Nonlinear dynamical systems

Home assignment nr 3

Question 1 Compute a reduction to the central manifold up to 4th. order (i.e., *including* the 4th. order) for the system:

$$\begin{aligned}\dot{x} &= -y + xz - x^4 \\ \dot{y} &= x + yz + xyz \\ \dot{z} &= -z - (x^2 + y^2) + z^2 + \sin(x^3),\end{aligned}$$

as well as the approximation to the dynamical equation on the manifold. Discuss which type of bifurcation is involved. Sketch the flow in the vicinity of the fixed point at the origin. You may diagonalize the linear part (and work with complex variables), but it is simpler to just write the linearisation matrix in blocked form (one diagonal block for eigenvalues with zero real part, another diagonal block for the eigenvalue with negative real part) and work with real varibles throughout. Is the fixed point stable? (Hint: Integrate numerically the equations on the manifold).

Question 2 Show that the system

$$\begin{aligned}x_{n+1} &= y_n \\ y_{n+1} &= \mu_1 y_n + \mu_2 - {x_n}^2\end{aligned}$$

can undergo saddle-node, period doubling and Hopf bifurcations when varying the parameters μ_1, μ_2. In other words, show curves in (μ_1, μ_2)-space along which the eigenvalues of the linearisation at the fixed point(s) become non-hyperbolic in the three mentioned ways. Analyse the "stability nearby" in the simplest cases[1] (the analysis can be done numerically if you want, but it is better to give it a try with pencil, paper and a computer algebra programme). **Side-hint:** If an eigenvalue of a 2×2 matrix is complex, then it will look like $\lambda = A \pm \sqrt{B}$, with A, B real and $B < 0$. In other words, $\lambda = A \pm i\sqrt{-B}$ and $|\lambda|^2 = A^2 - B$.

[1] Some choices of (μ_1, μ_2) lead to *resonant* (i.e., non-standard) Hopf bifurcations. Such values can be avoided in the excercise.