1 Combined approximation

The goal is to approximate $\mathbf{u}$ given $\mathbf{Ku} = \mathbf{f}$.

A reduced basis is generated using a previous factorization of $\mathbf{K}_0$. Let $\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}$, then the equilibrium equations can be written as

$$(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{u} = \mathbf{f}. \tag{1}$$

A recurrance equation can be found by moving terms to the right-hand side

$$\mathbf{K}_0\mathbf{u}_{k+1} = \mathbf{f} - \Delta\mathbf{K}\mathbf{u}_k. \tag{2}$$

Substitution gives $\mathbf{u}_{k+1}$ in terms of $\mathbf{u}_1 = \mathbf{K}_0^{-1}\mathbf{f}$

$$\mathbf{u}_{k+1} = \Big(\boldsymbol{I} + \mathbf{B} + \mathbf{B^2} + ... + \mathbf{B}^{\boldsymbol{k}}\Big)\mathbf{u}_1, \text{ where } \mathbf{B} = -\mathbf{K}_0^{-1}\Delta\mathbf{K}. \tag{3}$$

Thus, $\mathbf{u}$ can be expressed in the basis defined by

$$\begin{cases} \boldsymbol{\phi}_1 = \mathbf{K}_0^{-1}\mathbf{f} \\ \boldsymbol{\phi}_k = \mathbf{K}_0^{-1}\boldsymbol{\Delta}\mathbf{K}\phi_{k-1}. \end{cases}$$

If the change in the matrix $\mathbf{K}$ is small (given by $\Delta\mathbf{K}$), generating a small amount of basis vectors can yield an accurate approximation. Let $\mathbf{R_b}$ be the matrix consisting of the s first basis vectors, meaning $\mathbf{R_b} = (\boldsymbol{\phi_1}, \boldsymbol{...}, \boldsymbol{\phi_s})$. The approximation can be expressed as $\mathbf{R_b y} = \tilde{\mathbf{u}}$. Substitution into the equilibrium equation, and premultiplying by $\mathbf{R_b}^T$ from the left gives a reduces system of equations

$$\mathbf{R_b}^T\mathbf{K}\mathbf{R_b}\mathbf{y} = \mathbf{R_b}^T\mathbf{f}. \tag{4}$$

Since the basis vectors $\boldsymbol{\phi}_k$ may be almost parallel, orthogonalizing them can make the scheme more numerically stable. Furthermore, orthogonalizing with respect to the matrix $\mathbf{K}$ (meaning $\hat{\boldsymbol{\phi}}_i^T\mathbf{K}\hat{\boldsymbol{\phi}}_j = \delta_{ij}$) makes the system very easy to solve. Let $\mathbf{V_b}$ denote the orthogonal basis generated by the Gram-Schmidt scheme. The resulting system is given by

$$\mathbf{V_b}^T\mathbf{K}\mathbf{V_b}\mathbf{z} = \mathbf{V_b}^T\mathbf{f}.$$

Due to the orthogonalization the left-hand side reduces to the identity matrix, $\mathbf{V_b}^T\mathbf{K}\mathbf{V_b} = \boldsymbol{I}$, and the approximation is given by

$$\tilde{\mathbf{u}} = \mathbf{V_b}\mathbf{z} = \mathbf{V_b}\mathbf{V_b}^T\mathbf{f}.$$

2 Nonlinear FEM

The displacements $\mathbf{a}$ are found by solving the nonlinear system of equations

$$\mathbf{r} = \boldsymbol{f}_{int}(\mathbf{z}, \mathbf{a}) - \boldsymbol{f}_{ext} = \mathbf{0}.$$

The stresses in the structure are given by the change in the strain energy with respect to the strain, or

$$\mathbf{S} = \frac{\partial w}{\partial \mathbf{E}}.$$

Discretizing the structure using finite elements, the internal forces in the structure are given by

$$\boldsymbol{f}_{int} = \int_{V_e} \mathbf{B}^T\mathbf{S}dV_e.$$

A Newton-Raphson scheme is used to iteratively solve for the displacements $\mathbf{a}$,

$$\mathbf{K}\Delta\mathbf{a}_{k+1} = -\mathbf{r}_k$$

The displacements are then updated $\mathbf{a}_{k+1} = \mathbf{a}_k + \Delta\mathbf{a}_{k+1}$.

3 Optimization

As rate independant material models are used, the end compliance can be used as a measure of the stiffness of the structure. If a load controlled scheme is used, minimizing the compliance results in a stiff structure.

$$g_o = c = \boldsymbol{f}_{ext,f}^T \boldsymbol{u}_f.$$

The design variables $\boldsymbol{z}$ are filtered using a density filter to prevent a checkerboard pattern, meaning $\mathbf{z}_f = \mathbf{M}_f \mathbf{z}$. Finally a SIMP interpolation scheme gives elasticity modulus

$$E_e(\rho_e) = E_{min} + (E_{max} - E_{min})\rho_e^q,$$

where q is usually chosen as 3.

A Neo-Hookean material model is used to model the structure, meaning the strain energy is given by

$$w^{NH} = \frac{\kappa}{2}\left(\frac{1}{2}(J^2 - 1) - ln(J)\right) + \frac{1}{2}\mu\left(J^{-2/3} tr(\mathbf{C}) - 3\right),$$

where $J = \det(\mathbf{F})$, $\mathbf{C}$ is the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, $\kappa = \frac{E}{3(1-2\nu)}$ is the bulk modulus, and $\mu = \frac{E}{2(1+\nu)}$ is the shear modulus.

The strain energy is linear in the elasticity modulus $E_e(\rho_e)$, which is independent on the strain $\boldsymbol{E}$. This gives a very easy connection between the design variables and the internal forces, which can be evaluated choosing E = 1, and then multiplying by the interpolated elasticity modulus.

$$(\boldsymbol{f}_{int})_e = E_e(\rho_e) \int_{V_e} \mathbf{B}^T \boldsymbol{S}_e^{NH}|_{E=1} dV_e.$$

The sensitivity of the forces with respect to the element density is thus

$$\frac{d\boldsymbol{f}_{int,e}}{d\rho_e} = \frac{dE_e(\rho_e)}{d\rho_e} \int_{V_e} \mathbf{B}^T \boldsymbol{S}_e^{NH}|_{E=1} dV_e.$$

3.1 Sensitivities

Instead of computing the sensitivity of the goal function, a modified goal function is analysed in order to reduce comptational cost. Thus, consider the function

$$\tilde{g_0} = \boldsymbol{f}_{ext,f}^T \boldsymbol{u}_f + \boldsymbol{\lambda}^T(\boldsymbol{f}_{int,f} - \boldsymbol{f}_{ext,f}).$$

Assuming the structure is in equilibrium, this modified goal function is identical to the original goal function. The sensitivity is given by

$$\begin{aligned}\frac{d\tilde{g_0}}{dz_e} &= \boldsymbol{f}_{ext,f}^T \frac{d\mathbf{u}_f}{dz_e} + \boldsymbol{\lambda}^T\left(\frac{d\boldsymbol{f}_{int,f}}{dz_e} + \frac{d\boldsymbol{f}_{int,f}}{d\boldsymbol{u}_f}\frac{d\boldsymbol{u}_f}{dz_e}\right) = \\ &= \frac{d\boldsymbol{f}_{int,f}^T}{dz_e}\boldsymbol{\lambda} + \left(\boldsymbol{\lambda}^T \frac{d\boldsymbol{f}_{int,f}}{d\boldsymbol{u}_f} + \boldsymbol{f}_{ext,f}^T\right)\frac{d\boldsymbol{u}_f}{dz_e}.\end{aligned}$$

The derivative $\frac{d\boldsymbol{u}_f}{dz_e}$ can be eliminated by asserting

$$\left(\boldsymbol{\lambda}^T \frac{d\boldsymbol{f}_{int,f}}{d\boldsymbol{u}_f} + \boldsymbol{f}_{ext,f}^T\right) = \mathbf{0}$$

Using the notation $\frac{d\boldsymbol{f}_{int,f}}{d\mathbf{u}_f} = \boldsymbol{K}_{ff}$ we have

$$\begin{aligned}\frac{d\tilde{g_0}}{dz_e} &= \frac{d\boldsymbol{f}_{int,f}^T}{dz_e}\boldsymbol{\lambda} \\ \boldsymbol{K}_{ff}\boldsymbol{\lambda} &= -\boldsymbol{f}_{ext,f}.\end{aligned} \tag{5}$$

4 Pseudocode

Combined Approximation

- Initialize quantities
- $r_1 = \mathbf{K}_0^{-1}\mathbf{f},\ \Delta\mathbf{K} = \boldsymbol{K} - \boldsymbol{K}_0$
- $v_1 = (r_1^T\mathbf{K}r_1)^{-1/2}r_1$
- **for** k = 2 **to** s
  - compute basis $r_k = -\mathbf{K}_0^{-1}\Delta\mathbf{K}r_{k-1}$
  - initialize orthogonalization $v_k = r_k$
  - **for** j = 1 **to** k-1
    - $v_k = v_k - (r_k^T\mathbf{K}v_j)v_j$
  - normalize $v_k = (v_k^T\mathbf{K}v_k)^{-1/2}v_k$
- Solve $\boldsymbol{u} = \boldsymbol{V}(\boldsymbol{V}^T\boldsymbol{f})$

Figure 1: Pseudocode for the combined approximation. It's very simple.

Newton-Raphson (Load Controlled)

- Initialize quantities
- **for** k = 1 **to** # of loadsteps
  - Update force vector $\boldsymbol{f} = \boldsymbol{f} + \boldsymbol{\Delta f}$
  - **do while** $||\boldsymbol{r}_{free}|| > r_{tol}$
    - Compute stiffness matrix $\boldsymbol{K}$
    - **if** $\cos(\theta(\boldsymbol{z}, \boldsymbol{z}_{old})) > a_{tol}$ solve $\boldsymbol{K\Delta u} = -\boldsymbol{r}$ using CA
    - **else** factorize $\boldsymbol{K} = \boldsymbol{R'R}$ and solve $\boldsymbol{\Delta u} = -\boldsymbol{R}\backslash(\boldsymbol{R'}\backslash\boldsymbol{r})$
    - Update displacements and residual
  - Accept quantities
  - Save $\boldsymbol{K}^k$, $\boldsymbol{u}^k$ and $\boldsymbol{R}^k$

Figure 2: Newton Raphson scheme using the combined approximation for solving the linear equation system.

Optimization scheme

- **for** i = 1 **to** maxits

  Solve for displacements and forces using NR

  - **if** i == 1
    take n steps
  - **else**
    take 1 step with initial guess $\boldsymbol{u}_{i-1}^{n}$, and $\boldsymbol{K}_{i-1}^{n}$

  Compute sensitivities, solving equation 5 with CA

  Call mmasub

Figure 3: The complete optimzation scheme for the nonlinear optimization.

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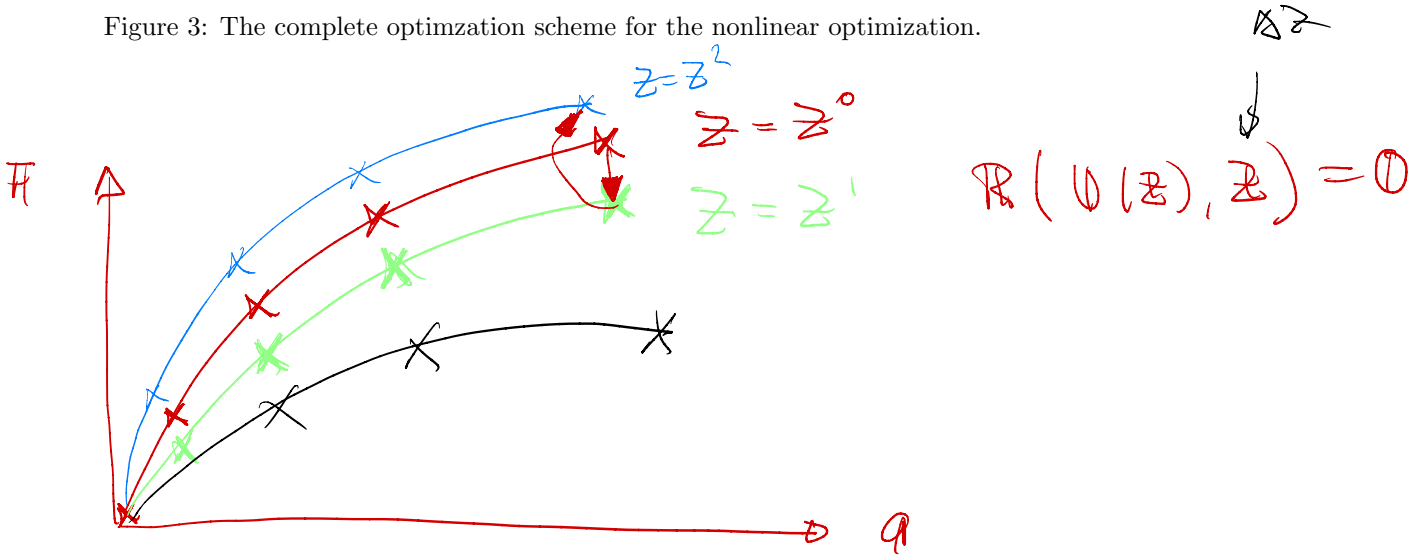

$R(\psi^0, z^0) = 0$

$R(\psi^1, z^1) = 0$

$R(\psi^2, z^2) = 0$ are known $z^3 = z^2 + \Delta z$

Solve $R(\psi, z^2 + \Delta z) = 0$ for $\psi(z^2 + \Delta z)$

$$R(\psi + \Delta\psi, z^3) = R(\psi, z^3) + \left.\frac{\partial R}{\partial \psi}\right|_{\psi, z^3} \Delta\psi \equiv 0 \Rightarrow$$

$$\Delta\psi = -\left[\left.\frac{\partial R}{\partial \psi}\right|_{\psi, z^3}\right] R(\psi, z^3) + \text{Line search, CA}$$

Initial guess for $\Delta\psi$

$$R(\psi^3, z^3) = R(\psi^2, z^2) + \left.\frac{\partial R}{\partial \psi}\right|_2 \Delta\psi + \left.\frac{\partial R}{\partial z}\right|_2 \Delta z \Rightarrow \Delta\psi^0 = -\bar{H}^{-1} \left.\frac{\partial R}{\partial z}\right|_2 \Delta z$$

$R(\psi^3, z^3) = 0$ — want this to be small

No convergence $\Rightarrow$ solve for $z^3 = z^2 + \zeta \Delta z$ $\zeta \in (0,1)$ $\zeta = 0.5 \Rightarrow \psi$

$\Rightarrow$ minimize move limits & wasted updates