# Deconstruction of Chess using a two-player Kalman tracking formulation with coupled noise

*A*

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*in Partial Fulfillment of the Requirements*

*for the Degree of*

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*By*

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**July 2018**

# Declaration

This is to certify that the thesis entitled "**Deconstruction of Chess using a two-player Kalman tracking formulation with coupled noise**", submitted by me to the *Indian Institute of Technology Guwahati*, for the award of the degree of Master of Technology, is a bonafide work carried out by me under the supervision of Dr. Kannan Karthik. The content of this thesis, in full or in parts, have not been submitted to any other University or Institute for the award of any degree or diploma. I also wish to state that to the best of my knowledge and understanding nothing in this report amounts to plagiarism.

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# CERTIFICATE

This is to certify that the thesis entitled "**Deconstruction of Chess using a two-player Kalman tracking formulation with coupled noise**", submitted by Vinaykumar V. Dubey (164102106), a master's student in the *Department of Electronics and Electrical Engineering*, *Indian Institute of Technology Guwahati*, for the award of the degree of Master of Technology, is a record of an original research work carried out by him under my supervision and guidance. The thesis has fulfilled all requirements as per the regulations of the institute and in my opinion has reached the standard needed for submission. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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**Vinaykumar V. Dubey**
**IIT Guwahati**
**July, 2018**

# Abstract

Games have always been stepping blocks of development toward better computation ability and Artificial Intelligence. A dynamic two-player non-cooperative game is based on better state estimation, in presence of white noise. Kalman Filter has always been one of the best state Estimator.

The Kalman filter is one of the best mathematical tool available for an algorithmic implementation of state estimation of a dynamic system which is corrupted by random noise with given a set of observations which are also corrupted by random noise. If state evolution and observation system are the linear time-invariant system with Gaussian noise, the filter can be proven to be an optimal estimator.

In this thesis, we have formulated a two-player game where both players are trying to achieve the same target. They are also generating mutual interference for each other. Here we have developed a state evolutionary model for target tracking using a Kalman filter. We have discussed the effects of white noise and mutual interference on their ability of convergence.

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# Chapter 1

# Introduction

Researchers have been using games to improve behaviour and decision making capability in the field of Artificial Intelligence. Real life problems have been modelled as game, to make it as a challenge for structured thinking computers. In 1921, Emile Borel addressed the challenges of various field which can be modelled in the game with imperfect information [2]. He theorized an organized system that is capable of playing games, can help in the field of economics and military applications and many more real-life scenarios. Later this idea was developed by John Von Neumann, who gave the way to Gaming theory, with his article from late 1928, "Theory of Parlor Games,", his legendary Minimax theorem. These discoveries have revolutionized the field of economics, military warfare, recreational games and many more. Thus started use of games as a structural informed modelled system to instruct computer into making the decision of its own, rather than the structured form of instructional operation.

Researchers have always considered chess as one of the complex problems for the computer to solve. Chess has always been a complex yet a well-structured problem. Chess is a two-player, zero-sum game, which initially gave direction to this thesis. As we know in chess, both players are trying to achieve a single objective, i.e. to win over other's king. During the process, they are creating the problem for adversary also. This can be also formulated as two players are trying to achieve a target while interfering with the adversary from winning.

Now coming to general two-player non-cooperative games, Taking this problem as a tracking scenario where two players are trying to track an object for some pay-off matrix, while disturbing another one so to prevent them from getting to close to object, i.e. to win. This thesis presents, a two-player non-cooperative game where the main objective is to track the object to the best of its capabilities and restrict adversary to do the same. As when it comes to tracking any object or state, Kalman Filter is one of the most used methods in this field.

## 1.1 Kalman Filter

The Kalman Filter (KF) is one of the best methods for tracking and estimation for its simplicity and optimality, since its discovery in 1970's. Kalman filter is an optimal linear quadratic estimator. Kalman filter works in an iterative loop of the predictor-corrector equation, that is the first stage deals with the estimation of a new state based on its past input and control variables. Next stage corrector, deals with adjustment based on current measured value and the newly predicted first stage. So basically Kalman filter takes your noisy observation and past states to create a new future estimate. [5] It is well equipped for computer implementation as it does finite states estimation in few steps algorithm architecture.

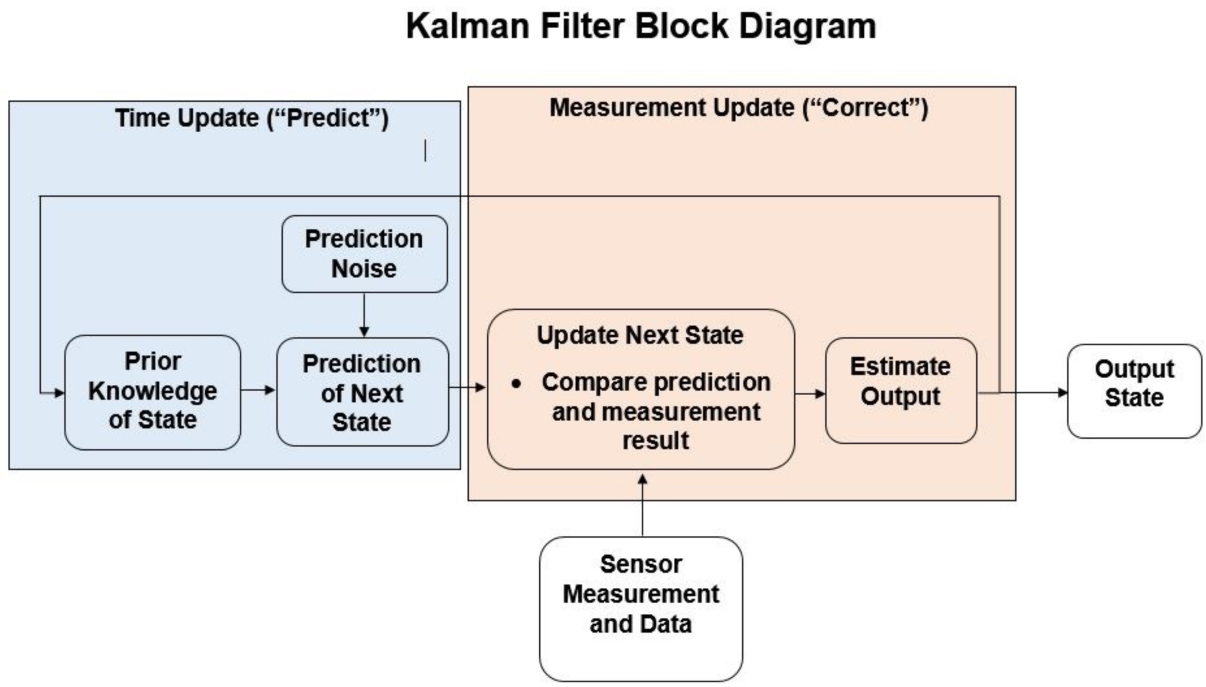


Figure 1.1: Significance of Kalman Gain

As we can see in Fig.1.1, prior knowledge of state is used to formulate prediction for next state with process noise covariance matrix. This stage is referred to as Update stage. In the next step, Kalman gain adjusts the prediction value according to the value of measurement obtained. This is called the Correction stage. The output data from this stage is propagated to form next prediction so on and so forth.

The main drawback of the Kalman filter to be optimal is the necessity that the dynamic system under consideration has to be linear. Researchers have made several different adaptations to this filter for the various condition over the years. [1] Its implementation with nonlinear problems have evolved to two main algorithms, Extended Kalman Filter (EKF) and much recently Unscented Kalman Filter (UKF).

Thus over the years, Kalman filter and its extended versions have found application in a variety of fields such as tracking and navigation in various military applications, signal processing, economics etc.

## 1.2 Motivation

Dynamic two-player games are related to most battlefield scenarios, with the sole objective is to out-do the other player. These two-player dynamic games have been one of the leading problems in recent times. Development of these games benefits several problems related to security and resource management. Patrol division, threat pursuits being some of the two-player examples which are majority studied for security purposes.

One of such problems is tracking scenario, where the target is the same for two parties if one wants to get the target he needs to make sure adversary can not approach the target. We have created a scenario where two Kalman tracking system are modelled for tracking of the random moving object.

## 1.3 Thesis Organization

The chapters in the dissertation are organized as follows:

Chapter (2) goes into details of the literature survey, explaining similar work done in past by the researchers. It also gives background information about the Kalman filter that is the main basis of the problem.

Chapter (3) describes the Singular State Evolutionary model that has been implemented for the above problem statement. It explains mathematical modelling and its structure for the problem. While creating the mathematical model several aspects of interference has also been discussed alongside.

In Chapter (4), we have discussed results of this arithmetical model described in chapter (3), with consideration of interference aspects also.

Chapter (5) concludes findings obtained in this thesis and their significance. We also have included several possible future endeavours, i.e. improvisations and improvement of the model and its variables.

# Chapter 2

# Literature Survey

## 2.1 Similar Work

In Two-player games, both players try to achieve the most favourable position that leads to winning or at least draw. Minimax filter has been a great help in this aspect for a long time. Yaesh [6] [7] and Simon [8] discussed the prospect of differential two-player, zero-sum games from $H_\infty$ estimation(minimax estimation) and Kalman filter based estimation respectively. They gave way to estimation problem formulation for these games. They have considered from player A's perspective with considering obstruction caused by player B.

They have considered the fundamental formulation of Minimax filter as the basis of an approach to state estimation using Kalman filter. In minimax filter, the player tries to minimize his estimation error function and adversary works to maximize this function. This has been the heart of many two-player zero-sum games in game theory.

- Yaesh [6] [7] at.al, have summarised that minimax filter optimizes error in the estimation of a linear system is better than that of the Kalman filter. They have not considered inference of mutual noise i.e. interference that your opponent gives you during the gameplay.

- They also have considered the differential game system matrix to be linear and static, in reality sometimes these systems can be linearized to some extent, often not the case. Even sometimes system matrix can be dynamic also.

- Dr Simon [8] have considered these assumptions while using constrained minimax state estimation. He has improved on perceived Interference from the adversary, he has tried to formulate this into a trade-off formulation between player and opponent gain. This estimation error cost function forms two way controlled (balanced) game from the observer's point of view.

This has led to our problem of state evolutionary two-player game formulation.

## 2.2 Problem Statement

While considering interference from the adversary in state estimation as noise, previously they have not considered it from adversary's perspective. This observer's perspective used for the formulation of estimation error cost function raises some fine questions.

- How this interference can affect the player?
- What is the reward adversary is getting from this interference?
- Can this one-sided formulation stand in the real-life scenario?

Thus we have created a two-player scenario, where their objective is to track a randomly moving object. They also have to object the adversary from getting to close to the target to best of their capability. We have used Singular state Evolutionary model Kalman filter to track this object from both player's perspective. In this thesis, we have considered two scenarios,

- Two trackers with an independent tracking system without mutual interference.
- Two trackers with an independent tracking system with mutual interference with random noise.

## 2.3 Background Theory

This section gives a brief but thorough review of the necessary methods that are needed to direct choices made in later chapters of the thesis.

### 2.3.1 The Kalman Filter

The Kalman filter's original formulation was described by Rudolf E. Kalman in 60's. [9] It was formulated to operate in the linear time-invariant state system.

$$
\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + w_t \\
y_t &= Cx_t + v_t
\end{aligned}
$$

where $w_t \sim N(0, \Sigma_w)$, $v_t \sim N(0, \Sigma_v)$, and $x_0 \sim N\left(x_{0|-1}, P_{0|-1}\right)$. Here $x_{t+1}$ is the present state to be estimated using past states and control variables input which are corrupted by white noise.

Note that for Kalman filter, states are system with mean $\mu$ and variance $\sigma$. We will use the following notation for describing this values.

$$
\begin{aligned}
\hat{x}_t &= E\left[x_t | y_{0:\,t}\right] \\
P_t &= E\left[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^{\mathsf{T}} | y_{0:\,t}\right]
\end{aligned}
$$

These mean and variance are the useful in estimation and update equation for estimation of the next state.

**Estimation equations :** These equation uses past state values to obtain a priori estimate.

$$
\begin{aligned}
\hat{x}_{t+1|t} &= A\hat{x}_{t|t} + Bu_t \\
P_{t+1|t} &= AP_{t|t}A^{\mathsf{T}} + \Sigma_w
\end{aligned}
$$

**Update equations :** Here priori estimates obtained from previous steps is compared with current measurements to obtain an improved posteriori estimate.

$$
\begin{aligned}
\hat{x}_{t+1|t+1} &= \hat{x}_{t+1|t} + K_{t+1}\left(y_{t+1} - C\hat{x}_{t+1|t}\right) \\
K_{t+1} &= P_{t+1|t}C^{\mathsf{T}}\left(CP_{t+1|t}C^{\mathsf{T}} + \Sigma_v\right)^{-1} \\
P_{t+1|t+1} &= P_{t+1|t} - P_{t+1|t}C^{\mathsf{T}}\left(CP_{t+1|t}C^{\mathsf{T}} + \Sigma_v\right)^{-1}CP_{t+1|t}
\end{aligned}
$$

Here Kalman gain works moderator between priori estimate and measurement as we can see in Fig.2.1, when Kalman gain is high, measurements are considered more accurate. As filter progresses, its Kalman gain goes down, indicating that estimated updates accuracy is increased as compared to measurement.

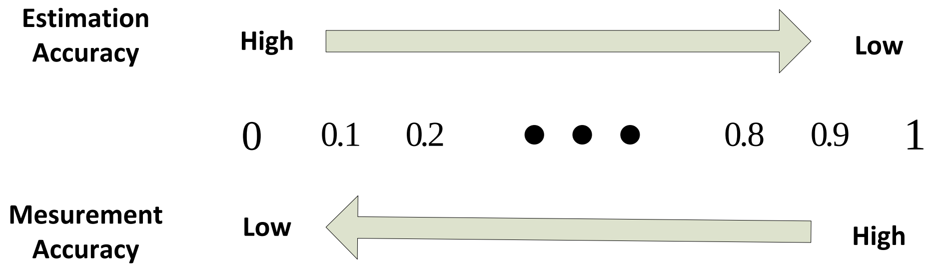


Figure 2.1: Significance of Kalman Gain

Observational Noise ($\sigma_v^2$) is the field of uncertainty region around the measurement, this error is solely dependent our sensor for measurement error. This additive noise is filtered out efficiently in later stages of prediction. Tracking becomes progressively more difficult when the internal noise becomes too large.

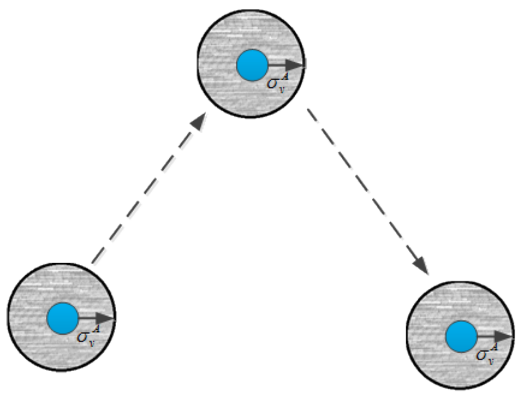

Figure 2.2: Sensitivity to Observational Noise

State Noise ($\sigma_w^2$) is uncertainty associated with state. When state is progressed to later stages, due to its state noise, state value diverges from the actual state value due to prediction stage. From Fig.2.3, we can see that state noise bring variation in every move, thus this uncertainty if large can make system random.

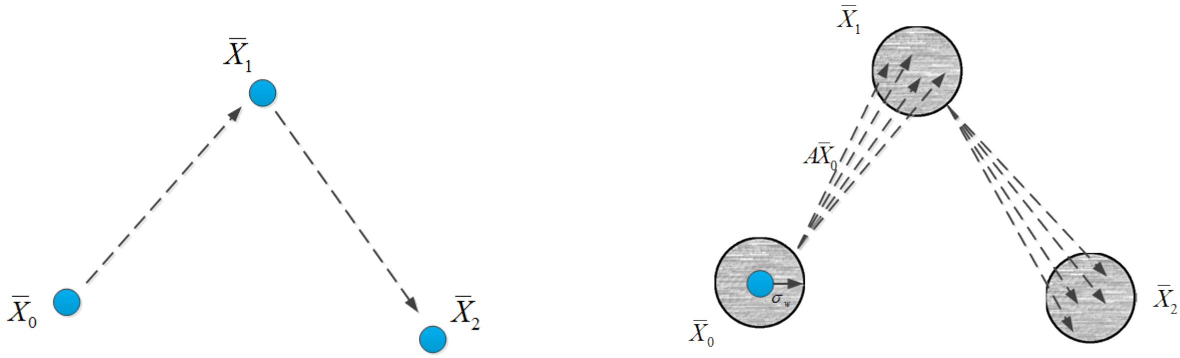


Figure 2.3: Sensitivity to State Noise(With and without)

In moderation these noises are helpful but if they gets out of control, they can very well sabotage the estimation. Thus in Kalman filter state estimation, these two set of noises control the accuracy of estimated state.

# Chapter 3

# Model Design

## 3.1 Basic Kalman Model

Let $\bar{X}_k = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix}$ be an unknown state of a particular system or an object. In the absence of any disturbance or errors,

The state evolves as

$$\bar{X}_{k+1} = A\bar{X}_k$$

Where $P = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a state matrix.

If there is an internal disturbance, the state evolves as,

State Evolution equation can be written as

$$\bar{X}_{k+1} = A\bar{X}_k + \bar{w}_k \tag{3.1}$$

To predict the next state $\hat{x}_{k+1}$ is based on measurement of object by the system. Here Observation equation can be written as

$$\bar{y}_k = \bar{x}_k + \bar{v}_k \tag{3.2}$$

Here $\bar{w}_k$ is state noise factor and $\bar{v}_k$ is observation noise factor. We have taken these value our state evolutionary model

$$A = r \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

$$\bar{w}_k = \begin{bmatrix} w_1^{(k)} \\ w_1^{(k)} \end{bmatrix}$$

with $w_i^{(k)} \sim N(0, \sigma_w^2)$, $\bar{v}_k^{(k)} \sim N(0, \sigma_v^2)$ Here we have taken following assumptions as

- State noise $\bar{w}_k$ is independent of state vector $\bar{x}_n$
- Observation noise $\bar{v}_k$ is independent of Measurement vector $\bar{y}_n$

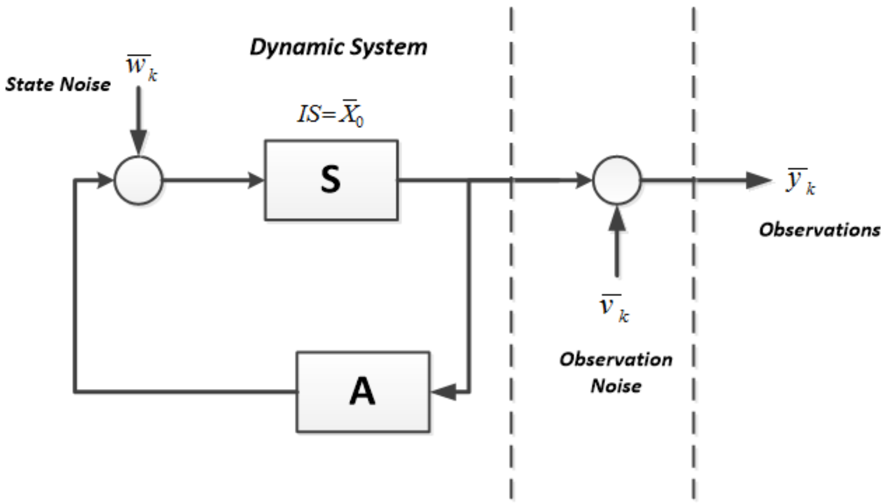


Figure 3.1: System Model

These equation evolves through iterative use of estimate-predict equations which are written as,

$$\hat{x}_{k/k} = A\hat{x}_{k-1/k-1} + G_k\tilde{y}_k \tag{3.3}$$

Here $\tilde{y}_k = \bar{y}_k - A\hat{x}_{k-1/k-1}$ is the innovation equation, which is difference in current observed value and current optimal estimate value. And $G_k$ is Kalman gain, which can be written as,

$$G_k = \frac{E[\bar{x}_k^T \tilde{y}_k]}{E[\tilde{y}_k^T \tilde{y}_k]}$$

and kalman gain[1] can be derived as,

$$G_k = \frac{Trace[A\Sigma_{k-1}A^T + \Sigma_w]}{Trace[A\Sigma_{k-1}A^T + \Sigma_w] + Trace[\Sigma_v]} \tag{3.4}$$

Kalman gain decides the extent of induction of observation value into estimation. It is dependent on past performance which in turn depends on $\Sigma_k$. $\Sigma_k$ is error covariance matrix, a measure of estimated accuracy of the state estimate value.

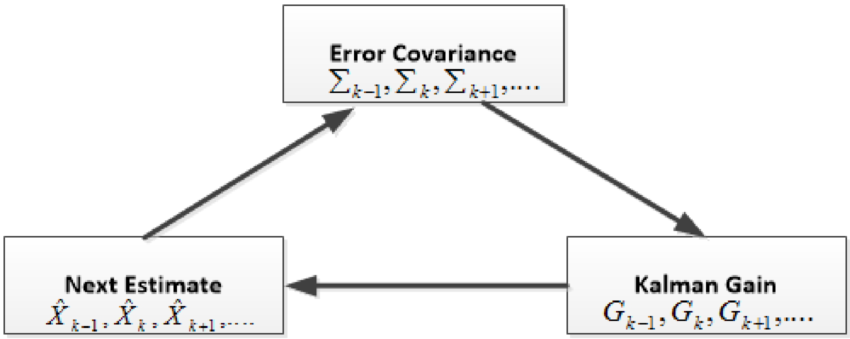


Figure 3.2: Cyclic Dependency of Kalman Gain

$$\Sigma_k = E[(\bar{x}_k - \hat{x}_{k|k})(\bar{x}_k - \hat{x}_{k|k})^T] \tag{3.5}$$

Error covariance updates can be written as,

$$\Sigma_{k|k} = (1 - 2G_{k-1})A\Sigma_{k-1|k-1}A^T + G_{k-1}^2 E[\tilde{y}_k \tilde{y}_k^T] + \Sigma_w \tag{3.6}$$

Noise variance factor $\Sigma_v$ is further estimated as,

$$\widehat{\Sigma}_{v_{current}} = \tilde{y}_k \tilde{y}_k^T - \Sigma_{k|k-1} \tag{3.7}$$

$$= \tilde{y}_k \tilde{y}_k^T - \left(A\Sigma_{k-1|k-1}A^T + \Sigma_w\right) \tag{3.8}$$

$$\widehat{\Sigma}_{v^k} = \left(\frac{k-1}{k}\right)\widehat{\Sigma}_{v^{k-1}} + \frac{1}{k}\widehat{\Sigma}_{v_{current}} \tag{3.9}$$

## 3.2 Case 1: Independent Tracker Systems

We have already seen the modelling for one Kalman tracker system earlier in this chapter. Now extend it to the system with two independent trackers, who are trying

[1]Proof for equation 3.4 can be seen in Appendix A.

to track a randomly moving object. As shown in Fig.3.3, they are just trying to track the object without any interaction.
Our State Evolution equation can be written as

$$\bar{X}_{k+1} = A\bar{X}_k + \bar{w}_k \tag{3.10}$$

Here $\bar{w}_k$ is state noise which is making path of the target random. We have two

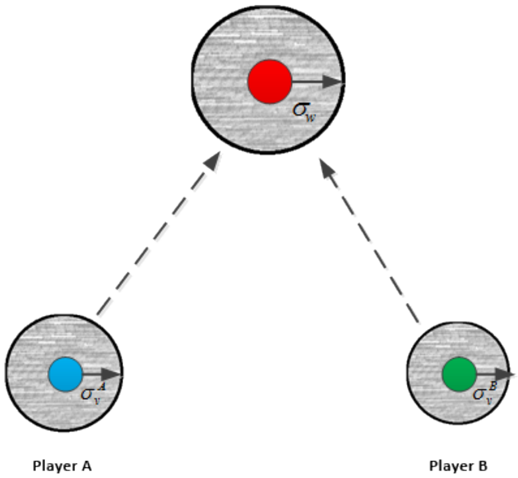


Figure 3.3: Model of two-player system with independent noise

observation system, which can be written as,

$$\bar{y}_k^A = \bar{x}_k + \bar{v}_k^A \tag{3.11}$$

$$\bar{y}_k^B = \bar{x}_k + \bar{v}_k^B \tag{3.12}$$

These systems have their own measurement noise factor, all the state estimate-update equation will be similar to previously developed. Their state estimation error will be solely dependent on their measurement noise factor $\bar{v}_k$ and state noise factor $\bar{w}_k$.

$$e_k = f\left(\tilde{e}_{k-1}, \sigma_w^2, \sigma_v^2\right) \tag{3.13}$$

let $e_A$ and $e_B$ be their state estimation error values, then measure of game can be decided as,

- If $e_A > e_B$ $\qquad fort \in [t_{current}, t_{current+\delta}]$

Player B is converging toward the target and player is currently winning.

- If $e_A < e_B$ $\quad\quad fort \in [t_{current}, t_{current+\delta}]$

  Player A is converging toward the target and player is currently winning.

- If $e_A \cong e_B$ $\quad\quad fort \in [t_{current}, t_{current+\delta}]$

  Currently game is in balanced mode, both players are aiming for convergence.

## 3.3 Case 2: Trackers With Mutual Interference

We have already seen the modelling for independent trackers system. Now we will be considering interference generated by both players toward each other. Our observation

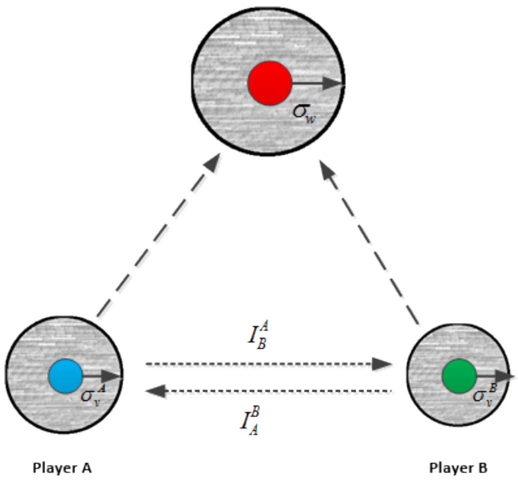


Figure 3.4: Model of two-player system with mutual interference noise

systems can be written as,

$$\bar{y}_k^A = \bar{x}_k + \bar{v}_k^A + I_B^A \tag{3.14}$$

$$\bar{y}_k^B = \bar{x}_k + \bar{v}_k^B + I_A^B \tag{3.15}$$

These systems now have mutual interference noise factor along with their own measurement noise factor, all the state estimate-update equation will be similar to previously developed but noise factor will be very much potent than that of in the previous case.

Now their state estimation error will be dependent on their measurement noise factor $\bar{v}_k$l state noise factor $\bar{w}_k$ and mutual noise interference which will differ based on time. These interference factors will be dependent on the estimation error of the opponent. As soon as system converges toward the target for player A, player B will be getting farther away from the target.

$$e_k^A = f\left(\tilde{e}_{k-1}, \sigma_w^2, \sigma_v^2, I_B^A\right) \tag{3.16}$$

where,

$$I_B^A = f\left(e_{k-1}^B, n_k\right)$$

here $n_i^{(k)} \sim N(0, \sigma_n^2)$ a random white noise factor.

This interference factor will create a push-pull effect between the player and target. As whenever any player is approaching the target, other player will induce interference noise factor proportional to the relative error between player and target. Thus due to this interference factor, convergence on the target will be difficult for both players.

# Chapter 4

# Result and Discussion

In this chapter, we have discussed our findings obtained from modelling done in chapter 3. In Fig.4.1 we have plotted original state evolution with a small state noise. As we can see, it runs in the circle with a varying radius depending on the state noise.

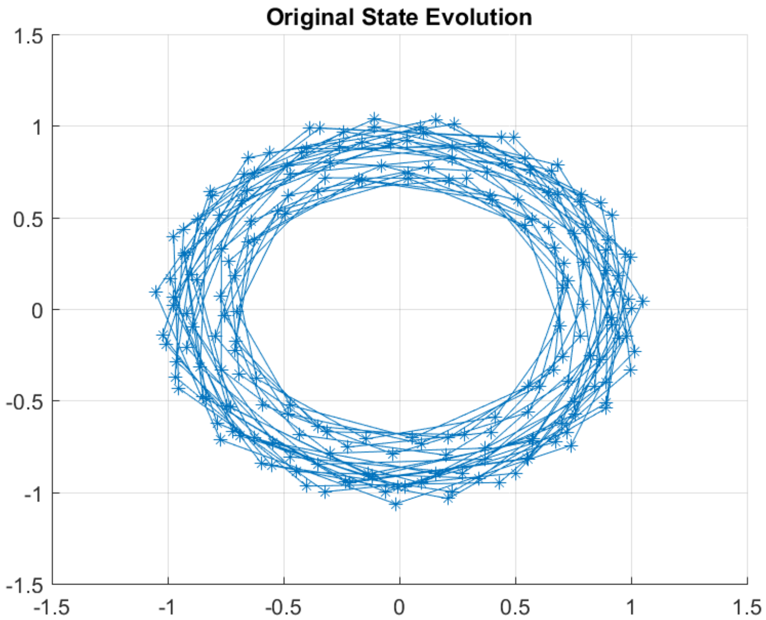


Figure 4.1: Original state evolution

## 4.1 Case 1: Independent Tracker Systems

Here, an independent tracker system, we have two players with different observation noise but without any influence from each other.

$$\sigma_v^A = 1, \qquad \sigma_v^B = 0.6$$

We can see that system for player A(i.e. Tracker 1) is much noisier that of system B. From Fig.4.2, we can see that due to large observational noise, player A is taking long time to track target, while player B which has relatively low observational noise is doing it much more smoothly.

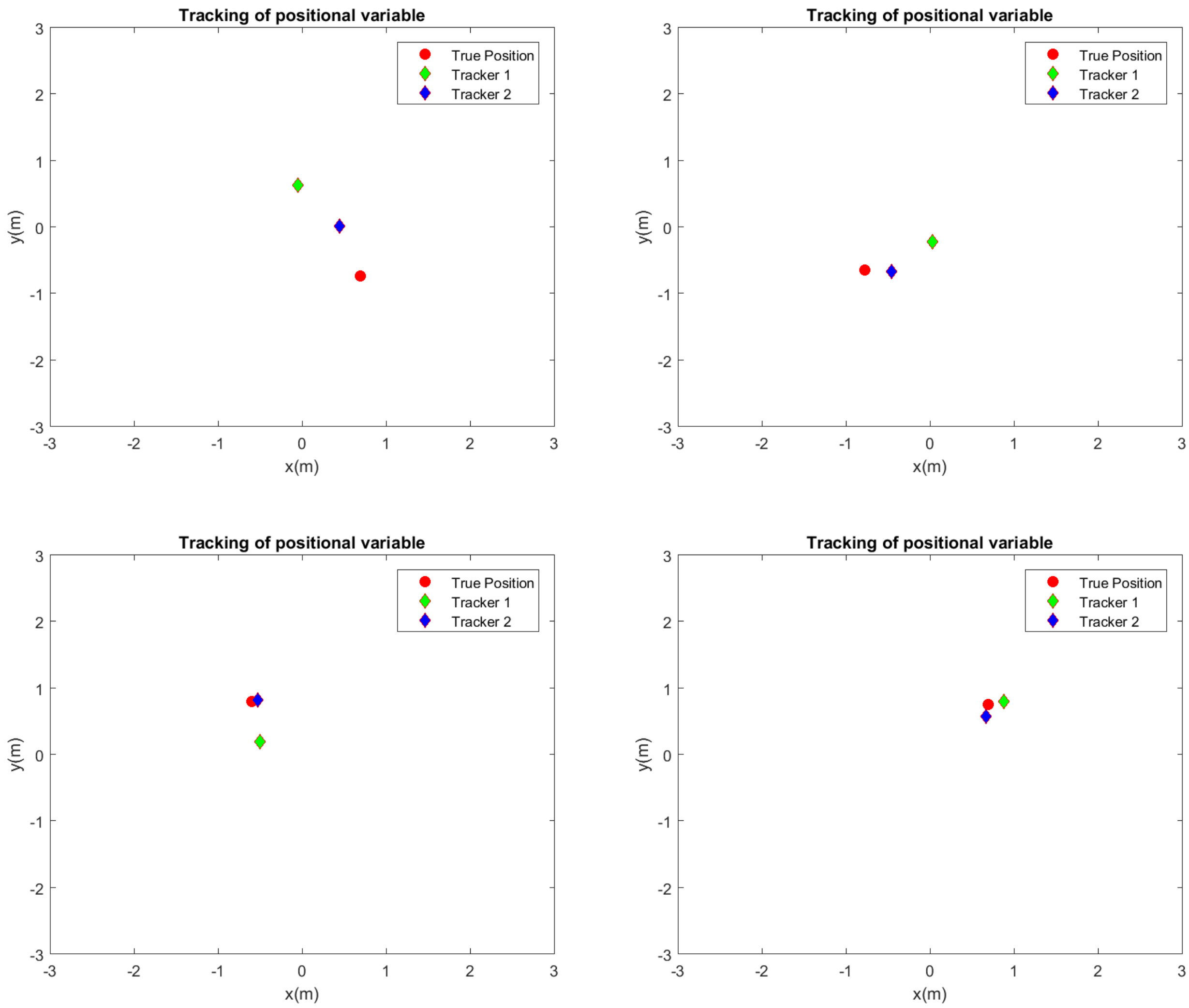


Figure 4.2: Both player tracking with independent estimation noise

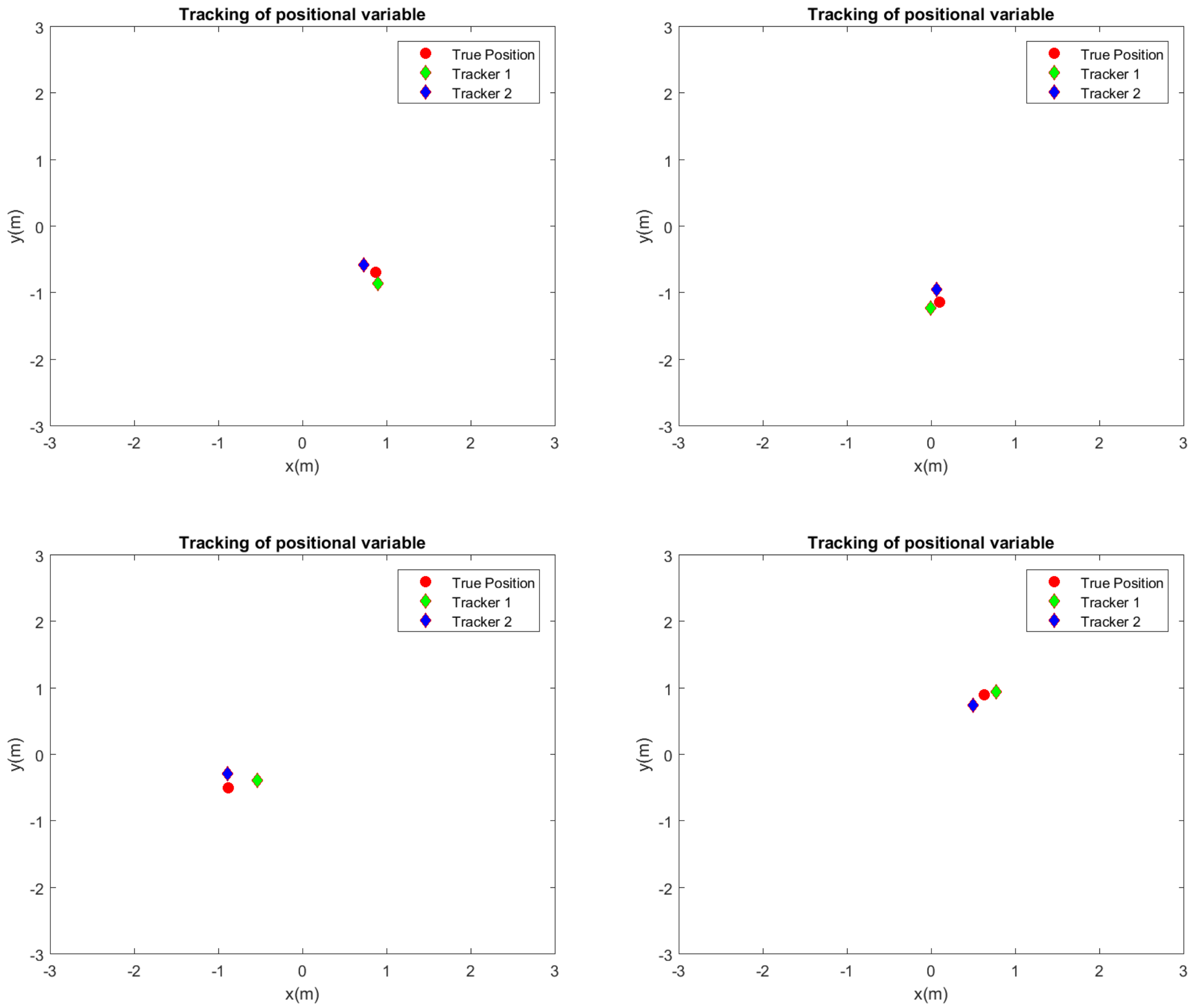


Figure 4.3: Both player tracking with independent estimation noise(Cont.)

As we can see in Fig.4.4 Estimation error of player A is taking large time to converge to target, That means player A will take on average large time to successfully track target than that of player B. In table 4.1 we can see, Average convergence steps required for player A is more than that of Player B.

Thus we can say, that observation error indeed works against the player.

| **Iterations** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **Average** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **TA** | 19 | 44 | 12 | 39 | 44 | 25 | 24 | 29 | 27 | 47 | **31** |
| **TB** | 20 | 17 | 20 | 19 | 16 | 19 | 24 | 40 | 25 | 23 | **22.3** |

Table 4.1: Average convergence of trackers in Independent system model

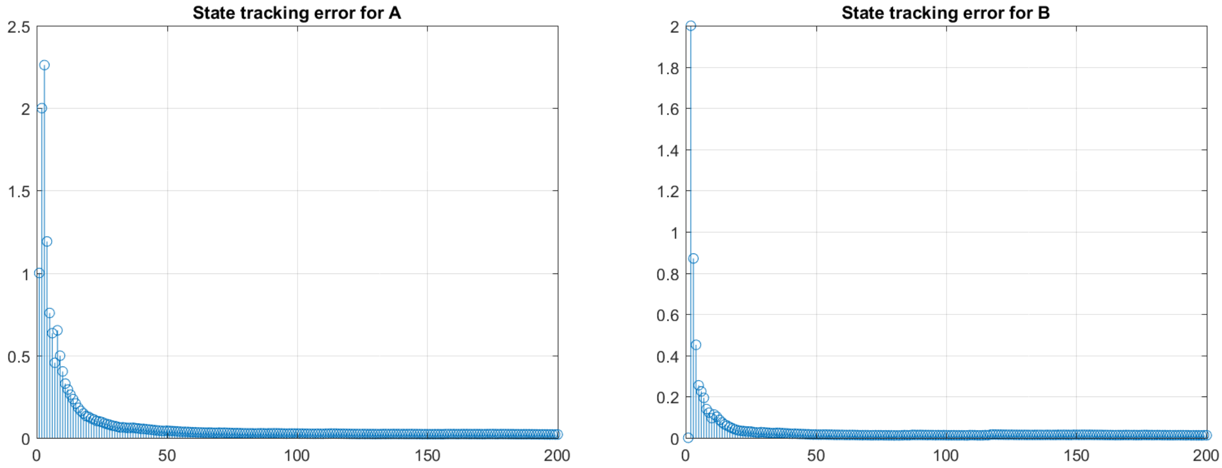


Figure 4.4: Estimation error comparison observed for both players

## 4.2 Case 2: Trackers With Mutual Interference

Here, an independent tracker system, we have two players with different observation noise but with a mutual interference factor from each other.

$$\sigma_v^A = 1, \quad \sigma_v^B = 0.6 \quad with\ \ I_A^B, \ \ \&\ \ I_B^A$$

As seen from Fig.4.5 that as soon as player B gets much closer to the target, it is being pushed away from the target. This is solely on basis of injection of Interference noise factor from the adversary. As soon as any player is converging to the target, it is been injected with a heavy dose of noise, to make it lose the convergence and get distracted. This sort of creates a push-pull kind of scenario, which usually happens in mid-games.

Thus it takes a really long time for both players to converge to target.

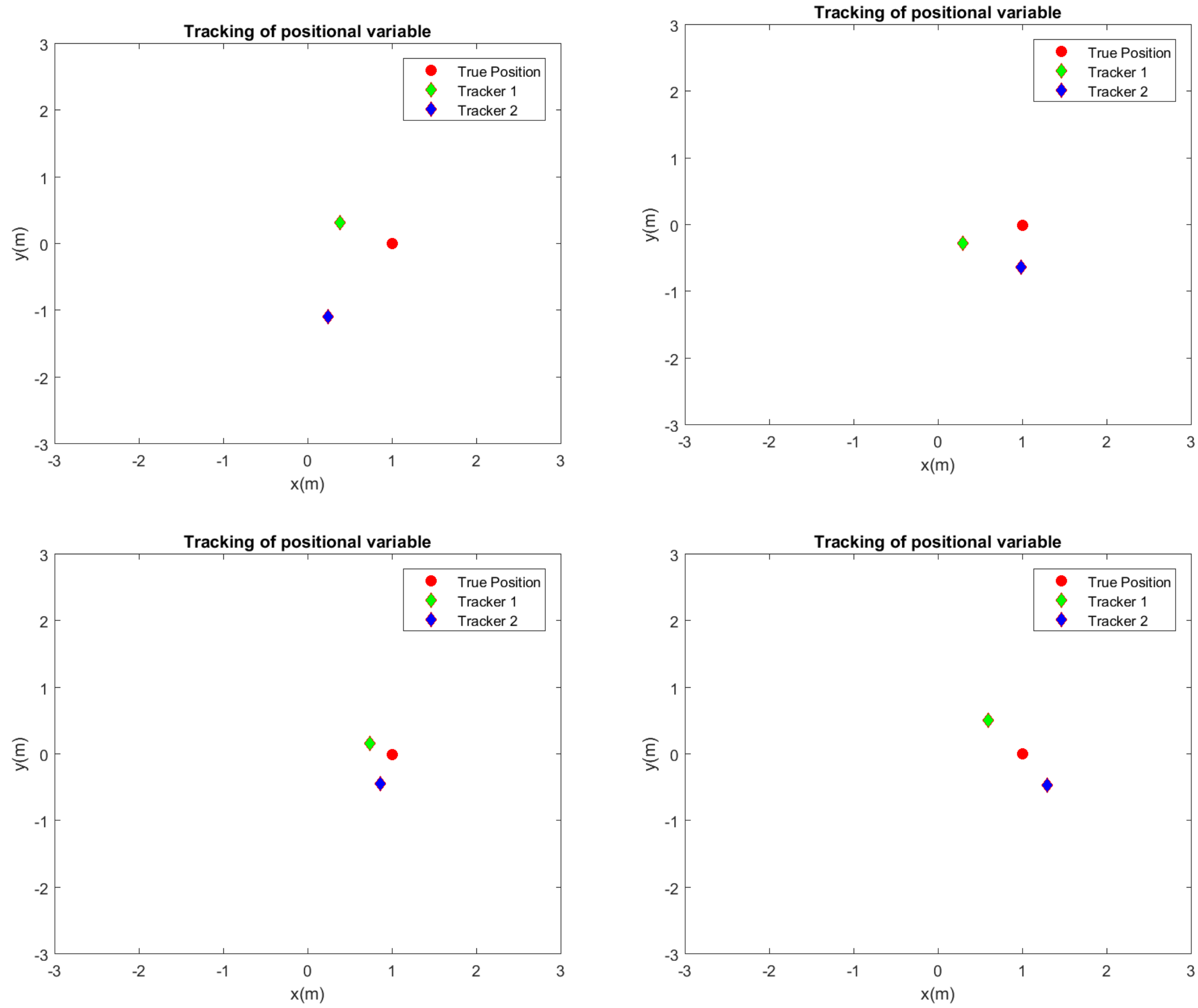


Figure 4.5: Both player tracking with Mutual Interference estimation noise

We can see that system for player A is suppose to be noisier that of system B. But From Fig.4.6, we can see that player 1 is converging to the target and as soon as it is nearer to target than that of player B, it is pushed away from target. This is due to interference factor which is high when player is nearer to target, adversary is pushed away to distance him from target. This interference factor is a due to intergain function[1] used in our simulations.

[1]working of Intergain function can be found in Appendix

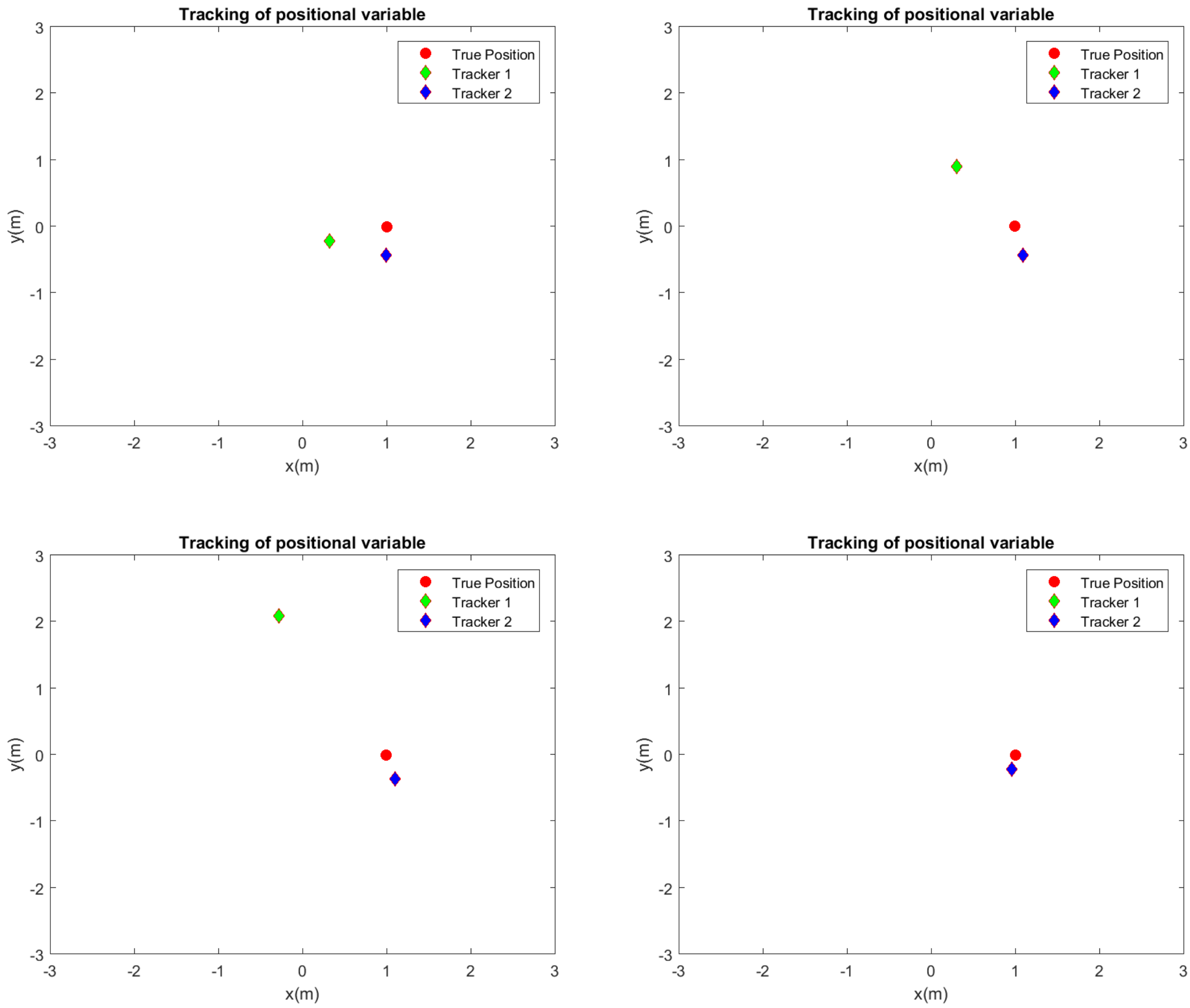


Figure 4.6: Both player tracking with Mutual Interference estimation noise (cont.)

As we can see in Fig.4.7 Estimation error of player A is taking the large time to converge to zero, which mean player A will take a large time to successfully track the target. But as compared to the previous case, we can see that Interference is making the game more even for both the player. In table 4.2 we can see, Average convergence steps required for player A is almost close to Player B, this indicates that Interference can make tracking system more challenging to both player.

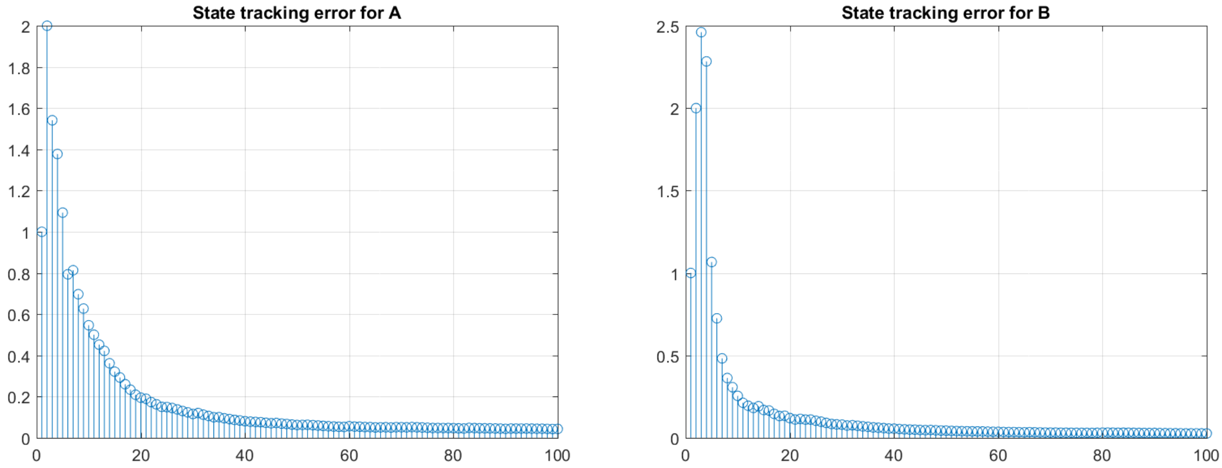


Figure 4.7: Estimation error comparison observed for both players

| **Iterations** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **Average** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **TA** | 28 | 42 | 22 | 99 | 29 | 34 | 24 | 31 | 23 | 68 | **40** |
| **TB** | 44 | 54 | 2 | 99 | 21 | 50 | 21 | 31 | 22 | 42 | **38.6** |

Table 4.2: Average convergence of trackers in Mutual Interference system model

# Chapter 5

# Conclusion and Future Works

This chapter summarizes the main results obtained in the previous chapter and points out futuristic improvement based on the discussion in the previous chapter which can make this model more robust and life-like. We have developed a linear state evolutionary model, which is path target is taking. Then this path is tracked by both players using the Kalman filter.

As we have seen in the previous chapter, In the two-player non-cooperative game system, an independent system is solely based on their observation and state noise. But when both players begin to interfere the tracking of other player, It leads to a much more balanced game than that of the previous case. Here Interference which is a function of estimation error of adversary can make player's estimation go out of control.

In Future works, following improvement can be made to this model, resulting in a better two-player non-cooperative game.

- Interference caused in real-life comes with its own cost, it can make an adversarial effect on the system. So we can include, the cost of this interference later in the model.

- Calculating the optimal cost function which can provide information about minima and maxima which can help with review of the game.

# Bibliography


[1] Grewal, Mohinder S., Angus P. Andrews, and R. W. Bass. "Kalman filtering: Theory and practice." IEEE Transactions on Automatic Control 40.11 (1995): 1983.

[2] Colman, Andrew M. "Game Theory and Experimental Games, The Study of Strategic Interaction, volume 4 of International Series in Experimental Psychology." (1982).

[3] Rapoport, Anatol. "Strategists' Guide.(Book Reviews: Two-Person Game Theory: The Essential Ideas)." Science 154 (1966): 642.

[4] Maschler, Michael. "Eilon Solan i Shmuel Zamir, Game Theory." (2013).

[5] Musoff, Howard, and Paul Zarchan. Fundamentals of Kalman filtering: a practical approach. American Institute of Aeronautics and Astronautics, 2009.

[6] Yaesh, I., and U. Shaked. "Game theory approach to optimal linear estimation in the minimum $H_\infty$-norm sense." Decision and Control, 1989., Proceedings of the 28th IEEE Conference on. IEEE, 1989.

[7] Yaesh, I., and U. Shaked. "Game theory approach to optimal linear state estimation and its relation to the minimum $H_\infty$-norm estimation." IEEE Transactions on Automatic Control 37.6 (1992): 828-831.

[8] Simon, Dan. "A game theory approach to constrained minimax state estimation." IEEE Transactions on Signal Processing 54.2 (2006): 405-412.

[9] Kalman, Rudolph Emil. "A new approach to linear filtering and prediction problems." Journal of basic Engineering 82.1 (1960): 35-45.

[10] Baar, Tamer, and Pierre Bernhard. H-infinity optimal control and related minimax design problems: a dynamic game approach. Springer Science & Business Media, 2008.

[11] Brudno, Michael. *"Competitions, controversies, and computer chess."* Unpublished paper, Department of Computer Science, University of Toronto.

# Appendix A

## Proof of Kalman Gain

Kalman gain $G_k$ can be written as,

$$G_k = \frac{E[\bar{x}_k^T \tilde{y}_k]}{E[\tilde{y}_k^T \tilde{y}_k]}$$

Lets break this proof into two parts,

**Step 1. Numerator Part**

$$\begin{aligned}
E[\bar{x}_k^T \tilde{y}_k] &= trace\left[E(\bar{x}_k \tilde{y}_k^T)\right] \\
E(\bar{x}_k \tilde{y}_k^T) &= E\left(\bar{X}_k(\bar{Y}_k - A\hat{X}_{k-1/k-1})^T\right) \\
&= E\left((A\bar{X}_{k-1} + \bar{w}_{k-1})(\bar{X}_k + \bar{v}_k - A\hat{X}_{k-1/k-1})^T\right) \\
&= E\left((A\bar{X}_{k-1} + \bar{w}_{k-1} + A\hat{X}_{k-1/k-1} - A\hat{X}_{k-1/k-1})(A\bar{X}_{k-1} + \bar{w}_{k-1} + \bar{v}_k - A\hat{X}_{k-1/k-1})^T\right) \\
&= E\left((A(\bar{X}_{k-1} - A\hat{X}_{k-1/k-1}) + \bar{w}_{k-1} + A\hat{X}_{k-1/k-1})(A(\bar{X}_{k-1} - \hat{X}_{k-1/k-1}) + \bar{w}_{k-1} + \bar{v}_k)^T\right) \\
&= A\Sigma_{k-1}A^T + \Sigma_W
\end{aligned}$$

$$E[\bar{x}_k^T \tilde{y}_k] = trace\left(A\Sigma_{k-1}A^T + \Sigma_W\right)$$

Now coming to second part of the equation

**Step 2. Denominator Part**

$$
\begin{aligned}
E[\tilde{y}_k^T \tilde{y}_k] &= trace[E(\tilde{y}_k \tilde{y}_k^T)] \\
E(\tilde{y}_k \tilde{y}_k^T) &= E\left[(\bar{Y}_k - A\hat{X}_{k-1/k-1})(\bar{Y}_k - A\hat{X}_{k-1/k-1})^T\right] \\
&= E\left[(A\bar{X}_{k-1} + \bar{w}_{k-1} + \bar{v}_k - A\hat{X}_{k-1/k-1})(A\bar{X}_{k-1} + \bar{w}_{k-1} + \bar{v}_k - A\hat{X}_{k-1/k-1})^T\right] \\
&= A\Sigma_{k-1}A^T + \Sigma_w + \Sigma_v
\end{aligned}
$$

$$
E[\tilde{y}_k^T \tilde{y}_k] = trace\left(A\Sigma_{k-1}A^T + \Sigma_w\right) + trace\left(\Sigma_v\right)
$$

Now combining both parts into main equation,

$$
G_k = \frac{Trace[A\Sigma_{k-1}A^T + \Sigma_w]}{Trace[A\Sigma_{k-1}A^T + \Sigma_w] + Trace[\Sigma_v]}
$$

# Appendix B

## Effect of Intergain function

This function is basis of interference part we created in our simulation. Here we have created several level of noise injection depending on error function of both player, i.e. $e_A$ and $e_B$. This function is

- when $e_A$ is greater than $e_B$, it means player B is nearer to target than that of A. In response, player A will inject a high amount of noise which will deviate the player B from target.
- In other case when $e_A$ is comparable or less than $e_B$, it will inject appropiate noise based on error function of both target.

This gain factor is part of function that creates interference noise factor based on estimation error of both players.