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MECHANICAL VIBRATIONS - Examination task 1

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Problem 1:1

Let $q = q_1, ..., q_n$ be generalized coordinates for a material system $\mathcal{B}$ subjected to the generalized forces $Q_i = \int_{\mathcal{B}} \boldsymbol{f}_a \cdot \frac{\partial \boldsymbol{r}}{\partial q_i} dm$, where $\boldsymbol{f}_a$ is the (specific) active accelerating force. The position vector of a material point $P \in \mathcal{B}$ is then given by $\boldsymbol{r} = \boldsymbol{r}(q; P)$. Let $\boldsymbol{a} = \boldsymbol{a}(q, \dot{q}, \ddot{q}; P)$ denote the acceleration of the material point.

a) Show that

$$\boldsymbol{a} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_j} \ddot{q}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \boldsymbol{r}}{\partial q_j \partial q_k} \dot{q}_j \dot{q}_k \tag{1}$$

b) Show that the equations of motion may be written

$$\sum_{j=1}^{n} a_{ij} \ddot{q}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} C_{ijk} \dot{q}_j \dot{q}_k = Q_i \quad i = 1, ..., n \tag{2}$$

where a_{ij} and C_{ijk} are defined by

$$a_{ij} = \int_{\mathcal{B}} \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial \boldsymbol{r}}{\partial q_j} dm, \quad C_{ijk} = \int_{\mathcal{B}} \frac{\partial \boldsymbol{r}}{\partial q_i} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial q_j \partial q_k} dm \tag{3}$$

(Hint: Use d’Alembert’s principle, equation (4.5) on page 31 in the Lecture Notes).

c) Show that if $\boldsymbol{f}_a = \boldsymbol{g} - cv\boldsymbol{v}$ where $\boldsymbol{g}$ is a constant vector, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, $v = |\boldsymbol{v}|$ and $c > 0$ is a constant, then

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial D}{\partial \dot{q}_i} \tag{4}$$

where

$$V = -m\boldsymbol{g} \cdot \boldsymbol{r}_c \quad \text{and} \quad D = \int_{\mathcal{B}} \frac{cv^3}{3} dm \tag{5}$$

Name:..

and $\boldsymbol{r}_c$ is the position vector of the *centre of mass c* of the body and m is its total mass. (Hint: Use relation (4.7) on page 32 in the Lecture Notes).

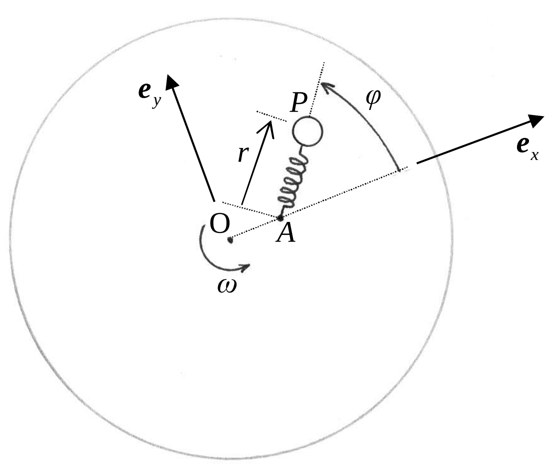
Solution:

Name:..

Problem 1:2

A particle P, which may slide *without friction* on a *horizontal* table, is connected to a fixed point A on the table with a linear elastic spring, see figure below. The table is rotating around a vertical axis through O with the *constant* angular velocity ω. Distance $OA = a$. Particle mass $= m$. Spring constant $= k$ and spring natural (unstressed) length $= r_n$. Using coordinates r and φ

a) Give an expression for the position vector of the particle P: $\boldsymbol{r} = \boldsymbol{r}(r,\varphi;P)$. (Hint: use the orthonormal base $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ fixed to the rotating table, see figure below).
b) Derive an expression for the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ of the particle. (Hint. use the kinematical relations $\dot{\boldsymbol{e}}_x = \boldsymbol{\omega}\times\boldsymbol{e}_x$, $\dot{\boldsymbol{e}}_y = \boldsymbol{\omega}\times\boldsymbol{e}_y$ where $\boldsymbol{\omega} = \boldsymbol{e}_z\omega$).
c) Derive the kinetic energy $T = T(r,\varphi,\dot{r},\dot{\varphi})$ for the system. Give the expressions for T_0, T_1 and T_2.
d) Derive the potential energy $V = V(r,\varphi)$ for the system.
e) Derive (by using Lagranges method) the equations of motion for the system.
f) Determine the (relative) *equilibrium states* ($r(t) = r_o$, $\varphi(t) = \varphi_o$, $r_o > 0$ and φ_o are constants) of the particle. (Note that the existence of equilibrium states will depend on the relation between the parameters: ω, a, m, k and r_n).
g) Consider the modified potential energy $V^* = V - T_0$. Characterize the modified potential energy at the equilibrium states (maximum, minimum or indifferent?).
h) Linearize the equations of motion at the equilibrium state where the modified potential energy has a minimum. Put the equation on matrix format and identify the mass matrix, the stiffness matrix, the damping matrix and the gyroscopic matrix. (Hint: For the linearization use the Maclaurin series for trigonometric functions, i.e. $\sin x = x - \frac{x^3}{3!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \ldots$)



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Solution:

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Problem 2:1

An n-dimensional mechanical system with positive definite mass matrix $\underline{M}$ and positive semi-definite stiffness matrix $\underline{K}$ has the eigenfrequencies $\omega_1,\ \omega_2,\ ...\ ,\omega_n$ and the modal matrix $\underline{X} = [\bar{x}_1\ ...\ \bar{x}_n]$ where

$$(-\omega_i^2 \underline{M} + \underline{K})\bar{x}_i = \bar{0}, \quad i = 1,...,n \tag{1}$$

and

$$\underline{X}^T \underline{M}\,\underline{X} = \underline{\mu} = diag(\mu_1\ ...\ \mu_n) \text{ and } \underline{X}^T \underline{K}\,\underline{X} = \underline{\kappa} = diag(\kappa_1\ ...\ \kappa_n) \tag{2}$$

where $\mu_1, ...\ ,\mu_n$ and $\kappa_1, ...\ ,\kappa_n$ are constants satisfying $\mu_i > 0$ and $\kappa_i \geq 0$. The kinetic and potential energies of the system are given by

$$T = \frac{1}{2}\dot{\bar{q}}^T \underline{M}\dot{\bar{q}} \quad \text{and} \quad V = \frac{1}{2}\bar{q}^T \underline{K}\bar{q} \tag{3}$$

respectively. Using *normal coordinates* $\bar{\eta} = (\eta_1 \quad \eta_2 \quad \cdot \quad \cdot \quad \cdot \quad \eta_n)^T$, (see Lecture Notes on p.100) defined by

$$\bar{q} = \underline{X}\bar{\eta} \tag{4}$$

Show that

a) the kinetic and potential energies may be written

$$T = \frac{1}{2}\dot{\bar{\eta}}^T \underline{\mu}\dot{\bar{\eta}} = \frac{1}{2}\sum_{i=1}^{n} \mu_i \dot{\eta}_i^2 \quad \text{and} \quad V = \frac{1}{2}\bar{\eta}^T \underline{\kappa}\bar{\eta} = \frac{1}{2}\sum_{i=1}^{n} \kappa_i \eta_i^2 \tag{5}$$

b) the equation for the free motion of the system may be written

$$\ddot{\bar{\eta}} + \underline{\omega}^2 \bar{\eta} = \bar{0}, \quad \text{where } \underline{\omega}^2 = \underline{\mu}^{-1}\underline{\kappa} \tag{6}$$

Solution:

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