Realistic Projectiles

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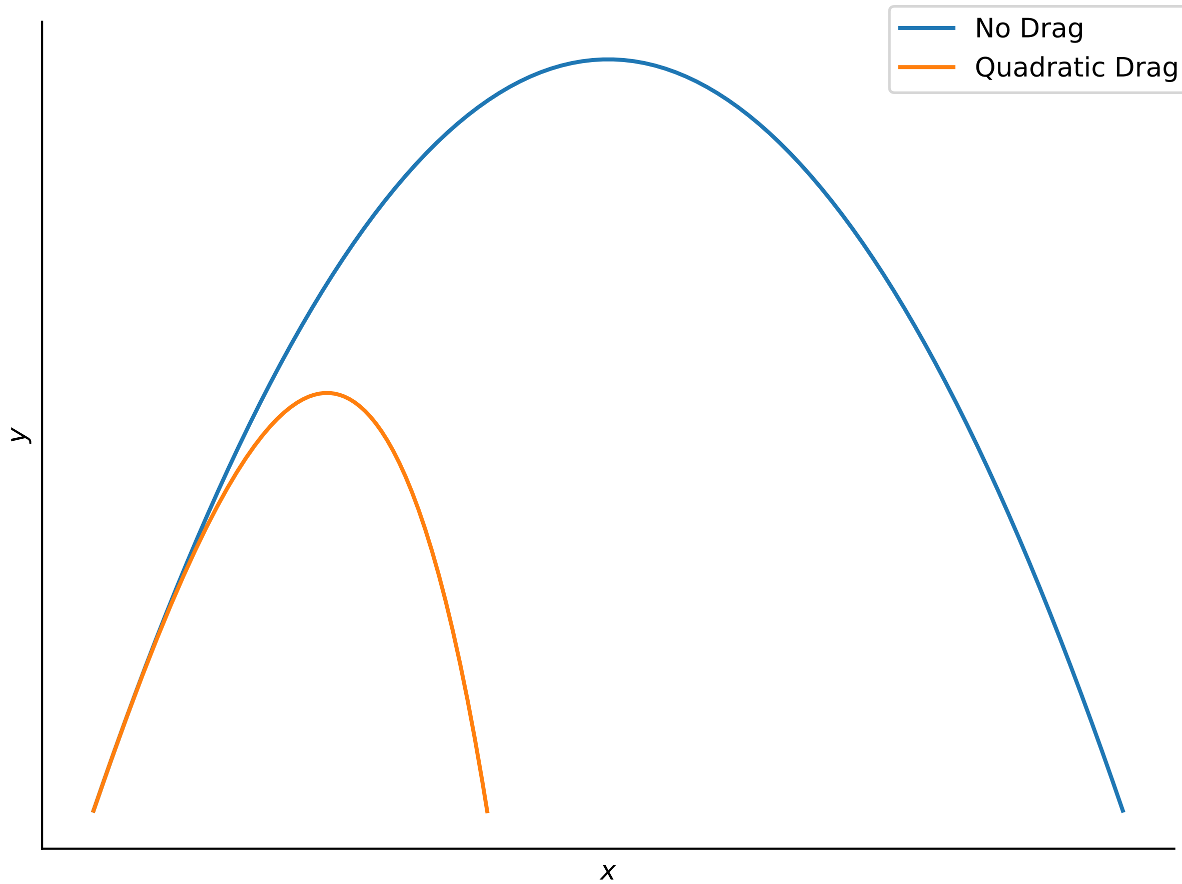

Contents

1. Quadratic Drag Equation

In introductory physics, projectiles are typically modeled as experiencing negligible air drag. For this project, projectiles were modeled as experiencing *quadratic drag*.

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} - kv^2\hat{v} \tag{1}$$

The terms in this equation are as follows:

$$\begin{aligned}
\vec{r} &= \begin{pmatrix} x \\ y \end{pmatrix} && \text{(position)} \\
\vec{v} &= \frac{d\vec{r}}{dt} && \text{(velocity)} \\
\vec{g} &= \begin{pmatrix} 0 \\ -g \end{pmatrix} && \text{(gravitation acceleration)} \\
k &= \text{"constant"} && \text{(drag constant)}
\end{aligned} \tag{2}$$

The y axis points straight up, and the x axis points horizontally along the plane of motion of the projectile. This keeps the motion in 2 dimensions. Projectiles were started on the ground at $(x, y) = (0, 0)$.

To focus on scale-independent features of the motion, units of distance and time were used such that $g = 1$ and $k = 1$. This makes the terminal speed $v_\infty = 1$.

2. Runge-Kutta Four (RK4) Method for Systems

To solve the system of differential equations, the RK4 method for systems was used.

$$\begin{aligned}
\frac{d\vec{u}}{dt} &= \vec{f}(t, \vec{u}) \\
\vec{k}_1 &= \vec{f}(t_i, \vec{u}_i) \\
\vec{k}_2 &= \vec{f}\left(t_i + \frac{h}{2}, \vec{u}_i + \frac{h}{2}\vec{k}_1\right) \\
\vec{k}_3 &= \vec{f}\left(t_i + \frac{h}{2}, \vec{u}_i + \frac{h}{2}\vec{k}_2\right) \\
\vec{k}_4 &= \vec{f}\left(t_i + h, \vec{u}_i + h\vec{k}_3\right) \\
\vec{u}_{i+1} &= \vec{u}_i + \frac{h}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right) \\
t_{i+1} &= t_i + h
\end{aligned} \tag{3}$$

The `rk4.py` file contains a `calculate()` function that implements the RK4 method for systems in Python. `calculate()` returns arrays containing t and $\vec{u}$ values, and it takes the following parameters:

- `t_0`: a starting t value
- `u_0`: an array containing the initial value for each variable in the system $\vec{u}_0$
- `h`: a step size

- `diff`: a function that takes `t` and `u` as inputs and returns an array containing the result of the differential equation $\vec{f}(t, \vec{u})$
- `should_exit`: a function that takes `t` and `u` as inputs and returns `True` when the iterations should stop

rk4.py

```
import numpy as np

def calculate(t_0, u_0, h, diff, should_exit):
    """
    Calculate t values & u vectors using the vector RK4 method on a system of 1st
order ODEs.
    Return an array of t values, and an array of u vectors.

    - t_0: starting t value
    - u_0: starting u vector
    - h: step size
    - diff: function that takes t and u as arguments and returns du/dt
    - should_exit: function that takes t and u as arguments and returns True if no
more steps should be taken
    """
    t = [t_0]
    u = [u_0]

    while not should_exit(t[-1], u[-1]):
        k_1 = diff(t[-1], u[-1])
        k_2 = diff(t[-1] + h/2, u[-1] + h/2 * k_1)
        k_3 = diff(t[-1] + h/2, u[-1] + h/2 * k_2)
        k_4 = diff(t[-1] + h, u[-1] + h * k_3)

        u_next = u[-1] + h/6 * (k_1 + 2*k_2 + 2*k_3 + k_4)
        u.append(u_next)
        t_next = t[-1] + h
        t.append(t_next)

    return np.array(t), np.array(u)
```

The `projectile.py` file contains functions to help simulate the motion of a projectile experiencing quadratic drag. The `u_prime()` function implements the system of differential equations that describe the motion of the projectile.

$$\begin{gathered} \vec{u} = \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} \\ \frac{d\vec{u}}{dt} = \begin{pmatrix} v_x \\ v_y \\ -kvv_x \\ -g - kvv_y \end{pmatrix} \end{gathered} \tag{4}$$

The `launch()` function simulates launching a projectile from the origin with the given initial velocity `v_0`, and it returns arrays containing t and $\vec{u}$ values. By default the `should_exit` parameter is set to the `below_ground()` function, which returns `True` when the projectile falls below the ground ($y < 0$).

projectile.py

```
import rk4
import numpy as np

def u_prime(t, u):
    """
    Return an array of derivatives with respect to t for each component of the vector
u.
    u consists of x, y, v_x, and v_y.
    """
    k = 1
    g = 1

    x, y, v_x, v_y = u
    speed = np.sqrt(v_x**2 + v_y**2);
    drag_part = k * speed
    if speed == 0:
        drag_x = 0
        drag_y = 0
    else:
        drag_x = drag_part * v_x
        drag_y = drag_part * v_y

    return np.array([
        v_x,
        v_y,
        -drag_x,
        -g - drag_y,
    ])

def below_ground(t, u):
    y = u[1]
    return y < 0

def launch(v_0, should_exit=below_ground):
    """
    Launch a projectile from the origin with the given launch velocity.
    By default, stop after the projectile hits the ground (when y < 0).
    If desired, an alternate function of t and u can be passed.
    This function should return True when the exit condition is met.

    Return the arrays of t and u.
    Note that u consists of x, y, v_x, and v_y.
    """
    t_0 = 0.0
    h = 0.001
    v_x, v_y = v_0
```

```
    u_0 = np.array([0, 0, v_x, v_y])

    t, u = rk4.calculate(t_0, u_0, h, u_prime, should_exit)

    return t, u

def horizontal_range(v_0):
    """
    Launch a projectile from the origin with the given launch velocity.
    Return the horizontal range of the projectile.
    Approximate the range as the x-intercept of the line connecting the last two
points of the projectile's path.
    """
    t, u = launch(v_0, below_ground)
    x = u[:, 0]
    y = u[:, 1]

    if x[-2] == x[-1]:
        distance = x[-1]
    else:
        slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
        distance = -y[-1]/slope + x[-1]
    return distance
```

3. Interdependence of Horizontal and Vertical Motion

When modeling projectiles with no drag or linear drag, one property that emerges is the independence of horizontal and vertical motion. This occurs because $\frac{dv_x}{dt}$ does not depend on y or v_y, and similarly $\frac{dv_y}{dt}$ does not depend on x or v_x.

With the quadratic drag model, $\frac{dv_x}{dt}$ depends on v_y.

$$\frac{dv_x}{dt} = -kvv_x = -kv_x\sqrt{v_x^2 + v_y^2} \tag{5}$$

Similarly, $\frac{dv_y}{dt}$ depends on v_x.

$$\frac{dv_y}{dt} = -g - kvv_y = -g - kv_y\sqrt{v_x^2 + v_y^2} \tag{6}$$

Increasing v_x or v_y causes the drag experienced in both the x and y directions to increase. This leads to the interdependence of horizontal and vertical motion.

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

# Plot horizontal position over time as vertical velocity changes
v_0x = 0.5
v_0y_values = np.linspace(0, 1.5, 7)
t_f = 2.0
```

```python
fig, ax = plt.subplots()
ax.set(ylabel="$x$", xlabel="$t$", title=f"$v_{{0x}}$ = {v_0x:.2f}")

for v_0y in v_0y_values:
    t, u = projectile.launch([v_0x, v_0y], lambda t, u: t >= t_f)
    x = u[:, 0]
    ax.plot(t, x)

fig.legend(v_0y_values, title="$v_{0y}$", loc="center right")
fig.tight_layout()
fig.savefig("media/x_vs_t.svg")


# Plot vertical position over time as horizontal velocity changes
v_0y = 0.5
v_0x_values = np.linspace(0, 1.5, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$t$", title=f"$v_{{0y}}$ = {v_0y:.2f}")

for v_0x in v_0x_values:
    t, u = projectile.launch([v_0x, v_0y])
    y = u[:, 1]
    ax.plot(t, y)

fig.legend(v_0x_values, title="$v_{0x}$")
fig.tight_layout()
fig.savefig("media/y_vs_t.svg")
```

Figure 1 plots the horizontal position of the projectile over time as the initial vertical velocity varies. The initial horizontal velocity was kept constant. Each launch was kept going for the same amount of time. If x and y motion were independent, then each plot for a different v_{0y} value would be identical. Since the plots vary as v_{0y} changes, this demonstrates the interdependence of x and y motion.

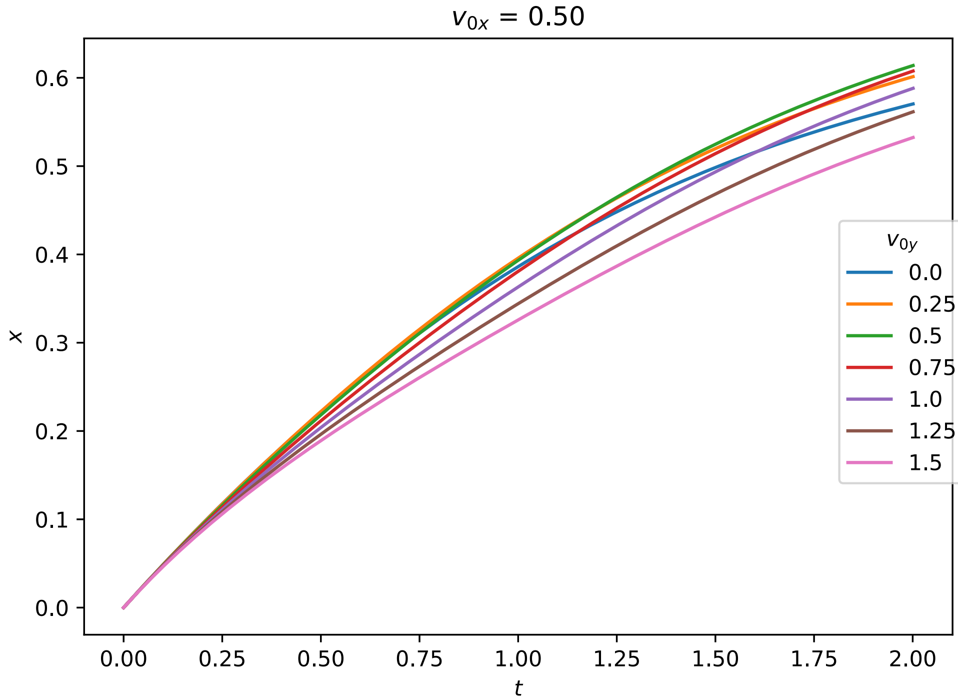

Figure 1: x vs t as v_{0y} Varies

Figure 2 plots the vertical position of the projectile over time as the initial horizontal velocity varies. The initial vertical velocity was kept constant. Each launch was kept going until the projectile hit the ground ($y = 0$). Notice that the max height and time in the air decrease as v_{0x} increases. If x and y motion were independent, then each plot for a different v_{0x} value would be identical. Since the plots vary as v_{0x} changes, this demonstrates the interdependence of x and y motion.

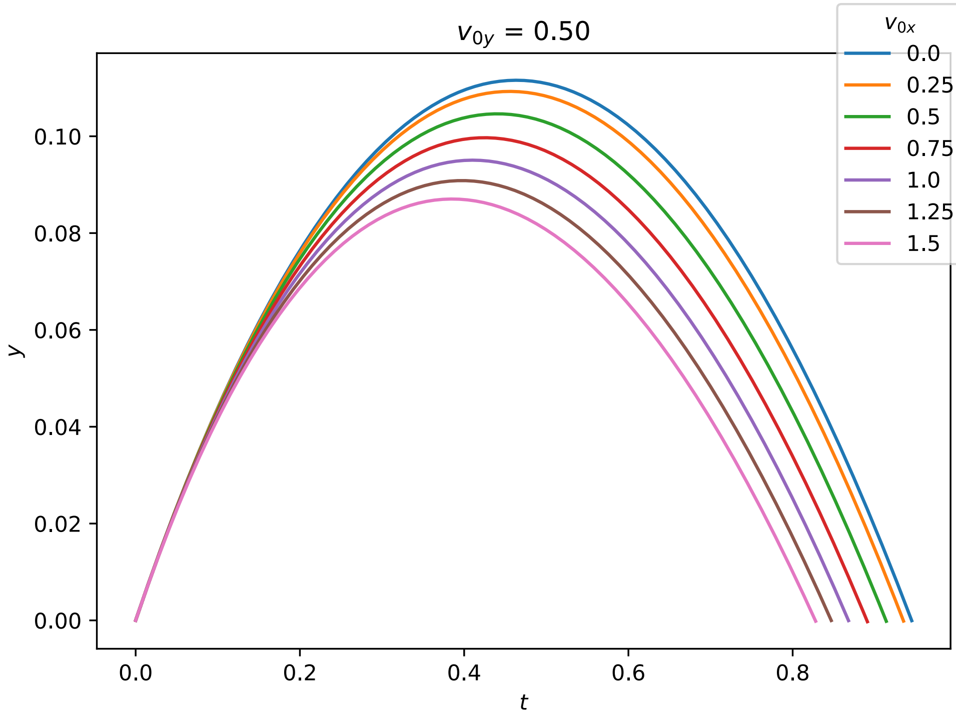

Figure 2: y vs t as v_{0x} Varies

4. Trajectory Shapes

Projectiles that experience no drag follow parabolic trajectories. In contrast, projectiles that experience quadratic drag follow trajectories that are approximately parabolic. These trajectories are asymmetric and drop more steeply than they rise. The horizontal velocity continuously decreases towards zero, since the drag always opposes the motion and decreases as the horizontal velocity decreases. The vertical velocity decreases to zero initially, then decreases towards the negative of the terminal speed. Thus, the angle of descent approaches −90° as the projectile falls for a long time.

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

# Plot trajectories as launch angle changes
v_0 = 1.5
deg_theta_values = np.linspace(0, 90, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$x$", title=f"$v_0$ = {v_0:.2f}")

for rad_theta in np.radians(deg_theta_values):
    v_x = v_0 * np.cos(rad_theta)
    v_y = v_0 * np.sin(rad_theta)
    t, u = projectile.launch([v_x, v_y])
    x = u[:, 0]
    y = u[:, 1]
```

```
    ax.plot(x, y)

fig.legend(deg_theta_values, title="$\\theta$ (°)")
fig.tight_layout()
fig.savefig("media/xy_vs_theta.svg")


# Plot trajectories as launch speed changes
deg_theta = 45
rad_theta = np.radians(deg_theta)
v_0_values = np.linspace(0, 2.0, 9)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$x$", title=f"$\\theta$ = {deg_theta}°")

for v_0 in v_0_values:
    v_x = v_0 * np.cos(rad_theta)
    v_y = v_0 * np.sin(rad_theta)
    t, u = projectile.launch([v_x, v_y])
    x = u[:, 0]
    y = u[:, 1]
    ax.plot(x, y)

fig.legend(v_0_values, title="$v_0$")
fig.tight_layout()
fig.savefig("media/xy_vs_v.svg")
```

Figure 3 plots the projectile's trajectory as the launch angle varies. The launch speed was kept constant. The trajectories appear more symmetric for launch angles that are closer to 0° or 90°. When the launch angle is smaller, the projectile does not stay in the air for very long, reducing the time drag has to impact the trajectory. When the launch angle is larger, the projectile has a relatively low horizontal velocity, so the drag will not reduce the horizontal velocity as much to shift the trajectory.

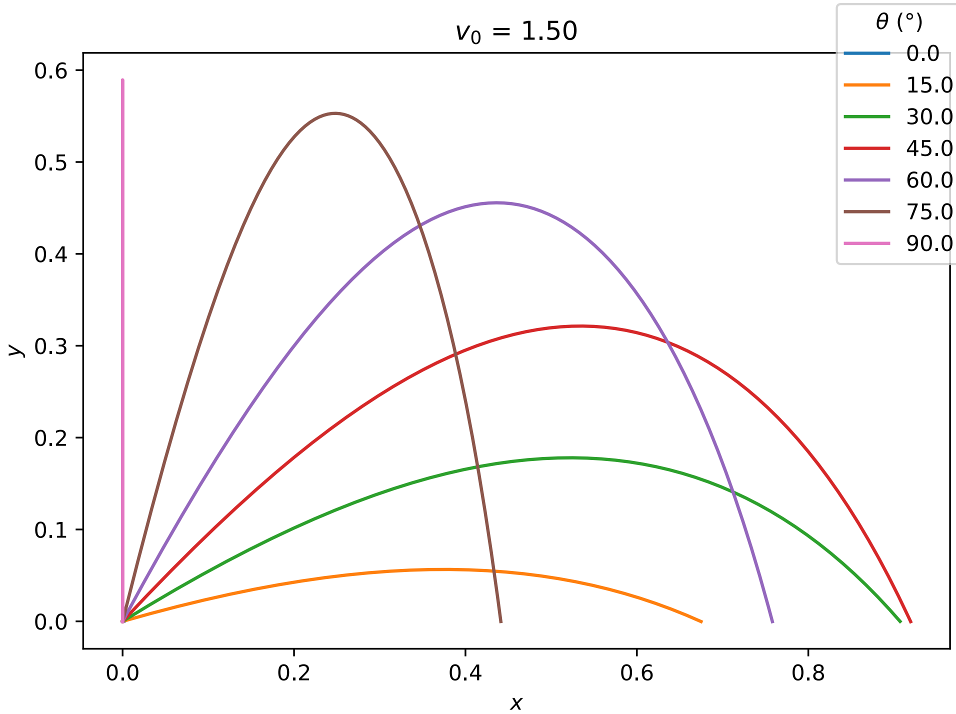

Figure 3: Trajectory as θ Varies

Figure 4 plots the projectile's trajectory as the launch speed varies. The launch angle was kept constant. The trajectories for smaller launch speeds appear more symmetric than for higher launch velocities. When the launch speed is higher, the projectile experiences higher drag force on average and its trajectory is more noticeably impacted. When the launch speed is lower, the projectile experiences lower drag force on average and its trajectory is less noticeably impacted.

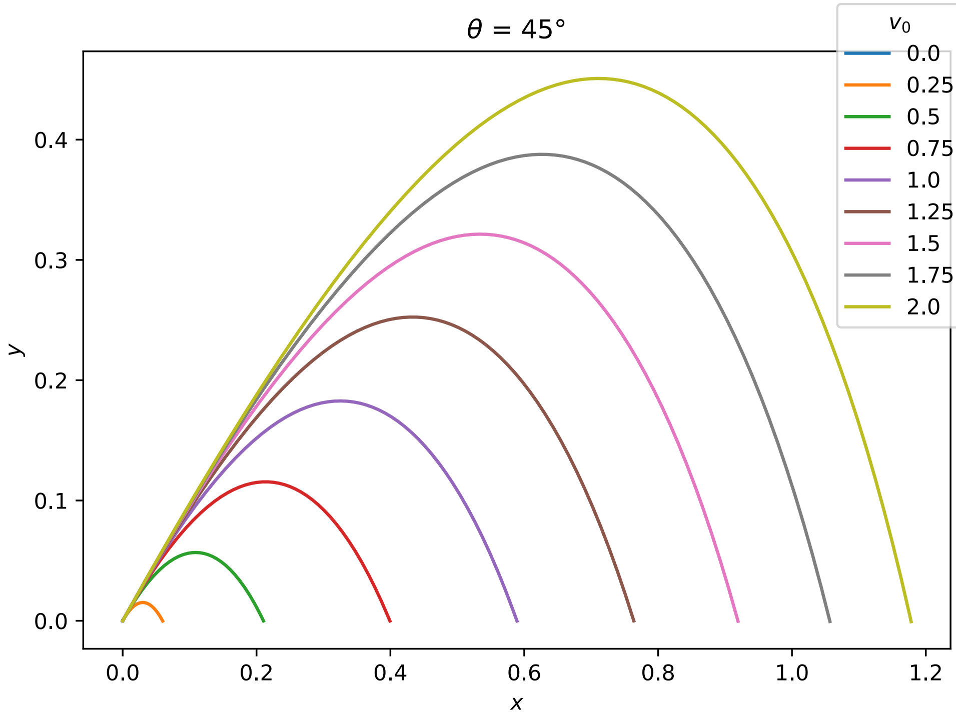

Figure 4: Trajectory as v_0 Varies

5. Firing Range

The `horizontal_range()` function in `projectile.py` was used to determine the horizontal range for a projectile launched with the given initial velocity `v_0`. It does so by calculating the intersection between the ground and the line connecting the last two points of the projectile's motion. If the last point is labeled (x_{-1}, y_{-1}) and the second to last point is labeled (x_{-2}, y_{-2}), then the slope of the line connecting them is

$$m = \frac{y_{-2} - y_{-1}}{x_{-2} - x_{-1}} \tag{7}$$

The equation for that line can be written as

$$y = m(x - x_{-1}) + y_{-1} \tag{8}$$

The x-value where the line intersects the ground ($y = 0$) corresponds to the range R. Solving for that intersection yields

$$\begin{aligned} 0 &= m(R - x_{-1}) + y_{-1} \\ R &= -\frac{y_{-1}}{m} + x_{-1} \end{aligned} \tag{9}$$

Note that if the line is vertical, which occurs when $x_{-2} = x_{-1}$, then the range is simply equal to the x-value of either of the last two points.

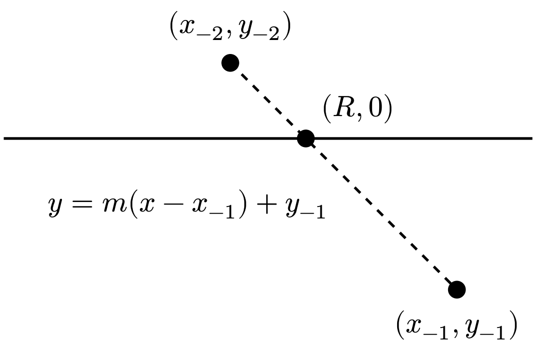

Figure 5: Intersection Between Ground and Last Two Points

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

N = 100

# Plot firing range vs launch angle as launch speed changes
v_0_values = np.linspace(0, 2.0, 9)
deg_theta_values = np.linspace(0, 90, N)

fig, ax = plt.subplots()
ax.set(ylabel="$R$", xlabel="$\\theta$ (°)")

for v_0 in v_0_values:
    firing_range_values = []
    for rad_theta in np.radians(deg_theta_values):
        v_x = v_0 * np.cos(rad_theta)
        v_y = v_0 * np.sin(rad_theta)
        firing_range = projectile.horizontal_range([v_x, v_y])
        firing_range_values.append(firing_range)
    ax.plot(deg_theta_values, firing_range_values)

fig.legend(v_0_values, title="$v_0$")
fig.tight_layout()
fig.savefig("media/R_vs_theta.svg")


# Plot firing range vs launch speed as launch angle changes
deg_theta_values = np.linspace(0, 90, 7)
v_0_values = np.linspace(0, 4, N)

fig, ax = plt.subplots()
ax.set(ylabel="$R$", xlabel="$v_0$")

for rad_theta in np.radians(deg_theta_values):
    firing_range_values = []
    for v_0 in v_0_values:
        v_x = v_0 * np.cos(rad_theta)
```

```
        v_y = v_0 * np.sin(rad_theta)
        firing_range = projectile.horizontal_range([v_x, v_y])
        firing_range_values.append(firing_range)
    ax.plot(v_0_values, firing_range_values)

fig.legend(deg_theta_values, title="$\\theta$ (°)", ncols=2, loc="upper center")
fig.tight_layout()
fig.savefig("media/R_vs_v.svg")
```

Figure 6 plots the firing range of the projectile versus the launch angle for different launch speeds. Regardless of launch speed, the range is zero if the launch angle is 0° or 90°. The projectile cannot move horizontally if does not have any initial horizontal velocity ($\theta = 90°$), nor can it do so if it has no time in the air ($\theta = 0°$). Each curve is concave downward. For low launch speeds, the optimal angle that achieves maximum range is close to 45°, which matches what is expected when air drag is negligible. As the launch speed increases, that optimal angle decreases. It becomes more efficient to reduce the air time slightly in exchange for greater horizontal velocity.

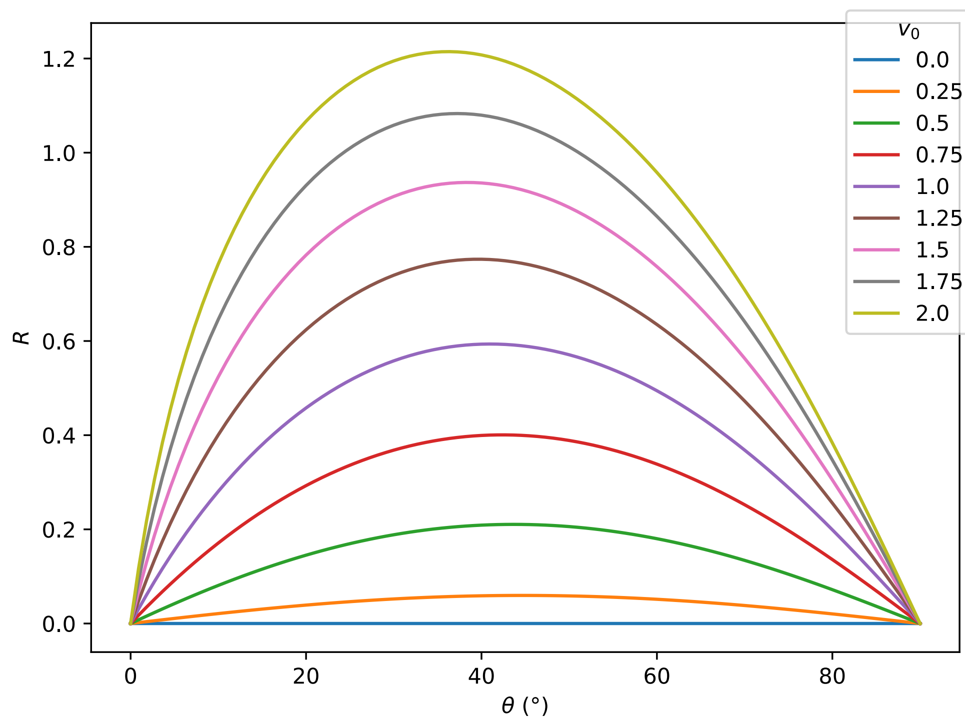

Figure 6: R vs θ as v_0 Varies

Figure 7 plots the firing range of the projectile versus the launch speed for different launch angles. For low launch speeds, projectiles launched at complementary angles achieve similar range, which matches what is expected when air drag is negligible. As the launch speed increases, the lower angle trajectories start to achieve greater range than the higher angle trajectories. As observed earlier, it becomes more efficient to prioritize having a greater initial horizontal velocity than trying to increase air time with a greater initial vertical velocity.

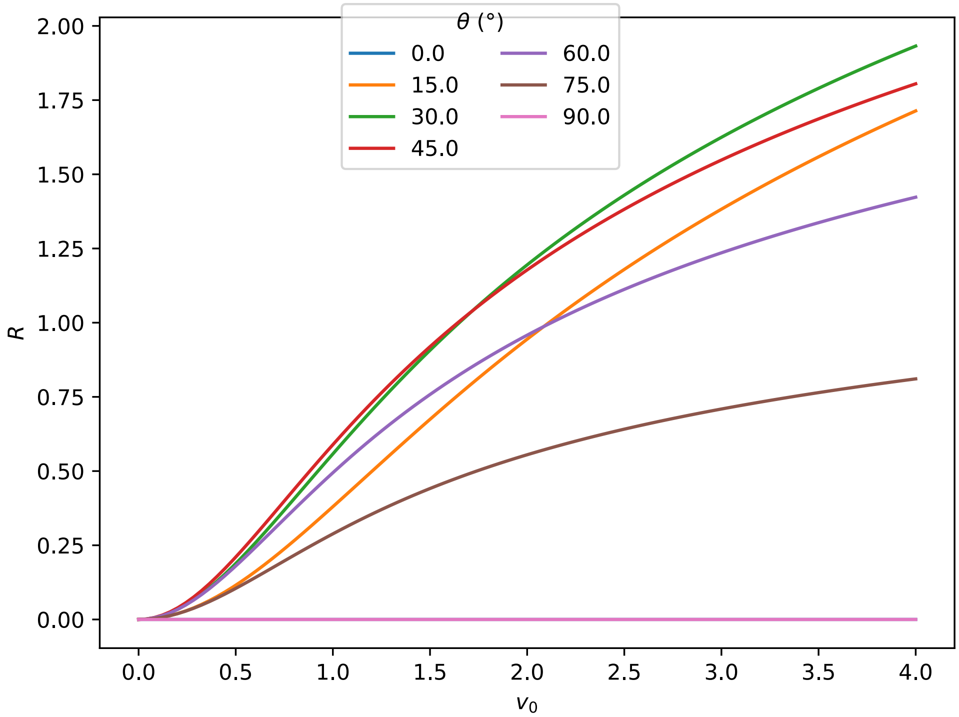

Figure 7: R vs v_0 as θ Varies

6. Hitting a Fixed Target

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

def distance_from_target(v_0, target_pos):
    """
    Launch a projectile from the origin with the given launch velocity.
    Stop simulating when the projectile falls below the line of sight to the target.
    Return position of the projectile relative to the target when it crossed the line
of sight.
    Approximate where the projectile crossed the line of sight as the intersection
between that line and the line connecting the last two points of the projectile's
path.

    If the target is straight up, then return the distance of the projectile from the
target when it was at its peak.
    If the target is straight down, then return the distance of the projectile from
the target when it fell below the target.
    """
    target_x, target_y = target_pos

    # If the target is straight up, exit when the projectile turns around
    # If the target is straight down, exit when the projectile falls below the target
    if target_x == 0:
```

```
        if target_y > 0:
            exit_condition = lambda t, u: u[3] < 0
        else:
            exit_condition = lambda t, u: u[1] < target_y

    # Otherwise, exit when the projectile falls below the line of sight
    else:
        target_slope = target_y / target_x
        exit_condition = lambda t, u: u[1] < target_slope * u[0]

    t, u = projectile.launch(v_0, exit_condition)
    x = u[:, 0]
    y = u[:, 1]

    if target_x == 0:
        if target_y > 0:
            distance_y = y[-1] - target_y
            distance = np.sign(distance_y) * np.sqrt((distance_y)**2 + (x[-1])**2)
        else:
            if x[-2] == x[-1]:
                distance = x[-1]
            else:
                projectile_slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
                distance = (target_y - y[-1])/projectile_slope + x[-1]
    else:
        target_slope = target_y / target_x
        if x[-2] == x[-1]:
            intersection_x = x[-1]
        else:
            projectile_slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
            intersection_x = (y[-1] - projectile_slope*x[-1]) / (target_slope -
projectile_slope)
        intersection_y = target_slope * intersection_x
        target_r = np.sqrt(target_x**2 + target_y**2)
        intersection_r = np.sqrt(intersection_x**2 + intersection_y**2)
        distance = intersection_r - target_r

    return distance

def bisection(a, b, f, atol=1e-8):
    """
    Use the bisection method to find a root of the given function on the interval [a,
b].
    The root returned will have the given absolute tolerance.
    f(a) and f(b) must have opposite signs.
    """
    f_a = f(a)
    f_b = f(b)
    assert max(f_a, f_b) > 0 and min(f_a, f_b) < 0, "f(a) and f(b) must have opposite
signs"
```

```
    while True:
        error_bound = (b - a) / 2
        mid = (a + b) / 2
        if error_bound < atol:
            return mid

        f_mid = f(mid)
        if (f_a >= 0 and f_mid >= 0) or (f_a <= 0 and f_mid <= 0):
            a = mid
            f_a = f_mid
        else:
            b = mid
            f_b = mid

def find_launch_speed(rad_launch_theta, target_pos, launch_speed_guess=1.0):
    """
    Find the launch speed required to hit the target at the given position using the
given launch angle.
    Optionally provide a launch speed guess to help pinpoint where the needed launch
speed might be.
    """
    target_x, target_y = target_pos
    min_target_theta = np.atan2(target_y, target_x)
    assert rad_launch_theta >= min_target_theta, "The launch angle must be greater
than the line of sight angle"

    v_hat = np.array([np.cos(rad_launch_theta), np.sin(rad_launch_theta)])
    distance_func = lambda v: distance_from_target(v * v_hat, target_pos)

    # Find an upper and lower bound for the needed launch speed
    d_guess = distance_func(launch_speed_guess)
    if d_guess <= 0:
        launch_speed_low = launch_speed_guess
        dist_low = d_guess
        launch_speed_high = launch_speed_guess
        while True:
            launch_speed_high *= 2
            dist_high = distance_func(launch_speed_high)
            if dist_high >= 0:
                break
            launch_speed_low = launch_speed_high
    else:
        launch_speed_high = launch_speed_guess
        dist_high = d_guess
        launch_speed_low = launch_speed_guess
        while True:
            launch_speed_low /= 2
            dist_low = distance_func(launch_speed_low)
            if dist_low <= 0:
                break
            launch_speed_high = launch_speed_low
```

```
    return bisection(launch_speed_low, launch_speed_high, distance_func)

N = 50
deg_pad_low = 3
deg_launch_values = np.linspace(0, 90, N)

# Plot launch velocity vs launch angle as the target angle varies
max_launch_speed = 5
deg_target_values = np.linspace(0, 45, 4)
target_distance = 1.0

fig, ax = plt.subplots()
ax.set(ylabel="$v_0$", xlabel="$\\theta$ (°)", title=f"Target Distance =
{target_distance}")

for deg_target in deg_target_values:
    deg_launch_above_target = deg_launch_values[deg_launch_values > deg_target +
deg_pad_low]
    rad_launch_above_target = np.radians(deg_launch_above_target)

    rad_target = np.radians(deg_target)
    target_pos = (target_distance * np.cos(rad_target), target_distance *
np.sin(rad_target))

    launch_speed_values = []
    launch_speed_guess = 1.0
    for rad_launch in rad_launch_above_target:
        launch_speed = find_launch_speed(rad_launch, target_pos,
launch_speed_guess=launch_speed_guess)
        launch_speed_values.append(launch_speed)
        launch_speed_guess = launch_speed

        if launch_speed > max_launch_speed:
            launch_speed_values.pop()
            deg_launch_above_target =
deg_launch_above_target[:len(launch_speed_values)]
            break

    ax.plot(deg_launch_above_target, launch_speed_values, ".")

fig.legend(deg_target_values, title="Target Angle (°)")
fig.tight_layout()
fig.savefig("media/hitting_target_varied_angle.svg")

# Plot launch velocity vs launch angle as the target distance varies
max_launch_speed = 30
deg_target = 15
target_distance_values = np.linspace(1, 3, 3)

fig, ax = plt.subplots()
```

```
ax.set(ylabel="$v_0$", xlabel="$\\theta$ (°)", title=f"Target Angle = {deg_target}°")

for target_distance in target_distance_values:
    deg_launch_above_target = deg_launch_values[deg_launch_values > deg_target +
deg_pad_low]
    rad_launch_above_target = np.radians(deg_launch_above_target)

    rad_target = np.radians(deg_target)
    target_pos = (target_distance * np.cos(rad_target), target_distance *
np.sin(rad_target))

    launch_speed_values = []
    launch_speed_guess = 1.0
    for rad_launch in rad_launch_above_target:
        launch_speed = find_launch_speed(rad_launch, target_pos,
launch_speed_guess=launch_speed_guess)
        launch_speed_values.append(launch_speed)
        launch_speed_guess = launch_speed

        if launch_speed > max_launch_speed:
            launch_speed_values.pop()
            deg_launch_above_target =
deg_launch_above_target[:len(launch_speed_values)]
            break

    ax.plot(deg_launch_above_target, launch_speed_values, ".")

fig.legend(target_distance_values, title="Target Distance")
fig.tight_layout()
fig.savefig("media/hitting_target_varied_distance.svg")
```

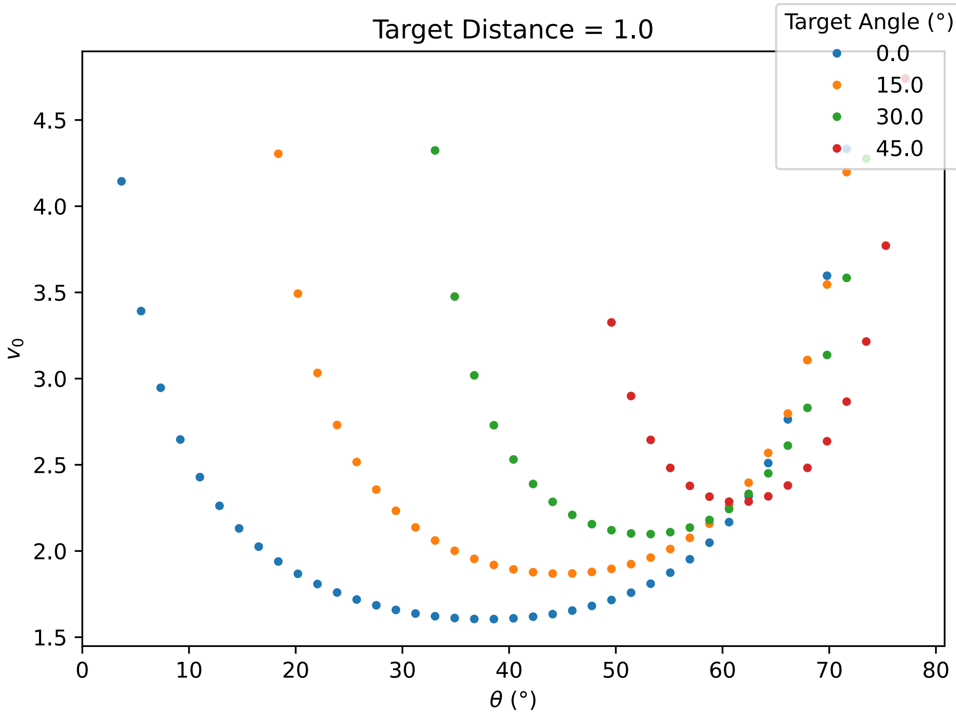

Figure 8: v vs θ as Target Angle Varies

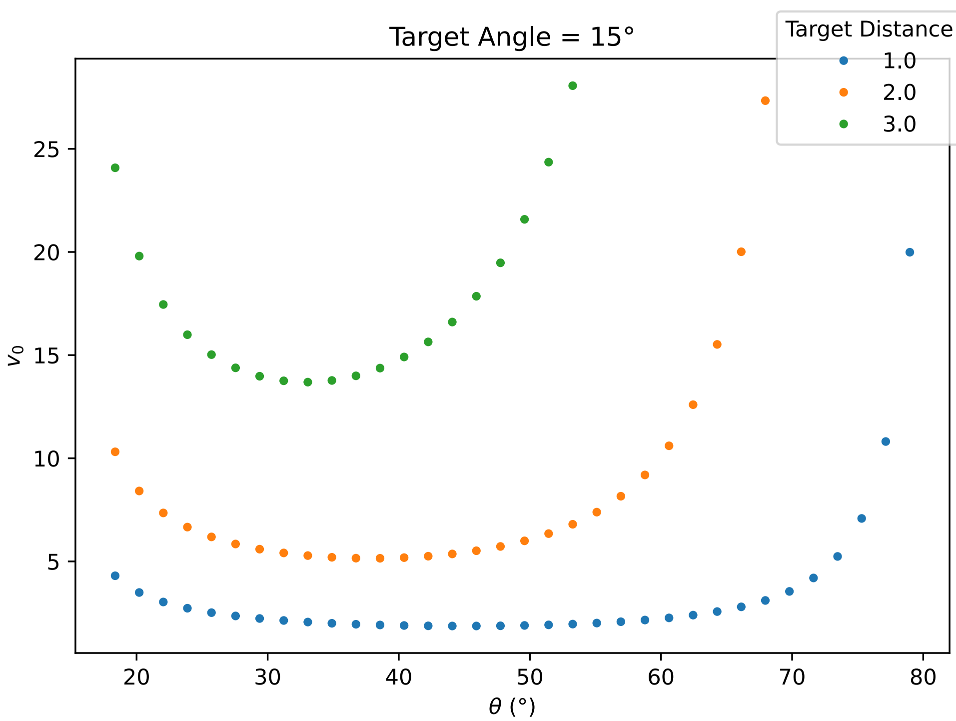

Figure 9: v vs θ as Target Distance Varies

7. Cover Image

The following script was use to create the cover image.

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

g = 1
v_0 = [2.0, 0.75]
v_0x, v_0y = v_0
N = 100

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$x$")
ax.tick_params(
    axis="both",
    which="both",
    labelbottom=False,
    bottom=False,
    labelleft=False,
    left=False,
)
ax.spines["top"].set_visible(False)
ax.spines["right"].set_visible(False)

# No drag
t_f = 2 * v_0y / g
t = np.linspace(0, t_f, N)
x = v_0x * t
y = v_0y * t - g * t**2 / 2
ax.plot(x, y, label="No Drag")

# Quadratic drag
t, u = projectile.launch(v_0)
x = u[:, 0]
y = u[:, 1]
ax.plot(x, y, label="Quadratic Drag")

fig.legend()
fig.tight_layout()
fig.savefig("media/thumbnail.svg")
```