Carnot Cycle Exercise

Vincent Edwards

Mt. San Antonio College, Physics 4B, CRN 42240
March 27, 2023

$T_H = 490$ K
$V_c = 1.90 \times 10^{-3}$ m^3

1. Purpose

The goal of the exercise is to perform various calculations related to the Carnot cycle.

2. Given

- $T_H = 490$ K
- $T_C = 300$ K
- $P_c = 1.01 \times 10^5$ Pa
- $V_c = 1.90 \times 10^{-3}$ m^3
- $Q_{a\to b} = 300$ J
- $\gamma = 1.40$
- d.o.f. $= 5$
- $C_v = \frac{5}{2}R$
- $C_p = \frac{7}{2}R$

3. Derivations

3.1. Pressure–Volume Relationship for Adiabatic Process

$$\begin{aligned}
\Delta U &= Q - W \\
\Delta U &= -W \\
dU &= -dW \\
nC_v dT &= -PdV \\
ndT &= -\frac{PdV}{C_v}
\end{aligned}$$

$$\begin{aligned}
PV &= nRT \\
PdV + VdP &= nRdT \\
PdV + VdP &= (C_p - C_v)ndT \\
PdV + VdP &= -(C_p - C_v)\frac{PdV}{C_v} \\
PdV + VdP &= \left(1 - \frac{C_p}{C_v}\right) PdV \\
VdP &= -\frac{C_p}{C_v}PdV \\
VdP &= -\gamma PdV \\
\frac{dP}{P} &= -\gamma\frac{dV}{V} \\
\int \frac{dP}{P} &= -\int \gamma\frac{dV}{V}
\end{aligned}$$

$$
\begin{aligned}
\ln(P) &= -\gamma \ln(V) + \text{const.} \\
\ln(P) + \ln(V^\gamma) &= \text{const.} \\
\ln(PV^\gamma) &= \text{const.} \\
PV^\gamma &= \text{const.} \\
P_i V_i^\gamma &= P_f V_f^\gamma
\end{aligned}
$$

3.2. Temperature–Volume Relationship for Adiabatic Process

$$
\begin{aligned}
P_i V_i^\gamma &= P_f V_f^\gamma \\
P_i V_i V_i^{\gamma-1} &= P_f V_f V_f^{\gamma-1} \\
nRT_i V_i^{\gamma-1} &= nRT_f V_f^{\gamma-1} \\
T_i V_i^{\gamma-1} &= T_f V_f^{\gamma-1}
\end{aligned}
$$

3.3. Work by Gas for Isothermal Process

$$
\begin{aligned}
W &= \int\limits_{i\to f} P dV \\
W &= \int\limits_{V_i}^{V_f} \frac{nRT}{V} dV \\
W &= nRT \ln(V)|_{V_i}^{V_f} \\
W &= nRT(\ln(V_f) - \ln(V_i)) \\
W &= nRT \ln\left(\frac{V_f}{V_i}\right)
\end{aligned}
$$

3.4. Work by Gas for Adiabatic Process

$$
\begin{aligned}
W &= \int\limits_{i\to f} P dV \\
W &= \int\limits_{V_i}^{V_f} (P_i V_i^\gamma) V^{-\gamma} dV \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} V^{1-\gamma}|_{V_i}^{V_f} \\
W &= \frac{P_i V_i^\gamma}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \\
W &= \frac{P_i V_i^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma} \\
W &= \frac{P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}}{1-\gamma}
\end{aligned}
$$

$$\begin{aligned} W &= \frac{P_f V_f - P_i V_i}{1-\gamma} \\ W &= \frac{P_i V_i - P_f V_f}{\gamma - 1} \end{aligned}$$

3.5. Carnot Cycle Heat Ratio Equals Temperature Ratio

$$\begin{aligned} \Delta U_{a\to b} &= Q_{a\to b} - W_{a\to b} \\ 0 &= Q_H - nRT_H \ln\left(\frac{V_b}{V_a}\right) \\ Q_H &= nRT_H \ln\left(\frac{V_b}{V_a}\right) \end{aligned}$$

$$\begin{aligned} \Delta U_{c\to d} &= Q_{c\to d} - W_{c\to d} \\ 0 &= Q_C - nRT_C \ln\left(\frac{V_d}{V_c}\right) \\ Q_C &= nRT_C \ln\left(\frac{V_d}{V_c}\right) \\ Q_C &= -nRT_C \ln\left(\frac{V_c}{V_d}\right) \end{aligned}$$

$$\begin{aligned} \frac{Q_H}{Q_C} &= \frac{nRT_H \ln(V_b/V_a)}{-nRT_C \ln(V_c/V_d)} \\ -\frac{Q_H}{Q_C} &= \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)} \end{aligned}$$

$$\begin{aligned} T_b V_b^{\gamma-1} &= T_c V_c^{\gamma-1} \\ \frac{T_b}{T_c} &= \left(\frac{V_c}{V_b}\right)^{\gamma-1} \\ \frac{T_H}{T_C} &= \left(\frac{V_c}{V_b}\right)^{\gamma-1} \end{aligned}$$

$$\begin{aligned} T_a V_a^{\gamma-1} &= T_d V_d^{\gamma-1} \\ \frac{T_a}{T_d} &= \left(\frac{V_d}{V_a}\right)^{\gamma-1} \\ \frac{T_H}{T_C} &= \left(\frac{V_d}{V_a}\right)^{\gamma-1} \end{aligned}$$

$$\begin{aligned} \left(\frac{V_c}{V_b}\right)^{\gamma-1} &= \left(\frac{V_d}{V_a}\right)^{\gamma-1} \\ \frac{V_c}{V_b} &= \frac{V_d}{V_a} \end{aligned}$$

$$\frac{V_c}{V_d} = \frac{V_b}{V_a}$$
$$\ln\left(\frac{V_c}{V_d}\right) = \ln\left(\frac{V_b}{V_a}\right)$$
$$1 = \frac{\ln(V_b/V_a)}{\ln(V_c/V_d)}$$

$$-\frac{Q_H}{Q_C} = \frac{T_H \ln(V_b/V_a)}{T_C \ln(V_c/V_d)}$$
$$-\frac{Q_H}{Q_C} = \frac{T_H}{T_C}$$
$$\left|-\frac{Q_H}{Q_C}\right| = \left|\frac{T_H}{T_C}\right|$$
$$\frac{|Q_H|}{|Q_C|} = \frac{T_H}{T_C}$$
$$\frac{T_C}{T_H} = \frac{|Q_C|}{|Q_H|}$$

4. Results

Table 1. Pressure, Volume, and Temperature for Key Points
Note: $T_a = T_b = T_H$ and $T_c = T_d = T_C$

Point	P (Pa)	V (m^3)	T (K)
a	1.46×10^6	2.14×10^{-4}	490
b	5.62×10^5	5.57×10^{-4}	490
c	1.01×10^5	1.90×10^{-3}	300
d	2.63×10^5	7.30×10^{-4}	300

Table 2. Heat Transferred, Change in Internal Energy, Work Done, and Change in Entropy for Key Processes
Note: $Q_{a\to b} = Q_H$ and $Q_{c\to d} = Q_C$

Process	Type	Q (J)	W (J)	ΔU (J)	ΔS (J/K)
$a \to b$	isothermal expansion	300	300	0	0.612
$b \to c$	adiabatic expansion	0	304	-304	0
$c \to d$	isothermal compression	-184	-184	0	-0.612
$d \to a$	adiabatic compression	0	-304	304	0

4.1. Moles of Gas (n)

$$P_c V_c = nRT_c$$

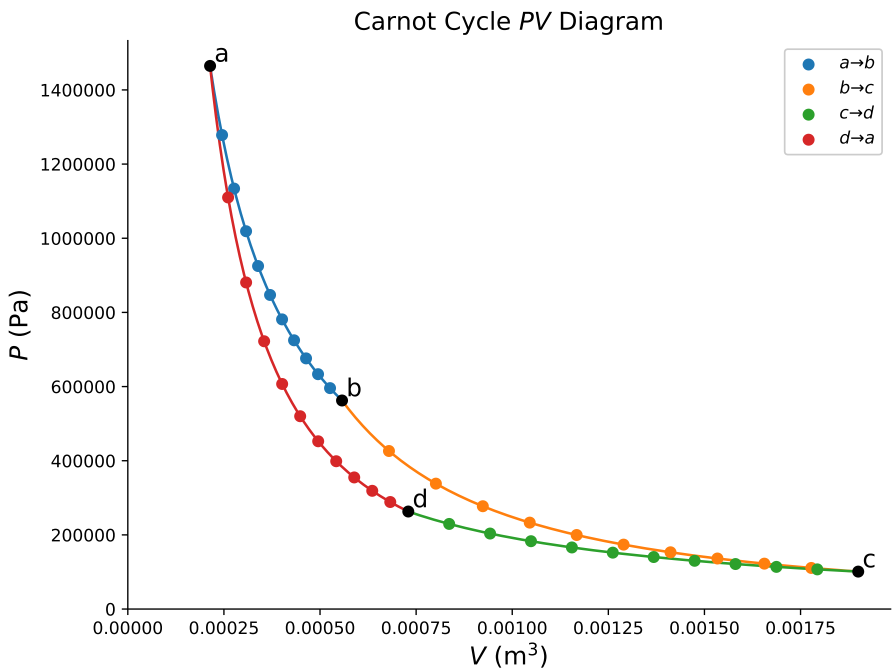

Figure 1. Carnot Cycle PV Diagram

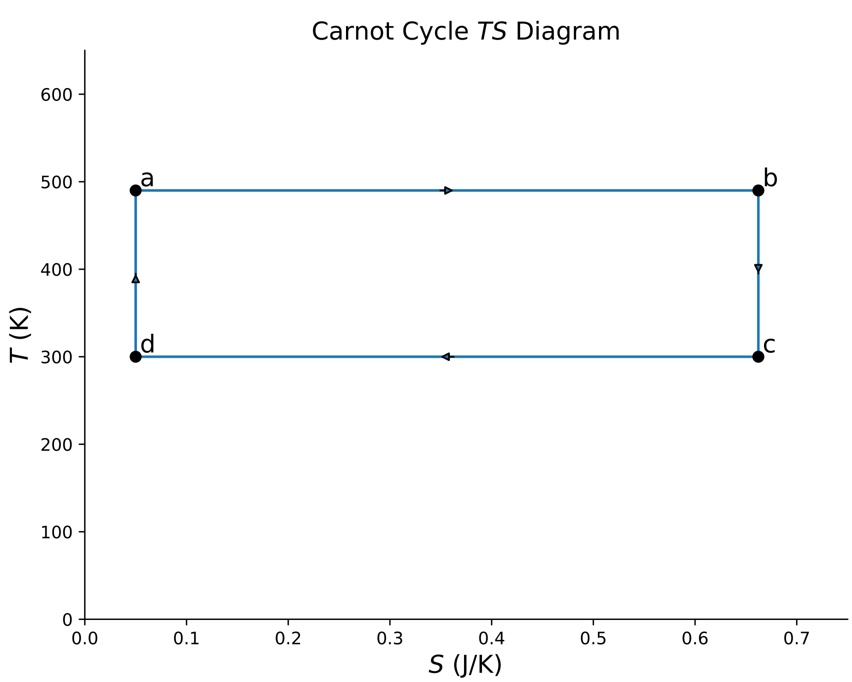

Figure 2. Carnot Cycle TS Diagram
Note: Assumed $S_a = 0.05$ J/K for initial condition

Table 3. Pressure and Volume Points on Carnot Cycle PV Diagram

Section		P (Pa)	V (m^3)
a		1.46×10^6	2.14×10^{-4}
$a \to b$	1	1.28×10^6	2.45×10^{-4}
$a \to b$	2	1.13×10^6	2.76×10^{-4}
$a \to b$	3	1.02×10^6	3.08×10^{-4}
$a \to b$	4	9.25×10^5	3.39×10^{-4}
$a \to b$	5	8.47×10^5	3.70×10^{-4}
$a \to b$	6	7.81×10^5	4.01×10^{-4}
$a \to b$	7	7.25×10^5	4.32×10^{-4}
$a \to b$	8	6.76×10^5	4.64×10^{-4}
$a \to b$	9	6.33×10^5	4.95×10^{-4}
$a \to b$	10	5.96×10^5	5.26×10^{-4}
b		5.62×10^5	5.57×10^{-4}
$b \to c$	1	4.26×10^5	6.79×10^{-4}
$b \to c$	2	3.38×10^5	8.01×10^{-4}
$b \to c$	3	2.77×10^5	9.23×10^{-4}
$b \to c$	4	2.33×10^5	1.05×10^{-3}
$b \to c$	5	2.00×10^5	1.17×10^{-3}
$b \to c$	6	1.74×10^5	1.29×10^{-3}
$b \to c$	7	1.53×10^5	1.41×10^{-3}
$b \to c$	8	1.36×10^5	1.53×10^{-3}
$b \to c$	9	1.22×10^5	1.66×10^{-3}
$b \to c$	10	1.11×10^5	1.78×10^{-3}
c		1.01×10^5	1.90×10^{-3}
$c \to d$	1	1.07×10^5	1.79×10^{-3}
$c \to d$	2	1.14×10^5	1.69×10^{-3}
$c \to d$	3	1.21×10^5	1.58×10^{-3}
$c \to d$	4	1.30×10^5	1.47×10^{-3}
$c \to d$	5	1.40×10^5	1.37×10^{-3}
$c \to d$	6	1.52×10^5	1.26×10^{-3}
$c \to d$	7	1.66×10^5	1.16×10^{-3}
$c \to d$	8	1.83×10^5	1.05×10^{-3}
$c \to d$	9	2.04×10^5	9.42×10^{-4}
$c \to d$	10	2.30×10^5	8.36×10^{-4}
d		2.63×10^5	7.30×10^{-4}
$d \to a$	1	2.89×10^5	6.83×10^{-4}
$d \to a$	2	3.19×10^5	6.36×10^{-4}
$d \to a$	3	3.55×10^5	5.89×10^{-4}
$d \to a$	4	3.99×10^5	5.42×10^{-4}
$d \to a$	5	4.52×10^5	4.95×10^{-4}
$d \to a$	6	5.20×10^5	4.48×10^{-4}
$d \to a$	7	6.07×10^5	4.01×10^{-4}
$d \to a$	8	7.22×10^5	3.55×10^{-4}
$d \to a$	9	8.81×10^5	3.08×10^{-4}
$d \to a$	10	1.11×10^6	2.61×10^{-4}

$$n = \frac{P_c V_c}{R T_c}$$
$$n = 0.0770 \text{ mol}$$

4.2. Pressure (P_b) and Volume (V_b) at b

$$T_b V_b^{\gamma-1} = T_c V_c^{\gamma-1}$$
$$V_b = V_c \left(\frac{T_c}{T_b}\right)^{\frac{1}{\gamma-1}}$$
$$V_b = 5.57 \times 10^{-4} \text{ m}^3$$

$$P_b V_b = nRT_b$$
$$P_b = \frac{nRT_b}{V_b}$$
$$P_b = 5.62 \times 10^5 \text{ Pa}$$

4.3. Pressure (P_a) and Volume (V_a) at a

$$\Delta U_{a\to b} = Q_{a\to b} - W_{a\to b}$$
$$0 = Q_{a\to b} - nRT_H \ln\left(\frac{V_b}{V_a}\right)$$
$$\ln\left(\frac{V_b}{V_a}\right) = \frac{Q_{a\to b}}{nRT_H}$$
$$\frac{V_b}{V_a} = e^{Q_{a\to b}/(nRT_H)}$$
$$V_a = V_b e^{-Q_{a\to b}/(nRT_H)}$$
$$V_a = 2.14 \times 10^{-4} \text{ m}^3$$

$$P_a V_a = nRT_a$$
$$P_a = \frac{nRT_a}{V_a}$$
$$P_a = 1.46 \times 10^6 \text{ Pa}$$

4.4. Pressure (P_d) and Volume (V_d) at d

$$T_d V_d^{\gamma-1} = T_a V_a^{\gamma-1}$$
$$V_d = V_a \left(\frac{T_a}{T_d}\right)^{\frac{1}{\gamma-1}}$$
$$V_d = 7.30 \times 10^{-4} \text{ m}^3$$

$$\begin{aligned}
P_d V_d &= nRT_d \\
P_d &= \frac{nRT_d}{V_d} \\
P_d &= 2.63 \times 10^5 \text{ Pa}
\end{aligned}$$

4.5. Process Variables $a \to b$

$$\Delta U_{a\to b} = 0$$

$$\begin{aligned}
\Delta U_{a\to b} &= Q_{a\to b} - W_{a\to b} \\
0 &= Q_{a\to b} - W_{a\to b} \\
W_{a\to b} &= Q_{a\to b} \\
W_{a\to b} &= 300 \text{ J}
\end{aligned}$$

$$\begin{aligned}
\Delta S_{a\to b} &= \int\limits_{a\to b} \frac{dQ}{T} \\
\Delta S_{a\to b} &= \frac{1}{T_H} \int\limits_{a\to b} dQ \\
\Delta S_{a\to b} &= \frac{Q_{a\to b}}{T_H} \\
\Delta S_{a\to b} &= 0.612 \text{ J/K}
\end{aligned}$$

4.6. Process Variables $b \to c$

$$Q_{b\to c} = 0$$

$$\begin{aligned}
W_{b\to c} &= \frac{P_b V_b - P_c V_c}{\gamma - 1} \\
W_{b\to c} &= 304 \text{ J}
\end{aligned}$$

$$\begin{aligned}
\Delta U_{b\to c} &= Q_{b\to c} - W_{b\to c} \\
\Delta U_{b\to c} &= -W_{b\to c} \\
\Delta U_{b\to c} &= -304 \text{ J}
\end{aligned}$$

$$\begin{aligned}
\Delta S_{b\to c} &= \int\limits_{b\to c} \frac{dQ}{T} \\
\Delta S_{b\to c} &= 0
\end{aligned}$$

4.7. Process Variables $c \to d$

$$\Delta U_{c\to d} = 0$$

$$\begin{aligned} W_{c\to d} &= nRT_C \ln\left(\frac{V_d}{V_c}\right) \\ W_{c\to d} &= -184 \text{ J} \end{aligned}$$

$$\begin{aligned} \Delta U_{c\to d} &= Q_{c\to d} - W_{c\to d} \\ 0 &= Q_{c\to d} - W_{c\to d} \\ Q_{c\to d} &= W_{c\to d} \\ Q_{c\to d} &= -184 \text{ J} \end{aligned}$$

$$\begin{aligned} \Delta S_{c\to d} &= \int\limits_{c\to d} \frac{dQ}{T} \\ \Delta S_{c\to d} &= \frac{1}{T_C} \int\limits_{c\to d} dQ \\ \Delta S_{c\to d} &= \frac{Q_{c\to d}}{T_C} \\ \Delta S_{c\to d} &= -0.612 \text{ J/K} \end{aligned}$$

4.8. Process Variables $d \to a$

$$Q_{d\to a} = 0$$

$$\begin{aligned} W_{d\to a} &= \frac{P_d V_d - P_a V_a}{\gamma - 1} \\ W_{d\to a} &= -304 \text{ J} \end{aligned}$$

$$\begin{aligned} \Delta U_{d\to a} &= Q_{d\to a} - W_{d\to a} \\ \Delta U_{d\to a} &= -W_{d\to a} \\ \Delta U_{d\to a} &= 304 \text{ J} \end{aligned}$$

$$\begin{aligned} \Delta S_{d\to a} &= \int\limits_{d\to a} \frac{dQ}{T} \\ \Delta S_{d\to a} &= 0 \end{aligned}$$

4.9. Pressure Calculation Given Volume $a \to b$

$$PV = P_a V_a$$
$$P = \frac{P_a V_a}{V}$$

Ex: $V = 2.45 \times 10^{-4}$ m$^3 \Rightarrow P = 1.28 \times 10^6$ Pa

4.10. Pressure Calculation Given Volume $b \to c$

$$PV^\gamma = P_b V_b^\gamma$$
$$P = P_b \left(\frac{V_b}{V}\right)^\gamma$$

Ex: $V = 6.79 \times 10^{-4}$ m$^3 \Rightarrow P = 4.26 \times 10^5$ Pa

4.11. Pressure Calculation Given Volume $c \to d$

$$PV = P_c V_c$$
$$P = \frac{P_c V_c}{V}$$

Ex: $V = 1.79 \times 10^{-3}$ m$^3 \Rightarrow P = 1.07 \times 10^5$ Pa

4.12. Pressure Calculation Given Volume $d \to a$

$$PV^\gamma = P_d V_d^\gamma$$
$$P = P_d \left(\frac{V_d}{V}\right)^\gamma$$

Ex: $V = 6.83 \times 10^{-4}$ m$^3 \Rightarrow P = 2.89 \times 10^5$ Pa

5. Conclusion

6. Citations

[1] Karen Schnurbusch, *Physics 4B Lab Book*, Mt. San Antonio College, 2023, pp. 35-38.
[2] Karen Schnurbusch, *Physics 4B Equations*, Mt. San Antonio College, 2023, pp. 1-3.