III. Entanglement

- If a pure state $|\psi\rangle_{AB}$ on systems A, B cannot be written as $|\varphi\rangle_A \otimes |\Phi\rangle_B$, it is entangled

Bell states

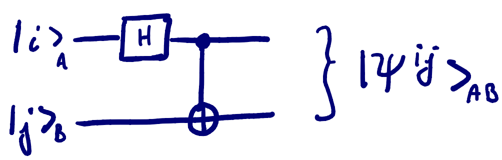
There are four so-called Bell states that are maximally entangled and build an orthonormal basis:

$$|\psi^{00}\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \qquad |\psi^{01}\rangle := \left(|01\rangle + |10\rangle\right)$$

$$|\psi^{10}\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right), \qquad |\psi^{11}\rangle := \left(|01\rangle - |10\rangle\right)$$

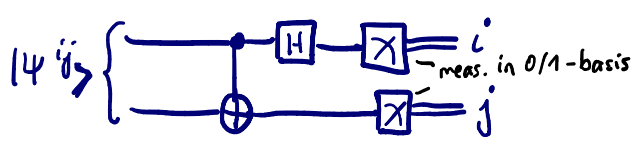
$\Rightarrow$ in general we can write $\quad |\psi^{ij}\rangle = (\mathbb{1} \otimes \sigma_x^j \cdot \sigma_z^i)\,|\psi^{00}\rangle$

Creation of Bell states



initial state $\lvert ij\rangle_{AB}$		$(H_A \otimes \mathbb{1}_B)\lvert ij\rangle_{AB}$		$\lvert\psi^{ij}\rangle$
$\lvert 00\rangle$		$(\lvert 00\rangle + \lvert 10\rangle)/\sqrt{2}$		$(\lvert 00\rangle + \lvert 11\rangle)/\sqrt{2} = \lvert\psi^{00}\rangle$
$\lvert 01\rangle$	$\xrightarrow{H_A}$	$(\lvert 01\rangle + \lvert 11\rangle)/\sqrt{2}$	$\xrightarrow{CNOT_{A,B}}$	$(\lvert 01\rangle + \lvert 10\rangle)/\sqrt{2} = \lvert\psi^{01}\rangle$
$\lvert 10\rangle$		$(\lvert 00\rangle - \lvert 10\rangle)/\sqrt{2}$		$(\lvert 00\rangle - \lvert 11\rangle)/\sqrt{2} = \lvert\psi^{10}\rangle$
$\lvert 11\rangle$		$(\lvert 01\rangle - \lvert 11\rangle)/\sqrt{2}$		$(\lvert 01\rangle - \lvert 10\rangle)/\sqrt{2} = \lvert\psi^{11}\rangle$

$\rightarrow$ opposite direction: **Bell measurement**



$\Rightarrow$ classical outcomes i,j correspond to a meas. of the state $|\psi^{ij}\rangle$

Teleportation

- **Goal**: Alice wants to send her (unknown) state $|\phi\rangle_S := \alpha|0\rangle_S + \beta|1\rangle_S$ to Bob.

  She can only send him two classical bits though. They both share the maximally entangled state $|\psi^{00}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$.

$\Rightarrow$ initial state of the total system:

$$|\phi\rangle_S \otimes |\psi^{00}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle_{SAB} + \alpha|011\rangle_{SAB} + \beta|100\rangle_{SAB} + \beta|111\rangle_{SAB}\right)$$

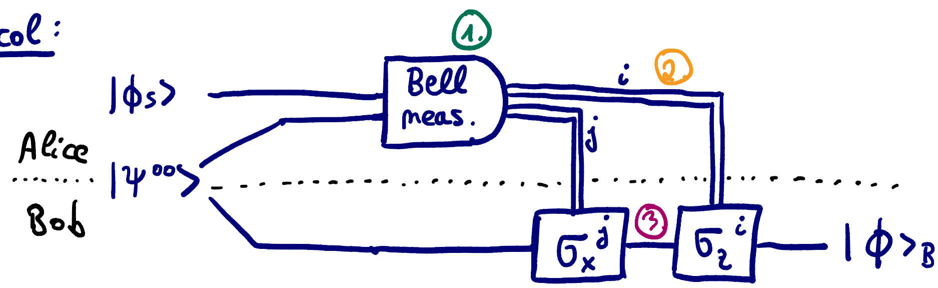
$$= \frac{1}{2\sqrt{2}}\Big[\left(|00\rangle_{SA} + |11\rangle_{SA}\right)\otimes\left(\alpha|0\rangle_B + \beta|1\rangle_B\right) + \left(|01\rangle_{SA} + |10\rangle_{SA}\right)\otimes\left(\alpha|1\rangle_B + \beta|0\rangle_B\right)$$

$$+ \left(|00\rangle_{SA} - |11\rangle_{SA}\right)\otimes\left(\alpha|0\rangle_B - \beta|1\rangle_B\right) + \left(|01\rangle_{SA} - |10\rangle_{SA}\right)\otimes\left(\alpha|1\rangle_B - \beta|0\rangle_B\right)\Big]$$

$$= \frac{1}{2}\Big[|\psi^{00}\rangle_{SA}\otimes|\phi\rangle_B + |\psi^{01}\rangle_{SA}\otimes\left(\sigma_x|\phi\rangle_B\right)$$

$$+ |\psi^{10}\rangle_{SA}\otimes\left(\sigma_z|\phi\rangle_B\right) + |\psi^{11}\rangle_{SA}\otimes\left(\sigma_x\sigma_z|\phi\rangle_B\right)\Big]$$

- **Protocol**:



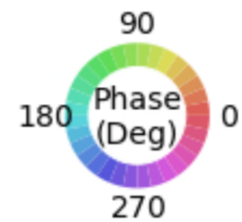
1. Alice performs a meas. on S & A in the Bell basis.
2. She sends her classical outputs i, j to Bob.
3. Bob applies $\sigma_z^i \sigma_x^j$ to his qubit and gets $|\phi\rangle$!

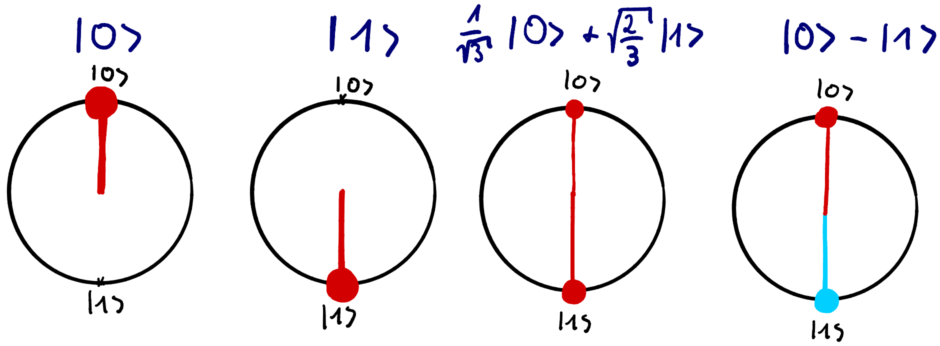
1. Alice's measurement		→ Bob's state	2. Alice sends i j	3. Bob applies	→ Bob's final state
$|\psi^{00}\rangle$		$|\phi\rangle_B$	0 0	$\mathbb{1}$	$|\phi\rangle_B$
$|\psi^{01}\rangle$		$\sigma_x|\phi\rangle_B$	0 1	σ_x	"
$|\psi^{10}\rangle$	$\otimes$	$\sigma_z|\phi\rangle_B$	1 0	σ_z	"
$|\psi^{11}\rangle$		$\sigma_x\sigma_z|\phi\rangle_B$	1 1	$\sigma_z\sigma_x$	"

Note, that Alice's state collapsed during the measurement, so she does not have the initial state $|\phi\rangle$ anymore. This is expected due to the no-cloning theorem, as she cannot copy her state, but just send her state to Bob when destroying her own.

Q-Sphere

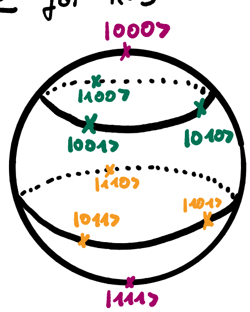
- Bloch sphere can only illustrate the state of 1 qubit $\Rightarrow$ for multiple qubits: Q-sphere
- for one qubit: - the "north pole" represents state $|0\rangle$, the "south pole" state $|1\rangle$
  - the size of the blob is proportional to the prob. of measuring the respective state
  - the color indicates the relative phase compared to state $|0\rangle$





- for n qubits, there are 2^n basis states, e.g. for $n=3$ we have 000, 001, 010, 100, 011, 101, 110, 111
  $\Rightarrow$ we plot those basis states as equally distributed points on a sphere,
  with $0^{\otimes n}$ on the "north pole", $1^{\otimes n}$ on the "south pole" and all other states aligned on parallels, s.t. the number of "1"s on each latitude is constant and increasing from North to South

example: for $n=3$:



$\rightarrow$ size & color of the blobs as before

e.g. $\frac{1}{2}\cdot\left(|000\rangle - |011\rangle + \sqrt{2}\cdot i\cdot|101\rangle\right)$