Discrete Grid Environment Formulation

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Abstract

A simplified discrete grid environment inspired by balloon navigation. The environment consists of a 3D grid with a spatially smooth stochastic wind field. Actors experience passive horizontal drift according to local wind distributions and controllable vertical movement. This formulation serves as a tractable abstraction for studying navigation and station-keeping policies under environmental uncertainty.

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1 Environment Formulation

The environment consists of two primary components: a **field** that governs environmental forces and an **actor** that moves according to both field dynamics and control actions.

1.1 Field Definition

Purpose: The field represents environmental force distributions that passively advect actors in the horizontal plane.

1.1.1 Spatial Domain

Grid Definition: Define a discrete 3D grid:

$$\mathcal{G} = \{(i, j, k) \mid i \in \{1, \ldots, N_x\}, j \in \{1, \ldots, N_y\}, k \in \{1, \ldots, N_z\}\}$$

with unit spacing, representing the operational spatial domain.

Note: More generally, the defined grid need not be unit spaced. Example: $\mathcal{G} = \{(x_i, y_j, z_k) \mid i \in \{1, ..., N_x\}, j \in \{1, ..., N_y\}, k \in \{1, ..., N_p\}\}$ where $x_i \in \mathcal{X}, y_j \in \mathcal{Y}, z_k \in \mathcal{Z}$ over some domain

Displacement Space: Define the set of possible horizontal displacements:

$$\mathcal{D} = \{(u, v) \mid u, v \in \mathbb{Z}, |u| \leq D_{\max}, |v| \leq D_{\max}\}$$

where $D_{\max} \in \mathbb{N}$ bounds the maximum displacement magnitude in each direction.

1.1.2 Local Field Distribution

Probability Mass Function: For each grid point $(i, j, k) \in \mathcal{G}$, define a joint probability mass function over horizontal displacements:

$$p_{i,j,k} : \mathcal{D} \to [0, 1]$$

$$p_{i,j,k}(u, v) = \mathbb{P}[U_{i,j,k} = u, V_{i,j,k} = v]$$

where $U_{i,j,k}, V_{i,j,k}$ are discrete random variables representing the horizontal displacement components experienced by an actor at grid point (i, j, k).

1.1.3 Field Properties and Constraints

The field must satisfy several mathematical constraints to ensure physical plausibility and computational tractability.

Local Constraints (for each $(i, j, k) \in \mathcal{G}$):

1. **Probability Mass Function Property**:

$$\sum_{(u,v)\in\mathcal{D}} p_{i,j,k}(u, v) = 1, \quad p_{i,j,k}(u, v) \geq 0$$

2. **Boundary Conditions**: Displacements that would move an actor outside the grid are forbidden:

$$p_{i,j,k}(u, v) = 0 \quad \text{if} \begin{cases} i + u \notin \{1, \ldots, N_x\} \\ j + v \notin \{1, \ldots, N_y\} \end{cases}$$

3. **Variance Bounds**: Local displacement variability is bounded and specified:

$$\mathrm{Var}(U_{i,j,k}) \leq \sigma_u^2, \quad \mathrm{Var}(V_{i,j,k}) \leq \sigma_v^2$$

4. **Covariance Structure**: The correlation between horizontal displacement components can be controlled:

$$\mathrm{Cov}(U_{i,j,k}, V_{i,j,k}) = \rho_{uv}\sigma_u\sigma_v$$

where $\rho_{uv} \in [-1, 1]$ is the correlation coefficient between u and v displacement components.

Global Constraints:

4. **Spatial Smoothness**: The displacement distributions must exhibit spatial coherence, meaning that neighboring grid points have similar statistical properties. This constraint ensures the field represents a physically plausible flow pattern rather than random noise.

   Formal Requirement: For neighboring grid points $(i, j, k), (i', j', k') \in \mathcal{G}$ with $\|(i, j, k) - (i', j', k')\|_1 = 1$, the probability mass functions $p_{i,j,k}$ and $p_{i',j',k'}$ must be "close" according to some distance metric $d(\cdot, \cdot)$:

$$d(p_{i,j,k}, p_{i',j',k'}) \leq \epsilon$$

   where $\epsilon > 0$ is a smoothness parameter. The specific choice of distance metric $d(\cdot, \cdot)$ depends on the implementation method:

   - **Total Variation**: $d_{TV}(p, q) = \frac{1}{2}\sum_{(u,v)} |p(u, v) - q(u, v)|$
   - **Wasserstein**: $d_W(p, q)$ measuring transport cost between distributions
   - **KL-Divergence**: $D_{KL}(p||q)$ or Symmetric KL-Divergence (Jensen-shannon)
   - **Continuous Field Induced**: When PMFs are derived from an underlying smooth continuous field $\mathbf{c}(\mathbf{x})$, smoothness is inherited through the continuity properties of $\mathbf{c}(\mathbf{x})$

5. **Approximate Flow Conservation**: The expected flow should satisfy discrete incompressibility to maintain physical plausibility. Using central differences for interior points:

$$\mathbb{E}\left[\frac{\partial U_{i,j,k}}{\partial x} + \frac{\partial V_{i,j,k}}{\partial y}\right] \approx 0$$

$$\implies |\mathbb{E}[U_{i+1,j,k}] - \mathbb{E}[U_{i-1,j,k}] + \mathbb{E}[V_{i,j+1,k}] - \mathbb{E}[V_{i,j-1,k}]| \leq \delta$$

   where $\delta > 0$ is a tolerance parameter accounting for discretization errors and compressible effects.

Parameterization: The field is characterized by hyperparameters:

$$\Theta = \{N_x, N_y, N_z, D_{\max}, \sigma_u, \sigma_v, \rho_{uv}, \epsilon, \delta, \mathrm{seed}\}$$

where seed enables reproducible stochastic generation of field realizations satisfying the above constraints.

1.2 Actor Dynamics

Control Actions: The actor receives discrete control inputs:

$$\mathcal{A} = \{-1, 0, +1\}$$

corresponding to descend, maintain altitude, and ascend commands.

Vertical Dynamics: Define action-dependent vertical displacement PMF $q_a(z)$ for action $a \in \mathcal{A}$:

$$q_a(z) = \mathbb{P}[Z_a = z], \quad z \in \{-Z_{\max}, \ldots, Z_{\max}\}$$

where Z_a is the vertical displacement random variable for action a.

Note: **Example Action PMFs**: consider a simple parameterized vertical displacement distributions:

$$q_{+1}(z) = \begin{cases} 1-\epsilon & \text{if } z = +1 \\ \epsilon/2 & \text{if } z = 0 \\ \epsilon/2 & \text{if } z = +2 \\ 0 & \text{otherwise} \end{cases}$$

$$q_0(z) = \begin{cases} 1-\epsilon & \text{if } z = 0 \\ \epsilon/2 & \text{if } z = -1 \\ \epsilon/2 & \text{if } z = +1 \\ 0 & \text{otherwise} \end{cases}$$

$$q_{-1}(z) = \begin{cases} 1-\epsilon & \text{if } z = -1 \\ \epsilon/2 & \text{if } z = 0 \\ \epsilon/2 & \text{if } z = -2 \\ 0 & \text{otherwise} \end{cases}$$

where $\epsilon \in [0, 1]$ is a noise parameter controlling the reliability of vertical control.

State Transition: Given actor position (i, j, k) and action a, the new position (i', j', k') is determined by:

1. **Horizontal Displacement**: $(u, v) \sim p_{i,j,k}(\cdot, \cdot)$
2. **Vertical Displacement**: $z \sim q_a(\cdot)$
3. **Position Update**:

$$i' = i + u, \quad j' = j + v, \quad k' = k + z$$

Boundary Handling: Ensure transitions remain within grid bounds:

$$(i', j', k') \in \mathcal{G}$$

Note: **Boundary Enforcement example:** If a transition would violate grid bounds, enforce hard boundary conditions:

$$i' = \max(1, \min(i + u, N_x))$$

$$j' = \max(1, \min(j + v, N_y))$$

$$k' = \max(1, \min(k + z, N_z))$$