COMS 3003A
HW 9

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In this HW, we will describe computations of Turing machines using first-order formulas.

When describing a Turing machine $M = (Q, \Sigma, \Gamma, q_0, q_1, q_2, \delta)$, we will use the following vocabulary (remember that every first-order language contains the equality symbol $=$):

- the constant symbol 0 ('zero');
- the unary function symbol S ('successor' on natural numbers);
- for every $k \in \{0, \ldots, |Q|\}$, a unary predicate letter $Q_k(x)$ ('at step x, the machine M is in state q_k');
- a binary predicate letter $C(x, y)$ ('at step x, the machine M scans cell y');
- for every $k \in \{0, \ldots, |\Gamma|\}$, a binary predicate letter $S_k(x, y)$ ('at step x, cell y contains symbol s_k').

When describing computations of Turing machines, we will be making the following assumptions:

- the tape is one-way infinite (hence the cells can be numbered 0, 1, 2, etc.);
- machines can move left and right (no other types of head movement are allowed);
- the left-most cell, i.e., cell 0, always contains an end-of-tape marker, which is a special tape symbol;
- states are always called q_0, q_1, q_2, etc.;
- the starting state is q_0, the accepting state is q_1, and the rejecting state is q_2;
- the tape symbols are always called s_0, s_1, s_2, etc.
- s_0 is the blank;
- s_1 is the end-of-tape marker.

Write formulas saying the following, in the vocabulary listed above:

1. **A formula describing the initial configuration of M on the empty word ϵ.**

$$C(0,0) \wedge Q_0(0) \wedge S_1(0,0) \wedge \forall x\,(\neg(x=0) \to S_0(0,x)).$$

2. **A formula describing the initial configuration of M on the word $s_2\, s_3\, s_2\, s_4$.**

$$C(0,0) \wedge Q_0(0) \wedge S_1(0,0) \wedge S_2(0,S(0)) \wedge S_3(0,S(S(0))) \wedge S_2(0,S(S(S(0)))) \wedge \\ S_4(0,S(S(S(S(0))))) \wedge \forall x\,(\neg(x=0) \to S_0(0,x)).$$

3. **A formula saying that, at every step of the computation, M is in exactly one of states from Q.**

$$\forall x \bigvee_{k=0}^{|Q-1|} (Q_k(x) \wedge \bigwedge_{j \neq k} \neg Q_j(x)).$$

4. **A formula saying that, at every step of the computation, M scans exactly one cell of the tape.**

$$\forall x\, \exists y\, \big(C(x,y) \wedge \forall z\,(C(x,z) \to z=y)\big).$$

5. **A formula saying that, at every step of the computation, each cell of M's tape contains exactly one symbol from Γ.**

$$\forall x \forall y \bigvee_{k=0}^{|\Gamma-1|} (S_k(x,y) \wedge \bigwedge_{j \neq k} \neg S_j(x,y)).$$

6. **Suppose that $q_i s_k \to q_j s_l R$ is an instruction from δ. Write a formula describing what happens when M executes this instruction.**

$$\forall x \forall y \Bigg(Q_i(x) \wedge S_k(x,y) \wedge C(x,y) \to \\ \big(Q_j(S(x)) \wedge S_l(S(x),y) \wedge C(S(x),S(y)) \wedge \\ \forall z\,(\neg(z=y) \to \bigwedge_{k=0}^{|\Gamma-1|} (S_k(x,z) \to S_k(S(x),z))))\big) \Bigg).$$

Then, do the same for an instruction $q_i s_k \to q_j s_l L$ from δ.

$$\forall x \forall y \Bigg(Q_i(x) \wedge S_k(x,y) \wedge C(x,y) \to \\ \big(Q_j(S(x)) \wedge S_l(S(x),y) \wedge \exists v (S(v)=y \wedge C(S(x),v)) \wedge \\ \forall z\,(\neg(z=y) \to \bigwedge_{k=0}^{|\Gamma-1|} (S_k(x,z) \to S_k(S(x),z))))\big) \Bigg).$$

7. **A formula saying that M halts.**

$$\exists x\,(Q_1(x) \vee Q_2(x)).$$

Hint: if you are struggling to answer $1-7$ in full generality, first write down a small Turing machine of your own and answer all the questions with regards to that Turing machine. Then, attempt to do the same for a specific, but generic, Turing machine M, as stated above.