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Realistic Projectiles

Quadratic Drag

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Contents

1 Background

1.1 Quadratic Drag Equation

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Let $g = 1$, $k = 1$, & $v_\infty = 1$

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$$\vec{u}_{i+1} = \vec{u}_i + \frac{h}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right) \quad t_{i+1} = t_i + h$$

1.2 RK4 Method for Systems


```python
def calculate(t_0, u_0, h, diff, should_exit):
    t = [t_0]
    u = [u_0]

    while not should_exit(t[-1], u[-1]):
        k_1 = diff(t[-1], u[-1])
        k_2 = diff(t[-1] + h/2, u[-1] + h/2 * k_1)
        k_3 = diff(t[-1] + h/2, u[-1] + h/2 * k_2)
        k_4 = diff(t[-1] + h, u[-1] + h * k_3)

        u_next = u[-1] + h/6 * (k_1 + 2*k_2 + 2*k_3 + k_4)
        u.append(u_next)
        t_next = t[-1] + h
        t.append(t_next)

    return np.array(t), np.array(u)
```

1.3 Projectile Equations

$$\vec{u} = \begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix}$$

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} v_x \\ v_y \\ -kvv_x \\ -g - kvv_y \end{pmatrix}$$

1.4 Equation Implementation

```python
def u_prime(t, u):
    k = 1
    g = 1

    x, y, v_x, v_y = u
    speed = np.sqrt(v_x**2 + v_y**2)
    drag_part = k * speed
    drag_x = drag_part * v_x
    drag_y = drag_part * v_y

    return np.array([
        v_x,
        v_y,
        -drag_x,
        -g - drag_y,
    ])
```

1.5 Launching Projectiles

```python
def below_ground(t, u):
    y = u[1]
    return y < 0

def launch(v_0, should_exit=below_ground):
    t_0 = 0.0
    h = 0.001
    v_x, v_y = v_0
    u_0 = np.array([0, 0, v_x, v_y])

    t, u = rk4.calculate(t_0, u_0, h, u_prime, should_exit)

    return t, u
```

2 Interdependence of Motion Components

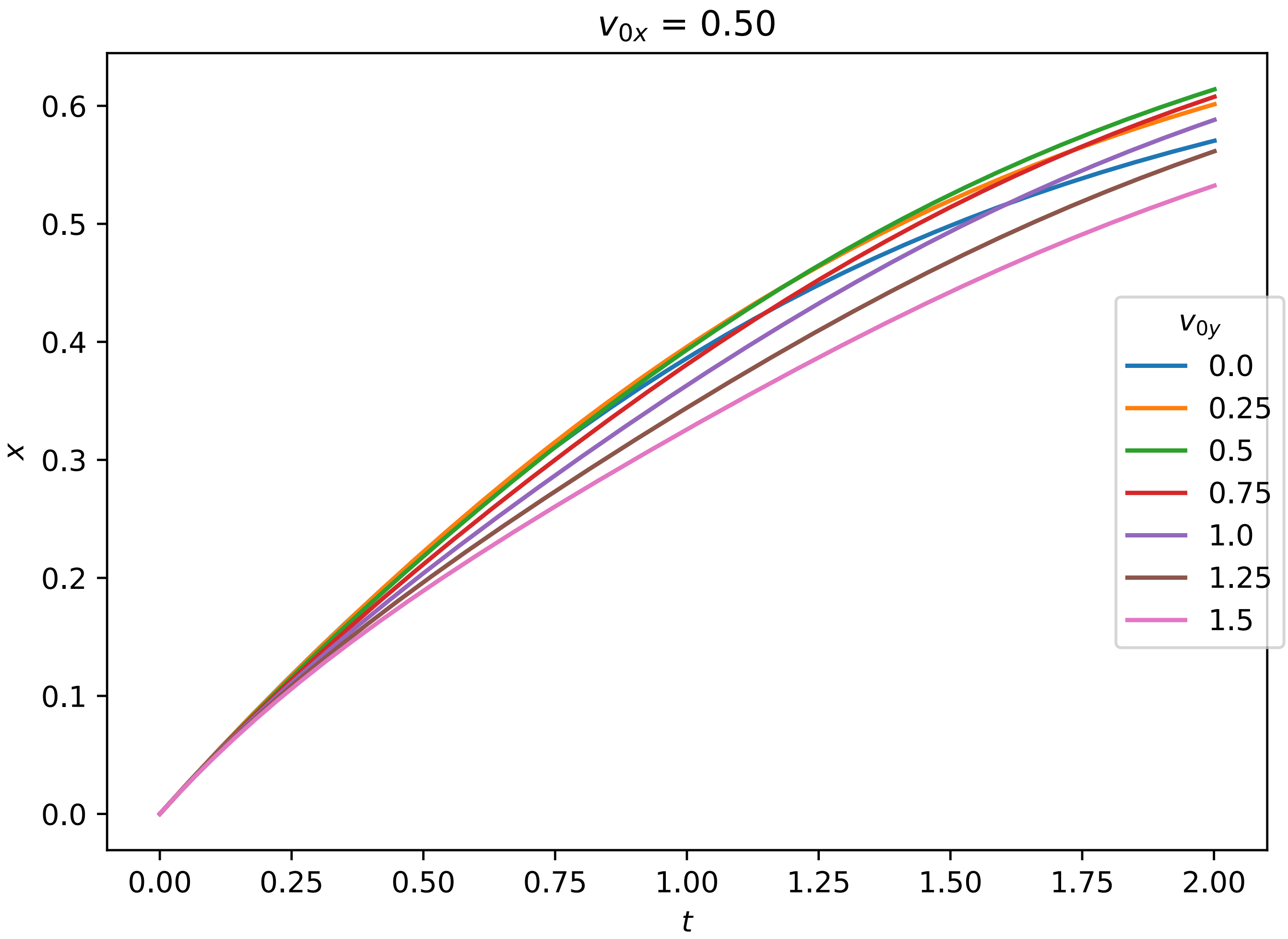
2.1 Equations

- Conditions for independence
  - $\frac{dv_x}{dt}$ does not depend on y or v_y
  - $\frac{dv_y}{dt}$ does not depend on x or v_x
- That is not the case for quadratic drag

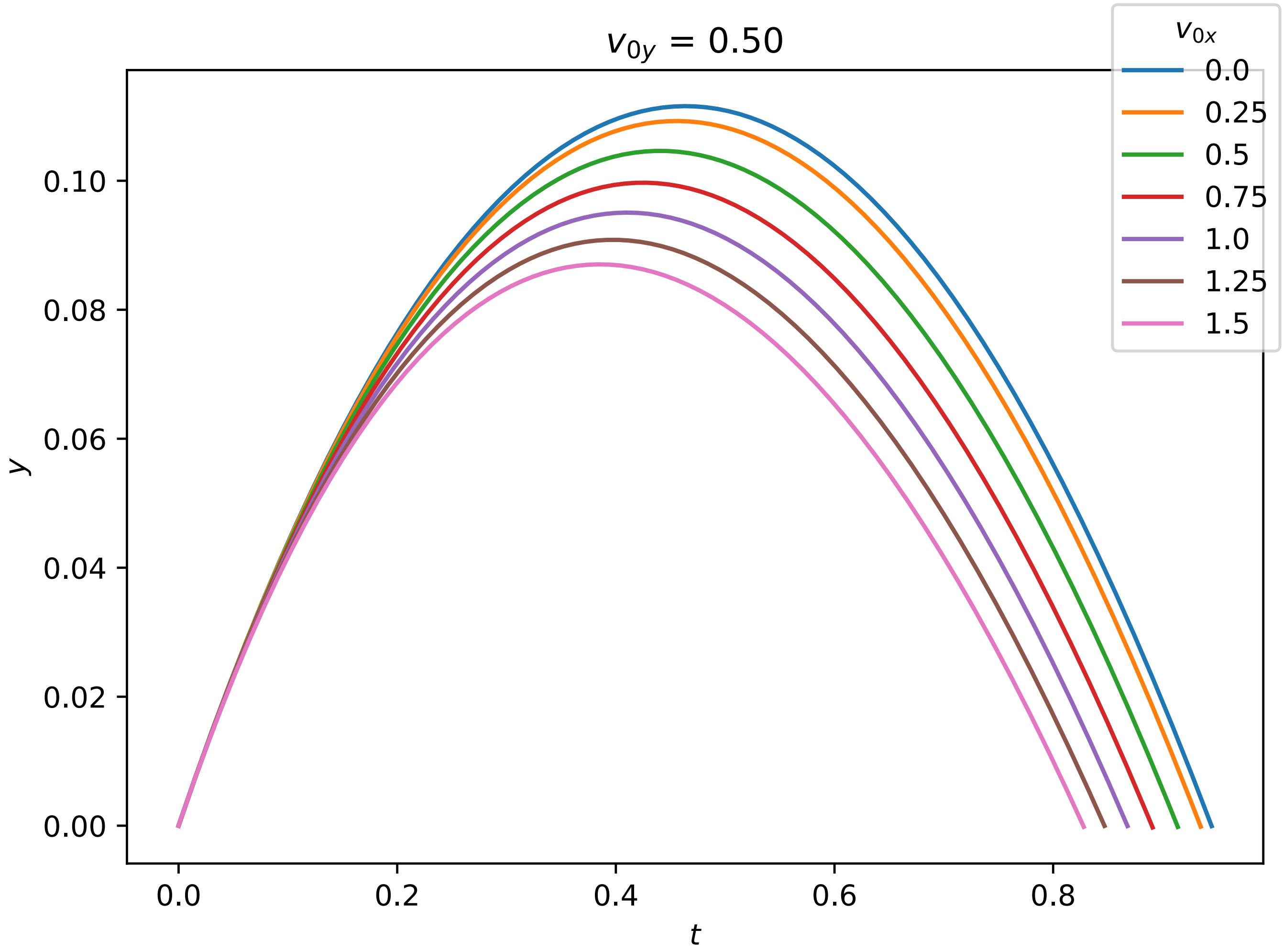
$$\frac{dv_x}{dt} = -kvv_x = -kv_x\sqrt{v_x^2 + v_y^2}$$

$$\frac{dv_y}{dt} = -g - kvv_y = -g - kv_y\sqrt{v_x^2 + v_y^2}$$

2.2 x vs t as v_{0y} Varies

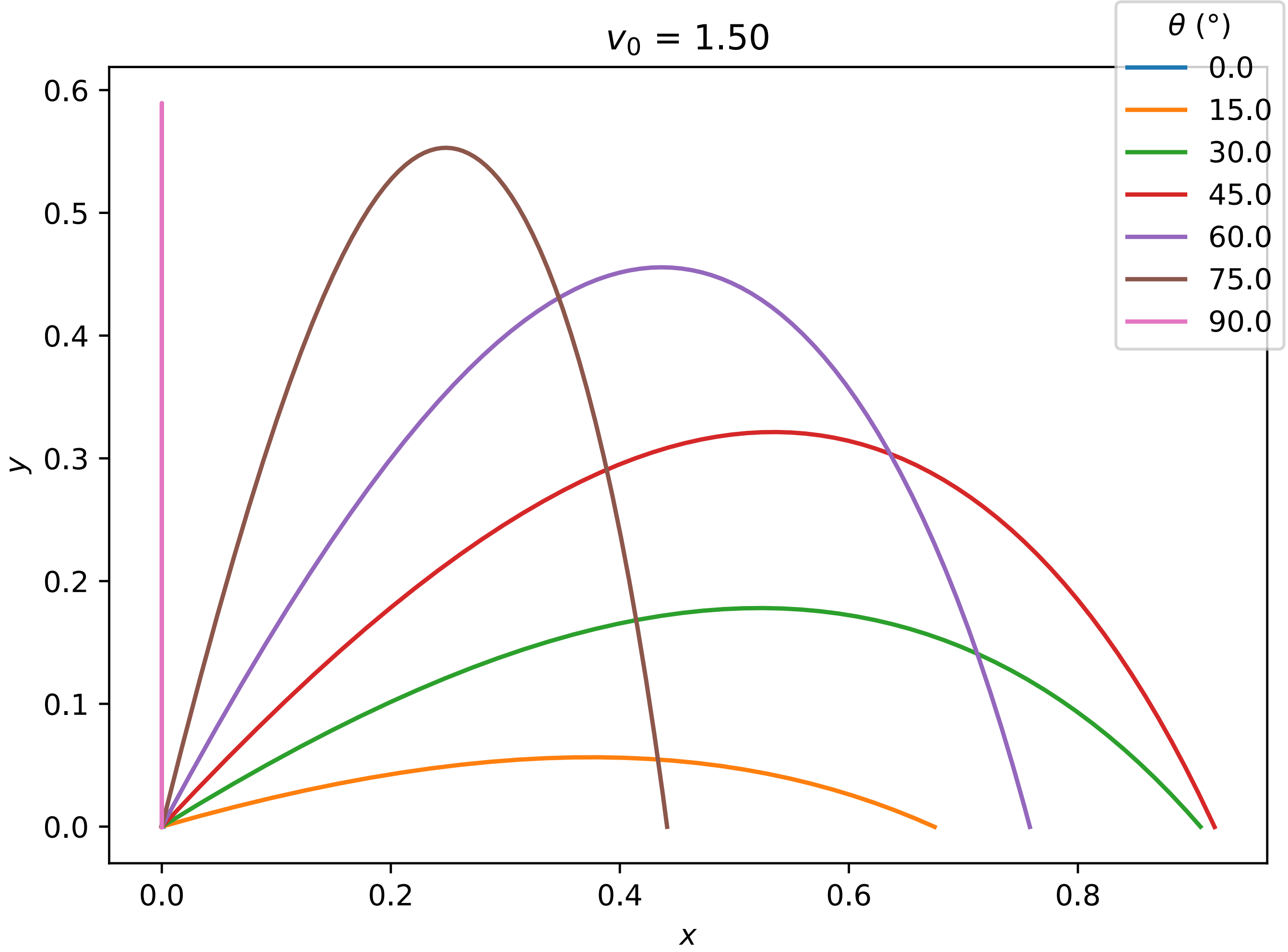


2.3 y vs t as v_{0x} Varies

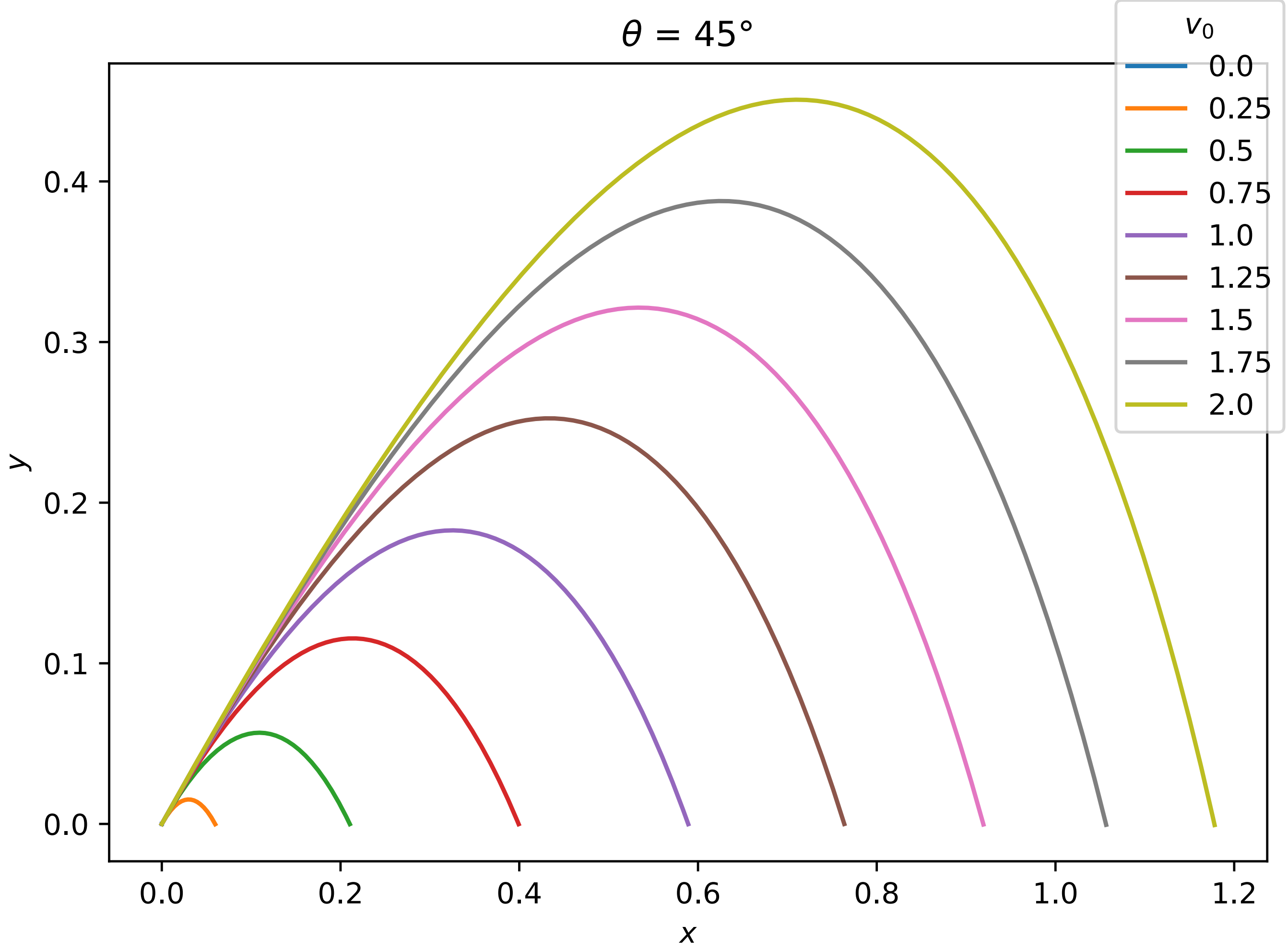


3 Trajectory Shapes

3.1 Trajectory as θ Varies

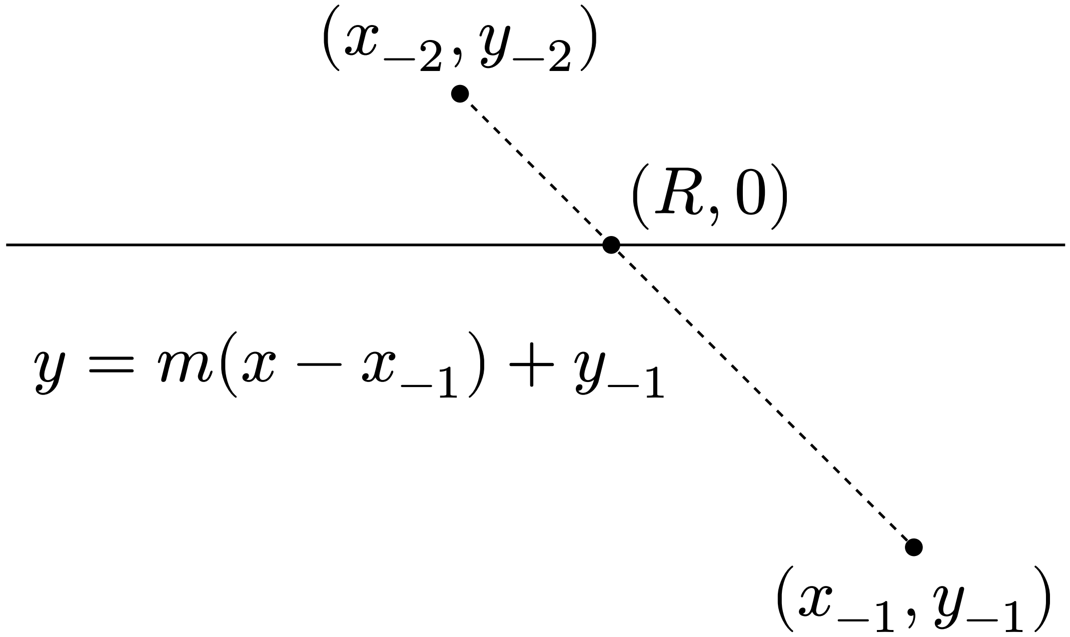


3.2 Trajectory as v_0 Varies

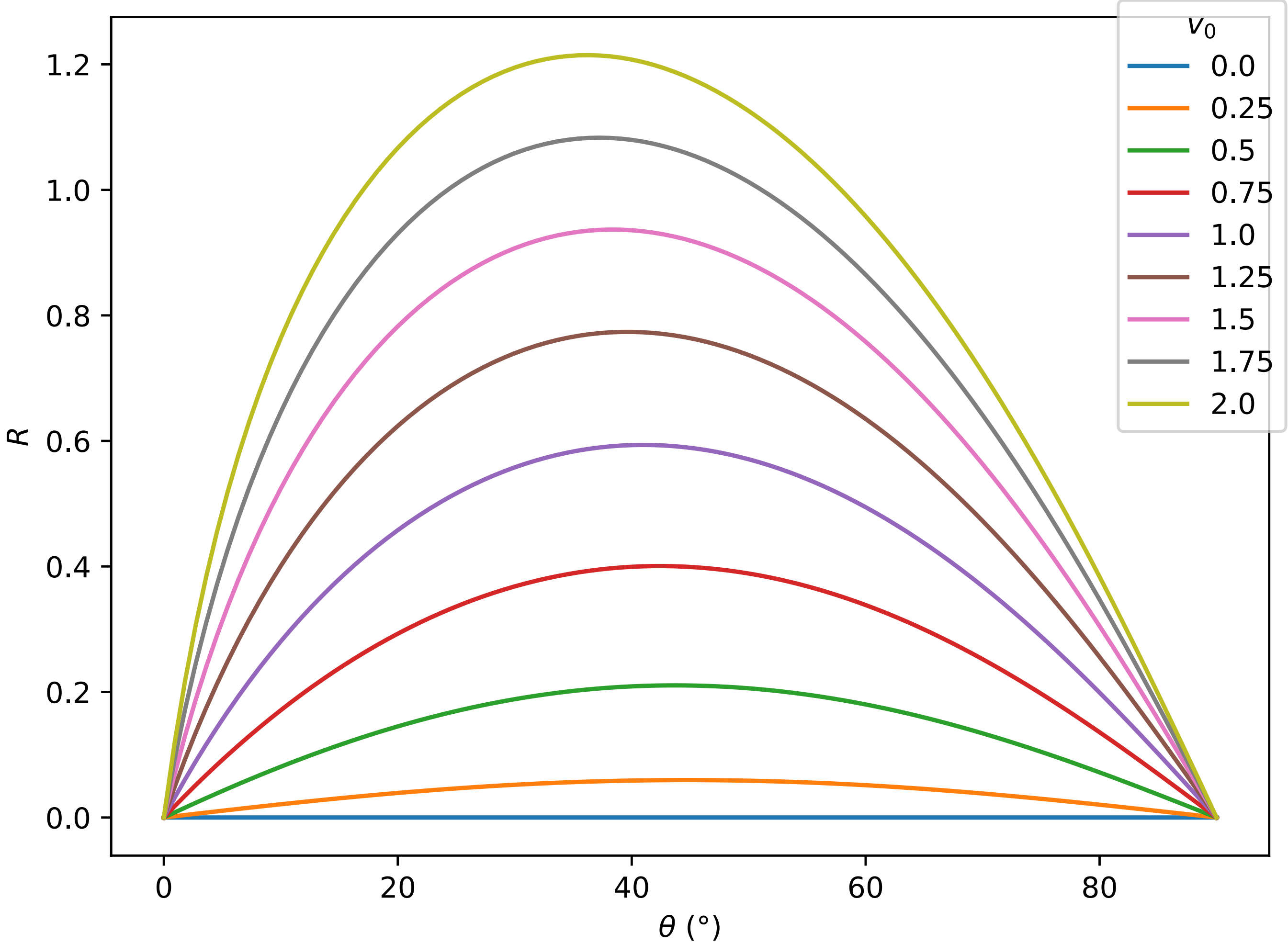


4 Firing Range

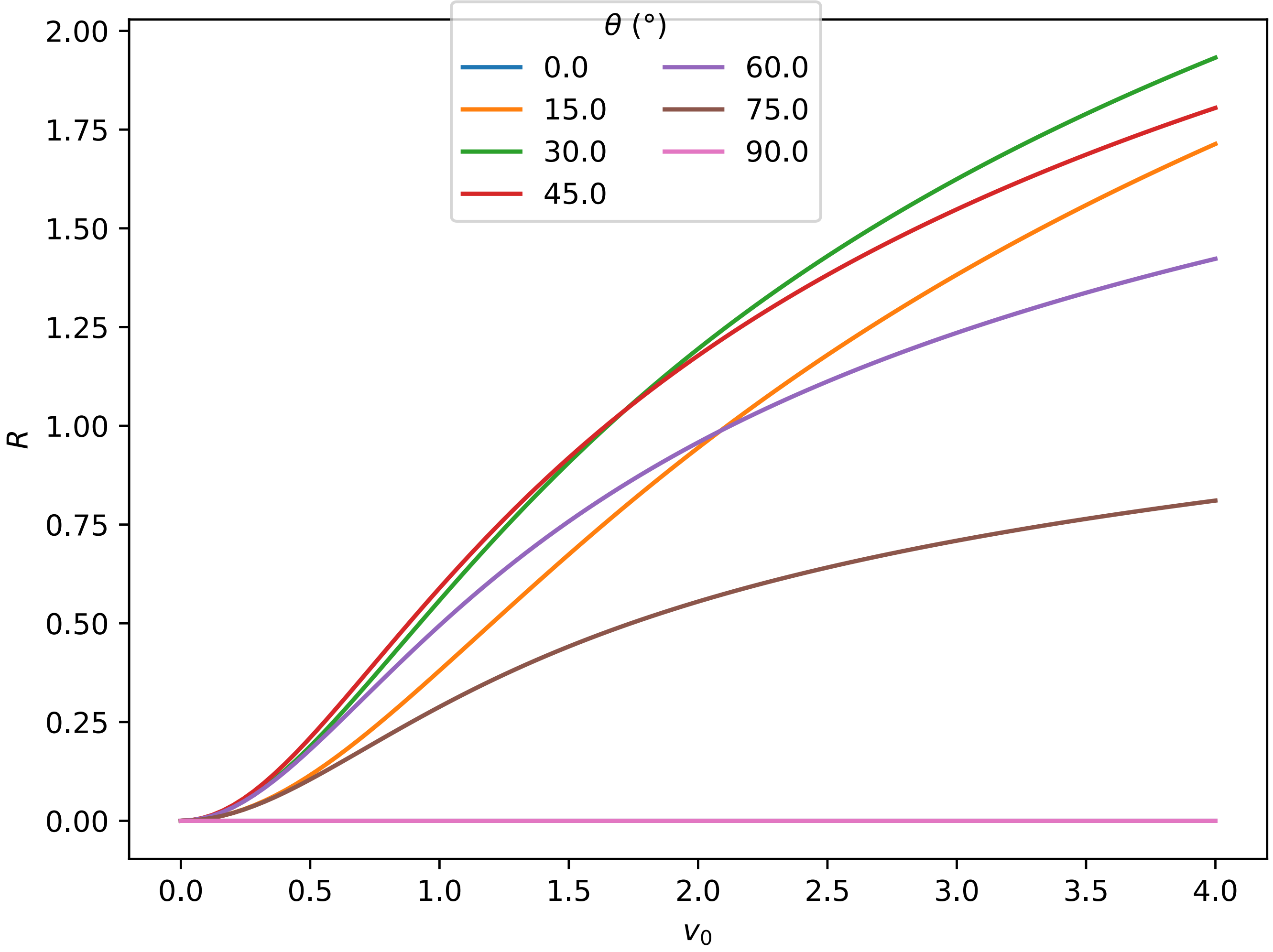
4.1 Diagram



4.2 R vs θ as v_0 Varies

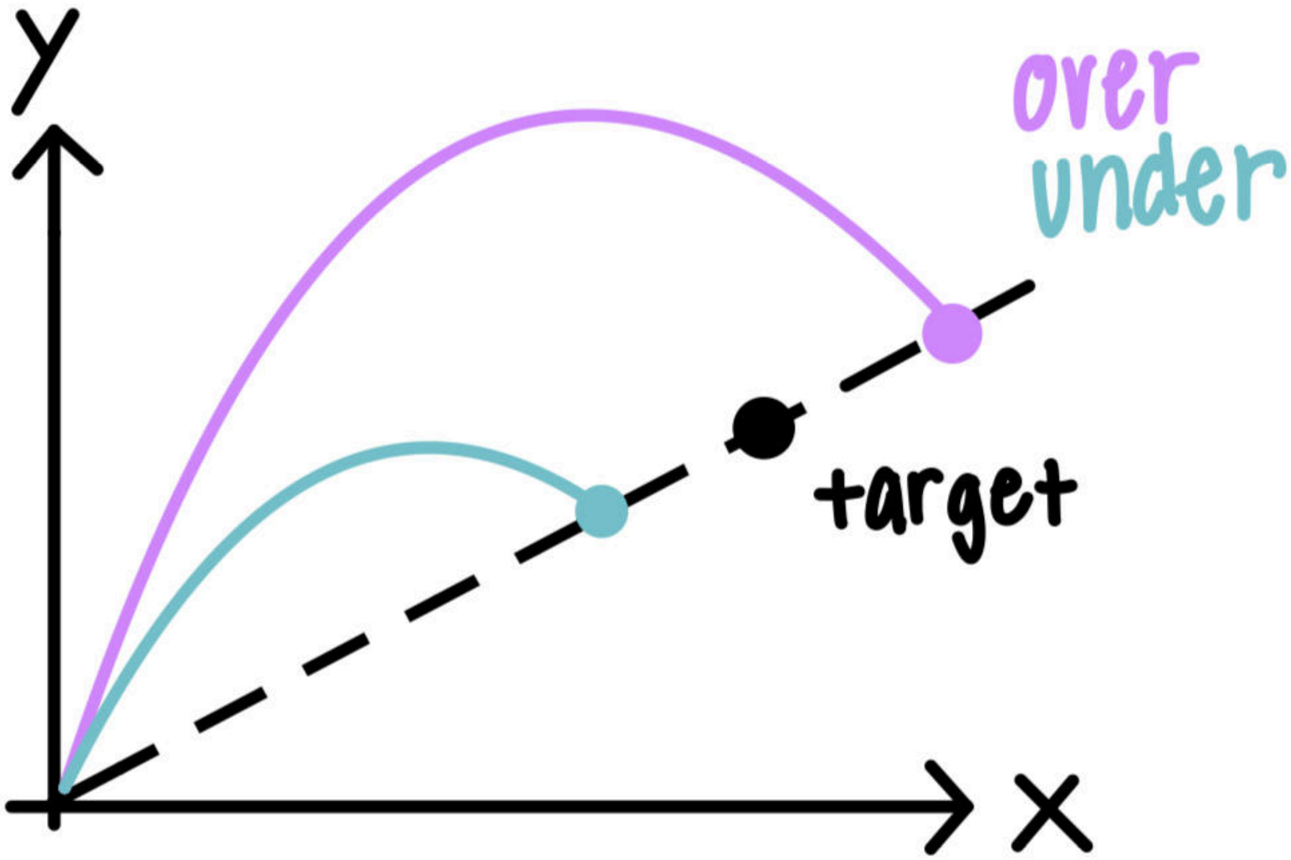


4.3 R vs v_0 as θ Varies

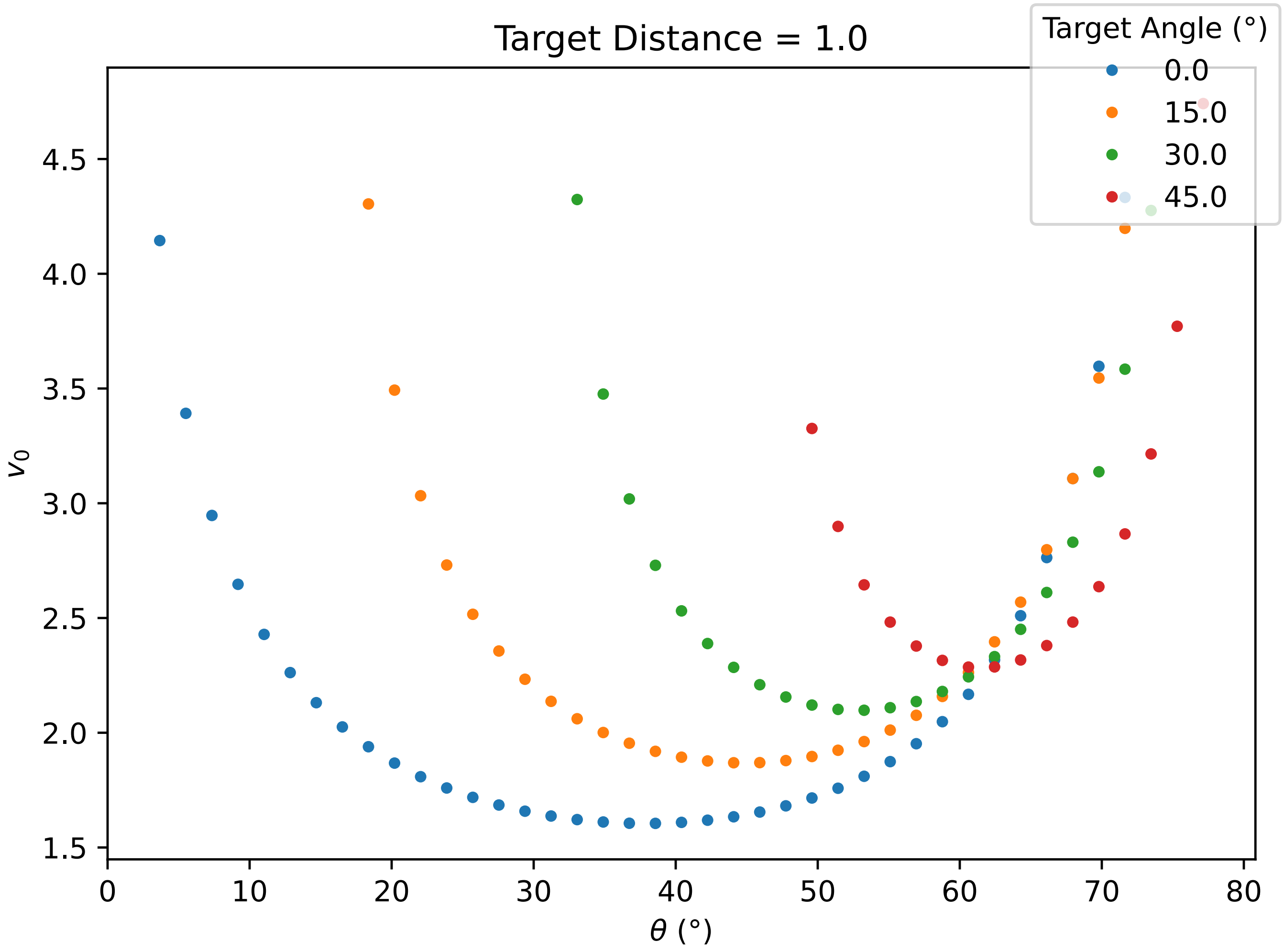


5 Hitting a Fixed Target

5.1 Over/Under-Shooting the Target



5.2 v vs θ as Target Angle Varies



5.3 v vs θ as Target Distance Varies