form is obtained and then the FE-approximations are introduced. As this task has already been performed in Section 3.8, we will only repeat the final relations. From (16.9) we have

$$
\begin{aligned}
&\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{\psi} = \boldsymbol{0} \\
&\textit{where} \\
&\boldsymbol{M} = \int_V \rho \boldsymbol{N}^T \boldsymbol{N}\, dV \\
&\boldsymbol{\psi} = \int_V \boldsymbol{B}^T \boldsymbol{\sigma}\, dV - \boldsymbol{f} \\
&\boldsymbol{f} = \int_S \boldsymbol{N}^T \boldsymbol{t}\, dS + \int_V \boldsymbol{N}^T \boldsymbol{b}\, dV
\end{aligned}
\tag{23.34}
$$

where $\boldsymbol{M}$ is the *mass matrix*, $\boldsymbol{\psi}$ represents the *out-of-balance forces* and $\boldsymbol{f}$ denotes the *external forces*.

To obtain the weak form of the heat equation (23.32), we proceed in a similar manner as in Section 3.8. Therefore, (23.32) is multiplied by the arbitrary weight function v and integrated over the region V, i.e.

$$
\int_V v \rho c_\varepsilon \dot{\theta}\, dV = \int_V vr\, dV - \int_V v q_{i,i}\, dV + \int_V vQ\, dV
\tag{23.35}
$$

From the divergence theorem of Gauss, it follows that

$$
\begin{aligned}
\int_V v q_{i,i}\, dV &= \int_V (v q_i)_{,i}\, dV - \int_V v_{,i} q_i\, dV \\
&= \int_S v q_n\, dS - \int_V v_{,i} q_i\, dV
\end{aligned}
$$

where the flux q_n is defined by $q_n = q_i n_i$; here n_i is the unit vector normal to the boundary S of the body and directed outwards. Use of the expression above in (23.35) gives

$$
\int_V v \rho c_\varepsilon \dot{\theta}\, dV = \int_V vr\, dV - \int_S v q_n\, dS + \int_V v_{,i} q_i\, dV + \int_V vQ\, dV
\tag{23.36}
$$

To rewrite this expression in matrix form, define the following matrices

$$
\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} ; \quad \nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}
\tag{23.37}
$$

where ∇ is the *gradient operator*, i.e. $v_{,i} = \nabla v$. Similarly to the finite element formulation of the mechanical part, the temperature within the body is approximated by

$$\theta(x_i, t) = N_\theta(x_i)a_\theta(t)$$

where N_θ denotes the global shape function matrix and a_θ is a column matrix that contains all the nodal values of the temperature. It follows that

$$\dot{\theta} = N_\theta \dot{a}_\theta \tag{23.38}$$

and that the temperature gradient $\theta_{,i} = \nabla \theta$ is given by

$$\nabla\theta = B_\theta a_\theta \quad \text{where} \quad B_\theta = \nabla N_\theta$$

In evident notation, Fourier's law (23.33) then reads

$$q = -k\nabla\theta = -kB_\theta a_\theta \tag{23.39}$$

Adopting again Galerkin's method, the arbitrary weight function v is chosen as

$$v = N_\theta c_\theta = c_\theta^T N_\theta^T ; \quad \nabla v = B_\theta c_\theta \tag{23.40}$$

where c_θ is a constant, but arbitrary column matrix. Insertion of (23.40) into (23.36), where $v_{,i}q_i = (\nabla v)^T q$, and c_θ is independent of the coordinates lead to

$$c_\theta^T [\int_V N_\theta^T \rho c_\varepsilon \dot{\theta}\, dV - \int_V N_\theta^T r\, dV$$
$$+ \int_S N_\theta^T q_n\, dS - \int_V B_\theta^T q\, dV - \int_V N_\theta^T Q\, dV] = 0$$

and since c_θ is arbitrary, it is concluded that

$$\int_V N_\theta^T \rho c_\varepsilon \dot{\theta}\, dV - \int_V N_\theta^T r\, dV$$
$$+ \int_S N_\theta^T q_n\, dS - \int_V B_\theta^T q\, dV - \int_V N_\theta^T Q\, dV = 0$$

Finally, insertion of (23.38) and (23.39) leads to the following finite element formulation

$$\boxed{\begin{aligned}
&C\dot{a}_\theta + K_\theta a_\theta - Q - f_\theta = 0 \\
&\text{where} \\
&C = \int_V N_\theta^T \rho c_\varepsilon N_\theta\, dV \\
&K_\theta = \int_V B_\theta^T k B_\theta\, dV \\
&Q = \int_V N_\theta^T Q\, dV \\
&f_\theta = \int_V N_\theta^T r\, dV - \int_S N_\theta^T q_n\, dS
\end{aligned}} \tag{23.41}$$

It appears that C is the *capacity matrix*, K_θ is the *thermal stiffness matrix* (or *conductivity matrix*) and f_θ is the *thermal external force*; moreover, Q expresses the amount by which the strain rate produces heat.

We will now discuss numerical techniques to solve the mechanical balance equation (23.34) and the thermal balance equation (23.41). For simplicity we will ignore inertia effects, and the equations of motion (23.34) then reduce to the static equilibrium equations given by

$$\boxed{\psi = 0}$$

The cases of uncoupled thermo-plasticity, coupled thermo-plasticity and adiabatic heating will now be dealt with.

23.5 Solution of uncoupled thermo-plasticity

For thermo-elasticity, we have shown in relation to (21.99) that very often the mechanical response influences the heat equation only insignificantly. For thermo-plasticity, if the external heat supply expressed by f_θ in (23.41) is significantly larger than the heat generation due to the mechanical response, we can ignore this latter part and a situation of uncoupled thermo-plasticity has arisen.

23.5.1 Thermal field equations

Since the heat generation due to the mechanical response is ignored, we have

$$Q = 0$$

and (23.41) then reduces to the uncoupled heat equation

$$\boxed{C\dot{a}_\theta + K_\theta a_\theta - f_\theta = 0} \tag{23.42}$$

Since the specific heat capacity c_ϵ as well as conductivity matrix k may depend on the temperature, C and K_θ may also depend on the temperature

$$\boxed{C = C(\theta); \quad K_\theta = K_\theta(\theta)}$$

As a result, (23.42) comprises in general a set of nonlinear ordinary differential equations. Like the discussion in Chapter 19, we will first transform this set of nonlinear ordinary differential equations into a set of nonlinear algebraic equations.

This transformation is achieved by introducing a *time integration scheme* and we will here adopt the *generalized Euler scheme* according to which

$$a_{\theta,n+1} = a_{\theta,n} + \Delta t[(1 - \gamma)\dot{a}_{\theta,n} + \gamma\dot{a}_{\theta,n+1})] \tag{23.43}$$

where $\Delta t = t_{n+1} - t_n$ is the time step. In this expression, $a_{\theta,n}$ and $\dot{a}_{\theta,n}$ are known quantities at state n and we seek $a_{\theta,n+1}$ and $\dot{a}_{\theta,n+1}$ at the new state $n + 1$; moreover, γ is a parameter we choose in the range $0 \le \gamma \le 1$.

Explicit scheme

To obtain the *explicit scheme*, the *Euler forward scheme*, we choose $\gamma = 0$ and (23.43) then gives

$$\dot{a}_{\theta,n} = \frac{1}{\Delta t}(a_{\theta,n+1} - a_{\theta,n}) \qquad (23.44)$$

The balance equation (23.42) is now evaluated at state n, i.e.

$$C_n \dot{a}_{\theta,n} + K_{\theta,n} a_{\theta,n} - f_{\theta,n} = 0$$

where C_n, $K_{\theta,n}$ and $f_{\theta,n}$ are the respective quantities evaluated at the known state n. Insertion of (23.44) gives

$$C_n a_{\theta,n+1} = (C_n - \Delta t K_{\theta,n}) a_{\theta,n} + \Delta t f_{\theta,n} \qquad (23.45)$$

Since all terms on the right-hand side of this expression are known, $a_{\theta,n+1}$ can be determined directly. This direct solution also holds for nonlinear heat problems where C and K_θ depend on the temperature. This very fortunate situation is further enhanced if the capacity matrix C is taken as *lumped*, i.e. diagonal. In that case the inversion of C_n is trivial, and (23.45) then gives the solution directly without, in reality, solving any equation system.

This situation is quite similar to the explicit method discussed for dynamic mechanical problems, cf. (19.13). However, we have to pay a corresponding price for this simplicity namely that of *conditional stability*. Instability means that any minor error introduced by truncation of numbers in the computer will jeopardize the solution in the sense that errors will accumulate with time and very quickly render the solution meaningless. On the other hand, conditional stability means that in order to maintain stability, the time step Δt must be within a certain limit. In the present case, it turns out that we must have

$$\Delta t \leq \frac{T_s}{\pi} \quad \Rightarrow \quad stability$$

where T_s is the shortest period for the *eigenmodes* of the finite element assemblage that comprises the discretized body, cf. for instance Bathe (1996) or Hughes (1977).

Implicit scheme

To obtain an *implicit scheme*, we choose $\gamma > 0$ and (23.43) then gives

$$\dot{a}_{\theta,n+1} = \frac{1}{\gamma \Delta t}(a_{\theta,n+1} - a_{\theta,n}) - \frac{1 - \gamma}{\gamma} \dot{a}_{\theta,n}$$

However, whereas $\gamma \geq 1/2$ implies *unconditional stability* for linear problems, only $\gamma = 1$, i.e. the *backward Euler scheme*, results in unconditional stability

for nonlinear problems, cf. Hughes (1977). Thus, $\gamma = 1$ will be assumed in the following and the relation above then reduces to

$$\dot{a}_{\theta,n+1} = \frac{1}{\Delta t}(a_{\theta,n+1} - a_{\theta,n})$$

(23.46)

We might mention that a time integration scheme (the 'α-method') closely related to (23.43) exists, which is unconditionally stable for $\gamma \geq 1/2$ even for nonlinear problems, see Hughes (1977) and Bathe (1996). From (23.46) and since $\theta = N_\theta a_\theta$ we also have

$$\dot{\theta}_{n+1} = \frac{1}{\Delta t}(\theta_{n+1} - \theta_n)$$

(23.47)

The balance equation (23.42) is now evaluated at the state $n + 1$, i.e.

$$C_{n+1}\dot{a}_{\theta,n+1} + K_{\theta,n+1}a_{\theta,n+1} - f_{\theta,n+1} = 0$$

Dropping the subscript $n + 1$ and insertion of (23.46) provides the sought for residual format for the heat equation

$$\psi_\theta(a_\theta) = 0$$
$$where$$
$$\psi_\theta(a_\theta) = \frac{1}{\Delta t}C(a_\theta - a_{\theta,n}) + K_\theta a_\theta - f_\theta$$

(23.48)

Whereas $a_{\theta,n}$ and the external force $f_\theta = f_{\theta,n+1}$ are known quantities, a_θ, C and K_θ are unknown. Clearly, (23.48) is a nonlinear equation system in a_θ and an iterative solution method must be applied; a natural procedure is to adopt the *Newton-Raphson scheme*, see Chapter 17. Therefore, assume that the approximation a_θ^{i-1} $(i = 1, 2, \dots)$ to the true solution a_θ has been established. Ignoring higher-order terms, a Taylor series expansion of ψ_θ about a_θ^{i-1} gives

$$\psi_\theta(a_\theta^i) = \psi_\theta(a_\theta^{i-1}) + (\frac{\partial \psi_\theta}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1})$$

According to the Newton-Raphson method, we require $\psi(a_\theta^i) = 0$, i.e.

$$0 = \psi_\theta(a_\theta^{i-1}) + (\frac{\partial \psi_\theta}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1})$$

(23.49)

In order to evaluate the partial derivative above, it is first noted that (23.41) implies

$$\frac{1}{\Delta t}C(a_\theta - a_{\theta,n}) = \int_V N_\theta^T \dot{\theta}\rho c_\varepsilon \, dV$$

(23.50)

where $\dot{\theta} = \dot{\theta}_{n+1}$ is defined by (23.47). By using $c_\varepsilon = c_\varepsilon(\theta)$, two useful results can now be established

$$\frac{\partial \dot{\theta}}{\partial a_\theta} = \frac{1}{\Delta t} N_\theta \; ; \qquad \frac{\partial c_\varepsilon}{\partial a_\theta} = \frac{dc_\varepsilon}{d\theta} \frac{\partial \theta}{\partial a_\theta} = \frac{dc_\varepsilon}{d\theta} N_\theta$$

From the results above it then follows that

$$\frac{1}{\Delta t} \frac{\partial [C(a_\theta - a_{\theta,n})]}{\partial a_\theta} = \tilde{C} + \frac{1}{\Delta t} C \tag{23.51}$$

where

$$\boxed{\tilde{C} = \int_V N_\theta^T \rho \dot{\theta} \frac{dc_\varepsilon}{d\theta} N_\theta \, dV} \tag{23.52}$$

Likewise, from (23.41) we have

$$K_\theta a_\theta = \int_V B_\theta^T k \nabla \theta \, dV \quad \text{where} \quad \nabla \theta = B_\theta a_\theta$$

Since $k = k(\theta)$, observation of (23.37) gives

$$\frac{\partial (k \nabla \theta)}{\partial a_\theta} = \frac{dk}{d\theta} \nabla \theta \frac{\partial \theta}{\partial a_\theta} + k \frac{\partial \nabla \theta}{\partial a_\theta} = \frac{dk}{d\theta} \nabla \theta N_\theta + k B_\theta$$

The above expressions then imply

$$\frac{\partial (K_\theta a_\theta)}{\partial a_\theta} = \tilde{K}_\theta + K_\theta \tag{23.53}$$

where

$$\boxed{\tilde{K}_\theta = \int_V B_\theta^T \frac{dk}{d\theta} \nabla \theta N_\theta \, dV} \tag{23.54}$$

Finally, use of (23.51) and (23.53) in (23.48b) provides

$$\frac{\partial \psi_\theta}{\partial a} = \tilde{C} + \frac{1}{\Delta t} C + \tilde{K}_\theta + K_\theta$$

and insertion into (23.49) yields the sought for Newton-Raphson scheme

$$\boxed{\begin{aligned} &(\tilde{C}^{i-1} + \frac{1}{\Delta t} C^{i-1} + \tilde{K}_\theta^{i-1} + K_\theta^{i-1})(a_\theta^i - a_\theta^{i-1}) = -\psi_\theta^{i-1} \\ &where \\ &\psi_\theta^{i-1} = \frac{1}{\Delta t} C^{i-1}(a_\theta^{i-1} - a_{\theta,n}) + K_\theta^{i-1} a_\theta^{i-1} - f_{\theta,n+1} \end{aligned}} \tag{23.55}$$

where $a_{\theta,n}$ and $f_{\theta,n+1}$ are constant given quantities. As in Chapter 17 we choose as starting vectors ($i = 1$) the values at the last accepted state n, that is, $a_\theta^0 = a_{\theta,n}$

and the iterations are stopped when the convergence criterion/criteria similar to those discussed in Section 17.7 are satisfied and then $a_{n+1} = a^i$.

It is evident that the computational effort to establish $\tilde{K}_\theta$ is significant. Moreover, as $\tilde{C}$ and $\tilde{K}_\theta$ are corrections to C and K_θ respectively, which vary slowly with temperature, cf. Figs. 23.6 and 23.7, it seems reasonable to ignore them in (23.55) and thereby obtain a modified Newton-Raphson scheme without significant loss of convergence speed, cf. Celentano *et al.* (1994). The slow temperature variation of C and K_θ also motivates accepting constant values and only occasionally making an update; this is, in fact, frequently adopted, cf. Bathe (1996). If the contributions $\tilde{C}$ and $\tilde{K}$ (which is unsymmetric) are ignored, the coefficient matrix in (23.55) becomes $\frac{1}{\Delta t}C^{i-1} + K_\theta^{i-1}$, which is symmetric and positive definite, cf. the definitions in (23.41). These are precisely the prerequisites for application of the quasi-Newton method in terms of the BFGS-method, cf. Section 17.8, and this route was pursued by Argyris *et al.* (1981).

23.5.2   Mechanical field equations

Having discussed the solution of the uncoupled heat equation in some detail, we next turn to the mechanical part of the problem. From (23.6) σ depends both on the strains and the temperature, that is, the stresses depend both on the nodal displacements a and the nodal temperatures a_θ. Assume that a_n and $a_{\theta,n}$ at state n are known. The external force is then changed to f_{n+1}. Again dropping for convenience the subscript $n + 1$ on quantities that depend on a_{n+1} and $a_{\theta,n+1}$, and as only static problems are considered, we have from (23.34)

$$\psi(a, a_\theta) = 0 \tag{23.56}$$

where the out-of-balance forces ψ are given by

$$\psi(a, a_\theta) = \int_V B^T \sigma \, dV - f \tag{23.57}$$

where f denotes the known external forces at state $n + 1$.

As the heat problem is solved independently to provide the temperature changes, the temperature field is now known when the mechanical problem is considered. Since the mechanical problem has already been treated in Chapter 17 we will here simply reiterate the equations for a Newton-Raphson scheme. From (17.26) and (17.32) the iteration scheme is given by

$$(K_t)^{i-1}(a^i - a^{i-1}) = -\psi(a^{i-1}, a_{\theta,n+1})$$

where

$$\psi(a^{i-1}, a_{\theta,n+1}) = \int_V B^T \sigma^{i-1} \, dV - f_{n+1}$$

and where

$$K_t = \int_V B^T D_t B dV$$

The starting vector for a^{i-1} for $i = 1$ is taken as $a^0 = a_n$, that is the value at the last accepted state n.

It appears that the only way in which the temperature field influences the mechanical field equations is through the stresses which, in turn, determine the internal forces $\int_V B^T \sigma dV$; this is the topic of the next section.

23.5.3 Numerical treatment of the constitutive relations

Evidently, we need to integrate the thermo-plastic constitutive relations to obtain the current stresses. Fortunately, it turns out that many of the results in Chapter 18 can be used. Here we will use methods that directly fulfill the yield criterion.

The first task is to consider whether plastic strains will develop during the load increment. Adopting the approach utilized in Section 18.1.1, a trial stress is defined assuming that no plastic strains will develop during the increment. Denote the current state and the last known state in equilibrium as state 2 and state 1 respectively. Evaluating (23.6) at state 2 and state 1 then results in

$$\sigma_{ij}^{(2)} = D_{ijkl}(\varepsilon_{kl}^{(2)} - \varepsilon_{kl}^{p(2)}) - \beta_{ij}(\theta^{(2)} - \theta_o)$$

and

$$\sigma_{ij}^{(1)} = D_{ijkl}(\varepsilon_{kl}^{(1)} - \varepsilon_{kl}^{p(1)}) - \beta_{ij}(\theta^{(1)} - \theta_o)$$

where for simplicity it was assumed that D_{ijkl} and β_{ij} are constant. Subtracting the above equations

$$\boxed{\sigma_{ij}^{(2)} = \sigma_{ij}^t - D_{ijkl}\Delta\varepsilon_{kl}^p} \qquad (23.58)$$

where the trial stress is defined as

$$\boxed{\sigma_{ij}^t = \sigma_{ij}^{(1)} + \Delta\sigma_{ij}^{te} \qquad where \qquad \Delta\sigma_{ij}^{te} = D_{ijkl}\Delta\varepsilon_{kl} - \beta_{ij}\Delta\theta} \qquad (23.59)$$

and

$$\Delta\varepsilon_{kl} = \varepsilon_{kl}^{(2)} - \varepsilon_{kl}^{(1)} \qquad \Delta\varepsilon_{kl}^p = \varepsilon_{kl}^{p(2)} - \varepsilon_{kl}^{p(1)} \qquad \Delta\theta = \theta^{(2)} - \theta^{(1)}$$

From (23.58) it can then be concluded that if no plastic strains develop, we have $\sigma_{ij}^{(2)} = \sigma_{ij}^t$.

The yield function is now evaluated for the trial stresses according to

$$f^t = f(\sigma_{ij}^t, K_\alpha^{(1)}, \theta^{(2)})$$

It then follows that

$$
\boxed{
\begin{array}{l}
f^t \le 0 \quad \Rightarrow \quad \textit{thermo-elastic response } \sigma_{ij}^{(2)} = \sigma_{ij}^t \\
f^t > 0 \quad \Rightarrow \quad \textit{thermo-plastic response}
\end{array}
}
$$

For a thermo-plastic response, the constitutive relations must be integrated numerically. In the uncoupled approach, the temperature is known when the integration of the constitutive relations is considered. This allows for the use of the methods described in Section 18.1.4. Let us summarize the set of equations considered in the integration procedure

$$
\sigma = \sigma^t - \Delta\lambda D \frac{\partial g}{\partial \sigma}
$$

$$
K = K(\kappa^{(1)} - \Delta\lambda \frac{\partial g}{\partial K}, \theta) \tag{23.60}
$$

$$
f = f(\sigma, K, \theta) = 0
$$

where, for simplicity, it is assumed that only one hardening parameter exists $K = K(\kappa, \theta)$. Moreover, in (23.60) the superscript 2 was omitted.

A comparison of (23.60) with (18.35) reveals a complete similarity. In fact, by replacing σ^t, which is a known quantity, in (18.35) with (23.59) and considering θ as a given quantity, the system in (18.35) can be directly used for the calculation of stresses. As an example, the solution for a thermo-plastic von Mises material directly follows the derivations in Section 18.1.4 and we will therefore not repeat the integration procedures here. Accordingly, the algorithmic tangent stiffness is unchanged and the results given in Section 18.1.5 hold.

23.6 Solution of coupled thermo-plasticity

We will now discuss solution strategies in the general case of thermo-plasticity where heat flow is accounted for and the heat generation due to plastic dissipation is significant. This problem has been the center of increasing interest during recent years due to its importance in metal-forming processes and development of shear bands; the latter topic is, for instance, treated by Lemonds and Needleman (1986a,b). In Section 23.8 an alternative solution method will be discussed: the staggered solution scheme.

Define $\tilde{a}$ and $\tilde{\psi}$ by

$$
\tilde{a} = \begin{bmatrix} a \\ a_\theta \end{bmatrix} ; \quad \tilde{\psi} = \begin{bmatrix} \psi \\ \psi_\theta \end{bmatrix}
$$

that is, the nodal quantities $\tilde{a}$ and the out-of-balance forces $\tilde{\psi}$ include both the mechanical and thermal parts. In the spirit of the general iteration format for

equilibrium iterations provided by the 'A-matrix' approach, cf. (17.18), we then have

$$A^{i-1}(\tilde{a}^i - \tilde{a}^{i-1}) = -\tilde{\psi}(\tilde{a}^{i-1}) \tag{23.61}$$

where $i = 1, 2, \ldots$ and $\tilde{a}^0 = \tilde{a}_n$. The only restrictions placed on the iteration matrix A are given by (17.16) and (17.19), i.e.

$$\det A \neq 0 \quad \text{and} \quad \det A^{-1} \neq 0$$

To find information on relevant choices of the A-matrix, we will adopt the Newton-Raphson scheme.

23.6.1 Thermal field equations

Adopt the backward Euler scheme

$$\dot{a}_{n+1} = \frac{1}{\Delta t}(a_{n+1} - a_n); \quad \dot{a}_{\theta,n+1} = \frac{1}{\Delta t}(a_{\theta,n+1} - a_{\theta,n})$$

Evaluation of the coupled heat equation given by (23.41) at state $n + 1$, and dropping the subscript $n + 1$ on quantities that depend on a_{n+1} and $a_{\theta,n+1}$, we obtain

$$\boxed{\begin{aligned} &\psi_\theta(a, a_\theta) = 0 \\ &where \\ &\psi_\theta(a, a_\theta) = \frac{1}{\Delta t}C(a_\theta - a_{\theta,n}) + K_\theta a_\theta - Q - f_\theta \end{aligned}} \tag{23.62}$$

The quantities a_n, $a_{\theta,n}$ and $f_{\theta,n+1}$ are known quantities, i.e. the thermal out-of-balance force ψ_θ is a function of a and a_θ as already indicated above.

For the continued treatment, the quantity Q needs some attention. From (23.41) we have

$$Q = \int_V N_\theta^T Q \, dV \tag{23.63}$$

where Q is evaluated at state $n + 1$. Moreover, from (23.23) we can identify

$$Q = \sigma^T \dot{\varepsilon}^p - K^T \dot{\kappa} - \theta P^T (\dot{\varepsilon} - \dot{\varepsilon}^p) + \theta (\frac{\partial K}{\partial \theta})^T \dot{\kappa} \tag{23.64}$$

where the different rates are evaluated at state $n + 1$ and approximated using the backward Euler scheme, i.e.

$$\dot{\varepsilon} = \frac{1}{\Delta t}(\varepsilon_{n+1} - \varepsilon_n); \quad \dot{\varepsilon}^p = \frac{1}{\Delta t}(\varepsilon_{n+1}^p - \varepsilon_n^p); \quad \dot{\kappa} = \frac{1}{\Delta t}(\kappa_{n+1} - \kappa_n) \tag{23.65}$$

Then we can write (23.64) as

$$Q = \frac{\Delta Q}{\Delta t} \tag{23.66}$$

where

$$\Delta Q = \sigma^T \Delta \varepsilon^p - K^T \Delta \kappa - \theta P^T (\Delta \varepsilon - \Delta \varepsilon^p) + \theta (\frac{\partial K}{\partial \theta})^T \Delta \kappa \tag{23.67}$$

With these results we are ready to consider the solution method for the non-linear equation system (23.62); the Newton-Raphson scheme will be adopted. Therefore, assume that the approximation a^{i-1} and a_θ^{i-1} ($i = 1, 2, \ldots$) to the true solution a_{n+1} and $a_{\theta, n+1}$ has been established. Ignoring higher order terms, a Taylor series expansion of ψ_θ about a^{i-1}, a_θ^{i-1} gives

$$\psi_\theta(a^i, a_\theta^i) = \psi_\theta(a^{i-1}, a_\theta^{i-1})$$
$$+ (\frac{\partial \psi_\theta}{\partial a})^{i-1}(a^i - a^{i-1}) + (\frac{\partial \psi_\theta}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1})$$

According to the Newton-Raphson method, we require $\psi_\theta(a^i, a_\theta^i) = 0$, i.e.

$$0 = \psi_\theta(a^{i-1}, a_\theta^{i-1}) + (\frac{\partial \psi_\theta}{\partial a})^{i-1}(a^i - a^{i-1}) + (\frac{\partial \psi_\theta}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1}) \tag{23.68}$$

Similar to the analysis given in Section 23.5.1, it is easily shown that

$$\frac{\partial \psi_\theta}{\partial a_\theta} = \tilde{C} + \frac{1}{\Delta t}C + \tilde{K}_\theta + K_\theta - \frac{1}{\Delta t}H \tag{23.69}$$

where $\tilde{C}$ and $\tilde{K}$ are defined by (23.52) and (23.54) respectively and

$$\boxed{H = \int_V N_\theta^T \frac{\partial \Delta Q}{\partial a_\theta} dV} \tag{23.70}$$

In the same fashion, it is easily shown that

$$\frac{\partial \psi_\theta}{\partial a} = \tilde{G} - \frac{1}{\Delta t}G \tag{23.71}$$

where

$$\boxed{\tilde{G} = \int_V N_\theta^T \dot{\theta} \rho \frac{\partial c_\varepsilon}{\partial a} dV \; ; \qquad G = \int_V N_\theta^T \frac{\partial \Delta Q}{\partial a} dV} \tag{23.72}$$

These partial derivatives will be addressed later on when the stress calculations are considered. We also note that according to (23.8), $c_\varepsilon = -\theta d^2 h(\theta)/d\theta^2 + \tilde{c}_\varepsilon$ where $\tilde{c}_\varepsilon$ depends on the deformation; however, it was shown in Section 23.2.1

that $\tilde{c}_\varepsilon / c_\varepsilon \approx 0$ so we expect the matrix $\tilde{G}$ to be very small compared with the other matrices entering the problem.

Finally, insertion of (23.71) and (23.69) into the Newton-Raphson scheme (23.68) results in

$$T_a^{i-1}(a^i - a^{i-1}) + T_\theta^{i-1}(a_\theta^i - a_\theta^{i-1}) = -\psi_\theta^{i-1} \tag{23.73}$$

where

$$\boxed{\begin{aligned} T_a &= \tilde{G} - \frac{1}{\Delta t} G \\ T_\theta &= \tilde{C} + \frac{1}{\Delta t} C + \tilde{K}_\theta + K_\theta - \frac{1}{\Delta t} H \end{aligned}} \tag{23.74}$$

23.6.2   Mechanical field equations

From (23.34) we have $\psi(a, a_\theta) = 0$ and again we seek a Newton-Raphson scheme for this nonlinear equation system. Assume that the approximation a^{i-1} and a_θ^{i-1} to the state $n + 1$ are known and make a Taylor series expansion about the state a^{i-1}, a_θ^{i-1}, i.e.

$$\begin{aligned} \psi(a^i, a_\theta^i) = \psi(a^{i-1}, a_\theta^{i-1}) \\ + (\frac{\partial \psi}{\partial a})^{i-1}(a^i - a^{i-1}) + (\frac{\partial \psi}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1}) \end{aligned}$$

and setting $\psi(a^i, a_\theta^i) = 0$ gives

$$\begin{aligned} 0 = \psi(a^{i-1}, a_\theta^{i-1}) \\ + (\frac{\partial \psi}{\partial a})^{i-1}(a^i - a^{i-1}) + (\frac{\partial \psi}{\partial a_\theta})^{i-1}(a_\theta^i - a_\theta^{i-1}) \end{aligned} \tag{23.75}$$

To obtain the Newton-Raphson scheme we must obviously evaluate the above partial derivatives. For the time being, let us observe that it is possible to evaluate these partial derivative using the algorithmic relation

$$d\sigma = D_{ats} d\varepsilon - \beta_{ats} d\theta$$

i.e.

$$d\sigma = D_{ats} B da - \beta_{ats} N_\theta da_\theta$$

The establishment of this relation will be performed later on in this section. From (23.34) it then follows that

$$\frac{\partial \psi}{\partial a} = K_t; \qquad \frac{\partial \psi}{\partial a_\theta} = -L$$

where

$$K_t = \int_V B^T D_{ats} B \, dV \qquad L = \int_V B^T \beta_{ats} N_\theta \, dV$$

Insertion into (23.75) results in

$$(K_t)^{i-1}(a^i - a^{i-1}) - L^{i-1}(a_\theta^i - a_\theta^{i-1}) = -\psi(a^{i-1}, a_\theta^{i-1}) \qquad (23.76)$$

where

$$\psi(a^{i-1}, a_\theta^{i-1}) = \int_V B^T \sigma^{i-1} \, dV - f_{n+1}$$

The starting vectors for a^{i-1} and a_θ^{i-1} are taken as $a^0 = a_n$ and $a_\theta^0 = a_{\theta,n}$ i.e. the values at the last accepted state n.

From (23.73) and (23.76) it is apparent that the Newton-Raphson formulation of the coupled problem becomes

$$\begin{bmatrix} (K_t)^{i-1} & -L^{i-1} \\ T_a^{i-1} & T_\theta^{i-1} \end{bmatrix} \begin{bmatrix} a^i - a^{i-1} \\ a_\theta^i - a_\theta^{i-1} \end{bmatrix} = - \begin{bmatrix} \psi^{i-1} \\ \psi_\theta^{i-1} \end{bmatrix}$$

where

$$\psi^{i-1} = \psi(a^{i-1}, a_\theta^{i-1}) = \int_V B^T \sigma^{i-1} \, dV - f_{n+1} \qquad (23.77)$$

$$\psi_\theta^{i-1} = \psi_\theta(a^{i-1}, a_\theta^{i-1})$$

$$= C \frac{1}{\Delta t}(a_\theta^{i-1} - a_{\theta,n}) + K_\theta^{i-1} a_\theta^{i-1} - Q^{i-1} - f_{\theta,n+1}$$

A comparison with (23.61) shows that the iteration matrix A for the Newton-Raphson method is given as the coefficient matrix in (23.77).

Unfortunately the coefficient matrix in (23.77) turns out to be unsymmetric and the bandwidth is very broad. In addition, several of the individual matrices are complex to derive. This is especially true for the matrices $\tilde{G}$, G and H, but even the more simple matrices $\tilde{C}$ and $\tilde{K}_\theta$ call for a substantial computational effort; it is noted that $\tilde{C}$ and $\tilde{K}_\theta$ are zero when the specific heat capacity c_ε and the conductivity matrix k are temperature independent. However, the contributions from $\tilde{G}$, H, $\tilde{C}$ and $\tilde{K}_\theta$ can be considered as slight and if they are ignored (23.77) reduces to

$$\begin{bmatrix} (K_t)^{i-1} & -L^{i-1} \\ -\frac{1}{\Delta t}G^{i-1} & \frac{1}{\Delta t}C^{i-1} + K_\theta^{i-1} \end{bmatrix} \begin{bmatrix} a^i - a^{i-1} \\ a_\theta^i - a_\theta^{i-1} \end{bmatrix} = - \begin{bmatrix} \psi^{i-1} \\ \psi_\theta^{i-1} \end{bmatrix} \qquad (23.78)$$

for $i = 1, 2, \ldots$. As usual, the starting vectors are taken as $a^0 = a_n$, $a_\theta^0 = a_{\theta,n}$ where n is the last state where the mechanical and thermal balance equations are fulfilled. Although solution of the fully coupled equation system (23.77) without introducing any simplifications is computationally very demanding, it was performed e.g. by Słuzalec (1988).

Algorithmic tangent stiffness

We note that in the iteration scheme both $\varepsilon^{(2)}$ and $\theta^{(2)}$ are known when the stresses are to be calculated. Accordingly, this topic has already been treated in Section 23.5.3 and will therefore not be repeated here.

Let us instead establish the algorithmic tangent stiffness. In order not to complicate the algebra, it will be assumed that $\partial g/\partial\sigma$ only depends on σ. In that case (23.60) becomes similar to (18.57). We will therefore utilize the same ideas as presented in Section 18.1.5; however, we now have to account for the temperature in the derivations.

Differentiation of (23.60a) and use of (23.59) result in

$$\dot{\sigma} = D\dot{\varepsilon} - \lambda D\frac{\partial g}{\partial\sigma} - \Delta\lambda D\frac{\partial^2 g}{\partial\sigma\partial\sigma}\dot{\sigma} - \beta\dot{\theta}$$

which leads to

$$\dot{\sigma} = D^a\dot{\varepsilon} - \lambda D^a\frac{\partial g}{\partial\sigma} - P^a\dot{\theta} \tag{23.79}$$

where

$$D^a = (D^{-1} + \Delta\lambda\frac{\partial^2 g}{\partial\sigma\partial\sigma})^{-1}$$
$$P^a = D^a D^{-1}\beta$$

It is interesting to note that D^a is identical to D^a given in (18.60). Considering the yield condition (23.60c), a differentiation provides

$$(\frac{\partial f}{\partial\sigma})^T\dot{\sigma} + \frac{\partial f}{\partial K}\dot{K} + \frac{\partial f}{\partial\theta}\dot{\theta} = 0 \tag{23.80}$$

Eventually, let us consider (23.60b). A straightforward differentiation results in

$$\dot{K} = d^a\dot{\lambda} + d^a_\theta\dot{\theta} \tag{23.81}$$

where

$$d^a = -(1 + \Delta\lambda\frac{\partial K}{\partial\kappa}\frac{\partial^2 g}{\partial K^2})^{-1}\frac{\partial K}{\partial\kappa}\frac{\partial g}{\partial K}$$
$$d^a_\theta = -(1 + \Delta\lambda\frac{\partial K}{\partial\kappa}\frac{\partial^2 g}{\partial K^2})^{-1}(\frac{\partial K}{\partial\kappa}\Delta\lambda\frac{\partial^2 g}{\partial K\partial\theta} - \frac{\partial K}{\partial\theta})$$

Insertion of (23.81) and (23.79) into (23.80) provides

$$\dot{\lambda} = \frac{1}{A^a}(\frac{\partial f}{\partial\sigma})^T D^a\dot{\varepsilon} + \frac{1}{A^a}[S^a - (\frac{\partial f}{\partial\sigma})^T P^a]\dot{\theta} \tag{23.82}$$

where

$$A^a = (\frac{\partial f}{\partial \sigma})^T D^a \frac{\partial g}{\partial \sigma} - \frac{\partial f}{\partial K} d^a$$

$$S^a = \frac{\partial f}{\partial K} d^a_\theta + \frac{\partial f}{\partial \theta}$$

Finally, insertion of (23.82) into (23.79) provides the result sought for

---

Algorithmic constitutive relation

$$\dot{\sigma} = D_{ats}\dot{\varepsilon} - \beta_{ats}\dot{\theta}$$

where

$$D_{ats} = D^a - \frac{1}{A^a} D^a \frac{\partial g}{\partial \sigma} (\frac{\partial f}{\partial \sigma})^T D^a$$

$$\beta_{ats} = P^a + \frac{1}{A^a} D^a \frac{\partial g}{\partial \sigma} [S^a - (\frac{\partial f}{\partial \sigma})^T P^a]$$

(23.83)

This relation is the algorithmic version of (23.20).

In (23.70) and (23.72) the matrices $\tilde{G}$, G and H were defined, but their specific content was left open. We have already argued that $\tilde{G}$ is expected to be small compared with other matrices, but it can be obtained by a linearization of c_ε; the derivations are lengthy but straightforward and will not be pursued here. To identify the matrices G and H a linearization of (23.67) provides

$$(\Delta \dot{Q}) = \bar{Q}^T_\varepsilon \dot{\varepsilon} + \bar{B}\dot{\theta} = \bar{Q}^T_\varepsilon B \dot{a} + \bar{B} N_\theta \dot{a}_\theta$$

The specific formats for Q_ε and B are straightforward to establish. From the above it is then possible to identify

$$\frac{\partial \Delta Q}{\partial a} = \bar{Q}^T_\varepsilon B \; ; \qquad \frac{\partial \Delta Q}{\partial a_\theta} = \bar{B} N_\theta$$

and the matrices G and H defined by (23.72) and (23.70) then become

$$G = \int_V N^T_\theta \bar{Q}^T_\varepsilon B dV \; ; \qquad H = \int_V N^T_\theta \bar{B} N_\theta dV$$

23.7   Adiabatic heating

If the loading is performed very rapidly, no heat exchange can occur with the surroundings of the body. Moreover, this rapid loading does not allow any heat flow to take place. Disregarding heat flow can be done by ignoring the heat flux vector $q_i = -k_{ij}\theta_{,j}$; likewise, since no heat exchange with the surroundings is

possible, we set $r = 0$. These conditions are called *adiabatic heating* and the heat equation (23.23b), cf. also (21.100), then reduces to

$$
\begin{array}{|l}
E = \rho c_\varepsilon \dot{\theta} - Q = 0 \\[4pt]
where \\[4pt]
Q = \sigma_{ij} \dot{\varepsilon}^p_{ij} - K_\alpha \dot{\kappa}_\alpha - \theta P_{ij}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij}) + \theta \dfrac{\partial K_\alpha}{\partial \theta} \dot{\kappa}_\alpha
\end{array}
\tag{23.84}
$$

It appears that it is only the mechanical coupling that produces heat and thereby change of temperature. It can also be noted that given the strain rate, plastic strain rate and the rate of the internal variables the change in temperature can be obtained without solving any boundary value problem. This implies that (23.84) should be treated together with the constitutive relations, i.e. on the material level. Adiabatic heating has been considered e.g. by Nemat-Nasser (1988), Batra and Wright (1988) and Wriggers *et al.* (1992).

The gain of not having to solve a boundary value problem must be paid for on the constitutive level, which now becomes more elaborate. Therefore, let us first consider the calculation of stresses and later return to the finite element formulation for the mechanical problem.

Before considering the algorithm for calculating the stresses, it is noted that (23.84) is in rate form. To obtain a suitable format for our purposes, (23.84) is integrated over the loading step. With the fully implicit method we obtain

$$
E = \rho c_\varepsilon \Delta\theta - \Delta Q = 0
$$
where
$$
\Delta Q = \sigma^T \Delta\varepsilon^p - K\Delta\kappa - \theta P^T(\Delta\varepsilon - \Delta\varepsilon^p) + \theta \frac{\partial K}{\partial \theta}\Delta\kappa
\tag{23.85}
$$

where a matrix formulation was adopted and where it was assumed that only one internal variable exists. Let us also for simplicity assume that D and β are constant quantities; this is in analogy with the assumptions in Section 23.5.3. With this in mind let us now recall the set of equations that needs to be considered for the calculation of stresses

$$
\begin{aligned}
R_\sigma &= \sigma + \Delta\lambda D \frac{\partial g}{\partial \sigma} - \sigma^t \\[6pt]
R_K &= K - \hat{K}(\kappa^{(1)} - \Delta\lambda \frac{\partial g}{\partial K}, \theta) \\[6pt]
R_f &= f(\sigma, K, \theta) \\[6pt]
R_E &= \rho c_\varepsilon \Delta\theta - \Delta Q(\sigma, K, \theta, \Delta\lambda)
\end{aligned}
\tag{23.86}
$$

where it is assumed that only one hardening parameter $K = K(\kappa, \theta)$ exists. Defining the residual vector as $V = [R_\sigma, R_K, R_f, R_E]^T$ and the vector containing the unknowns as $S = [\sigma, K, \Delta\lambda, \theta]^T$, it is evident that $V(S) = 0$ defines the solution sought for.

The solution of (23.86) can be found by adopting the Newton-Raphson method and the iterative procedure becomes

$$0 = S^{(i-1)} - \left[\frac{\partial V^{(i-1)}}{\partial S}\right]^{-1} V^{(i-1)}$$

which is found by considering a Taylor series expansion of V. The iteration procedure is stopped when the norm of V is sufficiently small. The iteration matrix is straightforward to identify and will therefore not be considered here.

As the last issue consider the algorithmic stiffness that will be used in the mechanical equilibrium iterations. The derivation follows the lines discussed in Section 23.6.2 and only one additional relation needs to be regarded, namely E given in (23.85). According to (23.83), the result from the first three relations in (23.86) is

$$\dot{\sigma} = D_{ats}\dot{\varepsilon} - \beta_{ats}\dot{\theta} \tag{23.87}$$

where it was assumed that $\partial g/\partial \sigma$ only depends on σ. The task is then to relate $\dot{\theta}$ to $\dot{\varepsilon}$ using (23.86). To maintain a compact formulation, a differentiation of (23.86d) provides after a little algebra

$$Q_\varepsilon^T \dot{\varepsilon} + B\dot{\theta} = 0$$

where Q_ε and B can be identified if advantage is taken of (23.81), (23.82) and (23.87). The above relation is the algorithmic counterpart of (23.25) for $r = 0$ as well as $q_{i,i} = 0$.

Use of these expressions in (23.87) results in the algorithmic tangent stiffness for adiabatic conditions, namely

Algorithmic tangent stiffness for adiabatic conditions

$$\dot{\sigma} = D_{ats}^{ad}\dot{\varepsilon}$$

where

$$D_{ats}^{ad} = D_{ats} + \frac{1}{B}\beta_{ats}Q_\varepsilon^T$$

In analogy with (23.76), it is then evident that a Newton-Raphson approach to the equilibrium equations (23.56) and (23.57) results in

Iteration format for adiabatic conditions

$$(K_t)^{i-1}(a^i - a^{i-1}) = -\psi^{i-1}$$

where

$$\psi^{i-1} = \int_V B^T \sigma^{i-1} \, dV - f_{n+1}$$

$$K_t = \int_V B^T D_{ats}^{ad} B \, dV$$

temperature

displacements

t_{n+2}

θ

t_{n+1} u t_{n+2} t_{n+1}

t_{n+1}

θ

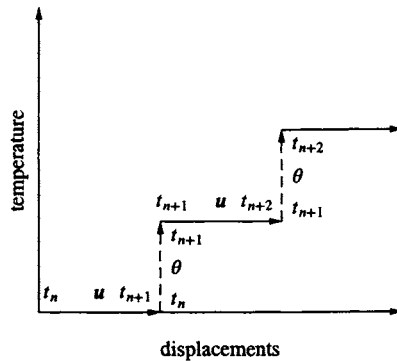t_n u t_{n+1} t_n

Figure 23.10: Illustration of the isothermal staggered approach.

The elimination of the temperature as described above results in a simple and convenient scheme.

23.8 Staggered solution scheme

As previously discussed, the computational effort relating to the solution of the fully coupled equation system (23.77) is substantial and one approach to avoid this complication is to make use of a so-called *staggered solution scheme*. Considering another type of coupled response, this approach was originately introduced by Park *et al.* (1977) and a review is given by Park and Felippa (1983). The procedure is as follows: the temperature field derived in the previous time step, i.e. at time t_n, is used as an input in the mechanical field equations (23.77a) and iterations are then performed - without changing the temperature field - until the mechanical balance equations are fulfilled at time t_{n+1}. Now the corresponding displacements at time t_{n+1} are used as input in the thermal field equations (23.77b) and iterations are then performed - without changing the displacement field - until the thermal balance equations are fulfilled at time t_{n+1}. Then the next time step Δt is performed. This approach is said to be based on the *isothermal split* and the process is illustrated in Fig. 23.10. The 'staggered' route is the motivation for the terminology of a 'staggered' solution scheme.

This solution scheme is the one most often adopted in coupled thermoplasticity and it was used in the pioneering work of Argyris *et al.* (1981) as well as in a number of other investigations, cf. for instance Simo and Miehe (1992), Słuźalec (1992), Tuğcu (1995), Celentano *et al.* (1996) and Håkansson *et al.* (2005). One advantage of the staggered scheme is that it permits different time steps to be used in the mechanical and in the thermal field equations, thereby considering the different time scales that control these two fields. How-

ever, the main disadvantage of the staggered scheme described above is that it is only conditionally stable, i.e. the time step cannot be taken arbitrarily large. This was shown by Armero and Simo (1992) who also concluded that the staggered scheme based on the isothermal split is not suitable for strongly coupled problems; instead they proposed alternative staggered schemes in terms of a so-called *adiabatic split* and an *isentropic split*, which are unconditionally stable. The reader is referred to Armero and Simo (1992) and Armero and Simo (1993) for further details.

Let us consider the staggered solution scheme based on the isothermal split and adopt the backward Euler scheme; the index $n + 1$ is as usual omitted. In the first step, the mechanical step, the temperature is kept constant and we are left with the mechanical problem

$$\sigma_{ij,j} + b_i = 0$$

alone. Since we have a purely mechanical problem, the solution procedures described in Chapter 17 can be used directly. For calculation of the stresses, and as the temperature is unchanged, the algorithms described in Chapter 18 are applicable.

In step 2, the thermal part, we consider

$$\rho c_\varepsilon \dot{\theta} = r - q_{i,i} + Q$$

and the configuration is kept constant. It should be noted that owing to the temperature changes this will have an effect on the constitutive relations as these depend on the temperature. In conclusion, even though the configuration (total strains) is kept fixed, changes in the constitutive relations must be considered, i.e. new stresses must be calculated.

It turns out that we have already carried out most of the derivation when we considered the fully coupled problem in Section 23.6. From this derivation we can extract our sought for relations by imposing that the configuration is fixed. Based on (23.62) we therefore obtain the finite element formulation for the thermal problem

$$
\boxed{
\begin{aligned}
&\boldsymbol{\psi}_\theta(\boldsymbol{a}_\theta) = \mathbf{0} \\
&\textit{where} \\
&\boldsymbol{\psi}_\theta(\boldsymbol{a}_\theta) = \frac{1}{\Delta t}\boldsymbol{C}(\boldsymbol{a}_\theta - \boldsymbol{a}_{\theta,n}) + \boldsymbol{K}_\theta \boldsymbol{a}_\theta - \boldsymbol{Q} - \boldsymbol{f}_\theta
\end{aligned}
}
\tag{23.88}
$$

Adopting the Newton-Raphson scheme and keeping the configuration fixed, the solution procedure is found from (23.73)

$$\boldsymbol{T}_\theta^{i-1}(\boldsymbol{a}_\theta^i - \boldsymbol{a}_\theta^{i-1}) = -\boldsymbol{\psi}_\theta^{i-1}$$

where

$$\boldsymbol{T}_\theta = \tilde{\boldsymbol{C}} + \frac{1}{\Delta t}\boldsymbol{C} + \tilde{\boldsymbol{K}}_\theta + \boldsymbol{K}_\theta - \frac{1}{\Delta t}\boldsymbol{H}$$

with $\tilde{C}$, $\tilde{K}_\theta$ and H defined by (23.52), (23.54) and (23.70) respectively.

As mentioned previously, the stresses need to be calculated in the thermal step. Once more it can be concluded that this topic has already been dealt with in Section 23.5.3. In this integration of the stresses, the total strain increment $\Delta \varepsilon$ is kept fixed and equal to the value found from step 1, but now we change the temperature. This will lead to a new value of the trial stress given by (23.59), and the integration procedure then follows the lines described in Section 23.5.3.

Let us finally consider the adiabatic split. As the name indicates one step is an adiabatic step similar to that dealt with in Section 23.7. Therefore, in the first step we consider

$$\sigma_{ij,j} + b_i = 0 \; ; \qquad \rho c_\varepsilon \frac{\tilde{\theta} - \theta_n}{\Delta t} = Q$$

where the last relation is the heat equation for adiabatic heating. The temperature field $\tilde{\theta}$ obtained influences the displacement increment. In the next step the configuration is fixed and the temperature field is achieved by the solution of the following heat equation

$$\rho c_\varepsilon \frac{\theta - \tilde{\theta}}{\Delta t} = r - q_{i,i}$$

and it appears that the $\tilde{\theta}$-field may be considered as the initial temperature field. This second step turns out to be a linear problem if c_ε is constant, otherwise an iterative scheme must be adopted. This can be found by considering (23.88) and adopting $\tilde{C} = 0$, $Q = 0$ as well as $H = 0$. As the derivation of this iterative scheme is straightforward, it will not be pursued further.