

Carnot Cycle Exercise

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March 27, 2023

$$T_H = 490 \text{ K}$$

$$V_c = 1.90 \times 10^{-3} \text{ m}^3$$

1. Purpose

The goal of the exercise is to perform various calculations related to the Carnot cycle.

2. Given

- $T_H = 490 \text{ K}$
- $T_C = 300 \text{ K}$
- $P_c = 1.01 \times 10^5 \text{ Pa}$
- $V_c = 1.90 \times 10^{-3} \text{ m}^3$
- $Q_{a \rightarrow b} = 300 \text{ J}$
- $\gamma = 1.40$
- d.o.f. = 5
- $C_v = \frac{5}{2}R$
- $C_p = \frac{7}{2}R$

3. Derivations

3.1. Pressure-Volume Relationship for Adiabatic Process

$$\Delta U = Q - W$$

$$\Delta U = -W$$

$$dU = -dW$$

$$nC_v dT = -PdV$$

$$ndT = -\frac{PdV}{C_v}$$

$$PV = nRT$$

$$PdV + VdP = nRdT$$

$$PdV + VdP = (C_p - C_v)ndT$$

$$PdV + VdP = -(C_p - C_v)\frac{PdV}{C_v}$$

$$PdV + VdP = \left(1 - \frac{C_p}{C_v}\right)PdV$$

$$VdP = -\frac{C_p}{C_v}PdV$$

$$VdP = -\gamma PdV$$

$$\frac{dP}{P} = -\gamma \frac{dV}{V}$$

$$\int \frac{dP}{P} = -\int \gamma \frac{dV}{V}$$

$$\begin{aligned}
\ln(P) &= -\gamma \ln(V) + \text{const.} \\
\ln(P) + \ln(V^\gamma) &= \text{const.} \\
\ln(PV^\gamma) &= \text{const.} \\
PV^\gamma &= \text{const.} \\
P_i V_i^\gamma &= P_f V_f^\gamma
\end{aligned}$$

3.2. Temperature–Volume Relationship for Adiabatic Process

$$\begin{aligned}
P_i V_i^\gamma &= P_f V_f^\gamma \\
P_i V_i V_i^{\gamma-1} &= P_f V_f V_f^{\gamma-1} \\
nRT_i V_i^{\gamma-1} &= nRT_f V_f^{\gamma-1} \\
T_i V_i^{\gamma-1} &= T_f V_f^{\gamma-1}
\end{aligned}$$

3.3. Work by Gas for Isothermal Process

$$\begin{aligned}
W &= \int_{i \rightarrow f} P dV \\
W &= \int_{V_i}^{V_f} \frac{nRT}{V} dV \\
W &= nRT \ln(V) \Big|_{V_i}^{V_f} \\
W &= nRT (\ln(V_f) - \ln(V_i)) \\
W &= nRT \ln \left(\frac{V_f}{V_i} \right)
\end{aligned}$$

3.4. Work by Gas for Adiabatic Process

$$\begin{aligned}
W &= \int_{i \rightarrow f} P dV \\
W &= \int_{V_i}^{V_f} (P_i V_i^\gamma) V^{-\gamma} dV \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} V^{1-\gamma} \Big|_{V_i}^{V_f} \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \\
W &= \frac{P_i V_i^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma} \\
W &= \frac{P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma}
\end{aligned}$$

$$W = \frac{P_f V_f - P_i V_i}{1 - \gamma}$$

$$W = \frac{P_i V_i - P_f V_f}{\gamma - 1}$$

3.5. Carnot Cycle Heat Ratio Equals Temperature Ratio

$$\Delta U_{a \rightarrow b} = Q_{a \rightarrow b} - W_{a \rightarrow b}$$

$$0 = Q_H - nRT_H \ln \left(\frac{V_b}{V_a} \right)$$

$$Q_H = nRT_H \ln \left(\frac{V_b}{V_a} \right)$$

$$\Delta U_{c \rightarrow d} = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$0 = Q_C - nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$Q_C = nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$Q_C = -nRT_C \ln \left(\frac{V_c}{V_d} \right)$$

$$\frac{Q_H}{Q_C} = \frac{nRT_H \ln(V_b/V_a)}{-nRT_C \ln(V_c/V_d)}$$

$$-\frac{Q_H}{Q_C} = \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)}$$

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$

$$\frac{T_b}{T_c} = \left(\frac{V_c}{V_b} \right)^{\gamma-1}$$

$$\frac{T_H}{T_C} = \left(\frac{V_c}{V_b} \right)^{\gamma-1}$$

$$T_a V_a^{\gamma-1} = T_d V_d^{\gamma-1}$$

$$\frac{T_a}{T_d} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\frac{T_H}{T_C} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\left(\frac{V_c}{V_b} \right)^{\gamma-1} = \left(\frac{V_d}{V_a} \right)^{\gamma-1}$$

$$\frac{V_c}{V_b} = \frac{V_d}{V_a}$$

$$\begin{aligned}\frac{V_c}{V_d} &= \frac{V_b}{V_a} \\ \ln\left(\frac{V_c}{V_d}\right) &= \ln\left(\frac{V_b}{V_a}\right) \\ 1 &= \frac{\ln(V_b/V_a)}{\ln(V_c/V_d)}\end{aligned}$$

$$\begin{aligned}-\frac{Q_H}{Q_C} &= \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)} \\ -\frac{Q_H}{Q_C} &= \frac{T_H}{T_C} \\ \left|-\frac{Q_H}{Q_C}\right| &= \left|\frac{T_H}{T_C}\right| \\ \frac{|Q_H|}{|Q_C|} &= \frac{T_H}{T_C} \\ \frac{T_C}{T_H} &= \frac{|Q_C|}{|Q_H|}\end{aligned}$$

4. Results

Table 1 contains the values for pressure, volume, and temperature at the key points of the Carnot cycle. Table 2 contains the values for heat transferred, work done, change in internal energy, and change in entropy for the key processes. Table 3 contains the pressure and volume points plotted on figure 1, the PV diagram for the process. Figure 2 is the TS diagram for the process. Various calculations are shown in the subsections below.

Table 1. Pressure, Volume, and Temperature for Key Points

Note: $T_a = T_b = T_H$ and $T_c = T_d = T_C$

Point	P (Pa)	V (m ³)	T (K)
a	1.46×10^6	2.14×10^{-4}	490
b	5.62×10^5	5.57×10^{-4}	490
c	1.01×10^5	1.90×10^{-3}	300
d	2.63×10^5	7.30×10^{-4}	300

4.1. Moles of Gas (n)

$$\begin{aligned}P_c V_c &= n R T_c \\ n &= \frac{P_c V_c}{R T_c} \\ n &= 0.0770 \text{ mol}\end{aligned}$$

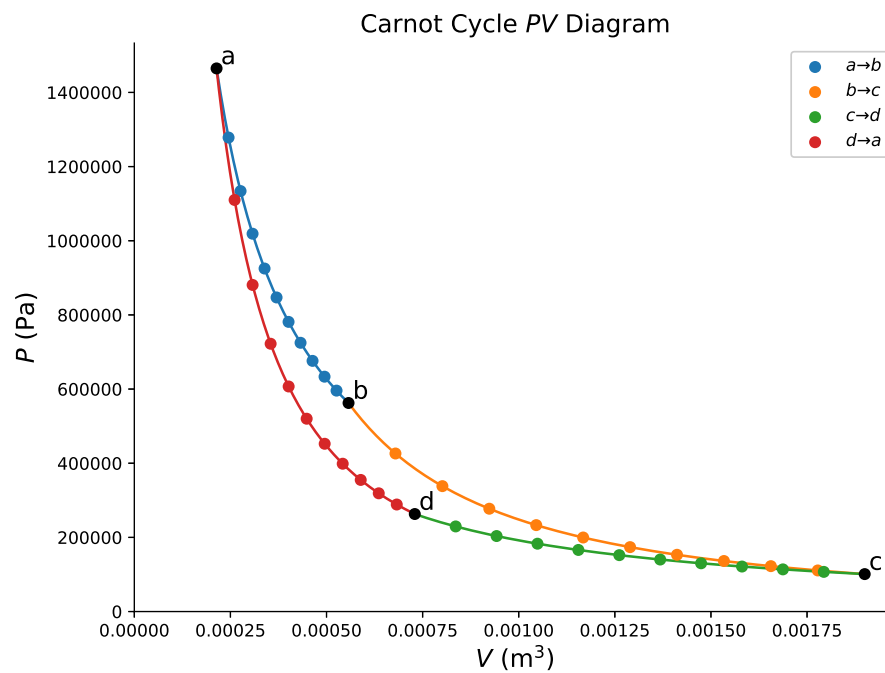


Figure 1. Carnot Cycle PV Diagram

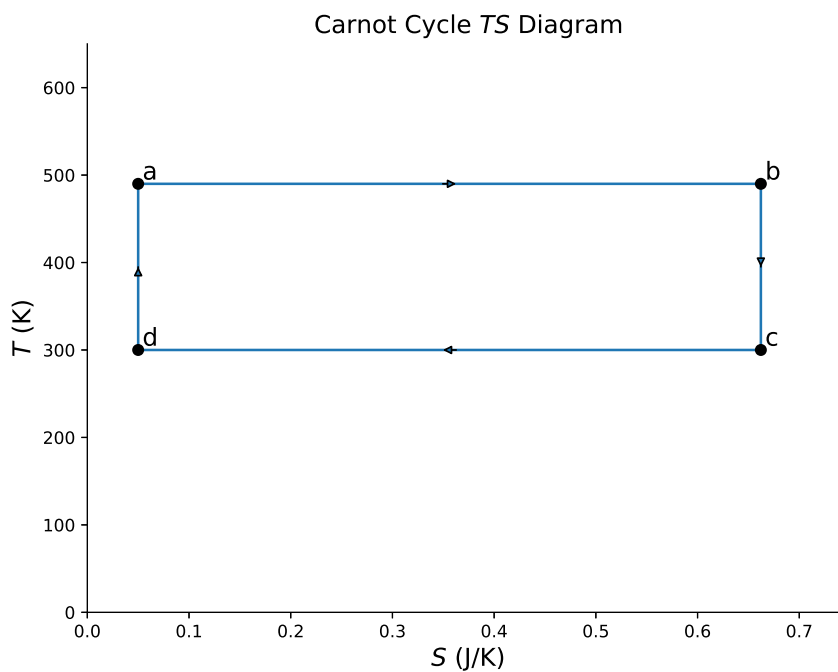


Figure 2. Carnot Cycle TS Diagram

Note: Assumed $S_a = 0.05 \text{ J/K}$ for initial condition

Table 2. Heat Transferred, Work Done, Change in Internal Energy, and Change in Entropy for Key ProcessesNote: $Q_{a \rightarrow b} = Q_H$ and $Q_{c \rightarrow d} = Q_C$

Process	Type	Q (J)	W (J)	ΔU (J)	ΔS (J/K)
$a \rightarrow b$	isothermal expansion	300	300	0	0.612
$b \rightarrow c$	adiabatic expansion	0	304	-304	0
$c \rightarrow d$	isothermal compression	-184	-184	0	-0.612
$d \rightarrow a$	adiabatic compression	0	-304	304	0
Total	Carnot cycle	116	116	0	0

4.2. Pressure (P_b) and Volume (V_b) at b

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$

$$V_b = V_c \left(\frac{T_c}{T_b} \right)^{\frac{1}{\gamma-1}}$$

$$V_b = 5.57 \times 10^{-4} \text{ m}^3$$

$$P_b V_b = n R T_b$$

$$P_b = \frac{n R T_b}{V_b}$$

$$P_b = 5.62 \times 10^5 \text{ Pa}$$

4.3. Pressure (P_a) and Volume (V_a) at a

$$\Delta U_{a \rightarrow b} = Q_{a \rightarrow b} - W_{a \rightarrow b}$$

$$0 = Q_{a \rightarrow b} - n R T_H \ln \left(\frac{V_b}{V_a} \right)$$

$$\ln \left(\frac{V_b}{V_a} \right) = \frac{Q_{a \rightarrow b}}{n R T_H}$$

$$\frac{V_b}{V_a} = e^{Q_{a \rightarrow b} / (n R T_H)}$$

$$V_a = V_b e^{-Q_{a \rightarrow b} / (n R T_H)}$$

$$V_a = 2.14 \times 10^{-4} \text{ m}^3$$

$$P_a V_a = n R T_a$$

$$P_a = \frac{n R T_a}{V_a}$$

$$P_a = 1.46 \times 10^6 \text{ Pa}$$

Table 3. Pressure and Volume Points on Carnot Cycle PV Diagram

Section		P (Pa)	V (m ³)
a		1.46×10^6	2.14×10^{-4}
$a \rightarrow b$	1	1.28×10^6	2.45×10^{-4}
$a \rightarrow b$	2	1.13×10^6	2.76×10^{-4}
$a \rightarrow b$	3	1.02×10^6	3.08×10^{-4}
$a \rightarrow b$	4	9.25×10^5	3.39×10^{-4}
$a \rightarrow b$	5	8.47×10^5	3.70×10^{-4}
$a \rightarrow b$	6	7.81×10^5	4.01×10^{-4}
$a \rightarrow b$	7	7.25×10^5	4.32×10^{-4}
$a \rightarrow b$	8	6.76×10^5	4.64×10^{-4}
$a \rightarrow b$	9	6.33×10^5	4.95×10^{-4}
$a \rightarrow b$	10	5.96×10^5	5.26×10^{-4}
b		5.62×10^5	5.57×10^{-4}
$b \rightarrow c$	1	4.26×10^5	6.79×10^{-4}
$b \rightarrow c$	2	3.38×10^5	8.01×10^{-4}
$b \rightarrow c$	3	2.77×10^5	9.23×10^{-4}
$b \rightarrow c$	4	2.33×10^5	1.05×10^{-3}
$b \rightarrow c$	5	2.00×10^5	1.17×10^{-3}
$b \rightarrow c$	6	1.74×10^5	1.29×10^{-3}
$b \rightarrow c$	7	1.53×10^5	1.41×10^{-3}
$b \rightarrow c$	8	1.36×10^5	1.53×10^{-3}
$b \rightarrow c$	9	1.22×10^5	1.66×10^{-3}
$b \rightarrow c$	10	1.11×10^5	1.78×10^{-3}
c		1.01×10^5	1.90×10^{-3}
$c \rightarrow d$	1	1.07×10^5	1.79×10^{-3}
$c \rightarrow d$	2	1.14×10^5	1.69×10^{-3}
$c \rightarrow d$	3	1.21×10^5	1.58×10^{-3}
$c \rightarrow d$	4	1.30×10^5	1.47×10^{-3}
$c \rightarrow d$	5	1.40×10^5	1.37×10^{-3}
$c \rightarrow d$	6	1.52×10^5	1.26×10^{-3}
$c \rightarrow d$	7	1.66×10^5	1.16×10^{-3}
$c \rightarrow d$	8	1.83×10^5	1.05×10^{-3}
$c \rightarrow d$	9	2.04×10^5	9.42×10^{-4}
$c \rightarrow d$	10	2.30×10^5	8.36×10^{-4}
d		2.63×10^5	7.30×10^{-4}
$d \rightarrow a$	1	2.89×10^5	6.83×10^{-4}
$d \rightarrow a$	2	3.19×10^5	6.36×10^{-4}
$d \rightarrow a$	3	3.55×10^5	5.89×10^{-4}
$d \rightarrow a$	4	3.99×10^5	5.42×10^{-4}
$d \rightarrow a$	5	4.52×10^5	4.95×10^{-4}
$d \rightarrow a$	6	5.20×10^5	4.48×10^{-4}
$d \rightarrow a$	7	6.07×10^5	4.01×10^{-4}
$d \rightarrow a$	8	7.22×10^5	3.55×10^{-4}
$d \rightarrow a$	9	8.81×10^5	3.08×10^{-4}
$d \rightarrow a$	10	1.11×10^6	2.61×10^{-4}

4.4. Pressure (P_d) and Volume (V_d) at d

$$\begin{aligned}
 T_d V_d^{\gamma-1} &= T_a V_a^{\gamma-1} \\
 V_d &= V_a \left(\frac{T_a}{T_d} \right)^{\frac{1}{\gamma-1}} \\
 V_d &= 7.30 \times 10^{-4} \text{ m}^3
 \end{aligned}$$

$$\begin{aligned}
 P_d V_d &= n R T_d \\
 P_d &= \frac{n R T_d}{V_d} \\
 P_d &= 2.63 \times 10^5 \text{ Pa}
 \end{aligned}$$

4.5. Process Variables $a \rightarrow b$

$$\Delta U_{a \rightarrow b} = 0$$

$$\begin{aligned}
 \Delta U_{a \rightarrow b} &= Q_{a \rightarrow b} - W_{a \rightarrow b} \\
 0 &= Q_{a \rightarrow b} - W_{a \rightarrow b} \\
 W_{a \rightarrow b} &= Q_{a \rightarrow b} \\
 W_{a \rightarrow b} &= 300 \text{ J}
 \end{aligned}$$

$$\begin{aligned}
 \Delta S_{a \rightarrow b} &= \int_{a \rightarrow b} \frac{dQ}{T} \\
 \Delta S_{a \rightarrow b} &= \frac{1}{T_H} \int_{a \rightarrow b} dQ \\
 \Delta S_{a \rightarrow b} &= \frac{Q_{a \rightarrow b}}{T_H} \\
 \Delta S_{a \rightarrow b} &= 0.612 \text{ J/K}
 \end{aligned}$$

4.6. Process Variables $b \rightarrow c$

$$Q_{b \rightarrow c} = 0$$

$$\begin{aligned}
 W_{b \rightarrow c} &= \frac{P_b V_b - P_c V_c}{\gamma - 1} \\
 W_{b \rightarrow c} &= 304 \text{ J}
 \end{aligned}$$

$$\Delta U_{b \rightarrow c} = Q_{b \rightarrow c} - W_{b \rightarrow c}$$

$$\Delta U_{b \rightarrow c} = -W_{b \rightarrow c}$$

$$\Delta U_{b \rightarrow c} = -304 \text{ J}$$

$$\Delta S_{b \rightarrow c} = \int_{b \rightarrow c} \frac{dQ}{T}$$

$$\Delta S_{b \rightarrow c} = 0$$

4.7. Process Variables $c \rightarrow d$

$$\Delta U_{c \rightarrow d} = 0$$

$$W_{c \rightarrow d} = nRT_C \ln \left(\frac{V_d}{V_c} \right)$$

$$W_{c \rightarrow d} = -184 \text{ J}$$

$$\Delta U_{c \rightarrow d} = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$0 = Q_{c \rightarrow d} - W_{c \rightarrow d}$$

$$Q_{c \rightarrow d} = W_{c \rightarrow d}$$

$$Q_{c \rightarrow d} = -184 \text{ J}$$

$$\Delta S_{c \rightarrow d} = \int_{c \rightarrow d} \frac{dQ}{T}$$

$$\Delta S_{c \rightarrow d} = \frac{1}{T_C} \int_{c \rightarrow d} dQ$$

$$\Delta S_{c \rightarrow d} = \frac{Q_{c \rightarrow d}}{T_C}$$

$$\Delta S_{c \rightarrow d} = -0.612 \text{ J/K}$$

4.8. Process Variables $d \rightarrow a$

$$Q_{d \rightarrow a} = 0$$

$$W_{d \rightarrow a} = \frac{P_d V_d - P_a V_a}{\gamma - 1}$$

$$W_{d \rightarrow a} = -304 \text{ J}$$

$$\Delta U_{d \rightarrow a} = Q_{d \rightarrow a} - W_{d \rightarrow a}$$

$$\Delta U_{d \rightarrow a} = -W_{d \rightarrow a}$$

$$\Delta U_{d \rightarrow a} = 304 \text{ J}$$

$$\Delta S_{d \rightarrow a} = \int_{d \rightarrow a} \frac{dQ}{T}$$

$$\Delta S_{d \rightarrow a} = 0$$

4.9. Process Variables Overall Cycle

$$Q_{\text{tot}} = Q_{a \rightarrow b} + Q_{b \rightarrow c} + Q_{c \rightarrow d} + Q_{d \rightarrow a}$$

$$Q_{\text{tot}} = 116 \text{ J}$$

$$W_{\text{tot}} = W_{a \rightarrow b} + W_{b \rightarrow c} + W_{c \rightarrow d} + W_{d \rightarrow a}$$

$$W_{\text{tot}} = 116 \text{ J}$$

$$\Delta U_{\text{tot}} = \Delta U_{a \rightarrow b} + \Delta U_{b \rightarrow c} + \Delta U_{c \rightarrow d} + \Delta U_{d \rightarrow a}$$

$$\Delta U_{\text{tot}} = 0 \text{ J}$$

$$\Delta S_{\text{tot}} = \Delta S_{a \rightarrow b} + \Delta S_{b \rightarrow c} + \Delta S_{c \rightarrow d} + \Delta S_{d \rightarrow a}$$

$$\Delta S_{\text{tot}} = 0 \text{ J/K}$$

4.10. Pressure Calculation Given Volume $a \rightarrow b$

$$PV = P_a V_a$$

$$P = \frac{P_a V_a}{V}$$

$$\text{Ex: } V = 2.45 \times 10^{-4} \text{ m}^3 \Rightarrow P = 1.28 \times 10^6 \text{ Pa}$$

4.11. Pressure Calculation Given Volume $b \rightarrow c$

$$PV^\gamma = P_b V_b^\gamma$$

$$P = P_b \left(\frac{V_b}{V} \right)^\gamma$$

$$\text{Ex: } V = 6.79 \times 10^{-4} \text{ m}^3 \Rightarrow P = 4.26 \times 10^5 \text{ Pa}$$

4.12. Pressure Calculation Given Volume $c \rightarrow d$

$$PV = P_c V_c$$

$$P = \frac{P_c V_c}{V}$$

Ex: $V = 1.79 \times 10^{-3} \text{ m}^3 \Rightarrow P = 1.07 \times 10^5 \text{ Pa}$

4.13. Pressure Calculation Given Volume $d \rightarrow a$

$$PV^\gamma = P_d V_d^\gamma$$

$$P = P_d \left(\frac{V_d}{V} \right)^\gamma$$

Ex: $V = 6.83 \times 10^{-4} \text{ m}^3 \Rightarrow P = 2.89 \times 10^5 \text{ Pa}$

5. Conclusion

For the particular Carnot cycle analyzed, Q_{tot} was 116 J. Thus, the system overall gained 116 J of thermal energy from the environment. W_{tot} was 116 J. Thus, the system overall did 116 J of work on the environment. ΔU_{tot} was equal to 0. Whatever energy gained from the environment was eventually returned to the environment, though the form of that energy might have transformed. This makes sense, as completing a closed loop on the PV diagram returns the gas to its initial state. Similarly, ΔS_{tot} was equal to 0. As stated before, this is expected since completing a closed loop on the PV diagram returns the gas to its initial state. The increases in entropy while heat was added to the system was later canceled about by decrease in entropy while heat was removed from the system.

The area enclosed by the Carnot cycle on the PV diagram represents W_{tot} , the total work done by the gas on the environment. Since $PdV = dW$, integrating PdV (finding the enclosed area) yields the work.

6. Citations

- [1] Karen Schnurbusch, *Physics 4B Lab Book*, Mt. San Antonio College, 2023, pp. 35-38.
- [2] Karen Schnurbusch, *Physics 4B Equations*, Mt. San Antonio College, 2023, pp. 1-3.