Part II: Quantum Phase Estimation

- Here, we will use QFT to do a very useful thing.

Problem: Recall that a unitary matrix has eigenvalues of the form $e^{i\theta}$ and that it has eigenvectors that form an orthonormal basis

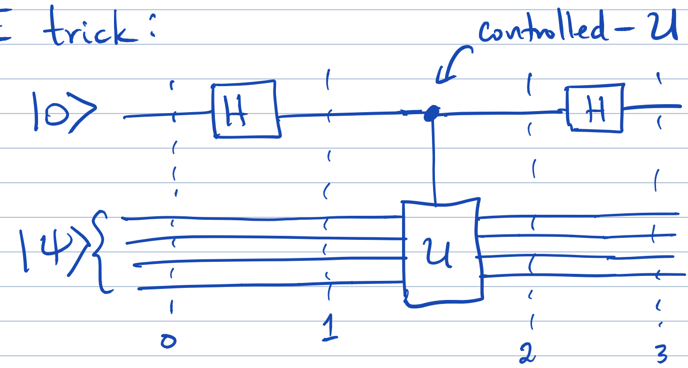
$$\mathcal{U}|\psi\rangle = e^{i\theta_\psi}|\psi\rangle$$

Can we extract θ_ψ given the ability to prepare $|\psi\rangle$?

Solution: Yes. Use QPE.

Why do we care? – Hamiltonian evolution operator is unitary
→ implications for q. simulation.

QPE trick:



step 0: $|0\rangle|\psi\rangle$

step 1: $\frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)$

step 2: $\frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle e^{i\theta_\psi}|\psi\rangle)$

step 3: $\frac{1}{2}\left[(|0\rangle + |1\rangle)|\psi\rangle + e^{i\theta_\psi}(|0\rangle - |1\rangle)|\psi\rangle\right]$

$$= \frac{1}{2}\left[|0\rangle(1+e^{i\theta_\psi}) + |1\rangle(1-e^{i\theta_\psi})\right]|\psi\rangle$$

measure qubit 0:

prob $\left|\frac{1+e^{i\theta_\psi}}{2}\right|^2 \Rightarrow$ measure 0

prob $\left|\frac{1-e^{i\theta_\psi}}{2}\right|^2 \Rightarrow$ measure 1

$\leftarrow$ almost $\frac{1}{2}$ each but with small shift

if $\theta_\psi > 0$, prob[0] = $\cos^2\left(\frac{\theta_\psi}{2}\right)$

prob[1] = $\sin^2\left(\frac{\theta_\psi}{2}\right)$ ~

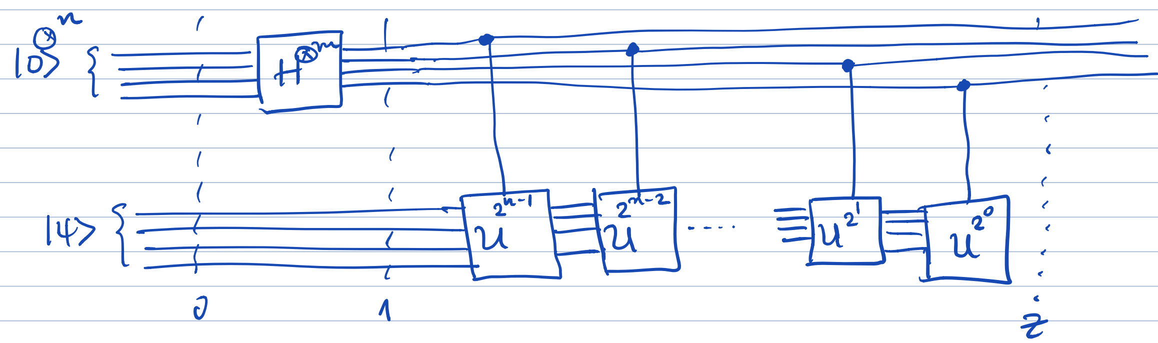
To measure θ precisely using this bit of information, need to do lots of measurements.

eg: $\theta_\psi = 1^\circ \Rightarrow$ prob[0], prob[1] = $\{0.9999, 7.615\times10^{-5}\}$

$\theta_\psi = 10^\circ \Rightarrow$ " $= \{0.9924, 0.007596\}$

$\uparrow$ not a good idea to try to measure with such low precision.

Better solution: use multiple qubits to measure the phase.



step 0: $|0\rangle^{\otimes n}\,|\psi\rangle$

step 1: $\frac{1}{(\sqrt{2})^{n}}\left(|0\rangle + |1\rangle\right)^{\otimes n}\,|\psi\rangle$

Before proceeding, note:

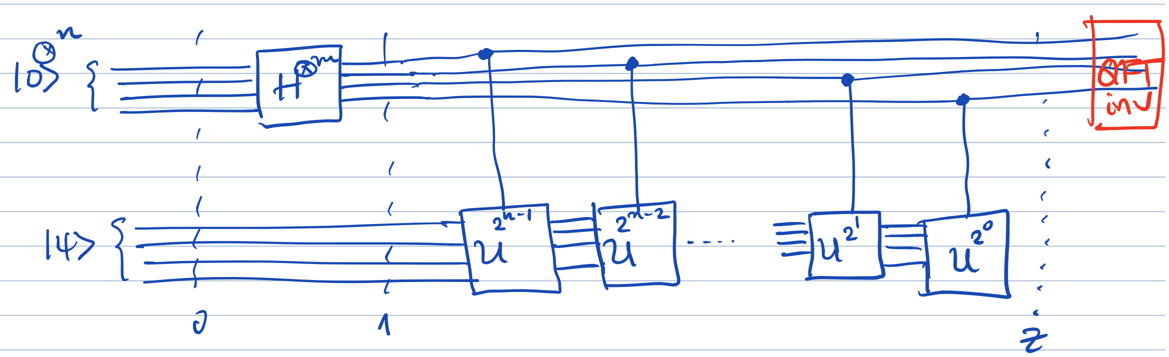
$$U^{2^x}|\psi\rangle = U^{2^x-1}\,U|\psi\rangle$$
$$= U^{2^x-1}e^{i\theta_\psi}|\psi\rangle$$
$$\vdots$$
$$= e^{i\theta_\psi 2^x}|\psi\rangle$$

step 2: $$\left(\frac{1}{\sqrt{2}}\right)^{n}\left(|0\rangle + e^{i\theta_\psi 2^{n-1}}|1\rangle\right)\otimes\left(|0\rangle + e^{i\theta_\psi 2^{n-2}}|1\rangle\right)\otimes\cdots\left(|0\rangle + e^{i\theta_\psi 2^{0}}|1\rangle\right)$$

vs QFT:

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{\frac{2\pi i x}{2^1}}|1\rangle\right)\otimes\left(|0\rangle + e^{\frac{2\pi i x}{2^2}}|1\rangle\right)\otimes\cdots\left(|0\rangle + e^{\frac{2\pi i x}{2^n}}|1\rangle\right)$$

the form is the same, but $\theta_\psi \to 2\pi\frac{\theta_\psi}{2^n}$, so do inverse QFT



Measurement should yield $2^n \times \left(\frac{1}{2\pi}\right)\theta$