

Carnot Cycle Exercise

Vincent Edwards

Mt. San Antonio College, Physics 4B, CRN 42240
March 27, 2023

$$T_H = 490 \text{ K}$$

$$V_c = 1.90 \times 10^{-3} \text{ m}^3$$

1. Purpose

The goal of the exercise is to perform various calculations related to the Carnot cycle.

2. Given

- $T_H = 490 \text{ K}$
- $T_C = 300 \text{ K}$
- $P_c = 1.01 \times 10^5 \text{ Pa}$
- $V_c = 1.90 \times 10^{-3} \text{ m}^3$
- $Q_{a \rightarrow b} = 300 \text{ J}$
- $\gamma = 1.40$
- d.o.f. = 5
- $C_v = \frac{5}{2}R$
- $C_p = \frac{7}{2}R$

3. Derivations

3.1. Pressure-Volume Relationship for Adiabatic Process

$$\Delta U = Q - W$$

$$\Delta U = -W$$

$$dU = -dW$$

$$nC_v dT = -PdV$$

$$ndT = -\frac{PdV}{C_v}$$

$$PV = nRT$$

$$PdV + VdP = nRdT$$

$$PdV + VdP = (C_p - C_v)ndT$$

$$PdV + VdP = -(C_p - C_v)\frac{PdV}{C_v}$$

$$PdV + VdP = \left(1 - \frac{C_p}{C_v}\right)PdV$$

$$VdP = -\frac{C_p}{C_v}PdV$$

$$VdP = -\gamma PdV$$

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

$$\int \frac{dP}{P} = -\int \gamma \frac{dV}{V}$$

$$\begin{aligned}
\ln(P) &= -\gamma \ln(V) + \text{const.} \\
\ln(P) + \ln(V^\gamma) &= \text{const.} \\
\ln(PV^\gamma) &= \text{const.} \\
PV^\gamma &= \text{const.} \\
P_i V_i^\gamma &= P_f V_f^\gamma
\end{aligned}$$

3.2. Temperature–Volume Relationship for Adiabatic Process

$$\begin{aligned}
P_i V_i^\gamma &= P_f V_f^\gamma \\
P_i V_i V_i^{\gamma-1} &= P_f V_f V_f^{\gamma-1} \\
nRT_i V_i^{\gamma-1} &= nRT_f V_f^{\gamma-1} \\
T_i V_i^{\gamma-1} &= T_f V_f^{\gamma-1}
\end{aligned}$$

3.3. Work by Gas for Isothermal Process

$$\begin{aligned}
W &= \int_{i \rightarrow f} P dV \\
W &= \int_{V_i}^{V_f} \frac{nRT}{V} dV \\
W &= nRT \ln(V) \Big|_{V_i}^{V_f} \\
W &= nRT (\ln(V_f) - \ln(V_i)) \\
W &= nRT \ln \left(\frac{V_f}{V_i} \right)
\end{aligned}$$

3.4. Work by Gas for Adiabatic Process

$$\begin{aligned}
W &= \int_{i \rightarrow f} P dV \\
W &= \int_{V_i}^{V_f} (P_i V_i^\gamma) V^{-\gamma} dV \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} V^{1-\gamma} \Big|_{V_i}^{V_f} \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \\
W &= \frac{P_i V_i^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma} \\
W &= \frac{P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma}
\end{aligned}$$

$$W = \frac{P_f V_f - P_i V_i}{1 - \gamma}$$

$$W = \frac{P_i V_i - P_f V_f}{\gamma - 1}$$

3.5. Carnot Cycle Heat Ratio Equals Temperature Ratio

$$\Delta U_{a \rightarrow b} = Q_{a \rightarrow b} - W_{a \rightarrow b}$$

$$0 = Q_H - nRT_H \ln \left(\frac{V_b}{V_a} \right)$$

$$Q_H = nRT_H \ln \left(\frac{V_b}{V_a} \right)$$

$$\Delta U_{c \rightarrow d} = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$0 = Q_C - nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$Q_C = nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$Q_C = -nRT_C \ln \left(\frac{V_c}{V_d} \right)$$

$$\frac{Q_H}{Q_C} = \frac{nRT_H \ln(V_b/V_a)}{-nRT_C \ln(V_c/V_d)}$$

$$-\frac{Q_H}{Q_C} = \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)}$$

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$

$$\frac{T_b}{T_c} = \left(\frac{V_c}{V_b} \right)^{\gamma-1}$$

$$\frac{T_H}{T_C} = \left(\frac{V_c}{V_b} \right)^{\gamma-1}$$

$$T_a V_a^{\gamma-1} = T_d V_d^{\gamma-1}$$

$$\frac{T_a}{T_d} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\frac{T_H}{T_C} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\left(\frac{V_c}{V_b} \right)^{\gamma-1} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\frac{V_c}{V_b} = \frac{V_d}{V_a}$$

$$\begin{aligned}\frac{V_c}{V_d} &= \frac{V_b}{V_a} \\ \ln\left(\frac{V_c}{V_d}\right) &= \ln\left(\frac{V_b}{V_a}\right) \\ 1 &= \frac{\ln(V_b/V_a)}{\ln(V_c/V_d)}\end{aligned}$$

$$\begin{aligned}-\frac{Q_H}{Q_C} &= \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)} \\ -\frac{Q_H}{Q_C} &= \frac{T_H}{T_C} \\ \left|-\frac{Q_H}{Q_C}\right| &= \left|\frac{T_H}{T_C}\right| \\ \frac{|Q_H|}{|Q_C|} &= \frac{T_H}{T_C} \\ \frac{T_C}{T_H} &= \frac{|Q_C|}{|Q_H|}\end{aligned}$$

4. Results

Table 1 contains the values for pressure, volume, and temperature at the key points of the Carnot cycle. Table 2 contains the values for heat transferred, work done, change in internal energy, and change in entropy for the key processes. Table 3 contains the pressure and volume points plotted on figure 1, the PV diagram for the process. Figure 2 is the TS diagram for the process. Various calculations are shown in the subsections below.

Table 1. Pressure, Volume, and Temperature for Key Points

Note: $T_a = T_b = T_H$ and $T_c = T_d = T_C$

Point	P (Pa)	V (m ³)	T (K)
a	1.46×10^6	2.14×10^{-4}	490
b	5.62×10^5	5.57×10^{-4}	490
c	1.01×10^5	1.90×10^{-3}	300
d	2.63×10^5	7.30×10^{-4}	300

4.1. Moles of Gas (n)

$$\begin{aligned}P_c V_c &= n R T_c \\ n &= \frac{P_c V_c}{R T_c} \\ n &= 0.0770 \text{ mol}\end{aligned}$$

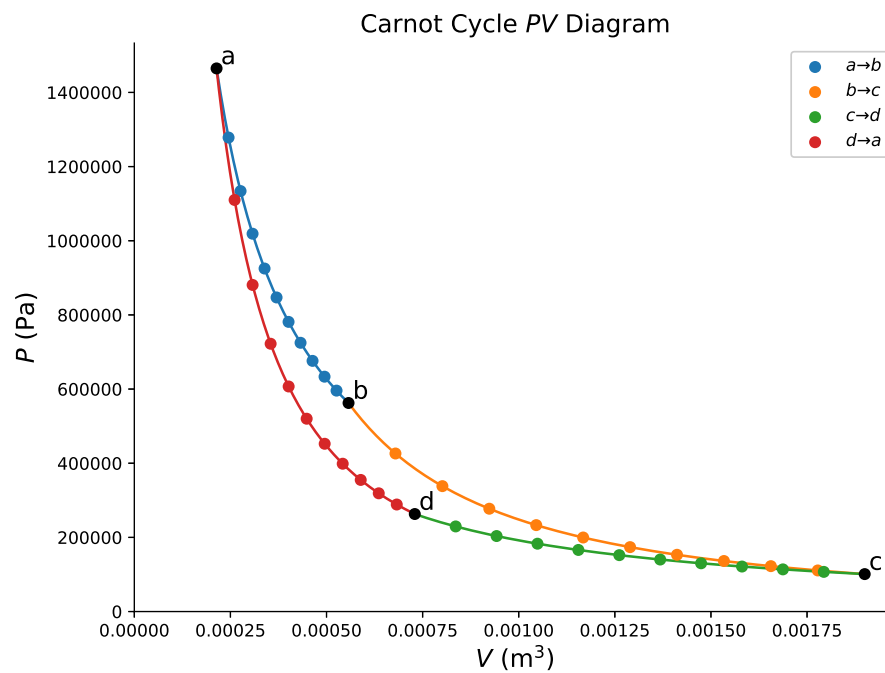


Figure 1. Carnot Cycle PV Diagram

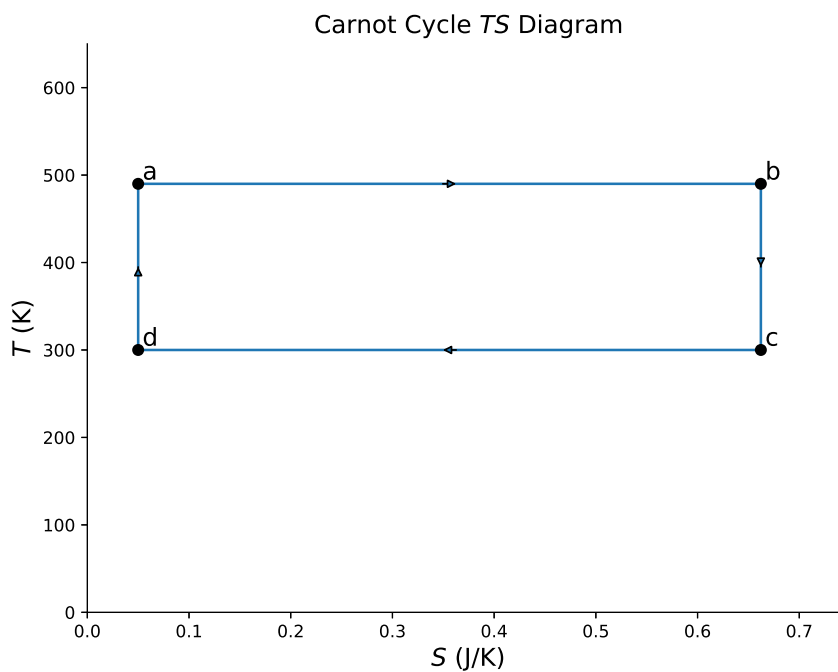


Figure 2. Carnot Cycle TS Diagram

Note: Assumed $S_a = 0.05 \text{ J/K}$ for initial condition

Table 2. Heat Transferred, Work Done, Change in Internal Energy, and Change in Entropy for Key ProcessesNote: $Q_{a \rightarrow b} = Q_H$ and $Q_{c \rightarrow d} = Q_C$

Process	Type	Q (J)	W (J)	ΔU (J)	ΔS (J/K)
$a \rightarrow b$	isothermal expansion	300	300	0	0.612
$b \rightarrow c$	adiabatic expansion	0	304	-304	0
$c \rightarrow d$	isothermal compression	-184	-184	0	-0.612
$d \rightarrow a$	adiabatic compression	0	-304	304	0
Total	Carnot cycle	116	116	0	0

4.2. Pressure (P_b) and Volume (V_b) at b

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$

$$V_b = V_c \left(\frac{T_c}{T_b} \right)^{\frac{1}{\gamma-1}}$$

$$V_b = 5.57 \times 10^{-4} \text{ m}^3$$

$$P_b V_b = nRT_b$$

$$P_b = \frac{nRT_b}{V_b}$$

$$P_b = 5.62 \times 10^5 \text{ Pa}$$

4.3. Pressure (P_a) and Volume (V_a) at a

$$\Delta U_{a \rightarrow b} = Q_{a \rightarrow b} - W_{a \rightarrow b}$$

$$0 = Q_{a \rightarrow b} - nRT_H \ln \left(\frac{V_b}{V_a} \right)$$

$$\ln \left(\frac{V_b}{V_a} \right) = \frac{Q_{a \rightarrow b}}{nRT_H}$$

$$\frac{V_b}{V_a} = e^{Q_{a \rightarrow b}/(nRT_H)}$$

$$V_a = V_b e^{-Q_{a \rightarrow b}/(nRT_H)}$$

$$V_a = 2.14 \times 10^{-4} \text{ m}^3$$

$$P_a V_a = nRT_a$$

$$P_a = \frac{nRT_a}{V_a}$$

$$P_a = 1.46 \times 10^6 \text{ Pa}$$

Table 3. Pressure and Volume Points on Carnot Cycle PV Diagram

Section		P (Pa)	V (m ³)
a		1.46×10^6	2.14×10^{-4}
$a \rightarrow b$	1	1.28×10^6	2.45×10^{-4}
$a \rightarrow b$	2	1.13×10^6	2.76×10^{-4}
$a \rightarrow b$	3	1.02×10^6	3.08×10^{-4}
$a \rightarrow b$	4	9.25×10^5	3.39×10^{-4}
$a \rightarrow b$	5	8.47×10^5	3.70×10^{-4}
$a \rightarrow b$	6	7.81×10^5	4.01×10^{-4}
$a \rightarrow b$	7	7.25×10^5	4.32×10^{-4}
$a \rightarrow b$	8	6.76×10^5	4.64×10^{-4}
$a \rightarrow b$	9	6.33×10^5	4.95×10^{-4}
$a \rightarrow b$	10	5.96×10^5	5.26×10^{-4}
b		5.62×10^5	5.57×10^{-4}
$b \rightarrow c$	1	4.26×10^5	6.79×10^{-4}
$b \rightarrow c$	2	3.38×10^5	8.01×10^{-4}
$b \rightarrow c$	3	2.77×10^5	9.23×10^{-4}
$b \rightarrow c$	4	2.33×10^5	1.05×10^{-3}
$b \rightarrow c$	5	2.00×10^5	1.17×10^{-3}
$b \rightarrow c$	6	1.74×10^5	1.29×10^{-3}
$b \rightarrow c$	7	1.53×10^5	1.41×10^{-3}
$b \rightarrow c$	8	1.36×10^5	1.53×10^{-3}
$b \rightarrow c$	9	1.22×10^5	1.66×10^{-3}
$b \rightarrow c$	10	1.11×10^5	1.78×10^{-3}
c		1.01×10^5	1.90×10^{-3}
$c \rightarrow d$	1	1.07×10^5	1.79×10^{-3}
$c \rightarrow d$	2	1.14×10^5	1.69×10^{-3}
$c \rightarrow d$	3	1.21×10^5	1.58×10^{-3}
$c \rightarrow d$	4	1.30×10^5	1.47×10^{-3}
$c \rightarrow d$	5	1.40×10^5	1.37×10^{-3}
$c \rightarrow d$	6	1.52×10^5	1.26×10^{-3}
$c \rightarrow d$	7	1.66×10^5	1.16×10^{-3}
$c \rightarrow d$	8	1.83×10^5	1.05×10^{-3}
$c \rightarrow d$	9	2.04×10^5	9.42×10^{-4}
$c \rightarrow d$	10	2.30×10^5	8.36×10^{-4}
d		2.63×10^5	7.30×10^{-4}
$d \rightarrow a$	1	2.89×10^5	6.83×10^{-4}
$d \rightarrow a$	2	3.19×10^5	6.36×10^{-4}
$d \rightarrow a$	3	3.55×10^5	5.89×10^{-4}
$d \rightarrow a$	4	3.99×10^5	5.42×10^{-4}
$d \rightarrow a$	5	4.52×10^5	4.95×10^{-4}
$d \rightarrow a$	6	5.20×10^5	4.48×10^{-4}
$d \rightarrow a$	7	6.07×10^5	4.01×10^{-4}
$d \rightarrow a$	8	7.22×10^5	3.55×10^{-4}
$d \rightarrow a$	9	8.81×10^5	3.08×10^{-4}
$d \rightarrow a$	10	1.11×10^6	2.61×10^{-4}

4.4. Pressure (P_d) and Volume (V_d) at d

$$\begin{aligned}
 T_d V_d^{\gamma-1} &= T_a V_a^{\gamma-1} \\
 V_d &= V_a \left(\frac{T_a}{T_d} \right)^{\frac{1}{\gamma-1}} \\
 V_d &= 7.30 \times 10^{-4} \text{ m}^3
 \end{aligned}$$

$$\begin{aligned}
 P_d V_d &= n R T_d \\
 P_d &= \frac{n R T_d}{V_d} \\
 P_d &= 2.63 \times 10^5 \text{ Pa}
 \end{aligned}$$

4.5. Process Variables $a \rightarrow b$

$$\Delta U_{a \rightarrow b} = 0$$

$$\begin{aligned}
 \Delta U_{a \rightarrow b} &= Q_{a \rightarrow b} - W_{a \rightarrow b} \\
 0 &= Q_{a \rightarrow b} - W_{a \rightarrow b} \\
 W_{a \rightarrow b} &= Q_{a \rightarrow b} \\
 W_{a \rightarrow b} &= 300 \text{ J}
 \end{aligned}$$

$$\begin{aligned}
 \Delta S_{a \rightarrow b} &= \int_{a \rightarrow b} \frac{dQ}{T} \\
 \Delta S_{a \rightarrow b} &= \frac{1}{T_H} \int_{a \rightarrow b} dQ \\
 \Delta S_{a \rightarrow b} &= \frac{Q_{a \rightarrow b}}{T_H} \\
 \Delta S_{a \rightarrow b} &= 0.612 \text{ J/K}
 \end{aligned}$$

4.6. Process Variables $b \rightarrow c$

$$Q_{b \rightarrow c} = 0$$

$$\begin{aligned}
 W_{b \rightarrow c} &= \frac{P_b V_b - P_c V_c}{\gamma - 1} \\
 W_{b \rightarrow c} &= 304 \text{ J}
 \end{aligned}$$

$$\Delta U_{b \rightarrow c} = Q_{b \rightarrow c} - W_{b \rightarrow c}$$

$$\Delta U_{b \rightarrow c} = -W_{b \rightarrow c}$$

$$\Delta U_{b \rightarrow c} = -304 \text{ J}$$

$$\Delta S_{b \rightarrow c} = \int_{b \rightarrow c} \frac{dQ}{T}$$

$$\Delta S_{b \rightarrow c} = 0$$

4.7. Process Variables $c \rightarrow d$

$$\Delta U_{c \rightarrow d} = 0$$

$$W_{c \rightarrow d} = nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$W_{c \rightarrow d} = -184 \text{ J}$$

$$\Delta U_{c \rightarrow d} = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$0 = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$Q_{c \rightarrow d} = W_{c \rightarrow d}$$

$$Q_{c \rightarrow d} = -184 \text{ J}$$

$$\Delta S_{c \rightarrow d} = \int_{c \rightarrow d} \frac{dQ}{T}$$

$$\Delta S_{c \rightarrow d} = \frac{1}{T_C} \int_{c \rightarrow d} dQ$$

$$\Delta S_{c \rightarrow d} = \frac{Q_{c \rightarrow d}}{T_C}$$

$$\Delta S_{c \rightarrow d} = -0.612 \text{ J/K}$$

4.8. Process Variables $d \rightarrow a$

$$Q_{d \rightarrow a} = 0$$

$$W_{d \rightarrow a} = \frac{P_d V_d - P_a V_a}{\gamma - 1}$$

$$W_{d \rightarrow a} = -304 \text{ J}$$

$$\Delta U_{d \rightarrow a} = Q_{d \rightarrow a} - W_{d \rightarrow a}$$

$$\Delta U_{d \rightarrow a} = -W_{d \rightarrow a}$$

$$\Delta U_{d \rightarrow a} = 304 \text{ J}$$

$$\Delta S_{d \rightarrow a} = \int_{d \rightarrow a} \frac{dQ}{T}$$

$$\Delta S_{d \rightarrow a} = 0$$

4.9. Pressure Calculation Given Volume $a \rightarrow b$

$$PV = P_a V_a$$

$$P = \frac{P_a V_a}{V}$$

$$\text{Ex: } V = 2.45 \times 10^{-4} \text{ m}^3 \Rightarrow P = 1.28 \times 10^6 \text{ Pa}$$

4.10. Pressure Calculation Given Volume $b \rightarrow c$

$$PV^\gamma = P_b V_b^\gamma$$

$$P = P_b \left(\frac{V_b}{V} \right)^\gamma$$

$$\text{Ex: } V = 6.79 \times 10^{-4} \text{ m}^3 \Rightarrow P = 4.26 \times 10^5 \text{ Pa}$$

4.11. Pressure Calculation Given Volume $c \rightarrow d$

$$PV = P_c V_c$$

$$P = \frac{P_c V_c}{V}$$

$$\text{Ex: } V = 1.79 \times 10^{-3} \text{ m}^3 \Rightarrow P = 1.07 \times 10^5 \text{ Pa}$$

4.12. Pressure Calculation Given Volume $d \rightarrow a$

$$PV^\gamma = P_d V_d^\gamma$$

$$P = P_d \left(\frac{V_d}{V} \right)^\gamma$$

$$\text{Ex: } V = 6.83 \times 10^{-4} \text{ m}^3 \Rightarrow P = 2.89 \times 10^5 \text{ Pa}$$

5. Conclusion

6. Citations

- [1] Karen Schnurbusch, *Physics 4B Lab Book*, Mt. San Antonio College, 2023, pp. 35-38.
- [2] Karen Schnurbusch, *Physics 4B Equations*, Mt. San Antonio College, 2023, pp. 1-3.