

Realistic Projectiles

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1. Quadratic Drag Equation

In introductory physics, projectiles are typically modeled as experiencing negligible air drag. For this project, projectiles were modeled as experiencing *quadratic drag*.

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} - kv^2\hat{v}$$

The terms in this equation are as follows:

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (\text{position})$$

$$\vec{v} = \frac{d\vec{r}}{dt} \quad (\text{velocity})$$

$$\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \quad (\text{gravitation acceleration})$$

$$k = \text{"constant"} \quad (\text{drag constant})$$

The y axis points straight up, and the x axis points horizontally along the plane of motion of the projectile. This keeps the motion in 2 dimensions. Projectiles were started on the ground at $(x, y) = (0, 0)$.

To focus on scale-independent features of the motion, units of distance and time were used such that $g = 1$ and $k = 1$. This makes the terminal speed $v_\infty = 1$.

2. Runge-Kutta Four (RK4) Method

rk4.py

```
import numpy as np

def calculate(t_0, u_0, h, diff, should_exit):
    """
    Calculate t values & u vectors using the vector RK4 method on a system of 1st
    order ODEs.
    Return an array of t values, and an array of u vectors.

    - t_0: starting t value
    - u_0: starting u vector
    - h: step size
    - diff: function that takes t and u as arguments and returns du/dt
    - should_exit: function that takes t and u as arguments and returns True if no
    more steps should be taken
    """
    t = [t_0]
    u = [u_0]

    while not should_exit(t[-1], u[-1]):
        k_1 = diff(t[-1], u[-1])
        k_2 = diff(t[-1] + h/2, u[-1] + h/2 * k_1)
        k_3 = diff(t[-1] + h/2, u[-1] + h/2 * k_2)
```

```

    k_4 = diff(t[-1] + h, u[-1] + h * k_3)

    u_next = u[-1] + h/6 * (k_1 + 2*k_2 + 2*k_3 + k_4)
    u.append(u_next)
    t_next = t[-1] + h
    t.append(t_next)

    return np.array(t), np.array(u)

```

projectile.py

```

import rk4
import numpy as np

def u_prime(t, u):
    """
    Return an array of derivatives with respect to t for each component of the vector
    u.
    u consists of x, y, v_x, and v_y.
    """
    k = 1
    g = 1

    x, y, v_x, v_y = u
    speed = np.sqrt(v_x**2 + v_y**2);
    drag = k * speed**2
    if speed == 0:
        drag_x = 0
        drag_y = 0
    else:
        drag_x = drag * v_x / speed
        drag_y = drag * v_y / speed

    return np.array([
        v_x,
        v_y,
        -drag_x,
        -g - drag_y,
    ])

def below_ground(t, u):
    y = u[1]
    return y < 0

def launch(v_0, should_exit=below_ground):
    """
    Launch a projectile from the origin with the given launch velocity.
    By default, stop after the projectile hits the ground (when y < 0).
    If desired, an alternate function t and u can be passed.
    This function should return True when the exit condition is met.
    """

```

```

    Return the arrays of t and u.
    Note that u consists of x, y, v_x, and v_y.
    """
    t_0 = 0.0
    h = 0.001
    v_x, v_y = v_0
    u_0 = np.array([0, 0, v_x, v_y])

    t, u = rk4.calculate(t_0, u_0, h, u_prime, should_exit)

    return t, u

def horizontal_range(v_0):
    """
    Launch a projectile from the origin with the given launch velocity.
    Return the horizontal range of the projectile.
    Approximate the range as the x-intercept of the line connecting the last two
    points of the projectile's path.
    """
    t, u = launch(v_0)
    x = u[:, 0]
    y = u[:, 1]

    if x[-2] == x[-1]:
        distance = x[-1]
    else:
        slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
        distance = -y[-1]/slope + x[-1]
    return distance

```

3. Interdependence of Horizontal and Vertical Motion

```

import projectile
import numpy as np
import matplotlib.pyplot as plt

# Plot horizontal position over time as vertical velocity changes
v_0x = 0.5
v_0y_values = np.linspace(0, 1.5, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$x$", xlabel="$t$", title=f"$v_{0x}$ = {v_0x:.2f}")

for v_0y in v_0y_values:
    t, u = projectile.launch([v_0x, v_0y])
    x = u[:, 0]
    ax.plot(t, x)

fig.legend(v_0y_values, title="$v_{0y}$", loc="center right")
fig.tight_layout()
fig.savefig("media/x_vs_t.svg")

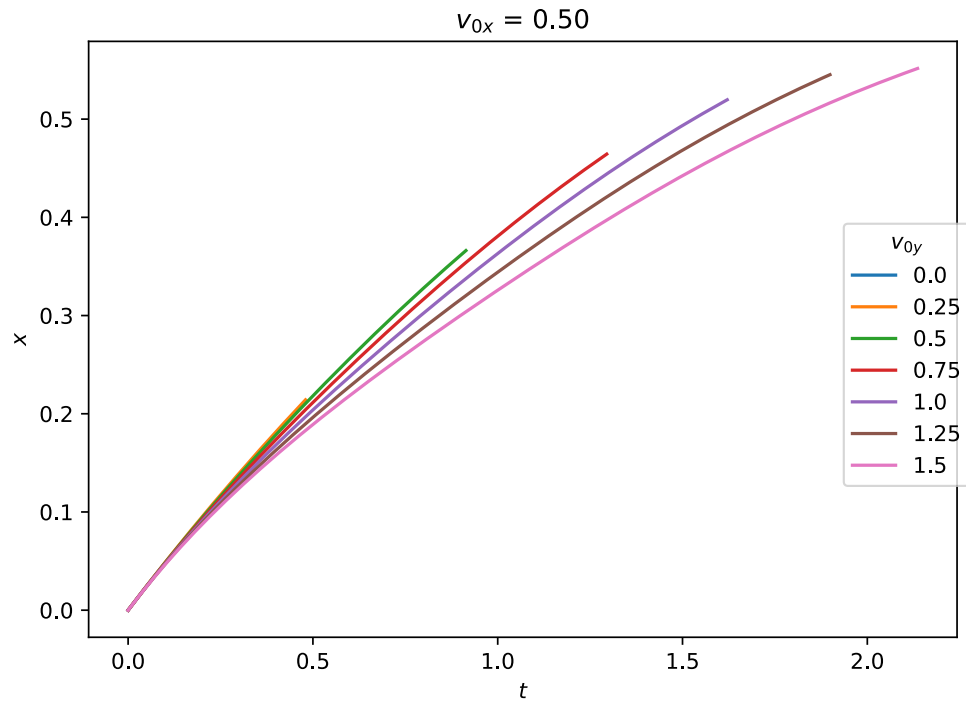
```

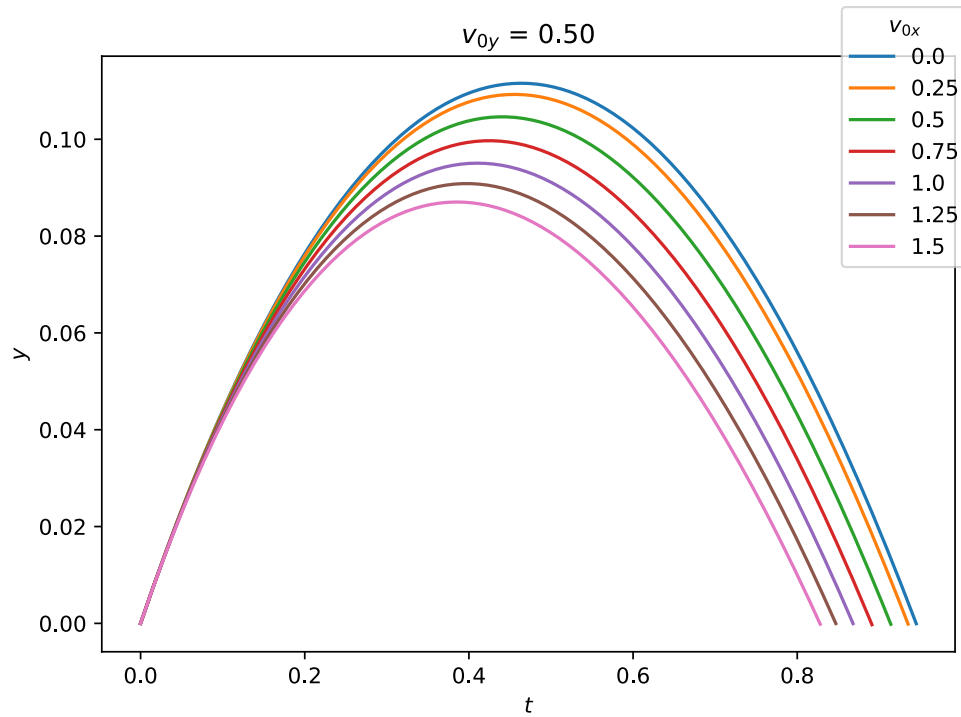
```
# Plot vertical position over time as horizontal velocity changes
v_0y = 0.5
v_0x_values = np.linspace(0, 1.5, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$t$", title=f"$v_{{0y}}$ = {v_0y:.2f}")

for v_0x in v_0x_values:
    t, u = projectile.launch([v_0x, v_0y])
    y = u[:, 1]
    ax.plot(t, y)

fig.legend(v_0x_values, title="$v_{{0x}}$")
fig.tight_layout()
fig.savefig("media/y_vs_t.svg")
```

Figure 1: x vs t as v_{0y} Varies

Figure 2: y vs t as v_{0x} Varies

4. Trajectory Shapes

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

# Plot trajectories as launch angle changes
v_0 = 1.5
deg_theta_values = np.linspace(0, 90, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$x$", title=f"$v_0$ = {v_0:.2f}")

for rad_theta in np.radians(deg_theta_values):
    v_x = v_0 * np.cos(rad_theta)
    v_y = v_0 * np.sin(rad_theta)
    t, u = projectile.launch([v_x, v_y])
    x = u[:, 0]
    y = u[:, 1]
    ax.plot(x, y)

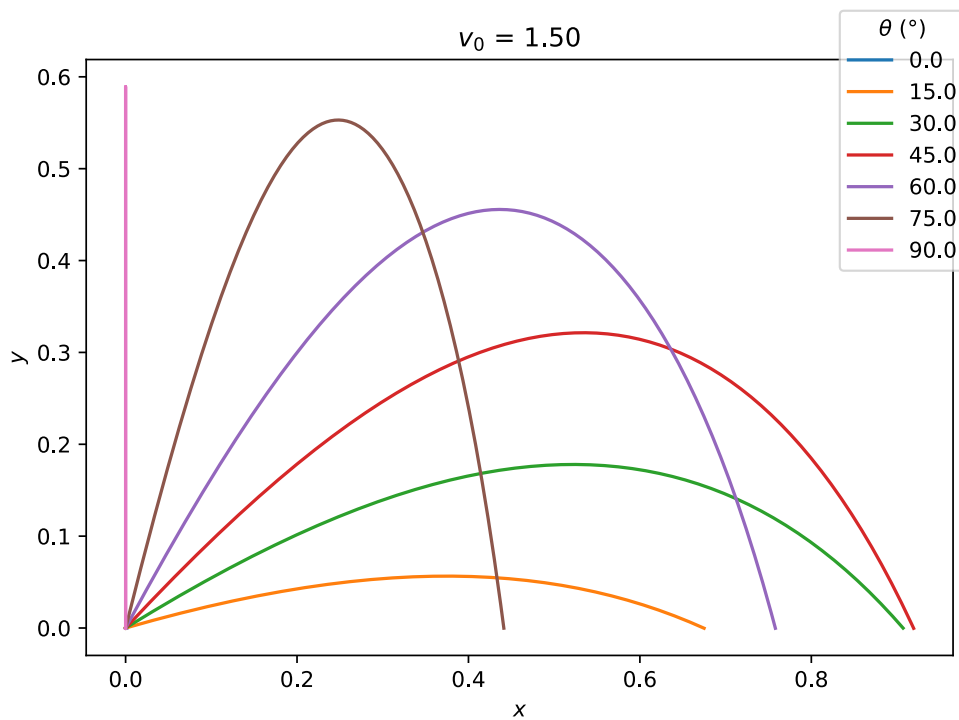
fig.legend(deg_theta_values, title="$\\theta$ ($^\circ$)")
fig.tight_layout()
fig.savefig("media/xy_vs_theta.svg")
```

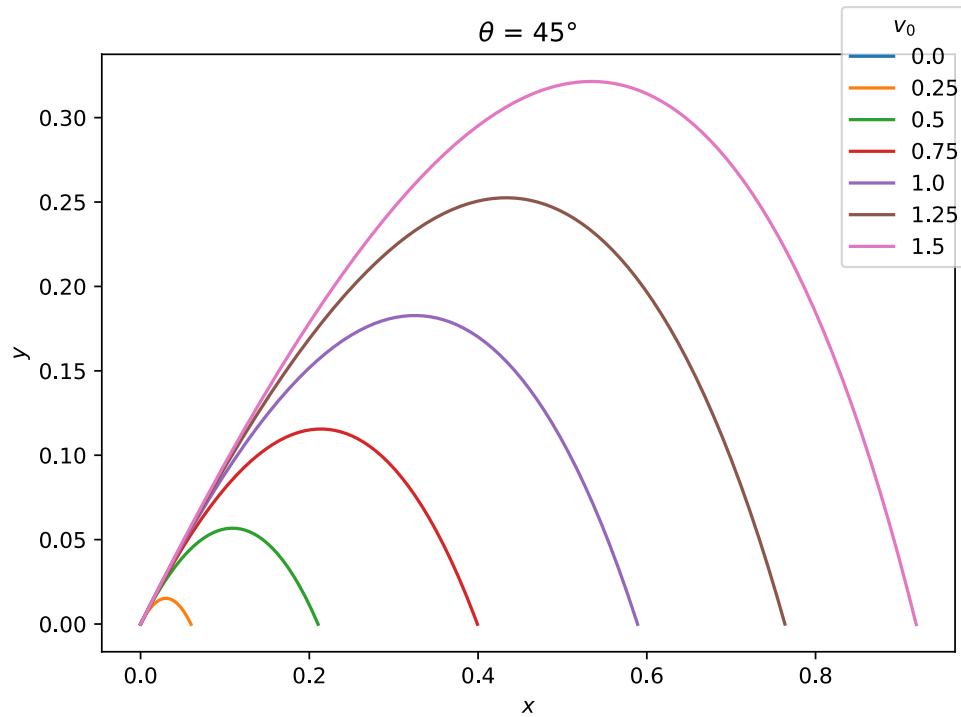
```
# Plot trajectories as launch speed changes
deg_theta = 45
rad_theta = np.radians(deg_theta)
v_0_values = np.linspace(0, 1.5, 7)

fig, ax = plt.subplots()
ax.set(ylabel="$y$", xlabel="$x$", title=f"$\\theta$ = {deg_theta}$^\circ$")

for v_0 in v_0_values:
    v_x = v_0 * np.cos(rad_theta)
    v_y = v_0 * np.sin(rad_theta)
    t, u = projectile.launch([v_x, v_y])
    x = u[:, 0]
    y = u[:, 1]
    ax.plot(x, y)

fig.legend(v_0_values, title="$v_0$")
fig.tight_layout()
fig.savefig("media/xy_vs_v.svg")
```

Figure 3: Trajectory as θ Varies

Figure 4: Trajectory as v_0 Varies

5. Firing Range

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

N = 100

# Plot firing range vs launch angle as launch speed changes
v_0_values = np.linspace(0, 2.0, 9)
deg_theta_values = np.linspace(0, 90, N)

fig, ax = plt.subplots()
ax.set(ylabel="$R$", xlabel="$\\theta$ ($^\circ$)")

for v_0 in v_0_values:
    firing_range_values = []
    for rad_theta in np.radians(deg_theta_values):
        v_x = v_0 * np.cos(rad_theta)
        v_y = v_0 * np.sin(rad_theta)
        firing_range = projectile.horizontal_range([v_x, v_y])
        firing_range_values.append(firing_range)
    ax.plot(deg_theta_values, firing_range_values)

fig.legend(v_0_values, title="$v_0$")
fig.tight_layout()
```

```

fig.savefig("media/R_vs_theta.svg")

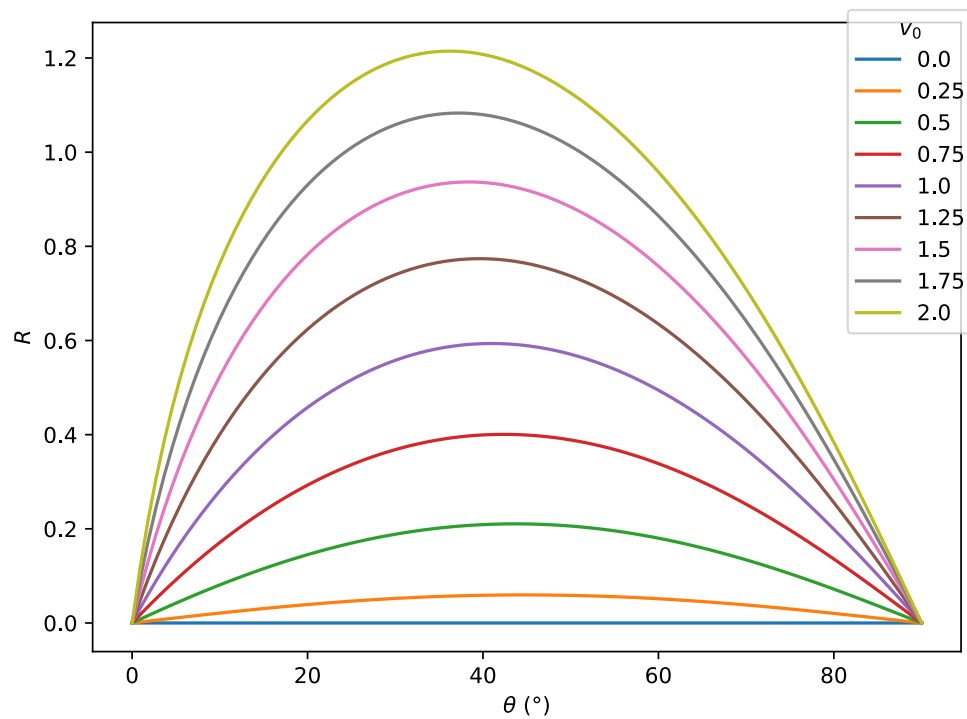
# Plot firing range vs launch speed as launch angle changes
deg_theta_values = np.linspace(0, 90, 7)
v_0_values = np.linspace(0, 4, N)

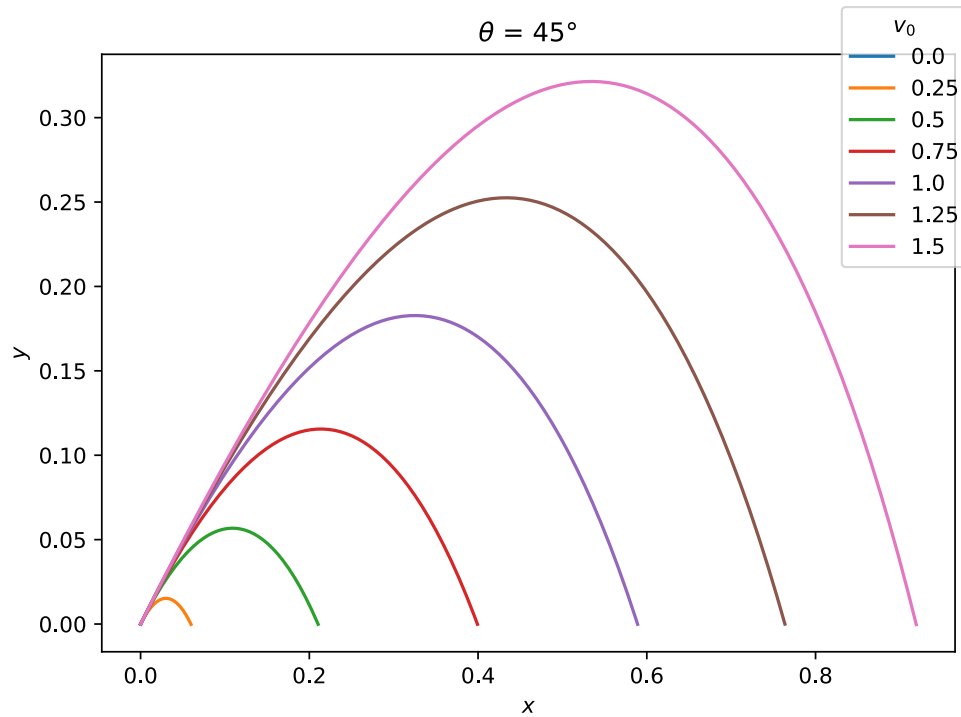
fig, ax = plt.subplots()
ax.set(ylabel="$R$", xlabel="$v_0$")

for rad_theta in np.radians(deg_theta_values):
    firing_range_values = []
    for v_0 in v_0_values:
        v_x = v_0 * np.cos(rad_theta)
        v_y = v_0 * np.sin(rad_theta)
        firing_range = projectile.horizontal_range([v_x, v_y])
        firing_range_values.append(firing_range)
    ax.plot(v_0_values, firing_range_values)

fig.legend(deg_theta_values, title="$\\theta$ ($^\circ$)", ncols=2, loc="upper center")
fig.tight_layout()
fig.savefig("media/R_vs_v.svg")

```

Figure 5: R vs θ as v_0 Varies

Figure 6: R vs v_0 as θ Varies

6. Extension

```
import projectile
import numpy as np
import matplotlib.pyplot as plt

def distance_from_target(v_0, target_pos):
    """
    Launch a projectile from the origin with the given launch velocity.
    Stop simulating when the projectile falls below the line of sight to the target.
    Return position of the projectile relative to the target when it crossed the line
    of sight.
    Approximate where the projectile crossed the line of sight as the intersection
    between that line and the line connecting the last two points of the projectile's
    path.

    If the target is straight up, then return the distance of the projectile from the
    target when it was at its peak.
    If the target is straight down, then return the distance of the projectile from
    the target when it fell below the target.
    """
    target_x, target_y = target_pos

    # If the target is straight up, exit when the projectile turns around
    # If the target is straight down, exit when the projectile falls below the target
    if target_x == 0:
```

```

    if target_y > 0:
        exit_condition = lambda t, u: u[3] < 0
    else:
        exit_condition = lambda t, u: u[1] < target_y

# Otherwise, exit when the projectile falls below the line of sight
else:
    target_slope = target_y / target_x
    exit_condition = lambda t, u: u[1] < target_slope * u[0]

t, u = projectile.launch(v_0, exit_condition)
x = u[:, 0]
y = u[:, 1]

if target_x == 0:
    if target_y > 0:
        distance_y = y[-1] - target_y
        distance = np.sign(distance_y) * np.sqrt((distance_y)**2 + (x[-1])**2)
    else:
        if x[-2] == x[-1]:
            distance = x[-1]
        else:
            projectile_slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
            distance = (target_y - y[-1]) / projectile_slope + x[-1]
else:
    target_slope = target_y / target_x
    if x[-2] == x[-1]:
        intersection_x = x[-1]
    else:
        projectile_slope = (y[-2] - y[-1]) / (x[-2] - x[-1])
        intersection_x = (y[-1] - projectile_slope * x[-1]) / (target_slope -
projectile_slope)
    intersection_y = target_slope * intersection_x
    target_r = np.sqrt(target_x**2 + target_y**2)
    intersection_r = np.sqrt(intersection_x**2 + intersection_y**2)
    distance = intersection_r - target_r

return distance

def bisection(a, b, f, atol=1e-8):
    """
    Use the bisection method to find a root of the given function on the interval [a,
    b].
    The root returned will have the given absolute tolerance.
    f(a) and f(b) must have opposite signs.
    """
    f_a = f(a)
    f_b = f(b)
    assert max(f_a, f_b) > 0 and min(f_a, f_b) < 0, "f(a) and f(b) must have opposite
signs"

```

```

while True:
    error_bound = (b - a) / 2
    mid = (a + b) / 2
    if error_bound < atol:
        return mid

    f_mid = f(mid)
    if (f_a >= 0 and f_mid >= 0) or (f_a <= 0 and f_mid <= 0):
        a = mid
        f_a = f_mid
    else:
        b = mid
        f_b = mid

def find_launch_speed(rad_launch_theta, target_pos, launch_speed_guess=1.0):
    """
    Find the launch speed required to hit the target at the given position using the
    given launch angle.
    Optionally provide a launch speed guess to help pinpoint where the needed launch
    speed might be.
    """
    target_x, target_y = target_pos
    min_target_theta = np.atan2(target_y, target_x)
    assert rad_launch_theta >= min_target_theta, "The launch angle must be greater
    than the line of sight angle"

    v_hat = np.array([np.cos(rad_launch_theta), np.sin(rad_launch_theta)])
    distance_func = lambda v: distance_from_target(v * v_hat, target_pos)

    # Find an upper and lower bound for the needed launch speed
    d_guess = distance_func(launch_speed_guess)
    if d_guess <= 0:
        launch_speed_low = launch_speed_guess
        dist_low = d_guess
        launch_speed_high = launch_speed_guess
        while True:
            launch_speed_high *= 2
            dist_high = distance_func(launch_speed_high)
            if dist_high >= 0:
                break
        launch_speed_low = launch_speed_high
    else:
        launch_speed_high = launch_speed_guess
        dist_high = d_guess
        launch_speed_low = launch_speed_guess
        while True:
            launch_speed_low /= 2
            dist_low = distance_func(launch_speed_low)
            if dist_low <= 0:
                break
        launch_speed_high = launch_speed_low

```

```

    return bisection(launch_speed_low, launch_speed_high, distance_func)

N = 50
angle_padding = 20
deg_theta_values = np.linspace(0 + angle_padding, 90 - angle_padding, N)

target_pos = (2, 1)
deg_target_theta = np.degrees(np.atan2(target_pos[1], target_pos[0]))
deg_theta_above_target = deg_theta_values[deg_theta_values > deg_target_theta]
rad_theta_above_target = np.radians(deg_theta_above_target)

launch_speed_values = []
launch_speed_guess = 1.0
for rad_theta in rad_theta_above_target:
    launch_speed = find_launch_speed(rad_theta, target_pos,
    launch_speed_guess=launch_speed_guess)
    launch_speed_values.append(launch_speed)
    launch_speed_guess = launch_speed

fig, ax = plt.subplots()
ax.set(ylabel="$v_0$", xlabel="$\\theta$ ($^\circ$)", title=f"Target at (x, y) =
{target_pos}")
ax.plot(deg_theta_above_target, launch_speed_values, ".")
fig.savefig("media/test_hitting_target.svg")

```

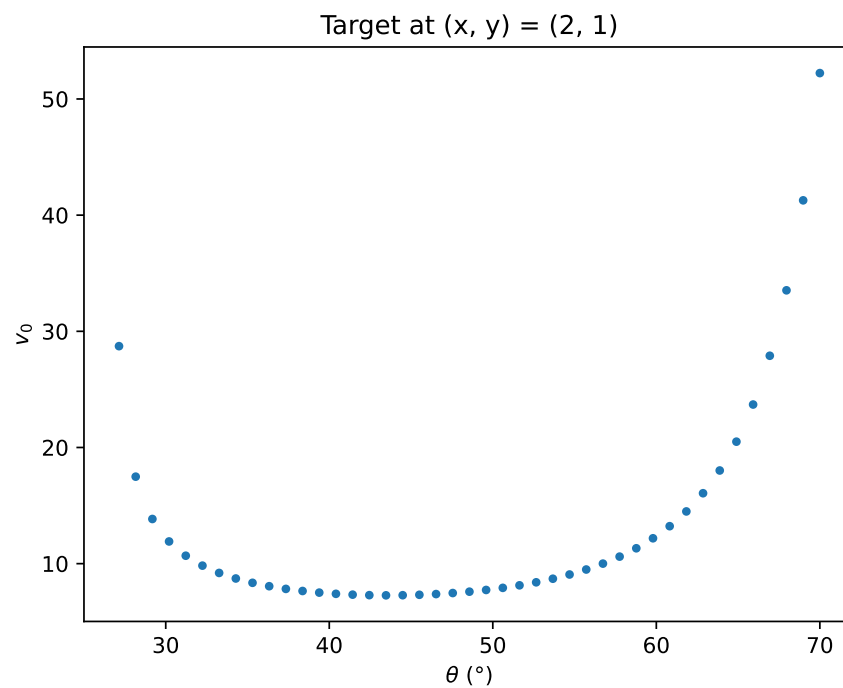


Figure 7: Test Hitting Target