Nonlinear dynamical systems.

Home assignment nr 2

Bakground: The goal of this excercise is to investigate theoretically and "experimentally" how the dynamical properties of a map or differential equation depend on the system parameters.

1. For a and b real parameters, consider the Hénon map:

$$\begin{cases} x_{n+1} = 1 + y_n - a{x_n}^2 \\ y_{n+1} = bx_n \end{cases}$$

Question 1 Where in (b, a)-plane do fixed points exist? Compute the region of stability in (b, a)-plane where at least one fixed point is stable. See a detailed hint at the end of this list.

Question 2 Let $b = 0.3$ throughout and investigate (numerically) what happens when a varies from 0 to 3.

Question 3 For $b = 0.3$, build up the so-called *bifurcation diagram* by plotting the x variable of the stationary solution against the parameter a. Plot just points, not lines joining the points.

2. For the differential equation:

$$\begin{cases} \dfrac{dx}{dt} = ax - by - x(x^2 + y^2) \\ \dfrac{dy}{dt} = bx + ay - y(x^2 + y^2) \end{cases}$$

Question 4 (a) Find the fixed point. Compute its stability region in (b, a)-plane. (b) For which (b, a)-values the system has a periodic orbit? Where is it stable? (Hint: Use polar coordinates).

Question 5 (a) Plot a few orbits of the system for $a = 1$ to get an idea of the overall dynamics. (b) Same for $a = -1$.

Question 6 Let $a = b = 1$ and find a *trapping region*. Note that in this case the origin is unstable, so you need to find a large region of phase space such that the flow points inwards on the surface of the region.

Hint to stability part of Question 1

1. First compute the x-coordinate of the fixed points $x_\pm$. Note that for $a > 0$ both fixed points have different signs on their x-coordinate.

2. Compute the jacobian as a function of $x_\pm$, note Trace and Determinant.

3. Appendix A.4.6.2(b) Compute the necessary condition for sability in terms of b.

4. Compute the eigenvalues of fixed points in terms of x: $\lambda_i(x) = -ax + \cdots$.

5. To get stability, in addition to the necessary condition on b, it is required that the eigenvalue of largest modulus for at least one fixed point satisfies $|\lambda_{MAX}(x)| < 1$. The border of the stability region is $|\lambda_{MAX}(x)| = 1$.

6. You get two conditions $|\lambda_{MAX}(x_+)| = 1$, $|\lambda_{MAX}(x_-)| = 1$. One of them can be discarded, while the other gives the border of a region in the (a, b)-plane. Check if the relevant part is the interior region or the exterior region (pick one "easy" point (a_0, b_0) and check fixed points, stability, eigenvalues, etc.).