Nonlinear dynamical systems.

Home assignement nr 1

When necessary, you should use a computer program that can produce pictures and plots (such as Maple, Matlab, etc). Give a short explanation of your results, if necessary with plots, etc. Kindly **avoid** handwritten answers; use a word-processor.

Bakground: A first-order autonomous system in discrete time is given by the difference equation

$$x_{n+1} = f(x_n), \quad n \geq 0$$

We are interested in the stationary state, i.e., what happens when transients have died out.

Question 1 Iterate the function $f(x) = \cos x$ with arbitrary initial conditions. What happens? Follow these steps to prove that the iteration has a fixed point, and compute it (perhaps numerically):

1. Note that after one iteration, the system enters a closed interval I and never leaves this interval in the sequel. Which interval is it?

2. Prove that $f(x)$ is a contraction in I. Use, e.g., the mean value theorem of differential calculus to estimate $f(x) - f(y)$.

3. Check that the interval I and the contraction property allow you to use Banach's theorem and compute the fixed point.

Question 2 Repeat the previous excercise with $f(x) = \arctan x$. You may start with positive x's to fix ideas. Banach's fixed point theorem does not directly apply, because the bounds to contraction are not strictly smaller than one. There is a way around using standard analysis results: What is the limit of a monotonically decreasing sequence bounded from below?

Question 3 Show that the function $f(x) = (\frac{1}{2})(x + a/x)$, $a > 0$ has a fixed point. Hint:

1. Show that if $0 < b \leq \sqrt{a}$ then $f(b) \geq b$. Show also that if $x > \sqrt{a}$ then $f(x) > \sqrt{a}$. This shows that there is an interval $[b, \infty)$ where all iterates lie after the first iteration. Now you need to find a "good" b.

2. Use again the mean value theorem of differential calculus or any other method you choose to estimate $f(x) - f(y)$ inside the interval, and choose now b so that the map is a contraction.

3. Check that now you can use Banach's theorem and compute the fixed point.

Question 4 Choose an arbitrary initial condition x_0 between 0 and 1 and iterate $f(x) = \mu x(1-x)$ (i.e., compute x_n) at least 200 times. Look at the values you get for iterates 201, 202 and so on and try to decide if the iterates approach some stationary value(s). Repeat this for the parameter values $\mu = 2$, 2.9, 3.1, 3.5, 3.72 and 3.83. What happens in each case? Describe the differences. Plot the "stationary" values of x_n (say the different outcome values for a few $n \geq 201$) against the corresponding values of μ. Plot just the points and **not** lines joining the points! An adequate μ-interval is from 2 to 4 with smaller stepsize near 4. Try to reason about an explanation for some of your observations. What happens in particular near $\mu = 3$? (A detailed explanation will arrive in Chapters 5 and 7).

Question 5 Describe all possible behaviours for the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where λ is a real number.
The system can be integrated analytically with elementary methods (it was done in one of your courses), so start by finding the exact solution in closed form. Then plot this solution for different initial conditions to support your description.