Quantum Teleportation

Name:	**Vimaleswar A**
Roll No:	20305
Instructor:	**Dr. Ankur Raina**
Institute/University Name:	IISER Bhopal
Stream:	BS
Department:	EECS

Introduction

We now walk through a popular quantum information technique known as **Quantum Teleportation**. While teleportation has been thought of as the stuff of sci-fi legend, we are going to prove that it is actually already possible today! The technique leverages many foundational principles of quantum computing, and it has lots of useful applications across the entire field. These principles include: the no-cloning theorem, quantum entanglement, and the principle of deferred measurement. We will be demonstrating how quantum teleportation can be quantified using Cluster States.

Single Qubit Quantum Teleportation

Suppose there are two communicating parties named Alice and Bob, and Alice wants to send her quantum state to Bob. The quantum teleportation protocol enables Alice to do exactly this in a very elegant manner, and it can be described in four steps:

1. State preparation: Alice initializes her qubit to the state (Single Qubit) she wishes to teleport.
2. Shared entanglement: A Bell state is created and distributed to Alice and Bob (one qubit each).
3. Change of basis: Alice converts her two qubits from the Bell basis to the computational basis.
4. Measurement: Alice measures her two qubits, then tells Bob how to convert his qubit to obtain the desired state. Note that it is only the quantum information being teleported, and not a physical particle. The overview of the protocol is shown in Figure 1.

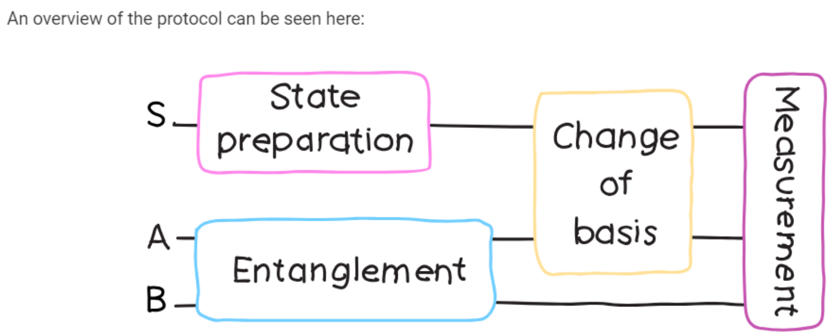

Figure 1: Figure 1: Overview of the protocol

In our toy example we define the single qubit quantum state $\Psi = \frac{j}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ which can be equivalently represented in the matrix form as follows:

$$\begin{bmatrix} 0 + 0.5j \\ -0.8660254 + 0j \end{bmatrix}$$

Result

Original density matrix at A1:

$$\begin{bmatrix} 0.25 + 0j & 0 - 0.4330127j \\ 0 + 0.4330127j & 0.75 + 0j \end{bmatrix}$$

Density matrix after teleportation at B:

$$\begin{bmatrix} 0.25 + 0j & 0 - 0.4330127j \\ 0 + 0.4330127j & 0.75 + 0j \end{bmatrix}$$

The original density matrix at Alice is same as the density matrix after teleportation at Bob. We have successfully been able to teleport the state Ψ from Alice to Bob !!!

Two Qubit Quantum Teleportation

Here, Alice (sender) sends her qubits information to Bob (receiver) under control, supervision of Charlie (controller) through the shared quantum channel between them. We use cluster states as quantum channels shared between three parties Alice, Bob and Charlie. The cluster state has been remotely prepared at the Alice place, where she performs all the necessary unitary operations including the deferred measurement. After performing all the operations, Alice sends the respective qubits to the respective parties. Then Alice immediate measures her qubits in computational basis, which destroys her qubits, making her incapable of any further communication. Now, after receiving qubits from Alice, Charlie measures his qubits in $|+-\rangle$ basis. If the measurement outcome is $|-\rangle$, then Charlie sends a classical bit of information to Bob within a certain time period. After receiving Charlie's classical information in a certain time period, Bob gets that Charlie's measurement outcome is $|-\rangle$, and he has to perform phase (Z-gate) change unitary operations on his qubits. If the measurement outcome is $|+\rangle$, then Charlie does not need to send any classical information to Bob. After waiting for a certain time period and have not received any classical information, Bob understands that Charlie's measurement outcome is $|+\rangle$ and he has to perform another set of unitary operations on his qubits. By the above analysis, we conclude that the classical communication is taking place between Charlie and Bob, only when Charlie's measurement outcome is $|-\rangle$. Hence, the average classical communication cost necessary for our protocols is 0.5 bit. The overview of the protocol is shown in Figure 2.

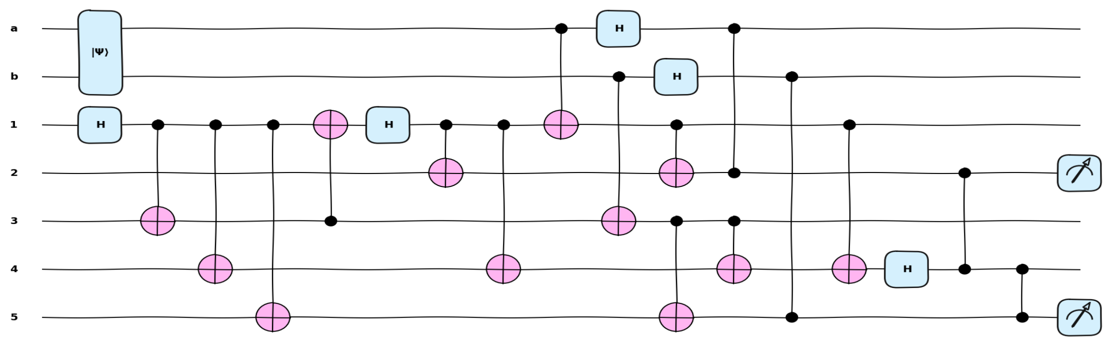

Figure 2: A generalized circuit for teleporting arbitrary two-qubit state using five-qubit cluster state

The cluster state is shared between three parties Alice, Bob and Charlie which are far apart from each other. Alice shares two qubits (1st and 3rd qubit), Charlie shares one qubit (4th qubit), and Bob

(2nd and 5th qubit) shares the remaining two qubits. Now, Bob takes the information of the classical channel into account and decides which set of unitary operations he has to perform on his qubits. Here, if Charlie wants to cheat and sends the wrong information to Bob through the classical channel, then after Bob's unitary operation and consulting with Alice, Bob finds out that Charlie cheated. Thus, even if Charlie wants to, he cannot cheat without getting caught, and this is the beauty of quantum communication. The overview of the Quantum circuit generating the five-qubit cluster state is shown in Figure 3.

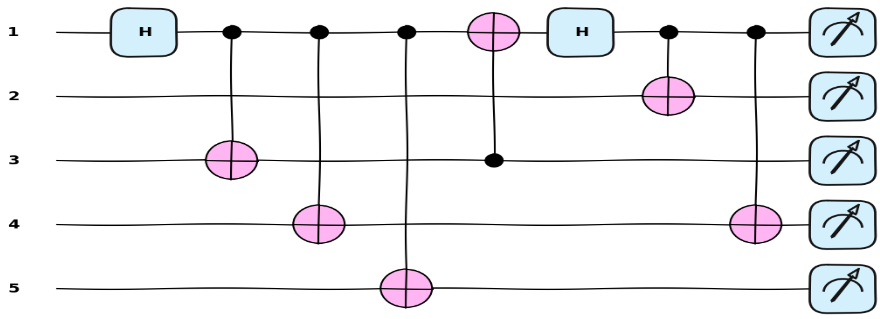

Figure 3: Quantum circuit generating the five-qubit cluster state

Calculation

The five-qubit cluster state from the state $|00000\rangle_{12345}$ is generated by the following circuit as shown in Figure 3. It is used as a quantum channel for quantum communication between Alice, Bob, and Charlie, given as:

$$|C5\rangle_{12345} = 1/2(|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)$$

In this scheme, we wish to teleport any two-qubit state $|\psi_{ab}\rangle = (\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)_{ab}$ through the five-qubit cluster state $|C5\rangle_{12345}$, where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. The qubits 1 and 3 belong to Alice, qubit 4 belongs to Charlie, and qubits 2 and 5 belong to Bob. Now, the joint state of the arbitrary two-qubit state and the five-qubit cluster state can be written as $|\psi_{ab12345}\rangle = |\psi_{ab}\rangle \otimes |C5\rangle_{12345}$.

Now, Alice implements arbitrary two-qubit operations on her share of entangled qubits in the following ways: First, Alice performs a CNOT gate on qubits $(a, 1)$ and $(b, 3)$, and the joint state $|\psi'_{ab12345}\rangle$ is changed to

$$|\psi'_{ab12345}\rangle = \frac{1}{2}\left[\alpha|00\rangle_{ab} \otimes |\phi_1\rangle + \beta|01\rangle_{ab} \otimes |\phi_2\rangle + \gamma|10\rangle_{ab} \otimes |\phi_3\rangle + \delta|11\rangle_{ab} \otimes |\phi_4\rangle\right]$$

Where,

$$\begin{aligned}
|\phi_1\rangle &= (|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)_{12345},\\
|\phi_2\rangle &= (|00100\rangle + |00011\rangle + |11110\rangle + |11001\rangle)_{12345},\\
|\phi_3\rangle &= (|10000\rangle + |10111\rangle + |01010\rangle + |01101\rangle)_{12345},\\
|\phi_4\rangle &= (|10100\rangle + |10011\rangle + |01110\rangle + |01001\rangle)_{12345}.
\end{aligned}$$

Next, Alice applies Hadamard gate on her qubits a and b and the new state is given as

$$\begin{aligned}
|\psi''_{ab12345}\rangle = \frac{1}{4}&\left[\alpha(|00\rangle + |01\rangle + |10\rangle + |11\rangle)_{ab} \otimes |\phi_1\rangle + \beta(|00\rangle - |01\rangle + |10\rangle - |11\rangle)_{ab} \otimes |\phi_2\rangle\right.\\
&\left.+ \gamma(|00\rangle + |01\rangle - |10\rangle - |11\rangle)_{ab} \otimes |\phi_3\rangle + \delta(|00\rangle - |01\rangle - |10\rangle + |11\rangle)_{ab} \otimes |\phi_4\rangle\right]
\end{aligned}$$

After expanding and rearranging the above equation, we have

$$|\psi''_{ab12345}\rangle = \frac{1}{4}\left[|0000\rangle_{ab13} \otimes |\psi_1\rangle + |0001\rangle_{ab13} \otimes |\psi_2\rangle + |0010\rangle_{ab13} \otimes |\psi_3\rangle + |0011\rangle_{ab13}\right.$$
$$\left. |\psi_4\rangle + \ldots + |1111\rangle_{ab13} \otimes |\psi_{16}\rangle\right.$$

where,

$$\begin{aligned}
|\psi_1\rangle &= (\alpha|000\rangle + \beta|011\rangle + \gamma|101\rangle + \delta|110\rangle)_{254},\\
|\psi_2\rangle &= (\alpha|011\rangle + \beta|000\rangle + \gamma|110\rangle + \delta|101\rangle)_{254},\\
|\psi_3\rangle &= (\alpha|101\rangle + \beta|110\rangle + \gamma|000\rangle + \delta|011\rangle)_{254},\\
|\psi_4\rangle &= (\alpha|110\rangle + \beta|101\rangle + \gamma|011\rangle + \delta|000\rangle)_{254},\\
|\psi_5\rangle &= (\alpha|000\rangle - \beta|011\rangle + \gamma|101\rangle - \delta|110\rangle)_{254},\\
|\psi_6\rangle &= (\alpha|011\rangle - \beta|000\rangle + \gamma|110\rangle - \delta|101\rangle)_{254},\\
|\psi_7\rangle &= (\alpha|101\rangle - \beta|110\rangle + \gamma|000\rangle - \delta|011\rangle)_{254},\\
|\psi_8\rangle &= (\alpha|110\rangle - \beta|101\rangle + \gamma|011\rangle - \delta|000\rangle)_{254},\\
|\psi_9\rangle &= (\alpha|000\rangle + \beta|011\rangle - \gamma|101\rangle - \delta|110\rangle)_{254},\\
|\psi_{10}\rangle &= (\alpha|011\rangle + \beta|000\rangle - \gamma|110\rangle - \delta|101\rangle)_{254},\\
|\psi_{11}\rangle &= (\alpha|101\rangle + \beta|110\rangle - \gamma|000\rangle - \delta|011\rangle)_{254},\\
|\psi_{12}\rangle &= (\alpha|110\rangle + \beta|101\rangle - \gamma|011\rangle - \delta|000\rangle)_{254},\\
|\psi_{13}\rangle &= (\alpha|000\rangle - \beta|011\rangle - \gamma|101\rangle + \delta|110\rangle)_{254},\\
|\psi_{14}\rangle &= (\alpha|011\rangle - \beta|000\rangle - \gamma|110\rangle + \delta|101\rangle)_{254},\\
|\psi_{15}\rangle &= (\alpha|101\rangle - \beta|110\rangle - \gamma|000\rangle + \delta|011\rangle)_{254},\\
|\psi_{16}\rangle &= (\alpha|110\rangle - \beta|101\rangle - \gamma|011\rangle + \delta|000\rangle)_{254}.
\end{aligned}$$

To make the teleportation successful and also to reduce the classical communication cost, we use the deferred measurement83. Afer using the deferred measurement, the average classical communication cost required for quantum teleportation is 0.5 bit. In the deferred measurement, CNOT gate is applied on qubits (1, 2), (3, 5), (3, 4), (1, 4) and controlled-Z (CZ) gate is applied on qubits (a, 2) and (b, 5). Afer applying the deferred measurements, the state (5) becomes,
—$\psi'''_{ab12345}\rangle = \frac{1}{4}(|0000\rangle_{ab13} + |0001\rangle_{ab13} + |0010\rangle_{ab13} + |0011\rangle_{ab13} + |0100\rangle_{ab13} + |0101\rangle_{ab13} + |0110\rangle_{ab13}$ + —$0111\rangle_{ab13} + |1000\rangle_{ab13} + |1001\rangle_{ab13} + |1010\rangle_{ab13} + |1011\rangle_{ab13} + |1100\rangle_{ab13} + |1101\rangle_{ab13} + |1110\rangle_{ab13} +$ $|1111\rangle_{ab13} \otimes (\alpha|000\rangle + \beta|011\rangle + \gamma|101\rangle + \delta|110\rangle)_{254}$.

Alice measured her sets of qubits in the computational basis, then the whole state is collapsed to $|\phi_{254}\rangle = (\alpha|000\rangle + \beta|011\rangle + \gamma|101\rangle + \delta|110\rangle)_{254}$ and this can be written in the form as

$$\begin{aligned}
|\phi_{254}\rangle &= (\alpha|000\rangle + \beta|011\rangle + \gamma|101\rangle + \delta|110\rangle)_{254}\\
&= \tfrac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle)_{25} \otimes |+\rangle_4\\
&+ \tfrac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle - \gamma|10\rangle + \delta|11\rangle)_{25} \otimes |-\rangle_4
\end{aligned}$$

Charlie measures his qubit in $|+\rangle_4$ or $|-\rangle_4$ basis, and when his measured state is $|-\rangle_4$, then Charlie sends his qubit information in one bit to Bob within a certain time period through a secure classical channel. After receiving the information from Charlie, Bob gets that Charlie's qubit is in $|-\rangle_4$ state, and he has to apply a phase-change unitary transformation on his qubits. If Charlie's measurement outcome is $|+\rangle_4$, then he does not have to send any classical information to Bob. Whereas Bob after waiting for a certain time period and have not received any information from Charlie, he understands that Charlie's qubit is in $|+\rangle_4$ state. And he has to apply an identity gate unitary transformation on his qubits (see Table 1). For example, let us say Alice measures her state and it comes out $|0111\rangle$, then the whole state is get collapsed to. After receiving qubits from Alice, Charlie measures his qubit state, and let us say that the measurement outcome is $|-\rangle_4$. Then Charlie sends a 1 bit of classical information to Bob. Then Bob understands that he has to perform a phase-change unitary transformation on his qubits, means he applies Z gate on both of his qubits 2 and 5 to get the state sent by Alice. So this protocol is deterministic, i.e. the probability of success is 100 percentange.

The Quantum State Ψ is:

$$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

Result

Original density matrix at Alice:

$$\begin{bmatrix} 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ -0.25+0j & -0.25+0j & -0.25+0j & 0.25+0j \end{bmatrix}$$

Density matrix after teleportation at Bob:

$$\begin{bmatrix} 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ 0.25+0j & 0.25+0j & 0.25+0j & -0.25+0j \\ -0.25+0j & -0.25+0j & -0.25+0j & 0.25+0j \end{bmatrix}$$

The original density matrix at Alice is same as the density matrix after teleportation at Bob for two-qubit teleportation.

Table 1: Classical communication and unitary operations

C.M.S	C.I	B.U.O 2nd qubit	B.U.O 5th qubit
$\|+\rangle$	No classical information has been sent	I	I
$\|-\rangle$	1 Bit	Z	Z

Three Qubit Quantum Teleportation

Here, we consider a seven-qubit cluster state which we used as a quantum channel for teleportation of three-qubit state. The cluster state is shared between as usual Alice (sender), Bob (receiver), and Charlie (controller) which are far apart from each other. More precisely, Alice and Bob each share three qubits and Charlie shares one qubit of the seven-qubit cluster state. The procedure is the same as for the previous scheme, i.e. Alice remotely prepared the cluster state at her place. After applying the required unitary operations between the cluster state and the three-qubit state, Alice sends the respective qubits to the respective parties. And after that, she immediately measured her qubits state and the whole state is get collapsed. Next, Charlie measures his qubit in $|+\rangle$ or $|-\rangle$ basis, and when his measured state is $|-\rangle$, then Charlie sends his qubit information in one bit to Bob within a certain time period through a secure classical channel. After receiving the information from Charlie, Bob gets that Charlie's qubit is in $|-\rangle$ state, and he has to apply a phase-change (Z-gate) unitary transformation on qubit 6 followed by CNOT operation on qubits (2,4) and (4,6). If Charlie measurement outcome is $|+\rangle$, then he does not have to send any classical information to Bob. Whereas Bob after waiting for a certain time period and have not received any information from Charlie, he gets that Charlie's qubit is in $|+\rangle$ state. And he has to apply an identity gate unitary transformation on qubit 6 followed by CNOT operation on qubits (2,4) and (4,6). Herein, the qubits 1, 3 and 5 belong to Alice, the qubits 2, 4 and 6 belong to Bob, and the qubit 7 belongs to Charlie.

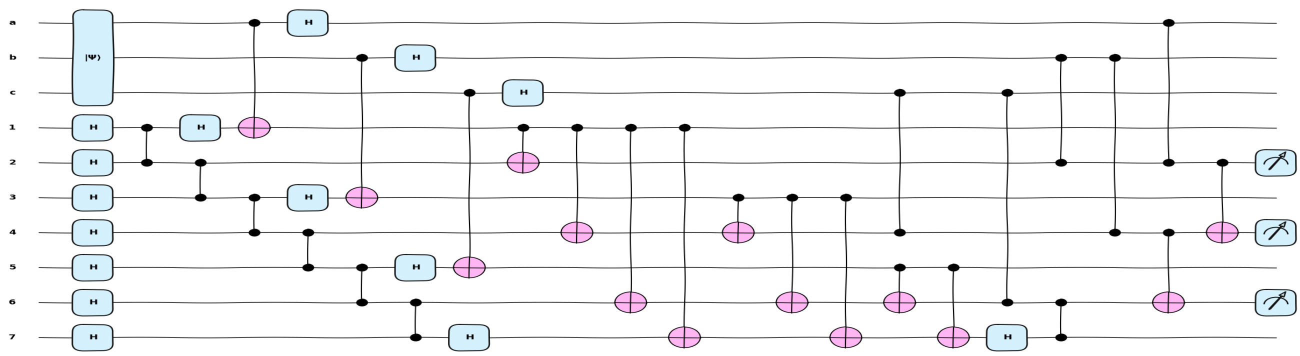

Figure 4: A generalized circuit for teleporting arbitrary three-qubit state

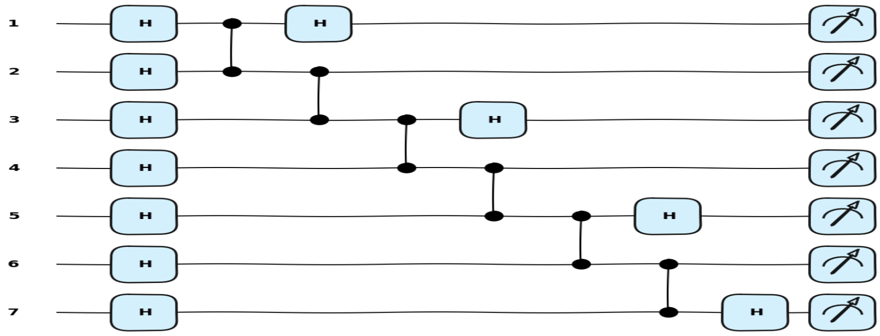

Figure 5: Quantum circuit generating the five-qubit cluster state

The Quantum State Ψ is:

$$\begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Result

Original density matrix at Alice:

$$\begin{bmatrix}
0.25+0j & 0.25+0j & -0.25+0j & -0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
0.25+0j & 0.25+0j & -0.25+0j & -0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
-0.25+0j & -0.25+0j & 0.25+0j & 0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
-0.25+0j & -0.25+0j & 0.25+0j & 0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\
0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j
\end{bmatrix}$$

Density matrix after teleportation at Bob:

$$\begin{bmatrix} 0.25+0j & 0.25+0j & -0.25+0j & -0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ 0.25+0j & 0.25+0j & -0.25+0j & -0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ -0.25+0j & -0.25+0j & 0.25+0j & 0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ -0.25+0j & -0.25+0j & 0.25+0j & 0.25+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \\ 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j & 0+0j \end{bmatrix}$$

The original density matrix at Alice is same as the density matrix after teleportation at Bob for three-qubit teleportation.

In the below table, C.M.S. Charlie's measured state of his qubit, C.I. classical information sent from Charlie to Bob, B.U.O. Bob applying unitary operations on his qubits.

Table 2: Classical communication and unitary operations

C.M.S	C.I	B.U.O 2nd qubit	B.U.O 4th qubit	B.U.O 6th qubit
$\|+\rangle$	No classical information has been sent	CNOT(2,4)	CNOT(4,6)	I
$\|-\rangle$	1 Bit	CNOT(2,4)	CNOT(4,6)	Z

References

1: Quantum Computation and Quantum Information by Michael A. Nielson Isaac L. Chuang
2: Pennylane website
3: Research paper