24 UNIQUENESS AND DISCONTINUOUS BIFURCATIONS

When solving boundary value problems, it is evidently of great importance to know whether the problem in question has a *unique* solution or not. Unique solutions are most often encountered in solid mechanics even though Euler buckling provides a trivial example of loss of uniqueness and where, above the Euler load, not only the *fundamental solution*, but also the *bifurcated solution* are possible. In this specific case, the finding is a consequence of equilibrium being related to the deformed configuration and not to the undeformed configuration.

Here, we will be concerned with uniqueness and bifurcations entirely related to the material behavior in itself; elasto-plasticity will be assumed. With the exceptions of the textbooks of Nguyen (2000) and Lubarda (2002), these topics are mostly addressed in the journal literature; before entering into a systematic analysis, we will present a simple example.

24.1 Simple illustration - Tension bar

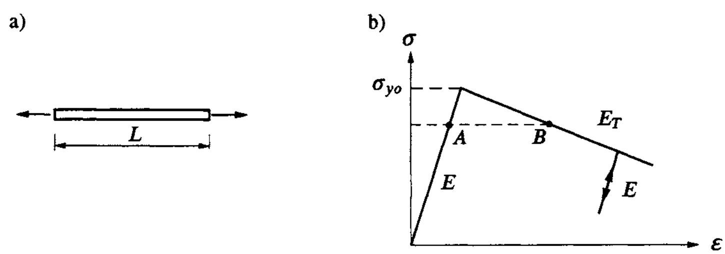

Figure 24.1: a) Homogeneous bar loaded in tension; b) bilinear stress-strain curve.

Consider the homogeneous bar of length L which is loaded uniaxially by a prescribed axial elongation into its postpeak regime, Fig. 24.1a); the material exhibits the bilinear stress-strain curve shown in Fig. 24.1b). Letting ε denote the total axial strain and ε_t the total transverse strain, then

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p\,; \qquad \dot{\varepsilon}_t = \dot{\varepsilon}_t^e + \dot{\varepsilon}_t^p \tag{24.1}$$

The incremental Hooke's law states that

$$\dot{\varepsilon}^e = \frac{\dot{\sigma}}{E}\,; \qquad \dot{\varepsilon}_t^e = -\nu\dot{\varepsilon}^e = -\frac{\nu}{E}\dot{\sigma} \tag{24.2}$$

For the plastic behavior, we adopt associated plasticity with the yield function f in terms of the maximum tensile stress criterion, i.e.

$$f(\sigma_{ij}, K) = \sigma - \sigma_y(K)\,; \qquad \sigma = \sigma_1 \tag{24.3}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses and σ_y is the current yield stress. The plastic strain rates are given by $\dot{\varepsilon}_{ij}^p = \dot{\lambda}\partial f/\partial\sigma_{ij}$ which in the present case become

$$\dot{\varepsilon}^p = \dot{\lambda}\,; \qquad \dot{\varepsilon}_t^p = 0 \tag{24.4}$$

According to Fig. 24.1b), we have

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_T} \tag{24.5}$$

where E_T is a constant. Use of this expression as well as (24.4) and (24.2) in (24.1) gives

$$\dot{\varepsilon}^p = \left(\frac{1}{E_T} - \frac{1}{E}\right)\dot{\sigma}\,; \qquad \dot{\varepsilon}_t = -\frac{\nu}{E}\dot{\sigma} \quad \text{softening regime} \tag{24.6}$$

For elastic unloading, we obtain from (24.2) and (24.1)

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E}\,; \qquad \dot{\varepsilon}_t = -\frac{\nu}{E}\dot{\sigma} \quad \text{elastic regime} \tag{24.7}$$

Evidently, when the bar is in the softening regime, a valid solution is to assume that the plastic zone occupies the entire bar. However, let us investigate the possibility that other solutions are realizable. For this purpose assume that the bar has been loaded up to its maximum load capacity σ_{yo}, see Fig. 24.1b). When the bar is further elongated, assume that two regions emerge in the bar as shown in Fig. 24.2: one region of length L_s exhibiting strain softening corresponding to point B in Fig. 24.1b) and another region of length $L - L_s$ is elastically unloading corresponding to point A in Fig. 24.1b). In each region, the strain state is assumed to be uniform and no dynamic effects are involved.

From equilibrium follows that the same axial stress rate $\dot{\sigma}$ applies in both regions. Referring to (24.6) and (24.7) we therefore have the same transverse

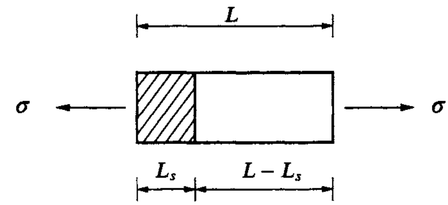

Figure 24.2: Tension bar in softening with one region in elastic unloading and the other exhibiting strain softening.

strain rates in the two regions; consequently, no shear stresses exist along the interface of the two regions. The deformation mode shown in Fig. 24.2 therefore fulfills all field equations and since we have said nothing about the length L_s we have lost uniqueness of the solution. Moreover, as the axial strain rate differs in the two regions we have obtained a situation of a (discontinuous) bifurcation.

If hardening plasticity occurs, i.e. $E_T > 0$, then it is evident that the situation shown in Fig. 24.2 cannot exist. This follows directly from equilibrium, which requires that the stress in both regions must exceed the initial yield stress σ_{yo}, cf. Fig. 24.1b). Therefore, hardening plasticity only allows one solution where the plastic region occupies the entire bar. For softening plasticity, however, uniqueness is lost and we have the situation in Fig. 24.2 where the length L_s is arbitrary. Let us evaluate this case more detailed.

For constant size L_s of the softening region, the elongation rate $\dot{\delta}$ of the bar becomes with (24.5) and (24.7a)

$$\dot{\delta} = (\frac{L_s}{E_T} + \frac{L - L_s}{E})\dot{\sigma} \tag{24.8}$$

If the prescribed elongation is assumed to always increase, we have $\dot{\delta} > 0$ and as $\dot{\sigma} \leq 0$ for $E_T \leq 0$ it follows that

$$L_{min} < L_s \leq L \quad \text{where} \quad L_{min} = \frac{L}{1 - \frac{E}{E_T}} \tag{24.9}$$

If L_s is written as $L_s = kL_{min}$ where k is a dimensionless positive number, (24.8) takes the format

$$\frac{\dot{\sigma}}{\sigma_{yo}} = \frac{1}{1-k}\frac{\dot{\delta}E}{L\sigma_{yo}}; \quad L_s = kL_{min} \tag{24.10}$$

which is illustrated in Fig. 24.3. It appears that $k > 1$ implies stability in the sense that the bar can be exposed to an increasing elongation, whereas $k < 1$ results in a snap-back response; these aspects were discussed by Bažant (1976)

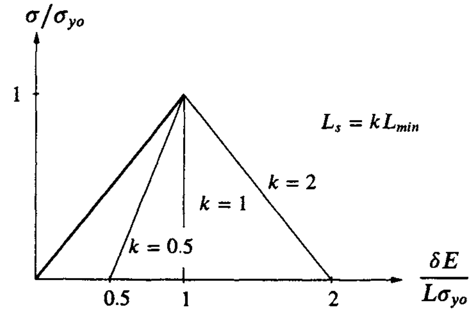

Figure 24.3: Behavior of bar for different k-values; $L_s = kL_{min}$.

and Sture and Ko (1978). Moreover, this stability criterion was combined by Ottosen (1986) with Gibbs' conditions for thermodynamic equilibrium, cf. Section 20.12. It was then shown that cracking of brittle materials like concrete will be characterized by a *cohesive zone* with a constitutive relation between stress and crack opening displacement much along the lines of the *fictitious crack model* of Hillerborg *et al.* (1976).

24.2 Equations of plasticity theory

Let us for convenience summarize the equations of plasticity theory. It is assumed that the stress state is located on the yield surface and then

$$\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl} \begin{cases} < 0 \Rightarrow & \text{elastic unloading} \\ = 0 \Rightarrow & \text{neutral loading} \\ > 0 \Rightarrow & \text{plastic loading} \end{cases} \tag{24.11}$$

When plastic loading occurs, the incremental stress-strain relation is given by

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\varepsilon}_{kl} \tag{24.12}$$

where

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A} D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} \tag{24.13}$$

and

$$A = H + \frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}} > 0 \tag{24.14}$$

The consideration of the following generalized eigenvalue problem turns out to be important

$$D^{ep}_{ijkl} z_{kl} = \lambda D_{ijkl} z_{kl} \tag{24.15}$$

where z_{ij} is the eigenvector and λ the eigenvalue. Use of (24.13) in this expression gives

$$(1-\lambda) D_{ijkl} z_{kl} - \frac{1}{A} D_{ijst} \frac{\partial g}{\partial \sigma_{st}} \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} z_{kl} = 0$$

It appears that a solution is given by

$$\lambda = 1 \quad \text{when} \quad \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} z_{kl} = 0 \tag{24.16}$$

This means that the eigenvalue $\lambda = 1$ holds when neutral loading occurs. On the other hand, if $\lambda \neq 1$ then the expression above shows that the eigenvector $z_{ij} \sim \partial g / \partial \sigma_{ij}$. With (24.14) this leads to

$$\frac{1}{A}(H - \lambda A) D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} = 0$$

and it is concluded that

$$\lambda = \frac{H}{A} \quad \text{when} \quad z_{ij} = \frac{\partial g}{\partial \sigma_{ij}} \tag{24.17}$$

With these results, it is useful to consider the conditions for a *limit point*. A limit point is defined by $\dot{\sigma}_{ij} = 0$ when $\dot{\varepsilon}_{ij} \neq 0$ and a limit point therefore corresponds to a peak stress. In view of the generalized eigenvalue problem (24.15) and the result (24.17) we conclude that

$$\boxed{\textit{A limit point } \dot{\sigma}_{ij} = 0\,;\, \dot{\varepsilon}_{ij} \neq 0 \textit{ requires } H^{lp} = 0} \tag{24.18}$$

where superscript 'lp' underlines that it is the plastic modulus at the limit point.

For later purposes, some further properties of the tangential stiffness tensor will now be investigated. Since D_{ijkl} is a positive definite tensor, there exists a positive definite tensor B_{ijkl} with the properties

$$B_{ijmn} B_{mnkl} = D_{ijkl} \tag{24.19}$$

cf. Strang (1980). It follows that B_{ijkl} possesses minor and major symmetries just like D_{ijkl}. Define the following quantities

$$p_{ij} = B_{ijmn} \frac{\partial f}{\partial \sigma_{mn}}; \qquad q_{ij} = B_{ijst} \frac{\partial g}{\partial \sigma_{st}} \tag{24.20}$$

Then D^{ep}_{ijkl} given by (24.13) can be written as

$$D^{ep}_{ijkl} = D_{ijkl} - \frac{1}{A} B_{ijst} q_{st} p_{mn} B_{mnkl}$$

The symmetric part $D^{ep,s}_{ijkl}$ of D^{ep}_{ijkl} is then defined by

$$D^{ep,s}_{ijkl} = D_{ijkl} - \frac{1}{2A}(B_{ijst} q_{st} p_{mn} B_{mnkl} + B_{ijst} p_{st} q_{mn} B_{mnkl}) \tag{24.21}$$

With these introductory remarks we will consider the following generalized eigenvalue problem

$$\boxed{D^{ep,s}_{ijkl} z_{kl} = \omega D_{ijkl} z_{kl}} \tag{24.22}$$

which will later prove useful when uniqueness properties are discussed. In analogy with (24.20) we define

$$\psi_{ij} = B_{ijkl} z_{kl}$$

and the eigenvalue problem then takes the form

$$(1 - \omega)\psi_{ij} - \frac{1}{2A}(q_{ij} p_{st} \psi_{st} + p_{ij} q_{st} \psi_{st}) = 0 \tag{24.23}$$

If we consider the symmetric second-order tensors p_{ij}, q_{ij} and ψ_{ij} in the six-dimensional space, we obtain the following solution to the eigenvalue problem

$$\boxed{\omega_2 = \omega_3 = \omega_4 = \omega_5 = 1 \quad \text{when} \;\; p_{st}\psi_{st} = 0 \;\text{ and }\; q_{st}\psi_{st} = 0} \tag{24.24}$$

i.e. $\omega = 1$ is an eigenvalue with multiplicity of four.

In view of (24.23) the last two eigenvectors are given by

$$\psi_{ij} = \alpha p_{ij} + \beta q_{ij}$$

where α and β are arbitrary constants. Insertion into (24.23) results in

$$\begin{aligned} &[(1-\omega)\alpha - \frac{1}{2A}(\alpha q_{mn} p_{mn} + \beta q_{mn} q_{mn})] p_{ij} \\ &+ [(1-\omega)\beta - \frac{1}{2A}(\alpha p_{mn} p_{mn} + \beta q_{mn} p_{mn})] q_{ij} = 0 \end{aligned} \tag{24.25}$$

Introduce the following quantities

$$m_{ij} = \frac{p_{ij}}{|p|}; \qquad n_{ij} = \frac{q_{ij}}{|q|} \quad \Rightarrow \quad m_{ij} n_{ij} = cos\theta \tag{24.26}$$

and it follows that $m_{ij}m_{ij} = 1$ and $n_{ij}n_{ij} = 1$. Then (24.25) takes the form

$$[(1-\omega)\alpha - \frac{1}{2A}(\alpha|q||p|cos\theta + \beta|q|^2)]p_{ij}$$
$$+ [(1-\omega)\beta - \frac{1}{2A}(\alpha|p|^2 + \beta|q||p|cos\theta)]q_{ij} = 0 \qquad (24.27)$$

The coefficients to p_{ij} and q_{ij} must be zero leading to a homogeneous equation system in α and β; a nontrivial solution requires that the determinant to the coefficient matrix be zero and this gives

$$\omega = 1 - \frac{|p||q|}{2A}(cos\theta \pm 1)$$

Since

$$A = H + |p||q|cos\theta$$

we finally obtain

$$\boxed{\left.\begin{matrix}\omega_1 \\ \omega_6\end{matrix}\right\} = \frac{1}{A}[H + \frac{1}{2}|p||q|(cos\theta \pm 1)]} \qquad (24.28)$$

and it follows that $\omega_1 \geq 1$ and $\omega_6 \leq 1$; these results will be of importance when uniqueness is investigated.

24.3 Uniqueness of elasto-plastic materials

It is of great importance to identify the conditions for which a boundary value problem possesses a unique solution. For this purpose we will follow the concept originately proposed by Kirchhoff (1859) for linear elastic bodies and later applied by Melan (1938) to elasto-plastic problems.

Assume firstly that the loading has brought the body to a certain state and secondly that uniqueness exists; then consider the response for a further incremental loading. From (3.29) we have

$$\dot{\sigma}_{ij,j} + \dot{b}_i = 0$$

Similar to (3.33), the weak format of this equation is given by

$$\int_V v_{i,j}\dot{\sigma}_{ij}\, dV = \int_S v_i\dot{t}_i\, dS + \int_V v_i\dot{b}_i\, dV \qquad (24.29)$$

where v_i is an arbitrary weight vector. For a given incremental loading, assume that there exist two different solutions to the problem. Similar to (24.29) we then obtain with evident notation

$$\int_V v_{i,j}\dot{\sigma}_{ij}^{(2)}\,dV = \int_S v_i \dot{t}_i^{(2)}\,dS + \int_V v_i \dot{b}_i^{(2)}\,dV$$

$$\int_V v_{i,j}\dot{\sigma}_{ij}^{(1)}\,dV = \int_S v_i \dot{t}_i^{(1)}\,dS + \int_V v_i \dot{b}_i^{(1)}\,dV$$

Subtraction of these expressions gives

$$\int_V v_{i,j}(\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)})\,dV = \int_S v_i(\dot{t}_i^{(2)} - \dot{t}_i^{(1)})\,dS + \int_V v_i(\dot{b}_i^{(2)} - \dot{b}_i^{(1)})\,dV \qquad (24.30)$$

The two solutions correspond to the same loading so $\dot{b}_i^{(2)} = \dot{b}_i^{(1)}$ holds. Moreover, along the outer boundary the displacements are prescribed along S_u and the traction vector is prescribed along the boundary S_t; therefore $\dot{t}_i^{(2)} = \dot{t}_i^{(1)}$ holds along S_t. With these remarks (24.30) becomes

$$\int_V v_{i,j}(\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)})\,dV = \int_{S_u} v_i(\dot{t}_i^{(2)} - \dot{t}_i^{(1)})\,dS$$

The weight vector v_i is arbitrary and it is now chosen as $v_i = \dot{u}_i^{(2)} - \dot{u}_i^{(1)}$ which implies that $v_i = 0$ along S_u. The expression above then reduces to

$$\int_V v_{i,j}(\dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)})\,dV = 0$$

Introduce the notation

$$[\dot{\varepsilon}_{ij}] = \dot{\varepsilon}_{ij}^{(2)} - \dot{\varepsilon}_{ij}^{(1)} \qquad [\dot{\sigma}_{ij}] = \dot{\sigma}_{ij}^{(2)} - \dot{\sigma}_{ij}^{(1)} \qquad (24.31)$$

which leads to

If two solutions exist, they fulfill

$$\boxed{\int_V [\dot{\varepsilon}_{ij}][\dot{\sigma}_{ij}]\,dV = 0} \qquad (24.32)$$

If both solutions correspond to elastic unloading or neutral loading then $[\dot{\sigma}_{ij}] = D_{ijkl}[\dot{\varepsilon}_{kl}]$ leading to

$$\int_V [\dot{\varepsilon}_{ij}]D_{ijkl}[\dot{\varepsilon}_{kl}]\,dV = 0 \qquad (24.33)$$

Since we required the strain energy to be positive and thereby D_{ijkl} to be positive definite, cf.(4.40), the expression above implies $[\dot{\varepsilon}_{ij}] = 0$ meaning that solutions to linear elastic boundary value problems are unique.

Before uniqueness of elasto-plastic boundary value problems are considered, we will derive an intermediate result. Assume that both solutions correspond to plastic loading then

$$\boxed{I_{pp} = [\dot{\varepsilon}_{ij}][\dot{\sigma}_{ij}] = [\dot{\varepsilon}_{ij}]D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}]} \tag{24.34}$$

where I refers to the integrand in (24.32) and the subscript 'pp' indicates that both solutions respond plastically. Use of (24.13) implies that we also have

$$I_{pp} = [\dot{\varepsilon}_{ij}]D_{ijkl}[\dot{\varepsilon}_{kl}] - \frac{1}{A}[\dot{\varepsilon}_{ij}]D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl}[\dot{\varepsilon}_{kl}] \tag{24.35}$$

Suppose that one solution (2) corresponds to plastic loading and the other solution (1) corresponds to elastic unloading/neutral loading, then

$$I_{pe} = [\dot{\varepsilon}_{ij}][\dot{\sigma}_{ij}] = [\dot{\varepsilon}_{ij}](D^{ep}_{ijkl}\dot{\varepsilon}^{(2)}_{kl} - D_{ijkl}\dot{\varepsilon}^{(1)}_{kl}) \tag{24.36}$$

Insertion of D_{ijkl} as determined by (24.13) results in

$$I_{pe} = [\dot{\varepsilon}_{ij}]D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}] - \frac{1}{A}[\dot{\varepsilon}_{ij}]D_{ijst}\frac{\partial g}{\partial \sigma_{st}}\underbrace{\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl}\dot{\varepsilon}^{(1)}_{kl}}_{\leq 0}$$

Assume that $[\dot{\varepsilon}_{ij}]D_{ijst}\frac{\partial g}{\partial \sigma_{st}} \geq 0$, then, since solution (1) was assumed to unload elastically, a comparison of the expression above with (24.34) shows that

$$\boxed{I_{pp} \leq I_{pe}} \tag{24.37}$$

Next assume that $[\dot{\varepsilon}_{ij}]D_{ijst}\frac{\partial g}{\partial \sigma_{st}} < 0$, then insertion of D^{ep}_{ijkl} as given by (24.13) into (24.36) gives

$$I_{pe} = [\dot{\varepsilon}_{ij}]D_{ijkl}[\dot{\varepsilon}_{kl}] - \frac{1}{A}\underbrace{[\dot{\varepsilon}_{ij}]D_{ijst}\frac{\partial g}{\partial \sigma_{st}}}_{<0}\underbrace{\frac{\partial f}{\partial \sigma_{mn}}D_{mnkl}\dot{\varepsilon}^{(2)}_{kl}}_{>0}$$

Since the elastic stiffness D_{ijkl} is positive definite, $I_{pe} > 0$ holds. Consequently, assume that we have reached a situation where $I_{pp} = 0$ which means loss of uniqueness, but at the same tine we have $I_{pe} > 0$. With this result, conclusion (24.37) is obtained once again.

Expression (24.37) shows that loss of uniqueness involving a situation of plastic loading and elastic unloading can never precede a situation of loss of uniqueness where both solutions exhibit plastic loading. Thus, the case of plastic loading/plastic loading is the most critical situation and this was used by Hill (1958, 1978) to introduce a so-called *linear comparison solid* which is a solid that only exhibits plastic loading.

We therefore conclude that as long as I_{pp} given by (24.34) is positive for $|\dot{\varepsilon}_{ij}| \neq 0$ then uniqueness exists, i.e.

$$\boxed{\begin{array}{l}\textit{Uniqueness exists if}\\ [\dot{\varepsilon}_{ij}]D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}] > 0\end{array}} \tag{24.38}$$

We also observe that in this quadratic expression, it is only the symmetric part $D^{ep,s}_{ijkl}$ of the tangential stiffness tensor that is of importance. When one of the eigenvalues of $D^{ep,s}_{ijkl}$ becomes zero then uniqueness is lost; considering the eigenvalue problem (24.22) with the solutions (24.24) and (24.28) it is concluded that this occurs when $\omega_6 = 0$. We then obtain the result

$$\boxed{\begin{array}{l}\textit{Loss of uniqueness:}\\ \quad H^{lu} = \dfrac{1}{2}|p||q|(1 - cos\theta)\\ \textit{where}\\ \quad |p| = (\dfrac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \dfrac{\partial f}{\partial \sigma_{kl}})^{1/2}; \qquad |q| = (\dfrac{\partial g}{\partial \sigma_{ij}} D_{ijkl} \dfrac{\partial g}{\partial \sigma_{kl}})^{1/2}\\ \quad cos\theta = \dfrac{1}{|p||q|} \dfrac{\partial f}{\partial \sigma_{ij}} D_{ijkl} \dfrac{\partial g}{\partial \sigma_{kl}}\end{array}} \tag{24.39}$$

where H^{lu} denotes the value of the plastic modulus when loss of uniqueness occurs.

For associated plasticity where $g = f$ and thereby $cos\theta = 1$ hold, we obtain $H^{lu} = 0$, i.e. uniqueness is assured in the hardening regime and this is in accordance with the previous discussion in the introductory example. For nonassociated plasticity, however, $cos\theta < 1$ and loss of uniqueness then occurs in the hardening regime where $H^{lu} > 0$. The result given in (24.39) was established by Maier and Hueckel (1979) as well as by Raniecki and Bruhns (1981) and the analysis presented here shares much in common with the approach suggested by Runesson and Mroz (1989).

It is of interest to compare the conditions for uniqueness, namely D^{ep}_{ijkl} being positive definite, with those of thermodynamic equilibrium as given by (21.109). For isothermal plasticity, Helmholtz' free energy ψ is given by (22.2) and we obtain

$$\rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = D_{ijkl}; \qquad \rho \frac{\partial^2 \psi}{\partial \varepsilon_{ij} \partial \kappa_\alpha} = 0$$

Insertion of these expressions into (21.109) gives

$$(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij}) D_{ijkl} (\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) + \dot{\kappa}_\alpha \rho \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \dot{\kappa}_\beta > 0$$

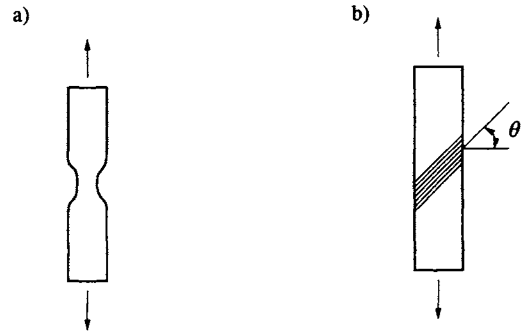

Figure 24.4: a) Continuous bifurcation; b) discontinuous bifurcation i.e. a shear band.

Since $\dot{\sigma}_{ij} = D^{ep}_{ijkl}\dot{\varepsilon}_{kl} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{p}_{kl})$ we obtain

$$\dot{\varepsilon}_{ij}D^{ep}_{ijkl}\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{p}_{ij}D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{p}_{kl}) + \dot{\kappa}_{\alpha}\rho\frac{\partial^2\psi^p}{\partial\kappa_\alpha\partial\kappa_\beta}\dot{\kappa}_\beta > 0 \tag{24.40}$$

For associated plasticity, the evolution laws are given by

$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\frac{\partial f}{\partial\sigma_{ij}}; \qquad \dot{\kappa}_\alpha = -\dot{\lambda}\frac{\partial f}{\partial K_\alpha}$$

Insertion into (24.40) and making use of (24.14) provide

$$\dot{\varepsilon}_{ij}D^{ep}_{ijkl}\dot{\varepsilon}_{kl} - \dot{\lambda}\frac{\partial f}{\partial\sigma_{ij}}D_{ijkl}\dot{\varepsilon}_{kl} + (\dot{\lambda})^2 A > 0$$

and since $\partial f/\partial\sigma_{ij}D_{ijkl}\dot{\varepsilon}_{kl} = A\dot{\lambda}$ we obtain $\dot{\varepsilon}_{ij}D^{ep}_{ijkl}\dot{\varepsilon}_{kl} > 0$. We have then reached the interesting observation that for associated plasticity, the conditions for uniqueness and thermodynamic equilibrium coincide.

24.4 Discontinuous bifurcations

When uniqueness is lost, more solutions are possible and apart from the fundamental solution, *bifurcations* are possible; these bifurcations can manifest themselves in terms of continuous bifurcations or *discontinuous bifurcations* as illustrated in Fig. 24.4.

The discontinuous bifurcations are also called *shear bands*, i.e. narrow bands across which a discontinuity occurs in the rate of the displacement gradient; for metals and steel, they can take the form of *Lüder's bands* as discussed, for instance, by Nadai (1950). In the pioneering works of Rudnicki and Rice (1975)

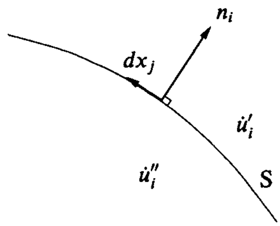

Figure 24.5: Singular surface i.e. characteristic surface.

and Rice (1976), the phenomenon was attributed directly to the properties of the constitutive model; however, the effects of large strains were also considered in these works.

The concept of a characteristic surface, across which a discontinuity of the rate of deformation gradient is permitted was already considered by Hill (1950) and Thomas (1961) and the classical argument for *localization* then implies a shear band bounded by two characteristic surfaces. We will see that the predictions of the directions of the characteristic surfaces made by plasticity theory are in close accuracy with experimental data; however, the width of the shear band is left unspecified. This is simply to say that conventional continuum theory lacks information on a length scale and this aspect has been used as an argument for invoking non-local continuum theories which contain a length scale that reflects the internal structure of the material, cf. Aifantis (1984), Belytschko *et al.* (1986), de Borst (1991), Strömberg and Ristinmaa (1996), Bassani (2001), Fleck and Hutchinson (2001) and Gurtin (2002).

Assume that the current state is characterized by continuous displacements, stresses and strains. With increased loading, we will consider the possibility that discontinuous bifurcations of the displacement rate $\dot{u}_i$ and the rate of the displacement gradient $\dot{u}_{i,j}$ can occur across a fixed *singular surface* S within the body, cf. Fig. 24.5. It is assumed that the difference between the value of $\dot{u}_i$ for the bifurcated and fundamental fields is preserved along S, i.e. $[\dot{u}_i]$ = constant along S, where the bracket denotes the difference of the two fields. It will appear that the strain rate and the stress rate then become discontinuous across S; note that here we have also allowed the velocity $\dot{u}_i$ to become discontinuous across S.

Let the orientation of this singular, or characteristic, surface S be defined by the unit normal vector n_i and denote the position of S by x_i. The assumption that $[\dot{u}_i]$ = constant along S implies that

$$d[\dot{u}_i] = [\dot{u}_{i,j}]\, dx_j = 0$$

where $[\dot{u}_{i,j}] = \partial[\dot{u}_i]/\partial x_j$ and dx_j is an arbitrary differential vector tangential to S, cf. Fig. 24.5. The general solution to the expression above is easily seen to become

$$[\dot{u}_{i,j}] = c_i n_j \quad \Rightarrow \quad [\dot{\varepsilon}_{ij}] = \frac{1}{2}(c_i n_j + n_i c_j) \tag{24.41}$$

where c_i is an arbitrary vector.

It follows from equilibrium considerations that the traction rate across the singular surface S must be unique

$$[\dot{\sigma}_{ij}]n_j = 0 \tag{24.42}$$

Assume that the material at both sides of the surface S responds plastically. As the field before bifurcation was assumed to be continuous, the tangential stiffness tensor takes the same value on either side of the surface S. Therefore, insertion of $[\dot{\sigma}_{ij}] = D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}]$ into (24.42) and taking advantage of (24.41) results in

Discontinuous bifurcation condition

$$Q^{ep}_{il} c_l = 0$$

where the elasto-plastic acoustic tensor Q^{ep}_{il} is defined by

$$Q^{ep}_{il} = n_j D^{ep}_{ijkl} n_k \tag{24.43}$$

Other names for the *acoustic tensor* are *characteristic stiffness tensor* and *polarization tensor* and we will later comment upon this terminology; we observe that O^{ep}_{il} is symmetric if D^{ep}_{ijkl} is likewise.

Consider now the situation where the material on one side of the surface S responds plastically ($\dot{\varepsilon}''_{ij}$) and the material on the other side unloads elastically ($\dot{\varepsilon}'_{ij}$). Then (24.43) leads to

$$n_j(D^{ep}_{ijkl}\dot{\varepsilon}''_{kl} - D_{ijkl}\dot{\varepsilon}'_{kl}) = 0 \tag{24.44}$$

Elimination of D_{ijkl} by means of (24.13) gives

$$Q^{ep}_{il} c_l = \frac{1}{A} n_j D_{ijst} \frac{\partial g}{\partial \sigma_{st}} \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}\dot{\varepsilon}'_{kl} \tag{24.45}$$

Expressions (24.43) and (24.45) are the classical bifurcation conditions discussed by Rice (1976).

We will now show that plastic/plastic bifurcation determined by (24.43) always occurs before the plastic/elastic bifurcation given by (24.45); this situation is analogous with that encountered in uniqueness, cf. (24.37), and we can therefore concentrate in the following on the plastic/plastic bifurcation. Before that, we must prove this result. For this purpose introduce the notations

$$a_i = n_j D_{ijst} \frac{\partial f}{\partial \sigma_{st}}; \qquad b_i = n_j D_{ijst} \frac{\partial g}{\partial \sigma_{st}} \tag{24.46}$$

and

$$\alpha = \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} \dot{\varepsilon}'_{kl} \leq 0 \, ; \qquad \beta = \frac{\partial f}{\partial \sigma_{mn}} D_{mnkl} \dot{\varepsilon}''_{kl} > 0 \tag{24.47}$$

where $\dot{\varepsilon}'_{ij}$ corresponds to elastic unloading and $\dot{\varepsilon}''_{ij}$ corresponds to plastic loading. Then (24.45) can be written as

$$Q^{ep}_{il} c_l = \frac{\alpha}{A} b_i \tag{24.48}$$

Elimination of D^{ep}_{ijkl} in (24.44) by means of (24.13) results in

$$Q_{il} c_l = \frac{\beta}{A} b_i \tag{24.49}$$

where

$$\boxed{Q_{il} = n_j D_{ijkl} n_k} \tag{24.50}$$

is the elastic acoustic tensor; as D_{ijkl} is symmetric and positive definite, so is Q_{il}. Determination of b_i from (24.49) and insertion into (24.48) lead to

$$Q^{ep}_{il} c_l = \frac{\alpha}{\beta} Q_{il} c_l \tag{24.51}$$

Consider this expression as a generalized eigenvalue problem with the eigenvalue α/β. According to (24.47), $\alpha/\beta < 0$ and as (24.43) corresponds to a zero eigenvalue, it follows immediately that plastic/plastic bifurcation occurs prior to plastic/elastic bifurcation.

The trivial solution $c_i = 0$ of the bifurcation condition (24.43) implies that the solution is continuous, cf. (24.41) whereas a nontrivial solution exists if Q^{ep}_{il} is singular. By analogy with the terminology used to classify scalar second-order partial differential equations, smooth solutions imply *ellipticity* and discontinuous solutions can only exist if ellipticity is lost. Therefore, when Q^{ep}_{il} becomes singular, ellipticity is lost, cf. Knops and Payne (1971). Clearly, only bifurcation solutions with the discontinuity defined by (24.41) are associated with loss of ellipticity and for linear elasticity where the acoustic tensor is positive definite, ellipticity always exists.

To identify when the elasto-plastic acoustic tensor becomes singular, we consider the following generalized eigenvalue problem

$$Q^{ep}_{il} z_l = \Lambda Q_{il} z_l \tag{24.52}$$

With the definition of the elasto-plastic acoustic tensor given by (24.43), we obtain

$$Q^{ep}_{il} = Q_{il} - \frac{1}{A} b_i a_l$$

and insertion into (24.52) results in

$$(1 - \Lambda)Q_{il}z_l = \frac{1}{A}b_i(a_l z_l) \tag{24.53}$$

Since the elastic acoustic tensor is positive definite and symmetric, it possesses an inverse S_{ki} that is also positive definite and symmetric, i.e.

$$\boxed{S_{ki}Q_{il} = \delta_{kl}} \tag{24.54}$$

Therefore, multiplication of (24.53) by S_{ki} gives

$$(1 - \Lambda)z_k = \frac{1}{A}S_{ki}b_i(a_l z_l) \tag{24.55}$$

It follows directly that

$$\Lambda_1 = \Lambda_2 = 1\,; \quad \text{when} \quad a_l z_l = 0 \tag{24.56}$$

is an eigenvalue of multiplicity of two. If $\Lambda \neq 1$ (24.55) shows that z_k must be proportional to $S_{ki}b_i$ and insertion of this eigenvector gives

$$\Lambda_3 = 1 - \frac{1}{A}a_i S_{il} b_l \tag{24.57}$$

From the results (24.56) and (24.57) appear that $\Lambda_3 = 0$ is the only possibility for O_{il}^{ep} being singular. With (24.14) and (24.46) we then obtain

$$\boxed{H(n_i) = -\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}\frac{\partial g}{\partial \sigma_{kl}} + n_j D_{ijst}\frac{\partial f}{\partial \sigma_{st}}S_{il}\frac{\partial g}{\partial \sigma_{mn}}D_{mnkl}n_k} \tag{24.58}$$

In this expression, all quantities except n_i are known and $H = H(n_i)$ then holds. The task is now to determine that shear band direction n_i which provides the maximum value of H. Evidently, this corresponds to the situation when a discontinuous bifurcation first becomes possible; this value is called H^{db} and we have

$$\boxed{\begin{array}{l}\textit{Discontinuous bifurcations occur for}\\ \quad H^{db} = \max H(n_i)\\ \textit{which corresponds to loss of ellipticity}\end{array}} \tag{24.59}$$

For nonassociated Drucker-Prager plasticity, analytical solutions to this problem were given by Rudnicki and Rice (1975) and later generalizations were established by Ottosen and Runesson (1991a), Runesson *et al.* (1991), Ottosen and Runesson (1991c), Bigoni and Hueckel (1991), Neilsen and Schreyer (1993) and Schreyer and Neilsen (1996). As shown by Rizzi *et al.* (1995, 1996) and Ekh and Runesson (2000), it is possible to rephrase elastic-damage models

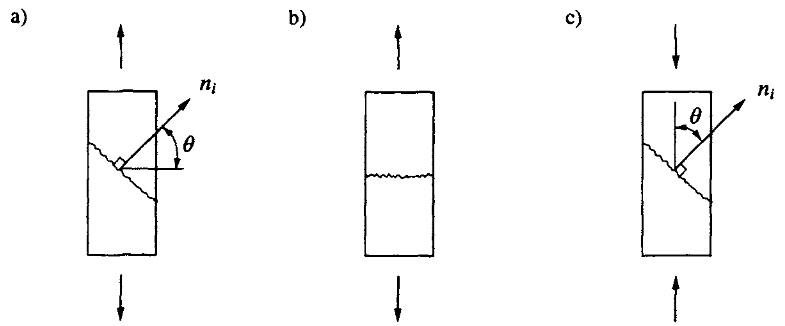

Figure 24.6: Predicted bifurcation directions for associated plasticity; a) von Mises material, b) Rankine material in tension and c) Coulomb material in compression.

in a plasticity-like format so that the results above can be used for bifurcation analysis in damage mechanics. In a finite element context, bifurcation results have been incorporated by the concepts proposed by Ortiz *et al.* (1987) and Larsson *et al.* (1993, 1998); we also refer to the regularization techniques used in localization analysis and discussed by Needleman (1988), Needleman and Tvergaard (1992) and Tomita (1994).

In order to illustrate a few of these results, the coordinate system is chosen collinear with the principal stress directions so that the x_1-axis is in the direction of the largest principal stress σ_1 where $\sigma_1 \geq \sigma_2 \geq \sigma_3$; the angle θ is measured from the unit vector n_i normal to the shear band to the σ_3-direction. For a von Mises material in plane stress, we then obtain

$$\tan^2\theta = -\frac{s_1}{s_2}; \qquad H^{db} = 0 \tag{24.60}$$

valid when $\sigma_1 \geq \sigma_2/2$ and $\sigma_1 \leq 2\sigma_2$, cf. Runesson *et al.* (1991). For uniaxial tension this gives $\theta = 54.7^0$ which is the classical result obtained by Nadai (1950) and Thomas (1961), cf. Fig. 24.6a). It is of considerable interest that experimental data reported by Nadai (1950) shows that $\theta \approx 57^0$. This demonstrates that the bifurcation analysis predicts shear band directions that are in close accuracy with experimental data. However, no information is given about the thickness of the shear band and, as discussed previously, such information requires the use of non-local theories. Returning to (24.60), it is observed that pure shear ($s_1 = -s_2$) results in $\theta = 45^0$.

For associated Coulomb plasticity, Ottosen and Runesson (1991c) derived the following results

$$\tan^2\theta = \frac{1+\sin\phi}{1-\sin\phi}; \qquad H^{db} = 0 \tag{24.61}$$

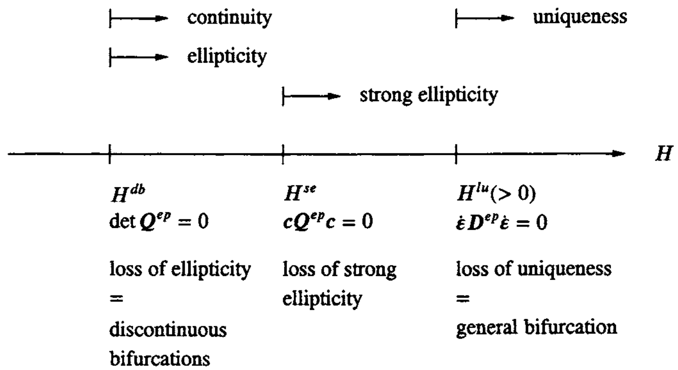

Figure 24.7: Identification of events for nonassociated plasticity.

where the angle θ for uniaxial compression is shown in Fig. 24.6c) and ϕ denotes the friction angle. It follows that $\theta = \pm(45^o + \phi/2)$ and it is of considerable interest that this is exactly the angle predicted by Mohr's failure mode criterion (8.52). The results for nonassociated Coulomb plasticity are also presented by Ottosen and Runesson (1991c). The fact that these results differ from (24.61), underlines that Mohr's failure mode criterion is a postulate that is in agreement with bifurcation analysis for associated Coulomb plasticity. If the friction angle is chosen as $\phi = 90^o$ then the Coulomb criterion reduces to the Rankine criterion and (24.61) provides $\theta = 90^o$. Recalling that θ measures the angle from the normal vector n_i to the σ_3-direction, this means that the bifurcation band is orthogonal to the maximum principal stress as illustrated in Fig. 24.6b); this is in close agreement with the emergence of cracks in brittle materials like concrete and ceramics.

As we have previously defined the concept of ellipticity, we now introduce the condition of *strong ellipticity* according to

Ellipticity is defined by

$$Q_{il}^{ep} c_l \neq 0$$

and strong ellipticity is defined by

$$c_i Q_{il}^{ep} c_l > 0 \tag{24.62}$$

cf. Knops and Payne (1971) and Bigoni and Zaccaria (1992).

Let H^{se} be the plastic modulus associated with loss of strong ellipticity. A comparison of (24.38) and (24.62) shows that uniqueness implies strong ellipticity, whereas the converse is not true; therefore $H^{lu} \geq H^{se}$. Moreover, loss of ellipticity $Q_{il}^{ep} c_l = 0$ implies $c_i Q_{il}^{ep} c_l = 0$, i.e. loss of strong ellipticity. On the other hand, loss of strong ellipticity does not necessarily mean loss of ellipticity.

$\longmapsto$ continuity $\qquad\longmapsto$ uniqueness

$\longmapsto$ ellipticity = strong ellipticity

H

$H^{db} = H^{se}$	$H^{lu} = H^{lp} = 0$
$\det Q^{ep} = 0,\ cQ^{ep}c = 0$	$\dot{\varepsilon}D^{ep}\dot{\varepsilon} = 0,\ D^{ep}\dot{\varepsilon} = 0$
loss of ellipticity	loss of uniqueness
=	=
discontinuous bifurcations	limit point
=	=
loss of strong ellipticity	general bifurcation

Figure 24.8: Identification of events for associated plasticity.

Thus, $H^{se} \geq H^{db}$ stating that strong ellipticity is a stronger requirement than ellipticity, as expected. With these observations as well as (24.39), we have arrived at the conclusions

$$\boxed{\begin{array}{l}\textit{Nonassocitated plasticity}\\ H^{lu} \geq H^{se} \geq H^{db}; \quad H^{lu} > H^{lp} = 0\end{array}} \tag{24.63}$$

For associated plasticity where loss of uniqueness occurs when $H^{lu} = 0$, cf. (24.39) and where the conditions for ellipticity and strong ellipticity coincide, we obtain

$$\boxed{\begin{array}{l}\textit{Associtated plasticity}\\ 0 = H^{lp} = H^{lu} \geq H^{db} = H^{se}\end{array}} \tag{24.64}$$

These conclusions are illustrated in Figs. 24.7 and 24.8.

24.5 Acceleration waves

Propagation of *acceleration waves* in solid bodies relates directly to the important issues of static discontinuous bifurcations, plane wave propagation and stability. Development within this fundamental field was initiated by Hadamard (1903) where elastic bodies were studied. Hill (1961, 1962) and Mandel (1962, 1964) extended this work to elasto-plasticity and further progress was made by Rice (1976); a comprehensive treatment is also given by Truesdell and Noll (1965). It turns out that propagation of acceleration waves leads to an eigenvalue problem where analytical solutions for associated plasticity were given by

Hill (1962); analytical solutions for nonassociated plasticity were established by Ottosen and Runesson (1991b). Before the problem can be formulated, some preliminary results need to be deduced.

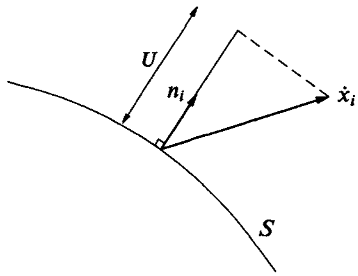

Figure 24.9: Surface S moving through the body at wave speed U.

For a given time t, the increment of the function $f_i(x_k, t)$ is given by $df_i = \partial f_i / \partial x_k \, dx_k$. Let $ds = |dx_k|$ denote the length of dx_k and $s_k = dx_k/ds$ is then the unit vector in the direction of dx_k. It follows that

$$\frac{df_i}{ds} = \frac{\partial f_i}{\partial x_k} s_k \tag{24.65}$$

Let us now study the motion of a surface S which moves through the body, cf. Fig. 24.9. At a given time, it is assumed that f_i is constant along the surface S. This means that $df_i = 0$ holds along S and in analogy with (24.41), it is concluded that

$$\frac{\partial f_i}{\partial x_k} = c_i n_k \tag{24.66}$$

where n_k is the unit vector normal to the surface S and c_i is an arbitrary vector. With this result and choosing in (24.65) s_k as n_k, we obtain

$$\frac{df_i}{dn} = c_i \tag{24.67}$$

which means that (24.66) can be written as

$$\frac{\partial f_i}{\partial x_k} = \frac{df_i}{dn} n_k \tag{24.68}$$

If, instead of the vector f_i, the second-order tensor f_{ij} is considered, we obtain in a similar manner

$$\frac{\partial f_{ij}}{dx_k} = \frac{df_{ij}}{dn} n_k \tag{24.69}$$

According to Fig. 24.9, the surface S moves through the body. A particle on this surface is at the velocity $\dot{x}_i$ and the component of the velocity in the direction of the unit normal vector n_i is by definition the so-called *wave speed* U

$$U = \dot{x}_i n_i \tag{24.70}$$

Assume now that the function $f_i(x_k, t)$ is constant along the surface S. Differentiation with respect to time gives

$$\dot{f}_i + \frac{\partial f_i}{\partial x_k}\dot{x}_k = 0 \tag{24.71}$$

where $\dot{f}_i = \partial f_i/\partial t$. Use of (24.68) in (24.71) gives

$$\dot{f}_i + U\frac{df_i}{dn} = 0 \tag{24.72}$$

Likewise, if the function $f_{ij}(x_k, t)$ is constant along the surface S, we obtain

$$\dot{f}_{ij} + U\frac{df_{ij}}{dn} = 0 \tag{24.73}$$

With these preliminary results, we are ready to investigate the situation where the surface S in Fig. 24.9 is moving through the body. The displacements, velocity, displacement gradient, strains, and stresses are assumed to vary continuously across the surface S

$$[u_i] = [\dot{u}_i] = 0\,; \qquad [u_{i,j}] = [\varepsilon_{ij}] = [\sigma_{ij}] = 0$$

The conditions for which the strain rate, stress rate and the acceleration become discontinuous across the moving surface S will now be investigated.

Since $[\dot{u}_i] = 0$ and $[\sigma_{ij}] = 0$ hold along the surface, use of (24.66) and (24.67) gives

$$\frac{\partial[\dot{u}_i]}{\partial x_k} = c_i n_k\,; \qquad c_i = \frac{d[\dot{u}_i]}{dn} \quad \Rightarrow \quad [\dot{\varepsilon}_{ij}] = \frac{1}{2}(c_i n_j + n_i c_j) \tag{24.74}$$

whereas (24.69) results in

$$\frac{\partial[\sigma_{ij}]}{\partial x_k} = \frac{d[\sigma_{ij}]}{dn}n_k \tag{24.75}$$

Similar to (24.72) and (24.73) we have

$$[\ddot{u}_i] + U\frac{d[\dot{u}_i]}{dn} = 0\,; \qquad [\dot{\sigma}_{ij}] + U\frac{d[\sigma_{ij}]}{dn} = 0 \tag{24.76}$$

Multiplication of the last equation by n_j and use of (24.75) give

$$[\dot{\sigma}_{ij}]n_j + U\frac{\partial[\sigma_{ij}]}{\partial x_j} = 0 \tag{24.77}$$

Since the fields on both sides of the surface S satisfy the equations of motion and since also the body force b_i and mass density ρ are identical, we obtain

$$\frac{\partial[\sigma_{ij}]}{\partial x_j} = \rho[\ddot{u}_i]$$

Use of this expression in (24.77) provides

$$[\dot{\sigma}_{ij}]n_j + U\rho[\ddot{u}_i] = 0 \tag{24.78}$$

With (24.74b), (24.76a) reads

$$[\ddot{u}_i] + Uc_i = 0$$

and insertion into (24.78) gives

$$[\dot{\sigma}_{ij}]n_j = \rho U^2 c_i \tag{24.79}$$

It is of interest to compare this expression with (24.42) applicable to the static situation. Since $[\dot{\sigma}_{ij}] = D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}]$ insertion into the expression above and use of (24.74c) give the following eigenvalue problem

$$\boxed{\begin{array}{l}\textit{Condition for acceleration waves}\\ Q^{ep}_{il}c_l = \rho U^2 c_i\end{array}} \tag{24.80}$$

which certainly reduces to the condition for static discontinuous bifurcations (24.43) when the wave speed $U = 0$.

Since the eigenvalue problem (24.80) only allows certain eigenvectors c_i, these eigenvectors are said to be polarized and this is the reason why Q^{ep}_{il} is occasionally termed the polarization tensor.

The eigenvalue problem above determines the condition for acceleration waves, i.e. the condition that stress rates and strain rates vary discontinuously across a surface S that moves through the body. However, the same condition also controls propagation of *plane waves*. By definition, a plane wave in direction n_m is given by

$$u_i = c_i f(n_m x_m \pm Ut) \tag{24.81}$$

where c_i, n_m and U are constants and f denotes an arbitrary function; U is the *phase speed*. It follows that

$$\ddot{u}_i = U^2 c_i \frac{\partial^2 f}{\partial(n_m x_m \pm Ut)^2}; \quad u_{k,lj} = c_k n_l n_j \frac{\partial^2 f}{\partial(n_m x_m \pm Ut)^2} \tag{24.82}$$

Let the material be stressed to a certain state in static equilibrium, i.e

$$\sigma_{ij,j} + \rho b_i = 0 \tag{24.83}$$

Assuming that the body forces are unchanged, we now investigate the existence of small vibrations about this equilibrium state. The additional small stresses and displacements caused by the vibrations are denoted by σ^*_{ij} and u_i respectively. Thus

$$(\sigma_{ij} + \sigma^*_{ij})_{,j} + \rho b_i = \rho \ddot{u}_i$$

Subtraction of (24.83) gives

$$\sigma^*_{ij,j} = \rho \ddot{u}_i$$

In the original equilibrium configuration, it is assumed that the material is in a homogeneous state. Consequently, the elasto-plastic stiffness tensor D^{ep}_{ijkl} is constant throughout the body and the expression above provides

$$D^{ep}_{ijkl} u_{k,lj} = \rho \ddot{u}_i$$

By using (24.82) in this expression, we are back to the acceleration wave condition given by (24.80). This demonstrates that even though acceleration waves and plane waves are physically distinct phenomena, they are controlled by the same equations. As the investigation of plane waves was based on small vibrations about an already stressed state, this is equivalent to the so-called acoustic approximation in fluid mechanics; for this reason, Q^{ep}_{il} is often referred to as the acoustic tensor.

With the interpretation of (24.80) in terms of plane waves, it is possible to draw certain conclusions about *stability*. If the eigenvalues ρU^2 are real and positive, both acceleration waves and plane waves exist. Since the amplitude of the function f in (24.81) is small, u_i will remain small; this signals a stable situation.

However, if the eigenvalue ρU^2 is real but negative, then the corresponding acceleration wave does not exist, but plane waves will still be possible. To see this, it is noted that any linear combination of solutions of the form (24.81) is a valid plane wave solution. Suppose that $U^2 = -\alpha^2$, i.e. $U = \pm i\alpha$, where α is positive. Since ρU^2 is real, the corresponding eigenvector c_i is also real and $U = \pm i\alpha$ corresponds to the same eigenvalue and therefore also to the same eigenvector. Choosing f in (24.81) as a sine-function, the following plane wave is possible

$$u_i = c_i[sin(n_m x_m + i\alpha t) + sin(n_m x_m - i\alpha t)]$$

Using Euler's formula, we obtain

$$u_i = c_i(e^{\alpha t} + e^{-\alpha t}) sin(n_m x_m) \tag{24.84}$$

Thus, this solution implies that the displacement increases with time; it indicates that any small disturbance can grow infinitely large with time and it certainly signals an unstable situation. For a fixed x_m-position, (24.84) shows that the displacement vector increases with time without any oscillations. Consequently, it is common to use the terminology of *divergence instability* in accordance with the notation adopted in aerodynamics, cf. Rice (1976) and Leipholz (1972).

For linear elasticity as well as associated plasticity, the acoustic tensor is symmetric and as a result the eigenvalues are always real. Before the static bifurcation condition (24.43) is fulfilled, the acoustic tensor only possesses positive eigenvalues. However, after this condition has been passed, a negative eigenvalue exists and the material then exhibits divergence instability.

For nonassociated plasticity, the acoustic tensor is nonsymmetric and, in principle, it may possess complex eigenvalues ρU^2; accordingly, U comprises both a real and an imaginary part. The corresponding u_i-solution can then be shown to consist of oscillations with increasing amplitude. Borrowing again the terminology from aerodynamics, this situation is called *flutter instability*, cf. Rice (1976) and Leipholz (1972). However, it was shown by Ottosen and Runesson (1991b) that for a very broad group of nonassociated plasticity, flutter instability cannot occur, cf. also the arguments presented by Loret (1992), Bigoni and Zaccaria (1994) and Bigoni (1995).