Intermittent Dynamics in 1D mappings (Types I,II and III)

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Non Linear Dynamics PYL711

Contents

1 Introduction

1.A Motivation

Intermittency has been identified in a diverse range of physical phenomena, including the nonlinear dynamics of transient periodic plasma and conducting fluids, fluid mechanics, and turbulent flows, as well as phenomena such as Rayleigh–Benard convection, electronic digital oscillators, logistic maps, Alfven wave-fronts, and derivative nonlinear Schrödinger equations. It is also evident in areas like premixed combustion, the Lorenz system, coupled oscillators, catalytic reactors, the Ginzburg–Landau equation, solar cycles, spatiotemporal chaos, thermoacoustic instability, and controlled chaos.

Moreover, chaotic intermittency has been observed beyond the realm of physics, making its presence known in economic systems, medicine, neuroscience, genetics, and marine biology[1].

1.B Preface

There exist three classical pathways through which continuous or discrete dynamical systems can transition from regular functioning to chaotic behaviors: the quasi-periodic route, period-doubling scenario, and intermittency.

Chaotic intermittency represents a specific route to chaos where the solutions of a dynamical system undergo transitions between regular or laminar phases and chaotic bursts or non-regular phases. The laminar phases manifest as regions of pseudo-equilibrium and/or pseudo-periodic solutions, while the burst phases represent regions where the evolution becomes chaotic.

Traditionally, intermittency has been categorized into three distinct types denoted as I, II, and III, based on the Floquet multipliers of the continuous-system periodic solution or the fixed-point eigenvalues in the local Poincaré map. Later works introduced other types of chaotic intermittency such as on–off, eyelet, ring, and in–out, type-X, and type-V.

In types I, II, and III intermittency, as a control parameter surpasses a threshold value, a fixed point of the local Poincaré map becomes unstable or may even vanish. It has been experimentally validated that the number of chaotic bursts intensifies with an external or control parameter[1].

2 Describing Intermittent Phenomenon

Chaotic intermittency is often investigated through the use of Poincaré maps. Two crucial elements define chaotic intermittency: 1) a specific local map and 2) a reinjection mechanism. The local map determines the type of intermittency, while the reinjection mechanism maps the system back from the chaotic zone to the local regular or laminar region. This mechanism is characterized by the reinjection probability density function (RPD), influenced by the chaotic dynamics of the system.

The RPD provides the probability of trajectories being reinjected into the laminar zone, situated close to the unstable fixed point. Together with the local map, the RPD governs all the dynamic properties of the system. However, the practical evaluation of the RPD, whether experimental or numerical, poses a challenge due to the substantial amount of data required.

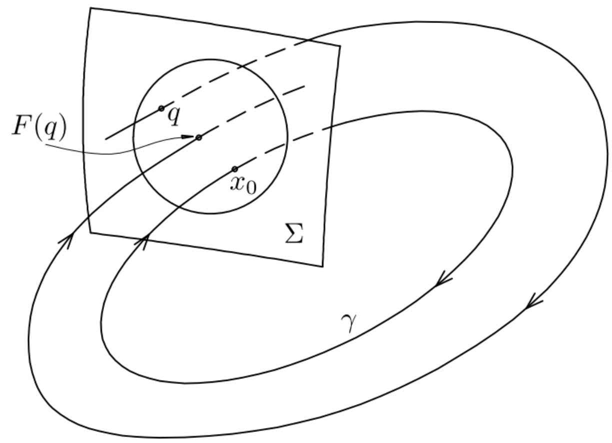


Figure 1: Poincare map. There are two orbits: 1 - The orbit γ is a limit cycle. 2 - The second orbit first crosses the surface in q and after in F(q); where F(x) describes the Poincare map.[1]

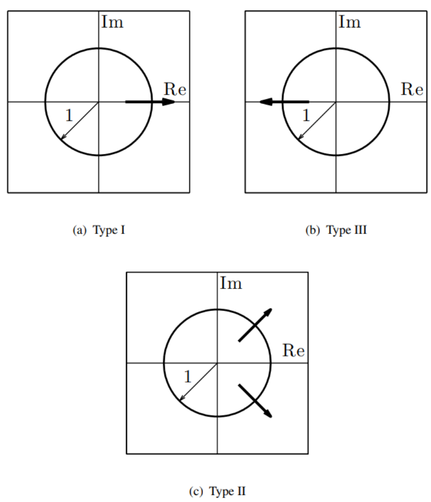
Various approaches have been employed to derive the intermittent RPD function. Classical studies on chaotic intermittency primarily relied on the assumption of uniform reinjection in the laminar region. Alternative methods construct the RPD using distinctive features of nonlinear processes. Nevertheless, these RPD functions may not be universally applicable to other nonlinear systems.

For a more precise description of the intermittency phenomenon, additional statistical functions are employed. These include the probability density of laminar lengths ($\psi(l)$), the average laminar length ($\bar{l}$), and the characteristic relation ($\bar{l} = \bar{l}(\varepsilon)$). However, it is important to note that these functions are dependent on the RPD. Both the RPD and other statistical functions utilized to characterize chaotic intermittency are susceptible to the influence of noise and the lower boundary of reinjection (LBR)[1].

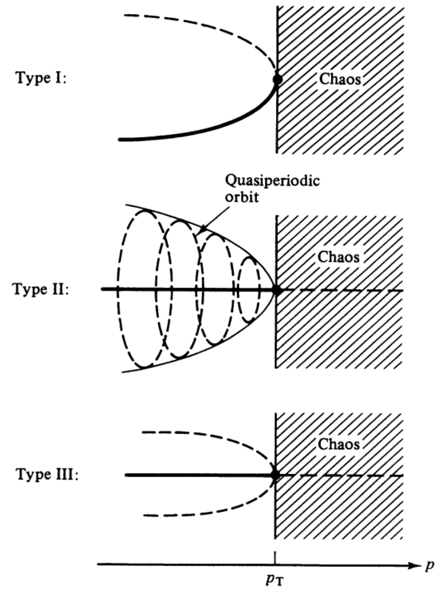
3 Types Of Chaotic Intermittency

Manneville and Pomeau[7] pioneered the classic classification of chaotic intermittency. They delineated this phenomenon into three distinct types labeled as I, II, and III. This classification was based on the analysis of either the eigenvalues of the fixed point in the local Poincaré map or the Floquet multipliers of the periodic solution in the continuous system.

The analysis of the stability of a closed orbit, denoted as γ, can be approached through various methods. One such approach is the Floquet theory, utilizing Floquet multipliers. An alternative method involves selecting a point x_0 on the closed orbit and passing a transverse hypersurface to the flow (Σ). Thus, we ensure that $f(x) \cdot$



(a) Floquet Multipliers[1]



(b) Bifurcations[6]

Figure 2: Illustration of three types of intermittency transitions to chaos

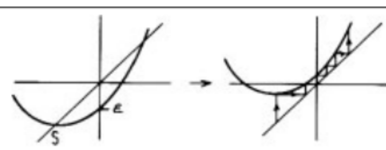
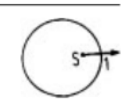
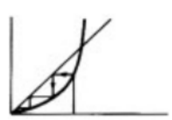
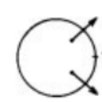
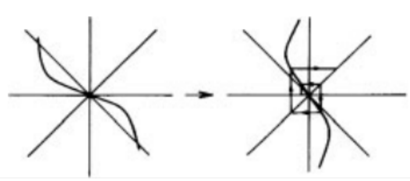
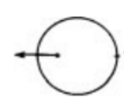
Type	Characteristic behavior and maps	Typical map ($\varepsilon < 0 \to \varepsilon > 0$)	Eigenvalues
I	A real eigenvalue crosses the unit circle at $+1$. $x_{n+1} = \varepsilon + x_n + ux_n^2$		
II	Two conjugate complex eigenvalues cross the unit circle simultaneously. $r_{n+1} = (1+\varepsilon)r_n + ur_n^3$ $\theta_{n+1} = \theta_n + \Omega$		
III	A real eigenvalue crosses the unit circle at -1. $x_{n+1} = -(1+\varepsilon)r_n - ur_n^3$		

Figure 3: Three types of intermittency[8]

$n(x) \neq 0$, where $n(x)$ is the unit normal vector to the hypersurface, and $f(x)$ is the vector flow (from $\frac{dx}{dt} = f(x)$, where $x \in R^n$). On this hypersurface, we derive the corresponding Poincaré map and linearize it at the point x_0. The resulting linear operator is termed the monodromy operator, with its eigenvalues referred to as multipliers. Notably, the monodromy operator's characteristic equation is real, requiring every nonreal multiplier to have a complex conjugate eigenvalue. Figure 1 illustrates the hypersurface Σ and the orbit γ for $f(x) \in R^3$. The multipliers, or Floquet multipliers, can depart from the unit circle in three distinct scenarios. One scenario occurs when a multiplier exits the unit circle along the positive real axis (by +1). Another scenario arises when the multiplier exits along the negative real axis (by -1). The final scenario unfolds when two multipliers are complex conjugates, departing from the unit circle away from the real axis. Figure 2a depicts these three scenarios.

The manifestation of a specific intermittency type hinges on the monodromy operator multiplier. Summarily, we can describe Types I, II, and III intermittencies based on these multipliers[6][7]:

Type I: Type I intermittency arises in the saddle node bifurcation (in which the stable and the unstable orbits coalesce and obliterate each other, also called the tangent bifurcation Figure2b), when a stable eigenvalue (inside the unit circle) and an unstable eigenvalue (from outside the unit circle) coalesce at the point +1.

Type II: Type II intermittency originates from a subcritical Hopf bifurcation (The word subcritical applied to the hopf bifurcation of a periodic orbit signifies that as the parameter is increased, an unstable two frequency quasiperiodic orbit (a closed curve on the surface in the section) collapses on to the stable periodic orbit, and the latter is rendered unusable as the parameter passes the bifurcation point Figure2b). Consequently, two complex-conjugate multipliers of the system move away from the unit circle.

Type III: Type III intermittency occurs when a multiplier exits the unit circle by -1, resulting in a an inverse period-doubling bifurcation (in which an unstable periodic orbit collapses onto a stable periodic orbit of one half it's period and the two are replaced by an unstable periodic orbit of the lower period Figure2b). A one-dimensional map $F(x)$ exhibiting an inverse period-doubling bifurcation is characterized by a positive Schwartzian derivative defined as:

$$S_F(x) = \frac{F'''(x)}{F'(x)} - 1.5\left(\frac{F''(x)}{F'(x)}\right)^2 > 0$$

Each of the three types of intermittency transitions display different characteristics near the critical transition value (say p_t). For example the scaling of the average time between bursts is given by

- $T(p) = (p - p_t)^{-1/2}$ for Type I
- $T(p) = (p - p_t)^{-1}$ for Type II

- $T(p) = (p - p_t)^{-1}$ for Type III

Note however while the scaling behaviour of the average interburst time is same for Type II and III, the characteristic probability distributions of the interburst times are different[7][6]. In general the characteristic relation can be expressed as a power law $T(p) \propto (p - p_t)^{\beta}$.

Note these derivations were taken from the review paper [1].

3.A Type I Intermittency

For type-I intermittency, the characteristic local map is[7][1][2]:

$$x_{n+1} = \epsilon + x_n + ax^2 \tag{1}$$

For $\epsilon < 0$, two fixed points exist. One is stable, and the other is unstable. When $\epsilon = 0$, both fixed points collide and meet in only one fixed point. Finally, for $\epsilon > 0$, the map does not have fixed points (tangent bifurcation). We assume the reinjection probability is the same for all points in the laminar interval $\phi(x) = k$. We can approximate the map 1 as

$$\frac{dx}{dl} = \epsilon + ax^2 \tag{2}$$

Here, we consider $x_{n+1} - x_n \approx \frac{dx}{dl}$, integrating which we get

$$l(x, c) = 1/\sqrt{a\varepsilon}\left[\arctan\left(\frac{c}{\sqrt{\varepsilon/a}}\right) - \arctan\left(\frac{x}{\sqrt{\varepsilon/a}}\right)\right] \tag{3}$$

where c is the upper limit of the laminar interval.Note than laminar length doesn't depend on RPD. The probability density of the laminar lengths, here called $\psi(l, c)$, outlines the probability of finding laminar lengths between l and l + dl:

$$\psi(l, c) = \phi[X(l, c)]\frac{dX(l, c)}{dl} \tag{4}$$

where, X(l,c) is the inverse of laminar length, given by-

$$X(l, c) = \sqrt{\frac{\varepsilon}{a}}\tan\left(\tan^{-1}\left(\sqrt{\frac{\varepsilon}{a}}c\right) - \sqrt{a\varepsilon}l\right) \tag{5}$$

the average laminar length $\bar{l}$ is given by (here we take note $k = \frac{1}{2c}$)

$$\bar{l} = \int_0^{lm} \psi(l, c)l(x, c)\, dl = \int_{-c}^{c} \phi(x)l(x, c)\, dx = \frac{1}{\sqrt{a\varepsilon}}\arctan\left(c\sqrt{\frac{a}{\varepsilon}}\right) \tag{6}$$

l_m is the max laminar length i.e l(-c,c). if $c\sqrt{\frac{a}{\varepsilon}} >> 1$

$$\bar{l} \propto \frac{1}{\sqrt{\varepsilon}} \tag{7}$$

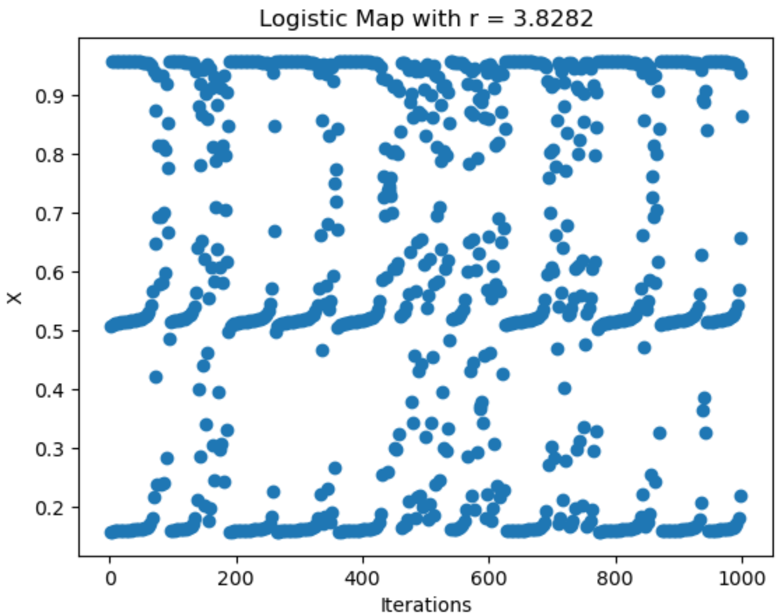


Figure 4: Third Iterate of the Logistic map for $\varepsilon = 0.0001$

3.B Type II Intermittency

Type-II intermittency happens in two or higher-dimensional maps. It occurs when two complex-conjugate eigenvalues of the fixed point go away from the unit circle, and a sub-critical Hopf bifurcation appears. The local dynamic can be described by a two-dimensional map-

$$r_{n+1} = (1+\varepsilon)r_n + ar_n^3 \tag{8}$$
$$\theta_{n+1} = \theta_n + b + qr_n^2 \tag{9}$$

where ε is the control parameter. a, b and q are constants. For $\varepsilon > 0$, Type II intermittency can occur. The classical formulation implements a constant RPD. for $0 < \varepsilon << 1$, equation 9 can approximated as:

$$\frac{dr}{dl} = \varepsilon r + ar^3 \tag{10}$$

Solving, we get

$$l(x,c) = \int_x^c \frac{dr}{\varepsilon r + ar^3} = \frac{1}{2\varepsilon} \ln\left(\frac{a+\varepsilon/x^2}{a+\varepsilon/c^2}\right) \tag{11}$$

The probability density of the laminar lengths, $\psi(l)$, can be obtained from Equation 4:

$$\psi(l) = \left(\frac{\varepsilon}{\left(\frac{\varepsilon}{c^2}+a\right)e^{2\varepsilon l} - a}\right)^{3/2} \left(\frac{\varepsilon}{c^2}+a\right)\frac{e^{2\varepsilon l}}{c} \tag{12}$$

The average laminar length:

$$\bar{l} = \int_0^c \phi(x) l(x)\, dx = \frac{1}{c}\int_0^c l(x)\, dx = \frac{1}{c\sqrt{a\varepsilon}} \arctan\left(c\sqrt{\frac{a}{\varepsilon}}\right) \tag{13}$$

where the RPD is constant, $\phi(x) = k = 1/c$. If $c\sqrt{\frac{a}{\varepsilon}} >> 1$,

$$\bar{l} \propto \varepsilon^{-1/2} \tag{14}$$

3.C Type III Intermittency

Type-III intermittency starts in an inverse period-doubling bifurcation. Therefore, an unstable period-2 orbit collides with a stable period-1 orbit and destabilizes it. A local map can be written as:

$$x_{n+1} = F(x_n) = -(1+\varepsilon)x_n + a_2 x_n^2 + a_3 x_n^3 + \dots \tag{15}$$

where $0 < \varepsilon << 1$. Note that the type-III intermittency behavior is different to that of type-I. The fixed point does not vanish; it only loses its stability. Furthermore, there is not a channel between the map and the bisector. It is stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$. To examine type-III intermittency, the second iteration of Equation 15 is built:

$$\begin{aligned} x_{n+2} &= F^2(x_n) \\ &= -(1+\varepsilon)\left[-(1+\varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3\right] \\ &\quad + a_2\left[-(1+\varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3\right]^2 \\ &\quad + a_3\left[-(1+\varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3\right]^3 \end{aligned} \tag{16}$$

for small values of $|x|$ and ε, the second iteration becomes-

$$x_{n+2} = F^2(x_n) = (1+2\varepsilon)x_n + a x_n^3 + \dots \tag{17}$$

where a $= -2(a_3 + a_2^2)$. Consider the behavior of the system for different values of the parameter a:

- When $a < 0$, there are three fixed points: $x_0 = 0$ and $x_{1,2} = \pm\left(\frac{2\varepsilon}{|a|}\right)^{0.5}$. When x_0 loses its stability, the other two fixed points become stable and attract trajectories. This transition is characterized by a supercritical pitchfork bifurcation, and intermittency does not occur.

- When $a > 0$, and for $\varepsilon > 0$, $x_0 = 0$ is the only real fixed point, which is unstable. If there is a reinjection mechanism, type-III intermittency occurs. This is linked to a sub-critical pitchfork bifurcation of $F^2(x)$ or associated with a sub-critical period-doubling bifurcation of $F(x)$. Notably, $a > 0$ implies $a^3 < -a$.

If $F(x)$ exhibits a subcritical period-doubling bifurcation, the Schwartzian derivative must be positive:

$$S_F(x) = \frac{F'''(x)}{F'(x)} - 1.5\left(\frac{F''(x)}{F'(x)}\right)^2 > 0$$

For $\varepsilon = 0$, this equation implies $a^3 < -a_2^2$. Accordingly, for one-dimensional maps, type-III intermittency occurs if $SF(x) > 0$. For the classic formulation of chaotic intermittency, the RPD is constant i.e $\phi(x) = \frac{0.5}{c}$ Following the procedure used above for Type I and II, we get

$$l(x,c) = \int_x^c \frac{dr}{\varepsilon r + ar^3} = \frac{1}{2\varepsilon}\ln\left(\frac{a+\varepsilon/x^2}{a+\varepsilon/c^2}\right) \tag{18}$$

here we've used $2\varepsilon \to \varepsilon$, noting l(x,c) is laminar length of $F^2(x)$ map, Consequently, the laminar length for the F(x) map is 2l(x, c). The probability density of the laminar lengths, $\psi(l)$:

$$\psi(l) = \left(\frac{\varepsilon}{\left(\frac{\varepsilon}{c^2}+a\right)e^{2\varepsilon l} - a}\right)^{3/2}\left(\frac{\varepsilon}{c^2}+a\right)\frac{e^{2\varepsilon l}}{2c} \tag{19}$$

The average laminar length:

$$\bar{l} = \int_0^c \phi(x)2l(x)\,dx = \frac{1}{c}\int_0^c l(x)\,dx = \frac{1}{c\sqrt{a\varepsilon}}\arctan\left(c\sqrt{\frac{a}{\varepsilon}}\right) \tag{20}$$

If $c\sqrt{\frac{a}{\varepsilon}} >> 1$,

$$\bar{l} \propto \varepsilon^{-1/2} \tag{21}$$

Kodama et al. 1991 [5] clarify that Two characteristic relations have been found for type III intermittency ([8] gives the characteristic length $\bar{l} \propto \varepsilon^{-1}$), for the case of linear perturbation the characteristic length id ε^{-1}, however maps with shift perturbation only have one characteristic length given by $\bar{l} = \varepsilon^{-\frac{z-1}{z}}$ (In the derivations I couldn't understand the scaling relations, so just stating the conclusion)

4 Renormalisation Group theory on Intermittent Chaos

Here we show the Renormalisation Group[3][4][5], treatment of Intermittency from the ref [8] To start we would like to consider maps which for $\varepsilon = 0$ and $x \to 0$ have the form

$$f(x \to 0) = x + u|x|^z, \quad z > 0 \tag{22}$$

Now we want to find fixed points of the doubling operator (T), such that-

$$Tf^*(x) = \alpha f^* \left(\frac{f^*(x)}{\alpha} \right) = f^*(x) \tag{23}$$

where f*(x) is the fixed point of T. This is subject to the boundary conditions f*(0)=0 and f'*(0)=1 so that we get back the form of generalised maps like 22. This formulation is based on the works by Hu and Rudnick [4]. Consider the recursion relation

$$x' = f(x)$$

in implicit form

$$G(x') = G(x) - a \tag{24}$$

i.e.,

$$x'(x) = G^{-1}[G(x) - a] = f(x) \tag{25}$$

where a is a free parameter. The fixed-point equation

$$\alpha f^*[f^*(x)] = f^*(\alpha x)$$

then becomes

$$\alpha x''(x) = x'(\alpha x)$$

or by operating on this with G:

$$G(\alpha x'') = G[x'(\alpha x)] = G(\alpha x) - a \tag{26}$$

Next, eq. 24 is used to obtain

$$G(x'') = G(x') - a = G(x) - 2a$$

i.e.,

$$\frac{1}{2}G(x'') = \frac{1}{2}G(x) - a \tag{27}$$

Comparison of 26 and 27 indicates that to solve the fixed-point equation, G^* (corresponding to f*(x)) must have the property (since a is a free parameter)

$$\frac{1}{2}G^*(x) = G^*(\alpha x) \tag{28}$$

The simple choice $G^*(x) = |x|^{-(z-1)}$ with $\alpha = 2^{\frac{1}{1-z}}$ yields the desired result. The fixed-point function therefore becomes

$$f^*(x) = G^{*-1}[G^*(x) - a] = \left(|x|^{-(z-1)} - a\right)^{-1/(z-1)} \tag{29}$$

which for $a = (z-1)u$ fulfills the boundary condition (f*(0)=0 and f'*(0)=1). This derivation shows that the fixed-point map for intermittency is mathematically related to a translation $G(x') = G(x) - a$; however, a simple physical explanation for this connection is not clear.

Now, one wants to classify the perturbations to f^* according to their relevance. We investigate, therefore, how the doubling transformation T acts (to linear order in ε) on a function

$$f_\varepsilon(x) = f^*(x) + \varepsilon h_\lambda(x) \text{ for } \varepsilon \ll 1$$

Using the definition 23 for T, we find:

$$\begin{aligned} Tf_\varepsilon =& \alpha f_\varepsilon \left[f_\varepsilon (\frac{x}{\alpha}) \right] \\ =& \alpha f^* \left[f^*\left(\frac{x}{\alpha}\right) + \varepsilon h_\lambda \left(\frac{x}{\alpha}\right)\right] + \varepsilon \alpha h_\lambda \left[f^*\left(\frac{x}{\alpha}\right) + h_\lambda \left[f^*\left(\frac{x}{\alpha}\right)\right]\right] \\ =& \alpha f^* \left[f^*\left(\frac{x}{\alpha}\right)\right] + \varepsilon\alpha \big\{ f^{*\prime} \left[f^*\left(\frac{x}{\alpha}\right)\right] h_\lambda\left(\frac{x}{\alpha}\right) + h_\lambda \left[f^*\left(\frac{x}{\alpha}\right)\right] \big\} + O(\varepsilon^2) \\ =& f(x) + \lambda\varepsilon h_\lambda(x) + O(\varepsilon^2) \end{aligned} \tag{30}$$

The last equation holds only if $h_\lambda(x)$ is an eigenfunction, with the eigenvalue λ, of the linearized doubling operator Lf^*:

$$Lf^*[h_\lambda(x)] \equiv \alpha \big\{ f^{*\prime}\left[f^*(x)\right] h_\lambda(x) + h_\lambda[f^*(x)] \big\} = \lambda h_\lambda(\alpha x) \tag{31}$$

We now show that the method used above (to find the fixed-point function f^*) allows us also to find the spectrum of eigenvalues λ and the corresponding eigenfunctions h_λ. First, we write $f_\varepsilon(x)$ in implicit form using eq. 25:

$$f_\varepsilon(x) = f^*(x) + \varepsilon h_\lambda(x) = x' = G_\varepsilon^{-1}[G_\varepsilon(x) - a] \tag{32}$$

If we expand

$$G_\varepsilon(x) = G^*(x) + \varepsilon H_\lambda(x)$$

then $h_\lambda(x)$ can be expressed in terms of $H_\lambda(x)$ (and vice versa) by comparing the factors linear in ε on both sides of 32. Next, we consider the second iterate,

$$x''(x) = f_\varepsilon[f_\varepsilon(x)]$$

and apply G_ε to this. This yields

$$G_\varepsilon(x'') = G_\varepsilon(x') - a = G_\varepsilon(x) - 2a$$

or more explicitly:

$$G^*(x'') + \varepsilon H_\lambda(x'') = G^*(x) + \varepsilon H_\lambda(x) - 2a. \tag{33}$$

Because $G^*(x)$ has the form of a simple power of x, we try a similar ansatz for $H_\lambda(x)$:

$$H_\lambda(x) = |x|^{-p} \tag{34}$$

Using the property 28 of $G^*(x)$, 33 then becomes (here, we used the form of H_λ and the expression of α to get λ)

$$G^*(\alpha x'') + \lambda\varepsilon H_\lambda(\alpha x'') = G^*(\alpha x) + \lambda\varepsilon H_\lambda(\alpha x) - a$$

or

$$\begin{aligned} G_{\lambda\varepsilon}(\alpha x'') =& G_{\lambda\varepsilon}(\alpha x) - a \\ \to \alpha x'' =& G_{\lambda\varepsilon}^{-1}[G_{\lambda\varepsilon}(\alpha x) - a] \end{aligned}$$

where

$$\lambda = 2^{\frac{p+1-z}{z-1}} \tag{35}$$

With 32, this translates into

$$\alpha f_\varepsilon[f_\varepsilon(x)] = f_{\lambda\varepsilon}(\alpha x) = f^*(\alpha x) + \lambda\varepsilon h_\lambda(\alpha x)$$

By comparing this result with eq. 30, we see that λ is indeed the eigenvalue of h_λ, which is determined by

$$f^*(\alpha x) + \lambda\varepsilon h_\lambda(\alpha x) = G_{\lambda\varepsilon}^{-1}[G_{\lambda\varepsilon}(\alpha x) - a]. \tag{36}$$

Solving 36 to linear order in ε one obtains

$$h_\lambda(x) = \frac{1}{up}\left[|x|^{-(z-1)} - u(z-1)\right]^{-\frac{z}{z-1}} \cdot \left\{|x|^{-p} - \left[|x|^{-(z-1)} - u(z-1)\right]^{-\frac{p}{z-1}}\right\} \tag{37}$$

Eqs. 35 and 37 represent the main results of this section. They provide the information as to how T acts (to linear order in the deviation $f - f^*$) on a function f that obeys the boundary condition (f*(0)=0 and f'*(0)=1), because we obtain by expanding $f(x) - f^*(x)$ into $h_\lambda(x)$:

$$T^n f(x) = T^n(f^*(x) + f(x) - f^*(x)) = T^n(f^*(x) + \sum_\lambda c_\lambda h_\lambda(x)) = f^*(x) + \sum_\lambda \lambda^n c_\lambda h_\lambda(x)$$

We now calculate the dependence of the duration $< l >$ of a laminar region on the shift ε of the map from tangency. The eigenfunction h_λ in eq. 37 has been normalized, so that its lowest order term in x is $|x|^{2z-1-p}$. We, therefore, see that a constant shift from tangency corresponds to a relevant perturbation with $p = 2z - 1$. The eigenvalue $\lambda\varepsilon$, that corresponds to this is

$$\lambda_\varepsilon = 2^{\frac{z}{z-1}} \tag{38}$$

With this information, we can determine $< l > (\varepsilon)$ by a simple scaling procedure. Because $< l >$ is related to the number of iterates of x_0, and $f^2(x) = f[f(x)]$ only requires half as many steps as $f(x)$, we arrive at the scaling relation

$$< l > [Tf(x_0)] = \frac{1}{2} < l > [f(x_0)]$$

Using 30 this becomes after many iterations

$$< l > [f(x_0)] = 2^n < l > [T^n f(x_0)] = 2^n < l > [f^*(x_0) + \varepsilon\lambda_\varepsilon^n h_\lambda(x_0)] = 2^n(??) \quad (39)$$

from which for $\varepsilon\lambda_\varepsilon^n = 1$ we obtain with 38:

$$< l > \propto \varepsilon^{-v} \quad \text{with} \quad v = \frac{z-1}{z}$$

In this I have a doubt understanding the last part of 39, I understand that $\varepsilon^{-v} = 2^n$ using 38.

I had a trouble understanding parts of what was done by Hirsch scalapino [3], since they skipped a lot of steps.

Although the renormalization-group method provides a good classification of universality of intermittency, it gives insufficient information to study the laminar length. The commonplace method of approximation difference equations by differential equations gives the probability density amongst other characteristic equations describing the intermittent phenomenon well[5].

5 Intermittency in the presence of Noise

Several techniques have been proposed to study the effect of noise in intermittent maps, most popular being the renormalisation group analysis on the other hand, by using the Fokker–Plank equation[2]. The common the assumption in most approaches is that they consider only the noise effect in the laminar region on the map, while we know that the noise affects the whole system, not just the local map;

6 Intermittency and 1/f-Noise

Skipped (see reference [8]).

7 Reinjection Probability Density (RPD)

Power law form [1].

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