# Nonlinear Dynamical Systems

Homework 3

November 29, 2021

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## Question 1

The goal of this task is to compute a fourth order reduction of the center manifold to the following system

$$\begin{aligned}\dot{x} &= -y + xz - x^4 \\ \dot{y} &= x + yz + xyz \\ \dot{z} &= -z - (x^2 + y^2) + z^2 + \sin(x^3)\end{aligned}$$

The fixed point which will be studied is origo, and the linearization around that point is given by the jacobian

$$J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Where it is clear that the non-hyperbolic parts are $x$ and $y$, and $z$ will decay to zero, so the center manifold is a curve $z = h(x, y)$. The system can then be written in the standard format

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + f(\hat{x}, \hat{y}) \\ \dot{\hat{y}} &= B\hat{y} + g(\hat{x}, \hat{y})\end{aligned}$$

Where $\hat{x} = [x,\ y]^T,\ \hat{y} = z$ and

$$\begin{aligned} A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad & f(\hat{x}, \hat{y}) = [xz - x^4,\ yz + xyz]^T \\ B = -1 \quad & g(\hat{x}, \hat{y}) = -(x^2 + y^2) + z^2 + \sin(x^3)\end{aligned}$$

The sequence of approximations to the center manifold $\psi_k$ are computed by first setting $\psi_1(x, y) = 0$ and then following the recipe below

i Set $\Delta_{k+1} = p_{k+1}(\hat{x}; c)$

ii $\psi_{k+1} = \psi_k + \Delta_{k+1}$

iii Find c such that $M\psi_{k+1} = \mathcal{O}(k + 2)$

where $p_k(\hat{x}; c) = p_k(x, y; c) = \sum_{j=0}^{k} x^{(k-j)} y^j c_j$ and $Mu = \nabla u \cdot (A\hat{x} + f(\hat{x}, u)) - Bu - g(\hat{x}, u)$.

The computations were done in wxMaxima, and the resulting polynomial is shown below. See the appendix for the code.

$$\psi_4(x, y) = \frac{1}{10}[-10y^4 + 3y^3 - 20x^2y^2 + 6xy^2 - 10y^2 + 6x^2y - 10x^4 + 4x^3 - 10x^2]$$

Using the center manifold approximation, the flow around the origin can be plotted in $x - y$ phase, see the figure below.

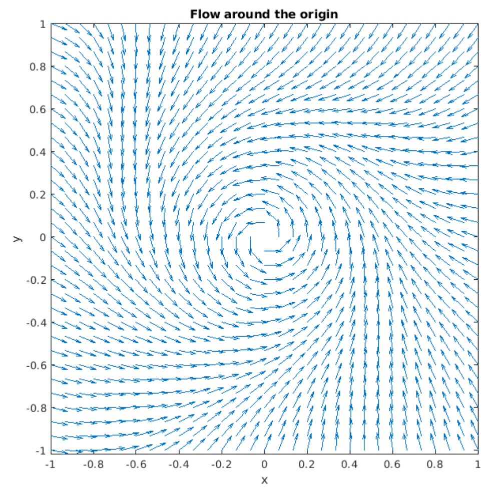



Figure 1: The figure shows the flow on the center manifold around the origin.

Near the origin it is clear that the system is non-hyperbolic, meaning that for small $x$ and $y$ the dynamics of the system on the center manifold resemble the dynamics of the linearized system. This looks like Hopf bifurcation, as we have a pair of complex conjugate eigenvalues.

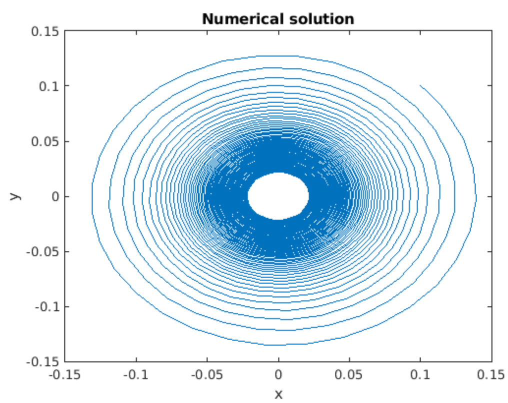



Figure 2: The figure shows a numerical solution to the equation on the manifold. The system seems to be stable, but not asymptotically stable.

## Question 2

The goal of this task is to study the behaviour of the following map

$$
\begin{aligned}
x_{n+1} &= y_n \\
y_{n+1} &= \mu_1 y_n + \mu_2 - x_n^2
\end{aligned}
$$

To start, the fixed points are computed

$$
\begin{aligned}
x &= y \\
y &= \mu_1 y + \mu_2 - x^2
\end{aligned}
$$

Where the second equation gives the fixed points

$$
x = y = \frac{1}{2}\left[(\mu_1 - 1) \pm \sqrt{(\mu_1 - 1)^2 + 4\mu_2}\right]
$$

The system is characterized by computing the Jacobian at the fixed points, and it's eigenvalues

$$
J = \begin{bmatrix} 0 & 1 \\ -2x & \mu_1 \end{bmatrix} \implies \lambda = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 - 8x}\right]
$$

For saddle node bifurcation one of the eigenvalues is 1

$$
2 = \mu_1 \pm \sqrt{\mu_1^2 - 8x} \implies (2 - \mu_1)^2 = \mu_1^2 - 4\left((\mu_1 - 1) \pm \sqrt{(\mu_1 - 1)^2 + 4\mu_2}\right)
$$

Expanding the square on the left side gives a relationship between $\mu_1$ and $\mu_2$

$$
4 - 4\mu_1 + \mu_1^2 = \mu_1^2 - 4\mu_1 + 4 \pm 4\sqrt{(\mu_1 - 1)^2 + 4\mu_2} \implies \mu_2 = -\frac{1}{4}(\mu_1 - 1)^2
$$

For period doubling bifurcations one of the eigenvalues is $-1$

$$
-2 = \mu_1 \pm \sqrt{\mu_1^2 - 8x} \implies (2 + \mu_1)^2 = \mu_1^2 - 4\left((\mu_1 - 1) \pm \sqrt{(\mu_1 - 1)^2 + 4\mu_2}\right)
$$

Expanding the square on the left side gives

$$
4 + 4\mu_1 + \mu_1^2 = \mu_1^2 - 4\mu_1 + 4 \pm 4\sqrt{(\mu_1 - 1)^2 + 4\mu_2} \iff 2\mu_1 = \pm\sqrt{(\mu_1 - 1)^2 + 4\mu_2}
$$

Squaring both sides gives a relationship between $\mu_1$ and $\mu_2$

$$
\mu_2 = \mu_1^2 - \frac{1}{4}(\mu_1 - 1)^2
$$

Lastly, for Hopf bifurcations we need two complex conjugate eigenvalues with modulus 1. To begin, we write

$$
\lambda = A \pm \sqrt{B^2} \quad \text{where} \quad A = \frac{1}{2}\mu_1,\ B^2 = \frac{1}{4}\mu_1^2 - 2x
$$

For complex conjugate eigenvalues, we have $B^2 < 0$. Then we may write $\lambda = A \pm i\sqrt{-B^2}$. The modulus of the complex conjugate eigenvalues is $|\lambda|^2 = A^2 - B^2 = 1$.

$$B^2 = \frac{1}{4}\mu_1^2 - 2x = A^2 - 1 = \frac{1}{4}\mu_1^2 - 1 \iff 2x = 1$$

Inserting the expression for x and some simplification

$$(\mu_1 - 1) \pm \sqrt{(\mu_1 - 1)^2 + 4\mu_2} = 1 \implies (2 - \mu_1)^2 = (\mu_1 - 1)^2 + 4\mu_2$$

Finally we get an expression for $\mu_2$

$$\mu_2 = \frac{1}{4}\left((\mu_1 - 2)^2 - (\mu_1 - 1)^2\right) = -\frac{1}{4}(2\mu_1 - 3)$$

To summarize

$$\begin{cases} \text{Saddle node :} & \mu_2 = -\frac{1}{4}(\mu_1 - 1)^2 \\ \text{Pitchfork :} & \mu_2 = -\frac{1}{4}(\mu_1 - 1)^2 + \mu_1^2 \\ \text{Hopf :} & \mu_2 = -\frac{1}{4}(2\mu_1 - 3) \end{cases}$$

Let us study the stability of fixed points on the lines in parameter space above. For the first case,

$$\mu_2 = -\frac{1}{4}(\mu_1 - 1)^2 \implies x = \frac{1}{2}(\mu_1 - 1)$$

the eigenvalues are then

$$\lambda = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 - 4\mu_1 + 4}\right] = \frac{1}{2}\left[\mu_1 \pm (\mu_1 - 2)\right] = \begin{cases} \mu_1 - 1 \\ 1 \end{cases}$$

meaning the fixed point is stable for $0 < \mu_1 < 2$.
For the second case,

$$\mu_2 = \mu_1^2 - \frac{1}{4}(\mu_1 - 1)^2 \implies x = \frac{1}{2}\left(\mu_1 - 1 \pm \sqrt{4\mu_1^2}\right) = \frac{1}{2}\begin{cases} 3\mu_1 - 1 \\ -\mu_1 - 1 \end{cases}$$

For the positive fixed point,

$$\lambda^+ = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 - 12\mu_1 + 4}\right] = \frac{1}{2}\left[\mu_1 \pm \sqrt{(\mu_1 - 6)^2 - 32}\right]$$

which at $\mu_1 = 0$ has the eigenvalues $-1, 1$, for other $\mu_1$ the fixed point is uninteresting. For the negative fixed point,

$$\lambda^- = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 + 4\mu_1 + 4}\right] = \frac{1}{2}\left[\mu_1 - 1 \pm (\mu_1 + 2)\right] = \begin{cases} \mu_1 + 1 \\ -1 \end{cases}$$

meaning the system is stable for $-2 < \mu_1 \leq 0$.

For the last case,

$$\mu_2 = -\frac{1}{4}(2\mu_1 - 3) \implies x = \frac{1}{2}\left(\mu_1 - 1 \pm \sqrt{(\mu_1 - 1)^2 - 2\mu + 3}\right) =$$
$$= \frac{1}{2}\left(\mu_1 - 1 \pm (\mu_1 - 2)\right) = \begin{cases} \mu_1 - 3/2 \\ 1/2 \end{cases}$$

For the positive fixed point,

$$\lambda^+ = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 - 8\mu_1 + 12}\right] = \frac{1}{2}\left[\mu_1 \pm \sqrt{(\mu_1 - 4)^2 - 4}\right]$$

For $2 < \mu_1 < 6$ these fixed points are complex conjugate with modulus

$$|\lambda^+|^2 = \frac{1}{4}(\mu_1^2 + (4 - (\mu_1 - 4)^2)) = \frac{1}{4}(4 + 8\mu_1 - 16) = 2\mu_1 - 3$$

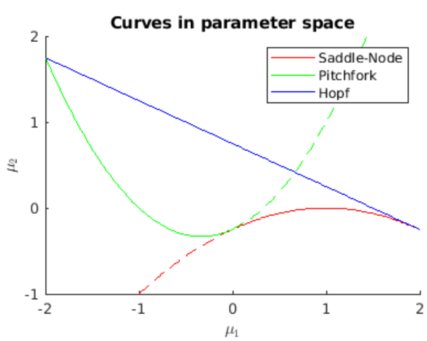
which is only 1 for $\mu_1 = 2$. For the negative fixed point,

$$\lambda^- = \frac{1}{2}\left[\mu_1 \pm \sqrt{\mu_1^2 - 4}\right]$$

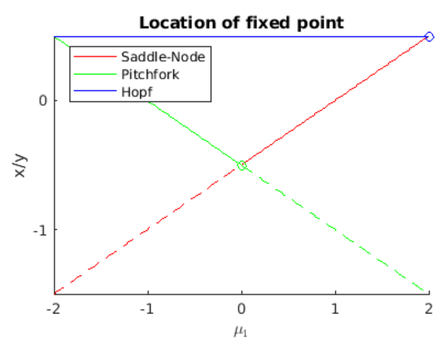
For $-2 < \mu_1 < 2$ these fixed points are complex conjugate with modulus

$$|\lambda^-|^2 = \frac{1}{4}(\mu_1^2 + (4 - \mu_1^2)) = 1$$

and the system is stable. Below are figures showing the curves defined above, and the respective fixed points.




(a) The figure shows curves in parameter space where the system exhibits the different bifurcations. The system has a stable fixed point on the filled lines, which becomes unstable on the dashed lines.




(b) The figure shows the fixed points location as $\mu_1$ changes. The same convention for stability as the previous figure is used.

```
(%i18) grad(F, x):=[diff(F(x), x[1]), diff(F(x), x[2])];
```

(%o18) $\mathrm{grad}(F,x):=[\frac{d}{d\,x_1}\mathrm{F}(x),\frac{d}{d\,x_2}\mathrm{F}(x)]$

```
(%i25) fpx:[x=0, y=0];
       xhat:[x, y];
       yhat:[z];
       A:matrix([0, −1], [1, 0]);
       f(xhat, yhat):=[xhat[1]·yhat − xhat[1]^4, xhat[2]·yhat + xhat[1]·xhat[2]·yhat];
       B:−1;
       g(xhat, yhat):=[−(xhat[1]^2 + xhat[2]^2) + yhat^2 +sin(xhat[1]^3)];
```

(fpx) $[x=0,y=0]$

(xhat) $[x,y]$

(yhat) $[z]$

(A) $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

(%o23) $\mathrm{f}(xhat,yhat):=[xhat_1\,yhat-xhat_1^4,xhat_2\,yhat+xhat_1\,xhat_2\,yhat]$

(B) $-1$

(%o25) $\mathrm{g}(xhat,yhat):=[-(xhat_1^2+xhat_2^2)+yhat^2+\sin(xhat_1^3)]$

```
(%i12) M(F):=grad(F, xhat).(A.xhat + f(xhat, F(xhat))) − B.F(xhat) − g(xhat, F(xhat));
       Mp(pk, dkp1):=grad(dkp1, xhat).(A.xhat) − B.dkp1(xhat) + (grad(pk, xhat).(A.xhat
       dk(w, α, k):=sum(α[i]·w[1]^(k − i)·w[2]^i, i, 0, k);
       generate_equations(poly, k):=makelist(at(ratcoeff(poly, x^(k − i)·y^i), fpx), i, 0, k)
```

(%o9) $\mathrm{M}(F):=\mathrm{grad}(F,xhat)\,.\,(A\,.\,xhat+\mathrm{f}(xhat,\mathrm{F}(xhat)))-B\,.\,\mathrm{F}(xhat)-\mathrm{g}(xhat,\mathrm{F}(xhat))$

(%o11) $\mathrm{dk}(w,\alpha,k):=\sum_{i=0}^{k}\alpha_i\,w_1^{k-i}\,w_2^i$

(%i16)
```
p(w):=0$
for i:1 thru 3 step 1 do (
    define(d(w), dk(w, c, i+1)),
    define(Mpe(x, y), expand(taylor(Mp(p, d)[1], x, 0, i+2))),
    eqs:generate_equations(Mpe(x, y), i+1),
    cs:solve(eqs, makelist(c[j], j, 0, i+1))[1],
    define(p(w), at(d(w), cs) + p(w))
)$
p(xhat);
ratsimp(taylor(subst(x, y, M(p)), x, 0, 5));
```

(%o15) $$-y^4+\frac{3\,y^3}{10}-2\,x^2\,y^2+\frac{3\,x\,y^2}{10}-y^2+\frac{3\,x^2\,y}{5}-x^4+\frac{2\,x^3}{5}-x^2$$

(%o16) $$\left[-\frac{18\,x^5}{5}\right]$$