Carnot Cycle Exercise

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$T_H = 490$ K
$V_c = 1.90 \times 10^{-3}$ m^3

1. Purpose

The goal of the exercise is to perform various calculations related to the Carnot cycle.

2. Given

- $T_H = 490$ K
- $T_C = 300$ K
- $P_c = 1.01 \times 10^5$ Pa
- $V_c = 1.90 \times 10^{-3}$ m^3
- $Q_{a\to b} = 300$ J
- $\gamma = 1.40$
- d.o.f. $= 5$
- $C_v = \frac{5}{2}R$
- $C_p = \frac{7}{2}R$

3. Derivations

3.1. Pressure–Volume Relationship for Adiabatic Process

$$
\begin{aligned}
\Delta U &= Q - W \\
\Delta U &= -W \\
dU &= -dW \\
nC_v dT &= -PdV \\
ndT &= -\frac{PdV}{C_v}
\end{aligned}
$$

$$
\begin{aligned}
PV &= nRT \\
PdV + VdP &= nRdT \\
PdV + VdP &= (C_p - C_v)ndT \\
PdV + VdP &= -(C_p - C_v)\frac{PdV}{C_v} \\
PdV + VdP &= \left(1 - \frac{C_p}{C_v}\right) PdV \\
VdP &= -\frac{C_p}{C_v} PdV \\
VdP &= -\gamma PdV \\
\frac{dP}{P} &= -\gamma \frac{dV}{V} \\
\int \frac{dP}{P} &= -\int \gamma \frac{dV}{V}
\end{aligned}
$$

$$\ln(P) = -\gamma \ln(V) + \text{const.}$$
$$\ln(P) + \ln(V^\gamma) = \text{const.}$$
$$\ln(PV^\gamma) = \text{const.}$$
$$PV^\gamma = \text{const.}$$
$$P_i V_i^\gamma = P_f V_f^\gamma$$

3.2. Temperature–Volume Relationship for Adiabatic Process

$$P_i V_i^\gamma = P_f V_f^\gamma$$
$$P_i V_i V_i^{\gamma-1} = P_f V_f V_f^{\gamma-1}$$
$$nRT_i V_i^{\gamma-1} = nRT_f V_f^{\gamma-1}$$
$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

3.3. Work by Gas for Isothermal Process

$$W = \int_{i\to f} P dV$$
$$W = \int_{V_i}^{V_f} \frac{nRT}{V} dV$$
$$W = nRT \ln(V)|_{V_i}^{V_f}$$
$$W = nRT(\ln(V_f) - \ln(V_i))$$
$$W = nRT \ln\left(\frac{V_f}{V_i}\right)$$

3.4. Work by Gas for Adiabatic Process

$$W = \int_{i\to f} P dV$$
$$W = \int_{V_i}^{V_f} (P_i V_i^\gamma) V^{-\gamma} dV$$
$$W = \frac{P_i V_i^\gamma}{1-\gamma} V^{1-\gamma}|_{V_i}^{V_f}$$
$$W = \frac{P_i V_i^\gamma}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma})$$
$$W = \frac{P_i V_i^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma}$$
$$W = \frac{P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma}$$

$$W = \frac{P_f V_f - P_i V_i}{1 - \gamma}$$
$$W = \frac{P_i V_i - P_f V_f}{\gamma - 1}$$

4. Results

Table 1. Pressure, Volume, and Temperature for Key Points
Note: $T_a = T_b = T_H$ and $T_c = T_d = T_C$

Point	P (Pa)	V (m^3)	T (K)
a	1.46×10^6	2.14×10^{-4}	490
b	5.62×10^5	5.57×10^{-4}	490
c	1.01×10^5	1.90×10^{-3}	300
d	2.63×10^5	7.30×10^{-4}	300

Table 2. Heat Transferred, Change in Internal Energy, Work Done, and Change in Entropy for Key Processes

Process	Type	Q (J)	W (J)	ΔU (J)	ΔS (J/K)
$a \to b$	isothermal expansion	300	300	0	0.612
$b \to c$	adiabatic expansion	0	304	-304	0
$c \to d$	isothermal compression	-184	-184	0	-0.612
$d \to a$	adiabatic compression	0	-304	304	0

4.1. Moles of Gas (n)

$$P_c V_c = nRT_c$$
$$n = \frac{P_c V_c}{RT_c}$$
$$n = 0.0770 \text{ mol}$$

4.2. Pressure (P_b) and Volume (V_b) at b

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$
$$V_b = V_c \left(\frac{T_c}{T_b}\right)^{\frac{1}{\gamma-1}}$$
$$V_b = 5.57 \times 10^{-4} \text{ m}^3$$

$$P_b V_b = nRT_b$$

Table 3. Pressure and Volume Points on PV Diagram

	P (Pa)	V (m^3)
a	1.46×10^6	2.14×10^{-4}
$a \to b$	1.28×10^6	2.45×10^{-4}
$a \to b$	1.13×10^6	2.76×10^{-4}
$a \to b$	1.02×10^6	3.08×10^{-4}
$a \to b$	9.25×10^5	3.39×10^{-4}
$a \to b$	8.47×10^5	3.70×10^{-4}
$a \to b$	7.81×10^5	4.01×10^{-4}
$a \to b$	7.25×10^5	4.32×10^{-4}
$a \to b$	6.76×10^5	4.64×10^{-4}
$a \to b$	6.33×10^5	4.95×10^{-4}
$a \to b$	5.96×10^5	5.26×10^{-4}
b	5.62×10^5	5.57×10^{-4}
$b \to c$	4.26×10^5	6.79×10^{-4}
$b \to c$	3.38×10^5	8.01×10^{-4}
$b \to c$	2.77×10^5	9.23×10^{-4}
$b \to c$	2.33×10^5	1.05×10^{-3}
$b \to c$	2.00×10^5	1.17×10^{-3}
$b \to c$	1.74×10^5	1.29×10^{-3}
$b \to c$	1.53×10^5	1.41×10^{-3}
$b \to c$	1.36×10^5	1.53×10^{-3}
$b \to c$	1.22×10^5	1.66×10^{-3}
$b \to c$	1.11×10^5	1.78×10^{-3}
c	1.01×10^5	1.90×10^{-3}
$c \to d$	1.07×10^5	1.79×10^{-3}
$c \to d$	1.14×10^5	1.69×10^{-3}
$c \to d$	1.21×10^5	1.58×10^{-3}
$c \to d$	1.30×10^5	1.47×10^{-3}
$c \to d$	1.40×10^5	1.37×10^{-3}
$c \to d$	1.52×10^5	1.26×10^{-3}
$c \to d$	1.66×10^5	1.16×10^{-3}
$c \to d$	1.83×10^5	1.05×10^{-3}
$c \to d$	2.04×10^5	9.42×10^{-4}
$c \to d$	2.30×10^5	8.36×10^{-4}
d	2.63×10^5	7.30×10^{-4}
$d \to a$	2.89×10^5	6.83×10^{-4}
$d \to a$	3.19×10^5	6.36×10^{-4}
$d \to a$	3.55×10^5	5.89×10^{-4}
$d \to a$	3.99×10^5	5.42×10^{-4}
$d \to a$	4.52×10^5	4.95×10^{-4}
$d \to a$	5.20×10^5	4.48×10^{-4}
$d \to a$	6.07×10^5	4.01×10^{-4}
$d \to a$	7.22×10^5	3.55×10^{-4}
$d \to a$	8.81×10^5	3.08×10^{-4}
$d \to a$	1.11×10^6	2.61×10^{-4}

$$P_b = \frac{nRT_b}{V_b}$$
$$P_b = 5.62 \times 10^5 \text{ Pa}$$

4.3. Pressure (P_a) and Volume (V_a) at a

$$\begin{aligned}
\Delta U_{a\to b} &= Q_{a\to b} - W_{a\to b} \\
0 &= Q_{a\to b} - nRT_H \ln\left(\frac{V_b}{V_a}\right) \\
\ln\left(\frac{V_b}{V_a}\right) &= \frac{Q_{a\to b}}{nRT_H} \\
\frac{V_b}{V_a} &= e^{Q_{a\to b}/(nRT_H)} \\
V_a &= V_b e^{-Q_{a\to b}/(nRT_H)} \\
V_a &= 2.14 \times 10^{-4} \text{ m}^3
\end{aligned}$$

$$\begin{aligned}
P_a V_a &= nRT_a \\
P_a &= \frac{nRT_a}{V_a} \\
P_a &= 1.46 \times 10^6 \text{ Pa}
\end{aligned}$$

4.4. Pressure (P_d) and Volume (V_d) at d

$$\begin{aligned}
T_d V_d^{\gamma-1} &= T_a V_a^{\gamma-1} \\
V_d &= V_a \left(\frac{T_a}{T_d}\right)^{\frac{1}{\gamma-1}} \\
V_d &= 7.30 \times 10^{-4} \text{ m}^3
\end{aligned}$$

$$\begin{aligned}
P_d V_d &= nRT_d \\
P_d &= \frac{nRT_d}{V_d} \\
P_d &= 2.63 \times 10^5 \text{ Pa}
\end{aligned}$$

4.5. Process Variables $a \to b$

$$\Delta U_{a\to b} = 0$$

$$\begin{aligned}
\Delta U_{a\to b} &= Q_{a\to b} - W_{a\to b} \\
0 &= Q_{a\to b} - W_{a\to b} \\
W_{a\to b} &= Q_{a\to b} \\
W_{a\to b} &= 300 \text{ J}
\end{aligned}$$

$$\Delta S_{a\to b} = \int\limits_{a\to b} \frac{dQ}{T}$$
$$\Delta S_{a\to b} = \frac{1}{T_H} \int\limits_{a\to b} dQ$$
$$\Delta S_{a\to b} = \frac{Q_{a\to b}}{T_H}$$
$$\Delta S_{a\to b} = 0.612 \text{ J/K}$$

4.6. Process Variables $b \to c$

$$Q_{b\to c} = 0$$

$$W_{b\to c} = \frac{P_b V_b - P_c V_c}{\gamma - 1}$$
$$W_{b\to c} = 304 \text{ J}$$

$$\Delta U_{b\to c} = Q_{b\to c} - W_{b\to c}$$
$$\Delta U_{b\to c} = -W_{b\to c}$$
$$\Delta U_{b\to c} = -304 \text{ J}$$

$$\Delta S_{b\to c} = \int\limits_{b\to c} \frac{dQ}{T}$$
$$\Delta S_{b\to c} = 0$$

4.7. Process Variables $c \to d$

$$\Delta U_{c\to d} = 0$$

$$W_{c\to d} = nRT_C \ln\left(\frac{V_d}{V_c}\right)$$
$$W_{c\to d} = -184 \text{ J}$$

$$\Delta U_{c\to d} = Q_{c\to d} - W_{c\to d}$$
$$0 = Q_{c\to d} - W_{c\to d}$$
$$Q_{c\to d} = W_{c\to d}$$
$$Q_{c\to d} = -184 \text{ J}$$

$$\Delta S_{c\to d} = \int\limits_{c\to d} \frac{dQ}{T}$$
$$\Delta S_{c\to d} = \frac{1}{T_C} \int\limits_{c\to d} dQ$$
$$\Delta S_{c\to d} = \frac{Q_{c\to d}}{T_C}$$
$$\Delta S_{c\to d} = -0.612 \text{ J/K}$$

4.8. Process Variables $d \to a$

$$Q_{d\to a} = 0$$

$$W_{d\to a} = \frac{P_d V_d - P_a V_a}{\gamma - 1}$$
$$W_{d\to a} = -304 \text{ J}$$

$$\Delta U_{d\to a} = Q_{d\to a} - W_{d\to a}$$
$$\Delta U_{d\to a} = -W_{d\to a}$$
$$\Delta U_{d\to a} = 304 \text{ J}$$

$$\Delta S_{d\to a} = \int\limits_{d\to a} \frac{dQ}{T}$$
$$\Delta S_{d\to a} = 0$$

5. Conclusion

6. Citations

[1] Karen Schnurbusch, *Physics 4B Lab Book*, Mt. San Antonio College, 2023, pp. 35-38.
[2] Karen Schnurbusch, *Physics 4B Equations*, Mt. San Antonio College, 2023, pp. 1-3.