Intermittent Dynamics in 1D mappings (Types I,II and III)

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Non Linear Dynamics PYL711

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Intermittency Overview

1. Identified in diverse physical phenomena:
   - Nonlinear dynamics (plasma, conducting fluids).
   - Fluid mechanics and turbulent flows.
   - Examples: Rayleigh–Benard convection, electronic oscillators, logistic maps.
2. Also observed in:
   - Premixed combustion, catalytic reactors.
   - Lorenz system, coupled oscillators, Ginzburg–Landau equation.
   - Solar cycles, spatiotemporal chaos, and thermoacoustic instability.
   - Controlled chaos.

Classical Pathways to Chaos

1. Three classical pathways for systems transitioning to chaos:
   - Quasi-periodic route.
   - Period-doubling scenario.
   - Intermittency.

Chaotic Intermittency

1. Chaotic intermittency: specific route to chaos.
2. Transitions between:
   - Regular/laminar phases.
   - Chaotic bursts/non-regular phases.
3. Laminar phases: pseudo-equilibrium or pseudo-periodic solutions.
4. Burst phases: chaotic evolution.

Traditional and New Intermittency Types

1. Traditional types (I, II, III) categorized by Floquet multipliers or fixed-point eigenvalues.
2. Later types include on–off, eyelet, ring, in–out, type-X, and type-V intermittency.

Threshold Behavior and Control Parameter

1. In types I, II, III:
   - Control parameter surpasses threshold.
   - Local Poincaré map fixed point becomes unstable or vanishes.
2. Experimental validation: Number of chaotic bursts intensifies with the external/control parameter.

Investigating Chaotic Intermittency with Poincaré Maps

1. Chaotic intermittency explored through Poincaré maps.
2. Two crucial elements:
    - Specific local map.
    - Reinjection mechanism.

Local Map and Reinjection Mechanism

1. Local map defines type of intermittency.
2. Reinjection mechanism maps system from chaotic to regular/laminar region.
3. Reinjection Probability Density Function (RPD) characterizes this mechanism, influenced by chaotic dynamics.

Challenges in Practical RPD Evaluation

1. RPD provides probability of reinjection into laminar zone.
2. Governs dynamic properties with the local map. Practical evaluation challenging due to substantial data requirements.

Deriving RPD Functions

1. Various approaches to derive intermittent RPD function.
2. Classical studies assumed uniform reinjection in laminar region.
3. Alternative methods use distinctive features of nonlinear processes. Note: Universality may vary across nonlinear systems.

Additional Statistical Functions

1. For precise description:
   - Probability density of laminar lengths ($\psi(l)$).
   - Average laminar length ($\bar{l}$).
   - Characteristic relation ($\bar{l} = l(\bar{\varepsilon})$).
2. Functions dependent on RPD.
3. Susceptible to noise and lower boundary of reinjection (LBR) influence[1].

Manneville and Pomeau's Classification

1. Pioneers: Manneville and Pomeau[3].
2. Three Types: I, II, III.
3. Based on eigenvalues of fixed point (local Poincaré map) or Floquet multipliers

Types I, II, III Intermittencies

1. Type I:
   - Saddle node bifurcation.
   - Coalescence at $+1$.
2. Type II:
   - Subcritical Hopf bifurcation.
   - Complex-conjugate multipliers move away from unit circle.
3. Type III:
   - Inverse period-doubling bifurcation.
   - Multiplier exits unit circle by -1.

Characteristics of Intermittency Transitions

Scaling near critical transition value (p_t):

- Type I: $T(p) = (p - p_t)^{-1/2}$
- Type II: $T(p) = (p - p_t)^{-1}$
- Type III: $T(p) = (p - p_t)^{-1}$

Further Insights

1. Scaling Behavior:
   - Same for Type II and III.
   - Characteristic probability distributions differ.
2. Characteristic Relation:
   - Expressed as power law: $T(p) \propto (p - p_t)^{\beta}$.
   - Note: Distinct probability distributions for interburst times[3][2].

Type-I Intermittency - Local Map

$$x_{n+1} = \epsilon + x_n + ax^2 \tag{1}$$

For $\epsilon < 0$, two fixed points exist. One is stable, and the other is unstable. When $\epsilon = 0$, both fixed points collide and meet in only one fixed point. Finally, for $\epsilon > 0$, the map does not have fixed points (tangent bifurcation).

Approximation and Laminar Length

- Reinjection probability assumed constant ($\phi(x) = k$).
- Map approximation: $\frac{dx}{dl} = \epsilon + ax^2$.
- Laminar length equation:
  $l(x, c) = 1/\sqrt{a\varepsilon}[\arctan(c/\sqrt{\varepsilon/a}) - \arctan(x/\sqrt{\varepsilon/a})]$.

Probability Density and Average Laminar Length

$$\psi(l, c) = \phi[X(l, c)]\frac{dX(l, c)}{dl} \tag{2}$$

where $X(l, c) = \sqrt{\frac{\varepsilon}{a}}\tan\left(\tan^{-1}\left(\sqrt{\frac{\varepsilon}{a}}c\right) - \sqrt{a\varepsilon}l\right)$.

Average Laminar Length ($\bar{l}$)

$$\bar{l} = \frac{1}{\sqrt{a\varepsilon}} \arctan(c\sqrt{\frac{a}{\varepsilon}}) \tag{3}$$

If $c\sqrt{\frac{a}{\varepsilon}} \gg 1$:

$$\bar{l} \propto \frac{1}{\sqrt{\varepsilon}} \tag{4}$$

Results Summary

- **Characteristic Map:** $x_{n+1} = \epsilon + x_n + ax^2$.
- **Laminar Length:** $l(x, c) = 1/\sqrt{a\varepsilon}[\arctan(c/\sqrt{\varepsilon/a}) - \arctan(x/\sqrt{\varepsilon/a})]$.
- **Probability Density:** $\psi(l, c) = \phi[X(l, c)]\frac{dX(l,c)}{dl}$.
- **Average Laminar Length:** $\bar{l} = \frac{1}{\sqrt{a\varepsilon}}\arctan(c\sqrt{\frac{a}{\varepsilon}})$ (if $c\sqrt{\frac{a}{\varepsilon}} \gg 1$: $\bar{l} \propto \frac{1}{\sqrt{\varepsilon}}$).

Type II Map

$$r_{n+1} = (1+\varepsilon)r_n + ar_n^3 \tag{5}$$

$$\theta_{n+1} = \theta_n + b + qr_n^2 \tag{6}$$

Approximation for $\varepsilon \ll 1$

$$\frac{dr}{dl} = \varepsilon r + ar^3 \tag{7}$$

Laminar Length and Probability Density

$$l(x,c) = \frac{1}{2\varepsilon} \ln\left(\frac{a+\varepsilon/x^2}{a+\varepsilon/c^2}\right) \tag{8}$$

$$\psi(l) = \left(\frac{\varepsilon}{\left(\frac{\varepsilon}{c^2}+a\right)e^{2\varepsilon l}-a}\right)^{3/2}\left(\frac{\varepsilon}{c^2}+a\right)\frac{e^{2\varepsilon l}}{2c} \tag{9}$$

Average Laminar Length

$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan\left(c\sqrt{\frac{a}{\varepsilon}}\right) \tag{10}$$

If $c\sqrt{\frac{a}{\varepsilon}} \gg 1$:

$$\bar{l} \propto \varepsilon^{-1/2} \tag{11}$$

Type III Intermittency - Overview

- Inverse period-doubling bifurcation.
- Unstable period-2 orbit collides with a stable period-1 orbit.
- Local map: $x_{n+1} = -(1+\varepsilon)x_n + a_2 x_n^2 + a_3 x_n^3 + \dots$

Second Iteration of Type III Map

$$x_{n+2} = (1 + 2\varepsilon)x_n + ax_n^3 + \dots \tag{12}$$

- When $a < 0$: Supercritical pitchfork bifurcation, no intermittency.
- When $a > 0$: Sub-critical pitchfork or period-doubling bifurcation, leading to intermittency.

Schwartzian Derivative Condition

$$S_F(x) = \frac{F'''(x)}{F'(x)} - 1.5\left(\frac{F''(x)}{F'(x)}\right)^2 > 0 \qquad (13)$$

Average Laminar Length (Type III)

$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan\left(c\sqrt{\frac{a}{\varepsilon}}\right) \tag{14}$$

If $c\sqrt{\frac{a}{\varepsilon}} \gg 1$:

$$\bar{l} \propto \varepsilon^{-1/2} \tag{15}$$

Renormalization Group Theory

- Classification of universality of intermittency.
- Description of laminar length.
- Scaling relation: $< l > \propto \varepsilon^{-v}$ with $v = \frac{z-1}{z}$.

Renormalization Group Equation

$$f_{\varepsilon}(x) = f^{*}(x) + \varepsilon h_{\lambda}(x) \tag{16}$$

Eigenvalue Equation (Linearized Doubling Operator)

$$Lf^*[h_\lambda(x)] = \lambda h_\lambda(\alpha x) \tag{17}$$

Scaling Relation for Laminar Length

$$< l > \propto \varepsilon^{-v} \quad \text{with} \quad v = \frac{z-1}{z} \tag{18}$$

Conclusion

- Renormalization Group theory provides a classification of intermittency.
- Laminar length described through probability density.
- Scaling relation for laminar length.

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