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Introduction to Quantum Computing

Overview:

I.) From bits to qubits: Dirac notation, measurements, Bloch sphere

II.) Quantum circuits: basic single-qubit & two-qubit gates, multipartite quantum states

III.) Entanglement: Bell states, Teleportation, Q-sphere

I. From bits to qubits

- classical states for computation are either "0" or "1"
- in quantum mechanics, a state can be in superposition, i.e., simultaneously in "0" and "1"

→ superpositions allow to perform calculations on many states at the same time

⇒ quantum algorithms with exponential speed-up

BUT: once we measure the superposition state, it collapses to one of its states

(→ we can only get one "answer" and not all answers to all states in the superposition)

⇒ it is not THAT easy to design quantum algorithms, but we can use interference effects

(→ "wrong answers" cancel each other out, while the "right answer" remai

Dirac notation

- used to describe quantum states: let $a, b \in \mathbb{C}^2$. (→ 2-dimensional vector with complex entries)
  - ket: $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
  - bra: $\langle b| = |b\rangle^{+} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}^{+} = (b_1^* \; b_2^*)$ ← complex conjugated & transposed
  - bra-ket: $\underbrace{\langle b|a\rangle}_{\text{inner product}} = a_1 b_1^* + a_2 b_2^* = \langle a|b\rangle^* \in \mathbb{C}$ (→ complex number)
  - ket-bra: $|a\rangle\langle b| = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{pmatrix}$ (→ 2×2-matrix) — not an "X" !!
- we define the states $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are orthogonal: $\langle 0|1\rangle = 1\cdot 0 + 0 \cdot 1 = 0$
- all quantum states are normalized, i.e., $\langle \psi|\psi\rangle = 1$, e.g. $|\psi\rangle = \frac{1}{\sqrt{2}} \cdot (|0\rangle + |1\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Measurements

- we choose orthogonal bases to describe & measure quantum states (→ projective measurement)
- during a meas. onto the basis $\{|0\rangle, |1\rangle\}$, the state will collapse into either state $|0\rangle$ or $|1\rangle$ → as those are the eigenstates of σ_z, we call this a z-measurement
- there are infinitely many different bases, but other common ones are
$\{|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$ and $\{|+i\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), |-i\rangle := \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\}$,
corresponding to the eigenstates of σ_x and σ_y, respectively.
- Born rule: the probability that a state $|\psi\rangle$ collapses during a projective meas. onto the basis $\{|x\rangle, |x^\perp\rangle\}$ to the state $|x\rangle$ is given by
$P(x) = |\langle x|\psi\rangle|^2$, $\sum_i P(x_i) = 1$
- examples: - $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle)$ is meas. in the basis $\{|0\rangle, |1\rangle\}$:
$\rightarrow P(0) = |\langle 0| \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle)|^2 = |\frac{1}{\sqrt{3}} \underbrace{\langle 0|0\rangle}_{1} + \sqrt{\frac{2}{3}} \underbrace{\langle 0|1\rangle}_{0}|^2 = \frac{1}{3} \rightarrow P(1) = \frac{2}{3}$
- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is measured in the basis $\{|+\rangle, |-\rangle\}$:
$\rightarrow P(+) = |\langle +|\psi\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \cdot \frac{1}{\sqrt{2}} \cdot (|0\rangle - |1\rangle)|^2$
$= \frac{1}{4} |\underbrace{\langle 0|0\rangle}_{1} - \underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|0\rangle}_{0} - \underbrace{\langle 1|1\rangle}_{1}|^2 = 0$ → expected, as $\langle +|\psi\rangle = \underbrace{\langle +|-\rangle}_{\text{orthogonal}} = 0$
↳ $P(-) = |\langle -|-\rangle|^2 =$

Bloch sphere:

We can write any normalized (pure) state as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$,

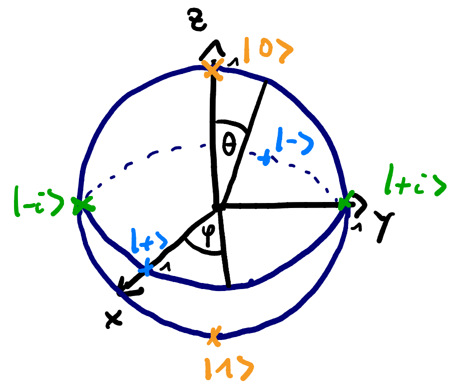
where $\varphi \in [0, 2\pi]$ describes the relative phase and $\theta \in [0, \pi]$ determines the probability to measure $|0\rangle / |1\rangle$: $p(|0\rangle) = \cos^2\frac{\theta}{2}$, $p(|1\rangle) = \sin^2\frac{\theta}{2}$.

⇒ all normalized pure states can be illustrated on the surface of a sphere with radius $|\vec{r}| = 1$, which we call the Bloch sphere

⇒ the coordinates of such a state are given by the Bloch vector: $\vec{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

examples: - $|0\rangle$: $\theta = 0$, φ arbitrary $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- $|1\rangle$: $\theta = \pi$, φ arb. $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
- $|+\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = 0$ $\rightarrow \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $|-\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \pi$ $\rightarrow \vec{r} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$
- $|+i\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- $|-i\rangle$: $\theta = \frac{\pi}{2}$, $\varphi = \frac{3\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



Be careful: On the Bloch sphere, angles are twice as big as in Hilbert space, e.g. $|0\rangle$ & $|1\rangle$ are orthogonal, but on the Bloch sphere their angle is 180°. For a general state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \ldots$ $\rightarrow \theta$ is the angle on the Bloch sphere, while $\frac{\theta}{2}$ is the actual angle in Hilbert space!

$\Rightarrow$ Z-measurement corresponds to a projection onto the Z-axis and analogously for X & Y!