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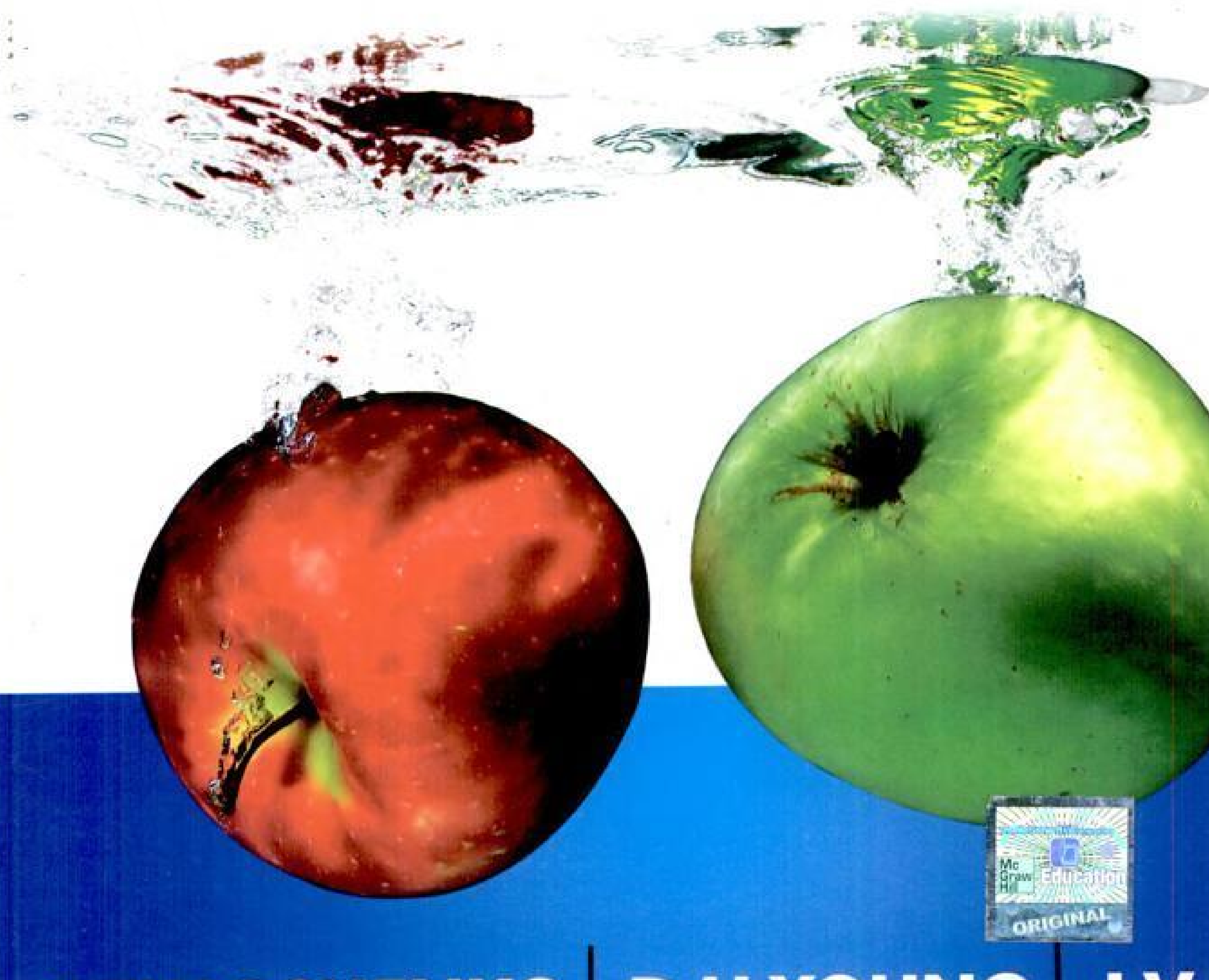
SPECIAL INDIAN EDITION

ENGINEERING MECHANICS

Revised Fourth Edition



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About the Adaptation Author

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Preface to the Adapted Edition

This book is the product of more than half a century of innovation in Engineering Mechanics education. When the first edition of Engineering Mechanics by S. Timoshenko and D.H. Young appeared in 1937, it was revolutionary among engineering mechanics textbooks with its emphasis on the fundamental principles of mechanics and how to apply them. The success of Engineering Mechanics with generations of students and educators throughout the world is a testament to the merits of this approach. In this revised fourth edition, SI units, which are most frequently used in mechanics, are introduced in Chapter 1 and are used throughout.

Objective

The main objective of a first course in mechanics should be to build a strong foundation, to acquaint the student with as many general methods of attack as possible, and to illustrate the application of these methods to practical engineering problems. However, it should avoid routine drill in the manipulation of standardized methods of solution. Such are the aims of this book. This text is designed for the first course in statics and dynamics offered in sophomore or junior year. It is hoped that this text will help the instructor achieve this goal.

General Approach

Scalar approach is used throughout the presentation of statics and dynamics. In Part One, statics, chapters are based on Force systems except principle of virtual work. In dynamics, emphasis is on rectilinear translation, curvilinear translation, rotation of a rigid body about a fixed axis and plane motion. Finally relative motion is dealt with in Part Two.

New to This Edition

While retaining the well-received approach and organization of the previous edition, the revised fourth edition offers the following new features and improvements:

- Each topic ends with a summary of the material covered in it.

- A set of review questions is included at the end of each topic. Short questions are added for the benefit of students which are useful for university examination.
- Multiple choice questions are given at the end of each topic to help the student prepare for competitive exams like IES, IAS (Prelims), GATE, etc.
- Important terms and concepts are added in each topic.
- Important formulae are added in each topic.
- Numerous new examples and new problems are added.

Acknowledgements

I want to extend my heartful thanks to my colleagues at Bapatla Engineering College, especially Asst. Prof. J. Girish, Prof. A.V. Narayanappa, for many stimulating discussions about engineering mechanics pedagogy and for their support and encouragement during the adaptation of this book. I am equally indebted to the Bapatla Engineering College students who have helped me learn what good teaching and good writing are, by showing me what works and what doesn't. I want to express special thanks to Sri Muppalaneni Sheshagiri Rao, President, Bapatla Educational Society and Prof. G. N. Rao, Principal, BEC, Prof. K. L. Prasad, H.O.D. Civil, BEC for their support while writing this book. I also want to express special thanks to Vibha Mahajan, Shukti Mukherjee and Mini Narayanan of Tata McGraw-Hill for their superb editorial guidance and vision. I want to thank my parents, wife and children for their support.

Feedback from professors and students, especially concerning errors or deficiencies in this edition are welcome. Comments and suggestions for further improvement of the text will be greatly appreciated.

J V Rao

Preface

The importance of mechanics in the preparation of young engineers for work in specialized fields cannot be overemphasized. The demand from industry is more and more for young men who are soundly grounded in their fundamental subjects rather than for those with specialized training. There is good reason for this trend: The industrial engineer is continually being confronted by new problems, which do not always yield to routine methods of solution. The man who can successfully cope with such problems must have a sound understanding of the fundamental principles that apply and be familiar with various general methods of attack rather than proficient in the use of anyone. It seems evident, then, that university training in such a fundamental subject as mechanics must seek to build a strong foundation, to acquaint the student with as many general methods of attack as possible, to illustrate the application of these methods to practical engineering problems, but to avoid routine drill in the manipulation of standardized methods of solution. Such are the aims of this book.

The content of the book is somewhat wider than can be covered in two courses of three semester hours or five quarter hours each. At the end of the discussion of statics, for example, there is a chapter on the principle of virtual work. The use of this principle results in great simplification in the solution of certain problems of statics, and it seems desirable to acquaint the student with its possibilities. At the end of the discussion of dynamics, there is a short chapter on relative motion, together with applications to engineering problems. These chapters can easily be omitted without introducing any discontinuity if there is insufficient time for them. Where time will not permit their consideration, they at least serve the purpose of indicating to the student that he has not exhausted the possibilities of the subject in his first encounter with it. Also, it is hoped that such material will be of value to those students especially interested in mechanics.

In many of our engineering schools, statics is given during the second semester of the sophomore year, before the student has studied integral calculus. For this reason Part One of this volume has been so written that, except for one or two sections that can easily be omitted, no knowledge of mathematics beyond the differential calculus is required. However, a free use of mathematics is made within these limits. Statics is probably the first course wherein the student has a chance to make practical use of his training in mathematics, and it seems important that he be not only given the opportunity but encouraged to use it to the full extent of its applicability.

The situation is usually quite different with dynamics. In some schools, for instance, this course does not immediately follow statics but is taken after strength of materials. Thus the students are more mature, and it seems justifiable in Part Two to make free use of the calculus and even some use of elementary differential equations. In this latter respect, however, the solutions are discussed in sufficient detail so that the student without special preparation in differential equations need have no difficulty.

Throughout Part Two the equations of motion are presented and handled as differential equations. Dynamics is not a subject to be handled superficially, and a too-arduous attempt to simplify its presentation can easily result in the fostering of false notions in the mind of the beginner. Besides helping to forestall such possible misconceptions, the use of the differential equation of motion, as such, possesses several other advantages: (1) It makes it possible, at the outset, to place proper emphasis upon the inherent difference between dynamical problems involving known motion and those involving known acting forces. (2) It makes practicable the discussion of certain problems of dynamics (such as vibration problems) which otherwise could be handled only in a very cumbersome manner, if at all. (3) It gives the student a foundation in dynamics upon which he can successfully build if he desires to pursue advanced study or to read current literature on the subject.

Since the student usually has his greatest difficulty in applying the principles and theorems that he has just learned to specific situations, special attention has been given to the selection and treatment of a series of illustrative examples at the end of each article. The purpose of these examples is twofold: (1) They are sometimes used as a medium of presentation of material not included in the text proper. (2) They are designed to set an example to the student in logical methods of approach to the solution of engineering problems. It is hoped that the examples will help the student to bridge the gap between mere cognizance of the general principles and the ability to apply them to concrete problems. Mastery in this respect is the. True goal of engineering education. The examples warrant as much attention from the student as the text material proper.

The solution of a problem in mechanics usually consists of three steps: (1) the reduction of a complex physical problem to such a state of idealization that it can be expressed algebraically or geometrically; (2) the solution of this purely mathematical problem; and (3) the interpretation of the results of the solution in terms of the given physical problem. It is too often the case that the student's attention is called only to the second step so that he does not see clearly the connection between this and the true physical problem. By successive development of these three steps in the solution of each illustrative example, it is hoped to lead the student to a realization of the full significance of mechanics, and also to encourage him to approach the solution of his own problems in a similar way.

Many of the illustrative examples are worked out in algebraic form, the answers being given simply as formulas. When numerical data are given, their substitution is made only in the final answer at the end. Such a procedure possesses several advantages, one of which is the training the student gets in reliable methods of checking answers. Two of the most valuable aids in checking the solution of a problem are the "dimensional check" and the consideration of certain limiting cases as logical extremes. The opportunity of making either of these checks is lost when given numerical data are substituted at the beginning of the solution. Another advantage of the algebraic solution is that it greatly enriches the possibilities of the third step in the solution of the problem, namely, significance of results. Finally, the algebraic solution is preferable if proper attention is to be given to numerical calculations, for only by having the result in algebraic form can it be seen with what number of figures any intermediate calculation must be made in order to obtain a desired degree of accuracy in the final result.

Since the first edition of "Engineering Mechanics" appeared in 1937, the authors' "Theory of Structures" and "Advanced Dynamics" have been published, and these later volumes now contain some of the more advanced material that was originally in "Engineering Mechanics." It is hoped that the three volumes taken together represent a fairly complete treatment of engineering mechanics and its applications to problems of modem structures and machines, at the same time leaving the present volume better suited to the undergraduate courses in statics and dynamics as given in our engineering schools today.

In the preparation of this fourth edition, the entire book has been thoroughly revised. In doing this, the authors have had these objectives: (1) simplification of the text proper, (2) improved arrangement of subject matter, and (3) deemphasis of the algebraic treatment of problems. Almost all problems throughout the book are now given with numerical data and numerical answers. Furthermore, the problem sets have been completely revised, and they contain a high percentage of new problems. The problems preceded by an asterisk present special difficulties of solution.

Various textbooks have been used in the preparation of this book, particularly in the selection of problems. In this respect, special acknowledgment is due the book "Collection of Problems of Mechanics," edited by J. V. Mestscherski (St. Petersburg, 1913), in the preparation of which the senior author took part. The authors also take this opportunity to thank their colleagues at Stanford University for many helpful suggestions in regard to this revision, in particular, Prof. Karl Klotter, who read some portions of the revision and made many valuable suggestions for improvement in this edition.

S Timoshenko
D H Young

Part One

STATICS

1
Introduction

1.1 ENGINEERING MECHANICS

The importance of mechanics in the preparation of young engineers for work in specialized fields cannot be overemphasized. Therefore, it is appropriate to begin with a brief explanation on the meaning of the term engineering mechanics and the role of this course in engineering education. Before defining engineering mechanics, we must consider the similarities and differences between science and engineering. In general terms, science is the knowledge that comes from observing facts about the universe carefully, carrying out experiments and making statements that are always true in particular conditions. On the other hand, engineering is the application of mathematics and science to the design and manufacture of items that benefit humanity. Design is the basic concept that distinguishes engineers from scientists. Accreditation Board of Engineering and Technology (ABET) defines engineering design as the process of devising a system, component or process to meet desired needs.

Mechanics may be defined as the science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Engineering mechanics is the branch of engineering that applies the principles of mechanics to design, which must take into account the effect of forces. The goals of engineering mechanics courses is to build a strong foundation, to acquaint the student with as many general methods of attack as possible, to illustrate the application of these methods to practical engineering problems, but to avoid routine drill in the manipulation of standardized methods of solution.

Depending upon the nature of the problems treated, mechanics is divided into statics and dynamics. Statics is the study of forces and conditions of equilibrium of material bodies subjected to the action of forces. Dynamics is the study of motion of rigid bodies and their correlation with the forces causing them. Dynamics is divided into kinematics and kinetics. Kinematics deals with the space–time relationship of a given motion of a body and not at all with the forces that cause the motion. Kinetics studies the laws of motion of material bodies under the action of forces or Kinetics is the study of the relationship between the forces and the resulting motion.

Of course, engineering mechanics is an integral component of the education of engineers whose disciplines are related to the mechanical behaviour of bodies or fluids. Such behaviour is of interest to aeronautical, civil, chemical, electrical, mechanical, metallurgical, mining and textile engineers. A sound training in engineering mechanics continues to be one of the most important aspects of engineering education due to the interdisciplinary character of many engineering applications (e.g., spaceships, robotics and manufacturing). It is appropriate to conclude that the subject of engineering mechanics lies at the core of all engineering analysis.

Idealisation in Mechanics

Mathematical models or idealisations are used in mechanics to simplify the application of theory. Now, we will define some of the important idealisations. Others will be discussed at points where they are needed.

Continuum Continuum may be defined as a continuous distribution of matter with no voids or empty spaces. Each body is made up of atoms and molecules. The matter is assumed as continuously distributed since the behaviour of atoms and molecules are too complex to deal with. It is used to study the measurable behaviour.

Particle In the abstract sense, a particle is a point mass or a material point. A body whose dimensions can be neglected in studying its motion or equilibrium may be treated as a particle.

Examples While studying the planetary motion, sun and planets; moon as seen from the earth; a satellite orbiting the earth and seen by an observer on the earth are considered as particles.

System of Particles A system of particles is an idealization of point masses. A system of particles is constituted, when two or more bodies are represented by particles and are dealt with together.

Examples: Planetary system; the structure of atom, i.e., electron–proton–neutron.

Rigid Body A rigid body is the one in which the distance between any two arbitrary points is invariant. Actually, solid bodies are not rigid, but deform under the action of forces. It is assumed to be rigid, if the deformation is negligible compared to the size of the body.

Examples: A wheel of a car; a lever supporting two weights at its ends.

Basic Concepts

For the investigation of problems of engineering mechanics we must introduce the concept of space, mass, time and force. The basic concepts used in mechanics are space, time, mass and force. These basic concepts cannot be truly defined but are developed for axiomatic thinking and mutual understanding.

Space Space refers to the geometric region occupied by bodies. The positions of bodies are described by linear and angular measurements relative to a coordinate system. The concepts of point, direction, length and displacement are required for measurements and locations in space. For example, a point is just an exact indication of a location in space, requiring no space at all for itself. Length is concept for describing the size of a body quantitatively by comparing it with a second body of known size. For two-dimensional problems, two independent co-ordinates are needed. For three-dimensional problems, three independent co-ordinates are required.

Mass Mass is the quantity of matter in a body. Matter refers to the substance of which physical bodies are composed. Each body is made up of atoms and molecules. Mass can also be regarded as a measure of the inertia of a body, which is its resistance to a change of motion.

Time Time is the measure of sequence of events. Time is related to the concepts of before, after and simultaneous occurrence of two or more events. Time is a basic quantity in dynamics and it is not directly involved in the analysis of statics problems.

Force Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction. Force is a vector quantity and its properties are discussed in detail in Chapter 2. In Newtonian mechanics, space, time and mass are absolute concepts, i.e., independent of each other. The concept of force is related to the mass of the body with Newton's Second law.

System of Forces

When several forces of various magnitudes and directions act upon a body, they are said to constitute a system of forces. The system of forces may be classified according to the orientation of the lines of action of the forces as follows: Force Systems in Plane: System of forces consists of a set of forces with their lines of action lying in the same plane. Force Systems in Space: System of forces consists of a set of forces with their lines of action lying in the space.

Both force systems in plane and force systems in space can further be classified into (i) concurrent force system, (ii) parallel force systems and (iii) non-concurrent or general force system. The classification of force systems is shown as in Fig. 1.1. In general, we can have six types of force systems.

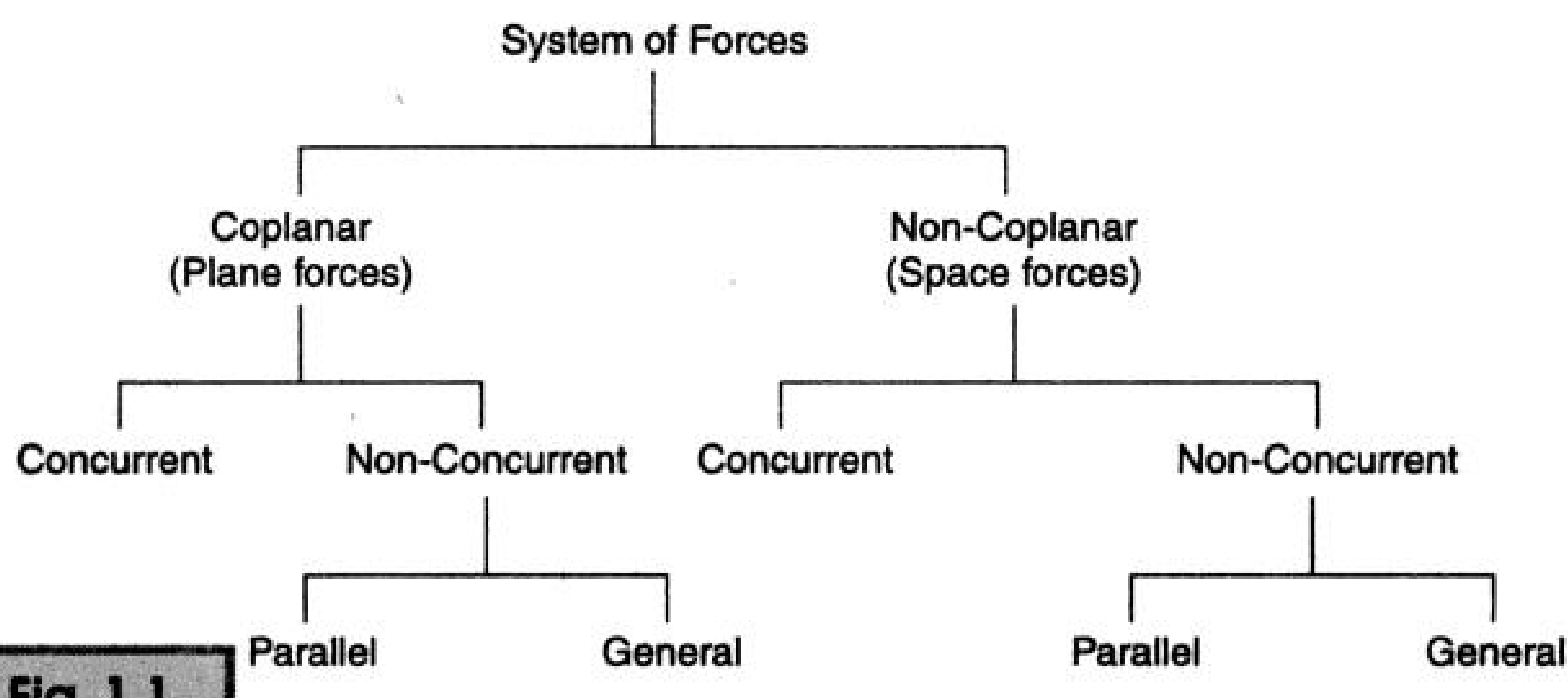

Fig. 1.1

Concurrent Force System in a Plane In this system, the lines of action of all forces pass through a single point and forces lie in the same plane (Fig. 1.2).

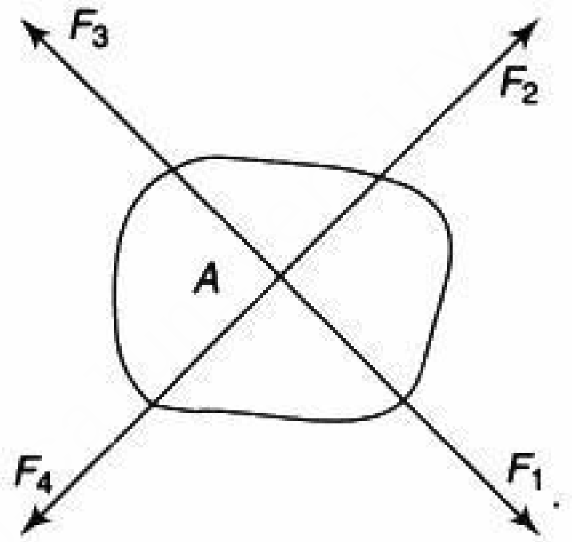

Fig. 1.2

Parallel Force System in a Plane In this system, the lines of action of all forces lie in the same plane and are parallel to each other (Fig. 1.3).

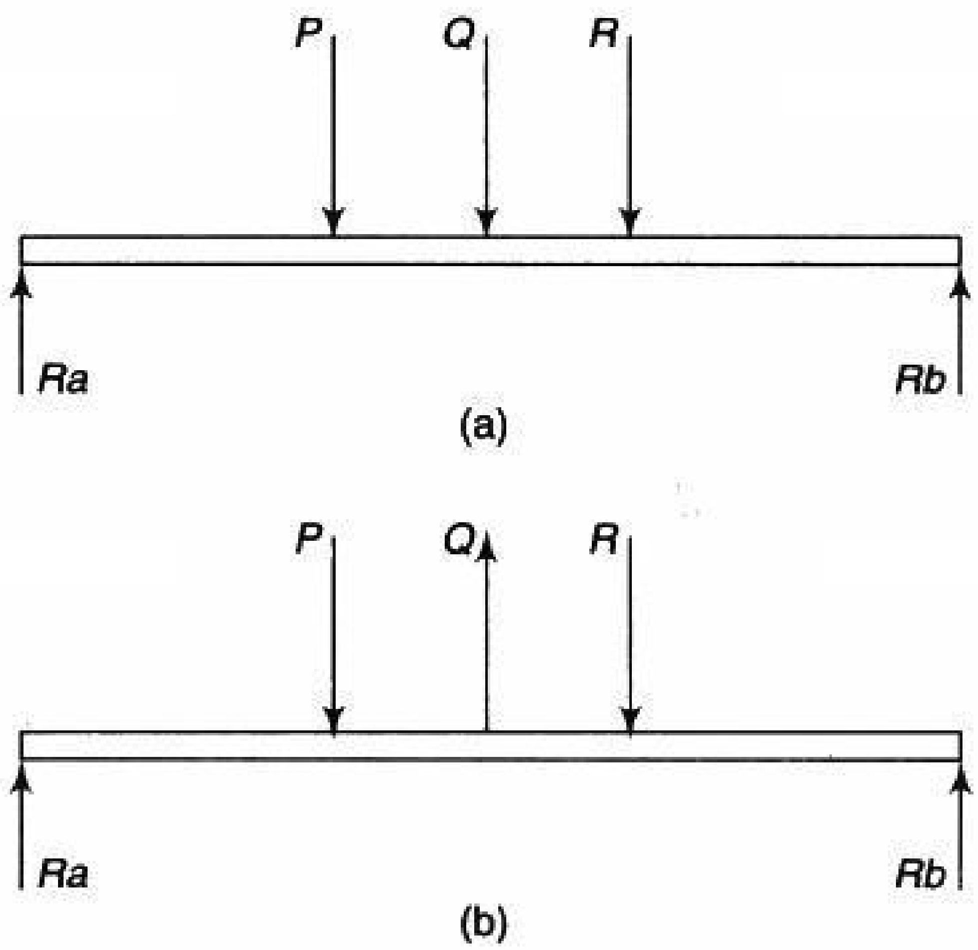

Fig. 1.3

General Force System in a Plane The lines of action of these forces lie in the same plane but they are neither parallel nor concurrent (Fig. 1.4).

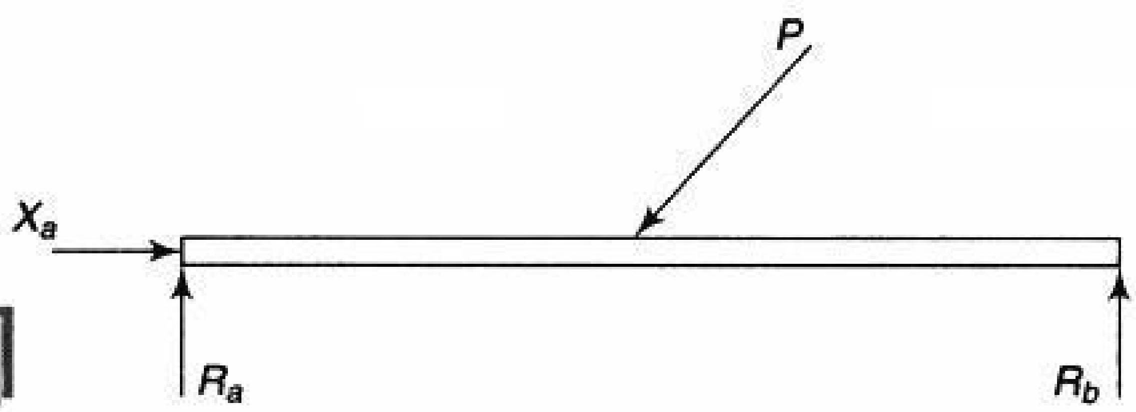

Fig. 1.4

Concurrent Force System in Space The lines of action of all forces pass through a single point but not lie in the same plane. Tripod carrying a camera as shown in Fig. 1.5 is an example.

Fig. 1.5

Parallel Force System in Space The lines of action of all forces are parallel to each other, but not lie in the same plane (Fig. 1.6).

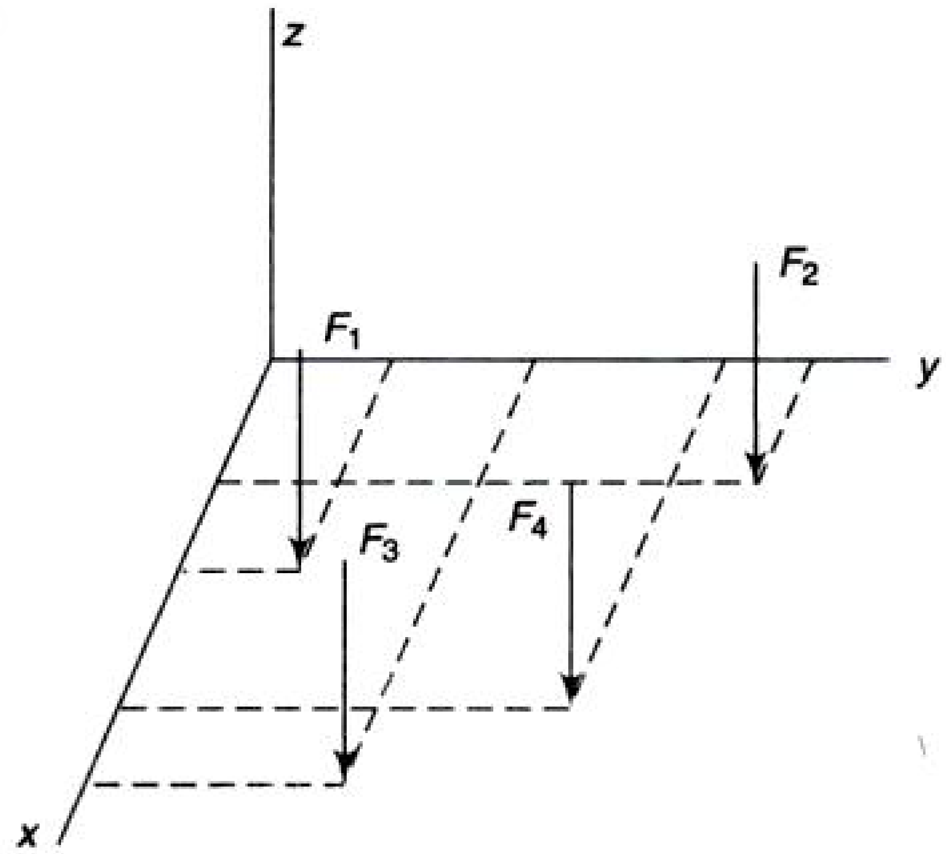

Fig. 1.6

General Force System in Space The lines of action of these forces do not lie in the same plane and they are neither parallel nor concurrent (Fig. 1.7).

Concurrent force systems can act on a particle or a rigid body. Parallel and

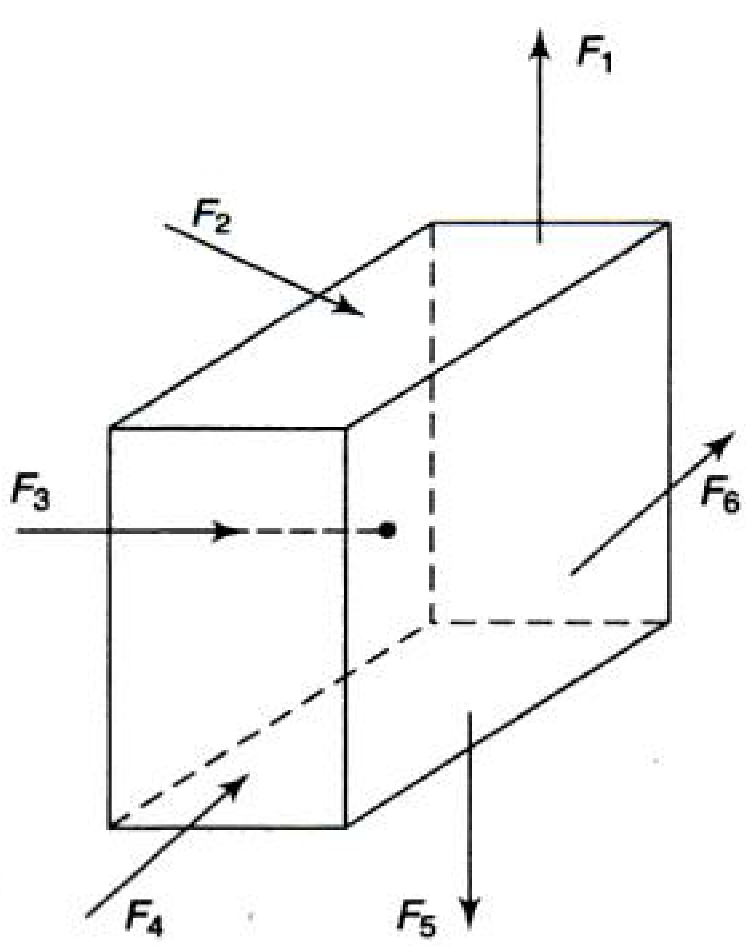

Fig. 1.7

general force systems can act only on a system of particles, a rigid body or a system of rigid bodies. In this book, we will study statics according to force systems.

Important Terms and Concepts

Engineeering	Mechanics	Scientists	Engineeers
Design	Statics	Dynamics	Kinematics
Kinetics	Idealizations	Continuum	Particle
System of particles	Rigid body	Basic concepts	Mass
Time	Space	Force	Specifications of a force
System of forces	Kinds of forces		

SUMMARY

- Engineering mechanics may be defined as the science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Depending upon the nature of the problems treated, mechanics is divided into statics and dynamics.
- Statics is the study of the forces and the conditions of equilibrium of material bodies subjected to the action of forces.
- Dynamics is the study of motion of rigid bodies and their correlation with the forces causing them. Dynamics is divided into kinematics and kinetics.
- Kinematics deals with the space–time relationship of a given motion of a body and not at all with the forces that cause the motion.
- Kinetics studies the laws of motion of material bodies under the action of forces.
- Mathematical models or idealizations are used in mechanics to simplify the application of theory.
- Continuum may be defined as a continuous distribution of matter with no voids or empty spaces.
- A particle is a point mass or a material point.
- A system of particles is an idealization of point masses. A system of particles is constituted, when two or more bodies are represented by particles and are dealt with together.
- A rigid body is the one in which the distance between any two arbitrary points is invariant.
- The basic concepts used in mechanics are space, time, mass and force. These basic concepts cannot be truly defined but are developed for axiomatic thinking and mutual understanding.
- Space refers to the geometric region occupied by bodies.
- Mass is the quantity of matter in a body.
- Time is the measure of sequence of events.
- Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction. Force is a vector quantity.
- In Newtonian mechanics, space, time and mass are absolute concepts, i.e., independent of each other. The concept of force is related to the mass of the body with Newton's Second law.
- When several forces of various magnitudes and directions act upon a body, they are said to constitute a system of forces.

PRACTICE SET 1.1

Review Questions

1. Define engineering mechanics.
2. State and explain the idealisations in mechanics.
3. What are the basic concepts used in engineering mechanics?
4. Explain the term 'system of forces'.
5. Distinguish between statics and dynamics.

Objective Questions

1. Select the incorrect statement.
   (a) Statics is the study of forces and the conditions of equilibrium of material bodies subjected to the action of forces.
   (b) Dynamics is the study of motion of bodies and their correlation with the forces causing them.
   (c) Kinematics deals with the space–time relationship of a given motion of a body.
   (d) Kinematics deals with the relationship between the forces and the resulting motion.

   [*Ans.* (d)]
2. Select the incorrect statement.
   (a) Continuum may be defined as a continuous distribution of matter with no voids.
   (b) A rigid body is the one in which the distance between any two arbitrary points is variant.
   (c) A particle is a point mass.
   (d) A system of particles is an idealization of point masses.

   [*Ans.* (b)]
3. While studying the planetary motion, sun and planets are considered as
   (a) Deformable body (b) Rigid body
   (c) Particles (d) None of the above.

   [*Ans.* (c)]
4. Which of the following is the basic concept of mechanics?
   (c) Charge (b) Power
   (c) Force (d) Energy

   [*Ans.* (c)]

1.2 UNITS AND DIMENSIONS

Unit is defined as the numerical standard used to measure the qualitative dimension of a physical quantity. When mass or force, length and time are accepted as basic quantities, then all other quantities are secondary or derived quantities in terms of these basic quantities.

Fundamental Units and Derived Units

The units in which the fundamental quantities are measured are called as fundamental or basic units. The three primary units basic to mechanics are length, time and mass.

The derived units are the units of derived physical quantities, which are expressed in terms of the fundamental units. Examples: Area, Volume, Force, Velocity, etc.

Coherent System of Units

A coherent system of units is the one in which the units of derived quantities are obtained as multiples or sub-multiples of certain basic units.

SI Units

The International System of Units, abbreviates SI (from the French, Systeme International d'Unites), has been accepted throughout the world and is a modern version of the metric system. In SI units, length in meters (m), mass in kilograms (kg) and time in seconds (s) are selected as the base units and force in newtons (N) is derived from Newton's Second law. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2.

Primary standards for the measurements of length, mass and time are as follows.

Length The meter is defined as 1 650 763.73 wavelengths of a certain radiation of the krypton-86 atom at 15°C and 76 cm of mercury.

Mass The kilogram is defined as the mass of a platinum Iridium cylinder of diameter equal to its height kept at the International Bureau of weights and measures near Paris France.

Time The second is defined as the duration of 9 192 631 770 periods of the radiation of a certain state of the cesium-133 atom.

Note ➻ SI units are absolute system of units (units of mass, length and time are basic units), which are independent of the location where the measurements are made.

The SI units consist of 7 base units, 2 supplementary units and a number of derived units. While studying engineering mechanics, it is sufficient if one knows about the three basic units given below in Table 1.1. The SI derived units with new names are given in Table 1.2. SI units of some common physical quantities are summarized in Table 1.3. When a numerical quantity is either very large or very small, units used to define its size may be modified by using a prefix. Some of the multiple or submultiples used in the SI system are shown in Table 1.4.

Table 1.1

Quantity	*SI units*	*Symbol*
Mass	Kilogram	kg
Length	Meter	m
Time	Second	s

Table 1.2

Derived unit	*Symbol*	*Physical quantity*
Newton	N = kg m/s^2	Force
Joule	J = N m = kg m^2/s^2	Energy, Work, Heat
Watt	W = J/s = N m/s = kg m^2/s^2	Power
Pascal	Pa = N/m^2 = kg m/s^2	Pressure, Stress
Hertz	Hz = s^{-1}	Frequency

Table 1.3

Physical quantity	*Unit*	*Symbol*
Acceleration	Metre/second^2	m/s^2
Angular acceleration	Radian/second^2	rad/s^2
Angular displacement	Radian	rad
Angular momentum	Kilogram metre^2/second	kg m^2/s
Angular velocity	Radian/second	rad/s
Area	Square metre	m^2
Couple, moment	Newton metre	N m
Density	Kilogram/metre^3	kg/m^3
Displacement	Metre	m
Energy	Joule	J
Force	Newton	N
Frequency	Per second	Hz
Length	Metre	m
Mass	Kilogram	kg
Momentum	Kilogram metre/second	kg m/s (=N s)
Moment of inertia of mass	Kilogram metre^2	kg m^2
Plane angle	Radian	rad
Power	Watt	W
Pressure	Pascal	Pa
Speed	Metre/second	m/s
Time	Second	s

Table 1.4

Multiplication factor	*Prefix*	*Symbol*
10^{12}	Tera	T
10^{9}	Giga	G
10^{6}	Mega	M
10^{3}	Kilo	k
10^{-3}	Milli	m
10^{-6}	Micro	μ
10^{-9}	Nano	n
10^{-12}	Pico	p

Rules for Use of SI Symbols

The following rules are given for the proper use of the various symbols in SI units.

1. A symbol is never written with a plural "s".
2. Symbols are always written in lowercase letters except the symbols named after an individual e.g., N and J.

3. Kilogram is written as kg and not as kgm, kgf, etc. Similarly, second as s, not sec. or sec, etc. No full stops, dots or dashes should be used. For example, moment is in N m, not N.m, N-m, etc.
4. It is permissible that one space be left between any two unit symbols, e.g., kg m^2/s, m s.
5. No space be left after a multiple or submultiple symbol, e.g., kJ/kg K.
6. Always leave a space between the number and the unit symbol, e.g., 3 m, 1500 N.
7. For numbers less than unity, zero must be put on the left of the decimal, e.g., 0.30 m. For large numbers exceeding five figures, one space after every three digits counting from the right end must be left blank without any commas, e.g., 1 500 375 is the correct way of writing the number.
8. The exponential power represented for a unit having a prefix refers to both the unit and its prefix. For example, $mm^2 = (mm)^2 = mm . mm$.
9. Represent the numbers in terms of the base or derived units by converting all prefixes to powers of, while performing calculations. The final result should then be expressed using a single prefix.
10. In general, avoid the use of a prefix in the denominator of composite units. Exception for this is the base unit, kilogram. For example, do not write N/μm, but rather MN/m.
11. Compound prefixes should not be used, e.g., GμN (giga-micro-newton) should be expressed as kN since 1 GμN = 1 (10^9) (10^{-6}) N = 1 (10^3) N = 1 kN.

Dimensions

Dimensional analysis deals with dimensions of quantities (Table 1.5).

Table 1.5

Base unit	*Dimension*
Mass	M
Length	L
Time	T
Temperature	K
Electric current	A
Luminous intensity	Cd
Amount of substance	mol

Dimensional Formula Dimensional formula is a formula in which the given physical quantity is expressed in terms of the fundamental quantities raised to suitable power.

In order to completely define a physical quantity, the following are to be known: (1) the unit of the quantity, (2) the number of times the unit contained in that quantity, i.e., the numerical value. Area is represented by L^2 and area has 2 dimensions in length. Volume is represented by L^3 and has 3 dimensions in length.

The dimensional formula of some of the derived quantities are given in Table 1.6.

Table 1.6

S.No.	Physical quantity	Expression	Dimensional formula
1.	Displacement (S)	Distance	L
2.	Velocity (V)	Distance/Time	LT^{-1}
3.	Acceleration (a)	Velocity/Time	LT^{-2}
4.	Force (F)	Mass × Acceleration	MLT^{-2}
5.	Momentum	Mass × Velocity	MLT^{-1}
6.	Impulse	Force × Time	MLT^{-1}
7.	Work or Energy	Force × Displacement	ML^2T^2
8.	Power	Work/Time	ML^2T^3
9.	Pressure	Force/Area	$ML^{-1}T^1$
10.	Frequency	No. of vibrations/Time	T^{-1}
11.	Angular Velocity	Angle/Time	T^{-1}

Uses of Dimensional Formula The dimensional formula is used

(i) to check the correctness of a given equation,

(ii) to derive equations for physical quantites involved in the problem.

Dimensionless Quantities Dimensionless quantities are the quantities that are the ratio of two quantities having the same dimensional formula.

Law of Dimensional Homogeneity

The law of dimensional homogeneity states that all equations which describe the physical processes must be dimensionally homogeneous. In other words, no equation in which the seperate terms have different dimensions can be physically valid. Dimensionally homogenous equations are also said to be dimensionally correct. In a dimensionally correct, all the terms on the left and right side will have the same dimensions.

Law of dimensional homogeity is used (a) to verify whether the given equation is dimensionally correct or not (b) to find the dimensions of certain terms in a dimensionally homogeneous equation.

Important Terms and Concepts

Unit	Fundamental unit	Derived unit	Coherent system of units
SI units	Rules for use of SI units	Dimensions	Dimensional formula
Law of dimensional homogeneity			

SUMMARY

- Unit is defined as the numerical standard used to measure the qualitative dimension of a physical quantity.
- The units in which the fundamental quantities are measured are called as fundamental or basic units. The three primary units basic to mechanics are length, time and mass.
- The derived units are the units of derived physical quantities, which are expressed in terms of the fundamental units (examples: Area, Volume, Force, Velocity, etc.).

- The International System of Units, abbreviates SI (from the French, Systeme International d'Unites), has been accepted throughout the world and is a modern version of the metric system. In SI units, length in meters (m), mass in kilograms (kg) and time in seconds (s) are selected as the base units.
- Dimensional formula is a formula in which the given physical quantity is expressed in terms of the fundamental quantities raised to suitable power.
- The law of dimensional homogeneity states that all equations which describe the physical processes must be dimensionally homogeneous.

PRACTICE SET 1.2

Review Questions

1. Define unit.
2. Differentiate between fundamental units and derived units.
3. Explain SI units.
4. Define dimensional formula.
5. What are the uses of dimensional formula?
6. State the law of dimensional homogeneity.

Objective Questions

1. Joule is the unit of
   (a) Power (b) Moment
   (c) Momentum (d) Work
   [*Ans.* (d)]
2. In SI units, the units of force and power are respectively
   (a) Newton and Watt (b) Newton and Joule
   (c) Newton and Pascal (d) Newton and Hertz
   [*Ans.* (a)]
3. Which of the following is dimensionless quantity?
   (a) Force (b) Velocity
   (c) Pure number (d) Frequency
   [*Ans.* (c)]
4. The dimension of linear momentum in MLT system is
   (a) MLT (b) MLT^{-1}
   (c) MT (d) LT^{-1}
   [*Ans.* (b)]
5. The dimension of pressure in MLT system is
   (a) $ML^{-1}T^{-1}$ (b) $ML^{-1}T^{2}$
   (c) $ML^{-1}T^{-2}$ (d) $ML^{2}T^{2}$
   [*Ans.* (c)]
6. Consider the following statements.
   I. The dimensional formula is used to derive the equations for physical quantities involved in the problem.
   II. Dimensionless quantities are the quantities that are the ratio of two quantities having the same dimensional formula.
   III. The law of dimensional homogeneity states that all equations which describe theprocesses must be dimensionally homogeneous.

   Out of these statements.
   (a) Only I is correct. (b) I and II are correct
   (c) All are incorrect (d) All are correct.
   [*Ans.* (d)]

1.3 METHOD OF PROBLEM SOLUTION AND THE ACCURACY OF SOLUTIONS

The solution of a problem in engineering mechanics usually consists of three steps: (1) the reduction of a complex physical problem to such a state of idealization that it can be expressed algebraically or geometrically; (2) the solution of this purely mathematical problem; and (3) the interpretation of the results of the solution in terms of the given physical problem.

The mastery of the principles of engineering mechanics will be reflected in the ability to formulate and solve problems. There is no simple method for teaching problem-solving skills. To develop the analytical skills that are so necessary for success in engineering, a considerable amount of practice in solving problems is necessary. For this reason, a relatively large number of examples and practice set are given throughout the text. An effective method of attack on statics problems is essential. Each solution should proceed with a logical sequence of steps from hypothesis to conclusion. Its representation should include a clear statement of the following parts, each clearly identified :

1. Given data: After carefully reading the problem statement, list all the data provided.
2. Results desired or find: State precisely the information that is to be determined.
3. Necessary diagrams: If a figure is required, sketch it neatly and approximately to scale.
4. Calculations or solution: Solve the problem, showing all the steps that you used in the analysis. Work neatly so that others can easily follow your work.
5. Answers and conclusions or validate: Many times, an invalid solution can be uncovered by simply asking yourself, "Does the answer make sense?"

The accuracy of the solution of a problem depends upon the two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed. The data given in a problem should be assumed known with a comparable degree of accuracy. The number of significant figures in an answer should be no greater than the number of figures, which can be justified by the accuracy of the given data. A practical rule is to use 4 figures to record numbers beginning with a "1" and 3 figures in all other cases.

2

Concurrent Forces in a Plane

2.1 PRINCIPLES OF STATICS

Statics deals with the conditions of equilibrium of bodies acted upon by forces and is one of the oldest branches of science. Some of its fundamental principles date back to the Egyptians and Babylonians who used them in the solution of problems encountered in building the famous pyramids and old temples.[1] The earliest writings on the subject were left by Archimedes (287–212 B.C.), who formulated the laws of equilibrium of forces acting on a lever and also some principles of hydrostatics. However, the principles from which the subject in its present form is developed were not fully stated until the later part of the seventeenth century and are mainly the work of Stevinus, Varignon and Newton, who were the first to use the principle of the parallelogram of forces.

Statics deals with the conditions of equilibrium of bodies acted upon by forces.

Rigid Body

We shall be mostly concerned in this book with problems involving the equilibrium of rigid bodies. Physical bodies, such as we have to deal with in the design of engineering structures and machine parts, are never absolutely rigid but deform slightly under the action of loads which they have to carry. Consider, for example, the lever shown in Fig. 2.1(a). Under the action of the two equal weights at the ends, the bar bends slightly over the fulcrum and the distance of each weight from the fulcrum is decreased by a very small amount. In discussing the equilibrium of the lever (equal weights at equal distances from the fulcrum are in equilibrium), we may safely ignore this deformation and assume that the lever is a rigid body which remains straight, as shown in Fig. 2.1(b). That is, we assume that the distance of each weight from the fulcrum is half the length of the bar. This illustrates the significance of the assumption of rigid bodies in dealing with static equilibrium.

[1]For historical data regarding the development of statics, see Ernst Mach, "Science of Mechanics," Open Court Publishing Company, Chicago, 1902.

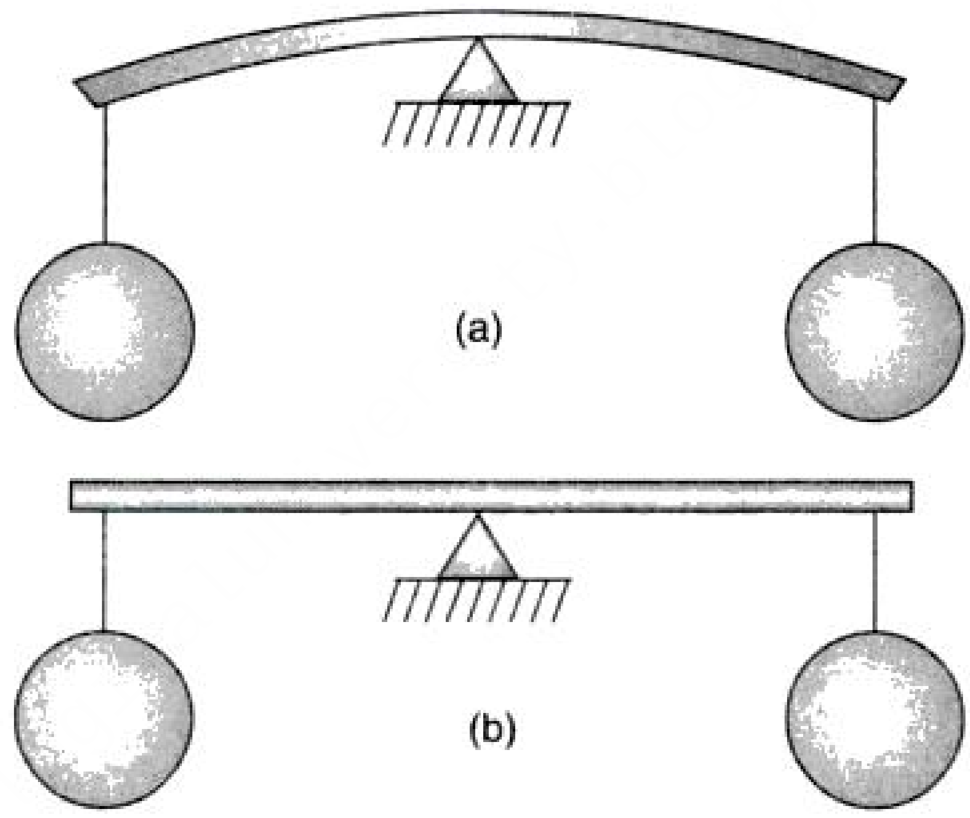

Fig. 2.1

*A **rigid body** is defined as a definite quantity of matter the parts of which are fixed in position relative to one another. The physical bodies are never absolutely rigid but deform slightly under the action of loads, which they have to carry. If the deformation is negligible when compared with the size of the body, it is assumed to be rigid.*

If we are interested in the *strength* of the lever in Fig. 2.1, or the amount of sag, the deformation represented by the bent form becomes important and must be taken into account. Problems in which the effect of small deformations of physical bodies must be taken into account are generally treated in books on *strength of materials* or *theory of elasticity*.

Problems dealing with the conditions of equilibrium of nonrigid bodies, such as liquids and gases, are usually treated in books on *fluid mechanics*. These will also not be considered here, except in so far as they may be involved in determining the pressure or loading exerted on rigid bodies that we do have under consideration.

Force

For the investigation of problems of statics we must introduce the concept of *force*, which may be defined as any action that tends to change the state of rest or motion of a body to which it is applied. A force or motion acting on a rigid body produces one or both of the following effects: (i) linear displacement, (ii) angular (rotating) displacement. These effects essentially result in a change in the state of rest or motion of a body.

***Force** may be defined as any action that tends to change the state of rest or motion of a body to which it is applied.*

There are many kinds of force, such as gravity force with which we are all familiar and the simple push or pull that we can exert upon a body with our hands. Other examples of force are the gravitational attraction between the sun and planets, the tractive effort of a locomotive, the force of magnetic attraction, steam or gas pressure in a cylinder, wind pressure, atmospheric pressure and frictional resistance between contiguous surfaces.

Weight is the force of gravitational attraction of the earth on a body.

Hydrostatic pressure: When a body such as a dam impounds water, the water exerts a force on th e imppounding bodywhich is distributed on the area of its contact with the body. This is known as hydrostatic pressure.

Gas pressure: Gas confined under pressure in a container exerts pressure over the entire area of the container, called gas pressure.

Earth pressure: Earth piled up against a wall exerts pressure on it called the earth pressure.

Wind pressure: Bodies exposed to wind are subjected to a force distributed over their exposed area, called the wind pressure.

The pull of gravity is one of the most common examples of force with which we shall have to deal. Given a ball that hangs by a string [Fig. 2.2(a)], we say that the ball pulls on the string with a force W equal to its weight. This force is applied to the string at point B and acts vertically downward.

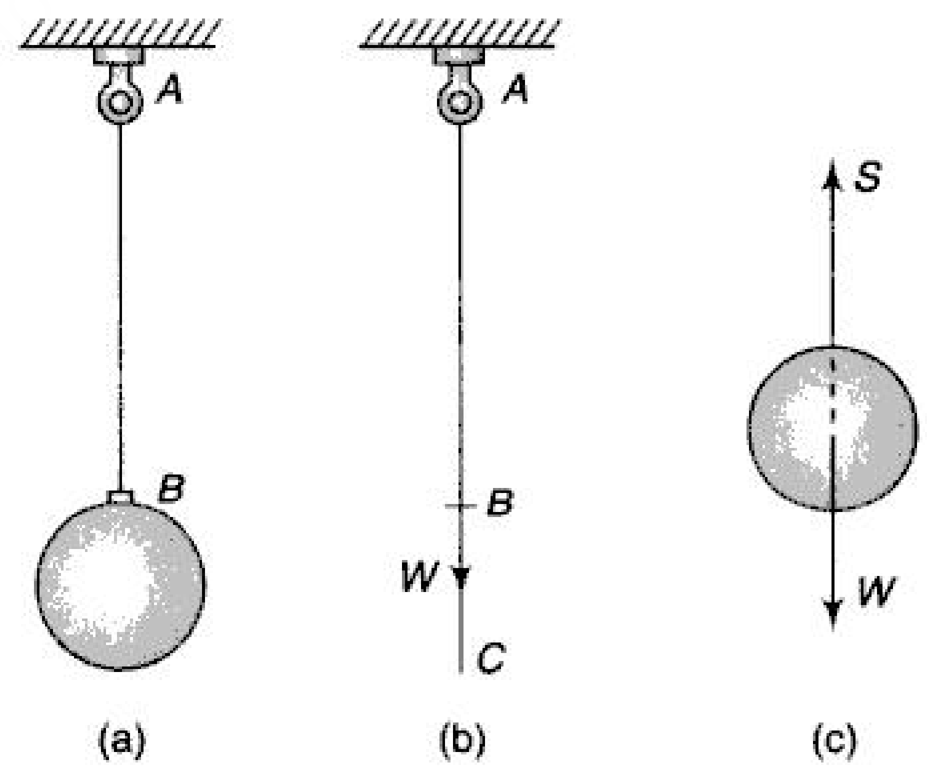

Fig. 2.2

From the above example, we see that for the complete definition of a force we must know (1) its magnitude, (2) its point of application, and (3) its direction. These three quantities which completely define the force are called its specifications or characteristics.

The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction.

The SI units used by engineers to measure the magnitude of a force are the Newton (N) and its multiple the kilonewton (kN), equal to 1000 N. The Newton is a derived unit. One Newton is defined as the force, which gives an acceleration of 1 m/s^2 to a mass of 1 kg.

The point of application of a force, acting upon a body is that point in the body at which the force can be assumed to be concentrated. Physically, it will be impossible to concentrate a force at a single point; i.e., every force must have some finite area or volume over which its action is distributed. For example, the force W exerted by the ball upon the string AB in Fig. 2.2 is in reality distributed over the small cross-sectional area of the string. Likewise, the gravity force, which the earth exerts on the ball, is distributed throughout the volume of the ball. However, we often find it convenient to think of such distributed force as being concentrated at a single point of application wherever this can be done without sensibly changing the effect of the force on the conditions of equilibrium. In the case of gravity force distributed throughout the volume of a body, the point of application at which the total weight can be assumed to be concentrated is called the *center of gravity* of the body.

Concentrated force or a point load is a force acting over a very small area.

Distributed force is a force distributed over a length or an area or a volume.

The direction of a force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied. This line is called the *line of action* of the force. The force of gravity, for example, is always directed vertically downward. Again, in the case of a force exerted upon a body by a flexible string, the string defines the line of action of the force. Thus the string *AB* in Fig. 2.2 pulls vertically downward on the hook at *A*.

Any quantity, such as force, that possesses direction as well as magnitude is called a *vector quantity* and can be represented graphically by a segment of a straight line, called a *vector.* For example, in Fig. 2.2(b), we can represent the force that the ball exerts on the string by the straight-line segment *BC*, the length of which, to some convenient scale, shows the magnitude of the force and the vertical downward direction of which, indicated by the arrow, shows the direction of the force. Point *B* is called the *beginning* of the vector; and point *C*, the *end*. Either the beginning or the end of a vector may be used to indicate the point of application of the force. With the beginning and the end of a vector indicated by letters [as *B* and *C* in Fig. 2.2(b)] we shall designate the vector by the symbol $\overline{BC}$, which defines it specifically as acting from *B* toward *C*.

Representation of Force Graphically a force may be represented by the segment of a straight line. The straight line represents the line of action of the force 1 kN and its length represents its magnitude. The direction (or sense) of the force is indicated by placing an arrow head on this straight line (Fig. 2.3). Either the head or the tail may be used to indicate the point of application of a force. Note that all the forces involved must be represented consistently.

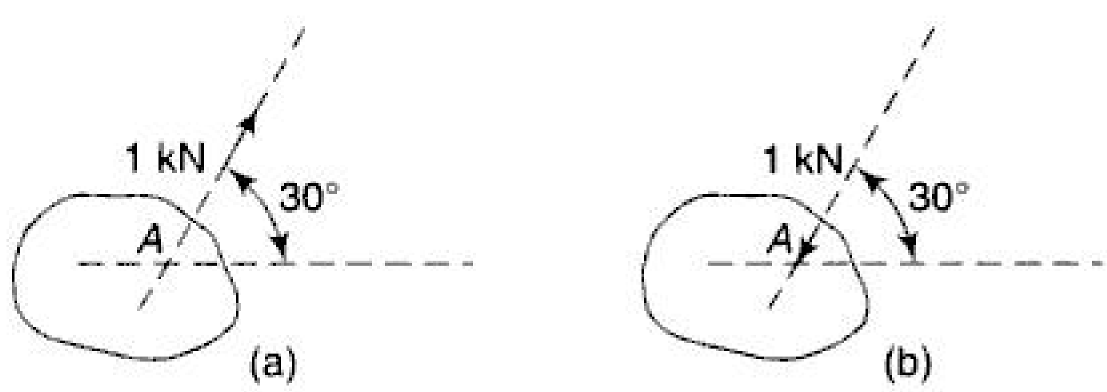

Fig. 2.3

Parallelogram of Forces When several forces of various magnitudes and directions act upon a body, they are said to constitute a *system of forces*. The general problem of statics consists of finding the conditions that such a system must satisfy in order to have equilibrium of the body. The various methods of solution of this problem are based on several axioms, called the *principles of statics*. We begin with the principle of the *parallelogram of forces*, first employed indirectly by Stevinus in 1586 and finally formulated by Varignon and Newton in 1687.

Composition of Two Forces The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces.* Here we will discuss the reduction of a given system of forces, i.e., two forces to the simplest system that will be its equivalent, i.e., resultant with the help of parallelogram law.

Parallelogram Law: *If two forces, represented by vectors $\overline{AB}$ and $\overline{AC}$, acting under an angle* α [Fig. 2.4(a)] *are applied to a body at point A, their action is equivalent to the action of one force, represented by the vector $\overline{AD}$, obtained as the diagonal of the parallelogram constructed on the vectors $\overline{AB}$ and $\overline{AC}$ and directed as shown in the figure.*

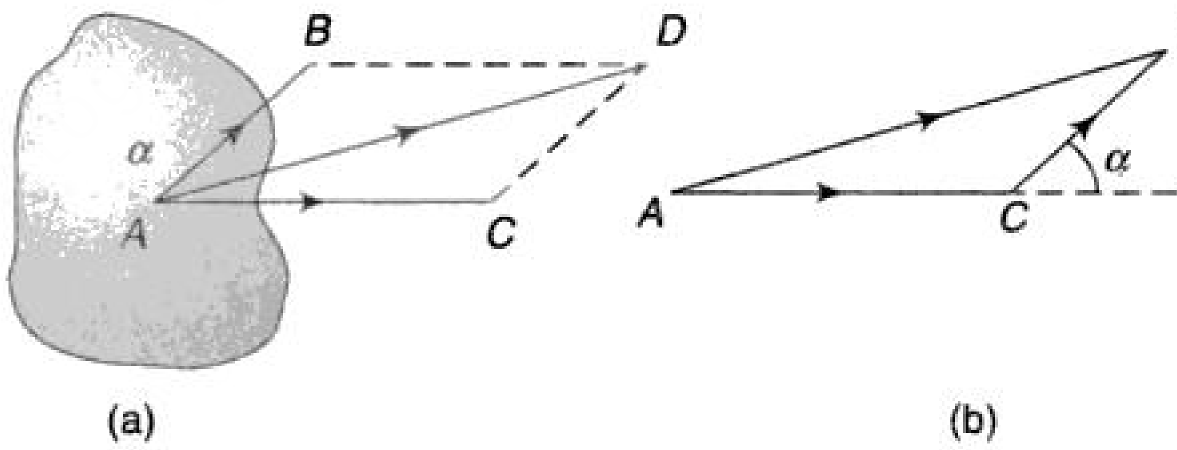

Fig. 2.4

The force $\overline{AD}$ is called the resultant of the two forces $\overline{AB}$ and $\overline{AC}$. The forces $\overline{AB}$ and $\overline{AC}$ are called *components* of the force $\overline{AD}$. Thus a force is equivalent to its components, and vice versa.

Instead of constructing the parallelogram of forces, the resultant can be obtained also by constructing the triangle *ACD*, as shown in Fig. 2.4(b). Here we take the vector $\overline{AC}$ and from its end *C* draw the vector $\overline{CD}$, equal and parallel to the vector $\overline{AB}$. Then the third side $\overline{AD}$ of the triangle gives the resultant, being directed from *A*, the beginning of the vector $\overline{AC}$, to *D*, the end of the vector $\overline{CD}$. The vector $\overline{AD}$, when obtained in this way, is called the *geometric sum of* the vectors $\overline{AC}$ and $\overline{CD}$. Thus, the magnitude and direction of the resultant of two forces, applied to a body at point *A*, may be obtained as the geometric sum of the two vectors representing these forces. Its point of application, of course, is also point *A*. Since the vectors in Fig. 2.4(b) do not show the points of application of the forces that they represent, they are called *free vectors.* The triangle *ACD* is called a *triangle of forces.*

If two forces $\overline{AB}$ and $\overline{AC}$ act under a very small angle [Fig. 2.5(a)], the triangle of forces [Fig. 2.5(b)] becomes very narrow and we conclude that, in the limiting case, where the two forces act along the same line and in the same direction, their resultant is equal to the sum of the forces and acts in the same direction. In the same manner it can be shown that, if two forces act along the same line in opposite directions, their resultant is equal to the difference between the forces and acts in the direction of the larger force. By taking one direction as positive and the other as negative along the common line of action of two forces and considering the forces themselves as positive or negative accordingly, we conclude that the resultant of two *collinear forces* is equal to their *algebraic sum.*

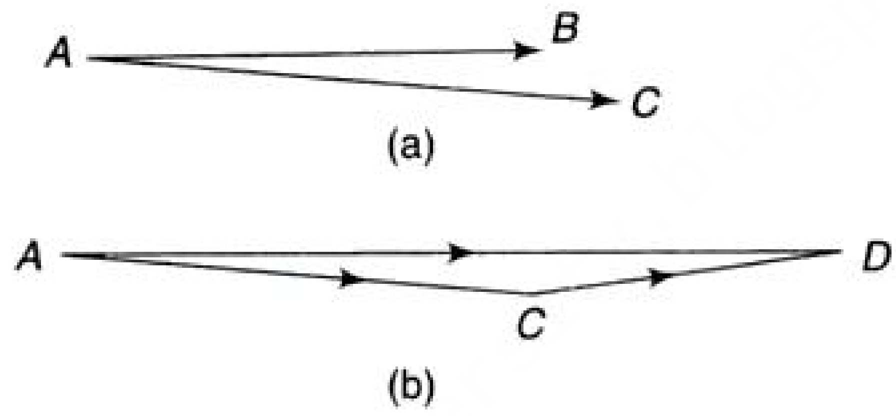

Fig. 2.5

Analytical Method If two given forces P and Q, acting under the angle α, are applied to a body at A, we will now find analytically the formulae for calculating the magnitude of their resultant R and the angles β and γ which its line of action makes with those of the given forces. Also give the formulae for the resultant in the special cases, where $\alpha = 0°$ and $\alpha = 180°$ and $P > Q$ and $\alpha = 90°$.

Figure 2.6(a) shows the parallelogram of forces constructed in the usual manner, while Fig. 2.6(b) shows the triangle of forces obtained by the geometric addition of their free vectors. From the triangle of forces we find

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \qquad \text{(a)}$$

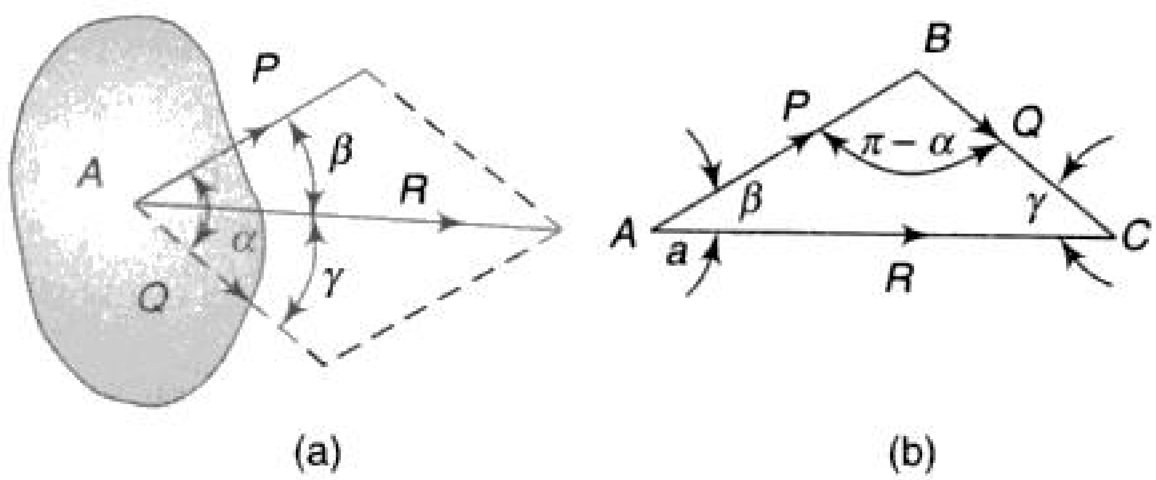

Fig. 2.6

The magnitude of the resultant R being known from Eq. (a), we may determine the angles β and γ by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha \qquad \sin\gamma = \frac{P}{R}\sin\alpha \qquad \text{(b)}$$

It is sometimes convenient to use these formulas for determining the resultant instead of making an accurate construction, to scale, of the triangle of forces.

For the special case (i) $\alpha = 0°$,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 0}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ(1)} = \sqrt{(P+Q)^2} = P + Q$$

$$\therefore \qquad R = P + Q$$

For the special case (ii) $\alpha = 180°$,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 180}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ(-1)} = \sqrt{(P-Q)^2} = P - Q$$

$$\therefore \qquad R = P - Q$$

In the special cases, the resultant of two forces became algebraic sum of the two forces.

Case (iii) if $\alpha = 90°$, i.e., rectangular components, then the resultant is given by the equation

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 90°} = \sqrt{P^2 + Q^2}$$

and $$\tan \gamma = \frac{P}{Q}$$

where γ= the angle between resultant R and the force Q.

Resolution of a Force The replacement of a single force by several components, which will be equivalent in action to the given force, is called the problem of *resolution of a force.* The case in which a single force is to be replaced by two components is the one most commonly encountered. By using the parallelogram law, we can resolve a given force R into any two components P and Q intersecting at a point on its line of action. We shall discuss two possible cases.

1. The directions of both components are given; their magnitudes, to be determined. Imagine, for example, that the force R, represented by the vector AB [Fig. 2.7(a)], is to be resolved into two components acting along the lines AC' and AD'. We proceed by drawing from point B the dotted lines BC and BD, parallel to the given lines of action of the desired components. The points C and D, where these lines intersect the given lines of action of the components, determine the vectors $\overline{AC}$ and $\overline{AD}$ which completely define the two components P and Q.

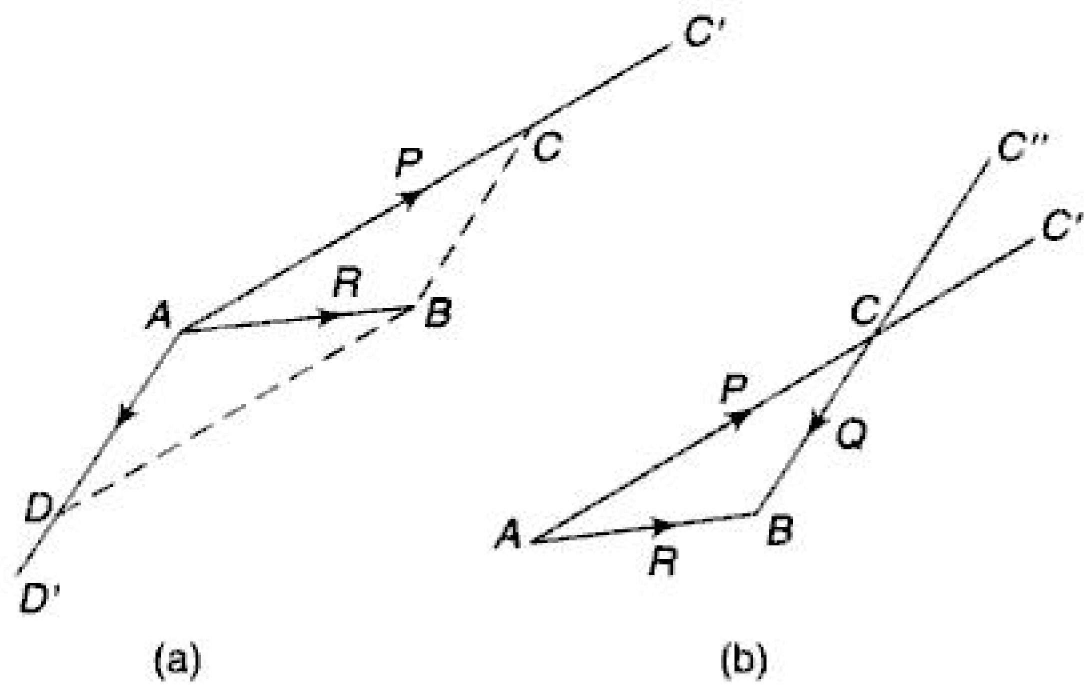

Fig. 2.7

We can obtain the same result by using the triangle of force ABC as shown in Fig. 2.7(b). Here the lines AC' and BC'', parallel to the given lines of action of the components, are extended from the beginning A and the end B of the vector AB representing the given force R and their point of intersection C determines the vectors $\overline{AC}$ and $\overline{CB}$, representing the components P and Q. These components, applied at any point on the line of action of the force R, will be its equivalent. In the particular case where the two components act at right angles to each other, they are called *rectangular components*.

2. Both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined. For example, imagine that the force R, represented by the vector $\overline{AB}$, and the component P, represented by the vector $\overline{AC}$ [Fig. 2.7(a)], have been given. Laying out these two vectors as shown in Fig. 2.7(b), the magnitude and direction of the other component Q are given by the vector $\overline{CB}$, obtained by joining the ends C and B of the two given vectors.

Analytical Method

1. The directions of both components are given; their magnitudes, to be determined. Imagine, for example, that the force R [Fig. 2.8(a)], is to be resolved into two components acting along the lines aa and bb.

   Figure 2.8(a) shows the parallelogram of forces constructed in the usual manner, while Fig. 2.8(b) shows the triangle of forces obtained by the geometric addition of their free vectors.

   The magnitude of the resultant R, angle β and angle γ being known, we may determine the magnitudes of forces P and Q by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha, \qquad \sin\gamma = \frac{P}{R}\sin\alpha$$

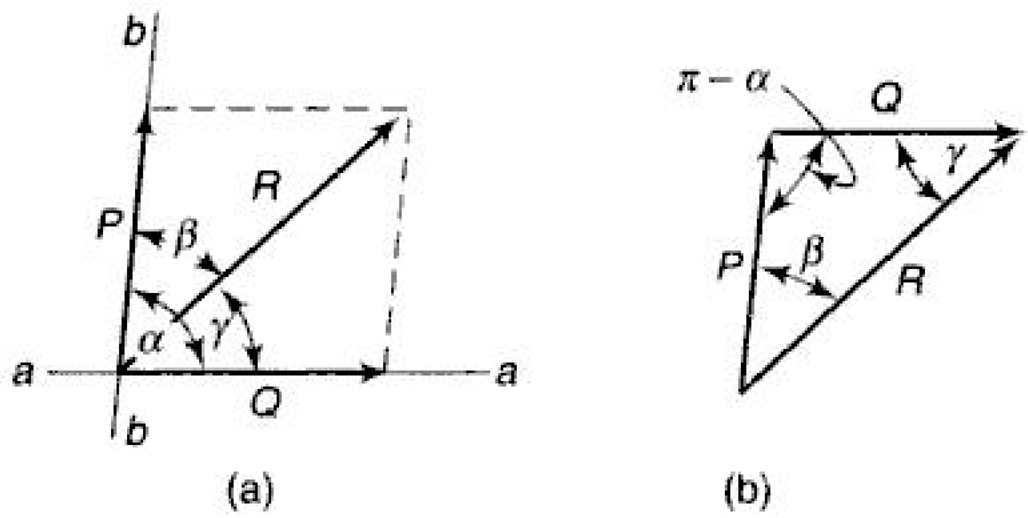

Fig. 2.8

2. Both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined.

The magnitude and direction of one force can be determined using the above equation.

Equilibrium of Collinear Forces From the principle of the parallelogram of forces, it follows that two forces applied at one point can always be replaced by their resultant which is equivalent to them. Thus, we conclude that two concurrent forces can be in equilibrium, only if their resultant is zero. From the discussion of the previous paragraph it follows that this will be the case if we have two forces of equal magnitude acting in opposite directions along the same line. We shall now generalize this conclusion as the second principle of statics.

Equilibrium Law: *Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction, and collinear in action.*

In engineering problems of statics we often have to deal with the equilibrium of a body in the form of a prismatic bar on the ends of which two forces are acting, as shown in Fig. 2.9. Neglecting their own weights, it follows from the principle just stated that either bar can be in equilibrium only when the forces are equal in magnitude, opposite in direction, and collinear in action, which means that they must act along the line joining their points of application. If these points of application can be assumed to be on the central axis of the bar (as is justifiable in many practical cases), the forces must act along this axis. When such central forces are directed as shown in Fig. 2.9(a), we say that the bar is in *tension*. When they act as shown in Fig. 2.9(b), the bar is said to be in *compression*.

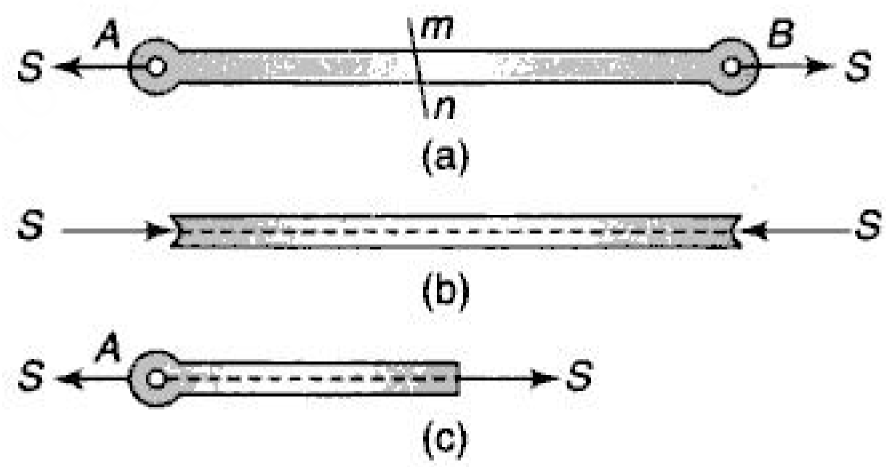

Fig. 2.9

Considering the equilibrium of a portion of the bar AB in Fig. 2.9(a), to the left of a section mn, we conclude that to balance the external force S at A the portion to the right must exert on the portion to the left an equal, opposite, and collinear force S, as shown in Fig. 2.9(c). The magnitude of this internal axial force which one part of a bar in tension exerts on another part is called the *tensile force in the bar* or simply the *force in the bar,* since in general it may be either a tensile force or a compressive force. Such internal force is actually distributed over the cross-sectional area of the bar, and its *intensity,* i.e., the force per unit of cross-sectional area is called the *stress* in the bar.

Internal forces are the forces which hold together the particles of a body. For example, if we try to pull a body by applying two equal, opposite and collinear forces, an internal force comes into play to hold the body together. Internal forces always occur in pairs and are equal in magnitude, opposite in direction and collinear. Therefore, the resultant of all of these internal forces is zero and does not affect the external motion of the body or its state of equilibrium.

External forces or applied forces are the forces that act on the body due to contact with other bodies or attraction forces from other, separated bodies. These forces may be surface forces (contact forces) or body forces (such as gravitational attraction).

Sometimes we have to deal with the equilibrium of a prismatic bar on each end of which two forces are acting as shown in Fig. 2.10(a), instead of a single force at each end as shown in Fig. 2.9(a) and discussed before. Then the forces at A and B are replaced with their respective resultants R_a and R_b as shown in Fig. 2.10(b). Now it is the same case as discussed before.

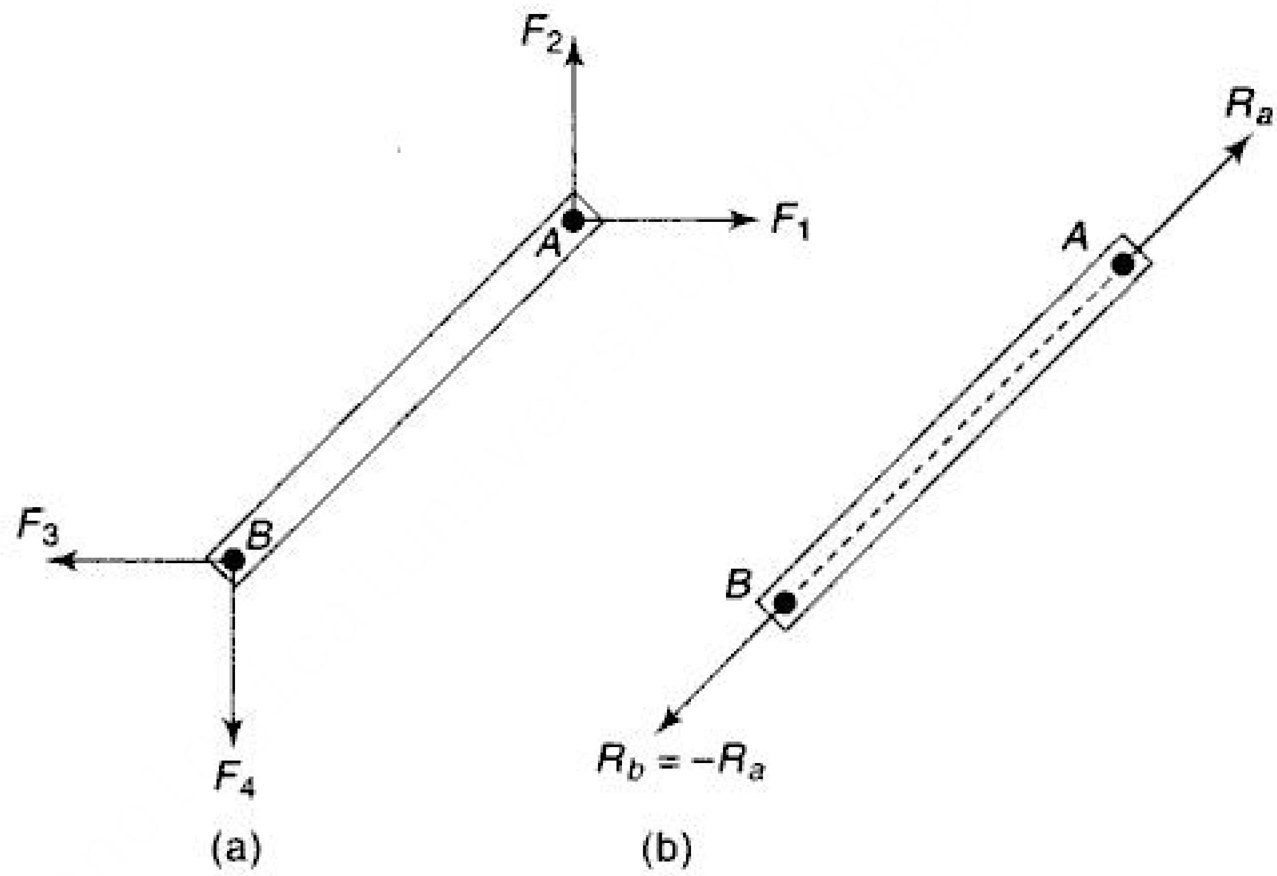

Fig. 2.10

Other examples of two force members held in equilibrium are shown in Fig. 2.11(a)–(c).

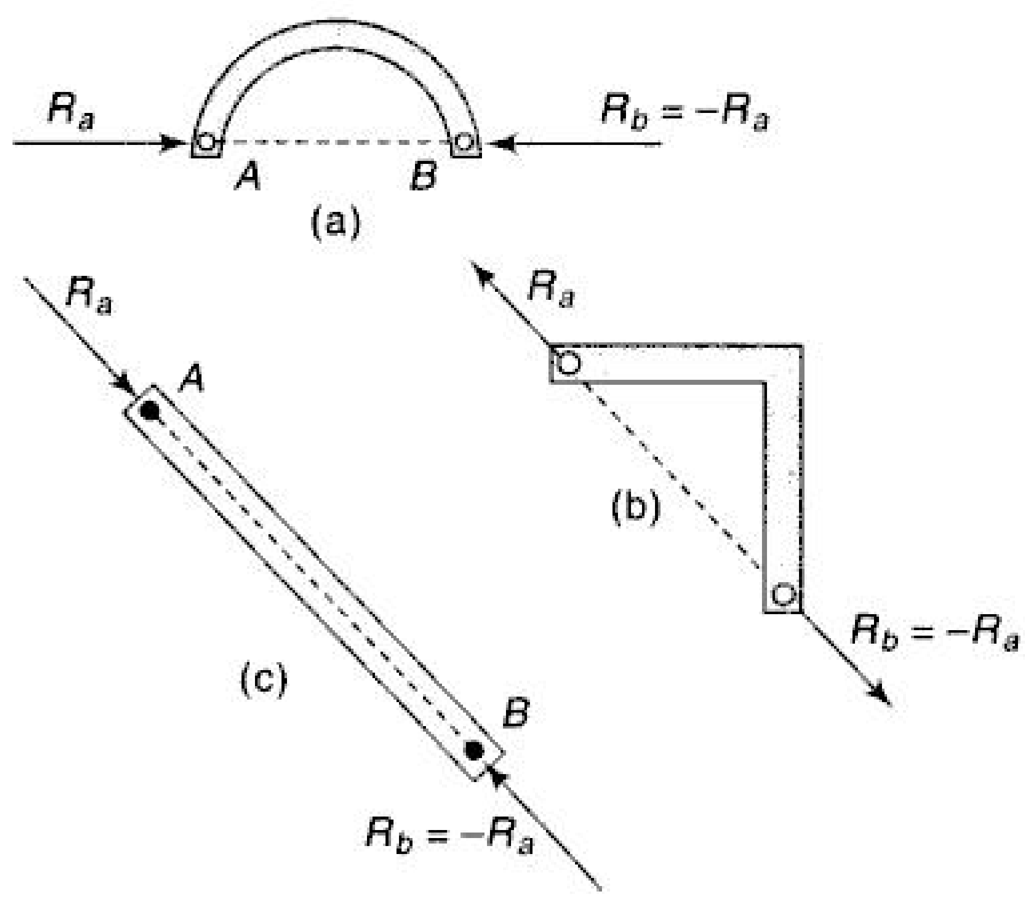

Fig. 2.11

We return now to the case of two forces under an angle α [Fig. 2.4(a)]. From the equilibrium law, we conclude that we can hold these two forces in equilibrium by applying, at point A, a force equal and opposite to their resultant. This force is called the *equilibrant* of the two given forces.

A force, which is equal, opposite and collinear to the resultant of the two given forces is known as the equilibrant of the given two forces.

Superposition and Transmissibility When two forces are in equilibrium (equal, opposite and collinear), their resultant is zero and their combined action on a rigid body is equivalent to that of no force at all. A generalization of this observation gives us the third principle of statics, sometimes called the *law of superposition.*

Law of Superposition: *The action of a given system of forces on a rigid body will in no way be changed if we add to or subtract from them another system of forces in equilibrium.*

Let us consider now a rigid body AB under the action of a force P applied at A and acting along BA as shown in Fig. 2.12(a). From the principle of superposition stated above, we conclude that the application at point B of two oppositely directed forces, each equal to and collinear with P, will in no way alter the action of the given force P. That is, the action on the body of the three forces in Fig. 2.12(b) is identical with the action of the single force P in Fig.2.12(a).

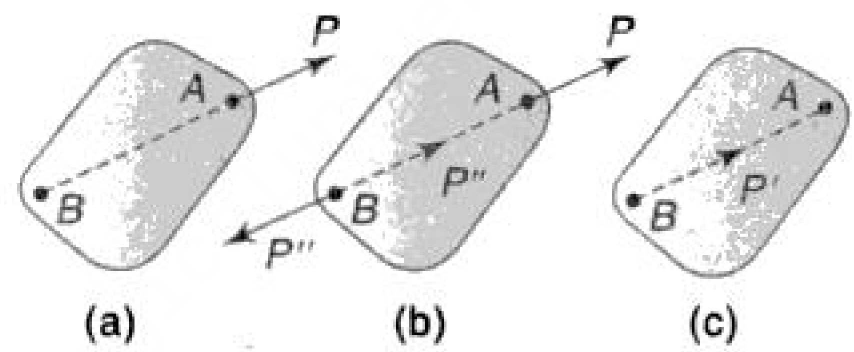

Fig. 2.12

Repeating the same reasoning again, we remove, from the system in Fig. 2.12(b), the equal, opposite, and collinear forces P and P'' as a system in equilibrium. Thus we obtain the condition shown in Fig. 2.12(c) where, instead of the original force P applied at A, we have the equal force P' applied at B. This proves that the point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied. This statement is called the *theorem of transmissibility of a force.*

Theorem of transmissibility of a force: *The point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied.*

By way of an example, let us consider the prismatic bar AB [Fig. 2.13(a)] which is acted upon by two equal and opposite forces P_1 and P_2, applied at the ends and acting along its axis. As discussed before, the bar is in equilibrium under the action of two such forces and is subjected to compression. Now in accordance with the theorem of transmissibility of a force, we transmit P_1 along AB until its point of application is at B and similarly we transmit P_2 along BA to act at A. The condition of the bar now is represented in Fig. 2.13(b), and we see that, while it is still in equilibrium under the action of these forces, the state of compression has been changed to one of tension. Again, imagine that we transmit the point of application of each force to the middle point C of the bar [Fig. 2.13(c)]. The two forces are again in equilibrium, but the bar is now subjected to no internal forces. From this example, we see that, while the transmission of the point of application of a force acting on a body does not change the condition of equilibrium, it may produce a decided change in the internal forces to which the body is subjected. Thus the use of the theorem of transmissibility of a force is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.

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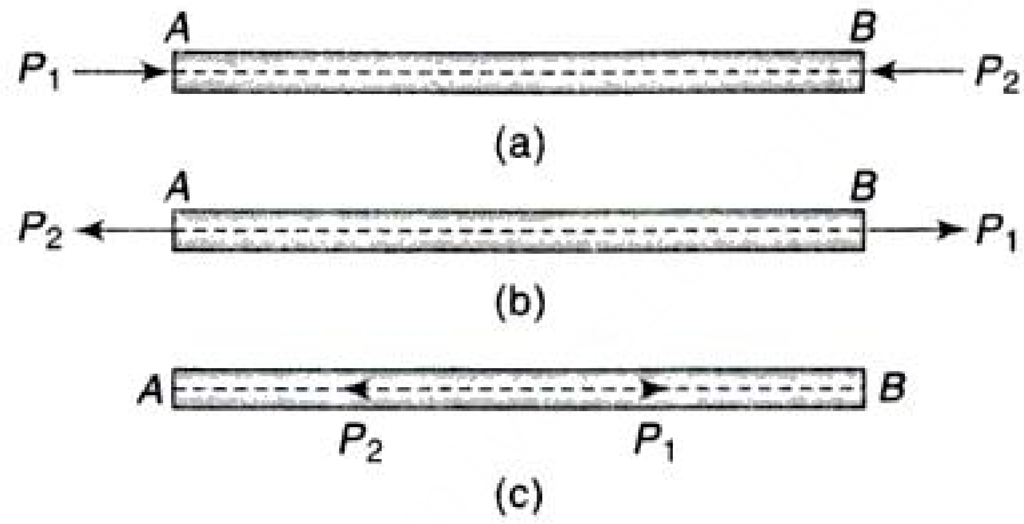


Fig. 2.13

From the theorem of transmissibility of a force it follows that, if two forces P and Q applied to a body at the points A and B [Fig. 2.14(a)] are acting along lines intersecting at point C, we can transmit the points of application of the forces to point C and replace them by their resultant [Fig. 2.14(b)].

If the intersection point C is outside the boundary of the body [Fig. 2.14(c)], we assume this point to be rigidly attached to the body by the imaginary extension, indicated in the figure by dotted lines, and then proceed as before.

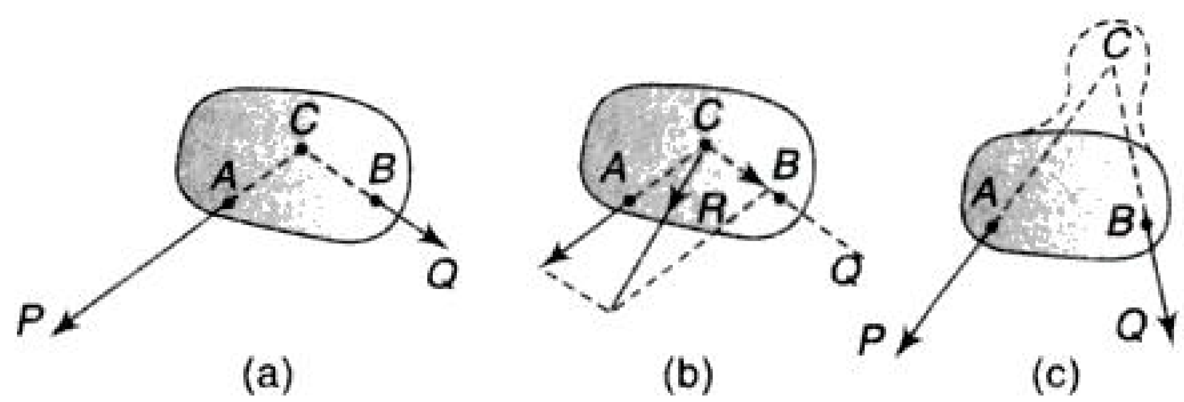


Fig. 2.14

Action and Reaction Very often we have to investigate the conditions of equilibrium of bodies that are not entirely free to move. Restriction to the free motion of a body in any direction is called *constraint.* In Fig. 2.15(a), for example, we have a ball resting on a horizontal plane such that it is free to move along the plane but cannot move vertically downward. Similarly, the ball in Fig. 2.2(a), although it can swing as a pendulum, is constrained against moving vertically downward by the string AB. In Fig. 2.16(a), we have a ball of weight W supported by a string BC and resting against a smooth vertical wall at A. With such constraints, all motion of the ball in the plane of the figure is prevented.[2] There are

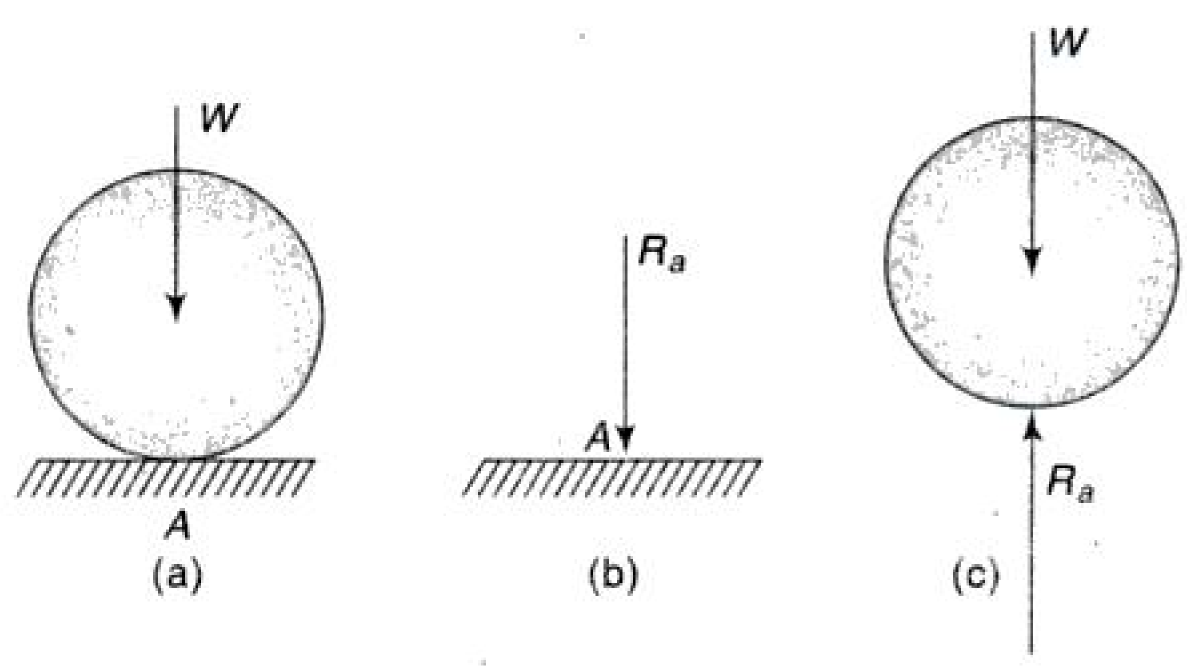


Fig. 2.15

[2]Since there is no tendency for the ball in this case to move upward or to swing away from the wall, we ignore the fact that the constraints as shown may not be able to prevent such motion.

many other kinds of constraint than those illustrated in Figs. 2.2(a), 2.15(a) and 2.16(a) but these are typical and will suffice as a basis for our present discussion.

A body that is not entirely free to move and is acted upon by some applied force (or forces) will, in general, exert *pressures* against its supports. For example, the ball in Fig. 2.2(a) exerts a downward pull on the end of the supporting string as shown in Fig. 2.2(b). Similarly, the ball in Fig. 2.15(a) exerts a vertical push against the surface of the supporting plane at the point of contact A as shown in Fig. 2.15(b). For the case in Fig. 2.16(a), the ball not only pulls downward on the string BC but also pushes to the left against the wall at A as shown in Fig. 2.16(b). Now in every case, these actions of a constrained body against its supports induce reactions from the supports on the body, and as the fourth principle of statics we take the following statement:

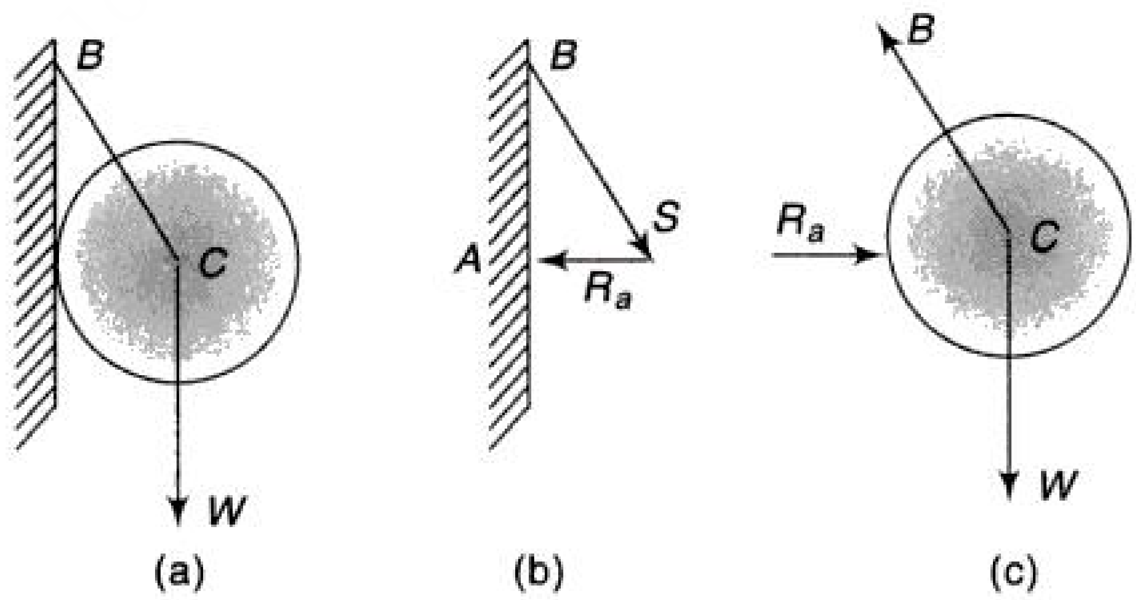


Fig. 2.16

Law of Action and Reaction *Any pressure on a support causes an equal and opposite pressure from the support so that action and reaction are two equal and opposite forces.* This last principle of statics is of course nothing more than Newton's third law of motion stated in a form suitable for the discussion of problems of statics.

Note ▸▸ The reaction of a constraint points away from the direction in which the given constraint prevents a body's displacement.

A free body is a body not connected with other bodies and which from any given position can be displaced in any direction in space.

Free-body Diagrams To investigate the equilibrium of a constrained body, we shall always imagine that we remove the supports and replace them by the *reactions* which they exert on the body. Thus, in the case of the ball in Fig. 2.2(a), we remove the supporting string and replace it by the reaction R_a that it exerts on the ball. We know that the point of application of this force must be the point of contact B, and from the law of equilibrium of two forces, we conclude that it must be along the string, i.e., vertical and equal to the weight W; thus it is completely determined. The sketch in Fig. 2.2(c) in which the ball is completely isolated from its support and in which all forces acting on it are shown by vectors is called a free-body diagram.

Free body diagram is a sketch of the isolated body, which shows the external forces on the body and the reactions exerted on it by the removed elements.

The general procedure for constructing a free-body diagram is as follows:

1. A sketch of the body is drawn, by removing the supporting surfaces.

2. Indicate on this sketch all the applied or active forces, which tend to set the body in motion, such as those caused by weight of the body or applied forces; etc.
3. Also indicate on this sketch all the reactive forces, such as those caused by the constraints or supports that tend to prevent motion. (The sense of unknown reaction should be assumed. The correct sense will be determined by the solution of the problem. A positive result indicates that the assumed sense is correct. A negative result indicates that the correct sense is opposite to the assumed sense.)
4. All relevant dimensions and angles, reference axes are shown on the sketch.

Similarly, in the case of the ball in Fig. 2.15(a), we remove the supporting surface and replace it by the reaction R_a that it exerts on the ball. We know that the point of application of this force must be the point of contact A, and from the law of equilibrium of two forces, we conclude that it must be vertical and equal to the weight W; thus it is completely determined. The free-body diagram of the ball in Fig. 2.15(a) is shown in Fig. 2.15(c).

In the case of the ball in Fig. 2.16(a), we again remove the supports and isolate the ball as a free body [Fig. 2.16(c)]. Then besides the weight W acting at C, we have two reactive forces to apply, one replacing the string BC and another replacing the wall AB. Since the string is attached to the ball at C and since a string can pull only along its length, we have the reactive force S applied at C and parallel to BC. Its magnitude remains unknown. Regarding the reaction R_a, we have for its point of application the point of contact A. Furthermore, we assume that the surface of the wall is perfectly smooth so that it can withstand only a normal pressure from the ball. Then, accordingly, the reaction R_a will be horizontal and its line of action will pass through C as shown. Again only the magnitude remains unknown and the free-body diagram is completed. The question of finding the magnitudes of S and R_a will not be discussed here, although it is only necessary to so proportion these vectors that their resultant is equal and opposite to the vertical gravity force W.

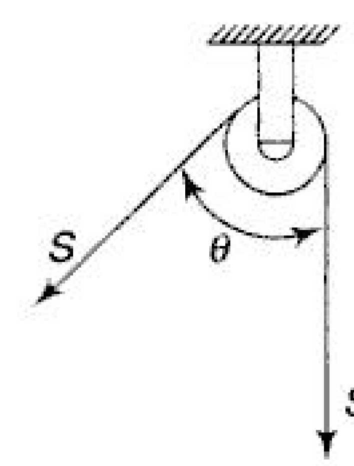


Fig. 2.17

From the above discussion, we come across two types of supports namely string support and a smooth surface or support. A flexible weightless and in-extensible string is a constraint prevents a body moving away, from the point of suspension of the string, in the direction of the string. The reaction of the string is directed along the string towards the point of suspension. So, string or cable can support only a tension and this force always acts in the direction of the string. The tension force developed in a continuous string, which passes over a frictionless pulley, must have a constant magnitude to keep the string in equilibrium (Fig. 2.17). Hence, the string or cord, for any angle θ, is subjected to a constant tension S throughout its length.

A smooth surface is one whose friction can be neglected. Smooth surface prevents the displacement of a body normal to both contacting surfaces at their point of contact. The reaction of a smooth surface or support is directed normal to both contacting surfaces at their point of contact and is applied at that point

(Fig. 2.18). If one of the contacting surfaces is a point, then the reaction is directed perpendicular or normal to the other surface (Fig. 2.15). If two of the contacting surfaces are points, then the reaction is directed perpendicular or normal to the tangent of contacting surfaces [Fig. 2.19(a) and 2.20(a)].

Fig. 2.18

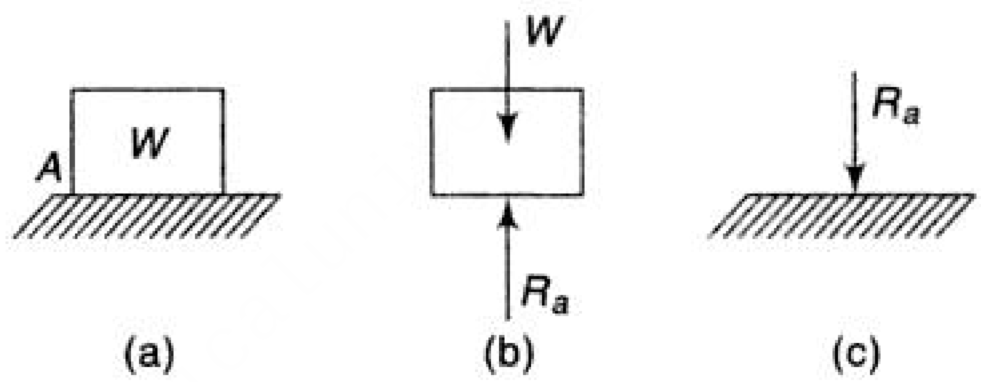


Fig. 2.19

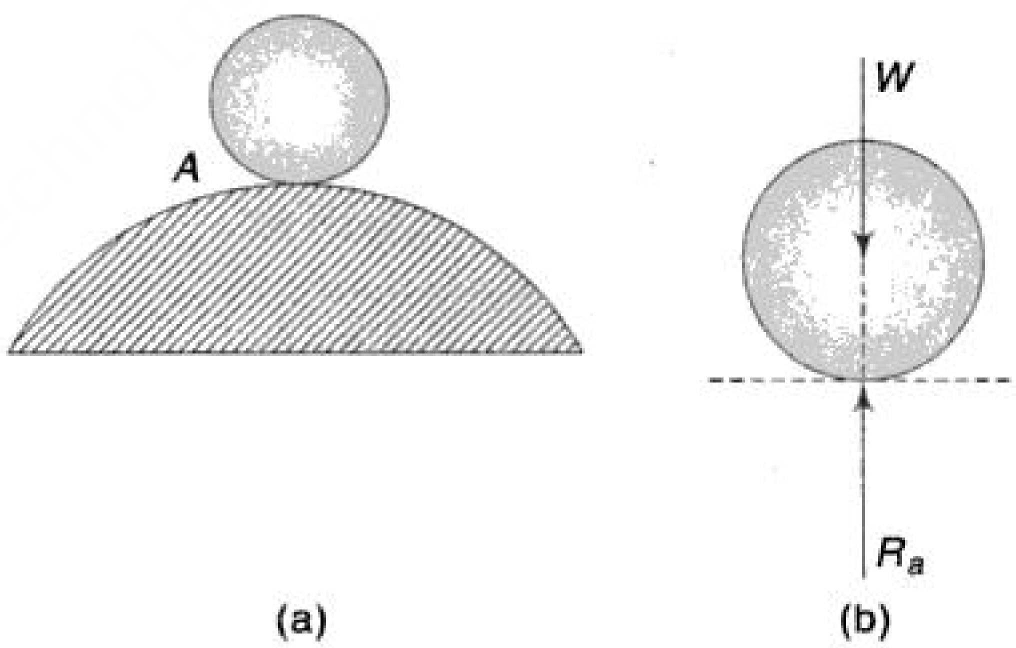
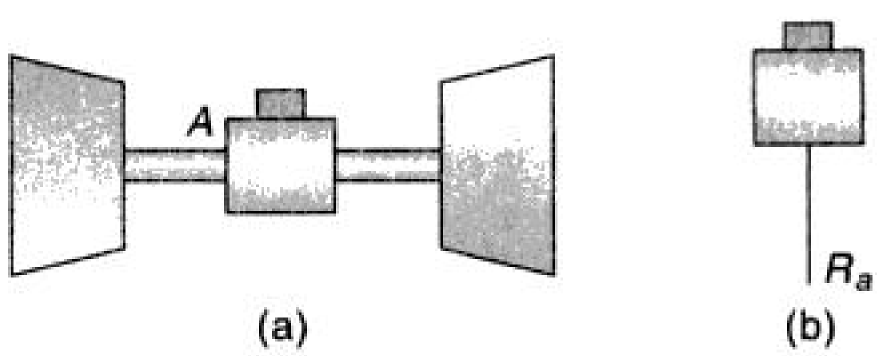


Fig. 2.20



The free body diagrams of the bodies are shown in Figs 2.15(b), 2.18(b), 2.19(b), 2.20(b), respectively.

Another type of support is linear elastic spring (Fig. 2.21). The magnitude of force developed by a linear elastic spring which has a stiffness k, and is deformed a distance x measured from its unloaded position, is

$$S = kx$$

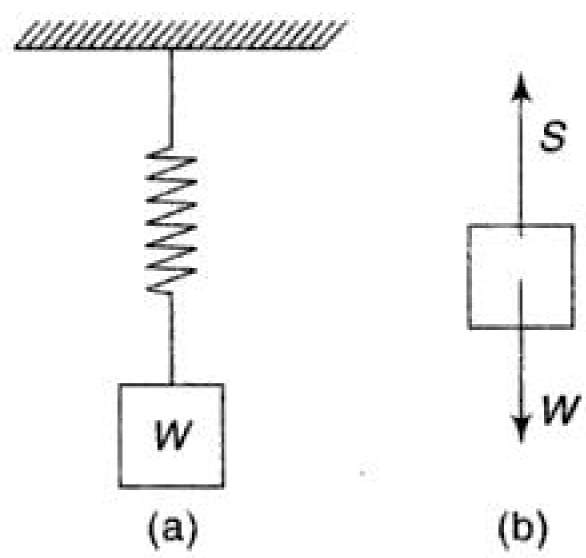


Fig. 2.21

Note ➛ x is determined from the difference in the spring's deformed length and its initial length. If x is positive, S 'pulls' on the spring; whereas if x is negative, S must 'push' on it.

In the case of the body in Fig. 2.21(a), we remove the supporting spring and replace it by the spring force S that it exerts on the body. We know that the point of application of this force must be the point of contact, and from the law of equilibrium of two forces, we conclude that it must be along the spring, i.e., vertical and equal to the weight W;

thus it is completely determined. The free body diagram of the body in Fig. 2.21(a) is shown in Fig. 2.21(b).

One more example of free-body diagram is considered here. The lawn roller, of weight W, being pushed up the inclined smooth plane as shown in Fig. 2.22(a). In the case of the lawn roller in Fig. 2.22(a), we again remove the support and isolate the body as a free body [Fig. 2.22(b)]. Then beside the weight W and push P acting at centre O, we have one reactive force R_a to apply, replacing the inclined plane. The reactive force R_a have its point of application at the point of contact A. We assume that the surface of the inclined plane is perfectly smooth so that it can withstand only a normal pressure from the roller. Then the reaction R_a will be normal to the inclined surface and its line of action will pass through O as shown here the magnitude remains unknown and the free body diagram is completed the question of finding the magnitudes of P and R_a will not be discussed here.

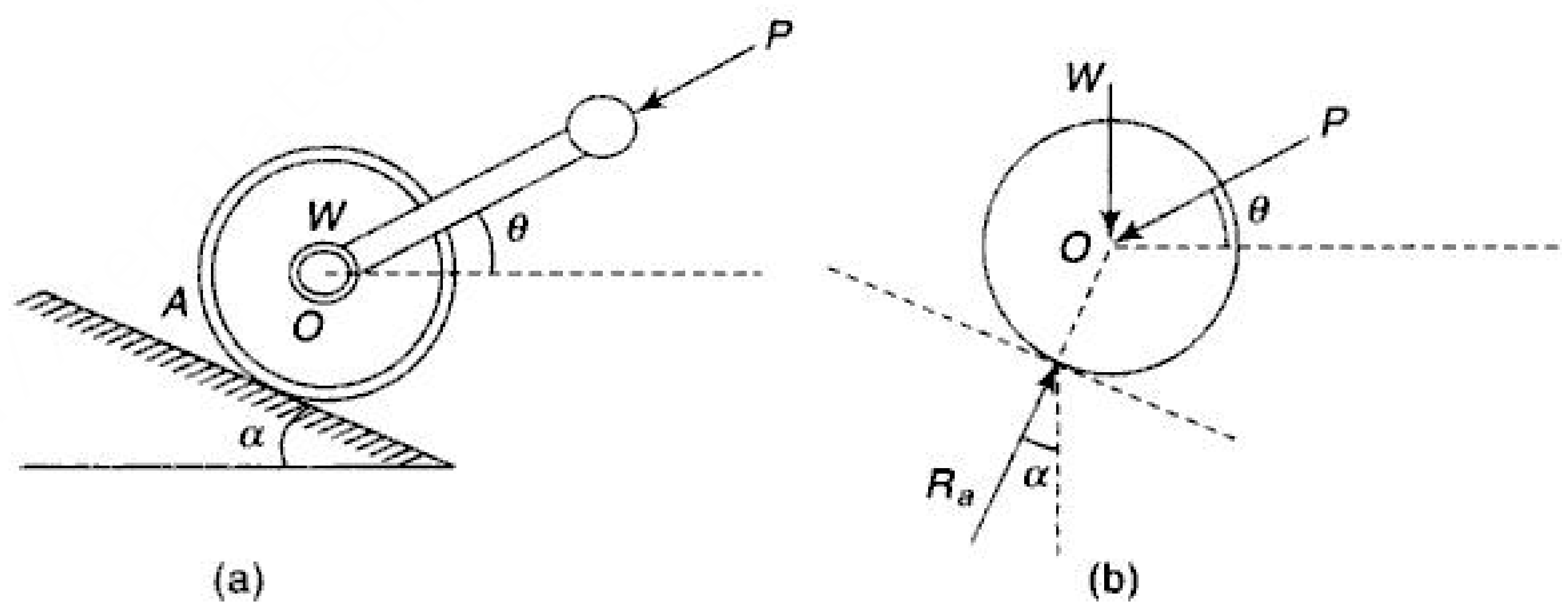


Fig. 2.22

Proceeding as above with constrained bodies, we shall always obtain two kinds of forces acting on the body: the given forces, usually called *active forces,* such as the gravity force W in Fig. 2.16(c), and *reactive forces,* replacing the supports, such as the forces S and R_a in Fig. 2.16(c). To have equilibrium of the body, it is necessary that the active forces and reactive forces together represent a system of forces in equilibrium. Thus it is by means of the free-body diagram that we define the system of forces with which we must deal in our investigation of the conditions of equilibrium of any constrained body. The construction of this diagram should be the first step in the analysis of every problem of statics, and it must be evident that any errors or omissions here will reflect themselves on all subsequent work.

The essential problem of statics may now be briefly recapitulated as follows: We have a body either partially or completely constrained which remains at rest under the action of applied forces. We isolate the body from its supports and show all forces acting on it by vectors, both active and reactive. We then consider what conditions this system of forces must satisfy in order to be in equilibrium, i.e., in order that they will have no resultant.

Examples Examples Examples Examples Examples

1. Two very nearly parallel forces P and Q are applied to a rigid body at points A and B, as shown in Fig. 2.23. Find their resultant R graphically.

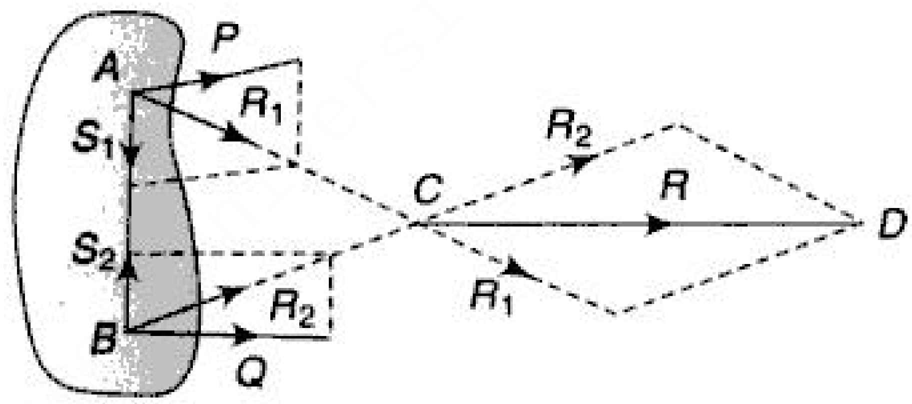


Fig. 2.23

Solution: Since the point of intersection of the given forces P and Q is not well defined and does not occur within the limits of the drawing, we begin by adding to the system two equal, opposite and collinear forces S_1 and S_2 of any convenient magnitudes at points A and B, as shown in the figure. It follows from the law of superposition that two such forces, being in equilibrium, do not change the action of the given forces P and Q. Hence the resultant R_1 of P and S_1 together with the resultant R_2 of Q and S_2, obtained as shown, are statically equivalent to the given forces P and Q and their resultant R will be the one required. To find this resultant R, we transmit R_1 and R_2 along their lines of action to point C, which is a well-defined point, and complete the parallelogram of forces as shown. The vector $\overline{CD}$ represents the required resultant and if all constructions have been made to scale, its magnitude may be measured directly from the drawing.

2. Draw a free body diagram of the body of weight W, the string BD and the ring shown in Fig. 2.24(a).

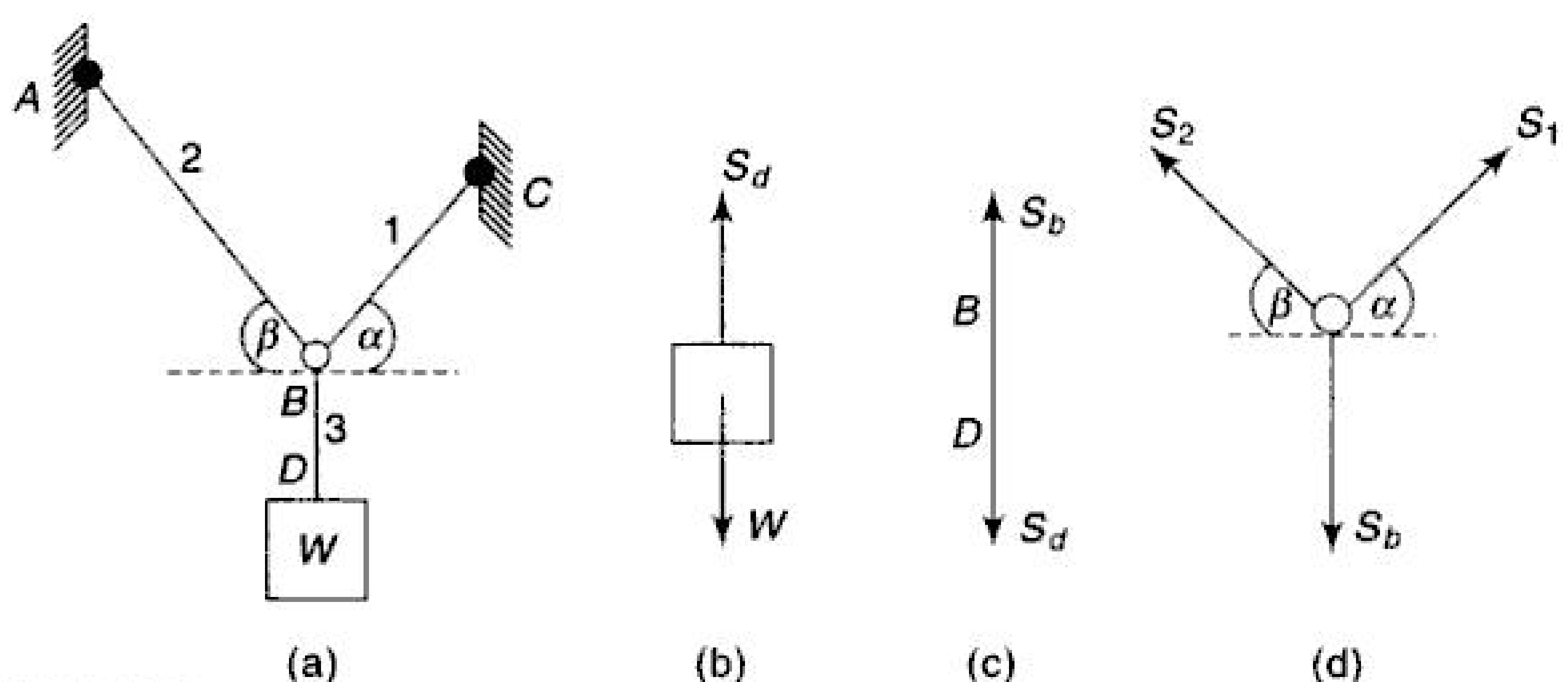


Fig. 2.24

Solution: In the case of the body in Fig. 2.24(a), remove the support, i.e., string and isolate the body as a free body [Fig. 2.24(b)]. Replace the string support by the tension S_3 or S_d it exerts on the body. We know that the point of application of this force must be the point of contact D, and from the law of equilibrium of two forces, we conclude that it must be along the string, i.e., vertical and equal to the weight W; thus it is completely determined.

If the string BD isolated from its supports, then there are only two forces acting on it, Fig. 2.24(c), namely the tensile force of the string S_d and the tensile force S_b caused by the string. S_d shown here is equal but opposite to that shown in Fig. 2.24(b) from the law of action and reaction.

From the law of equilibrium of two forces S_d and S_b are equal, i.e., $S_d = S_b = S_3 = W$.

Now isolate the ring at B from its supports. On the ring, three forces are acting. All the forces on the ring are reactive forces from the strings and shown as in Fig. 2.24(d).

Note ▸▸ S_b shown here is equal but opposite to that shown in Fig. 2.24(c) from the law of action and reaction.

3. The body of weight W is supported as shown in Fig. 2.25(a). Draw a free body diagram of the body and the knot at C.

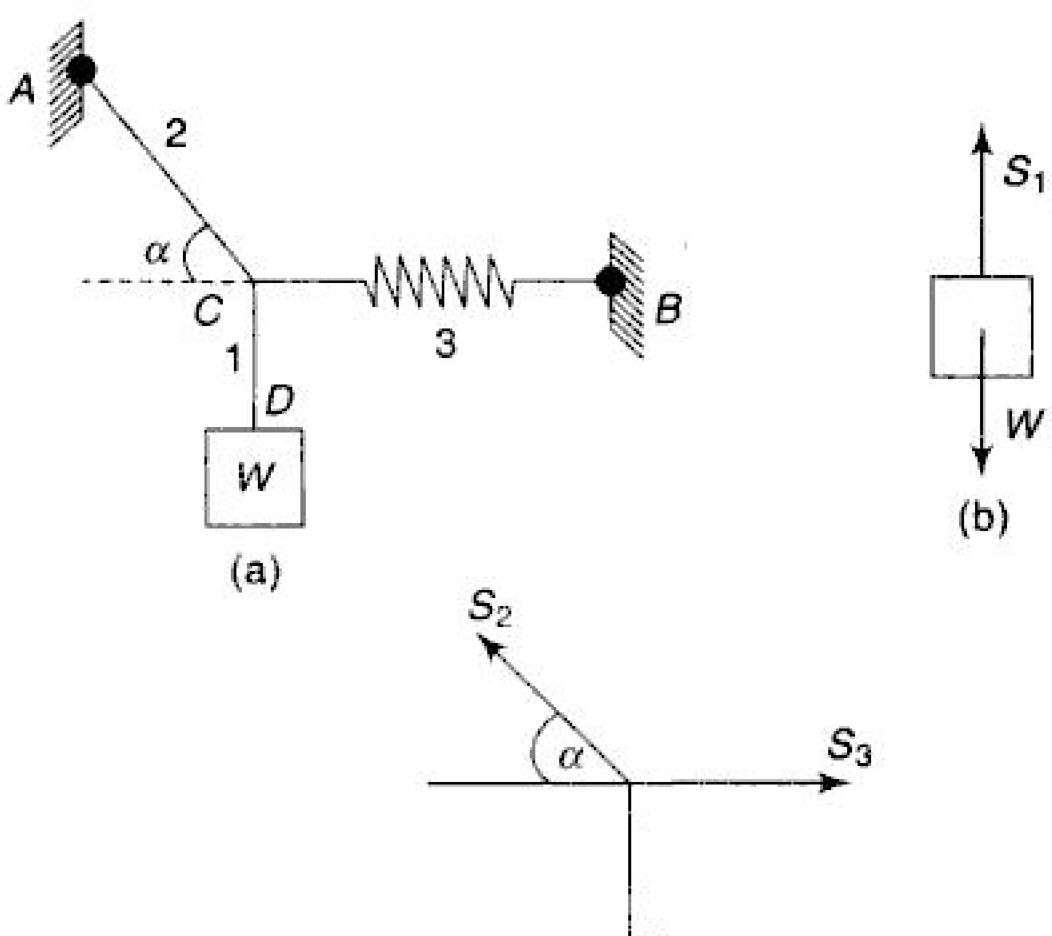


Fig. 2.25

Solution: Isolate the body from its supports. There are two forces acting on the body, namely, its weight and the tensile force S_1 of the string CD. The free body diagram of the body shown in Fig. 2.25(b). From the law of equilibrium of two forces, we have $S_1 = W$ and S_1 is acting vertically upward.

If we isolate the knot from its supports, there are three forces acting on it. They are reactive forces from the spring and the tensile forces in the string. The free body diagram of the knot is shown in Fig. 2.25(c) as usual.

4. A body of weight W is supported on a frictionless pulley as shown in Fig. 2.26(a). Draw a free body diagram of the body and the pulley, if the radius of the pulley and the weight of the pulley are neglected.

Solution: If we isolate the body from its supports, there are two forces acting on the body. One is weight and the other is tension of the string. The free body diagram is as shown in Fig. 2.26(b).

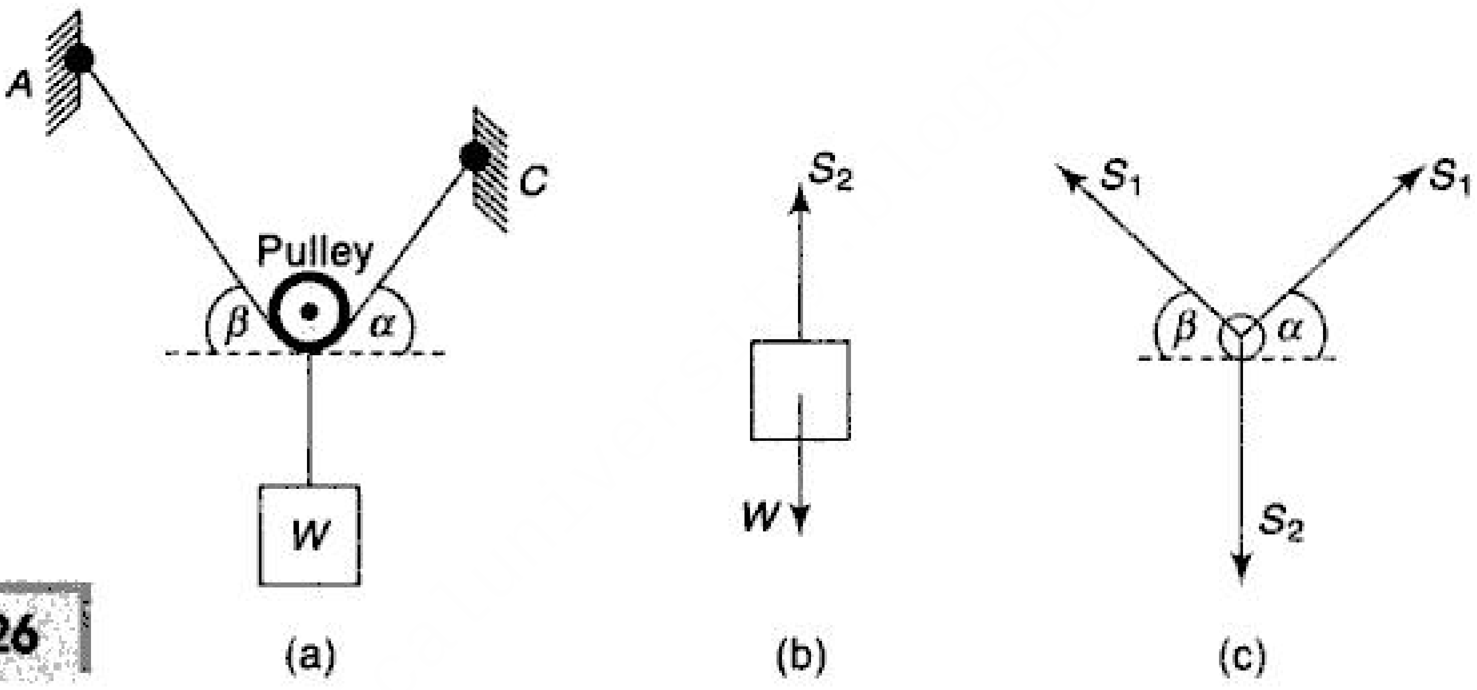


Fig. 2.26 (a) (b) (c)

From the law of equilibrium of two forces, we obtain $S_2 = W$ and S_2 is acting vertically upward. Now isolate the pulley from its supports. As the pulley is frictionless, the forces exerted at B by the portions AB and BC of the cord must be equal. These two forces can balance the vertical resultant, this condition requires that AB and BC be equally inclined to the horizontal, i.e., $\beta = \alpha$. The free body diagram of the pulley is as shown in Fig. 2.26(c).

5. Two spheres of weight P and Q rest inside a hollow cylinder, which is resting on a horizontal plane as shown in Fig. 2.27. Draw the free body diagram/s of both the spheres taken together and both the spheres taken separately.

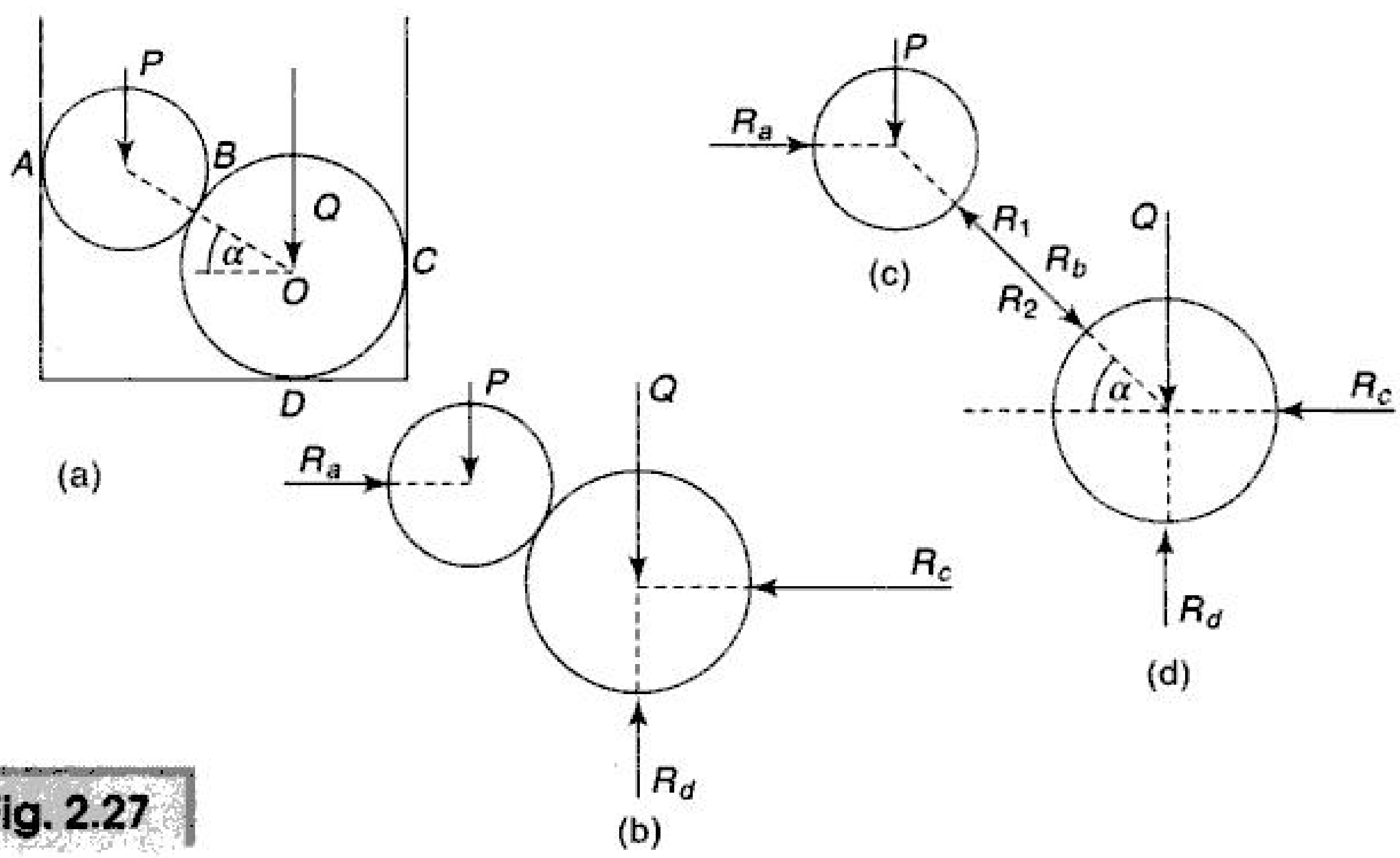


Fig. 2.27

Solution: Since the spheres are smooth, the pressures at the various points of contact must be normal to the surfaces. Removing the supporting walls and floor and replacing them by their reactions R_a, R_c and R_d, we obtain the free body diagram for both spheres as shown in Fig. 2.27(b). These reactions are equal and opposite to the required pressures exerted by the spheres on the walls and floor.

Isolate the spheres taken separately from its supports. On the sphere P, three forces are acting, namely, weight, the reaction from the wall of the cylinder. The free body diagram is as shown in Fig. 2.27(c). On the sphere Q, four forces are acting, namely, weight, the reaction from the wall of the cylinder, reaction from the floor and the reaction from the sphere P. The free body diagram is as shown in Fig. 2.27(d).

At the point of contact between the two spheres, we have two equal and opposite forces R_a, R_2 which must act along the line OE joining the centres of the spheres. When considering the free body diagram of the upper sphere, we take only the force R_1, representing the reaction exerted by the lower sphere, likewise when considering the lower sphere, we take only the force R_2.

6. Find the force with which the 1000 N press against the floor shown in Fig. 2.28(a).

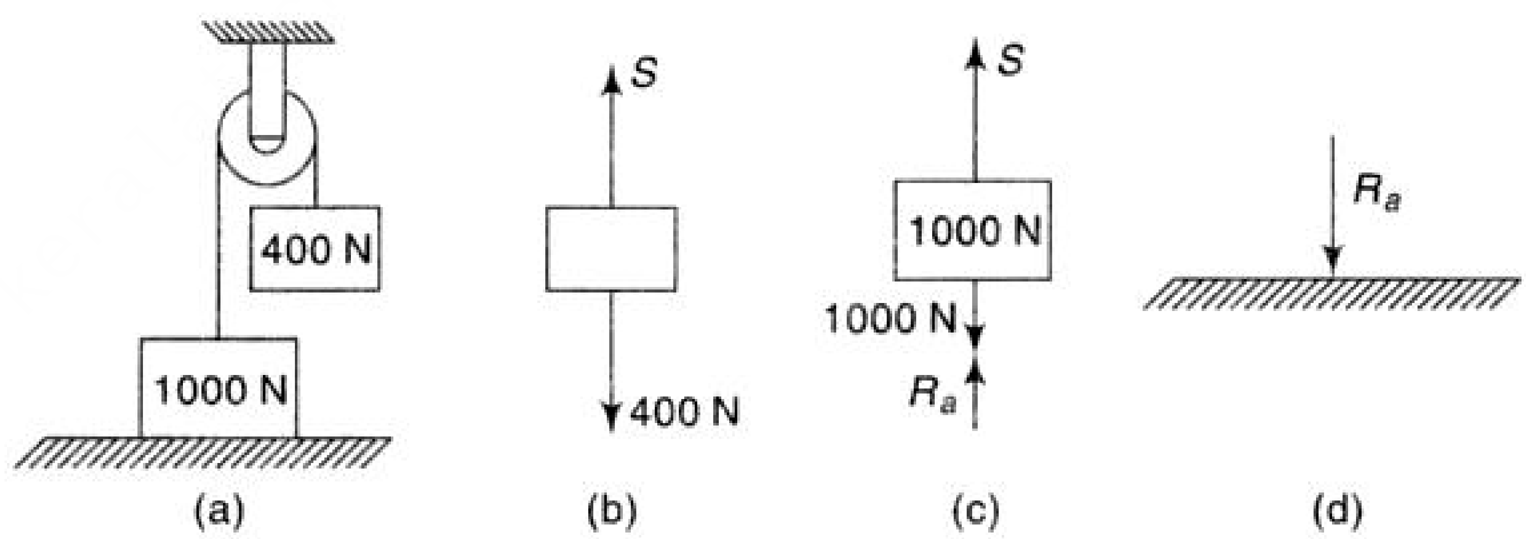


Fig. 2.28

Solution: The free body diagram of 400 N weight is as shown in Fig. 2.28(b).

From the law of equilibrium of two forces, we get

$$S = 400 \text{ N}$$

Removing the supporting string and the floor and replacing them by their reactions S and R_a we obtain the free body diagram of the body as shown in Fig. 2.28(c). The reactions S and R_a are equal and opposite to the required pressures exerted by the body on the string and floor. The free body diagram of the floor is shown in Fig. 2.28(d).

The three forces S, R_a and W are all acting along one line, the sides of the polygon of forces will all lie along one line and the geometric summation will be replaced by algebraic summation. Therefore, the resultant is the algebraic sum of the three forces, i.e.,

$$R = S + R_a - 1000$$

The resultant is equal to zero, since the body is in equilibrium. Equating the resultant zero, we get

$$R_a = 600 \text{ N}$$

Therefore, the force with which the 1000 N press against the floor = 600 N.

7. Determine the components of the 1000 N force shown along the aa' and bb' axes shown in Fig. 2.29(a).

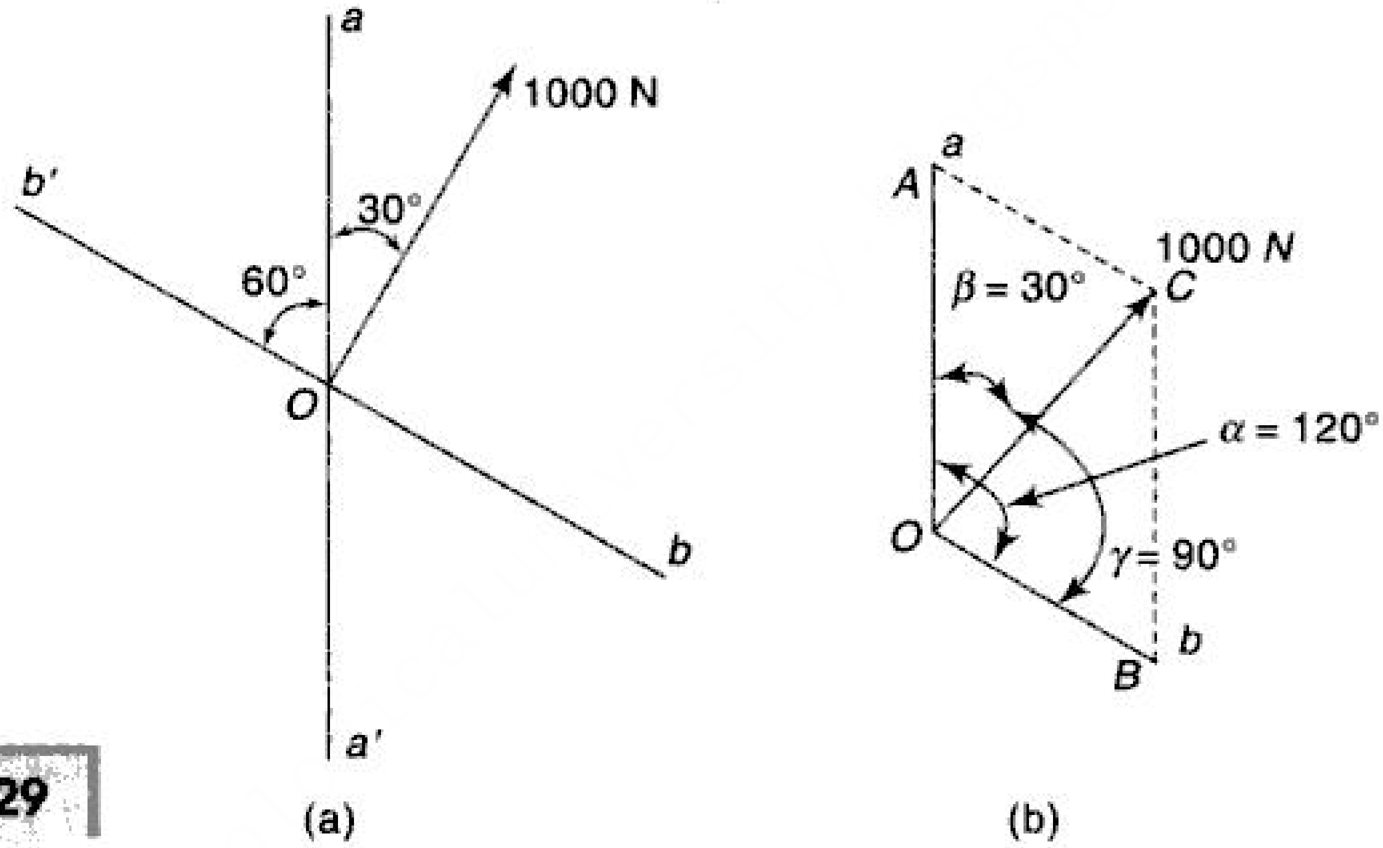


Fig. 2.29

Solution: Figure 2.29(b) shows the parallelogram of forces *OACB* constructed in the usual manner. The angles β and γ are given by the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha, \qquad \sin\gamma = \frac{P}{R}\sin\alpha \tag{c}$$

Substituting the values $\alpha = 120°$, $\beta = 30°$, $\gamma = 90°$ and $R = 1000$ N, we get from Eq. (c)

$$P = 1154.7 \text{ N}$$
$$Q = 577.35 \text{ N}$$

8. The 500 N force is to be resolved into components along the *aa* and *bb* axes as shown in Fig. 2.30(a). Determine the angles β and γ knowing that the component along *aa′* is to be 400 N. Also, find the corresponding value of the component along *bb′* axis.

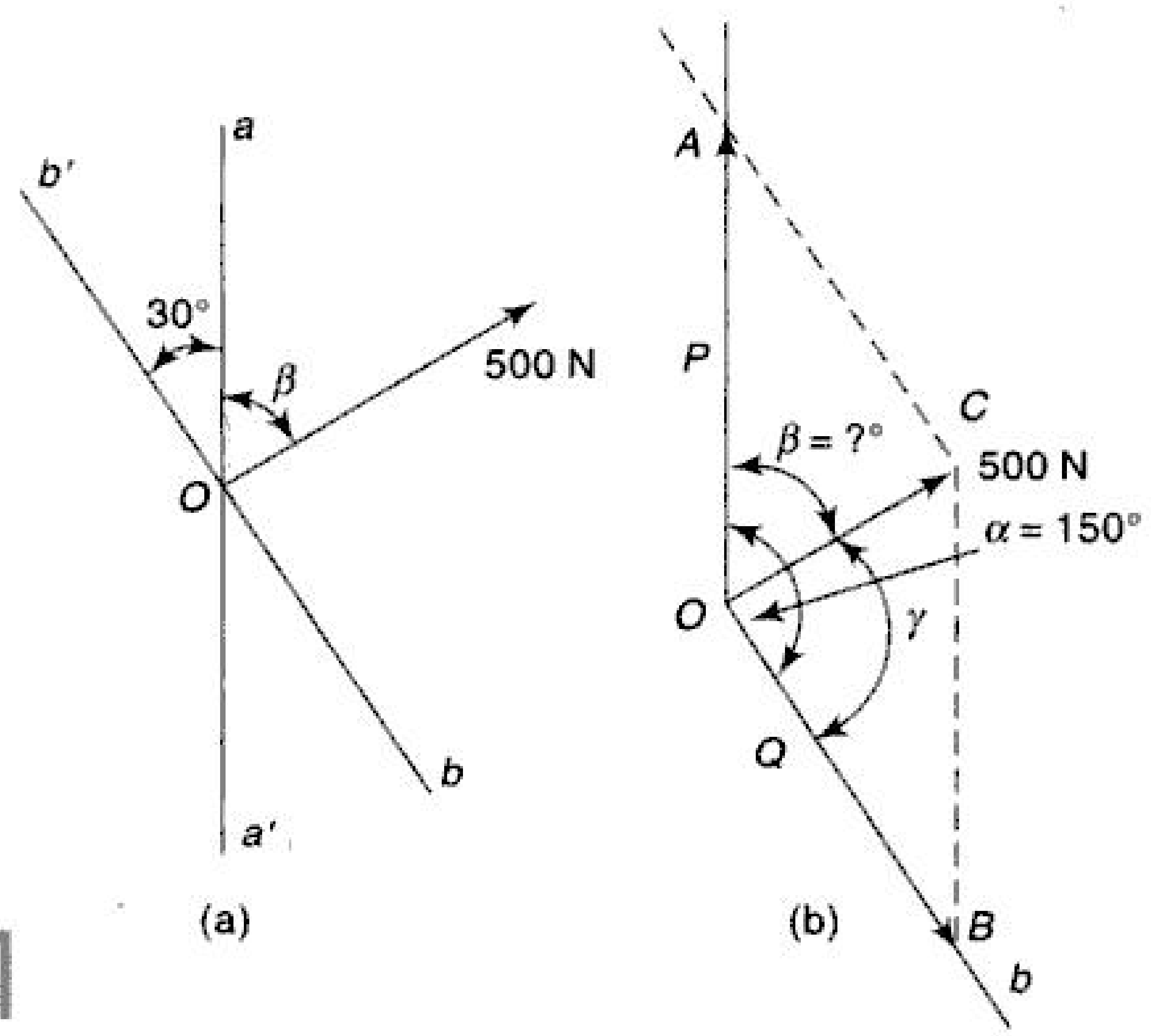


Fig. 2.30

Solution: In this problem both the direction and the magnitude of one component are given; the direction and the magnitude of the other, to be determined. Using Fig. 2.30(b) and the relation

$$\sin \gamma = \frac{P}{R} \sin \alpha \qquad \text{(d)}$$

and substituting the values $\alpha = 150°$, $P = 400$ N and $R = 500$ N, we will get from Eqn. (d)

$$\gamma = 23.58°$$

and $\beta = \alpha - \gamma = 150 - 23.58 = 126.42°$

Using Fig. 2.30(b) and the relation

$$\sin \beta = \frac{Q}{R} \sin \alpha \qquad \text{(i)}$$

and substituting the values $\alpha = 150°$, $\beta = 126.42°$, and $R = 500$ N, we will get from Eq. (e).

$$Q = 804.7 \text{ N}$$

Important Terms and Concepts

Statics	Rigid Body	Strength of materials
Fluid mechanics	Force	The specifications of a force
Concentrated load	Vector	System of forces
Free vectors	Composition of forces	Parallelogram law
Resultant	Triangle law	Resolution of forces
Rectangular components	String	Smooth surface
Equilibrium law	Tension	Compression
Stress	Equilibrant	Law of superposition
Theorem of transmissibility	Constraint	Action and reaction
Law of action and reaction	Active forces	Reactive forces
Free body diagram		

SUMMARY

- Statics deals with the conditions of equilibrium of bodies acted upon by forces.
- A rigid body is defined as a definite quantity of matter, the parts of which are fixed in position relative to one another. The physical bodies are never absolutely rigid but deform slightly under the action of loads, which they have to carry. If the deformation is negligible when compared with the size of the body, it is assumed to be rigid.
- Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The three quantities, which completely define the force, are called its *specifications*. The specifications of a force are (1) its magnitude, (2) its point of application, and (3) its direction. The SI units used by engineers to measure the magnitude of a force are the newton (N). The point of application of a force acting upon a body is that point in the body at which the force can be assumed to be concentrated. The direction of a force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied.
- The point of application at which the total weight can be assumed to be concentrated is called the center of gravity of the body.

- Sometimes a linear elastic spring is used as a support. The reactive force of a spring may either be tensile or compressive in nature, according to the load acting on the spring.
- Free body diagram is a sketch of the isolated body, which shows the external forces on the body and the reactions exerted on it by the removed elements.
- The essential problem of statics: Consider a body, either partially or completely constrained, which remains at rest under the action of applied forces. Isolating the body from its supports and showing all forces acting on it by vectors, both active and reactive. Then considering what conditions this system of forces must satisfy in order to be in equilibrium, i.e., in order that they will have no resultant.

Important Formulae

The magnitude of the resultant R of the two forces P and Q having an angle α between them and acting at a point, is given by the equation

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

The angles β and γ may be determined by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha \qquad \sin\gamma = \frac{P}{R}\sin\alpha$$

where, β = angle between the forces R and P
γ = angle between the forces R and Q

PRACTICE SET 2.1

Review Questions

1. What do you study in statics?
2. Define rigid body.
3. State and explain the concept of force.
4. Give the examples of force.
5. What is meant by specifications of a force?
6. Explain the term concentrated force.
7. Define vector and free vector.
8. State and explain the parallelogram law of forces.
9. State and explain the triangle law of forces.
10. Explain rectangular components of a force.
11. State the law of equilibrium of two forces.
12. Discuss about two force body or two force members.
13. Define tension and compression in a bar.
14. State the law of superposition of forces.
15. Explain the theorem of transmissibility of a force.
16. What is the limitation of the theorem of transmissibility of a force?
17. State the law of action and reaction.
18. Define constraint and explain the constraints string, smooth surface and linear elastic spring .
19. Differentiate between active and reactive forces.
20. What is a 'free body diagram'? Explain it with the help of figures.
21. State the essential problem of statics.

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PROBLEM SET 2.1

1. A man of weight W = 712 N holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight Q = 534 N. Find the force with which the man's feet press against the floor. (*Ans.* 178 N)
2. A boat is moved uniformly along a canal by two horses pulling with forces P = 890 N and Q = 1068 N acting under an angle $\alpha = 60°$ (Fig. A). Determine the magnitude of the resultant pull on the boat and the angles β and γ as shown in the figure. (*Ans.* R=1698 N; $\beta = 33°$; $\gamma = 27°$)

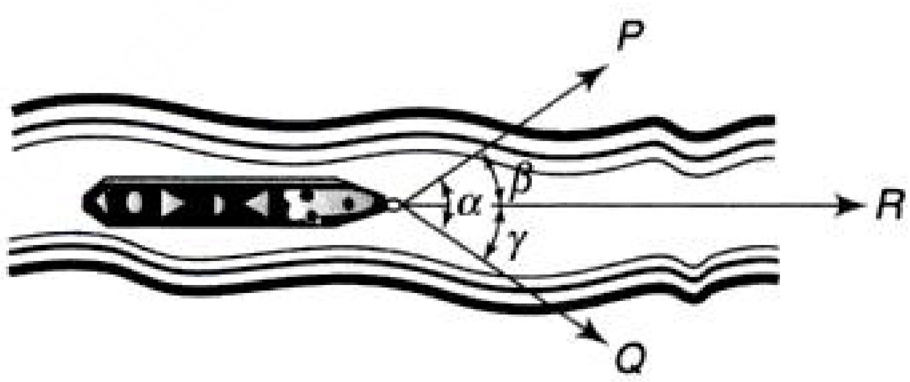

Fig. A

3. What force Q combined with a vertical pull P = 27 N will give a horizontal resultant force R = 36 N? (*Ans.* 45 N inclined by 36° 52′)
4. To move a boat uniformly along a canal at a given speed requires a resultant force R = 1780 N. This is accomplished by two horses pulling with forces P and Q on two ropes, as shown in Fig. A. If the angles that the two ropes make with the axis of the canal are $\beta = 35°$ and $\gamma = 25°$, what are the corresponding tensions in the ropes? (*Ans.* P = 868 N; Q = 1179 N)
5. If, in Fig. A, the horses pull with the forces P = 1068 N and Q = 890 N, what must be the angles β and γ to give the resultant R = 1780 N ? (*Ans.* $\beta = 22°\ 22'$; $\gamma = 27°\ 12'$)
6. Draw the free body diagram of the boom BC and point B shown in Fig. B.

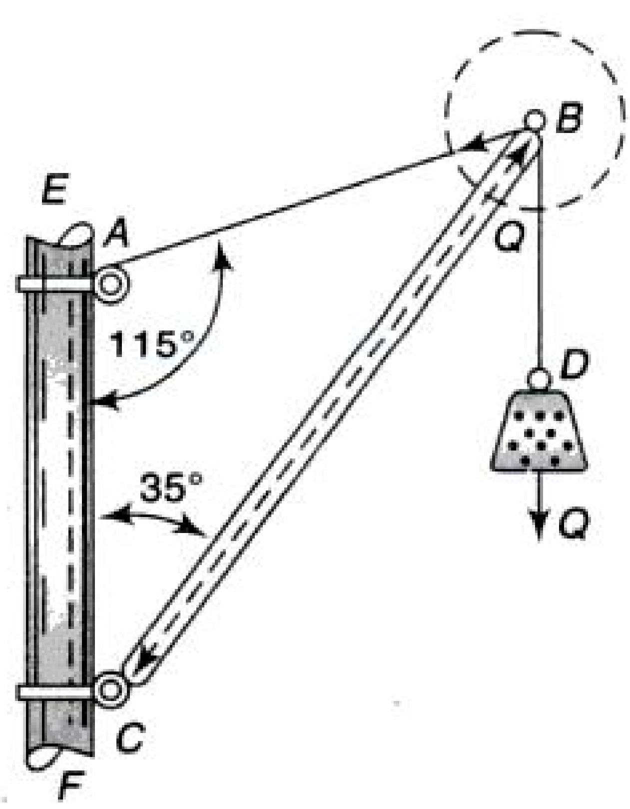

Fig. B

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resultant $\overline{AC}$ of the two forces F_1 and F_2. Combining this resultant with the force F_3, we obtain the resultant $\overline{AD}$ which must be equivalent to F_1, F_2 and F_3. Finally, combining the forces $\overline{AD}$ and F_4, we obtain the resultant R of the given system F_1, F_2, F_3, F_4. This procedure may be carried on for any number of given forces acting at one point in a plane.

It is evident, in the above case, that exactly the same resultant R will be obtained by successive geometric addition of the free vectors representing the given forces [Fig. 2.31(b)]. In this case we begin with the vector $\overline{AB}$ representing the force F_1. From the end B of this vector we construct the vector $\overline{BC}$, representing the force F_2, and afterward, the vectors $\overline{CD}$ and $\overline{DE}$, representing the forces F_3 and F_4. The polygon $ABCDE$ obtained in this way is the same as the polygon $ABCDE$ in Fig. 2.31(a), and the vector $\overline{AE}$, from the beginning A of the vector

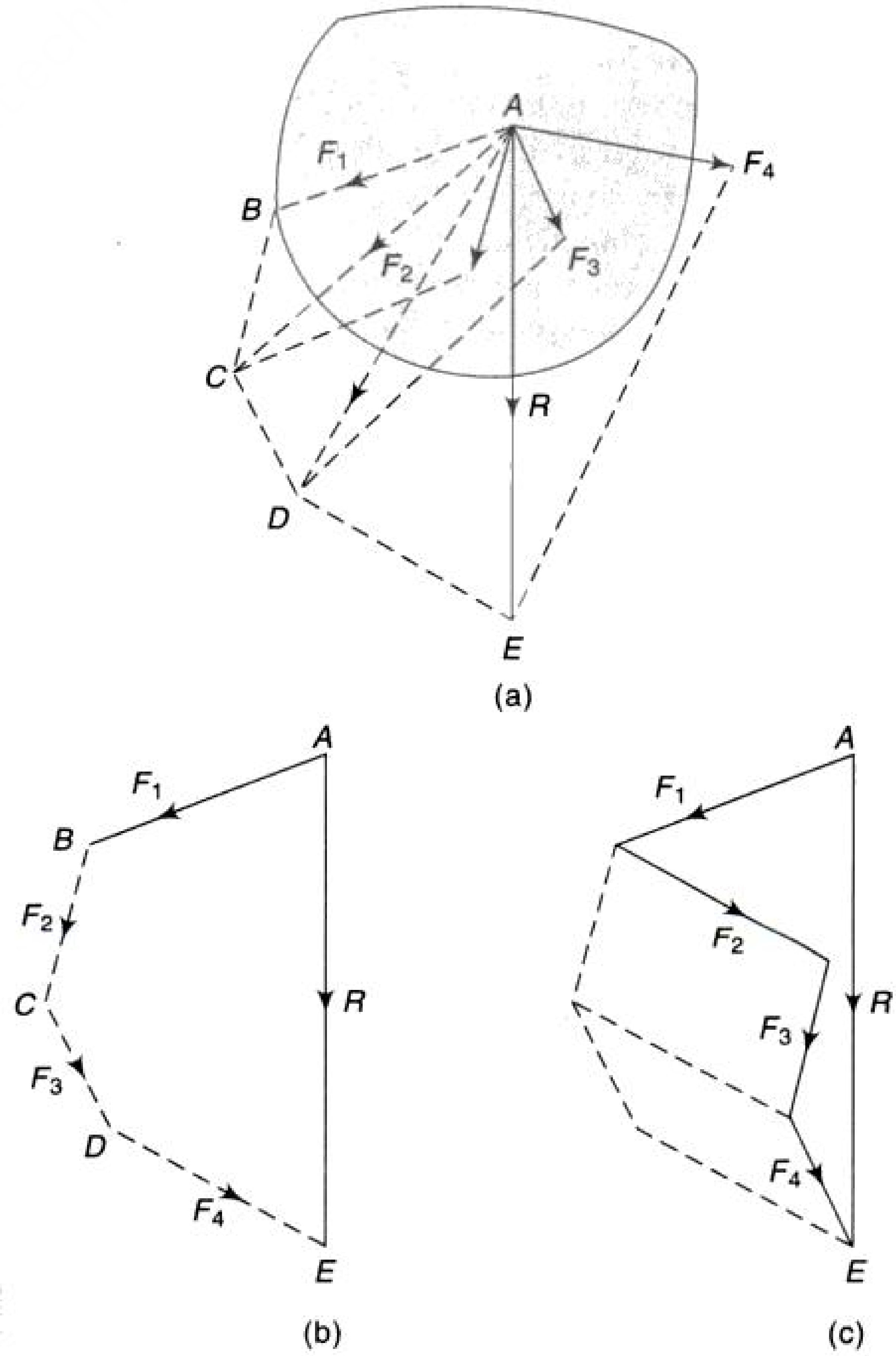

Fig. 2.31

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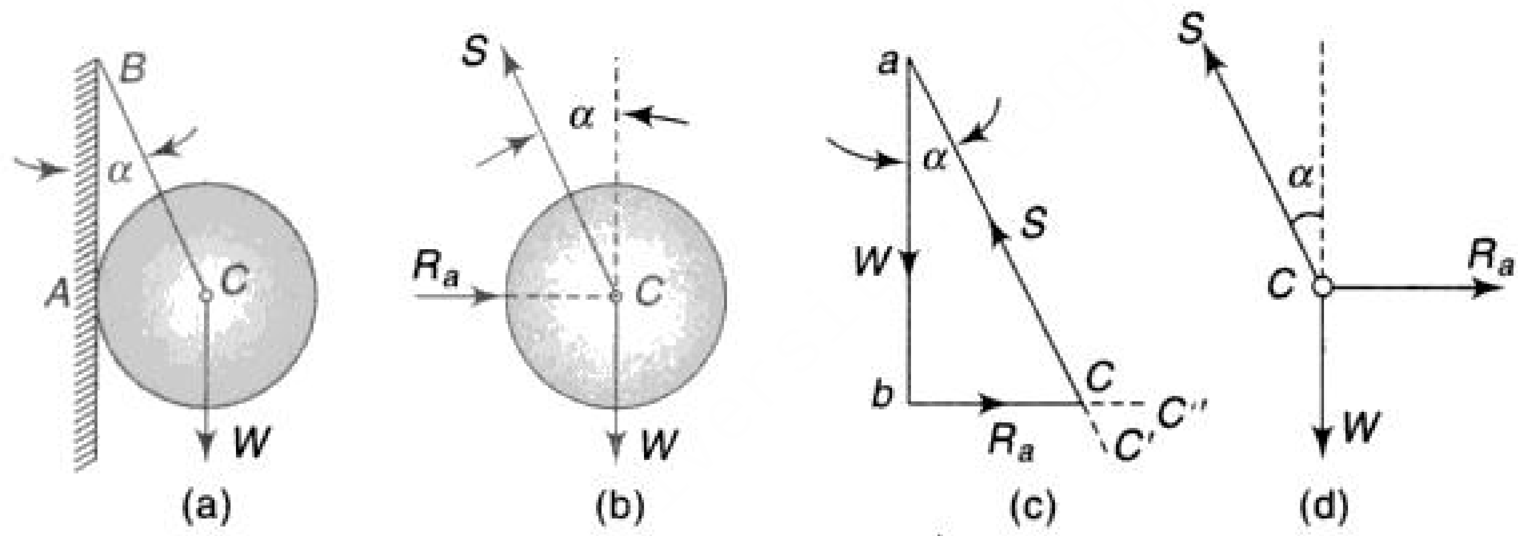

Fig. 2.32

tion C of these two lines determines the required magnitudes of R_a and S and the arrows show the directions that these forces must have to build a closed triangle. It should be noted that in any closed polygon of forces, the vectors must follow one another tail to head around the polygon. The magnitudes of R_a and S may now be scaled from the drawing and the problem is solved.

The foregoing procedure, in which the closed polygon of forces is constructed to scale and the magnitudes of the reactions measured directly from the drawing, is called a graphical solution of the problem. In order to make such a solution, of course, numerical values would have to be given for the magnitude of W and α, the angle between the string and the wall.

If numerical data are not given, we can still sketch the closed triangle of forces as shown in Fig. 2.32(c) and then express the magnitudes of R_a and S in terms of W and α by trigonometry. In the present case, for example, we see from the triangle of forces that

$$R_a = W \tan \alpha \qquad S = W \sec \alpha \tag{a}$$

Then for any given numerical data, the magnitudes of R_a and S can be computed from expressions (a). Such an analysis of the problem is called a *trigonometric solution*.

Lami's Theorem: *If three concurrent forces are acting on a body, kept in an equilibrium, then each force is proportional to the sine of the angle between the other two forces and the constant of proportionality is the same.*

Consider forces P, Q and R acting at point O as shown in Fig. 2.33(a). Mathematically, Lami's theorem is given by the following equation,

$$\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma} = k \tag{b}$$

Since the forces are in equilibrium, the triangle of forces should close. Draw the triangle of forces ΔABC as shown in Fig. 2.33(b) corresponding to the forces P, Q and R acting at a point O. The angles of triangle are

$$\angle A = \pi - \alpha$$
$$\angle B = \pi - \beta$$
$$\angle A = \pi - \gamma$$

From the sine rule for the triangle, we get

$$\frac{P}{\sin(\pi - \alpha)} = \frac{Q}{\sin(\pi - \beta)} = \frac{R}{\sin(\pi - \gamma)}$$

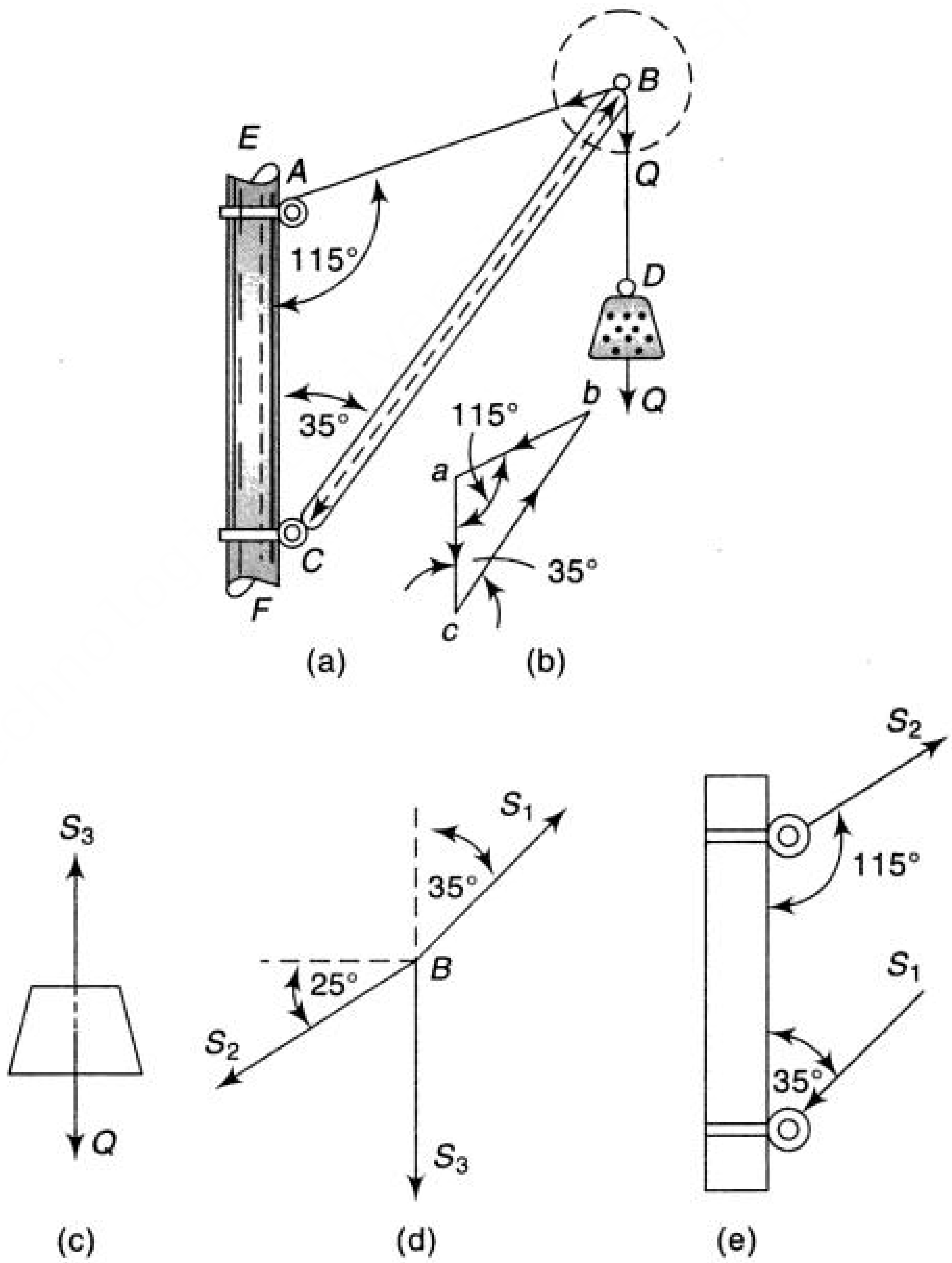

Fig. 2.36

BC. Since each of these members is a body acted upon by forces only at its ends and since we are assuming an ideal hinge at C, we conclude that the direction of each of these reactions must coincide with the axis of the member that produces it. The free-body diagram for the pin at B is as shown in the circle around this joint [Fig. 2.36(a)].

Now having given the magnitude and direction of one of the three forces in equilibrium and the lines of action of the other two, the magnitudes of these latter two forces are obtained by constructing the triangle of forces [Fig. 2.36(b)]. Knowing that the vector $\overline{ac}$, representing the weight Q, acts downward, the arrows on the other two vectors $\overline{cb}$ and $\overline{ba}$ must be directed as shown on the triangle of forces, since all arrows must follow each other tail to head around any closed polygon of forces. Considering the vector $\overline{cb}$ which represents the reaction of the boom on the pin at B, we see that the boom pushes against this pin and hence is in compression. Similarly, the arrow on the vector $\overline{ba}$ indicates tension in the cable AB. In general, if the directions of any unknown reactions are assumed incorrectly in the free-body diagram, they may be corrected after the construction of the polygon of forces.

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tal. From this it follows that by continuing the line BC down to point D, we obtain an isosceles triangle ACD . Thus, it is evident that ΔBFD is a 3 : 4 : 5 triangle. Now from the similarity of ΔBGH and ΔBDF we may write $GH : BH = DF : BF = 4{:}3$ or, using the given dimensions, $(4.88 - 2x) : 0.915 = 4 : 3$ from which = 1.83 m.

The triangle of forces [Fig. 2.38(b)] for the three forces in equilibrium at C is similar, by construction, to ΔACD. Hence, $S : P = 5 : 6$, from which we conclude that the tensile force in the cord is 133.33 N.

Alternate Solution: After drawing the free body diagram as above, using the free body diagram of point C as shown in Fig. 2.38(a) and applying Lami's theorem, we get

$$\frac{S}{\sin\left(\dfrac{\pi}{2}+\alpha\right)} = \frac{P}{\sin(\pi - 2\alpha)} \tag{e}$$

and $$\tan\alpha = \frac{BH}{GH} = \frac{3}{4} \tag{f}$$

Simplifying Eq. (e) for S,

$$\Rightarrow \qquad S = P\frac{\cos\alpha}{\sin 2\alpha} = P\frac{\cos\alpha}{2\sin\alpha\cos\alpha} = \frac{P}{2\sin\alpha} \tag{g}$$

Solving for α from Eq. (f) and substituting this in Eq. (g), we obtain

$$S = 133.33 \text{ N}$$

as before.

5. Determine the magnitude and direction of the smallest force P, which will maintain the body of weight $W = 300$ N on an inclined smooth plane as shown in Fig. 2.39(a), is in equilibrium.

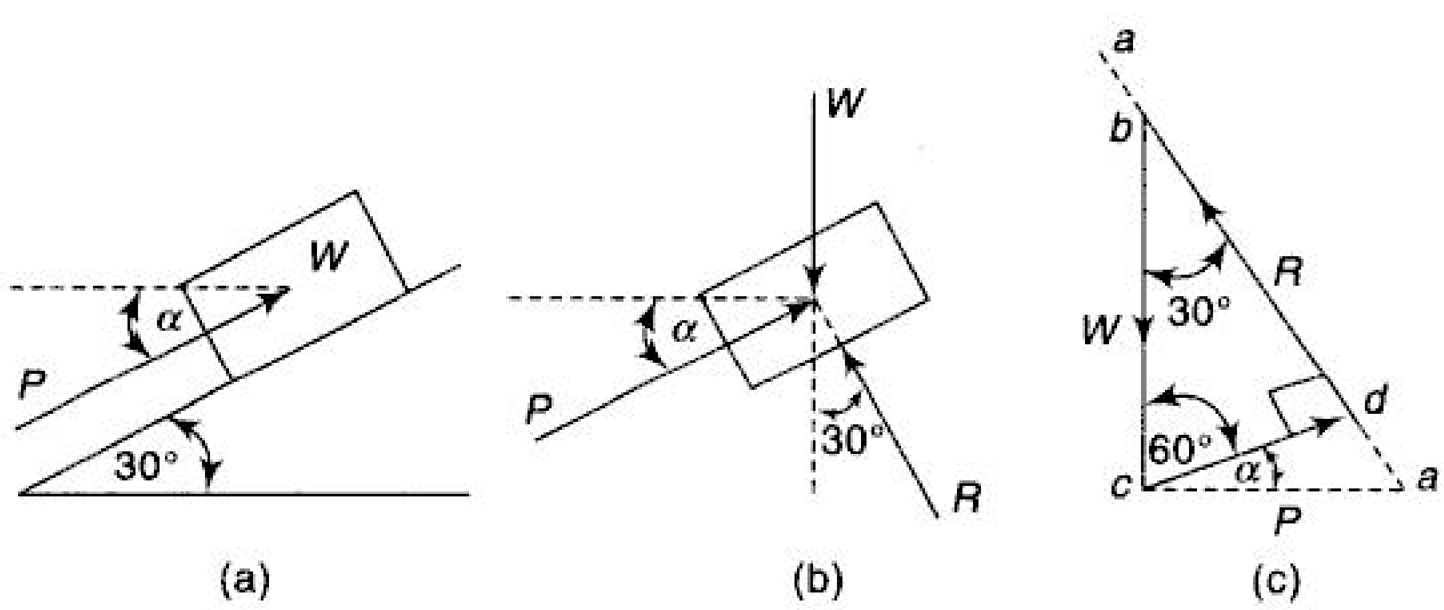

Fig. 2.39

Solution: Since the inclined plane is smooth, the pressure at the point of contact between body and inclined plane must be normal to the surface. Removing supporting inclined plane and replacing it by its reaction R, we obtain the free body diagram of the body as shown in Fig. 2.39(b).

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Simplifying the above equation, we will get

$$\Rightarrow \qquad S_1 = \frac{S_3 \cos\beta}{\sin(\alpha+\beta)} = \frac{W\cos\beta}{\sin(\alpha+\beta)}$$

$$S_2 = \frac{S_3 \cos\alpha}{\sin(\alpha+\beta)} = \frac{W\cos\alpha}{\sin(\alpha+\beta)}$$

Substituting the values of W, α and β in the above equations, we obtain

$$\Rightarrow \qquad S_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)} = \frac{40\text{N}\times\cos 36.87^\circ}{\sin(45^\circ+36.87^\circ)} = 32.33 \text{ N}$$

$$S_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)} = \frac{40\text{N}\times\cos 45^\circ}{\sin(45^\circ+36.87^\circ)} = 28.57 \text{ N}$$

The spring force is given by the formula

$$S = kx$$

$$\Rightarrow \qquad S_1 = k_1x_1 \qquad \text{(k)}$$

$$S_2 = k_2x_2$$

$$S_3 = k_3x_3$$

Substituting the values of S_1, S_2, S_3, k_1, k_2 and k_3, in the above equations, we get the stretch in each spring as below:

$$x_1 = \frac{S_1}{k_1} = \frac{32.33}{40} = 0.81 \text{ m}$$

$$x_2 = \frac{S_2}{k_2} = \frac{28.57}{50} = 0.57 \text{ m}$$

$$x_3 = \frac{S_3}{k_3} = \frac{40}{60} = 0.67 \text{ m}$$

8. Determine the axial forces S_1 and S_2 induced in the bars AC and BC in Fig. 2.42(a) due to the action of the horizontal force P applied load at C. The bars are hinged together at C and to the foundation at A and B.

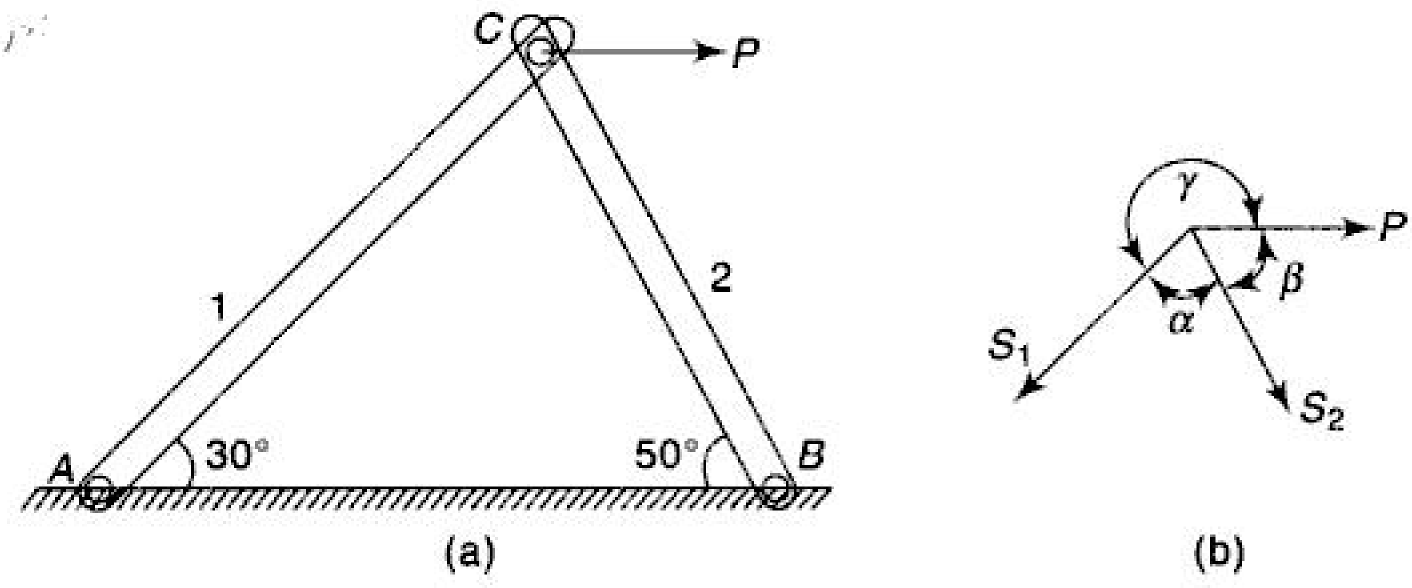

Fig. 2.42

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2. The force that cancels the effect of the force system acting on the body is known as
   (a) Resultant (b) Neutral force (c) Balancing force (d) Equilibrant
   [*Ans.* (d)]
3. Consider the following statements :
   I. The resultant of any number of concurrent forces in a plane is given by the closing side of the polygon of forces obtained by successive geometric addition of their free vectors.
   II. If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces, or rather their free vectors, when geometrically added must form a closed polygon.

   Of these statements
   (a) 1alone is correct (b) 2 alone is correct
   (c) 1 and 2 are correct (d) Neither 1 nor 2 is correct [*Ans.* (c)]
4. Match the list I with the list II using the codes given below:

I	II
A. Equilibrium	1. A force, which is equal, opposite and collinear to the resultant of a concurrent force system.
B. Equilibrant	2. The resultant vanishes.
C. Statically indeterminate	3. Supports in excess of those necessary and sufficient to completely constrain the body in the plane of the figure.
D. Redundant constraints	4. In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane contains more than two reactive forces.

Codes:

(a)	A-1	B-2	C-3	D-4
(b)	A-2	B-3	C-4	D-1
(c)	A-2	B-1	C-4	D-3
(d)	A-4	B-3	C-2	D-1

[*Ans.* (c)]

5. Assertion (A): In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we cannot determine definitely the magnitudes of more than two reactive forces and the problem is said to be statically indeterminate.

   Reason (R): The resolution of a given force into more than two coplanar concurrent components is an indeterminate problem.

   Select the answer from the following codes:
   (a) A and R are true and R is the correct explanation of A.
   (b) A and R are true and R is not the correct explanation of A.
   (c) A is true and R is false.
   (d) Both A and R are false. [*Ans.* (a)]

PROBLEM SET 2.3

1. An electric-light fixture of weight $Q = 178$ N is supported as shown in Fig. A. Determine the tensile forces S_1 and S_2 in the wires BA and BC if their angles of inclination are as shown. (*Ans.* $S_1 = 130.3$ N; $S_2 = 92.14$ N)
2. A ball of weight $Q = 53.4$ N rests in a right-angled trough, as shown in Fig. B. Determine the forces exerted on the sides of the trough at D and E if all surfaces are perfectly smooth. (*Ans.* $R_d = 46.25$ N; $R_e = 26.7$ N)

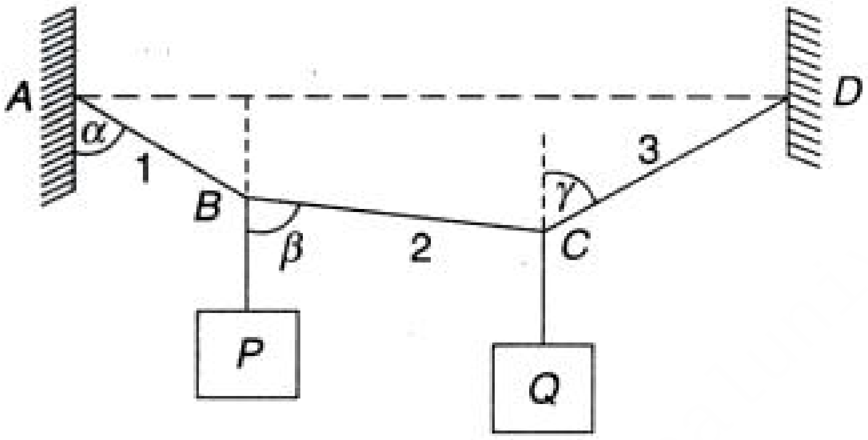

Fig. M

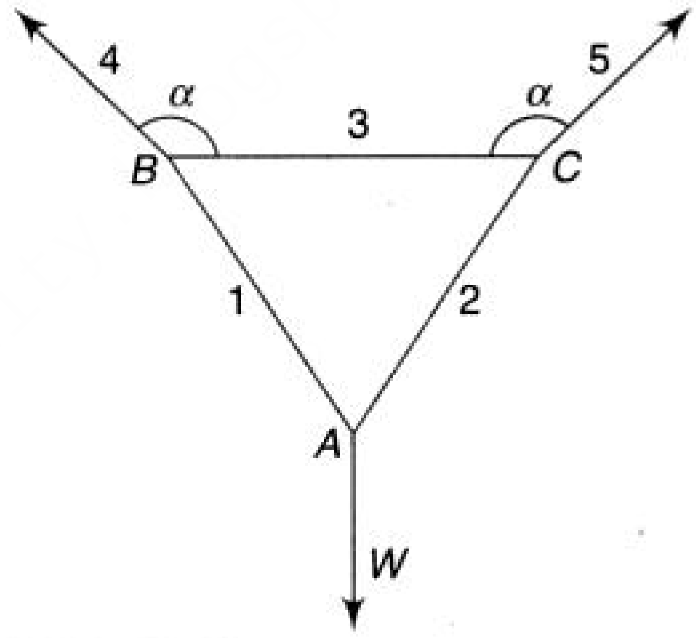

Fig. N

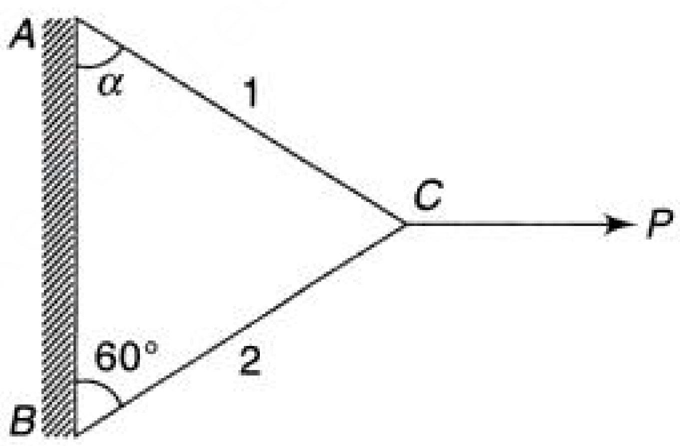

Fig. O

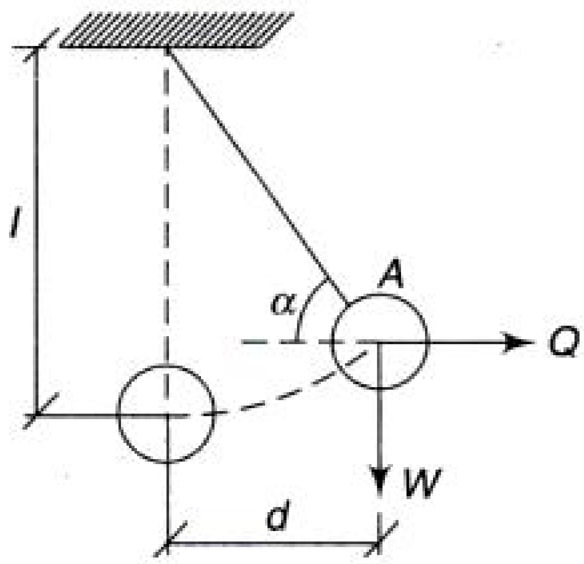

Fig. P

$$\left(Ans.\ \alpha = \cos^{-1}\frac{d}{l}, Q = \frac{Wd}{\sqrt{l^2 - d^2}}, S = \frac{Wl}{\sqrt{l^2 - d^2}} \right)$$

17. A weight 100 N hangs by an inextensible string from a fixed point A. The string is drawn out of the vertical by applying a force 50 N to the weight at point B as shown in Fig. Q. In what direction must this force be applied in order that, in equilibrium, the direction of the string from the vertical may have its greatest value. What is the amount of greatest deflection. Find also the tension in the string.
(*Ans.* $\alpha = 90°$, $\beta = 30°$, $S = 86.6$ N)

18. Three bars in one plane, hinged at their ends as shown in Fig. R, are submitted to the action of a force $P = 44.5$ N applied at the hinge as shown. Determine the magnitude of the force that it will be necessary to apply at the hinge in order to keep the system of bars in equilibrium if the angles between the bars and the lines of action of the forces are as given in the figure. (*Ans.* $Q = 72.54$ N)

*19. A rigid bar with rollers of weights $P = 222.5$ N and $Q = 445$ N at its ends is supported inside a circular ring in a vertical plane as shown in Fig. S. The radius of the ring and the length AB are such that the radii AC and BC form a right angle at C; that is, $\alpha + \beta = 90°$. Neglecting friction and the weight of the bar AB, find the configuration of equilibrium as defined by the angle $(\alpha - \beta)/2$ that makes with the horizontal. Find also the reactions R_a and R_b and the compressive force S in the bar AB.
(*Ans.* $(\alpha - \beta)/2 = 18°26'$; $R_a = 298.5$ N; $R_b = 597$ N; $S = 281.5$ N)

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$$\alpha = \arc\tan \frac{Y}{X} = \arc\tan \frac{+265.8}{-361.6} = 143°41'$$

We see from the signs of X and Y that the resultant lies in the second quadrant; hence the angle α is measured out in the counterclockwise direction from the positive end of the x axis, as shown.

2. A load $P = 4450$ N is bracketed from a vertical wall by two bars AB and AC hinged together at A and to the wall at B and C as shown in Fig. 2.46. Using the method of projections, compute the axial forces S_1 and S_2 induced in these bars.

Solution: We first make a free body of the pin A, replacing the bars AB and AC by the reactions S_1 and S_2 directed as shown in the figure. Then choosing coordinate axes x and y as shown, the equations of equilibrium (3) become

$$-S_1 + 0.500\,P = 0$$
$$+\,S_2 - 0.866\,P = 0$$

giving $S_1 = 0.500P = 2250$ N tension and

$S_1 = 0.866P = 3853.\ 7$ N compression

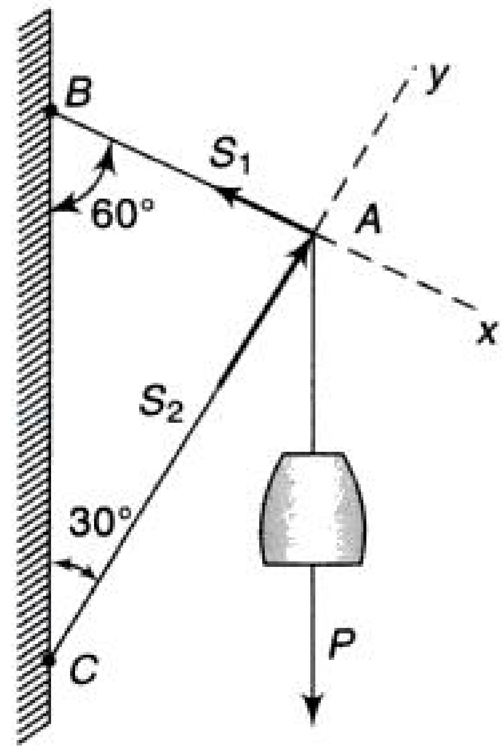

Fig. 2.46

3. A small ring B carries a vertical load P and is supported by two strings BA and BC, the latter of which carries at its free end a weight $Q = 44.5$ N, as shown in Fig. 2.47. Find the magnitude of the load P and the tension S in the string AB if the angles that the strings AB and BC make with the vertical are as shown in the figure and the system is in equilibrium.

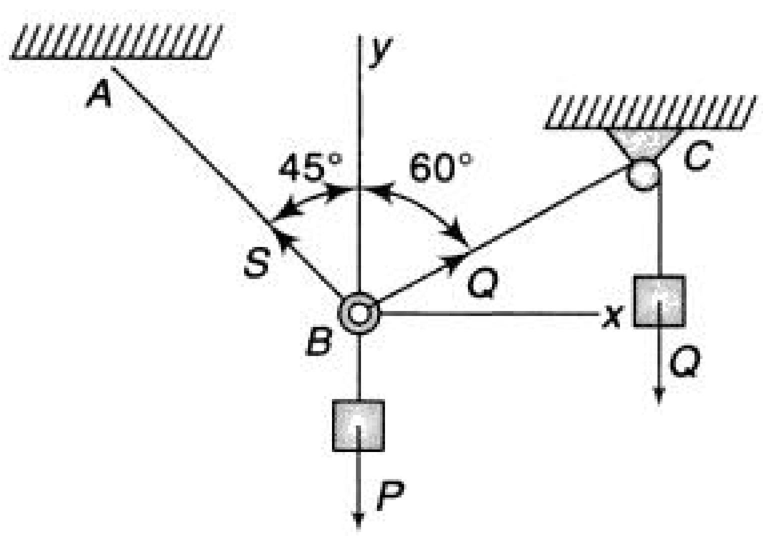

Fig. 2.47

Solution: Neglecting friction in the pulley at C, it is evident that the tension in the string BC is equal to the load Q . Thus, acting on the ring B, we have three concurrent forces in a plane that are in equilibrium. Taking coordinate axes x and y as shown, Eq. (3) become

$$Q \cos 30° - S \cos 45° = 0$$
$$Q \cos 60° + S \cos 45° = P$$

From the first of these equations, we find $S = 22.25\ \sqrt{6} = 54.50$ N. Then substituting in the second equation, we obtain

$$P = 22.25\ (1 + \sqrt{3}) = 60.8\text{N}$$

$$S_4 - Q = 0 \qquad \text{giving } S_4 = Q \text{ tension}$$

As the pulleys at D and E are frictionless, it is evident that the tensions in the string CD and CE are equal to weights P and Q. Thus, acting on the ring C, we have four concurrent forces in a plane that are in equilibrium. The free body diagram of the ring C is shown in Fig. 2.50(d). The forces acting on the ring are string forces 1, 2, 3 and 4. Taking the coordinate axes x and y as shown in the figure, Eq. (3) become

$$S_2 \cos \alpha + Q - S_1 \cos \beta - P \cos \gamma = 0$$
$$S_2 \sin \alpha + S_1 \sin \beta - P \sin \gamma = 0$$

Substituting the values for P and Q; angles α, β and γ in these equations, and solving for S_1 and S_2, we obtain

$$S_1 = 482.46 \text{ N Tension}$$
$$S_2 = 92.82 \text{ N Tension}$$

7. Two smooth cylinders each of weight P and Q, respectively, rest in a horizontal channel having one inclined wall and one vertical wall, the distance between them at bottom which is a [Fig. 2.51(a)]. Find the pressures exerted on the walls and floor at the points of contact A, B and D. The following numerical data are given: $P = 2000$ N and $Q = 800$ N; $r_0 = 100$ mm, $r_2 = 50$ mm and $a = 200$ mm; $\alpha = 60°$.

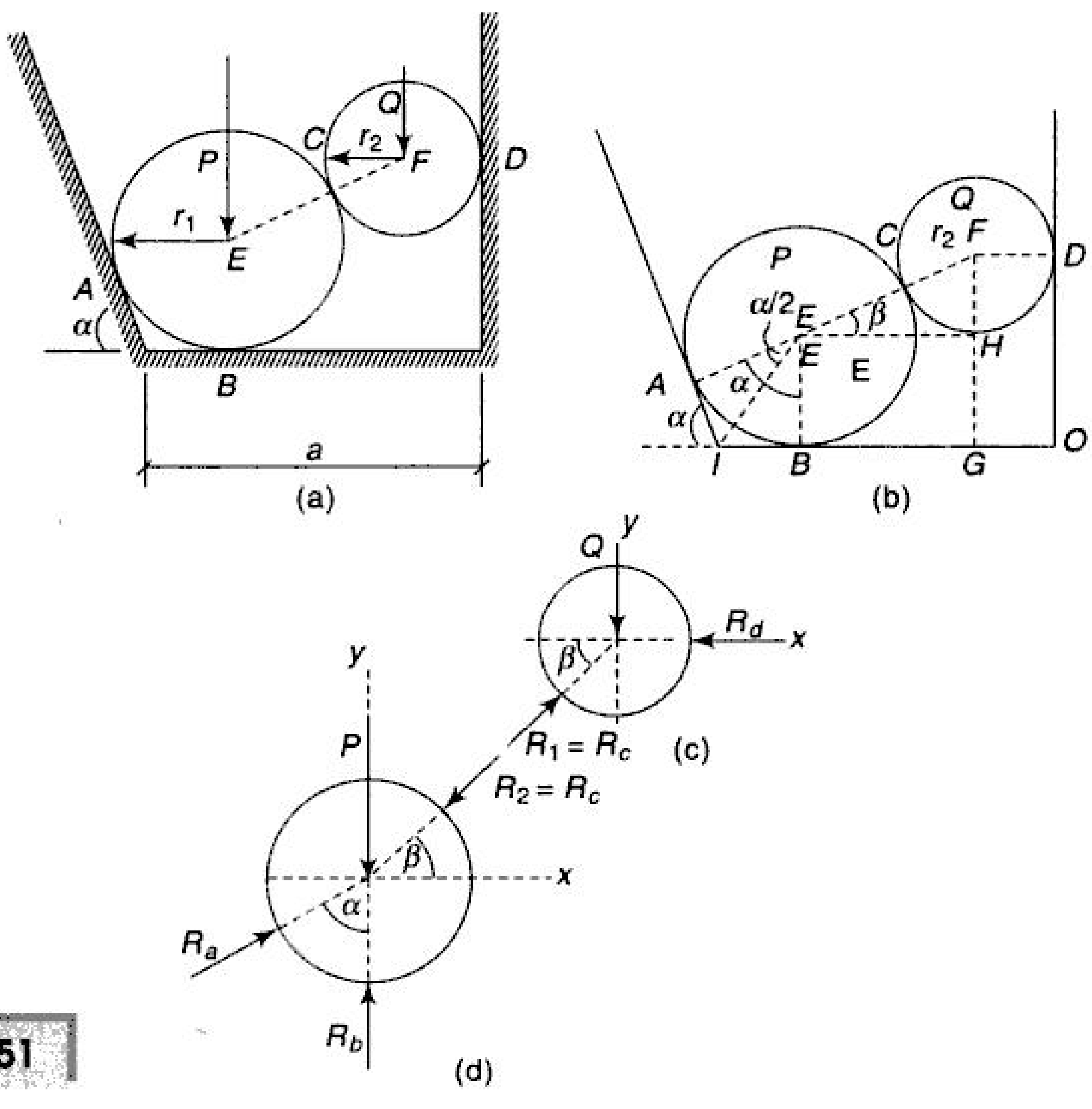

Fig. 2.51

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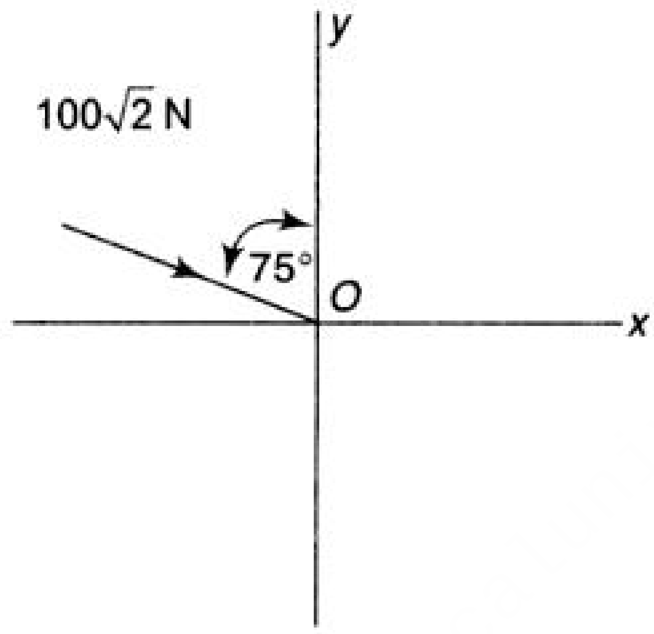
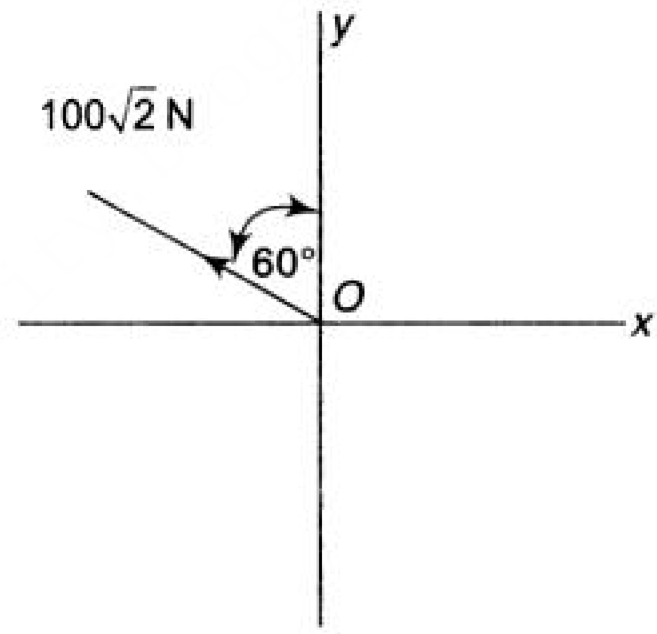


(c)



(d)

[*Ans.* (c)]

4. A lamp of weight $W = 100$ N is supported by two cables CA and CB as shown in the Fig. C. The equation for analysing the cable system is given by

   (a) $T_{CA}\dfrac{2}{\sqrt{4.563}} + T_{CB}\dfrac{1.5}{\sqrt{2.813}} = 0$ (b) $T_{CA}\sin\alpha + T_{CA}\sin\beta + 100 = 0$

   (c) $T_{CB}(3.5)\sin\alpha = 100(1.5)$ (d) $T_{CA}(3.5)\sin\alpha = 100(1.5)$ [*Ans.* (a)]

PROBLEM SET 2.4

1. Using the method of projections, find the magnitude and direction of the resultant R of the four concurrent forces shown in Fig. A and having the magnitudes F_1 = 1500 N, F_2 = 2000 N, F_3 = 3500 N and F_4 = 1000 N.

   (*Ans.* $R = 1842.6$ N and $\alpha = 227°$)

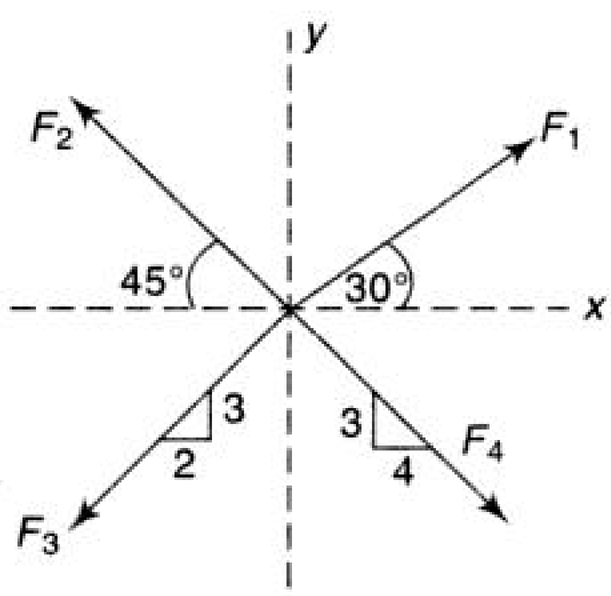


Fig. A

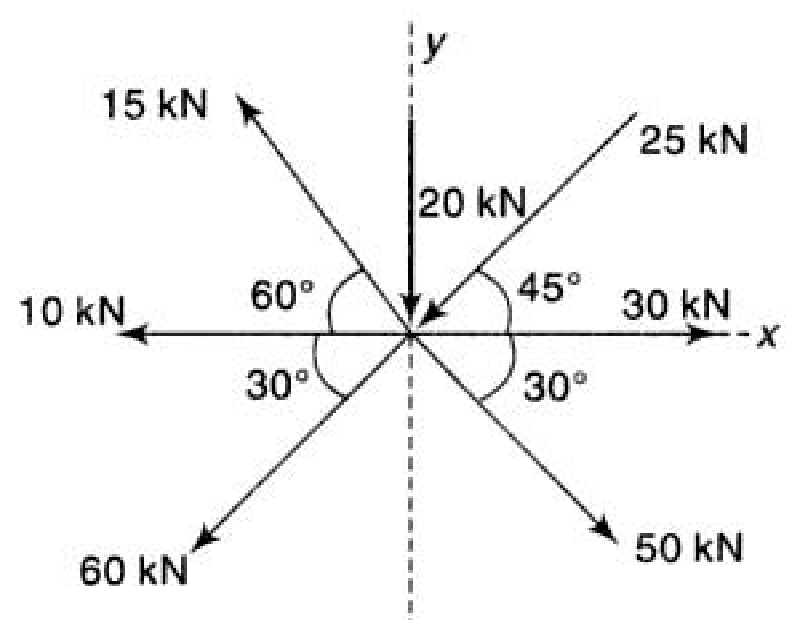


Fig. B

2. Forces of 2, 3, 4, 5 and 6 kN are acting at one of the angular points of a regular hexagon towards the other angular points taken in order. Find the resultant of the system of forces. (*Ans.* $R = 15.6$ kN; $\alpha = 76.7°$)
3. Find the magnitude and direction of the force F to be added to the system of coplanar concurrent forces shown in Fig. B to maintain equilibrium.

   (*Ans.* $F = 91$ kN, $\alpha = 61.18°$)
4. Referring to Fig. C, calculate the tensions S_1 and S_2 in the two strings AB and AC that support the lamp of weight $Q = 178$ N. Use the method of projections [Eq (3)].

   (*Ans.* $S_1 = 133.5$ N; and $S_2 = 222.5$ N)

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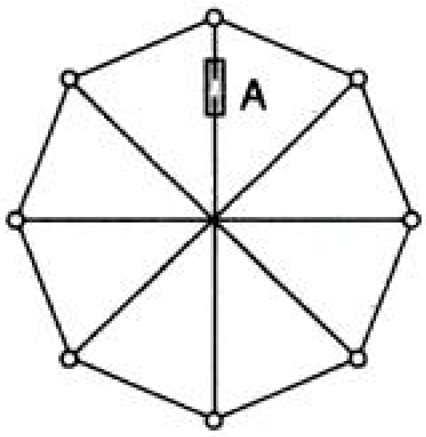

Fig. O

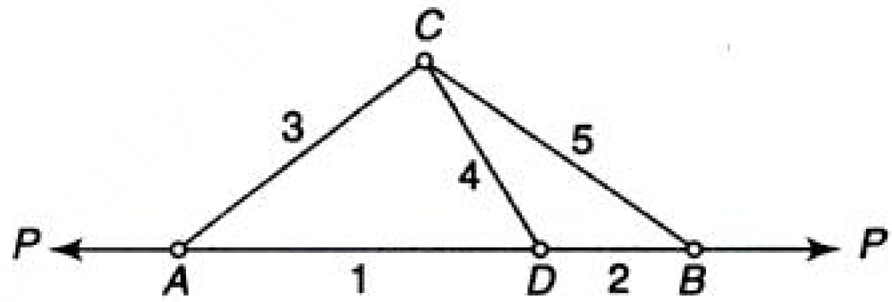

Fig. P

18. Determine the axial force induced in each bar of the system shown in Fig. P due to the action of the applied forces P. (*Ans.* $S_1 = S_2 = P$ tension; $S_3 = S_4 = S_5 = 0$)
19. The smooth cylinders rest in a horizontal channel having vertical walls, the distance between which is a (Fig. Q). Find the pressures exerted on the walls and floor at the points of contact A, B, D and F. The following numerical data are given: P = 200 N, Q = 400 N, R = 300 N, r_1 = 120 mm, r_2 = 180 mm, r_3 = 150 mm and a = 540 mm. (*Ans.* R_a = 525 N, R_b = 900N, R_d = 772.48 N and R_f = 247.48 N)

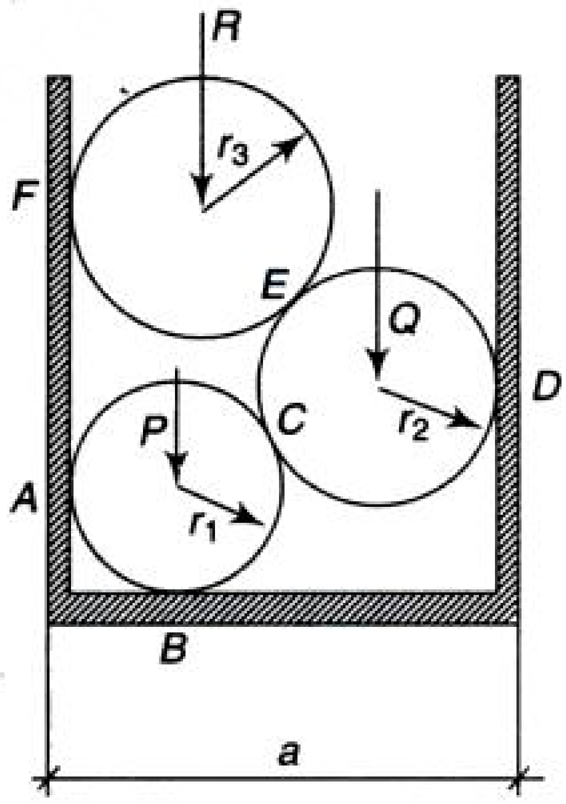

Fig. Q

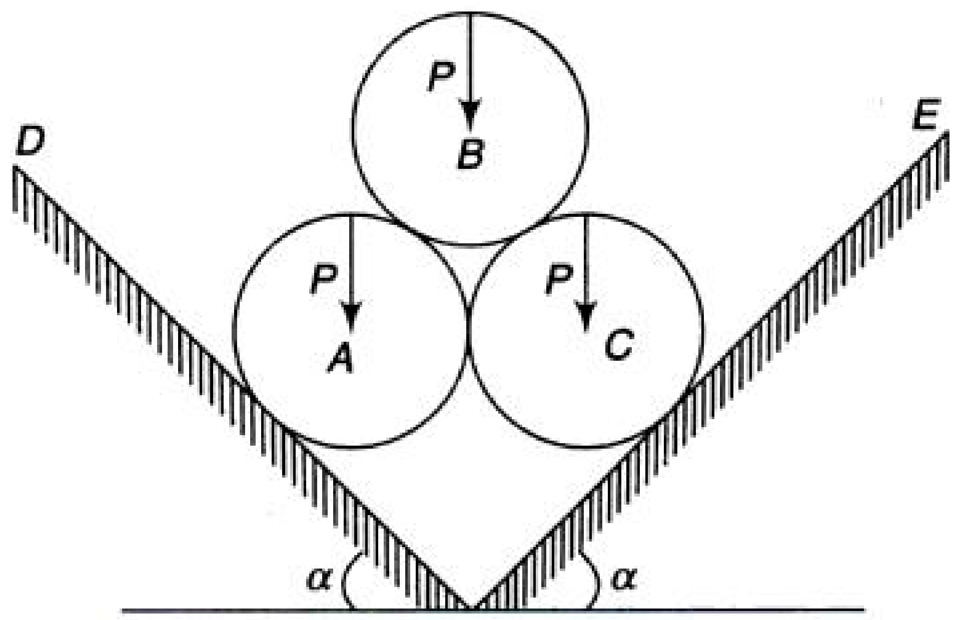

Fig. R

20. In Fig. R, three smooth right circular cylinders, each of radius r and weight P, are arranged on smooth inclined surfaces as shown. Determine the least value of angle α that will present the arrangement from slipping. (*Ans.* α = 10. 9°)
21. Two smooth cylinders of weights P and Q are placed in a smooth trough as shown in Fig. S. Determine the reactions at contact surfaces A, B and C. The following numerical data are given: P = 200 N and Q = 800 N; r_1 = 100 mm, r_2 = 200 mm, and a = 400 mm; α = 45°. (*Ans.* R_a = 70.7 N; R_b = 1414.21 N, R_c = 1070.71 N)
22. Three smooth spheres of weights P, P and Q are placed in a smooth trench as shown in Fig. T. Find the pressures exerted on the walls and floor at the points of contact A, B, C and D. The following numerical data are given: P = 0.3 kN, Q = 0.6 kN and R = 0.3 kN; r_1 = 0.4 m, r_2 = 0.6 m and r_3 = 0.4 m; α = 30°. (*Ans.* R_a = 61N, R_b = 635 N, R_c = 1100N, R_d =291 N)

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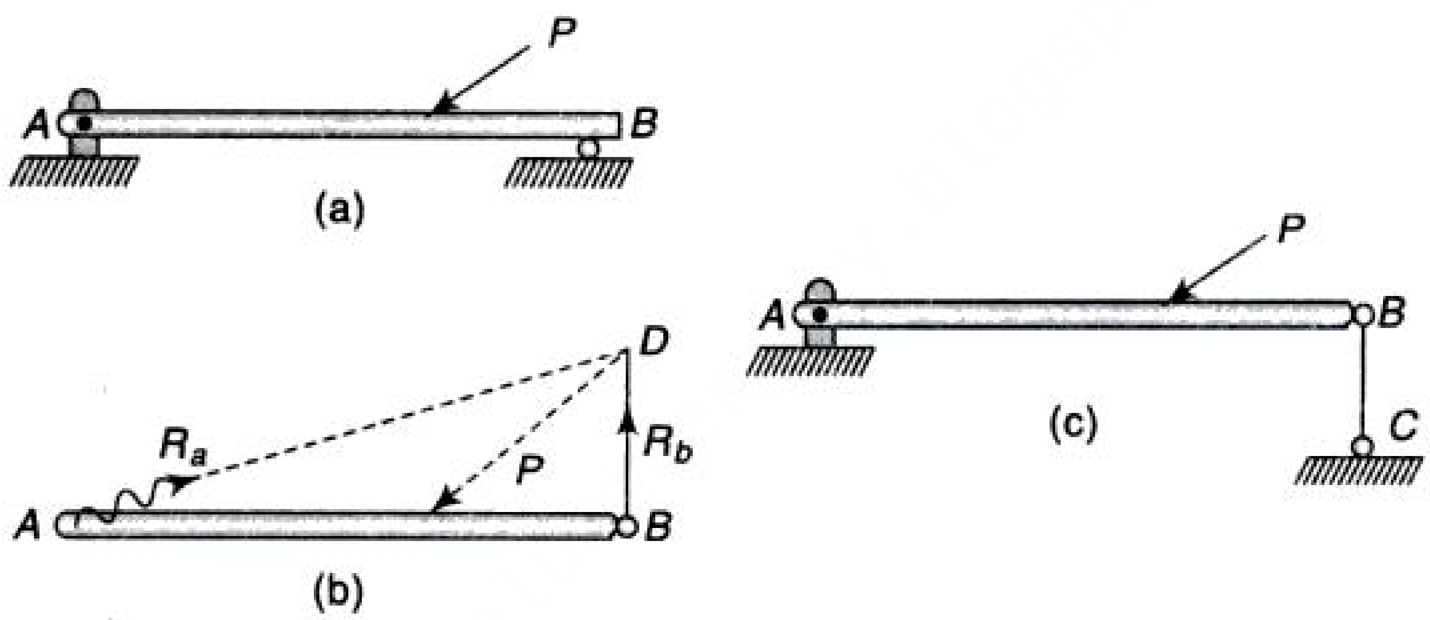

Fig. 2.55

the roller rests; in this case, vertically. Such rollers allowing free horizontal movement of one end of a beam are often used for bridge supports to prevent damage due to contraction or expansion of the beam resulting from changes in temperature. This *simple roller* is a common type of constraint and always exerts its reaction normal to the surface on which it rolls. To complete the free-body diagram in Fig. 2.55(b), we now observe that the known lines of action of P and R_b determine the point of concurrence D of the system of three forces in equilibrium; hence the true line of action AD of the reaction R_a is finally determined.

If the roller at B in Fig. 2.55(a) is replaced by a vertical bar BC of negligible weight and hinged at both ends [Fig. 2.55(c)], we shall obtain the same free-body diagram as before. Since forces act on this bar only at its two ends, their lines of action must coincide with the axis BC, as previously explained. Consequently, the reactive forces that the bar exerts on the beam must be directed along BC. Such a *simple strut* (or *tie bar* if it is in tension) is another common type of constraint used as a support.

Examples Examples Examples Examples Examples

1. The vertical axis AB of a crane is supported by a guide at A and a socket at B as shown in Fig. 2.56(a). Determine the reactions R_a and R_b produced at A and B by the load P = 35.6 kN. Friction at the supports should be neglected.

   Solution: Considering the entire crane as a free body [Fig. 2.56(a)], we imagine the supports at A and B removed and replace them by the reactions R_a and R_b which they exert on the crane. Thus we have the case of equilibrium of three forces, P, R_a and R_b in a plane and they must intersect in one point. Strictly speaking, the reaction exerted by the guide at A will be distributed over the surface of contact between the guide and the mast, but since the dimensions of this surface are small compared with the dimensions of the entire structure, we can assume this pressure to be concentrated at one point, say, the center of the guide, and further, neglecting friction, it

- The simple roller is a common type of constraint and always exerts its reaction normal to the surface on which it rolls.
- The simple bar is another common type of constraint used as a support. The reactive forces that the bar exerts on the body must be directed along the bar. A bar under tension is called tie. A bar under compression is called a strut.

PRACTICE SET 2.5

Review Questions

1. State the theorem of three forces in a plane.
2. Explain the reactions exerted by the supports namely roller support, hinge support, simple bar.
3. The reaction from an ideal smooth surface must be directed along the ______ at the point of contact.
4. Sketch the different types of supports and the reactions developed in each type.

Objective Questions

1. Consider the following statements.
   I. Three nonparallel forces can be in equilibrium only when they lie in one plane, intersect in one point, and their free vectors build a closed triangle.
   II. The reaction from an ideal smooth surface must be directed along the normal at the point of contact.
   Of these statements
   (a) I alone is correct (b) II alone is correct
   (c) I and II are correct (d) Neither I nor II is correct [*Ans.* (c)]

PROBLEM SET 2.5

1. A boat is suspended on two identical davits like *ABC* which is pivoted at *A* and supported by a guide at *B* (Fig. A). Determine the reactions R_a and R_b at the points of support *A* and *B* if the vertical load transmitted to each davit at *C* is 4272 N. Friction in the guide at *B* should be neglected.
   (*Ans.* R_a = 7121.73N; R_b = 5696.87 N)
2. A prismatic bar *AB* of weight Q = 17.8 kN is hinged to a vertical wall at *A* and supported at *B* by a cable *BC* (Fig. B). Determine the magnitude and direction of the

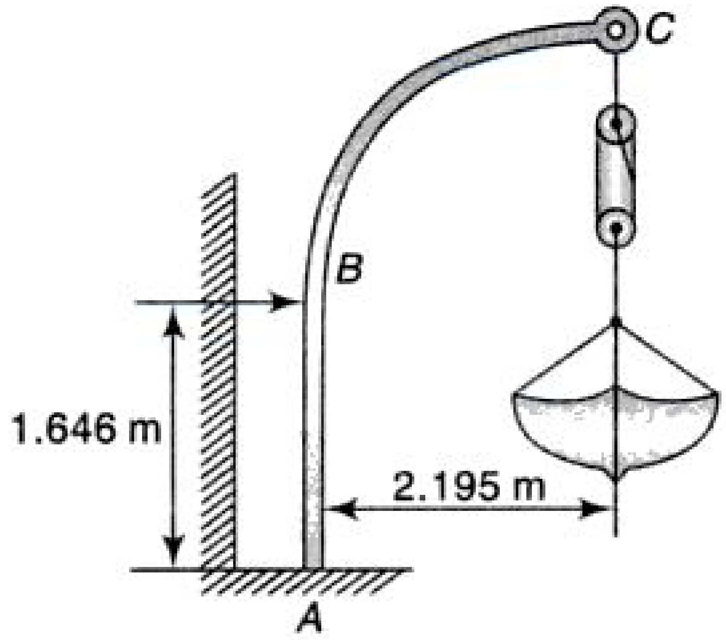

Fig. A

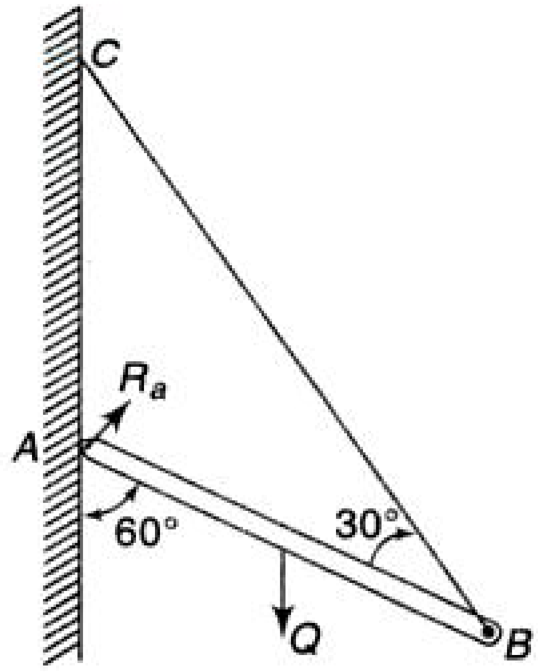

Fig. B

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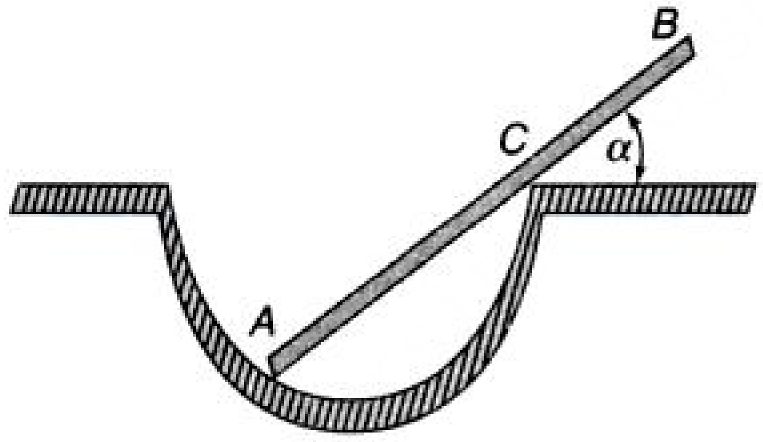

Fig. O

2.6 METHOD OF MOMENTS

Moment of a Force with Respect to a Point. The concept that a force tends to produce rotation about a fixed point in any body to which it is applied is useful in the solution of problems of statics. Consider, for example, the wrench shown in Fig. 2.59, to the handle of which two equal forces P and Q are applied as shown. It is a matter of common experience that the force P acting at right angles to the handle is more effective in tending to turn the nut to which the wrench is fitted than is the force Q, even though they are of equal magnitude. The effectiveness or importance of a force, as regards its tendency to produce rotation of a body about a fixed point, is called the *moment* of the force with respect to that point, and this moment can be measured by the product of the magnitude of the force and the distance from the point to the line of action of the force. Thus, the magnitude of the moment of the force Q with respect to the point O (Fig. 2.59) depends not only upon the magnitude of the force itself but also upon the distance OD from the point O to its line of action. The point O is called the *moment center*, and the distance OD is called the *arm of the force.*

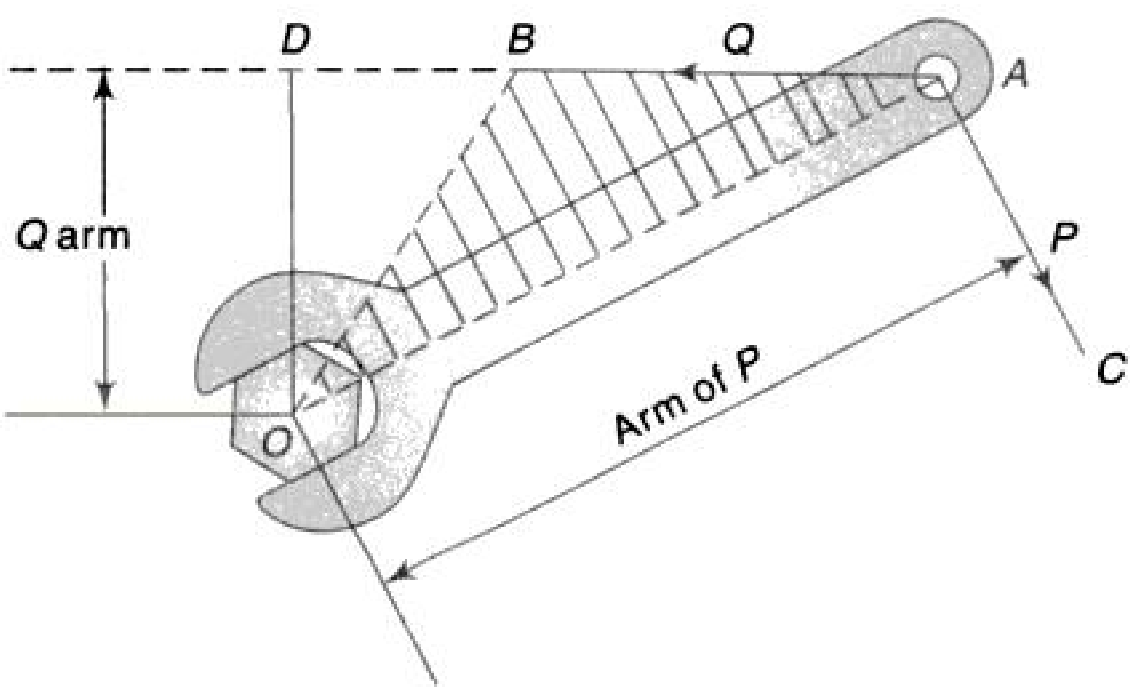

Fig. 2.59

From the preceding definition, it follows that the moment of the force Q is numerically equal to the doubled area of ΔABO, constructed on the vector $\overline{AB}$ representing the force, and having its vertex at the moment center. In this calculation the vector $\overline{AB}$ should be measured to the scale used for representing force, while the arm OD should be measured to the scale used for length. Thus it is seen that the unit of moment of force is the unit of length times the unit of force. For example, taking the *newton* as the unit of force and the *metre* as the unit of length,

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from which

$$R_b = \frac{l}{2} Q \cos\alpha$$

2. Figure 2.64 represents the cross-section of a retaining wall supporting an earth fill. The earth pressure per meter of length of wall can be replaced by its resultant H = 58.4 kN/m acting as shown in the figure. Determine the minimum thickness b of the wall required to prevent overturning about the front edge A if h = 4.57 m and the specific gravity of the wall is $2\frac{1}{2}$.

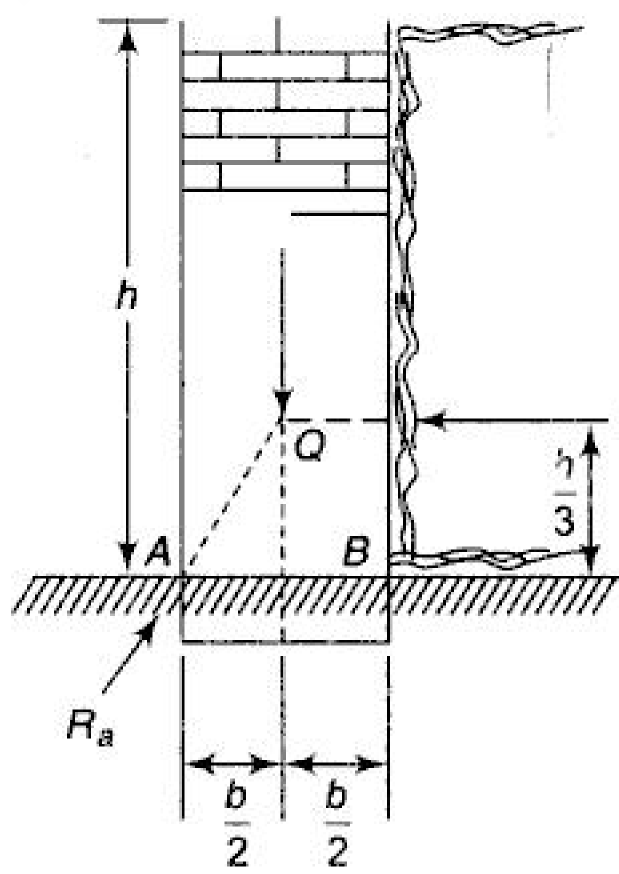

Fig. 2.64

Solution: Let us assume that we are dealing with a unit section of the wall 1 m in length normal to the plane of the figure. If such a section is made stable, then a similar wall of any length will be equally stable. When conditions are such that overturning of the wall impends the reaction R_a exerted by the foundation on the bottom of the wall will be concentrated at the front edge A as shown in the figure. Thus the 1-m section of wall is in equilibrium under the action of three forces: the weight

$$Q = 2\frac{1}{2} \times 9.8 \times 4.57b = 112b \text{ kN}$$

acting at the center of the cross-section, the earth pressure

$$H = 58.4 \text{ kN}$$

and the reaction R_a. The algebraic sum of the moments of these forces with respect to any center in the plane of the cross-section must be equal to zero. Taking point A as the moment center, we obtain

$$H\frac{h}{3} - Q\frac{b}{2} = 0$$

Substituting the numerical data as given and solving for b gives

$$b = \sqrt{1.589} = 1.261 \text{ m}$$

3. A slender prismatic bar AB of weight Q and length $2l$ rests on a very small frictionless roller at D and against a smooth vertical wall at A, as shown in Fig. 2.65. Find the angle α that the bar must make with the horizontal in the condition of equilibrium.

Solution: Isolating the bar AB, we obtain the free-body diagram as shown in the figure. The reaction at A is normal to the wall, that is, horizontal, and the reaction at D is normal to AB. When the bar is in a condition of equilibrium, the three forces Q, R_a and R_d meet in one point and the algebraic sum of their projections on any axis must be zero [Eq. (3)]. Likewise, the alge-

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6. A prismatic bar AB of negligible weight and length l is hinged at A and supported at B by a string that passes over a pulley C and carries a load P at its free end (Fig. 2.68). Assuming that the distance h between the hinge A and the pulley C is equal to the length l of the bar, find the angle α at which the system will be in equilibrium.

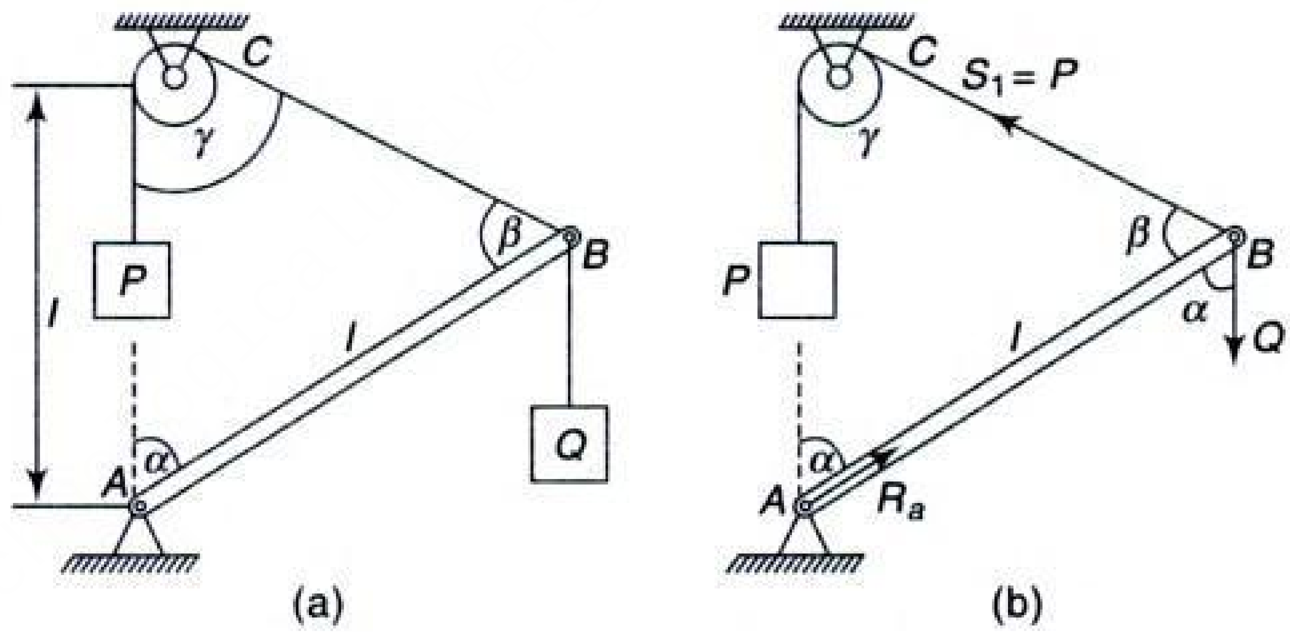

Fig. 2.68

Solution: We begin with a consideration of the equilibrium of the bar AB which is acted upon by three forces: A force Q representing the pull of the string at B, a force P representing the pull of the string at B, and a reaction at the hinge A. As the forces P and Q meet at B, we conclude that the reaction Ra must be directed along the line AB as shown.

The triangle ABC is isosceles

$$\beta = \gamma = \frac{\pi - \alpha}{2} = 90° - \left(\frac{\alpha}{2}\right)$$

Taking point A as the moment center (thus eliminating consideration of the unknown reaction at A), we obtain

$$(l)P \sin\left(90° - \frac{\alpha}{2}\right) - (l)(Q \sin \alpha) = 0$$

$$P \cos \frac{\alpha}{2} - (Q \sin \alpha) = 0$$

$$P \cos \frac{\alpha}{2} - \left(Q \times 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}\right) = 0$$

$$\Rightarrow \qquad \sin \frac{\alpha}{2} = \frac{P}{2Q}$$

$$\Rightarrow \qquad \alpha = 2 \sin^{-1}\left(\frac{P}{2Q}\right)$$

Important Terms and Concepts

Moment of a force with respect to a point   Moment center   Arm of the force
Newton-metre   Theorem of Varignon   Equilibrium equations

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7. A vertical load P is supported by a triangular bracket as shown in Fig. G. Find the forces transmitted to the bolts A and B. Assume that the bolt B fits loosely in a vertical slot in the plate. (*Ans.* $R_a = 1.25P$; $R_b = 0.75\ P$)

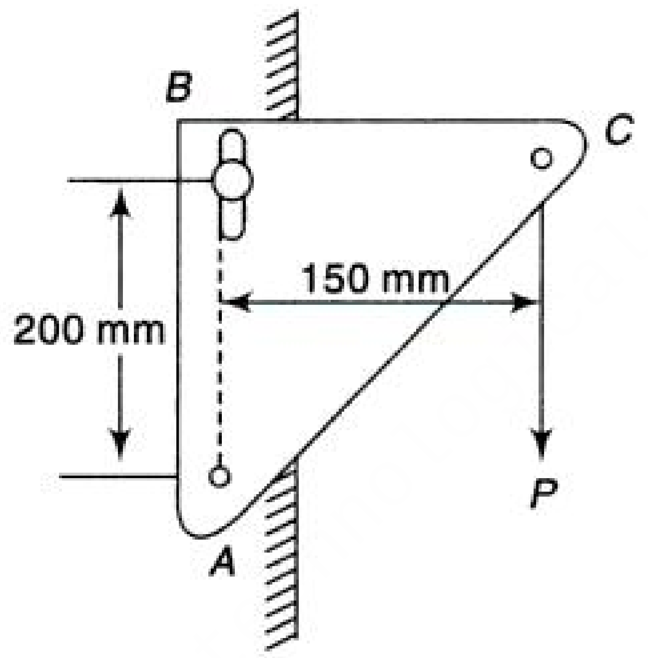

Fig. G

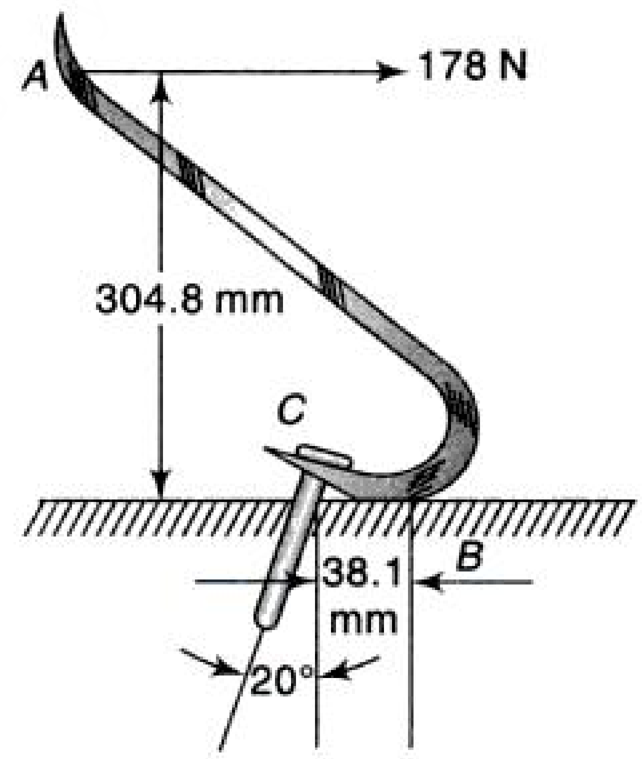

Fig. H

8. Find the magnitude of the pull P exerted on the nail C in Fig. H if a horizontal force of 178 N is applied to the handle of the wrecking bar as shown. (*Ans.* $P = 1515.4$ N)
9. Determine the forces exerted on the cylinder at B and C by the spanner wrench shown in Fig. I due to a vertical force of 222.5 N applied to the handle as shown. Neglect friction at B. (*Ans.* $R_b = 1068$ N; $R_c = 1091$ N)

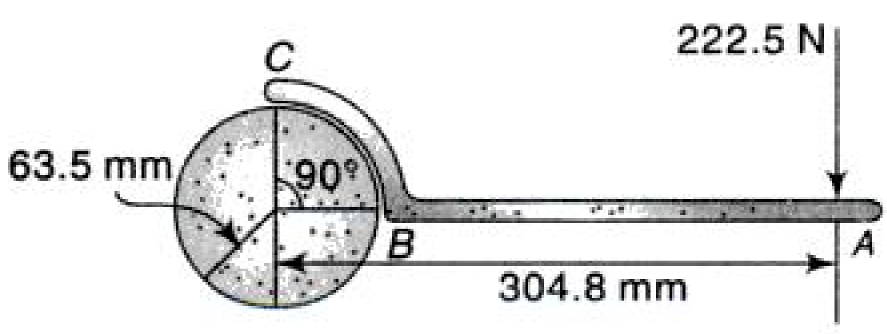

Fig. I

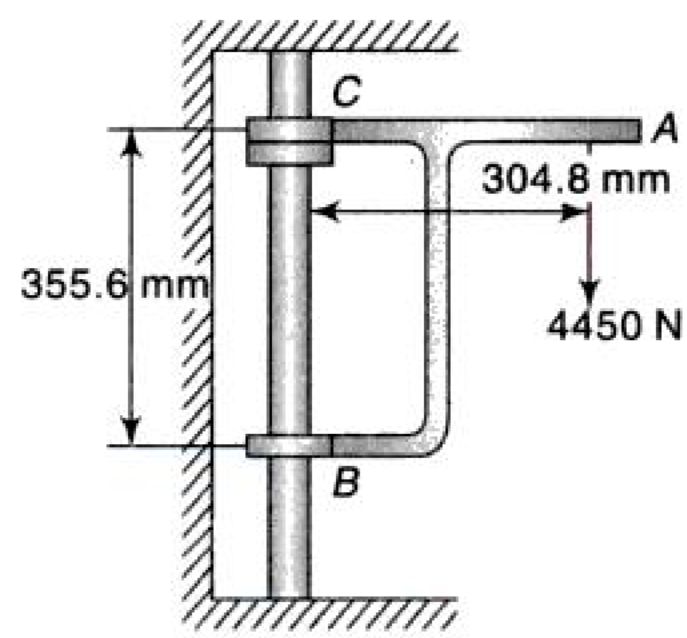

Fig. J

10. A bracket ACB can slide freely on the vertical shaft BC but is held by a small collar attached to the shaft as shown in Fig. J. Neglecting all friction, find the reactions at B and C for the vertical load shown. (*Ans.* $R_b = 3814.3$ N; $R_c = 5861$ N)
11. Two beams AB and DE are arranged and supported as shown in Fig. K. Find the magnitude of the reaction R_e at E due to the force $P = 890$ N applied at B as shown. (*Ans.* $R_e = 445$ N)
12. A smooth right circular cylinder of radius r rests on a horizontal plane and is kept from rolling by an inclined string AC of length $2r$ (Fig. L). A prismatic bar AB of length $3r$ and weight Q is hinged at point A and leans against the roller as shown. Find the tension S that will be induced in the string AC. (*Ans.* $S = 0.433\ Q$)

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These laws of friction may be expressed by the simple formula,

$$F = \mu N \tag{7}$$

where μ is called the coefficient of friction. If F is taken as the force necessary to start sliding, μ is called the coefficient of static friction. If F is taken as the somewhat smaller force necessary to maintain sliding, once it has been started, μ is called the coefficient of kinetic friction. The coefficients of static and kinetic friction vary greatly for different materials and for different conditions of their surfaces. Table 2.1 lists approximate values of coefficients of friction for various materials.

Table 2.1 *Coefficients of Friction*

Materials	*Static friction*		*Kinetic friction*	
	μ	φ^0	μ	φ^0
Leather on wood	0.5 – 0.6	27 – 31	0.3 – 0.5	17 – 27
Leather on metal	0.3 – 0.5	17 – 27	About 0.3	About 17
Masonry on dry clay	About 0.5	About 27		
Metal on metal	0.15 – 0.25	8 – 14	About 0.1	About 6
Metal on wood	0.4 – 0.6	22 – 31	0.3 – 0.5	17 –27
Rope on wood	0.5 – 0.8	27 – 39	About 0.5	About 27
Stone on stone	0.6 – 0.7	31 – 35		
Stone on wood	About 0.4	About 22		
Wood on wood	0.4 – 0.7	22 – 35	About 0.3	About 17
Steel on ice	About 0.03	About 2	0.015	About 1

To see how friction affects the reactions exerted by supporting surfaces, let us consider the simple case of a small block resting upon a horizontal plane surface and acted upon by a force P making the angle α with the vertical [Fig. 2.70(a)]. We shall assume, for simplicity, that the force P is large in comparison with the weight of the block so that the gravity force can be neglected or, if preferred, P may be considered as the resultant of the gravity force and some other force not shown. The actual distribution of pressure over the area of contact between the block and the plane will depend upon the point of application of the force P and

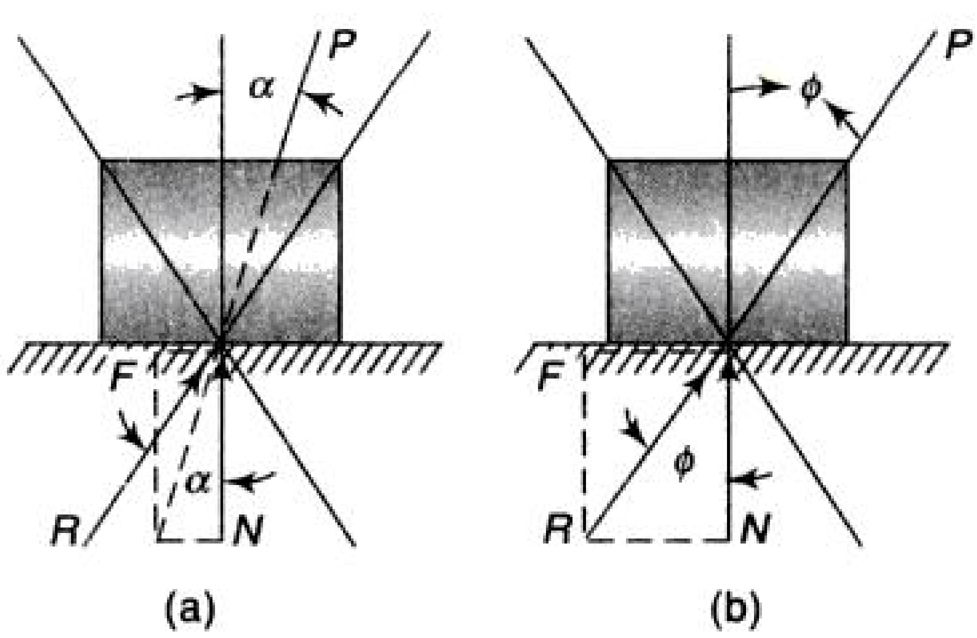

Fig. 2.70

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Solution: When conditions are such that sliding of the block impends, it will be in equilibrium under the action of three forces: the gravity force W, the pull P_{min}, and the reaction R which is the resultant of the distributed pressure exerted on the block by the floor [Fig. 2.73(a)]. These three forces must meet in one point and must build a closed triangle when geometrically added. To construct this triangle of forces, we begin with the known vector W, and from the end of this we draw the line that makes the angle of friction φ with it and that is the known limiting direction of the reaction R when motion impends. It is now evident that the shortest vector P_{min} which will make the closing side of the triangle is one at right angles to the reaction R. Thus we conclude that the least force that will cause sliding of the block to impend will be one making the angle of friction φ with the plane of the floor. That is, $\alpha = \varphi$.

From the triangle of forces [Fig. 2.73(b)], the magnitude of this least force with which the block can be made to slide is found to be

$$P_{min} = W \sin \varphi \tag{g}$$

Comparing Eqs (g) and (f), it is seen that, for cases in which the coefficient of friction is fairly large, considerable effort in sliding a heavy block over a rough surface will be saved by pulling along a line that makes the angle of friction with the plane of sliding. For example, in the case of stone sliding on concrete (assuming $\mu = 0.6$) the least force P_{min} as given by Eq. (g), will be only 86 per cent of the horizontally applied force P as given by Eq. (f).

4. To raise a heavy stone block weighing 8.9 kN, the arrangement shown in Fig. 2.74(a) is used. What horizontal force P will it be necessary to apply to

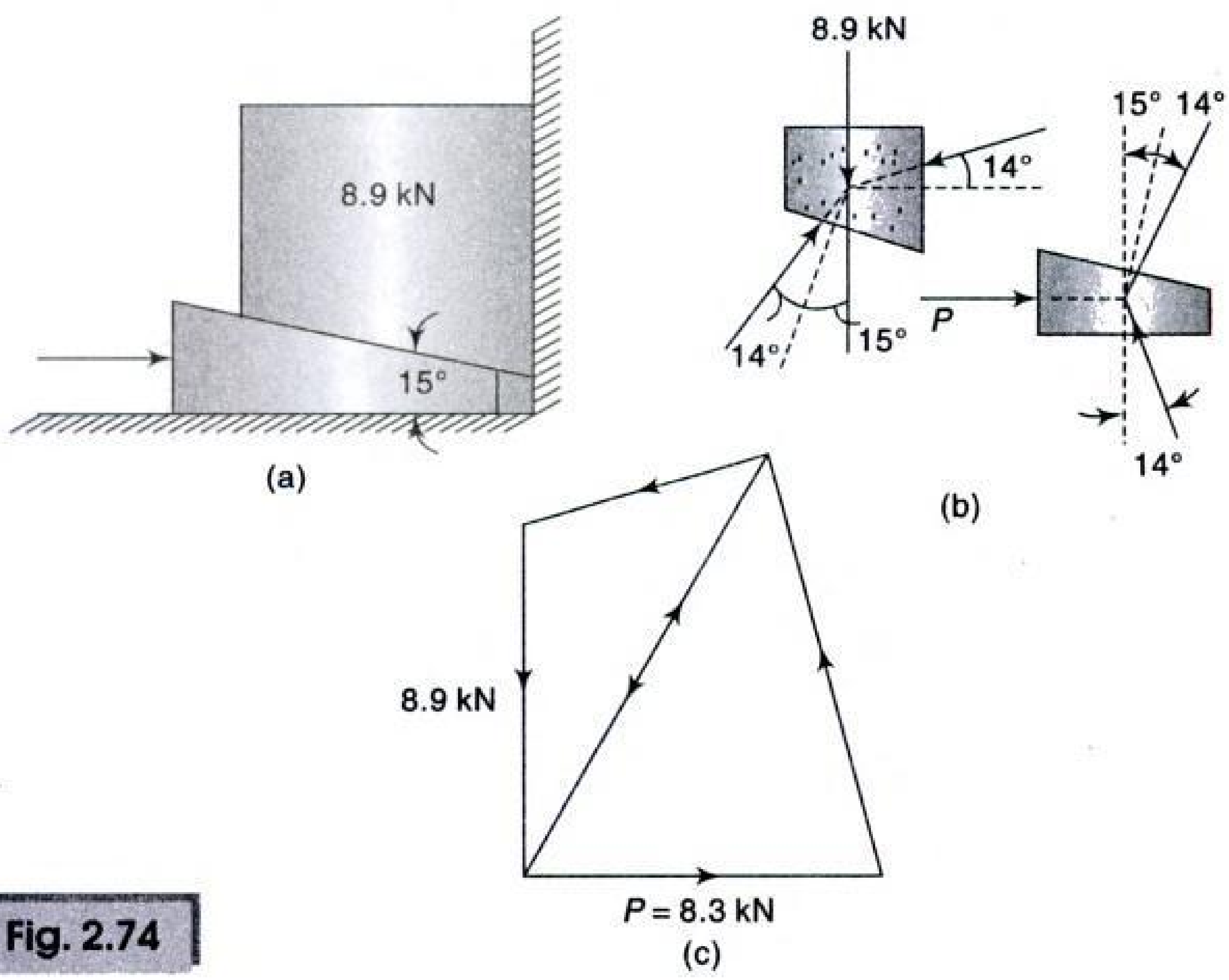

Fig. 2.74

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6. A body of weight W is placed on an inclined plane. The angle made by the inclined plane with the horizontal, when the body is on the point of moving down is called
   (a) Angle of inclination (b) Angle of repose
   (c) Angle of friction (d) Angle of limiting friction
   [*Ans.* (b)]
7. A body of weight Q is placed on an inclined rough plane. The inclination of the plane with the horizontal is less than the angle of friction. The body will
   (a) be in motion (b) move downwards
   (c) be in equilibrium (d) move upwards
   [*Ans.* (c)]
8. On a ladder resting on a smooth ground and leaning against a rough vertical wall, the force of friction acts
   (a) towards the wall at its upper end (b) away from the wall at its upper end
   (c) upwards at its upper end (d) downwards at its upper end
   [*Ans.* (c)]
9. If a ladder is not in equilibrium against a smooth vertical wall, then it can be made in equilibrium by
   (a) increasing the angle of inclination with horizontal
   (b) increasing the area of the ladder
   (c) decreasing the angle of inclination with horizontal
   (d) decreasing the area of the ladder [*Ans.* (a)]

PROBLEM SET 2.7

1. What must be the angle α between the plane faces of a steel wedge used for splitting logs if there is to be no danger of the wedge slipping out after each blow of the sledge? (*Ans.* $\alpha \le 2\varphi$)
2. A flat stone slab rests on an inclined skidway that makes an angle α with the horizontal. What is the condition of equilibrium if the angle of friction is φ? (*Ans.* $\alpha \le 2\varphi$)
3. What is the necessary coefficient of friction between tires and roadway to enable the four-wheel-drive automobile in Fig. A to climb a 30 per cent grade? (*Ans.* $\mu \ge 0.3$)

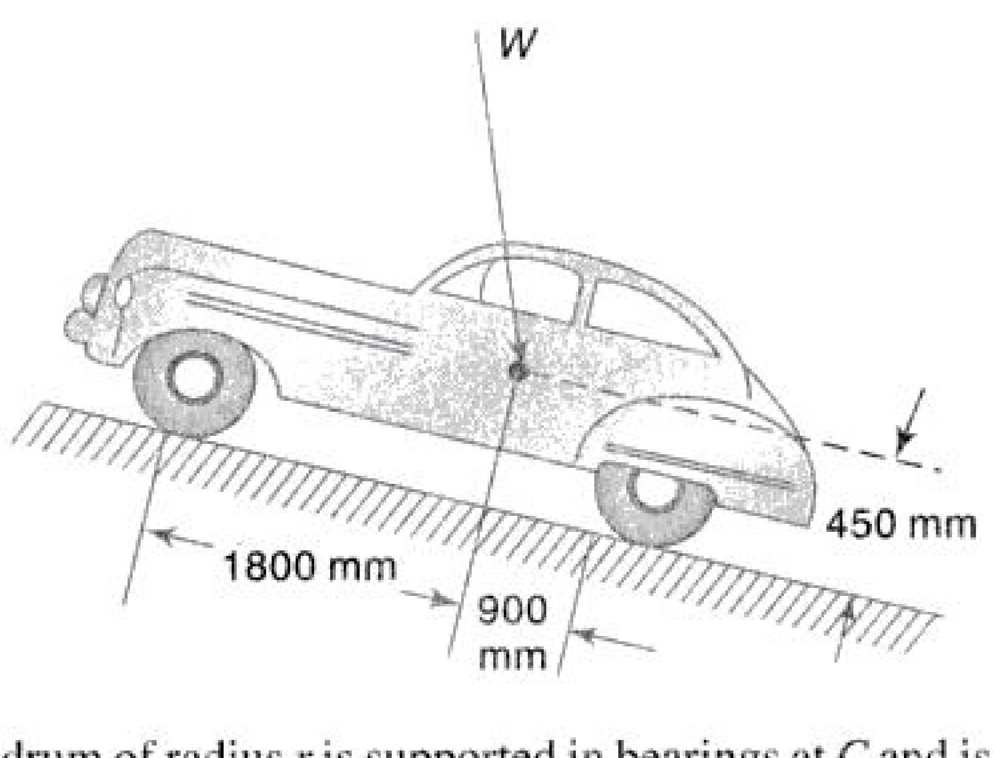

Fig. A

4. A heavy rotating drum of radius r is supported in bearings at C and is braked by the device shown in Fig. B. Calculate the braking moment M_c with respect to point C if the coefficient of kinetic friction between drum and brake shoe is μ. (*Ans.* $\mu Plr/a$)
5. To determine experimentally the coefficient of friction for steel on steel, flat plates of negligible weight compared with the large top weight W, are stacked on a hori-

cient of friction μ has the same value for all surfaces of contact, determine the necessary condition under which the large roller can be pulled over the small one.

(*Ans.* $\mu \geq \sqrt{d/D}$)

*17. A solid right circular cone of altitude $h = 304.6$ mm and radius of base $r = 76.2$ mm has its center of gravity C on its geometric axis at the distance $h/4 = 76.2$ mm above the base. This cone rests on an inclined plane AB, which makes an angle of 30° with the horizontal and for which the coefficient of friction is $\mu = 0.5$ (Fig. O). A horizontal force P is applied to the vertex O of the cone and acts in the vertical plane of the figure as shown. Find the maximum and minimum values of P consistent with equilibrium of the cone if the weight $W = 44.5$ N.

(*Ans.* $P_{max} = 20.52$ N; $P_{min} = 2.67$ N)

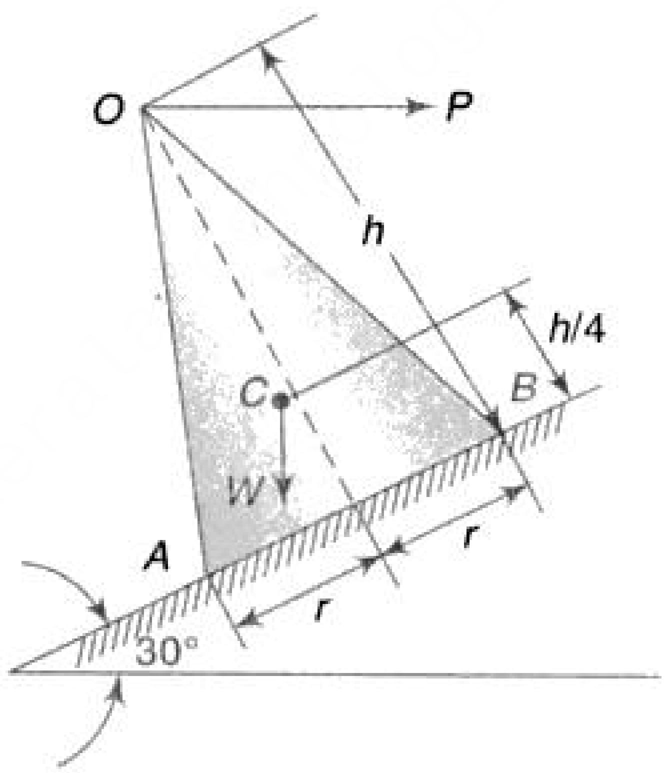

Fig. O

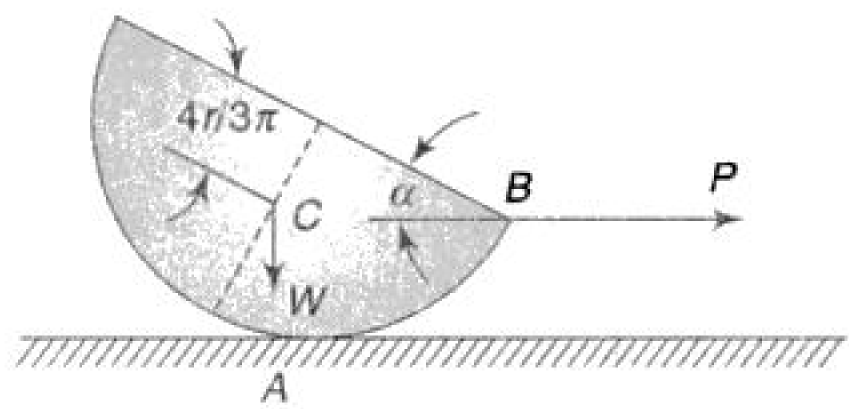

Fig. P

*18. A short semicircular right cylinder of radius r and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied at right angles to its geometric axis by a horizontal force P applied at the middle B of the front edge (Fig. P). Find the angle α that the flat face will make with the horizontal plane just before sliding begins if the coefficient of friction at the line of contact A is μ. The gravity force W must be considered as acting at the center of gravity C as shown in the figure. (*Ans.* $\sin \alpha = 3\mu\pi/(4 + 3\mu\pi)$)

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$$Q \cdot CB' - P \cdot CA' = 0$$

from which

$$\frac{CB'}{CA'} = \frac{P}{Q} \tag{h}$$

Thus again the distances of the components from the line of action of the resultant are inversely proportional to their magnitudes, but the line of action of the resultant lies outside the space between the components on the side of the larger force.

Two Equal Parallel Forces Acting in Opposite Directions

A system of two equal parallel forces acting in opposite directions cannot be reduced to one resultant force. This can readily be seen from Fig. 3.3. If the forces P and Q are equal in magnitude, the two parallelograms APP_1S and BQQ_1S will be equal and corresponding sides will be parallel. Hence P_1 and Q_1 will still be equal and oppositely directed parallel forces, and we never obtain a point of intersection C. Thus, we cannot reduce two equal and opposite but noncollinear forces to any simpler system. Two such forces are called a *couple*, the plane in which they act is called *the plane of the couple*, and the distance between their lines of action is called the *arm of the couple*.

The algebraic sum of the moments of the two forces of a couple is independent of the position, in the plane of the couple, of the moment center and is always equal to the product of the magnitude of either force and the arm of the couple. Consider, for example, the couple PP with arm AB of length a in Fig. 3.4, and let O be any arbitrary moment center. Then the algebraic sum of moments is

$$P \cdot OD - P \cdot OC = P(OD - OC) = Pa$$

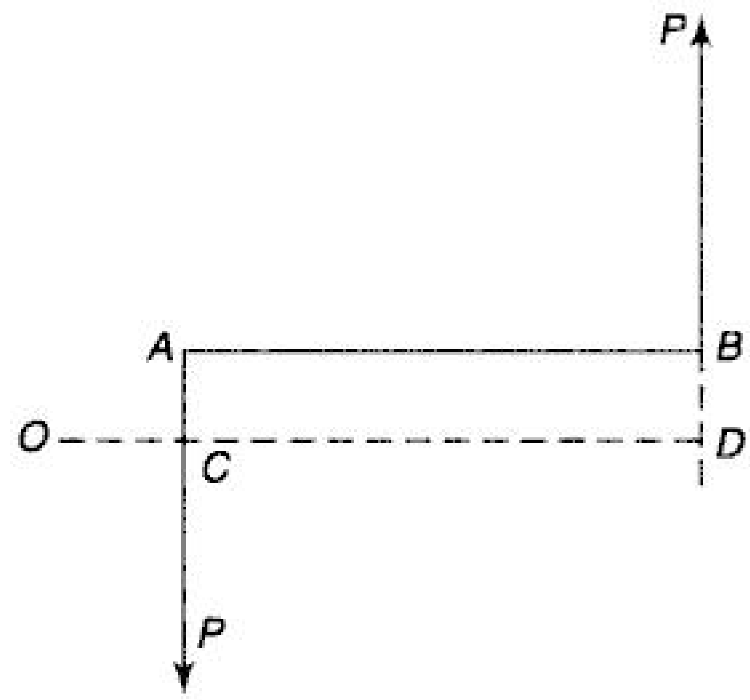

Fig. 3.4

The same result will be obtained if O lies between the lines of action of the forces. The moment Pa is called the *moment of the couple*. It is positive when the couple tends to produce counter-clockwise rotation as shown, and it is negative for clockwise rotation.

The action of a couple on a rigid body will not be changed if its arm is turned in the plane of the couple through any angle α about one of its ends. To prove this statement, let us consider a couple PP with the arm AB, as shown in Fig. 3.5. We take under any desired angle α with the arm AB a straight-line segment AC equal in length to the arm AB of the given couple. At each of the ends A and C of this line we apply two equal and opposite forces Q and Q' perpendicular to AC and equal in magnitude to P. From the principle of superposition it follows that the addition of these forces which are in equilibrium does not change the action of the given couple. The forces Q together with the forces P give the resultant SR,

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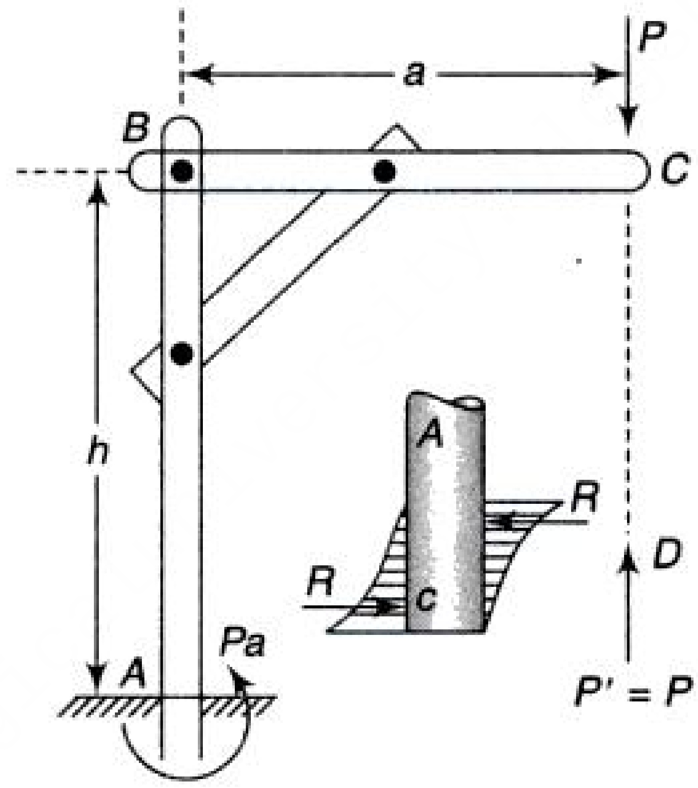

Fig. 3.10

Solution: Clearly, the reaction at A must be statically equivalent to the equilibrant P' of the active force P, that is, a vertical upward force acting along the line DC. To represent this force as a reaction at A, we resolve it into an equal parallel force P at A and a couple of moment Pa acting as shown in the figure. To understand physically just how the foundation exerts this couple on the bottom of the mast, we must consider the detail of the bottom of the mast as shown in Fig. 3.10. In trying to tip over to the right, the mast presses against the walls of the hole in which it is seated and, of course, equal and opposite reactive pressure will be exerted on the mast. This distribution of pressure will be something like that indicated in the figure and is sensibly equivalent to a couple RR acting as shown. We see that this kind of constraint, a so-called *built-in end*, is considerably more complex than any we have previously discussed. This example illustrates one of the many useful applications of the resolution of a force into a force and a couple.

3. Two gears having pitch diameters d_1 and d_2 are connected as shown in Fig. 3.11(a). If a couple of moment M_1 is applied to the upper gear as shown, what is the moment M_2 of the couple that must be applied to the lower gear to maintain equilibrium?

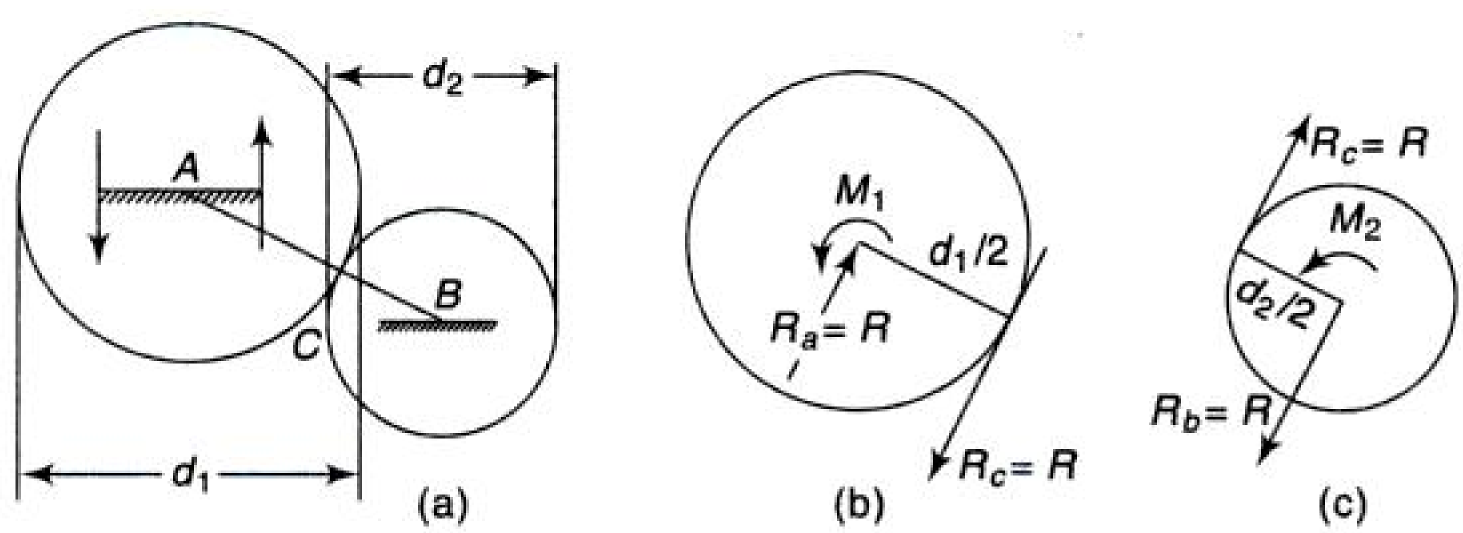

Fig. 3.11

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4. Consider the following statements regarding the resultant of two unequal parallel forces acting in opposite directions.
   I. It is equal to the subtraction of smaller force from larger force.
   II. The direction of the resultant force is acting in the direction of smaller force.
   III. The distances of the components from the line of action of the resultant are inversely proportional to their magnitudes.
   IV. The lines of action of the resultant force lies outside the space between the components on the side of the smaller force.

   Of these
   (a) I is true (b) Both I and III are true
   (c) I, II and IV are true (d) All are true [*Ans.* (b)]
5. Two equal parallel forces acting in opposite directions is known as
   (a) Couple (b) Moment
   (c) Equilibrant (d) Resultant [*Ans.* (a)]
6. A couple is completely defined by its
   I. Plane of action
   II. Magnitude
   III. Sign of its moment

   Of these
   (a) Only I is true (b) Only III is true
   (c) Both II and III are true (d) I, II and III are true [*Ans.* (d)]
7. Consider the following statements regarding the properties of a couple.
   I. The algebraic sum of the moments of the two forces of a couple is independant of the position, in the plane of the couple, of the moment center and is always equal to the product of the magnitude of either force and the arm of the couple.
   II. We can transpose a couple in its plane without changing its action on a body.
   III. Two couples acting in the same plane are equivalent if they have equal moments.

   Of these statements:
   (a) I and III are true (b) II and III are true
   (c) I and II are true (d) All are true [*Ans.* (d)]
8. Consider the following two statements:
   Assertion (A): We can replace several couples in one plane by a single resultant couple acting in the same plane, the moment of which is equal to the algebraic sum of the moments of the given couples.
   Reason (R): A couple can be balanced only by another couple which is equal in moment, opposite in sign, and coplanar in action with the given couple.
   Choose the correct answer:
   (a) A and R are true R is the correct explanation of A.
   (b) A and R are true but R is not the correct explanation of A.
   (c) A is true but R is false.
   (d) A is false but R is true. [*Ans.* (b)]
9. Consider the following two statements.
   Assertion (A): A system of couples acting in one plane is in equilibrium if the algebraic sum of their moments is equal to zero.
   Reason (R): Resolution of a force into a force and a couple is not possible.
   Choose the correct answer:
   (a) A and R are true R is the correct explanation of A.
   (b) A and R are true but R is not the correct explanation of A.

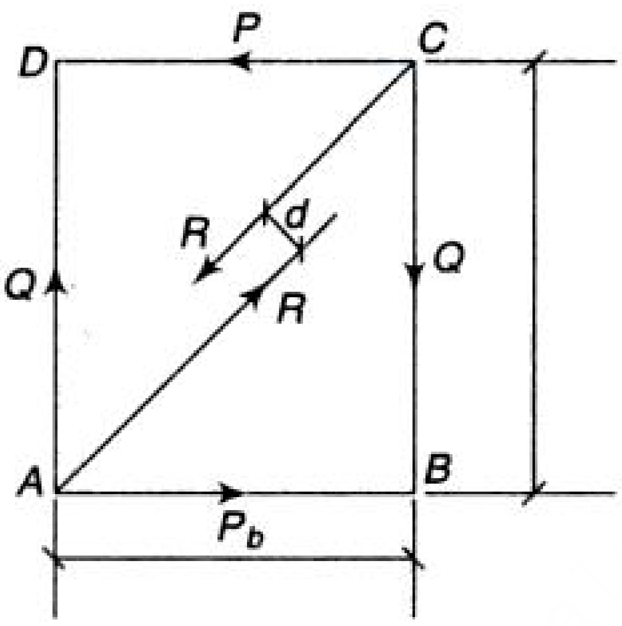

Fig. F

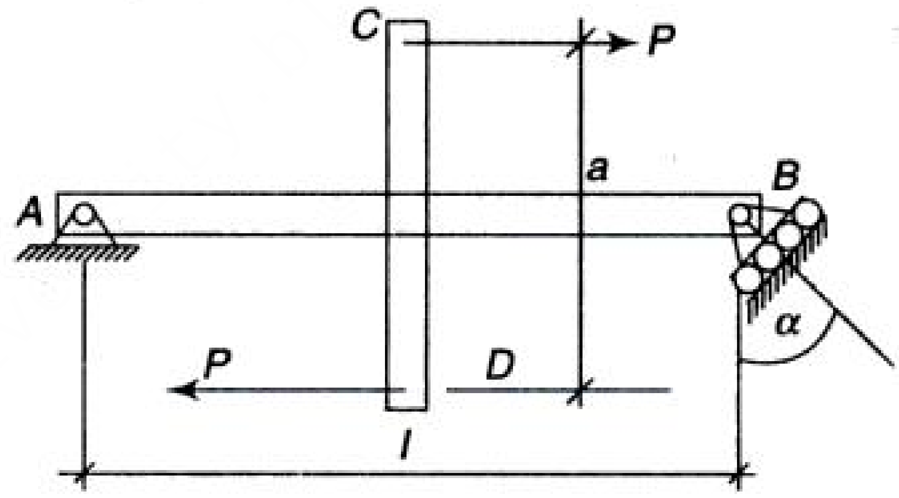

Fig. G

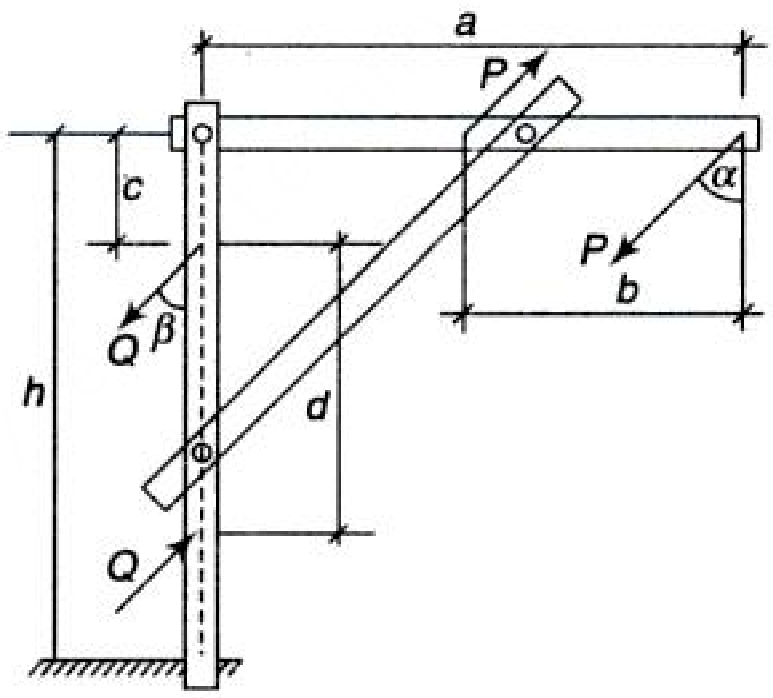

Fig. H

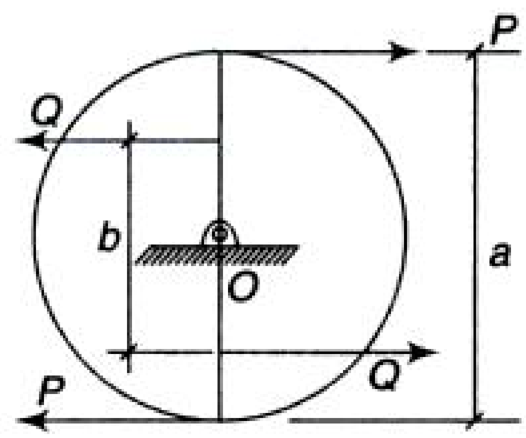

Fig. I

9. Two couples are acting on the disc as shown in Fig. I. If the resultant couple moment is to be zero. Determine the magnitude of force Q. $\left(Ans.\ Q = \dfrac{Pa}{b}\right)$

3.2 GENERAL CASE OF PARALLEL FORCES IN A PLANE

Let us consider now the case of several parallel forces in one plane (Fig. 3.12). In such a case we may find the resultant R_1 of the forces F_1, F_2, F_3 acting in one

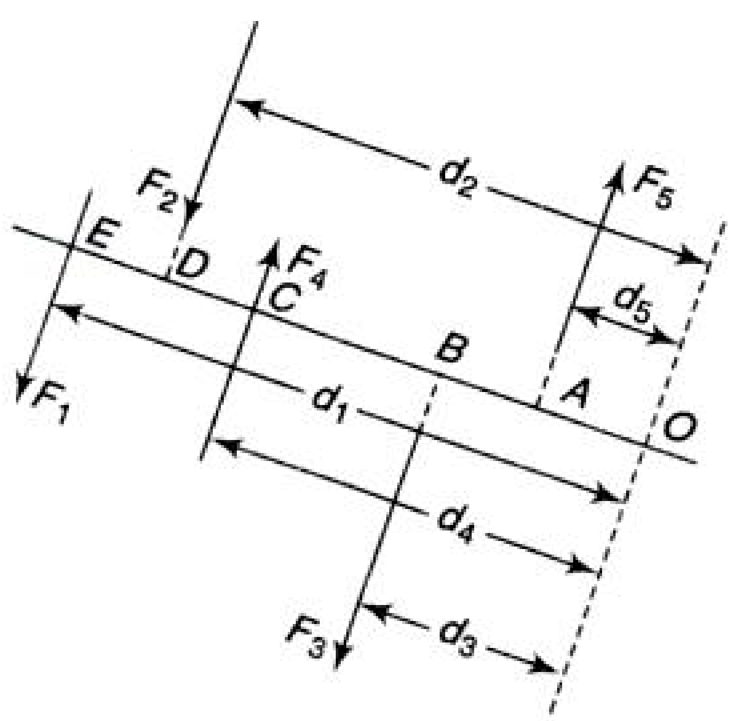

Fig. 3.12

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action of the beam on the hinge at A is equal and opposite to the reaction R_a. Resolving this force into two components acting along the axes of the bars AD and AE, we find the compressive forces in these bars to be $P(l - a)/(2l \cos \alpha)$ each.

3. Two rollers C and D produce vertical forces P and Q on the horizontal beam AB, as shown in Fig. 3.16. Determine the distance x of the load P from the support A if the reaction R_a is twice as great as the reaction R_b. The weight of the beam is to be neglected, and the following numerical data are given: $P = 18$ kN, $Q = 9$ kN, $l = 3.6$ m, $c = 0.9$ m.

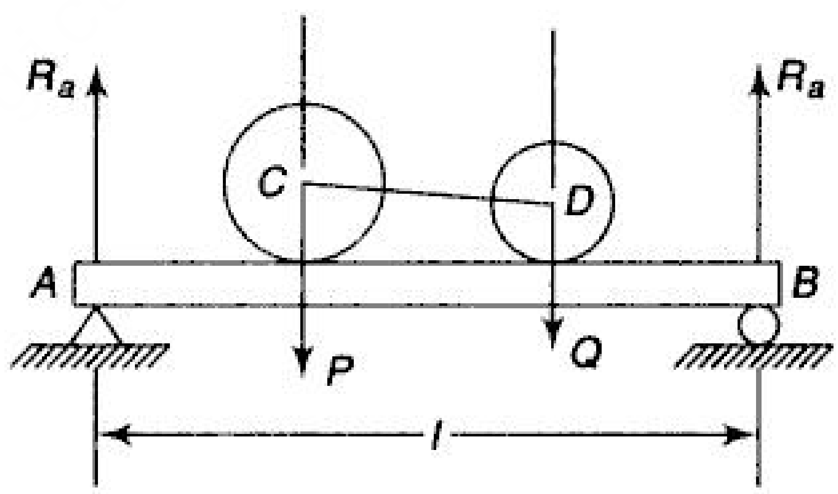

Fig. 3.16

Solution: Considering the beam as a free body and equating to zero the sum of the moments of all forces with respect to points A and B, we obtain

$$R_b l - Q(x + c) - Px = 0$$
$$Q(l - c - x) + P(l - x) - R_a l = 0$$

The third necessary equation is

$$R_a = 2R_b$$

Eliminating R_a and R_b from these three equations, we obtain

$$x = \frac{l}{3} - \frac{Qc}{P + Q}$$

Substituting the given numerical data, we find $x = 0.9$ m.

4. A crane $CDEF$ (Fig. 3.17) of weight $Q = 45$ kN applied at G supports at F a load $P = 9$ kN and can move along the horizontal girder AB. For the position of the crane shown in the figure, determine the reactions produced at the supports A and B of the girder. Assume that all forces act in one plane, and neglect the weight of the girder. Numerical data are given as follows: $l = 9$ m, $x = 2.7$ m, $c = 3.6$ m.

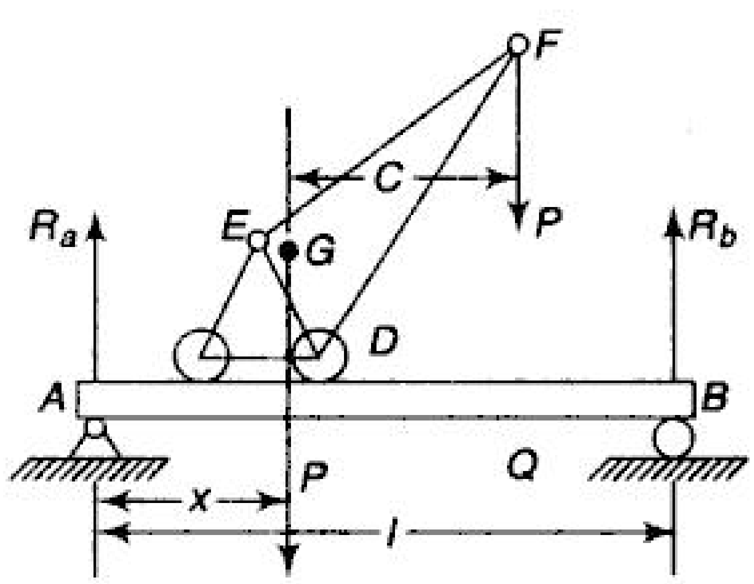

Fig. 3.17

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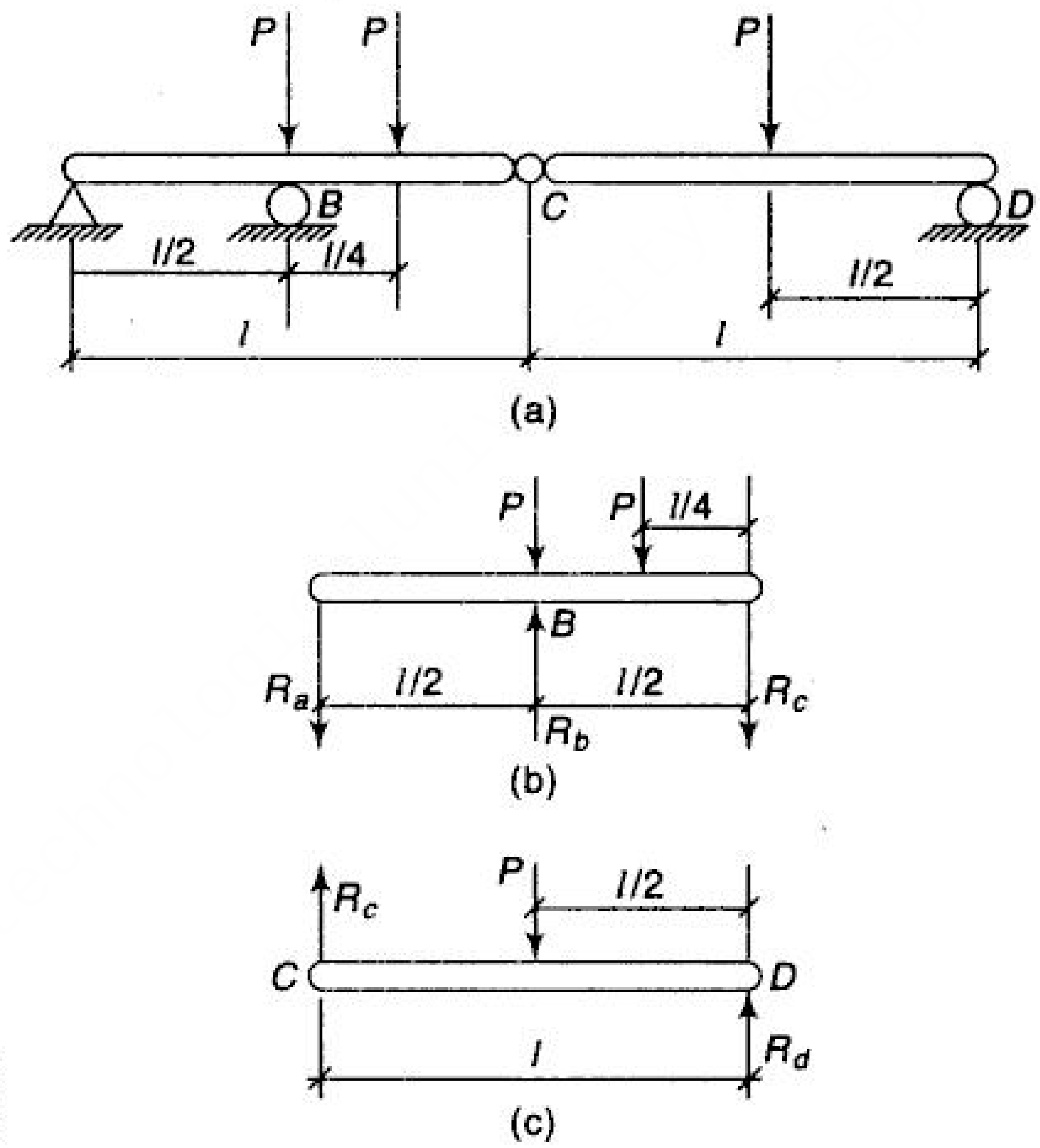

Fig. 3.21

Thus acting on the beam CD, we have the vertical forces R_c, P and R_d in equilibrium. The simplest way of calculating the reactions R_c and R_d is by using Eq. (12). Taking moments, first with respect to point C and then with respect to point B, we obtain the equations.

$$R_d l - \frac{Pl}{2} = 0 \quad -R_c l + \frac{Pl}{2} = 0$$

from which

$$R_d = \frac{P}{2} \quad \text{and } R_c = \frac{P}{2}$$

Now consider the free body of beam ABC. The loads acting on this beam are two active forces P, the reaction R_b from the roller support at B and the reaction R_c from the hinge C acting downward. We can reduce R_c and P into a resultant force R vertical. Since R and R_b are parallel forces, so the reaction R_a at A must be vertical for the system to be in equilibrium as discussed for the previous beam. Taking moments, first with respect to point A and then with respect to point B, we obtain the equations

$$R_b \frac{l}{2} - \frac{Pl}{2} - P\left(\frac{3l}{4}\right) - R_c l = 0$$

$$R_b \frac{l}{2} - \frac{Pl}{4} - \frac{R_c l}{2} = 0$$

Substituting the value of R_c in the above equations, we get the reactions R_a and R_b as shown below:

$$R_a = P \text{ and } R_b = 3.5\, P$$

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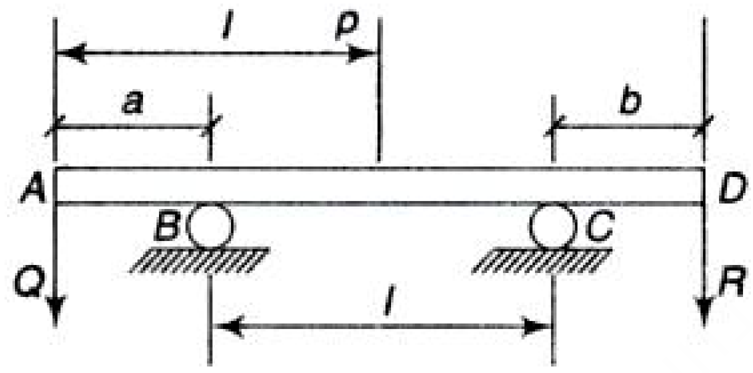


Fig. H

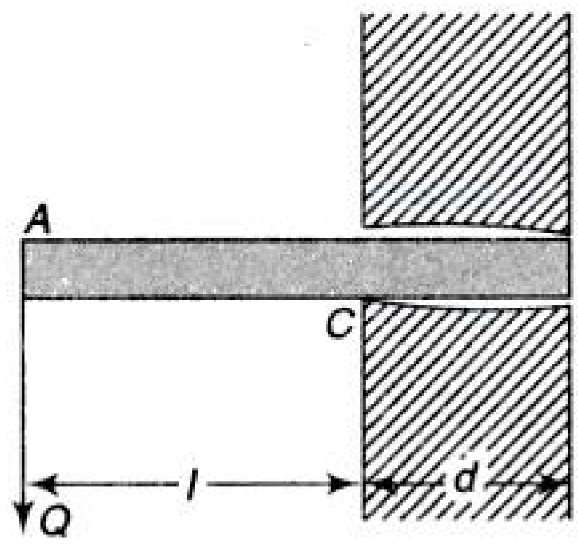


Fig. I

10. Determine the reactions at A for the cantilever beam AB subjected to the loads as shown in Fig. J. Numerical data are given: $P = 1500$ N, $Q = 1000$ N, $R = 1500$ N and

$$a = b = c = \frac{l}{3}, l = 3 \text{ m}.$$

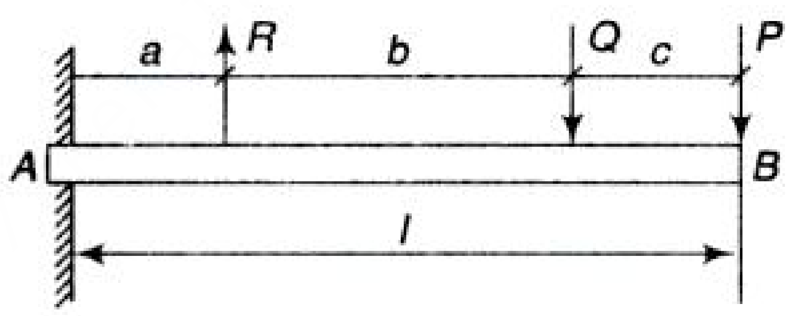


Fig. J

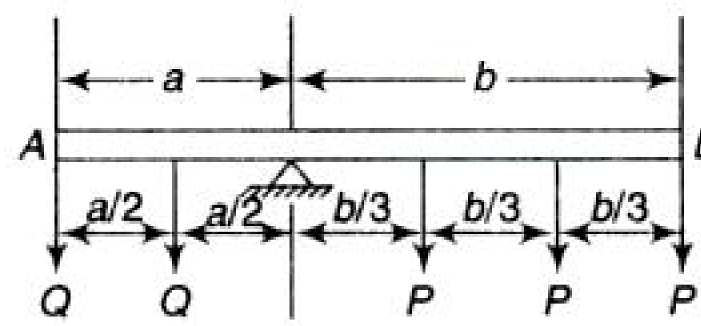
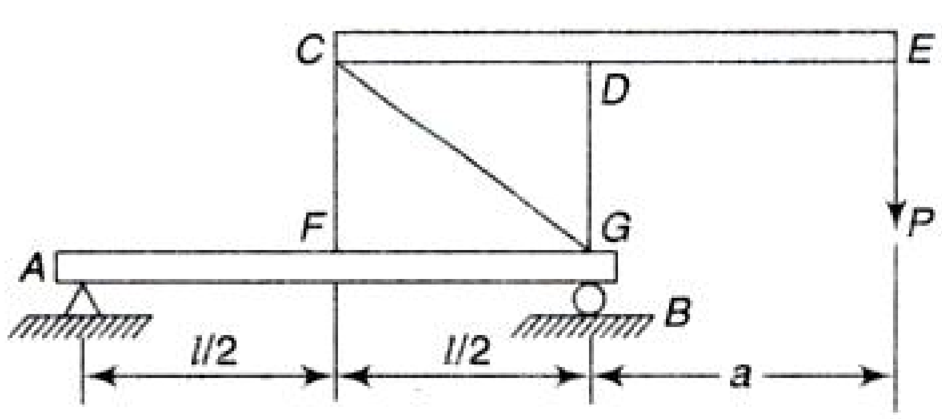


Fig. K

11. Along a lever AB loads Q and P are distributed, as shown in Fig. K. If $Q = 2P$ and the weight of the lever is negligible, determine the ratio $a : b$ of the arms of the lever if it is in equilibrium. (*Ans.* $a : b = 2 : 3$)
12. The beam CE in Fig. L is supported on the beam AB by the three bars CF, DG and CG, as shown. Find the reactions that will be produced at the points of support A and B of the lower beam due to the action of a load P applied at the free end E of the upper beam if the span $l = 3.6$ m and $a = 1.2$ m. (*Ans.* up; down)

Fig. L



13. Find the load P required to maintain the system of levers in equilibrium with AB in horizontal position, if $Q = 9.6$ kN (Fig. M). (*Ans.* $P = 200$ N)
14. Two identical prismatic bars AB and CD are welded together in the form of a rigid T and suspended in a vertical plane as shown in Fig. N. Calculate the angle α that the bar CD will make with the vertical when a vertical load $P = 44.5$ N is applied at B. The weight of each bar is $Q = 22.25$ N as shown. (*Ans.* $a = 15°57'$)

$$x_c = \frac{\Sigma(F_i x_i)}{\Sigma F_i} \tag{13a}$$

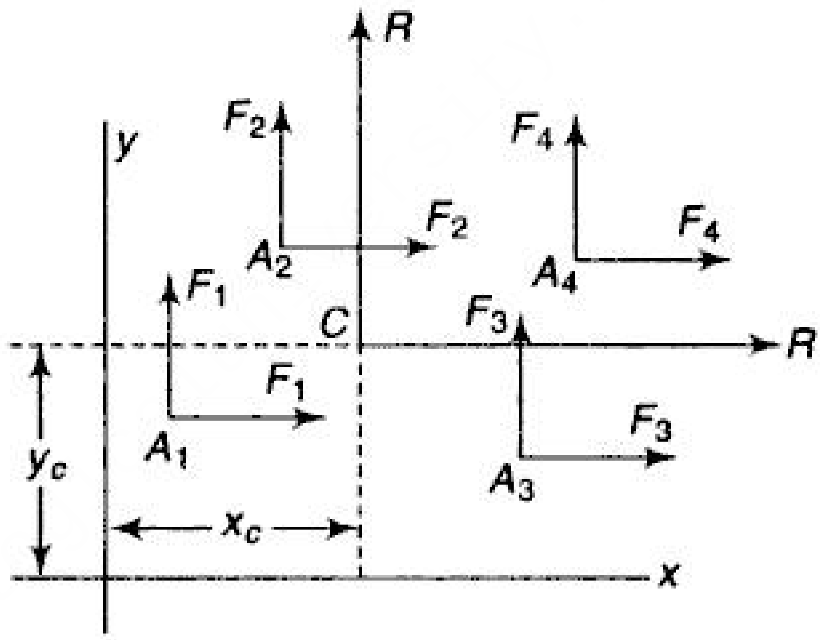


Fig. 3.24

Now let the forces all be rotated in the plane of the figure until they act parallel to the x-axis. In this case the arm y_c of the resultant may be found by again using the second of Eq. (9), and we obtain

$$y_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i} \tag{13b}$$

We have already seen that the center of parallel forces of any number of forces applied at a given system of points is independent of the direction in which the forces act. Hence, we conclude that the moment arms x_c and y_c as defined by Eq. (13) represent the coordinates of this point.

The center of parallel forces for a given system of forces $F_1, F_2, \ldots, F_n$ applied at given points $A_1, A_2, \ldots, A_n$ in one plane will not be changed if the magnitudes of the forces are all multiplied by the same constant factor. This statement follows at once from the form of Eq. (13) from which we see that such a factor will appear n times in both the numerator and the denominator of either of these expressions and will therefore cancel out. Thus we conclude that the center of parallel forces for any given system of forces applied at a given system of points in one plane depends only upon the positions of the points and upon the relative magnitudes of the forces.

Center of Gravity

The center of gravity of a body is that point through which the resultant of the distributed gravity force passes regardless of the orientation of the body in space. From this definition it follows that the center of gravity of a rigid body is the center of parallel gravity forces acting on the various particles of the body. Since gravity forces always act vertically downward, it is evident that some rotation of a body through an angle α is equivalent to a corresponding rotation through the same angle of all the gravity forces about their points of application as discussed above.

All physical bodies are, of course, three-dimensional as a consequence of which the gravity forces acting on the various particles of the body represent a system of

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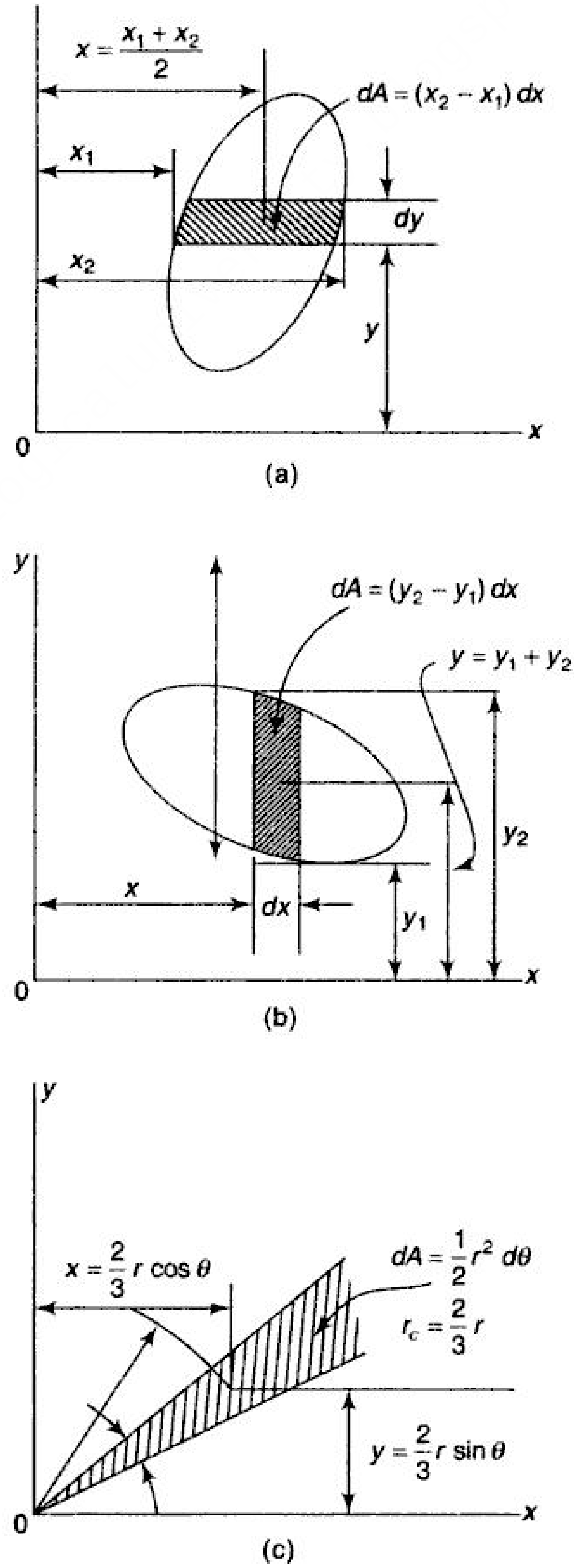

Fig. 3.27

The expressions for the differential length dL for rectangular and polar coordinates are shown in Fig. 3.28.

Sometimes the position of the centroid of a plane figure or curve can be seen by inspection. For example, if a figure has two axes of symmetry, its centroid lies at their intersection. This statement follows at once from the form of expressions (16) and (17).

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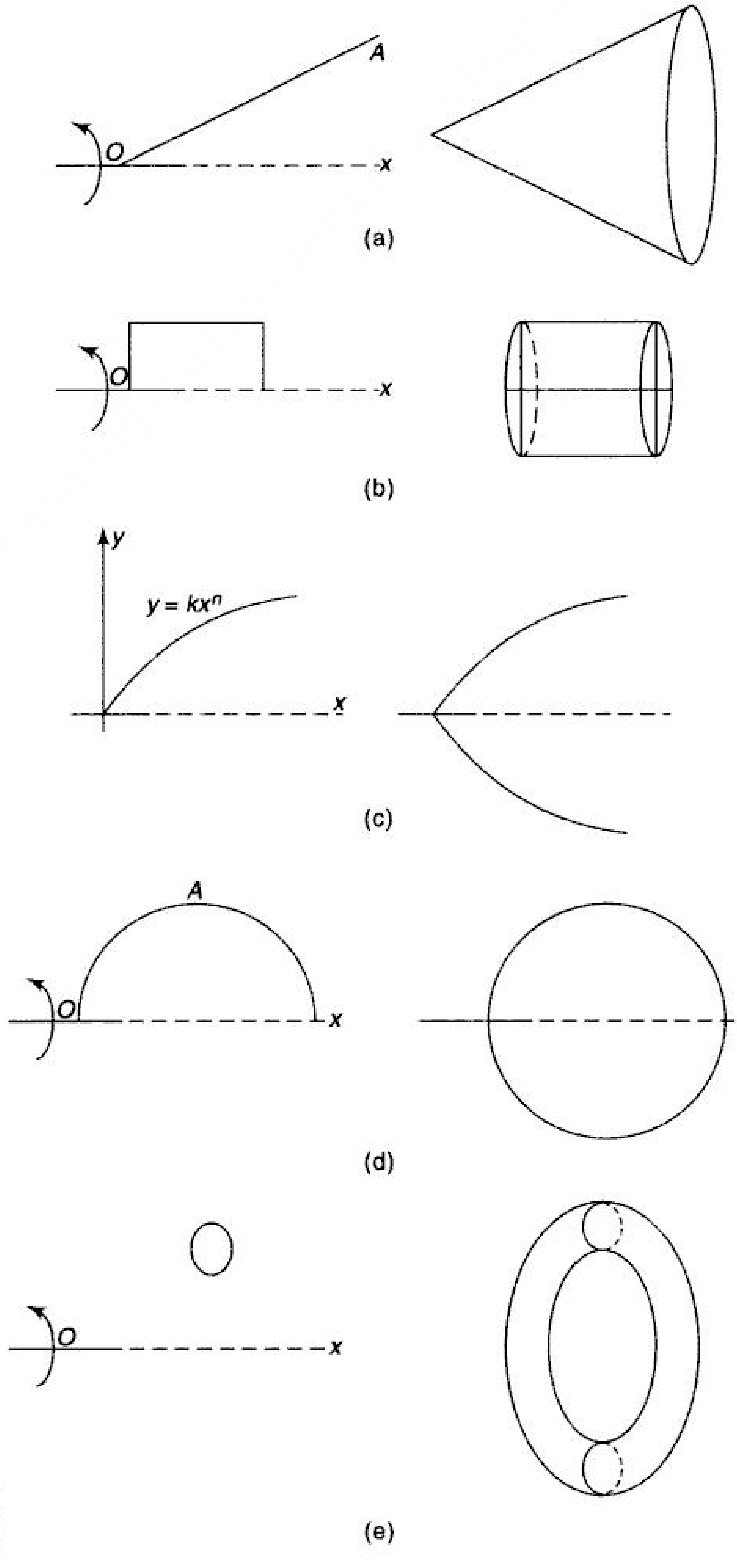

Fig. 3.36

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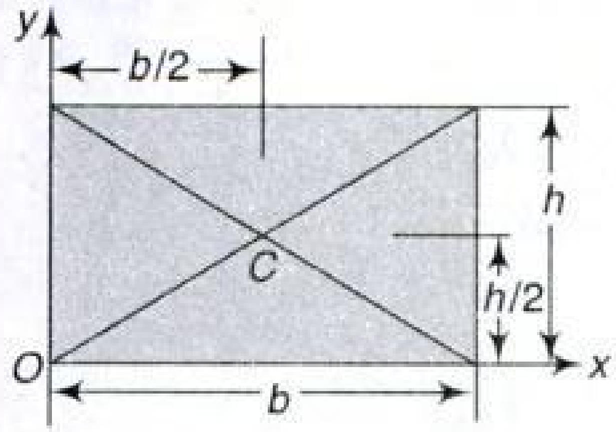
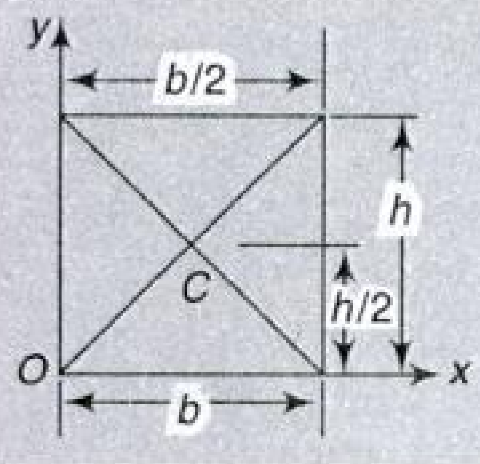
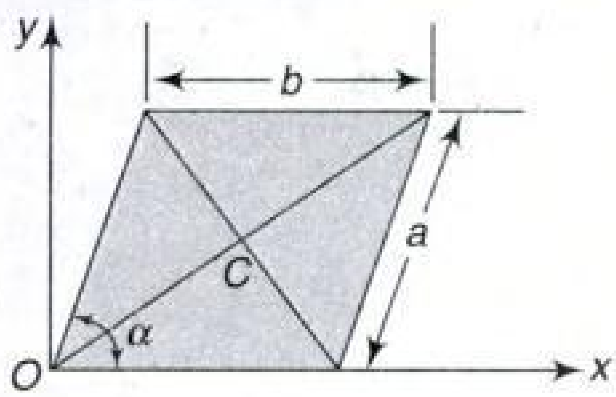
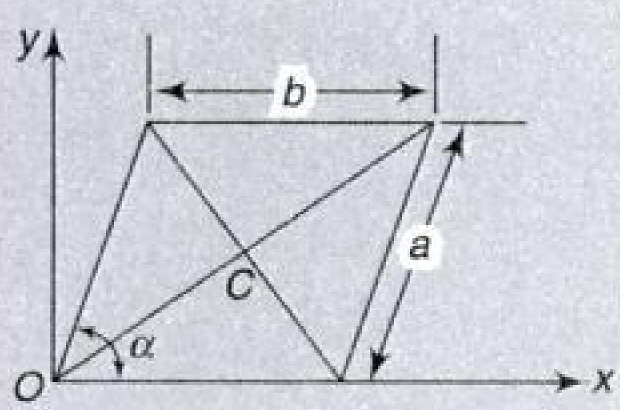
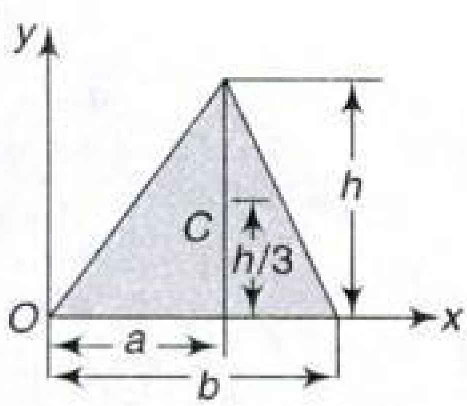
Various Centroids of Plane Areas

Description	*Shape*	*Area*	x_c	y_c
Rectangle		bh	$b/2$	$h/2$
Square ($h = b = a$)		a^2	$a/2$	$a/2$
Parallelogram		$ab \sin \alpha$	$\dfrac{b + a\cos\alpha}{2}$	$\dfrac{a \sin \alpha}{2}$
Rectangle ($a = \pi/2$)		ab	$b/2$	$a/2$
Triangle		$1/2bh$	$1/3(a + b)$	$h/3$

(*Contd.*)

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$$x_c = \frac{\int x dA}{dA} = \frac{2\int_0^{\alpha/2} \frac{2}{3} r\cos\theta \frac{r^2 d\theta}{2}}{2\int_0^{\alpha/2} \frac{r^2}{2} d\theta} = \frac{\frac{r^3}{3}\sin\frac{\alpha}{2}}{\frac{\alpha r^2}{4}} = \frac{4}{3}\frac{r}{\alpha}\sin\frac{\alpha}{2} \qquad \text{(f)}$$

From symmetry, we conclude that $y_c = 0$. Substituting $\alpha = \pi$ in Eq. (f) gives, for the case of a semicircle, $x_c = 4r/3\pi$.

4. Using the first theorem of Pappus, calculate the area of the bell-shaped shell shown in Fig. 3.42(a), which is obtained by rotating a quarter-circular arc about x-axis.

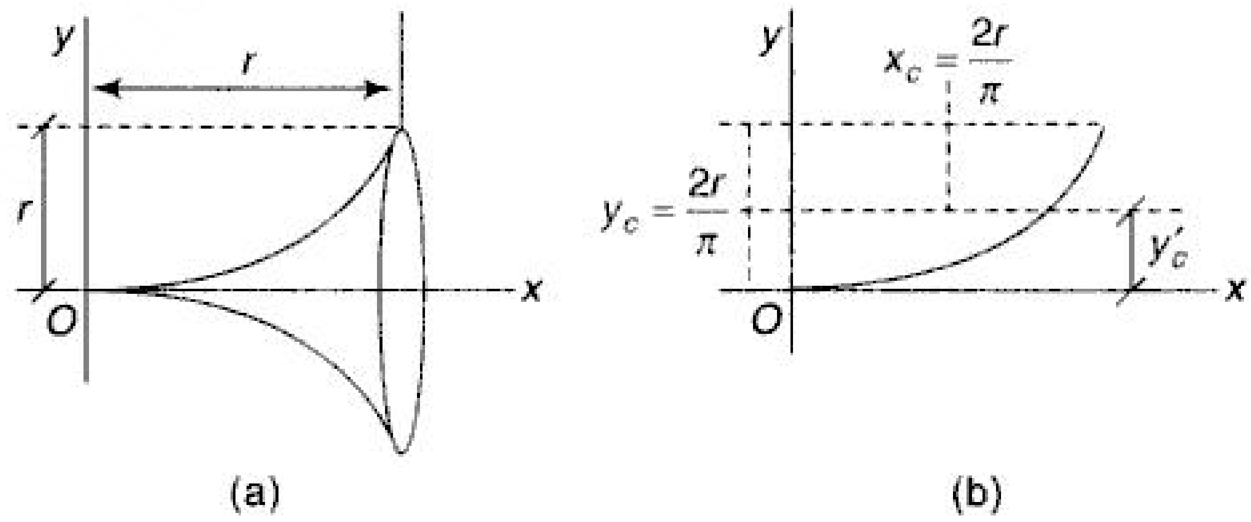

Fig. 3.42

Solution: From the first theorem of Pappus, the area generated is equal to the product of the length of the arc and the distance travelled by its centroid in one revolution about the x-axis.

$$\therefore \qquad A = 2\pi y_c' L$$

$$= 2\pi\left(r - \frac{2r}{\pi}\right)\left(\frac{\pi r}{2}\right)$$

$$= \pi r^2(\pi - 2)$$

5. Find the surface area of the annular torus formed by revolving the circle about the y-axis as shown in Fig. 3.43(a) using the first theorem of Pappus, if $R = 250$ mm and $r = 100$ mm.

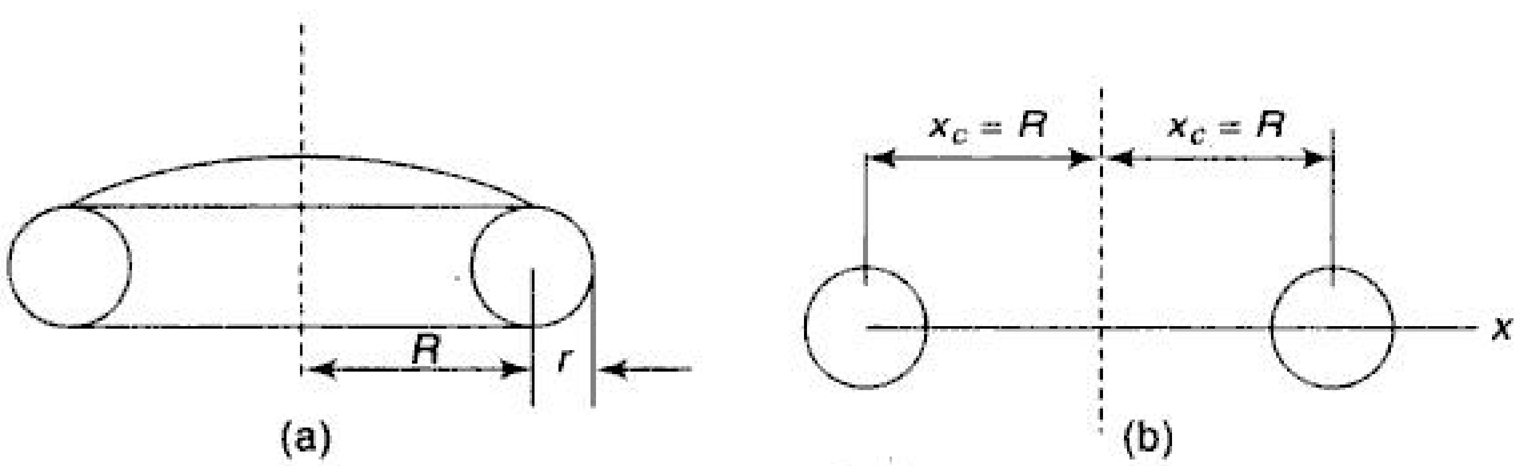

Fig. 3.43

Solution: From the first theorem of Pappus, the area generated is equal to the product of the length of the circle and the distance travelled by its centroid in one revolution.

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L of the curve and the distance travelled by its centroid. 2. The volume of the solid generated by rotating any plane figure about a nonintersecting axis in its plane is equal to the product of the area A of the figure and the distance travelled by its centroid.

- A surface of revolution is a surface generated curve by rotating a plane curve about a fixed axis.
- A body of revolution is a body generated by rotating a plane area about a fixed axis.
- The theorems of Pappus are very useful in calculating the surface areas and volumes of various bodies of revolution encountered in engineering, particularly in machine design.
- The theorems of Pappus can also be used to determine the centroid of a plane curve when the area of the surface generated by the curve is known or to determine the centroid of a plane area when the volume of the body generated by the area is known.

Important Formulae

1. The center of paralle‌l forces for a given system of forces applied at the given point in one plane can be obtained by the following equations:

$$x_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i}, \qquad y_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i}$$

2. Analytically, the position of the centroid of area of a plane figure may be defined by the formulae and the summations are understood to include all elements of area within the boundary of the figure.

$$x_c = \frac{\Sigma(\Delta A_i x_i)}{\Sigma \Delta A_i}, \qquad y_c = \frac{\Sigma(\Delta A_i y_i)}{\Sigma \Delta A_i}$$

3. The position of the centroid of length of a plane curve may be defined by the formulas

$$x_c = \frac{\Sigma(\Delta L_i x_i)}{\Sigma \Delta L_i}, \qquad y_c = \frac{\Sigma(\Delta L_i y_i)}{\Sigma \Delta L_i}$$

where L_i denotes the length of an element and the x_i, y_i coordinates of its mid-point.

4. The coordinates of the centroid of any plane figure or curve can be calculated, provided the integrals appearing therein can be evaluated.

$$x_c = \frac{\int x dA}{\int dA}, \qquad y_c = \frac{\int y dA}{\int dA}$$

for the case of area and

$$x_c = \frac{\int x dL}{\int dL}, \qquad y_c = \frac{\int y dL}{\int dL}$$

for the case of length.

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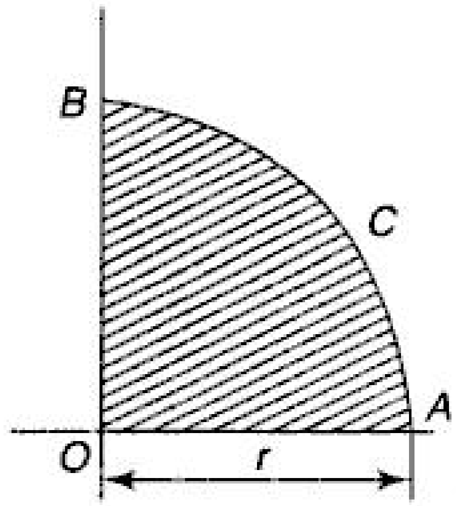

Fig. L

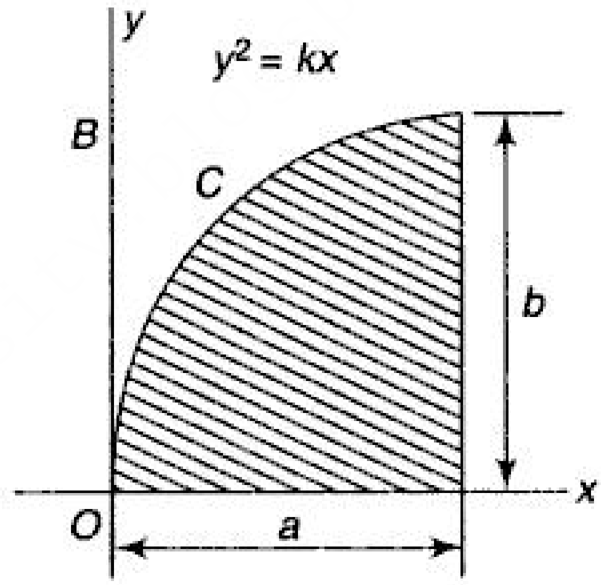

Fig. M

14. Using the first theorem of Pappus, calculate the area of the surface of revolution shown in Fig. N, which is obtained by rotating a semicircular arc about a vertical axis. (*Ans.* $A = 4\pi r^2(\pi - 1)$

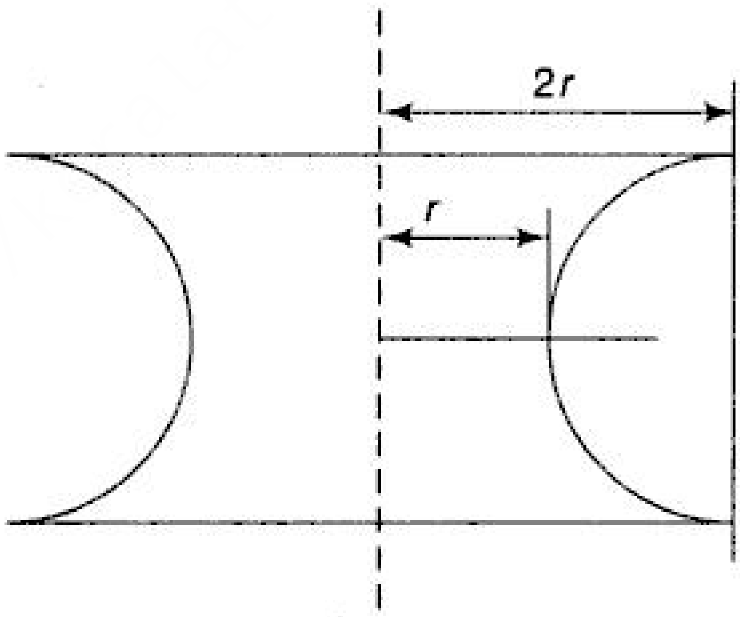

Fig. N

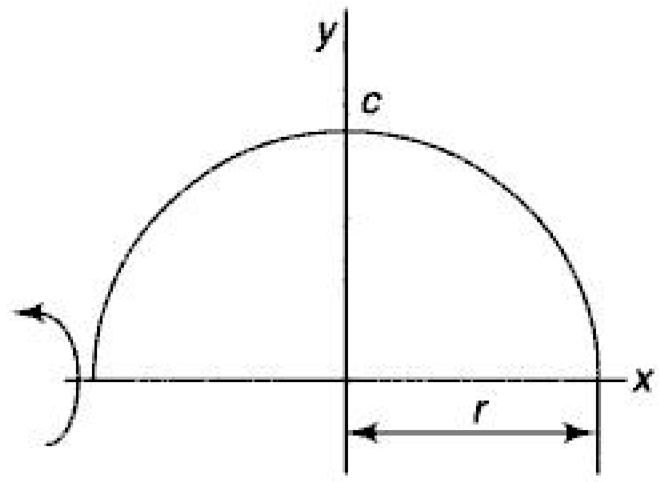

Fig. O

15. Using the first theorem of Pappus, calculate the centroid of a semicircular arc of radius r, if the surface area of sphere is $4\pi r^2$ (Fig. O). $\left(Ans.\ x_c = 0,\ y_c = \dfrac{2r}{\pi}\right)$

16. Using the second theorem of Pappus, determine the centroid of a semicircular area of radius r, if the volume of the sphere $V = \dfrac{4}{3}\pi r^3$? (Fig. P).

$$\left(Ans.\ x_c = 0,\ y_c = \frac{4r}{3\pi}\right)$$

17. Using the second theorem of Pappus, determine the volume of the ring shown in Fig. Q, if $R = 2r$. (*Ans.* $V = 3r^3$)

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Solution: Note that the shaded triangles mpC and nqC are geometrically similar. Hence, we may write

$$\frac{y_c}{h - y_c} = \frac{a/2 + b}{a + b/2}$$

which is readily brought to the form $y_c = \dfrac{a(a+2b)}{3(a+b)}$

as previously obtained in Example 1 above.

3. Locate the centroid C of the shaded area of the figure BDE (Fig. 3.51) which is obtained by cutting the quadrant of a circle of radius a from a square $OBDE$ of the same dimensions.

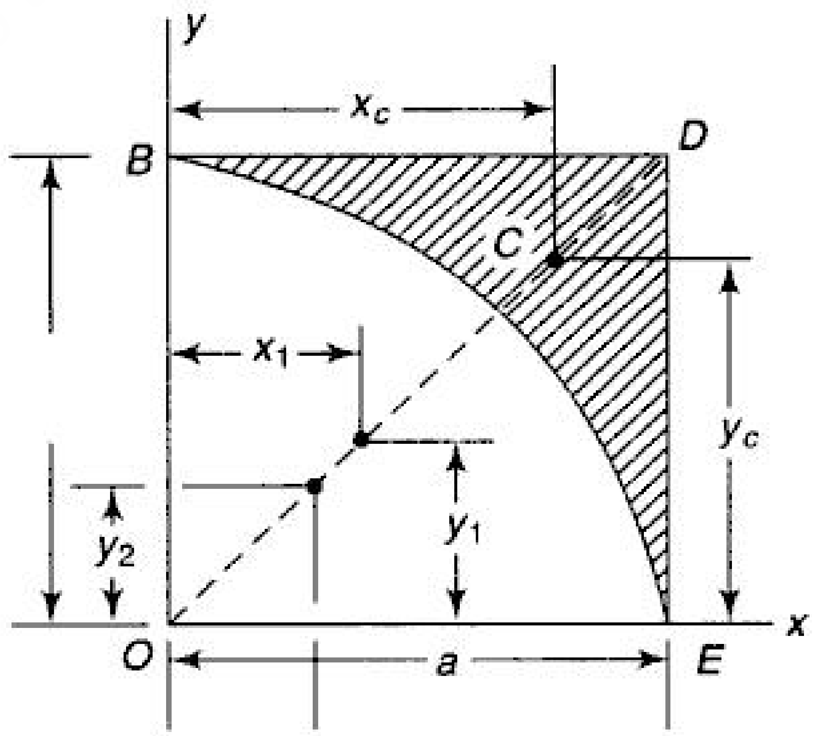

Fig. 3.51

Solution: From symmetry of the figure, it is evident that the desired centroid C lies on the diagonal OD of the square. Hence, choosing coordinate axes x and y as shown in the figure, it will only be necessary to determine one coordinate of the centroid, since $x_c = y_c$.

To determine x_c, let us denote by A_1 the area of the square, by x_1 the x coordinate of its centroid C_1, and by A_2, x_2 the corresponding quantities for the quadrant of the circle. Then the first of Eq. (14) gives

$$xc = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2}$$

For the given dimensions and remembering that $x_2 = y_2 = 4a/3\pi$ this becomes

$$x_c = \frac{a^2(a/2) - (\pi a^2/4)(4a/3\pi)}{a^2 - \pi a^2/4}$$

from which

$$y_c = x_c = \frac{2a}{3(4-\pi)} = 0.777a \qquad \text{(e)}$$

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Solution: In dealing with this figure, divide the lengths L_1, L_2, L_3, L_4 and L_5 with the centroids C_1, C_2, C_3, C_4 and C_5 as shown. Using the given dimensions, we may fill in the horizontal lines of the table given on next page:

No.	$\Delta A_i\,(mm^2)$	$x_i\,(mm)$	$y_i\,(mm)$	$\Delta A_i x_i\,(mm^2)$	$\Delta A_i y_i\,(mm^2)$
1	78.54	31.83	81.83	2500.00	6427.00
2	78.54	68.17	68.17	5354.07	5354.07
3	50.00	0.0	25.00	0.00	1250.00
4	50.00	100	25.00	500.00	1250.00
5	100	0.0	0.00	0.00	0.00
Σ	6696			12854.07	14281.07

Equation (15) gives,

$$y_c = \frac{12854.07}{357.08} = 36 \text{ mm}$$

$$y_c = \frac{14281.07}{357.08} = 40 \text{ mm}$$

8. A homogeneous prismatic bar *ABC* of negligible diameter, is bent into a straight section and a semicircular arc as shown in Fig. 3.56. It is attached to a hinge at *A*. Determine the value of α for which the bar is in equilibrium for the indicated position.

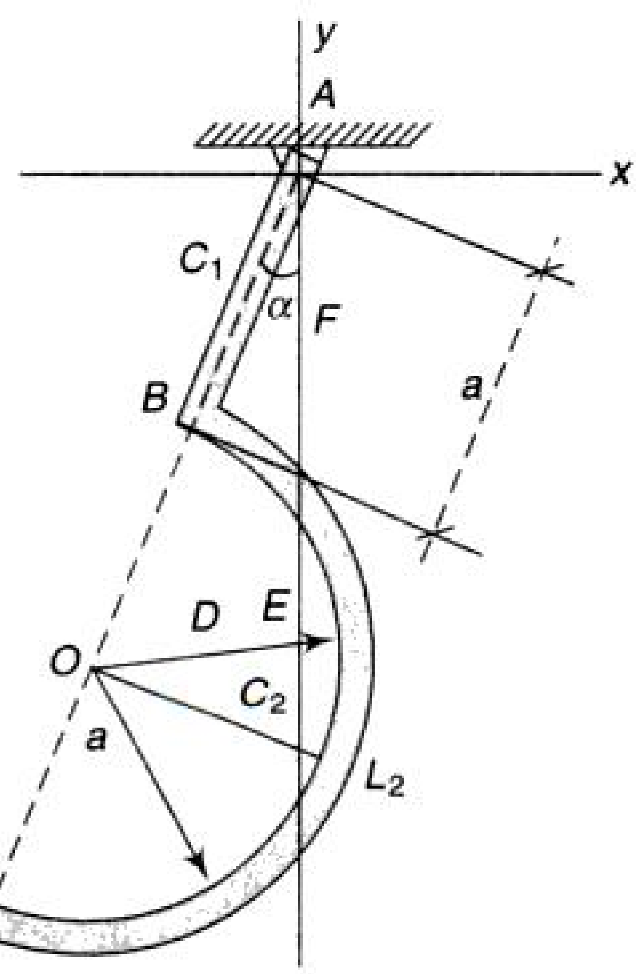

Fig. 3.56

Solution: Since *ABC* is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. For the body to be in equilibrium, the center of gravity should be on the y-axis, i.e., $x_c = 0$. Dividing the lengths L_1 and L_2 with the centroids C_1 and C_2 as shown. Now with the help of list of Eq. (15),

$$x_c = \frac{L_1 x_1 + L_2 x_2}{L_1 + L_2}$$

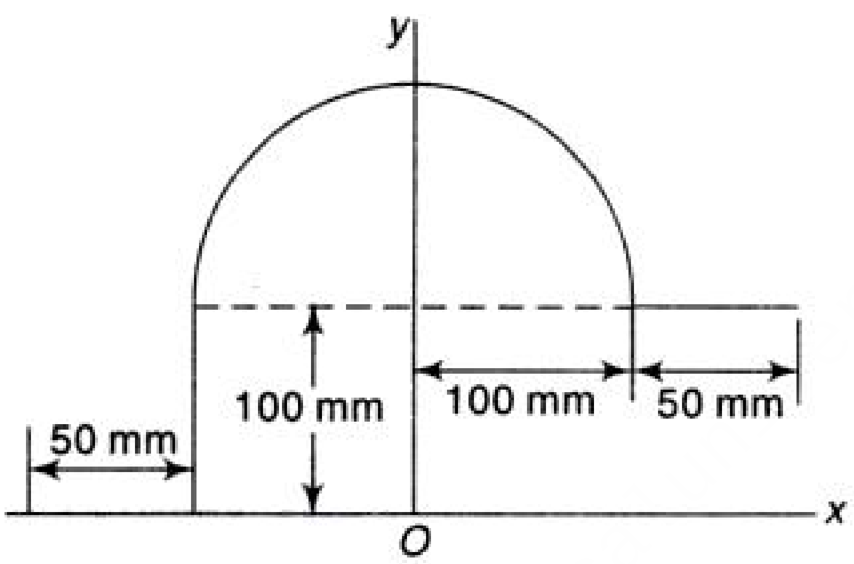

Fig. E

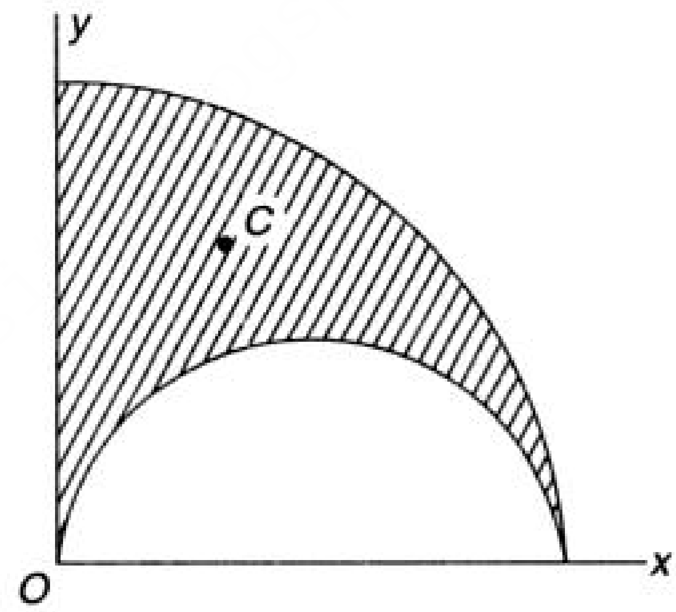

Fig. F

7. Locate the centroid C of the shaded area obtained by cutting a semicircle of diameter a from the quadrant of a circle of radius a as shown in Fig. F.

(*Ans.* $x_c = 0.349a$; $y_c = 0.636a$)

8. A slender homogeneous wire of uniform cross-section is bent into the shape shown in Fig. G. If the dimension a is fixed, find the dimension b so that the center of gravity of the wire will coincide with the center C of the semicircular portion.

(*Ans.* $b = 0.618a$)

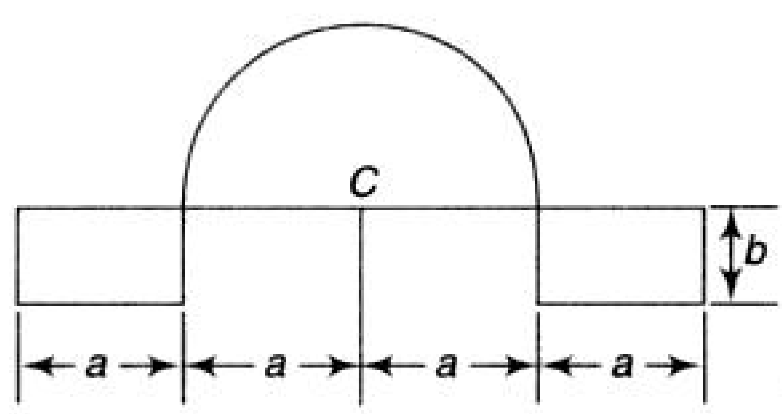

Fig. G

9. Locate the centroid C of the shaded area $OABD$ shown in Fig. H.

(*Ans.* $x_c = 67.75$ mm; $y_c = 59$ mm)

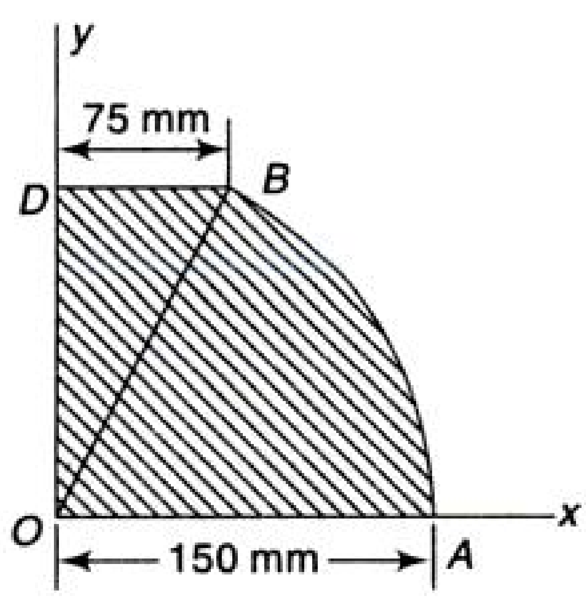

Fig. H

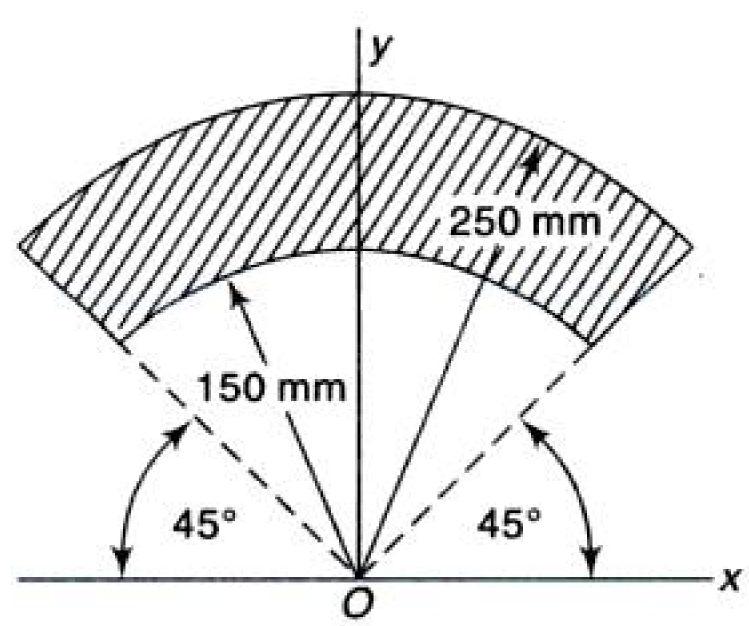

Fig. I

10. Locate the centroid C of the shaded sector of a ring subtending a 90° central angle and symmetrical about the y-axis, as shown in Fig. I.

(*Ans.* $x_c = 0$; $y_c = 183.75$ mm)

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Eq. (12) of Section 3.2. Taking moments with respect to points A and B, these equations become

$$-p_b h + Q\frac{h}{3}h = p_a h - Q\frac{h}{3} = 0$$

from which we find $p_a = Q/3$ and $p_b = 2/3Q$. Substituting for Q its value from Eq. (e), we find

$$p_a = \frac{wh}{2}h \quad \text{and} \quad p_b = \frac{wh^2}{2} \tag{f}$$

2. One end of a cantilever beam AC is built into a wall of thickness a as shown in Fig. 3.62. Owing to the action of a load P applied at the free and C, distributed reactions, as represented by the load diagrams AaB' and $A'Bb$, are produced. Find the maximum intensities q_a and q_b of these reactions. Neglect the weight of the beam, and assume all forces to act in its vertical axial plane of symmetry.

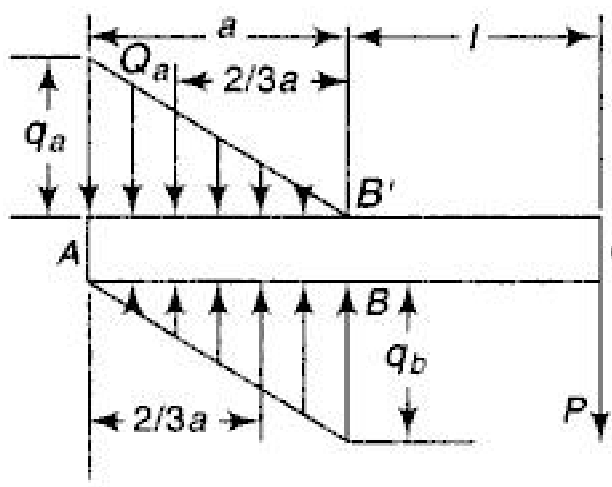

Fig. 3.62

Solution: Since we assume a linear variation in the intensities of pressure along the lines AB' and $A'B$, it follows that the lines of action of the resultants, Q_a and Q_b are as shown in the figure. Replacing the distributed reactions by these resultants, we obtain a system of three parallel forces in a plane, which are in equilibrium. Hence, taking moments of the three forces p, Q_a and Q_b with respect to points D and E and using Eq. (12), we obtain

$$Q_b\frac{a}{3} - P\left(l + \frac{2}{3}a\right) = 0, \qquad Q_a\frac{a}{3} - P\left(l + \frac{a}{3}\right) = 0$$

form which

$$Q_a = \frac{3p}{a}\left(l + \frac{a}{3}\right) \quad \text{and} \quad Q_b = \frac{3p}{a}\left(l + \frac{2}{3}a\right)$$

From the fact that the resultant of a distributed force in a plane is equal to the area of its load diagram, we obtain,

$$Q_a = \frac{q_a a}{2} \quad \text{and} \quad Q_b = \frac{q_b a}{2}$$

from which

$$q_a = \frac{2Qb}{2} \quad \text{and} \quad q_b = \frac{2Qb}{2}$$

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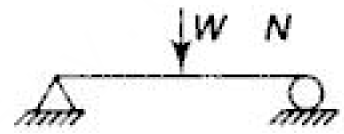
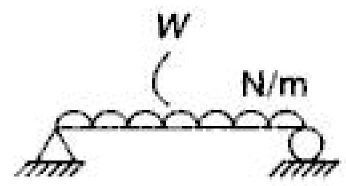
2. Match the List I with the List II using the codes given below:

List I | *List II*

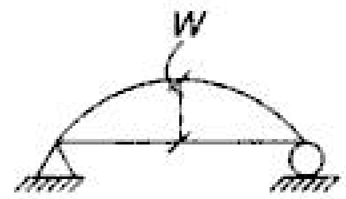
A. Uniformly distributed load



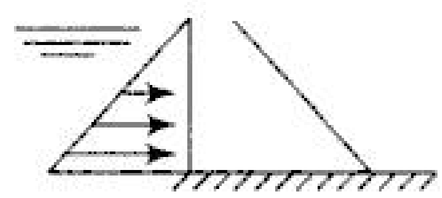
B. Parabolic loading



C. Hydrostatic pressure



D. Concentrated load



(a) A – 1, B – 2, C – 3, D – 4
(b) A – 2, B – 3, C – 4, D – 1
(c) A – 2, B – 3, C – 1, D – 4
(d) A – 3, B – 4, C – 1, D – 2

[*Ans.* (b)]

3. Match the List I with the List II using the codes given below:

List I	*List II*
A. Distributed force	1. The diagram showing the variation of the load along the length of the body.
B. Intensity of distributed force	2. The area under the load diagram.
C. The magnitude of the resultant force	3. The load per unit length of the line of application.
D. Load diagram	4. A large number of parallel forces acting over the small elements of the length of the body.

Codes:

(a) A – 1, B – 2, C – 3, D – 4
(b) A – 2, B – 3, C – 4, D – 1
(c) A – 4, B – 3, C – 1, D – 1
(d) A – 4, B – 2, C – 3, D – 1

[*Ans.* (c)]

4. Consider the following statements:

Assertion (A): The magnitude of the single concentrated resultant force of the distributed force is equal to the area under the load diagram.

Reason (R): The line of action of the single concentrated resultant force of the distributed force passes through the centroid of the area under the load diagram.

Select your answer using the codes given below:

Codes:

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not a correct explanation of A.

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4

General Case of Forces in a Plane

4.1 COMPOSITION OF FORCES IN A PLANE

If several coplanar forces applied to a body are not parallel and do not intersect in one point, we have the general case of forces in a plane. Let us consider such a system as represented by the forces $F_1, \ldots, F_4$, applied, respectively, at points A, B, C and D of the body as shown in Fig. 4.1(a). To find the resultant of these forces, we being with any two forces, say, F_1 and F_2 and determine their resultant R_1 by using the parallelogram law as indicated in the figure. Treating next the forces R_1 and F_3 in the same manner, we find their resultant R_2 which evidently is the resultant of the forces F_1, F_2 and F_3. In the same way again, the resultant R of the forces R_2 and F_4, and consequently of the given system of forces $F_1, \ldots, F_4$ is found applied at point G as shown. The point of application of this resultant R may be transmitted to any other point along its line of action if desired. From this discussion, we see that the magnitude and direction of the resultant R are determined by the closing side of the polygon of forces [Fig. 4.1(b)] and are independent of the points of application of the given forces.

If, in a more general case of n forces $F_1, \ldots, F_n$ in a plane, we find, by successive applications of the parallelogram law that the partial resultant of the first k forces $F_1, \ldots, F_k$ is parallel to the remaining $n - k$ forces $F_{k+1}, \ldots, F_n$. we use the method of addition of parallel forces as discussed in Section 3.2. On the basis of that discussion we conclude that, in general there are three possibilities: (1) The system of n forces in a plane reduces to a resultant force, (2) the system reduces to a resultant couple, or (3) the system is in equilibrium.

To distinguish between these three cases, we begin with the construction of the polygon of forces. If this polygon is not closed [Fig. 4.1(b)], the system reduces to a resultant force, the magnitude and direction of which are given by the closing side of the polygon and the line of action which may be located by using the construction indicated in Fig. 4.1(a) or the method of adding parallel forces.

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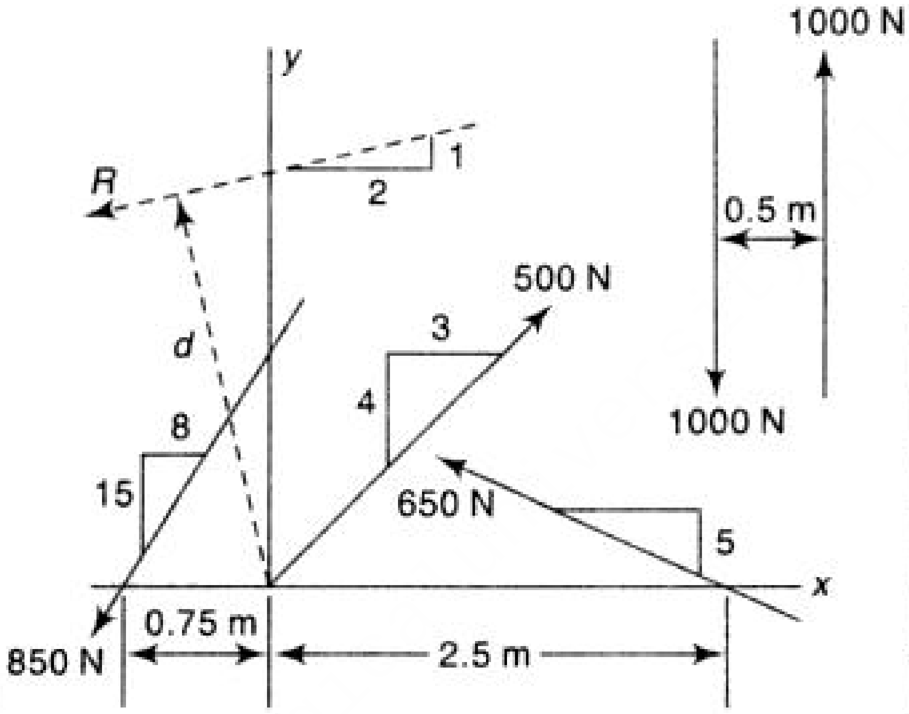


(a)

F_i	X_i	Y_i	$(M_o)_i$
170	−80	−150	+300
100	+60	+80	0
130	−120	+50	+400
200	0	−200	+300
200	0	+200	
Σ	−140	−20	+1000

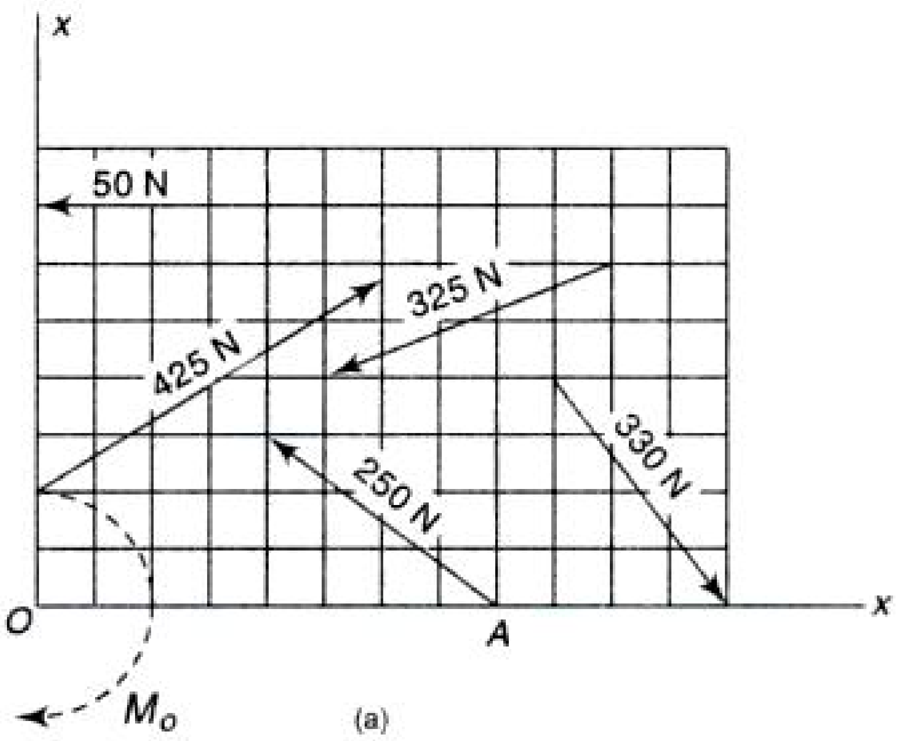
(b)

Fig. 4.4

$$\tan\alpha = \frac{X}{Y} = \frac{-100}{-700} = \frac{1}{7}, \qquad d = \frac{M_0}{R} = \frac{1687.5}{707} = 2.4 \text{ m}$$

Thus the resultant is located as shown in Fig. 4.4.

2. Determine the resultant of the system of coplanar forces shown in Fig. 4.5. Each division of the superimposed grid is 1 cm square.



(a)

F_i	X_i	Y_i	$(M_o)_i$
10	−10.0	0	+70.0
85	+70.7	+47.1	−141.4
65	−60.3	−24.3	+120.6
50	−40.0	+30.0	+240.0
66	+39.6	−52.8	−633.6
Σ	0	0	−344.4

(b)

Fig. 4.5

Solution: We first compute and record the projections X_i and Y_i of each force as shown in Fig. 4.5(b). Summing these projections for all forces, we obtain $X = 0$ and $Y = 0$. Therefore the resultant is not a force.

To see if there is resultant couple, we compute and record the moment $(M_o)_i$ of each force about the origin O. The calculation of these moments is greatly simplified if we resolve each force into its components X_i and Y_i at the point where its line of action intersects either the x- or y-axis, so that one component will pass through O. Take for example, the 250 N force. Resolving it at A into components X_i and Y_i we have

$$(M_o)_i = 200 \times 0 + 150 \times 8 = 1200 \text{ N cm} = 12 \text{ N m}$$

Moments of the forces are calculated in a similar manner. Summing moments in accordance with Eq. (c), we get $M_0 = -1732.2$ N cm $= -17.32$ N m. Thus the resultant is a clockwise couple in the plane of action of the forces as shown in Fig. 4.5(a).

resultant is a force acting along the line AB. Finally, if the algebraic sums of the moments of all forces, not only with respect to A and B but also with respect to a third point C, are zero and C is not on the line AB, the possibility of a resultant force falls completely and we must have equilibrium. Expressing these three conditions of equilibrium algebraically, we have

$$\Sigma(M_A)_i = 0, \qquad \Sigma(M_b)_i = 0, \qquad \Sigma(M_c)_i = 0 \tag{19}$$

Reconsidering the set of Eqs (18) or (19), we observe that in both cases, three independent conditions are not only necessary but also sufficient to ensure equilibrium of a system of coplanar force. Naturally, with three equations we can determine only three unknowns. This means that in dealing with constrained bodies where unknown reactions are to be evaluated, we shall not be able to determine the magnitudes of more than three such forces, or possibly the magnitude and direction of one and the magnitude of another. For this reason a system of physical constraints of rigid body in a plane which gives rise to just three unknown is said to be statically determinate.

Common systems of constraints which satisfy the above condition are shown in Fig. 4.6. In Fig. 4.6(a), we have a body supported by a hinge A and a roller B, which completely constrain the body in the plane of the figure. The roller at B determines physically the direction of R_b nut, the direction of R_a remains unknown as indicated by the wavy-line vector. Thus the three unknowns in this case are the magnitude and direction of R_a and the magnitude of R_b. In Fig. 4.6(b), a body is constrained in one plane by three hinged bars which are not parallel and do not intersect in one point[1]. In this case, the lines of action of three reactive forces S_1, S_2 and S_3 (coinciding with the axes of the bars producing them) are known and again we have only three unknown magnitudes to determine.

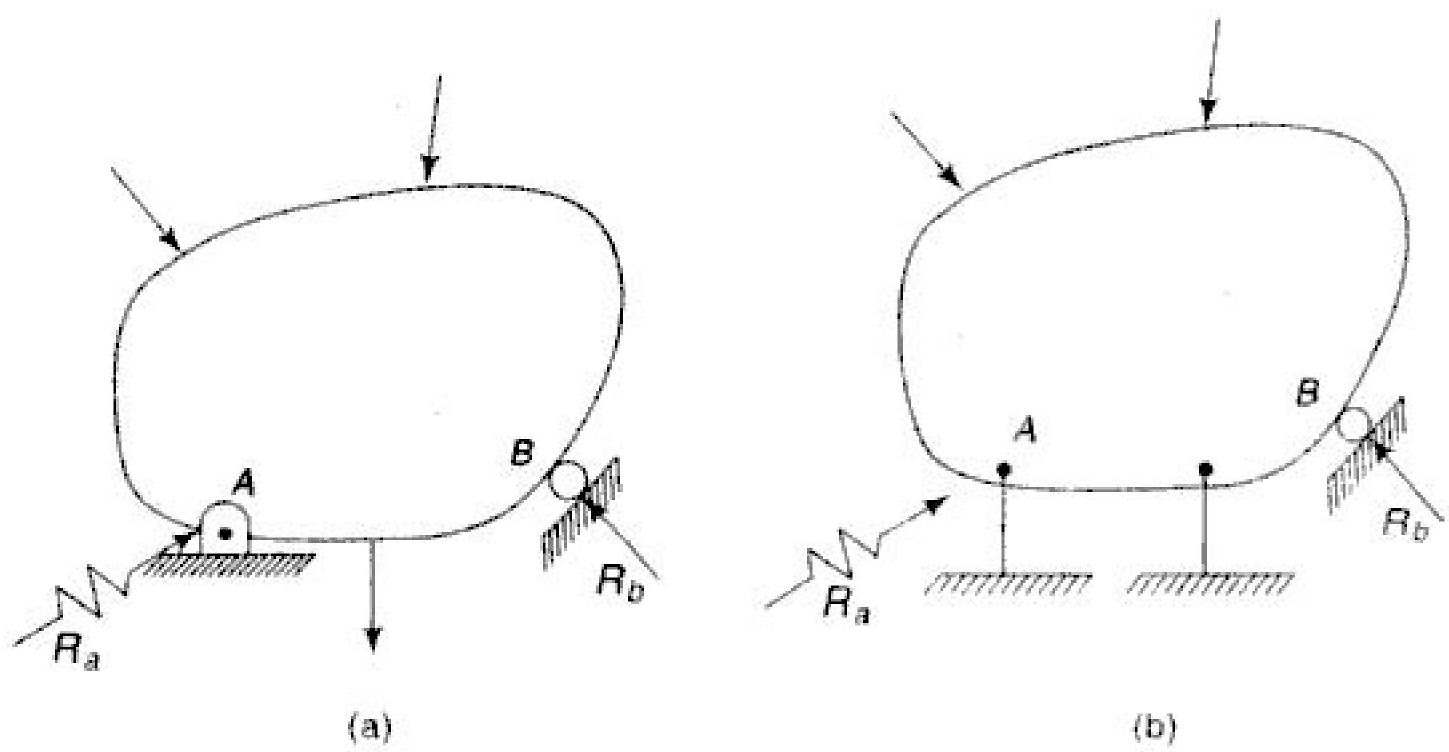


Fig. 4.6

Any system of supports of a rigid body in one plane which contains more than three degrees of constraint will set up reactive forces involving more than three

[1]If the three bars in Fig. 4.6(b) are parallel or intersect in one point, the body will always have some freedom of motion in the plane of the figure and is not adequately constrained, i.e., it will move under the action of applied loads.

Referring back to Fig. 4.10(a), we see that

$$b = 2a - r_1 - r_2$$

and expression (d) becomes

$$Q = W_1 \frac{(2a - r_1 - r_2)}{a} \tag{e}$$

$$\Rightarrow \qquad Q = W\left(2 - \frac{r_1 + r_2}{a}\right)$$

In the particular cases of two identical spheres each of weight W and radius r, we have

$$b = 2(a - r)$$

and expression (d) becomes

$$Q = 2W\frac{(a-r)}{2} = 2W\left(1 - \frac{r}{a}\right) \tag{f}$$

It must be kept in mind that expressions (e) and (f) represent in each case the minimum value of the weight Q for which equilibrium can exist.

Important Terms and Concepts

Eqations of equilibrium for the general case of forces acting in a plane
Equations of equilibrium for the general case of forces acting in a plane in terms of moments
Statically determinate   Hinge   Roller
Simple bar   Statically indeterminate   Redundant supports

SUMMARY

- A system of forces in one plane can reduce to a resultant force, a resultant couple, or a state of equilibrium.
- The body can be in equilibrium if the algebraic sum of all the external forces and their moments about any point in their plane is zero.
- A system of physical constraints of a rigid body in a plane which gives rise to just three unknowns is said to be statically determinate.
- Any system of supports of a rigid body in one plane which contains more than three degrees of constraint will set up reactive forces involving more than three unknowns. Then the system is said to be statically indeterminate.
- Supports in excess of those just necessary and sufficient to prevent motion are called redundant supports.

Important Formulae

1. The equations of equilibrium for the general case of forces acting in one plane are
$$\Sigma X_i = 0, \qquad \Sigma Y_i = 0, \qquad \Sigma(M_o)_i = 0$$
2. The equations of equilibrium for the general case of forces acting in one plane in terms of moments are
$$\Sigma(M_A)_i = 0, \qquad \Sigma(M_b)_i = 0, \qquad \Sigma(M_c)_i = 0$$

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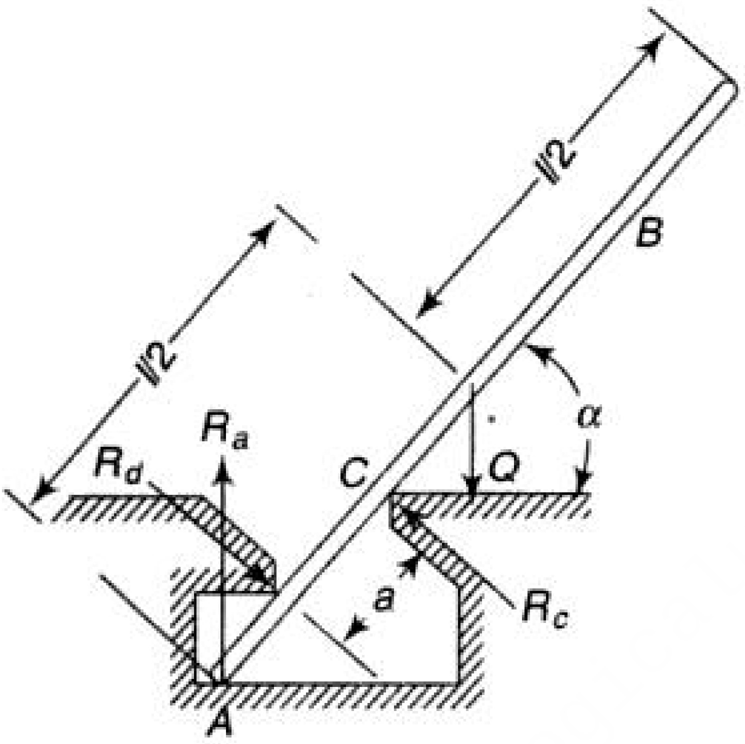

Fig. I

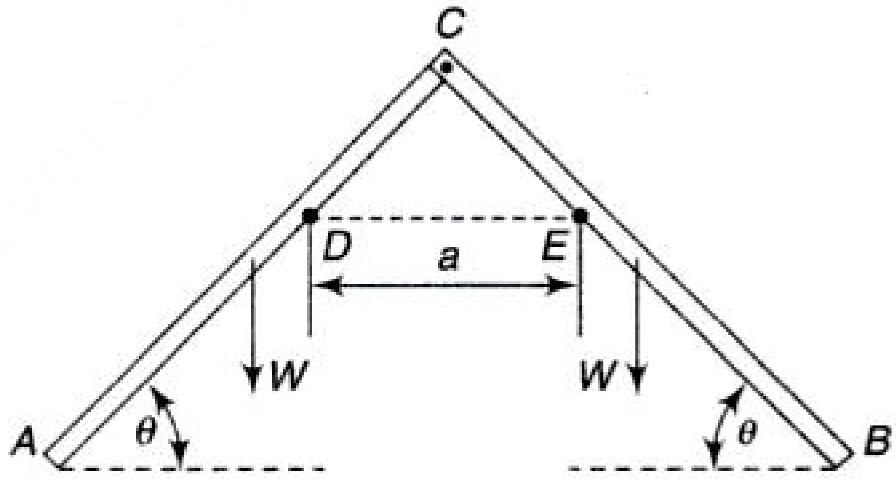

Fig. J

11. A heavy prismatic timber of weight W and length l is supported horizontally between two fixed fulcrums A and B, as shown in Fig. K. If the coefficient of friction between the timber and each fulcrum is μ, find the magnitude of a horizontal force P applied as shown that will cause impending sliding of the timber to the right. The following numerical data are given: $W = 500$ N; $a = 500$ mm, $h = 300$ mm, $l = 1.75$ m, $d = 300$ mm and $\mu = 1/3$. (*Ans.* $P = 250$ N)

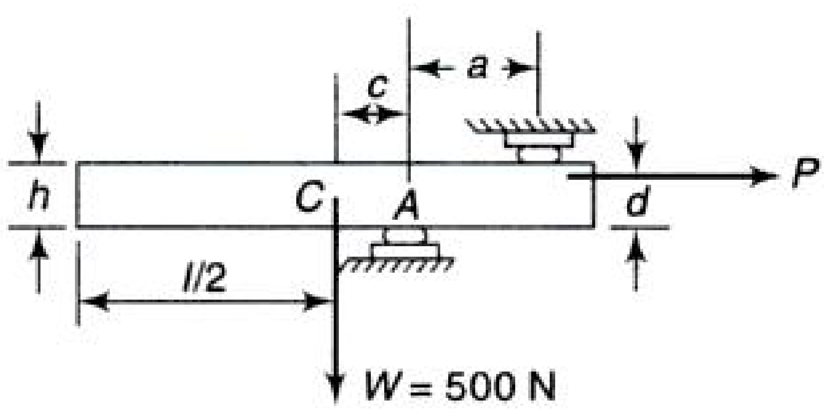

Fig. K

*12. Find the magnitude and direction of the least force P necessary to cause impending sliding of the timber in Fig. K. Compare this result with the value of the corresponding horizontal force as obtained in the preceding problem.

(*Ans.* $P_{min} = 226$ N, inclined downward by 25°17 to the horizontal)

*13. A heavy prismatic timber of weight W is supported in vertical plane as shown in Fig. L. If the coefficient of friction between the timber and each of the supports A and B is $\mu = 1/3$ and the dimensions are as shown, find the maximum value of the angle α consistent with equilibrium. (*Ans.* $\alpha \leq 43°$)

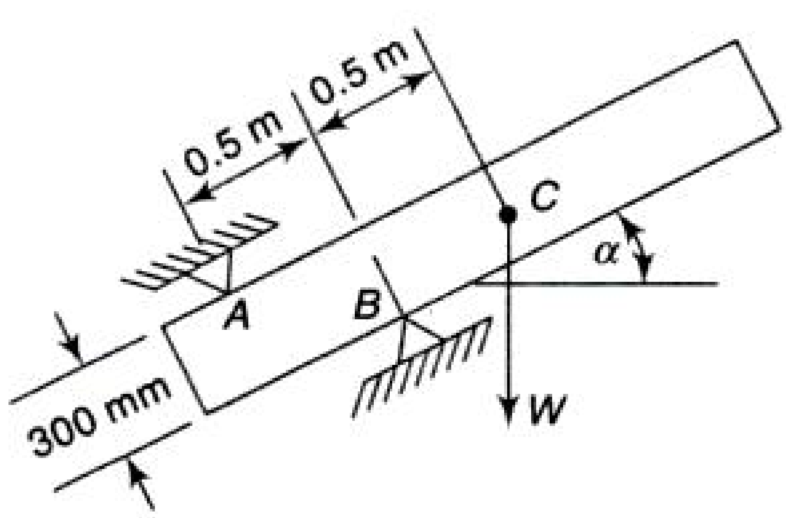

Fig. L

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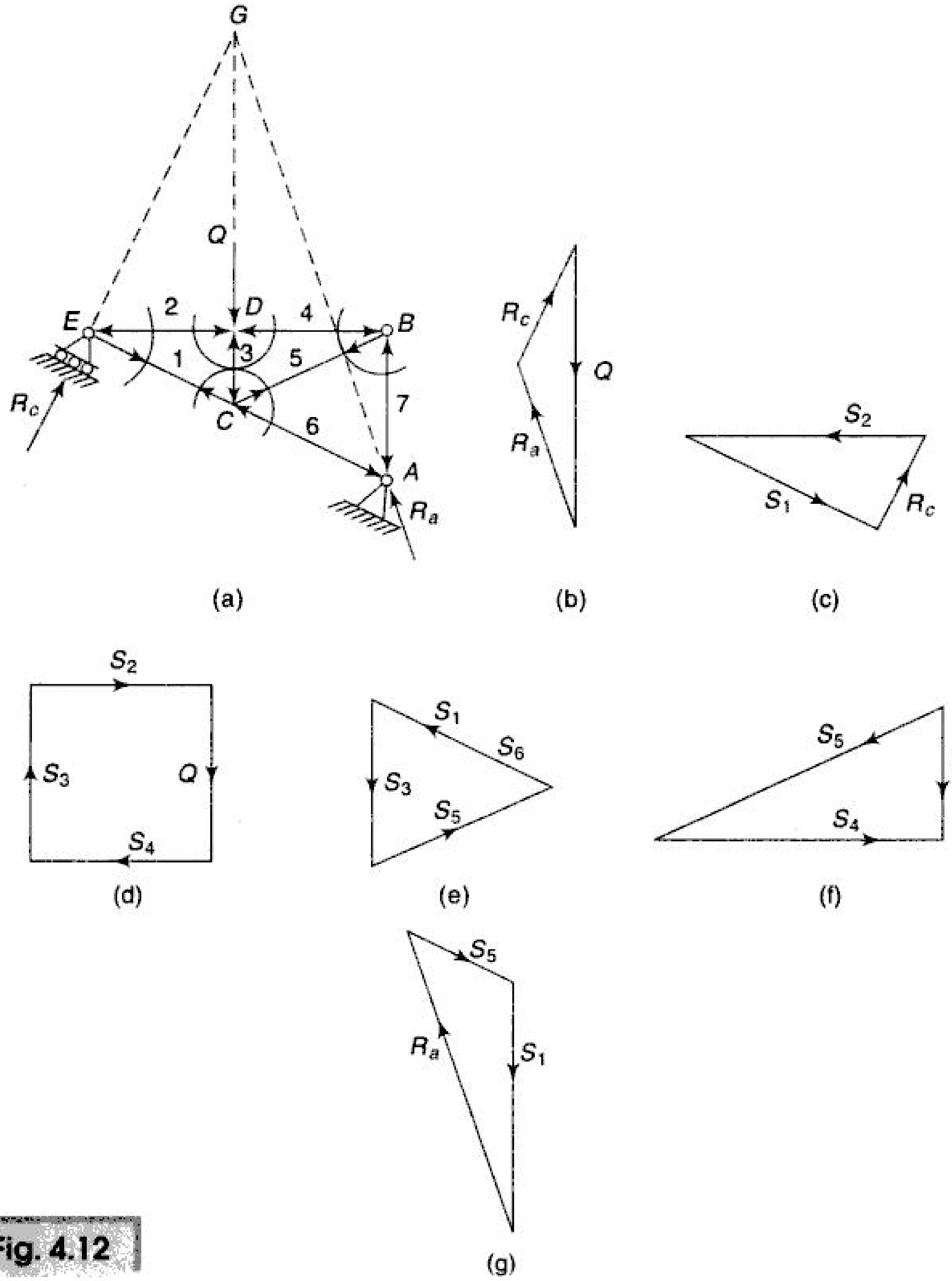

Fig. 4.12

Let us consider next the equilibrium of the hinge D. We first replace the bar 2 by its reaction S_2 equal but opposite to the previously considered reaction of this bar on the hinge E. There remain then but two unknown forces at D, representing the reactions on this hinge of the bars 3 and 4. Again we do not know whether these last two forces are directed toward the hinge or away from it, but knowing the directions of their lines of action we can construct the polygon of forces shown in Fig. 4.13, from which the forces, S_3 and S_4 are determined as before. In this case we see from the arrows on the sides of the polygon of forces that both bars are pressing on the hinge and hence are in compression. Arrows indicating such reactions can now be placed on the axes of the bars 3 and 4 at D, as shown in Fig. 4.12(a).

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PRACTICE SET 4.3

Review Questions

1. Define plane truss.
2. State the assumptions made in the analysis of trusses.
3. Explain the method of joints for analysis of the truss.
4. Explain the importance of the assumptions in the analysis of truss.

Objective Questions

1. Minimum number of members required to form a simple truss
   (a) 2 (b) 3 (c) 4 (d) 5
   [*Ans.* (b)]
2. In the method of joints for the analysis of forces in the members of the truss, the number of equilibrium equations available at each joint are
   (a) 2 (b) 3 (c) 4 (d) 5
   [*Ans.* (a)]
3. Which of the following assumptions are necessary in the analysis of truss?
   I. The bars are connected at their ends by frictionless hinges.
   II. The bars are weightless.
   III. The axes of all bars lie in one plane, called the middle plane of the truss.
   IV. All forces acting on the truss are applied at the hinges only and also lie in the plane of the truss.

   The correct answer is
   (a) Only I (b) Only II
   (c) I, II and III (d) All of the above
   [*Ans.* (d)]

PROBLEM SET 4.3

1. Calculate the axial force S_i in each bar of the simple truss supported and loaded as shown in Fig. A. The triangle ACB is isosceles with 30° angles at A and B and $P = 5$ kN. (*Ans.* $S_1 = -3.34$ kN, $S_2 = -6.67$ kN; $S_3 = +\ 5.77$ kN; $S_4 = +\ 2.89$ kN; $S_5 = +\ 5.77$ kN)

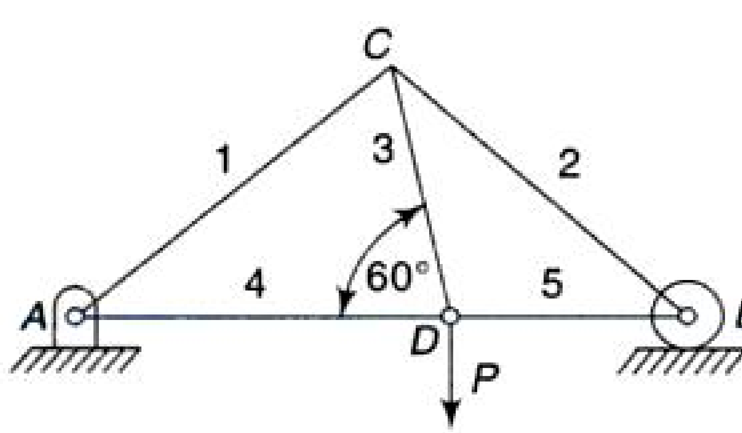

Fig. A

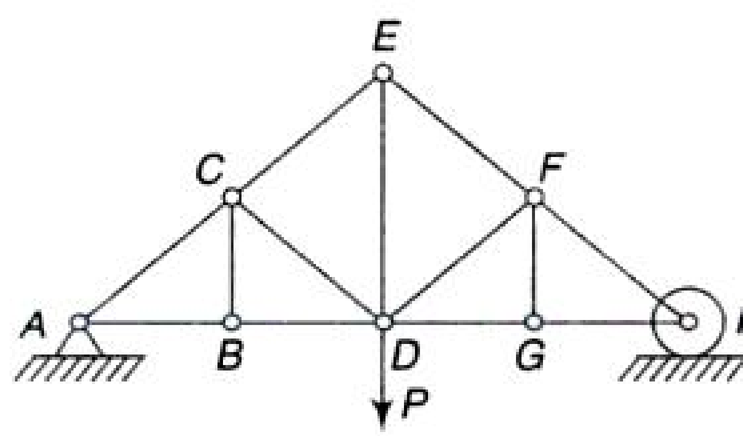

Fig. B

2. Prove that a tensile force equal to the applied load P is produced in the bar DE of the truss shown in Fig. B.
3. Determine the axial forces in the bars 1, 2, 3, 4 and 5 of the plane truss supported and loaded as shown in Fig. C.
   (*Ans.* $S_1 = -P$; $S_2 = +\ P$; $S_3 = -\ 0.5\ P$; $S_4 = +\ 0.442\ P$; $S_5 = -0.333\ P$)

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The foregoing procedure in the analysis of trusses is called the *method of sections*. It consists essentially in the isolation of a portion of the truss by a section is such a way as to cause those internal forces that we wish to evaluate to become external forces on the isolated free body. By this procedure, we usually arrive at the case of equilibrium of general system of forces in a plane and the usual conditions of equilibrium [Eqs (18) or (19)] may be employed to evaluate the unknown forces as was done above. The success or failure of the method rests entirely upon the choice of section. In general, a section should cut only three bars, since only three unknowns can be determined from three equations of equilibrium. However, there are special cases where we may successfully cut more than three bars, and some of these will be illustrated in the following examples. Here we shall always assume tension to be positive and compression, negative.

Examples Examples Examples Examples Examples

1. Using the method of sections, determine the axial forces in bars for which 1, 2 and 3 of the tower loaded as shown in Fig. 4.15(a).

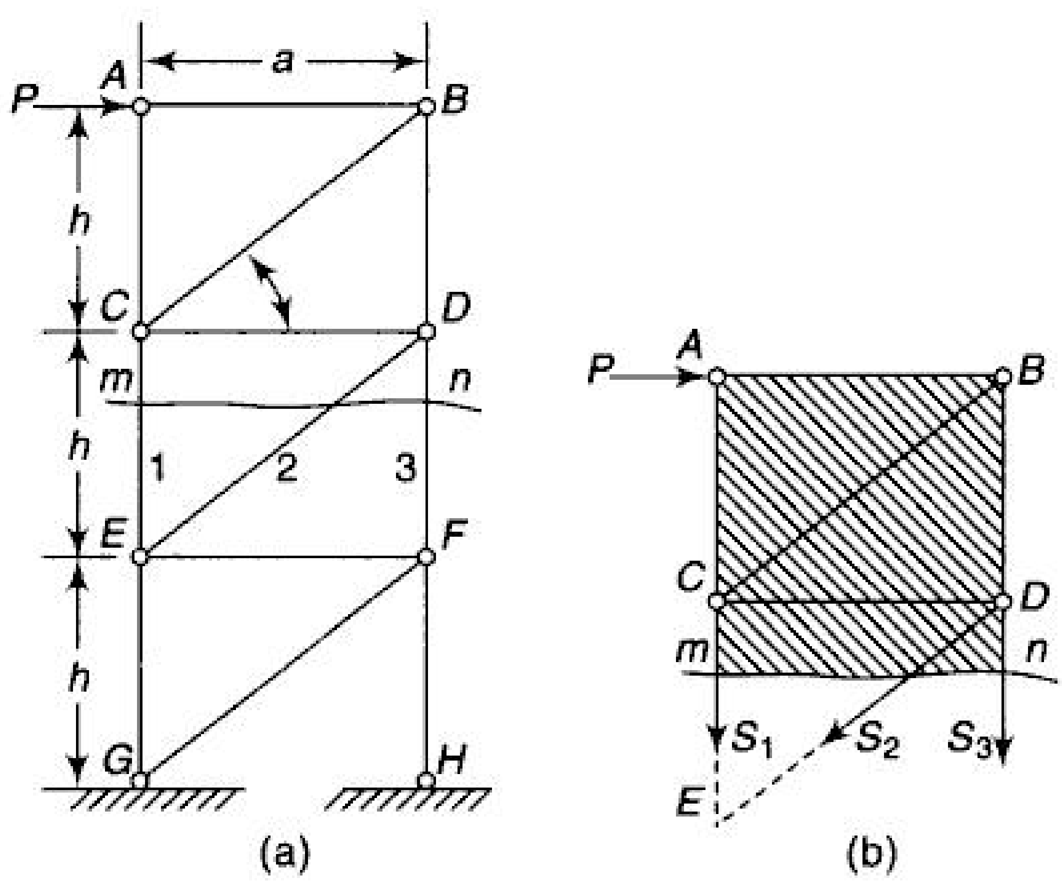

Fig. 4.15

Solution: We begin by making a section *mn*, cutting the three bars for which the axial forces are required and consider the portion of the tower above this section as a free body [Fig. 4.15(b)]. Acting on this free body, we have the applied force P and the forces S_1, S_2, and S_3 representing the axial forces in the cut bars 1, 2 and 3.

Equating to zero the algebraic sum of moments of these forces with respect to point D, we obtain

$$S_1 a - 2Ph = 0$$

From which $S_1 = +Ph/a$, tension. Similarly, with E as a moment center, we have

$$-S_3 a - 2Ph = 0$$

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6. Determine, by the method of sections, the axial force in each of the bars, 1, 2 and 3 of the plane truss shown in Fig. F.

(*Ans.* $S_1 = -1.5$ kN; $S_2 = -3.33$ kN; $S_3 = +2.665$ kN)

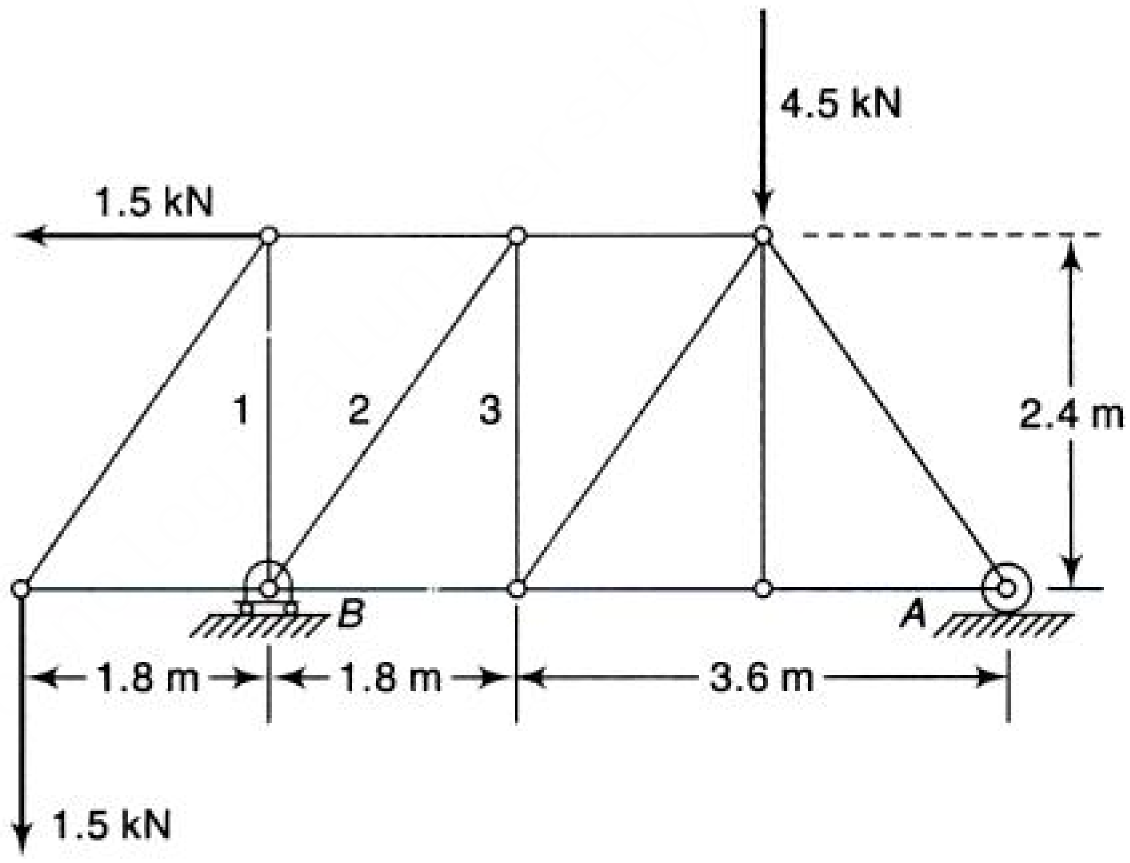

Fig. F

7. Using the method of sections, calculate the axial force in each of the bars 1, 2 and 3 of the plane cantilever truss loaded as shown in Fig. G.

(*Ans.* $S_1 = -5.33P$; $S_2 = +2P$; $S_3 = -1.67P$)

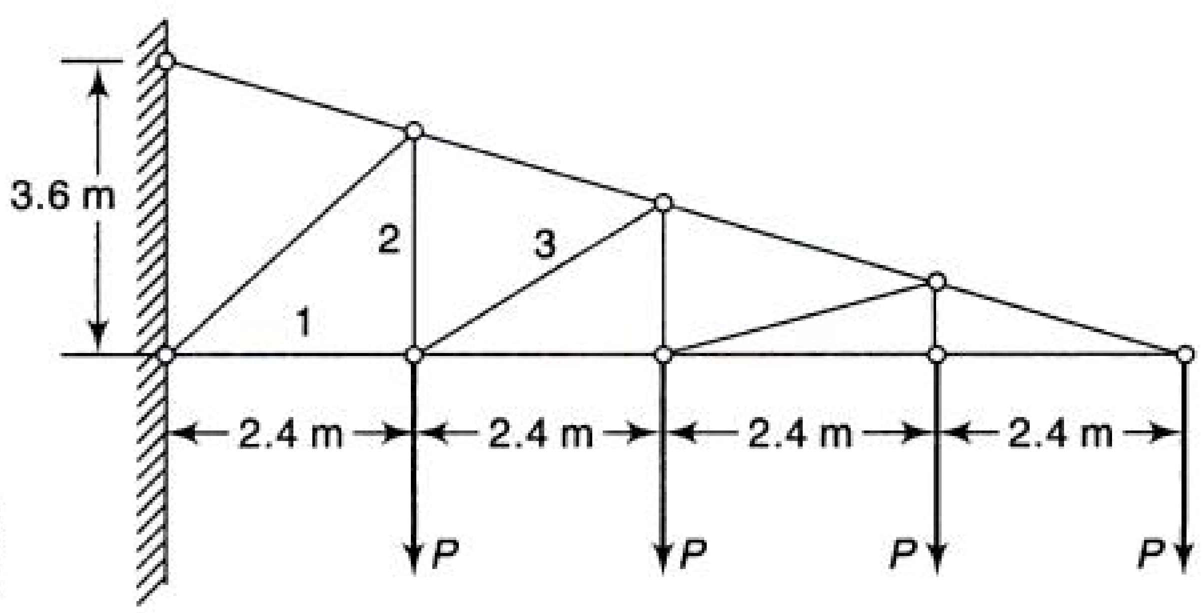

Fig. G

8. Determine the forces bars 1, 2 and 3 of the plane truss loaded and supported as shown in Fig. H. (*Ans.* $S_1 = -4Pa/h$; $S_2 = -P/2$; $S_3 = +4Pa/h$)

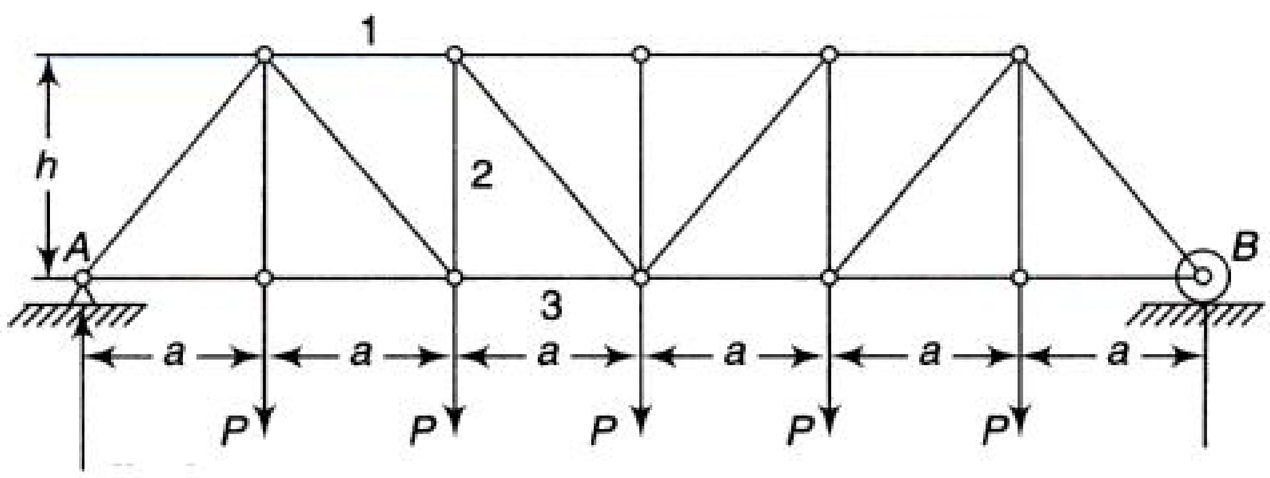

Fig. H

9. Determine the forces bars 1, 2 and 3 of the plane truss loaded and supported as shown in Fig. I. (*Ans.* $S_1 = S_2 = S_3 = -3Q\,a/2h$)

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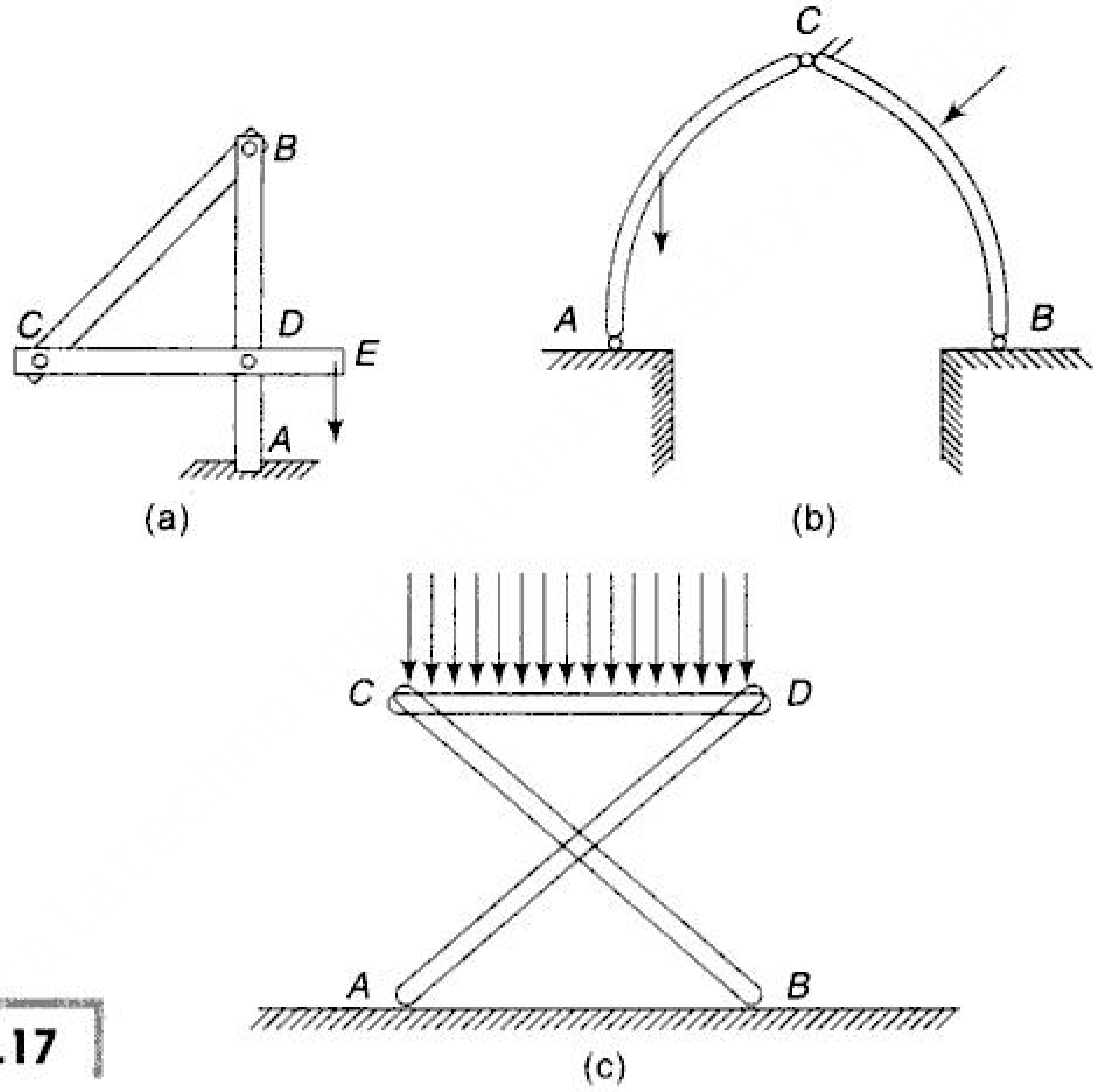

Fig. 4.17

This structure consists of two rigid bars or ribs AC and BC hinged together at C and to the foundation at A and B. Since each member is acted upon by a force at some intermediate point, it is subjected to bending, and the forces at its ends will not be directed along the axis of the member.

Accordingly, we do not know the direction of the reaction at either A or B; therefore we represent these reactions by rectangular components X_a, Y_a and X_b, Y_b as shown. Considering the equilibrium of the entire structure, we encounter a system of coplanar forces with four unknowns and with only three equations of equilibrium and the problem appears to be indeterminate. To avoid this difficulty, we disconnect the two bars at C and isolate each one as a free body, Fig. 4.18(b) and (c). Then on each of these free bodies we have, in addition to the forces already considered, a pair of rectangular components X_c, Y_c, representing the force transmitted from one bar to the other through the hinge C. If we assume these forces on the free body AC to be directed as shown in Fig. 4.18(b), then if follows from the law of action and reaction that the corresponding forces on BC must be oppositely directed, as shown in Fig. 4.18(c). It makes no difference which way we direct the forces X_c, Y_c, as long as they are opposite on the two free bodies. If either of them is incorrectly assumed, we shall simply obtain a result with negative sign after calculation.

We now have two simultaneous free-body diagrams [Fig. 4.18(b) and (c)] involving altogether six unknowns. Since for each free body we have three equations of equilibrium, Eqs (18) or (19), it follows that the problem is statically determinate, i.e., we have six equations and six unknowns. With points A and B as moment centers, Eq. (18) for the two systems will be as follows:

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Solution: We begin with a free-body diagram of the entire frame, Eq. (18) may be written as follows:

$$-X_a + 2.5 = 0$$
$$Y_a - R_b - 2.5 = 0$$
$$-10\,R_b + 2.5 \times 5 + 2.5 \times 5 = 0$$

from which we obtain $X_a = 2.5$ kN, $Y_a = 7.5$ kN, $R_b = 5$ kN.

We now make a separate free-body diagram for each of the three members, as shown in Fig. 4.20(b) to (d). On these free-body diagrams, we place the given loads at D and C and the numerical values of the previously found reactions at A and B, as shown. At each of the point E, F, G, we indicate two rectangular components of force representing the interactions between members at these points of connection, but we do not immediately attach any arrows or symbols to these vectors.

Now let us consider the equilibrium of the vertical bar AC in Fig. 4.20(b). At this time none of the four forces acting at E and G is known, but, nevertheless, we can find X_a and X_g and by writing $\Sigma M_g = 0$, which give

$$2.5 \times 5 + 2.5 \times 10 - 5\,X_e = 0$$
$$2.5 \times 5 + 2.5 \times 10 - 5\,X_g = 0$$

From these two equations, we obtain $X_c = 7.5$ kN, $X_g = 7.5$ kN, directed as assumed. Accordingly, we at once record 7.5 kN to the left at E in Fig. 4.20(c) and 7.5 kN to the right at G in Fig. 4.20(d).

Next, we consider the equilibrium of the horizontal bar DF in Fig. 4.20(c). With F as a moment center, the equations of equilibrium Eq. (18) for this free body become

$$X_f - 7.5 = 0$$
$$Y_e - Y_f - 2.5 = 0$$
$$5\,Y_e - 2.5 \times 10 = 0$$

from which $X_f = 7.5$ kN, $Y_e = 5$ kN and $Y_f = 2.5$ kN, all directed as assumed. These numerical values may now be recorded at E in Fig. 4.20(b) and at F in Fig. 4.20(d), as shown.

Returning to the free body in Fig. 4.20(d), and writing $\Sigma Y_i = 0$, we obtain

$$7.5 - 5 - Y_g = 0$$

from which $Y_g = 2.5$ kN directed down. We immediately place a corresponding 2.5 kN force up at G in Fig. 4.20(d), and we see that all forces have now been evaluated.

As a check on our arithmetic, we observe that the forces on the inclined bar in Fig. 4.20(d), which have all been found from considerations of the other free-body diagrams, satisfy the three conditions of equilibrium for this bar.

Important Terms and Concepts

Plane frame Flexure members Method of members Three hinged frame

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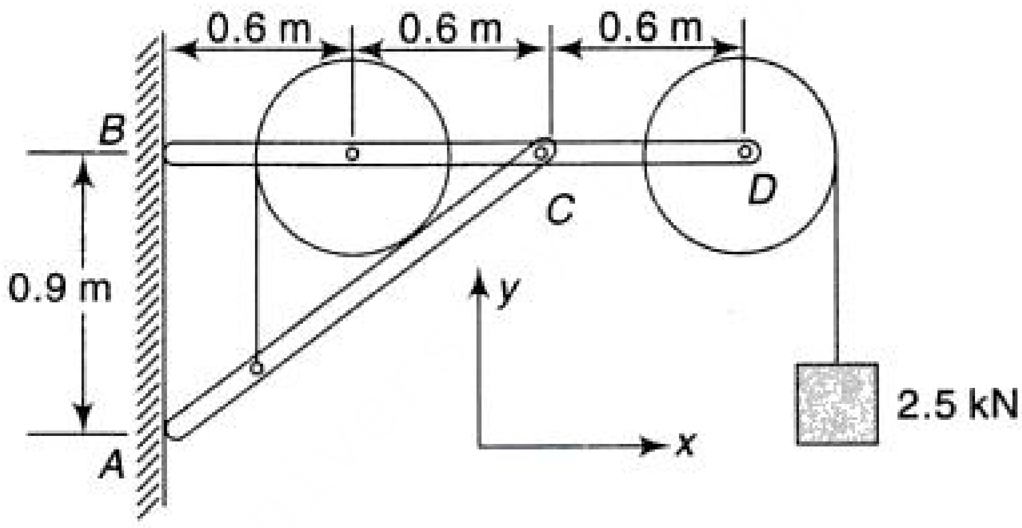

Fig. H

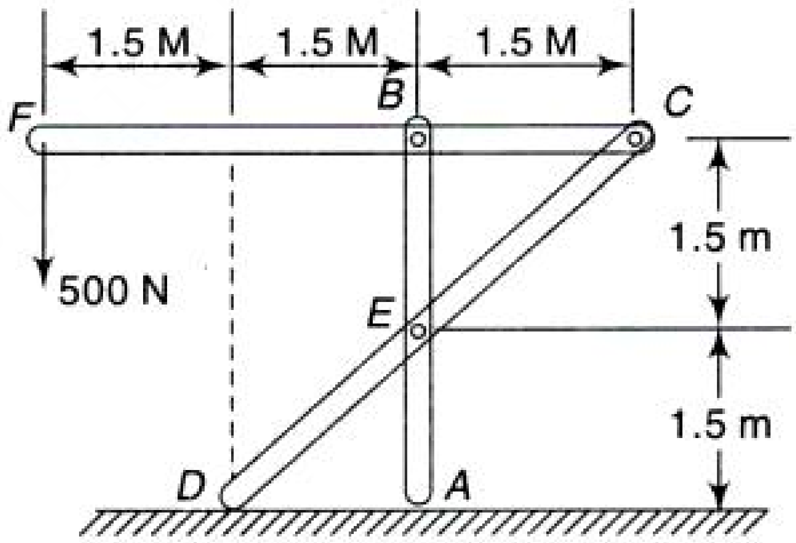

Fig. I

10. Calculate the pressure exerted against the sides of an ice cube of weight $W = 500$ N by the points A and B of the tongs which support it (Fig. J). The dimensions of the tongs are as follows: $a = 100$ mm, $b = 200$ mm, $c = 200$ mm, $d = 400$ mm.

(*Ans.* $R_a = R_b = 475$ N)

11. Calculate the shear force R_c on the pin C of the ice tongs in Fig. J.

(*Ans.* $R_c = 837$ N)

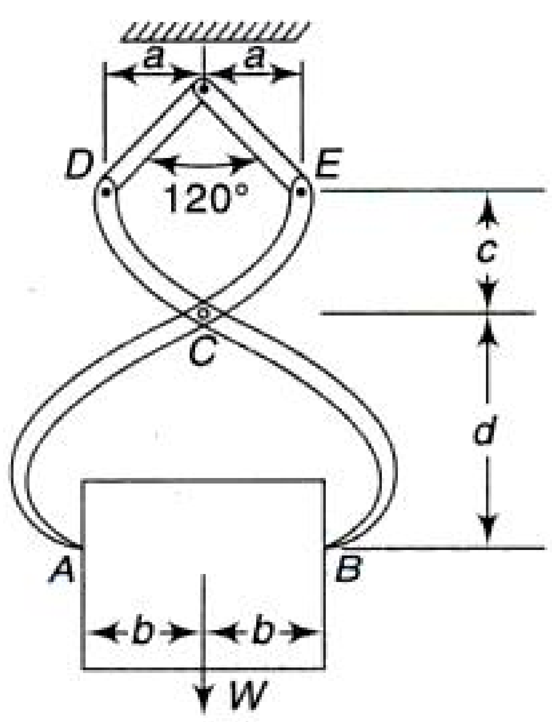

Fig. J

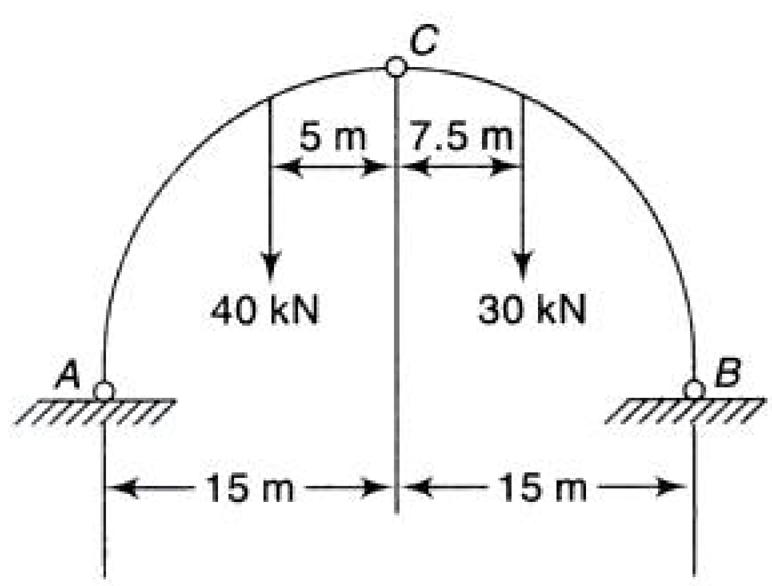

Fig. K

12. Find the reactions at the supports A and B of the semicircular three-hinged arch loaded as shown in Fig. K. (*Ans.* $R_a = 40.1$ kN, $R_b = 41.5$ kN)

13. Calculate the horizontal and vertical components of the reactions at A and B of the frame structure loaded as shown in Fig. L.

(*Ans.* $X_a = 315$ N; $Y_a = 210$ N; $X_b = 885$ N; $Y_b = 390$ N)

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The same procedure may be used in the case of two unequal parallel forces acting in opposite directions, as shown in Fig. 4.23(a). In this case again, we begin with the polygon of forces ABC, select a pole O, and draw rays 1, 2 and 3. Starting with any point a in the plane of action of the given forces, the funicular polygon $abcd$ with its sides parallel, respectively, to rays OA, OB, and OC is obtained as shown [Fig. 4.23(a)]. The magnitude and direction of the resultant are given by the closing side $\overline{AC}$ of the polygon of forces [Fig. 4.23(b)], and a point on its line of action, by the intersection e of the first and last sides of the funicular polygon [Fig. 4.23(a)].

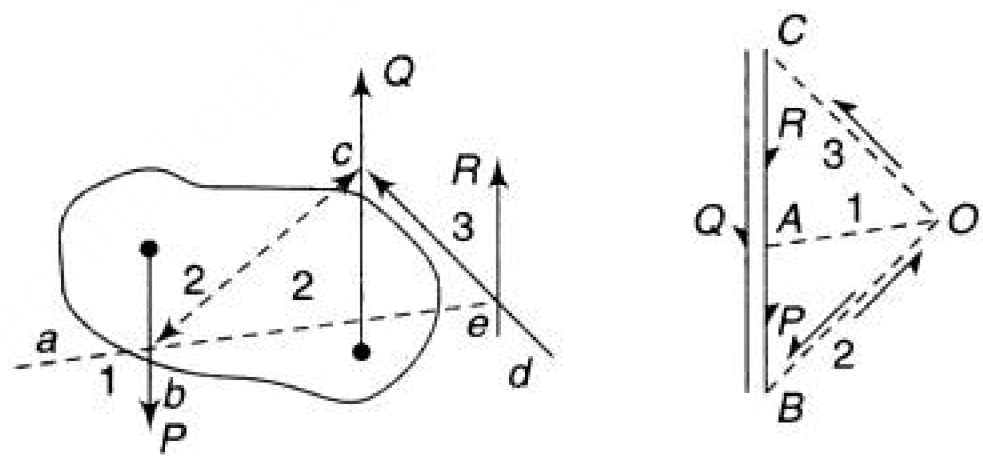

Fig. 4.23 (a) (b)

Let us consider now a general case of several coplanar forces $F_1, \ldots F_5$ applied to a body as shown in Fig. 4.24(a). We begin with the construction of the polygon of forces $ABCDEF$. Choosing an arbitrary pole O, drawing the rays 1, 2, 3, 4, 5, 6 and constructing, in the plane of action of the forces, the lines ab, bc, … fg, parallel to these rays, we obtain the funicular polygon $abcdefg$ as shown. At the apexes of this polygon, each of the given forces $F_1, \ldots F_5$ is replaced by its two components as represented to rays in Fig. 4.24(b). The forces acting along the sides bc, cd, de, and ef are pairs of equal and opposite forces and may be removed from the system. There remain only the forces 1 and 6 acting at points b and f which are equivalent to the given forces $F_1, \ldots F_5$. The magnitude and direction of the resultant of these forces are given by the closing side $\overline{AF}$ of the polygon of forces [Fig. 4.24(b)], and a point on its line of action, by the intersection h of the first and last sides of the funicular polygon [Fig. 4.24(a)].

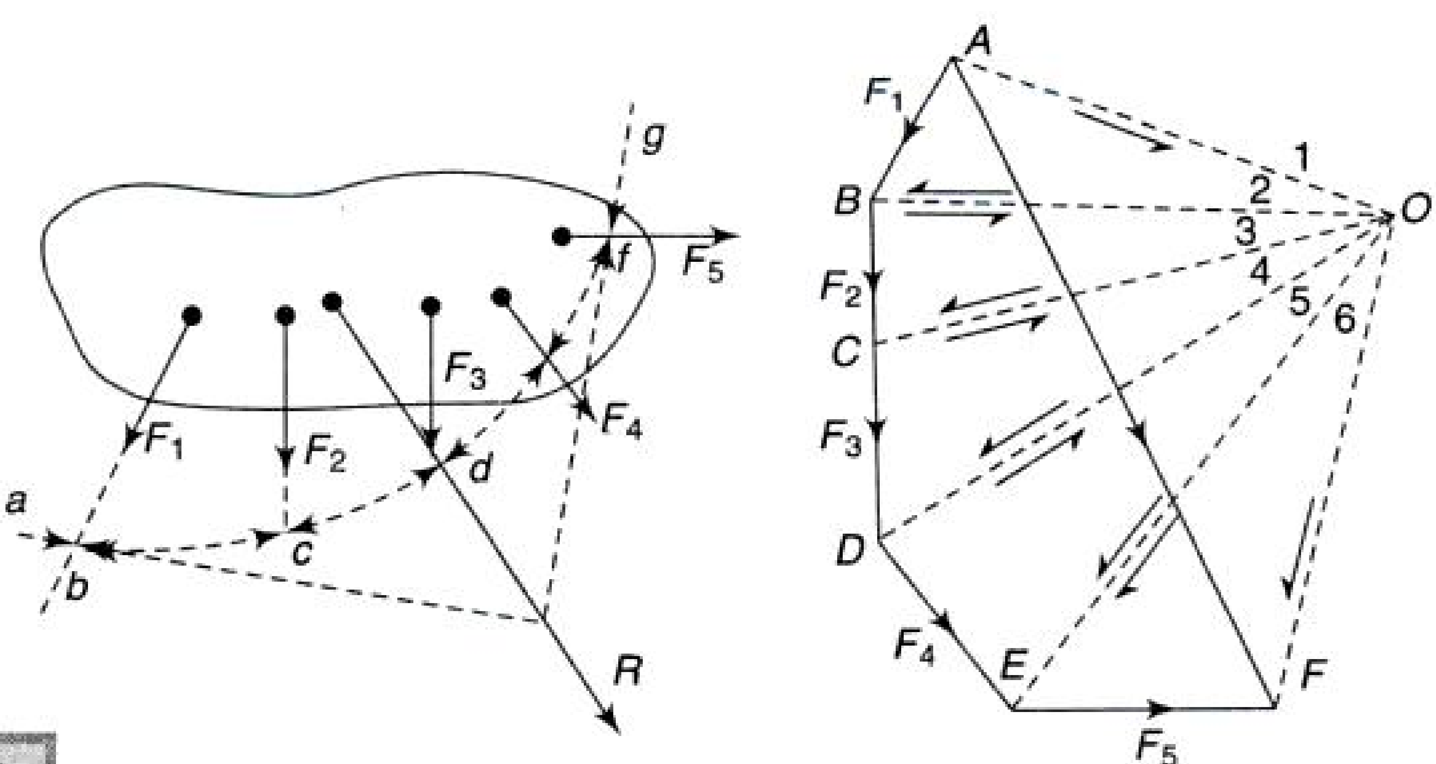

Fig. 4.24 (a) (b)

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R_c and R_a, respectively, we obtain the apexes a and c of the funicular polygon and can draw its closing side ac after which the missing ray 3 in the polygon of forces can be drawn through point O parallel to this closing side ac of the funicular polygon. The intersection of ray 3 with the vertical line EF' determines the apex F of the closed polygon of forces. The vectors $\overline{EF}$ and $\overline{FD}$ represent, respectively, the reactions R_a and R_c.

It should be noted that, while any position of the point F along the line EF' [Fig. 4.27(b)] can satisfy the condition of a closed polygon of forces, only the point F as obtained above can simultaneously satisfy the condition of a closed funicular polygon.

3. Determine graphically the reactions R_a and R_b at the supports A and B of the truss loaded and supported as shown in Fig. 4.28(a).

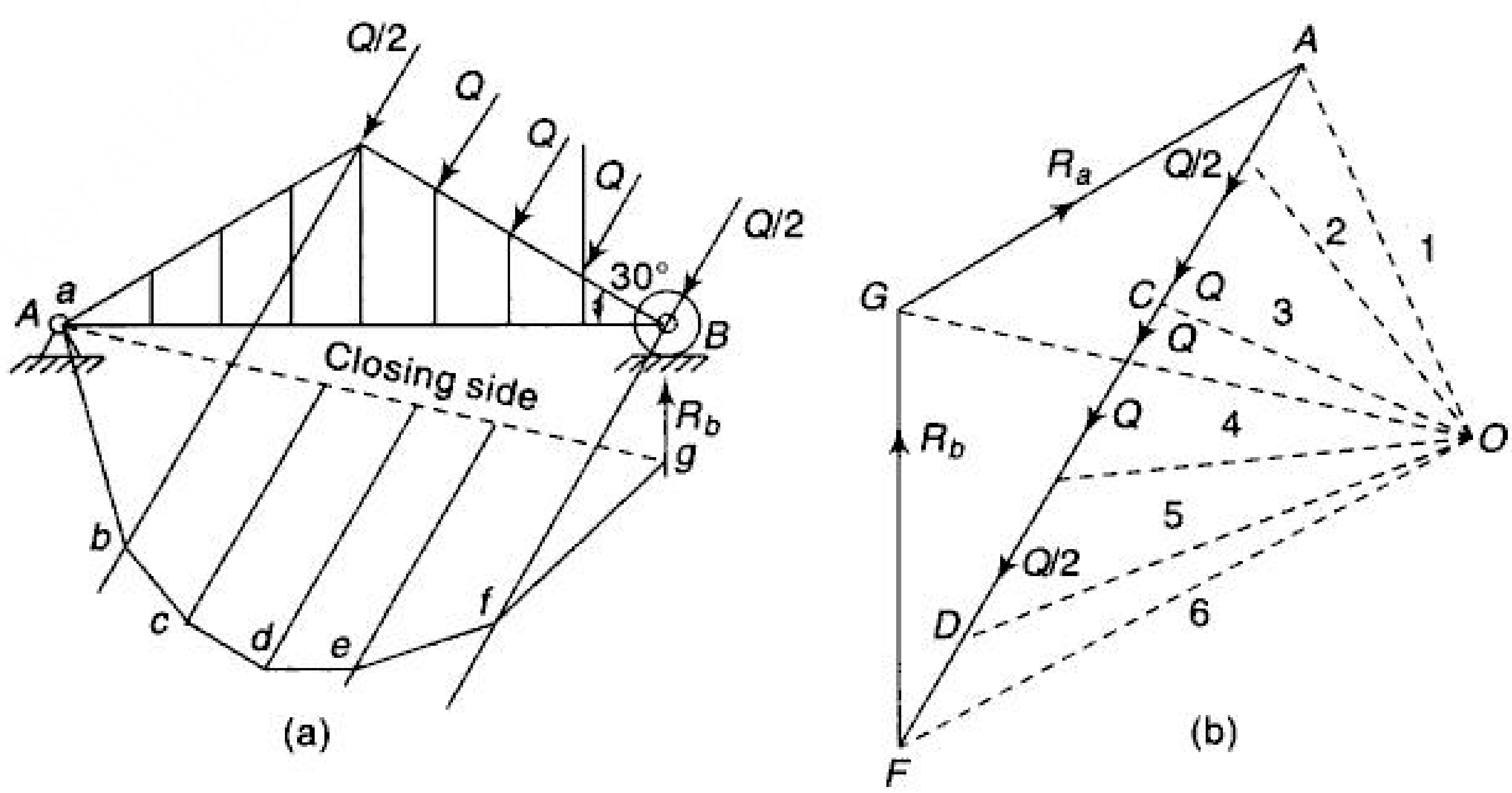

Fig. 4.28

Solution: Since the truss is in equilibrium under the action of the applied forces Q together with the reactions R_a and R_b, all these forces considered together must build a closed polygon of forces and their funicular polygon must be closed also. Proceeding on this basis, we begin with the construction of the polygon of forces [Fig. 4.28(b)]. The vectors representing the active forces Q are laid out in order, and from point F the vertical line FG' is extended in the known direction of the reaction R_b. Since, however, we know neither the magnitude of this reaction nor the direction of the reaction R_a, the closed polygon of forces cannot yet be completed. We know only that from some point G on the vertical line FG' the vector $\overline{GA}$ representing the reaction R_a must close the polygon. An arbitrary pole O is now selected and the rays 1, 2, 3, 4, 5 and 6 drawn, as shown in Fig. 4.28(b). The ray 7 cannot yet be drawn, since the apex G of the polygon of forces is not known at this stage of the construction.

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2. Find the resultant of the coplanar forces acting on the gravity dam section shown in Fig. B by constructing a funicular polygon. Forces are shown in kN.

(1 kN = 1000 N)

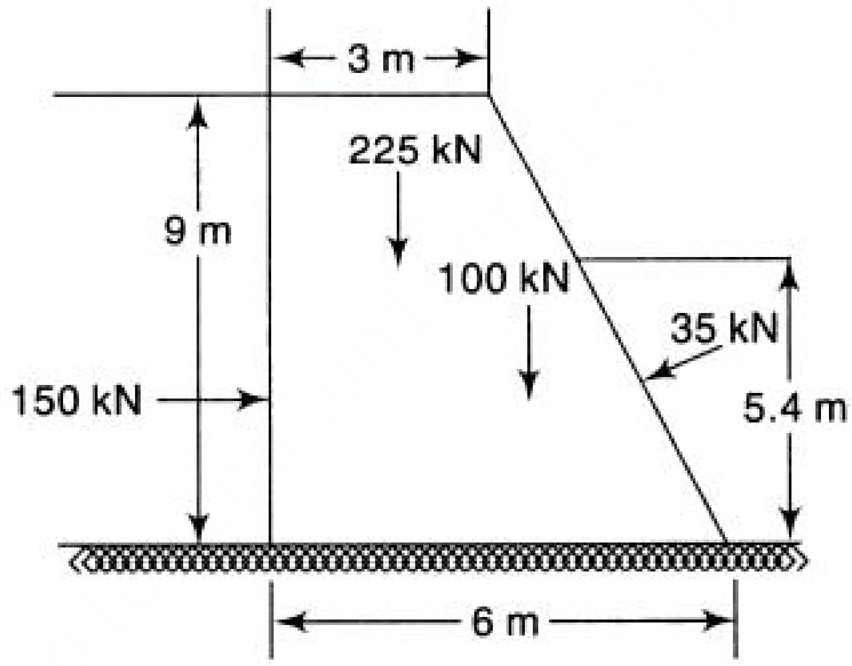

Fig. B

3. Determine, graphically, the reactions at the supports of a girder AB due to the locomotive loading shown in Fig. C. (*Ans.* R_a = 1042 kN, R_b = 1108 kN)

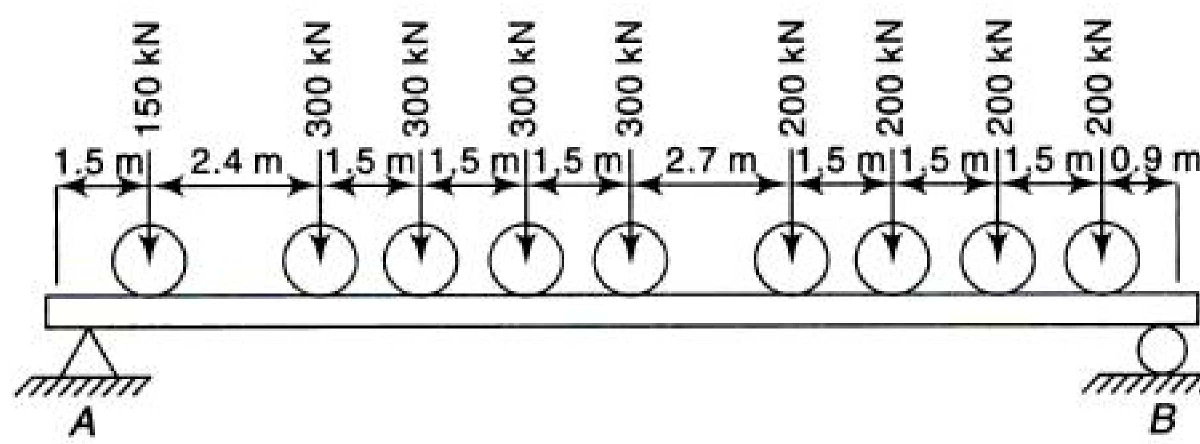

Fig. C

4. Treating the flexible cable overrunning the pulleys in Fig. D as a funicular polygon, find graphically the forces exerted on the members of the frame by the axles of the pulleys B and D. Each stationary pulley is 300 mm in diameter.

(*Ans.* P_b = 23.4 kN ; P_d = 10.3 kN)

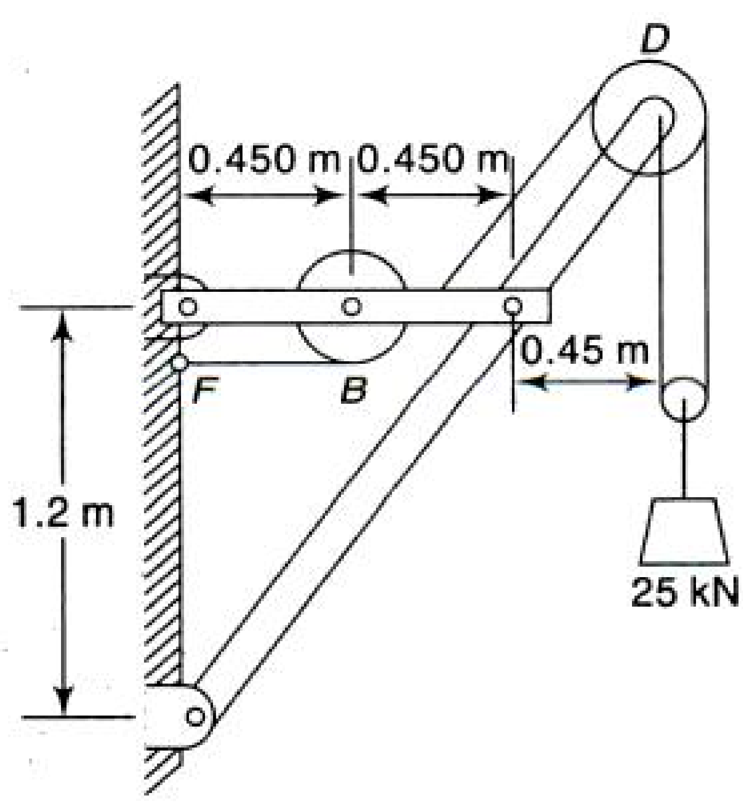

Fig. D

5. Find graphically the reactions at A and B for the beam loaded as shown in Fig. E.

(*Ans.* R_a = 430 N ; R_b = 385 N)

6. Determine graphically the reactions R_a and R_b at the supports A and B of the horizontal beam AB due to the action of the vertical loads applied as shown in Fig. F.

(*Ans.* R_a = 2405 N ; R_b = 1745 N)

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4.7 MAXWELL DIAGRAMS

We shall discuss here a graphical method of analysis of simple trusses based on the method of joints previously explained in Section 4.3. Referring to Fig. 4.30(a), we have a simple truss ABC supported at A and B and carrying a vertical load P at C. Thus the vertical reactions at A and B are each $P/2$ and we have the entire truss in equilibrium under the action of three parallel forces as shown. Denoting the axial forces in the bars by S_1, S_2, S_3 and considering the equilibrium of each of the joints A, B, C, in succession, we obtain the closed triangles of forces shown in Fig. 4.30(b) from which the magnitudes of the axial forces S_1, S_2, S_3 can be scaled.

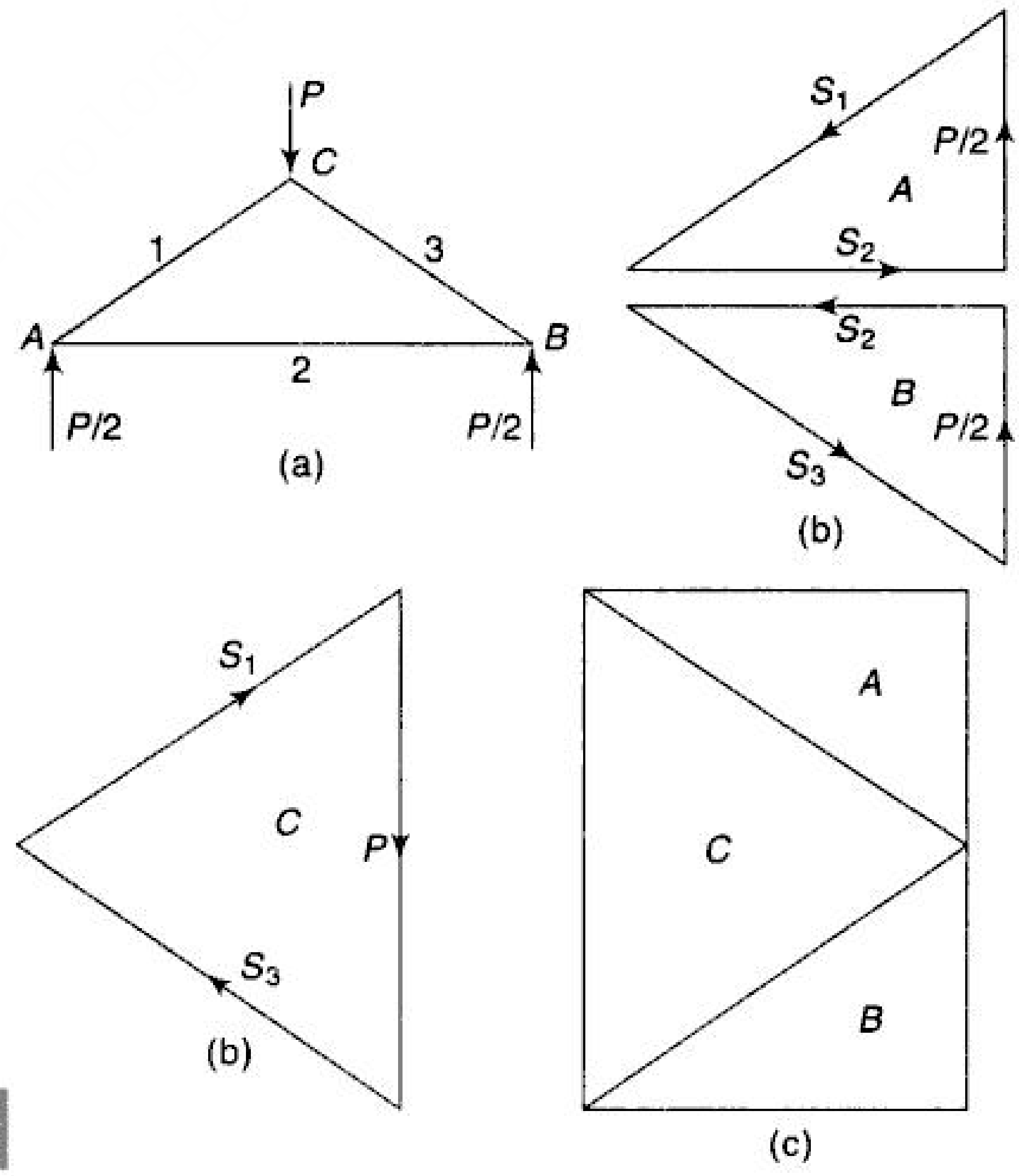

Fig. 4.30

We note now that each of these axial force vectors appears in two different polygons, once for each of the joints at the ends of that member. To avoid this duplication of vectors, the separate polygons of forces, under certain conditions, can be superimposed to form one composite diagram called a Maxwell diagram[1] for the truss. For example, the polygons of forces in Fig. 4.30(b), when superimposed, make the composite diagram shown in Fig. 4.30(c). Such superposition is desirable, since it reduces the amount of necessary construction and makes a more compact record of the analysis.

In constructing separate polygons of forces for the various hinges of a truss, it makes no difference in what order we add the vectors. Each hinge is in equilibrium, and the forces acting on it must build a closed polygon in any order. However, if we wish to superimposed these polygons as in Fig. 4.30(c), they must be constructed in a definite manner. It may be noted now that, for each polygon in

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explained, and we obtain finally the composite diagram, as shown in Fig. 4.32(b). This diagram is drawn to the scale km = 40 kN and from it the axial force in any bar of the truss may be found. The student will find it a worthwhile exercise to follow through the diagram and ascertain for each bar whether it is in tension or compression.

2. A crane supports a load P which hangs on a flexible cable overrunning small pulleys attached to the top chord joints, as shown in Fig. 4.33(a). Find graphically the axial forces produced in the various bars.

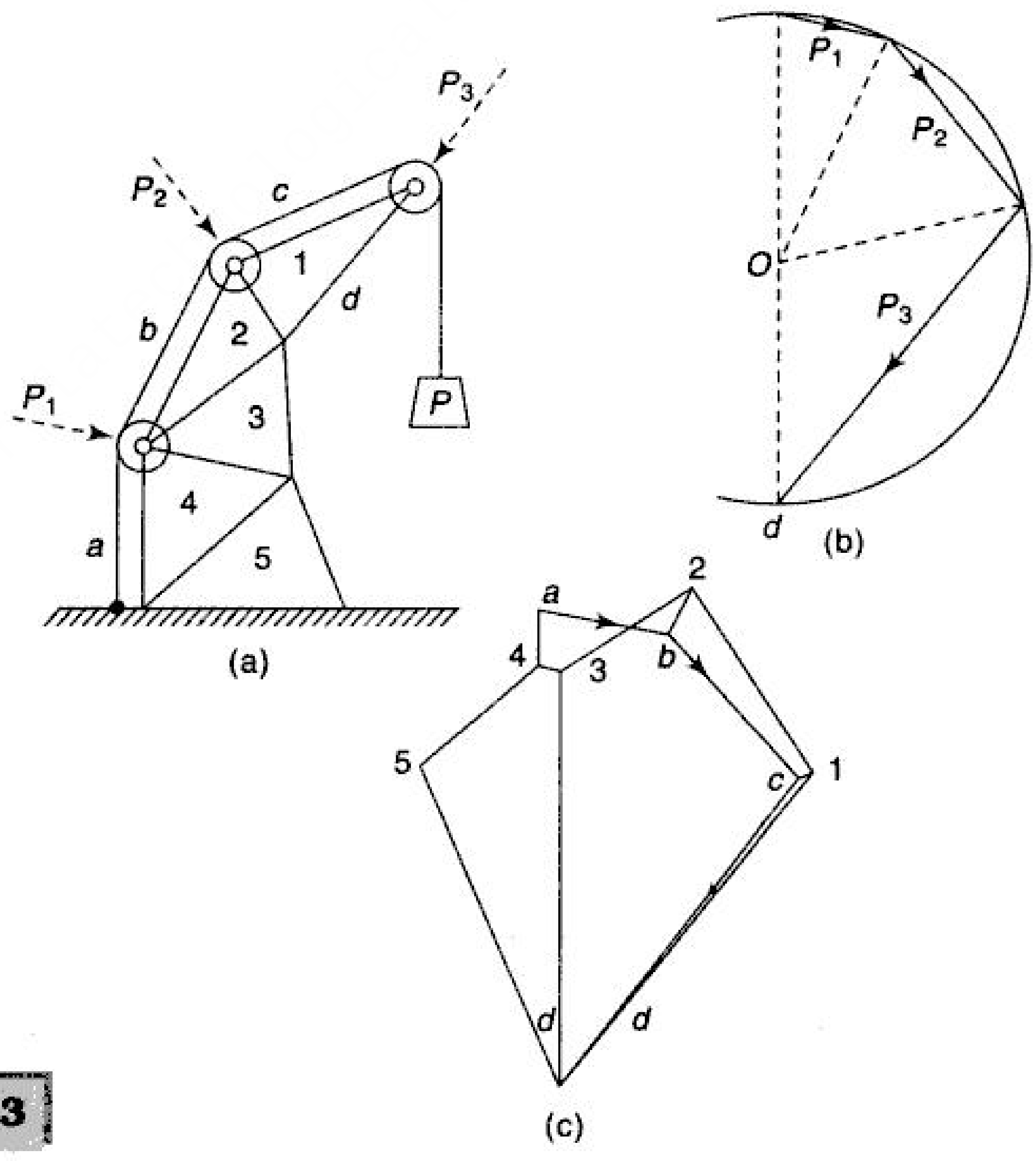

Fig. 4.33

Solution: We assume that the diagram of the truss in Fig. 4.33(a) has been drawn to scale. Then to determine the internal forces in the bars, we need to know first the magnitudes and directions of the external forces P_1, P_2, P_3 exerted on the joints by the axles of the pulleys. These, we find by observing that the cable, under uniform tension P, represents a funicular polygon for the desired forces, as already discussed in Example 1 of Section 4.6. Accordingly, in Fig. 4.33(b), we construct a circle the radius of which represents, to some convenient scale, the magnitude of the load P. Then drawing rays Oa, Ob, Oc and Od, parallel to the several portions of the cable, we find the forces P_1, P_2, P_3 exerted on the joints of the truss, as shown. We place these forces on the truss (thereafter ignoring the cable) and letter (or

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which, by using Eqs (e) and (f), becomes,

$$dS = \mu S\, d\theta$$

or

$$\frac{dS}{S} = \mu\, d\theta \tag{g}$$

This expresses the ratio between the increment of tension over the length of the element to the total tension in the belt at the point defined by the angle θ. Integrating Eq. (g) over the entire line of contact AB, of length $r\beta$, we obtain

$$\ln\frac{S_1}{S_2} = \mu\beta \text{ or } \quad \frac{S_1}{S_2} = e^{\mu\beta} \tag{h}$$

We see that the ratio between the tensions S_1 and S_2 in the belt on the two sides of the pulley increases very rapidly with the magnitude of the central angle β of the line of contact AB. This explains how a man hold a great load on the end of a rope simply by taking a turn of two of rope around a post. It will also be noted that the ratio S_1/S_2 is independent of the radius r of the pulley.

Center of Pressure

Another problem involving distributed force is illustrated in Fig. 4.38. The opening to a penstock through the base of a gravity dam is closed by a circular plate AB and it is desired to define the resultant water pressure P on the plate. This is really a problem involving force distributed over an area, but we can easily bring it to an equivalent distribution in the vertical plane of the diameter AB of the plate.

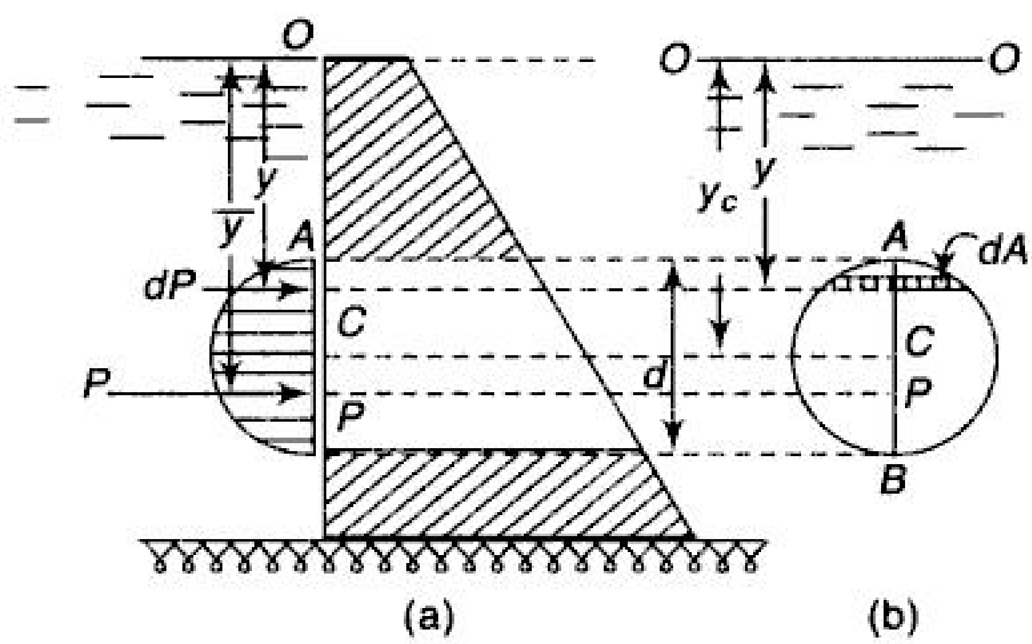

Fig. 4.38

Looking directly at the plane of the plate [Fig. 4.38(b)], we consider one element of area dA in the form of a horizontal strip at depth y as shown. Then since the intensity of water pressure $p = wy$ is constant across the length of this strip, the corresponding element of force $dP = wy\, dA$ acts at the mid-point of the strip, i.e. on the vertical diameter AB. This conclusion holds for each such horizontal strip, and we obtain a series of elemental forces distributed along AB and all lying in the same vertical plane, Fig. 4.38(a). The point P through which the resultant pressure acts on the plate is called the *center of pressure* and we need only to find the depth $\bar{y}$ to completely define its position on the vertical diameter AB.

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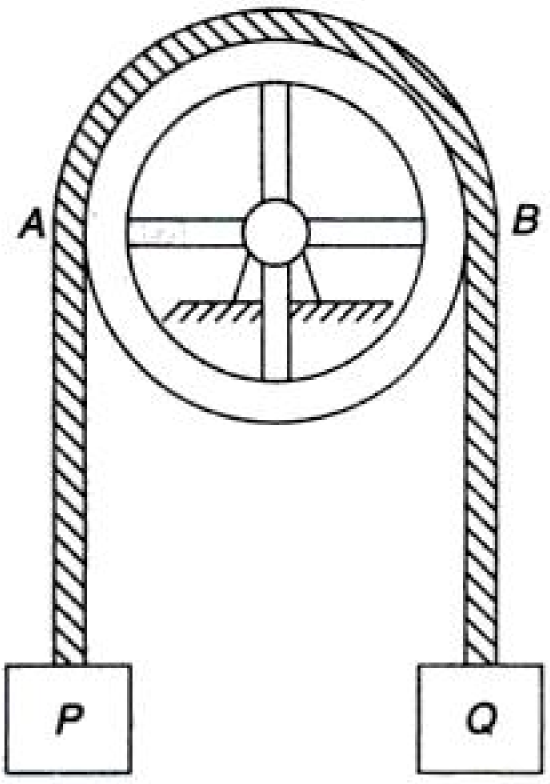

Fig. C

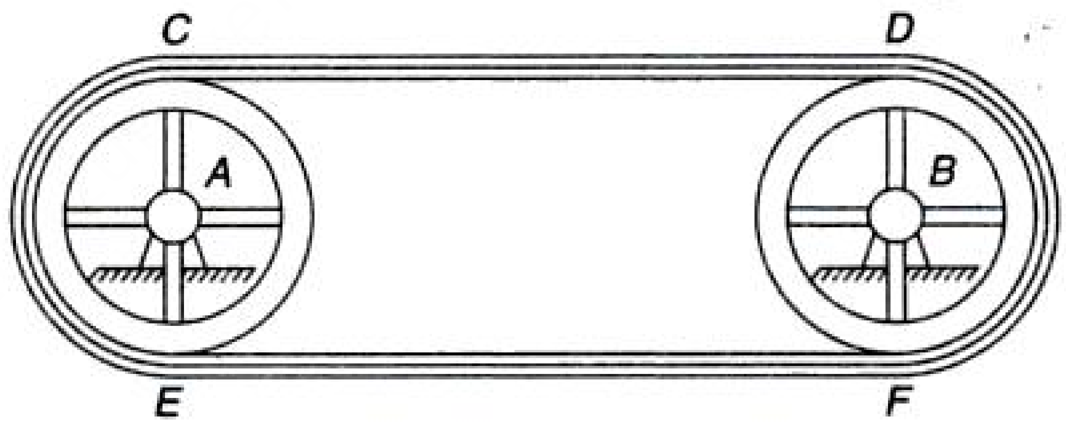

Fig. D

4.9 FLEXIBLE SUSPENSION CABLES

In engineering structures, we sometimes encounter flexible cables or chains suspended between two supports at their ends and subjected to the action of vertical load continuously distributed along their lengths. The distribution of load may, in general, be uniform or otherwise. Two of the most commonly encoutered cases of loading are illustrated in Fig. 4.39. Figure 4.39(a) represents the case of a flexible chain freely suspended in the gravity field and subjected to the action of its own distributed weight only. Such loading is, of course, uniformly distributed with respect to the curve of the chain itself. Figure 4.39(b) represents the case of a thin wire cable or cord subjected to the action of a uniformly distributed load attached to it by vertical *hangers*. In the event that this loading is large compared with the weight of the cable itself, we may assume in this case that the load is uniformly distributed with respect to the horizontal span.

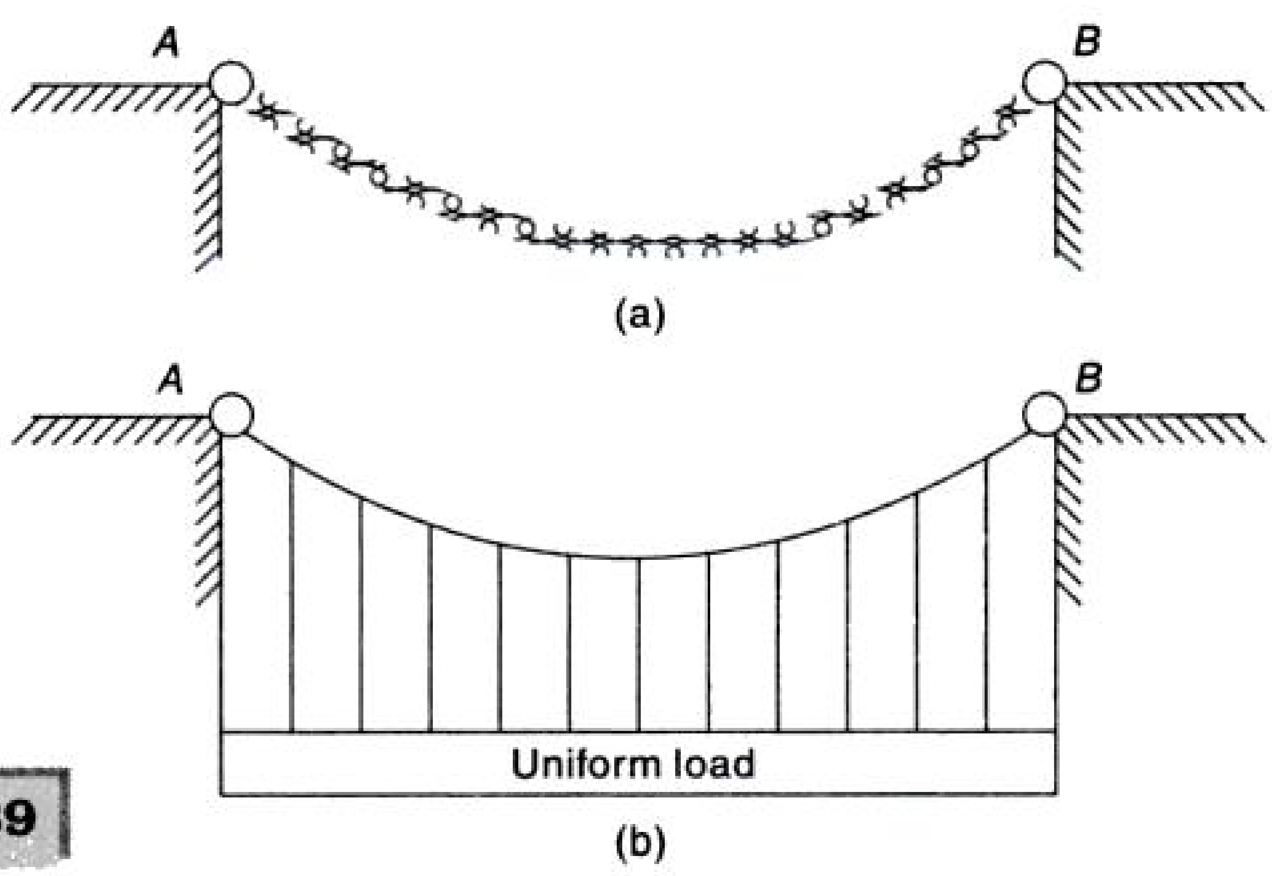

Fig. 4.39

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tension H in the cable may be found at once, after which any of the quantities previously defined in terms of H can be calculated without difficulty.

Catenary Cable

Assuming that the cable in Fig. 4.42 hangs freely in the gravity field and is subjected only to its own weight uniformly distributed along the curve, Eq. (a) becomes

$$\frac{dy}{dx} = \frac{qs}{H} \tag{k}$$

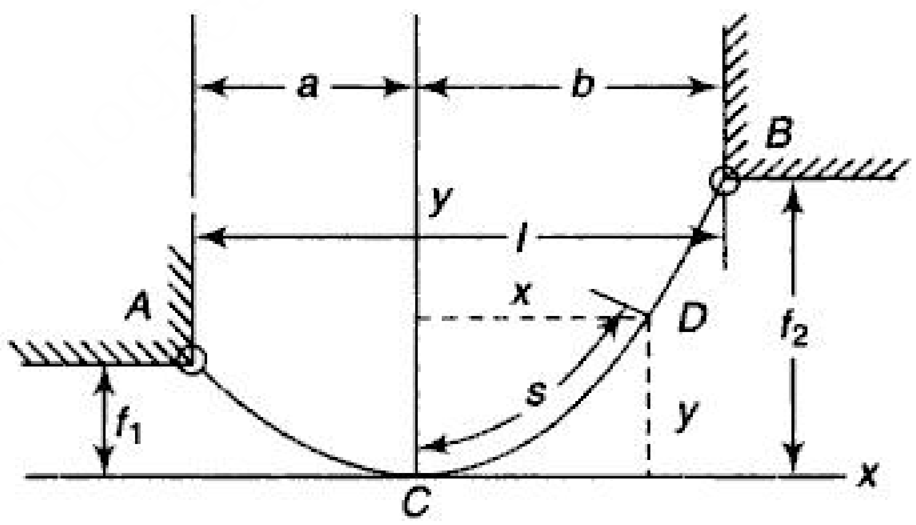

Fig. 4.42

where q is the weight per unit length of the cable and s is the length of the arc CD. Before this equation can be integrated, it will be necessary to express the length S as a function of the coordinates x and y. To do this, we use the relationship,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which, with the above value of dy/dx, becomes

$$ds = \sqrt{1 + \left(\frac{qx}{H}\right)^2}\, dx$$

Integration of this last equation gives

$$\frac{H}{q} \text{arc sinh } \frac{qs}{H} = x + C_1$$

where C_1 is a constant.

Since, for the coordinate axes as shown in Fig. 4.42, we have $s = 0$ when $x = 0$, it is evident that $C_1 = 0$ and the above equation becomes

$$s = \frac{H}{q} \sinh \frac{qx}{H} \tag{l}$$

Substituting the value of s from Eq. (1) into Eq. (k), we obtain

$$dy = \sinh \frac{qx}{H}\, dx \tag{m}$$

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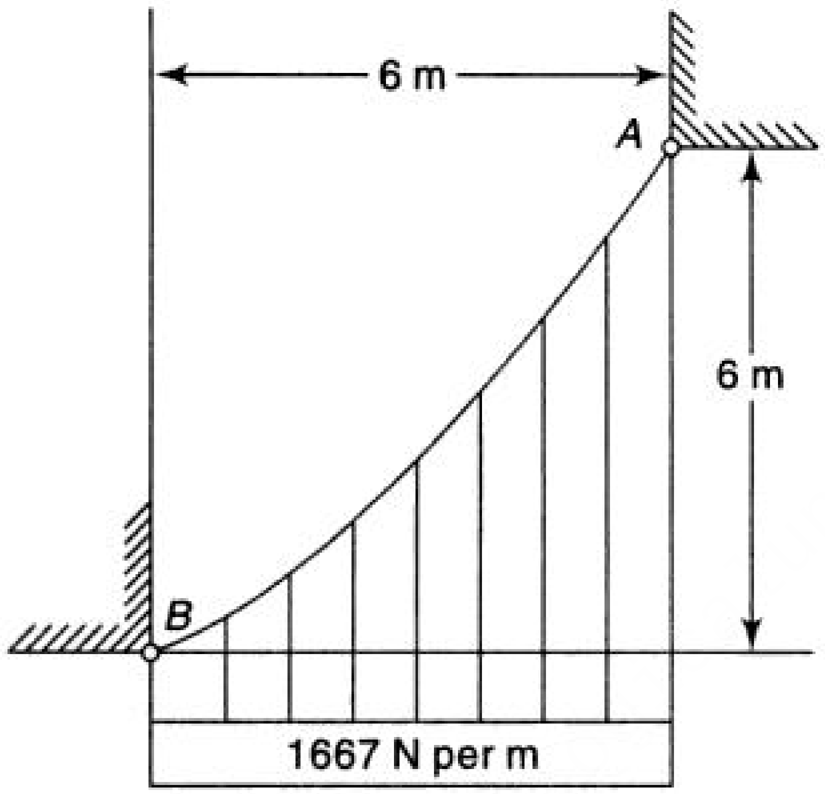


Fig. D

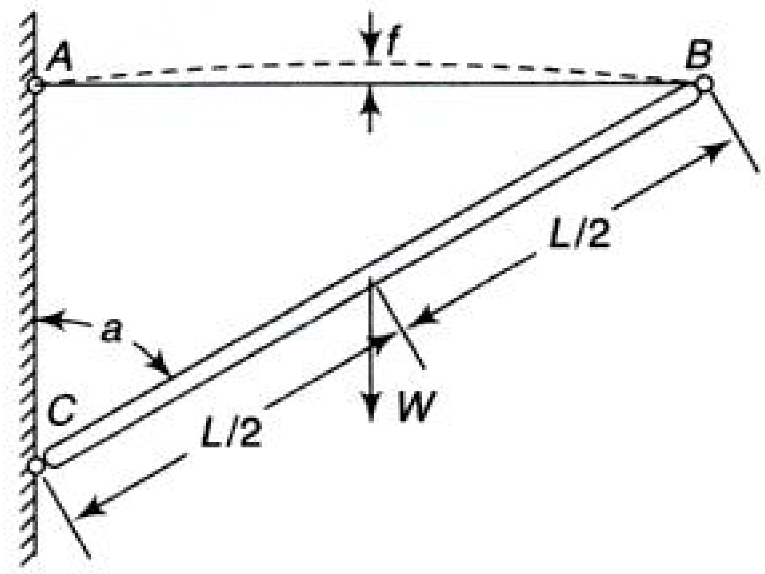
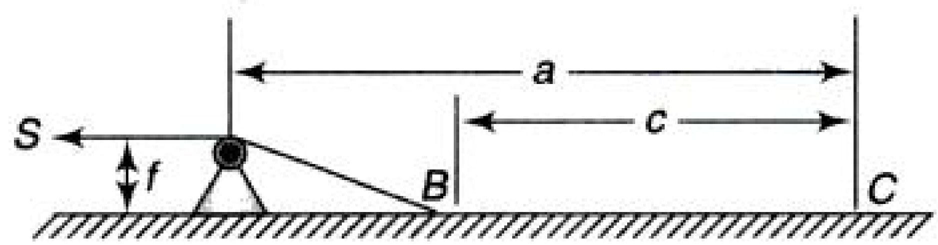


Fig. E

8. A flexible cable 30 m long and weighing 38 N per metre of length is freely suspended at its ends from two supports 15 m apart and having the same elevation. Find the sag f at the middle of the span. (*Ans.* $f = 11.7$ mm)
9. A flexible cable of uniform weight per unit length rests partly on a horizontal plane and passes over a small pulley at A, as shown in Fig. F. By gradually increasing the force S applied to the end of the cable, the length of contact BC with the plane diminishes to a certain limiting value c at which sliding of the cable along the plane impends. Find this limiting value c if $a = 60$ m, $f = 6$ m and the coefficient of friction between the cable and the plane is $\mu = 0.5$.

(*Ans.* $c = 44.1$ m)

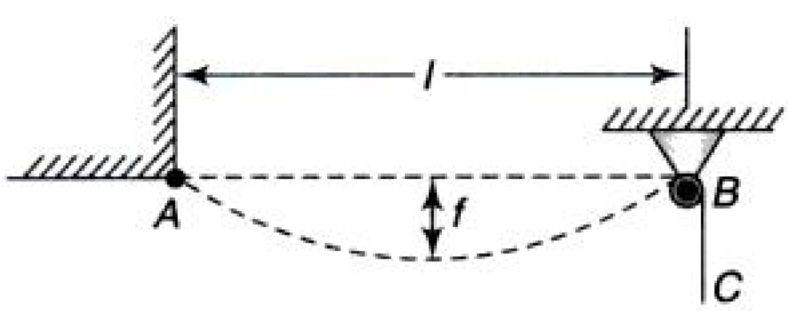
Fig. F



10. Determine the minimum length L of a flexible cable AC of uniform weight per unit length which can hang in equilibrium as shown in Fig. G. Neglect friction and the dimensions of the pulley B. What is the sag-span ratio?

(*Ans.* $L_{m/n} = 1.14\, l + 0.80\, l = 1.94l; f / l = 0.238$)

Fig. G



lines of action of all but one of any number of concurrent forces in space lie in one plane, then equilibrium can exist only if this one force is zero. (4) If the known lines of action of all but two of any number of concurrent forces in space lie in one plane and one of these two forces is known in magnitude, then the magnitude of the other can always be found without difficulty. The proof of each of these statements is left to the student.

Examples Examples Examples Examples Examples

1. Three concurrent forces F_1, F_2 and F_3 have the lines of action OA, OB and OC as shown in Fig. 5.4, and the magnitudes shown in the following table. Find the magnitude and direction of their resultant R.

   Solution: We begin by computing the lengths of the lines OA, OB and OC, from the observed coordinates of the points A, B and C, as follows:

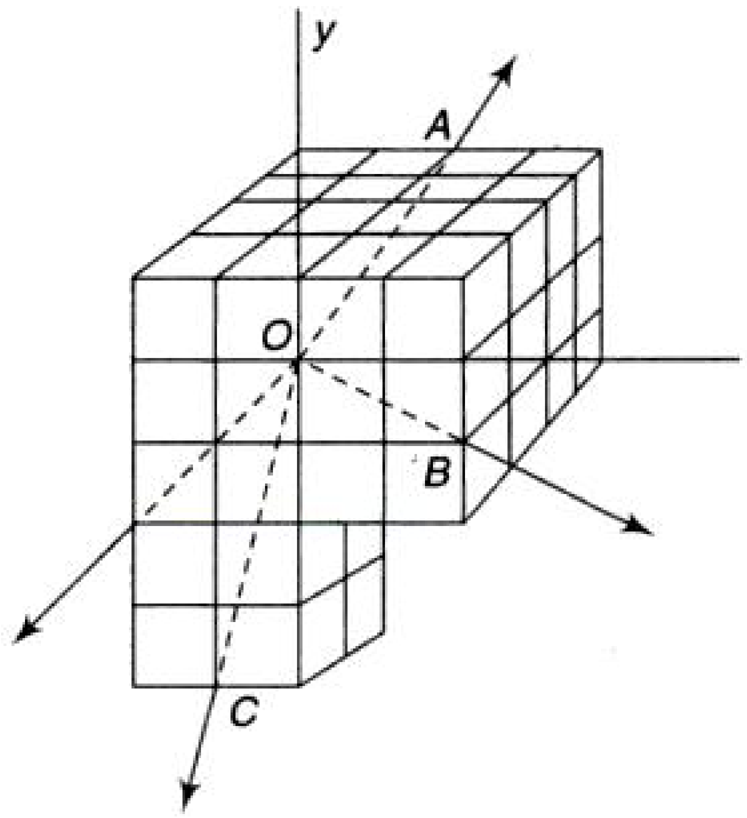


Fig. 5.4

$$OA = \sqrt{(2)^2 + (3)^2 + 0} = \sqrt{13}$$

$$OA = \sqrt{(4)^2 + (1)^2 + (4)^2} = \sqrt{33}$$

$$OA = \sqrt{(1)^2 + (-2)^2 + (4)^2} = \sqrt{21}$$

Using these values, the direction cosines of the lines of action of the forces are easily computed and tabulated as shown in the table. Then using Eq. (b), we find the projections of the given forces as shown in the last three columns of the table.

Finally, making the summations of projections and using Eqs (c) and (d), we obtain

F_i	lb	$\cos\alpha_i$	$\cos\beta_i$	$\cos\gamma_i$	X_i	Y_i	Z_i
F_1	40	0.555	0.822	0	22.2	33.3	0
F_2	10	0.696	0.174	0.696	7.0	1.7	7.0
F_3	30	0.218	−0.436	0.873	6.5	−13.1	26.2
Σ					35.7	21.9	33.2

$$R = \sqrt{(158.9)^2 + (97.60)^2 + (147.60)^2} = 237.83 \text{ N}$$

$$\alpha = \cos^{-1}\frac{158.9}{237.8} = 48°04'$$

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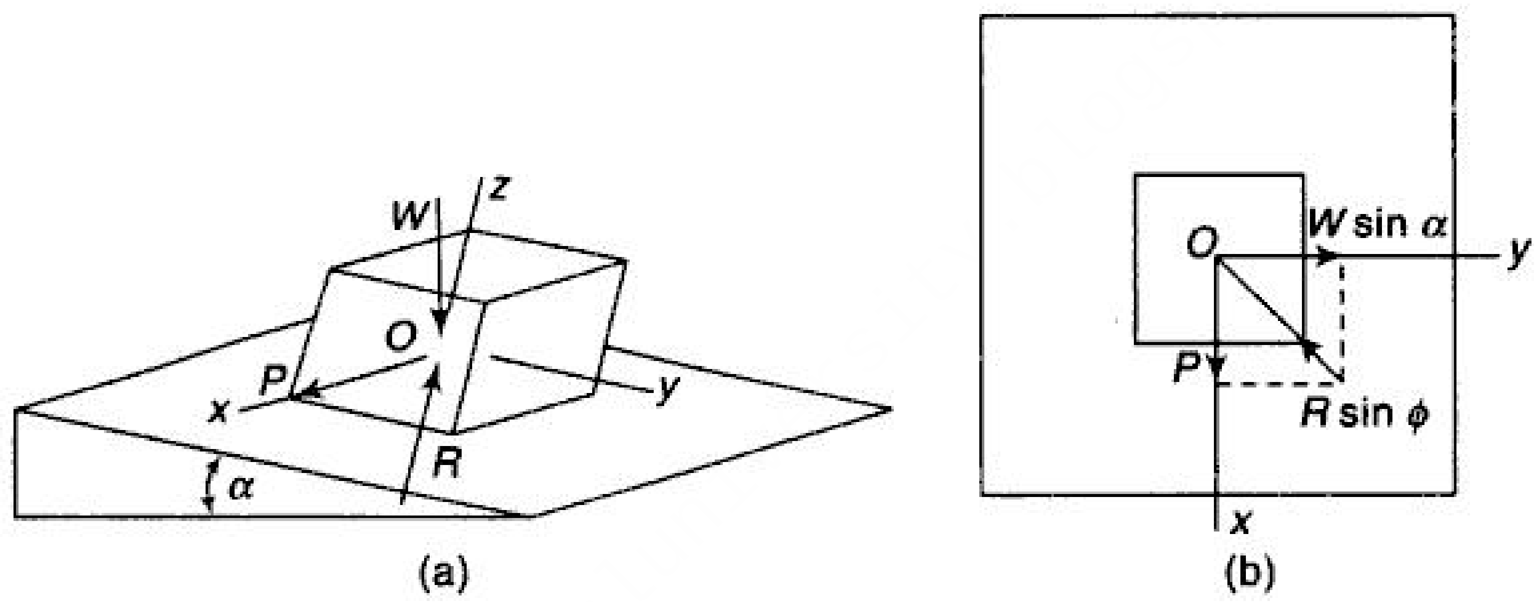

Fig. 5.8

Solution: At the instant of impending, slipping the block is in equilibrium under the action of the gravity force W, the applied force P, and a reaction R exerted by the inclined plane. Since these three forces are in equilibrium, we conclude that they must intersect in one point and also lie in one plane. Further, when sliding of the block impends, we know that the reaction R is inclined to the normal to be inclined plane by the angle of friction φ.

Choosing O as the origin of coordinates with the z-axis normal to the inclined plane and the x-axis horizontal and equating to zero the algebraic sum of the projections of all forces on the z-axis, we obtain

$$R \cos \varphi = W \cos \alpha \tag{f}$$

Projecting the entire system of forces onto the inclined plane, we obtain the system of coplanar forces in equilibrium as shown in Fig. 5.8(b), from which we conclude that

$$R^2 \sin^2 \varphi = P^2 + W^2 \sin^2 \alpha \tag{g}$$

Eliminating R between Eqs. (f) and (g) gives

$$P = W\sqrt{\mu^2 \cos^2 a - \sin^2 \alpha} \tag{h}$$

where $\mu = \tan \varphi =$ coefficient of friction. It will be noted that for the limiting case where $\alpha = \varphi$, Eq. (h) gives $P = 0$. Also for the limiting case where $\alpha = 0$ and the inclined plane becomes a horizontal plane, Eq. (h) gives $P = \mu W$.

We see from the first limiting case that when all available friction is already being used to resist sliding of the block down the plane, then there is no resistance to lateral slipping. This explains, for example, why a rear-wheel-drive automobile can skid so freely from side to side when climbing a grade on wet or icy pavement. For the same reason, a car loses lateral stability if the brakes are too suddenly applied so as to cause the tires to slip.

6. The small pulley in Fig. 5.9(a) drives the large one in a counter-clockwise direction by a V belt overrunning their rims, as shown. The angle of the V in both pulleys is 2α, the total angle of contact on the small pulley is β, and the coefficient of friction between the belt and rim is μ. Find the ratio S_1/S_2 between the tensions in the two branches of the belt when slipping impends

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4. A mast AB supported by spherical socket at A and horizontal guy wires BC and BD carries a vertical load P at B as shown in Fig. B. Find the axial force induced in each of the three members of this system.(*Ans.* $S_1 = +0.8P$; $S_2 = +0.6P$; $S_3 = -1.4P$)
5. Repeat the solution of Prob. 4 if point D is 0.3 m vertically below the position shown in Fig. B and all other dimensions remain unchanged.

(*Ans.* $S_1 = +1.03P$; $S_2 = +0.75P$; $S_3 = -1.77P$)

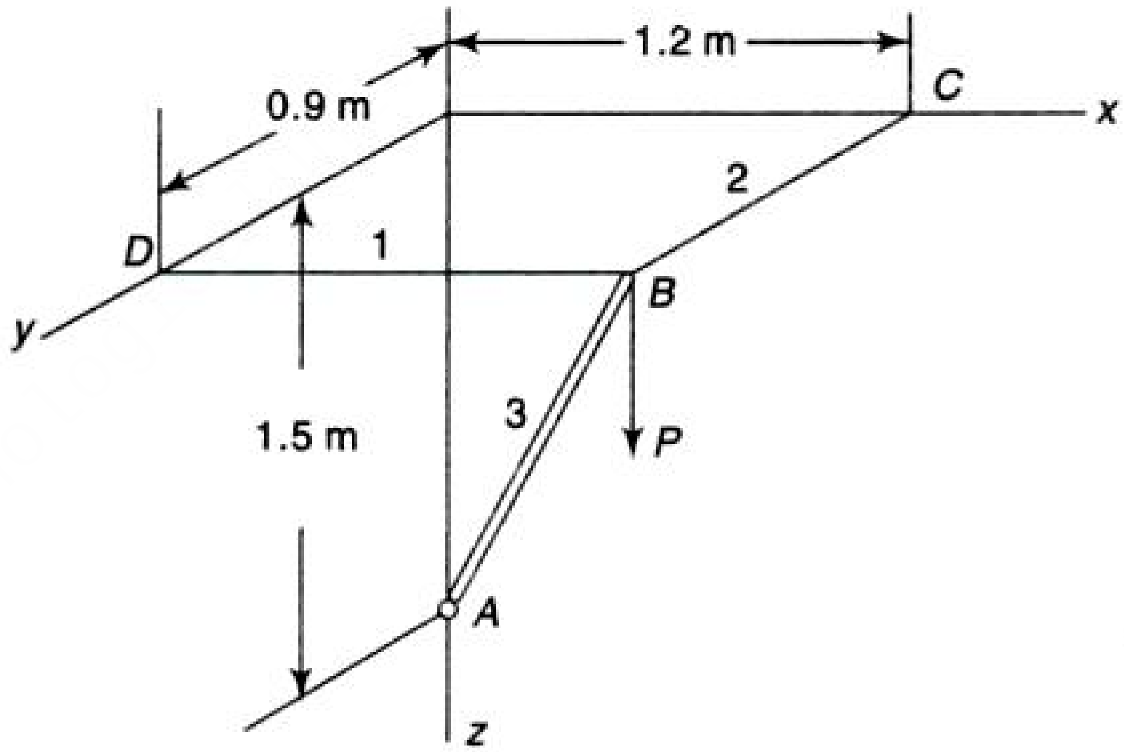

Fig. B

6. In the case of the tripod shown in Fig. C there is no friction between the ends of the legs and the floor on which they rest. To prevent slipping of the legs, their ends are connected by strings along the lines AB, BC and AC. Determine then the tensile force S in each of these strings if each leg makes 30° with the vertical and P is a vertical load. $\left(Ans. S + \frac{1}{2} P\right)$

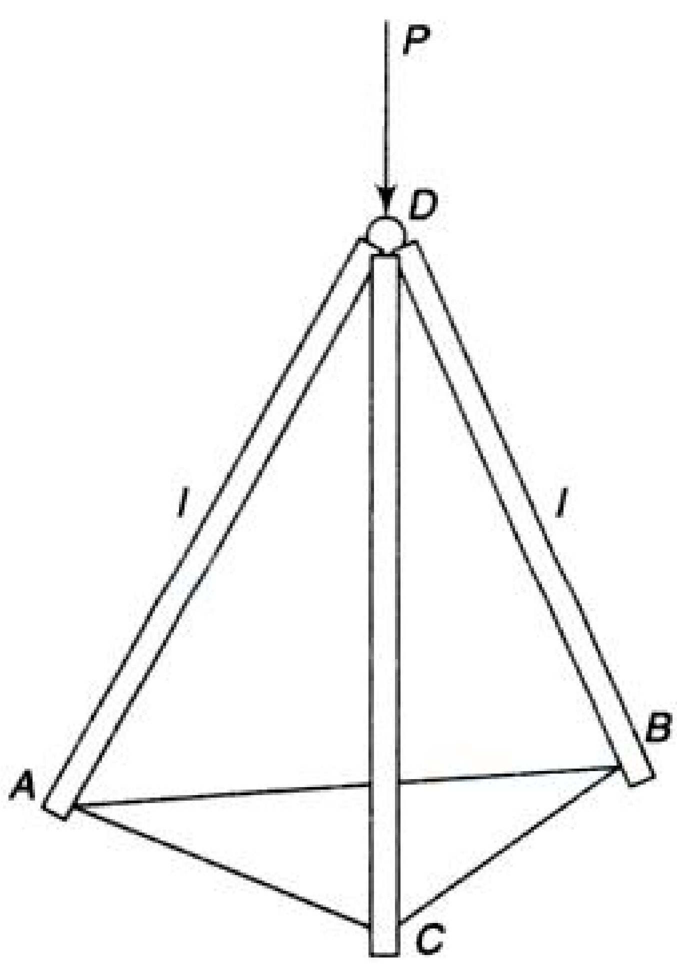

Fig. C

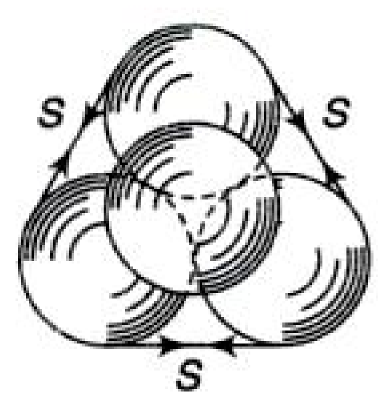

Fig. D

7. Four balls of equal radii r form a pyramid supported by a smooth horizontal plane surface (Fig. D). The three lower balls are held together by an encircling string as shown. Determine the tensile force S in this string if the weight of each ball is Q and the surfaces of the balls are perfectly smooth. Neglect any initial tension that may be in the string before the top ball is placed upon the other three.

(*Ans.* $S = 0.136\,Q$)

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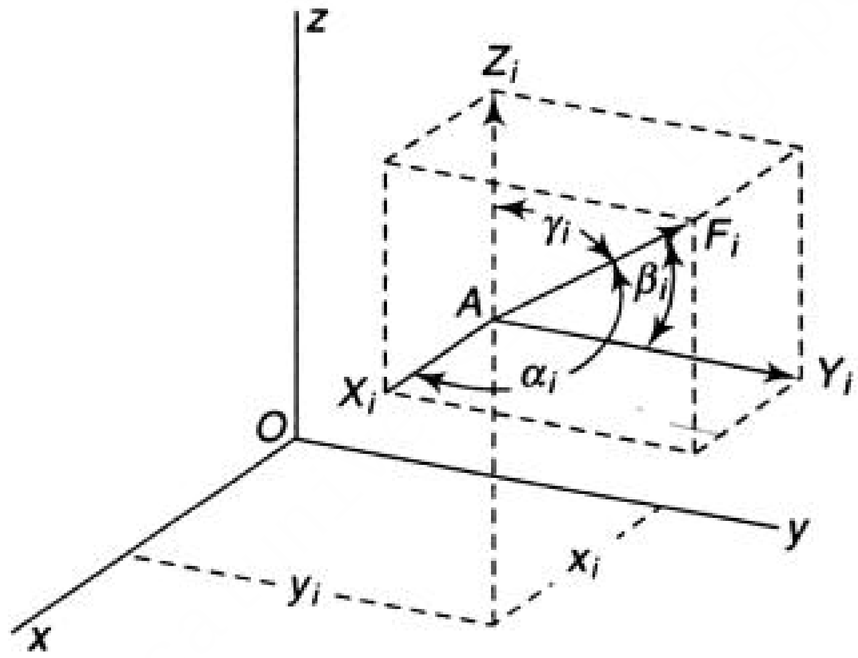

Fig. 5.12

Although derived on the assumption that the force F_i has positive projections, Eq. (21) are always valid if proper attention is paid to the signs of the projections as defined by Eq. (b).

Equations of Equilibrium

We have seen above that the algebraic sum of moments of a system of concurrent forces in space with respect to a given axis is equal to the moment of their resultant with respect to the same axis. Further, this sum of moments is zero if the resultant force intersects the axis, is parallel to it, or vanishes itself. These three possibilities are sometimes useful in deciding the question of equilibrium of a system of concurrent forces in space.

Consider, for example, the force system concurrent at a point A in space, as shown in Fig. 5.13, and let 1-1, 2-2, 3-3, be three non-parallel axes defining a plane abc that does not contain point A.

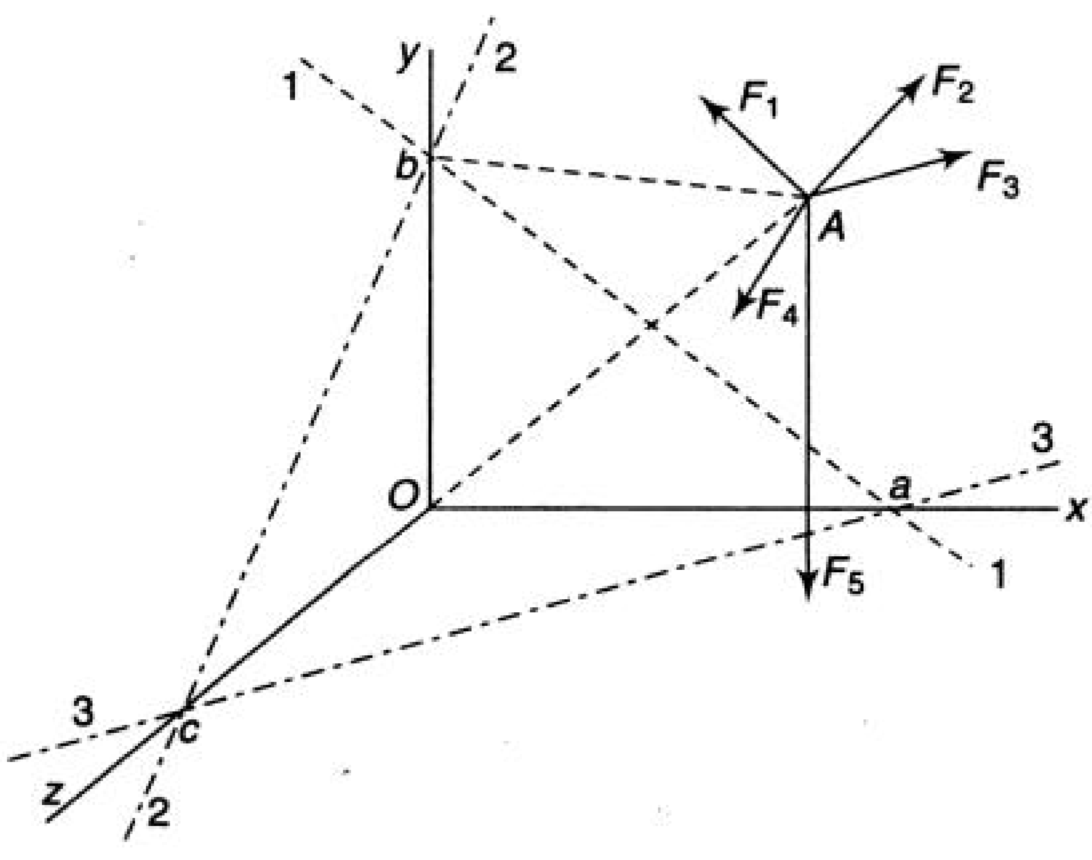

Fig. 5.13

Now suppose that the algebraic sum of moments of the forces about the axis 1-1 is zero. Then if there is a resultant, it must either be parallel to, or intersect, this axis. If the sum of moments about the axis 2-2 is also zero, the resultant, if one exists, can only lie along the line Ab, since it cannot be parallel to both 1-1 and 2-2 which are non-parallel. Finally, if the sum of moments about the axis 3-3 is likewise zero, the resultant has to vanish completely, because it was already

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4. The sum of moments is zero if the resultant force intersects the axis, is parallel to it, or vanishes itself. These three possibilities are sometimes useful in deciding the question of equilibrium of a system of concurrent forces in space. Explain.
5. Is it necessary to write the three conditions of equilibrium for concurrent force system in space in terms of forces or moment equations? Explain.

Objective Questions

1. Given that the moment of all the concurrent force system in space acting on the body about some axis is zero and also given that forces are concurrent, implies that
   (a) Resultant force intersects the axis
   (b) Resultant force is zero
   (c) Resultant forces line of action parallel to the axis
   (d) Any of (a), (b), and (c) can be true

[*Ans.* (d)]

PROBLEM SET 5.2

1. A pulley A of radius a is supported from the face of a vertical wall by two braces AB and AC together with a tie bar AD, as shown in Fig. A. A flexible cord EAF is fastened to the wall at E, passes over the pulley, and carries at its end F a load Q. Find the tensile force S produced in the tie bar AD if Q = 100 *lb*, a = 150 mm, b = 1.2 m, c = 2.25 m. (*Ans.* S = 236 N)

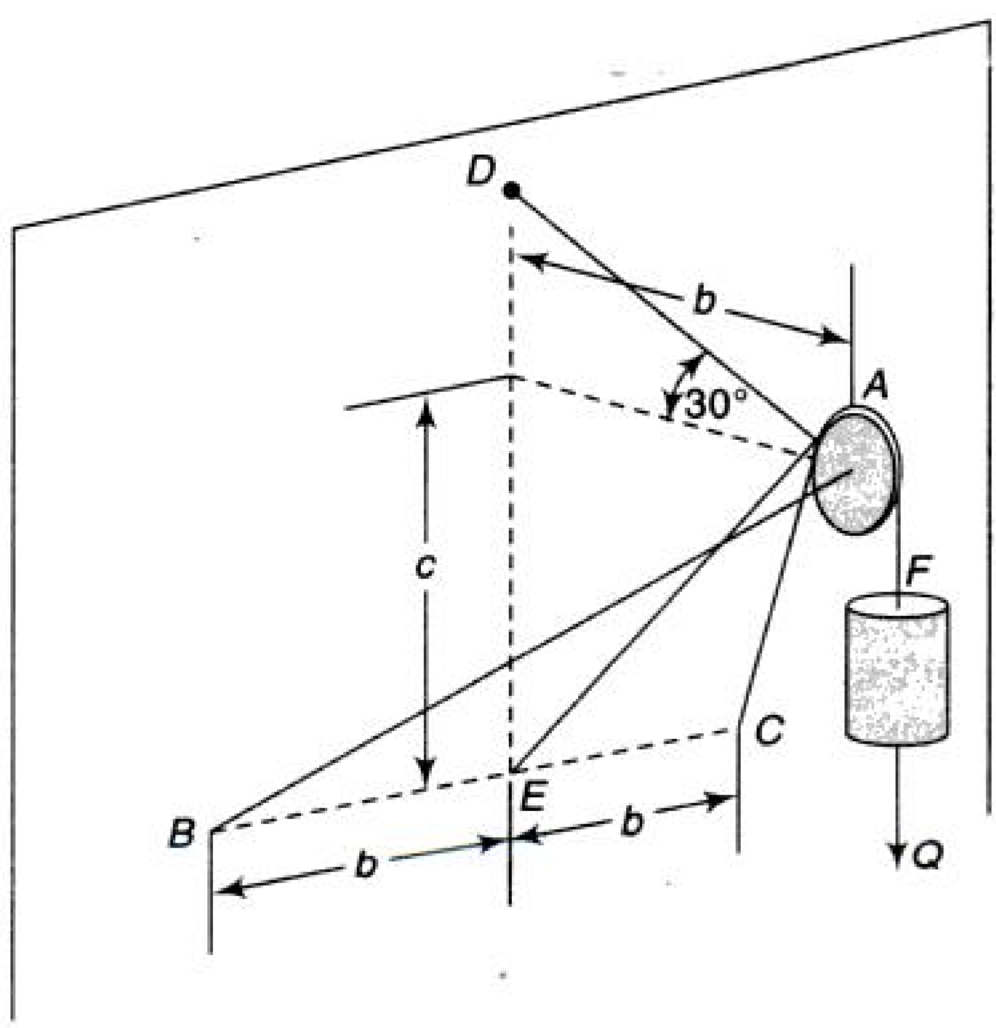

Fig. A

2. A ball of radius r and weight Q is hung from the corner of a room by a string AB of length l as shown in Fig. B. Neglecting friction between the ball and the walls, find the tensile force S in the string AB, which in the absence of friction will pass through C. The following numerical data are given: r = 150 mm, l = 250 mm, Q = 445 N. (*Ans.* S = 525 N)
3. Using the method of moments, find the axial force S produced in the vertical mast AB of the crane which supports a vertical load P = 50 kN as shown in Fig. C. (*Ans.* S = 60 kN)

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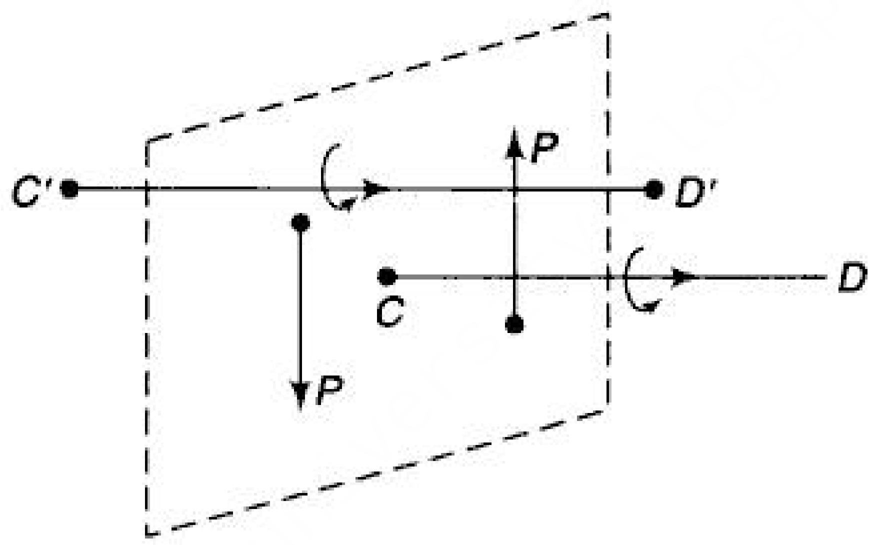

Fig. 5.17

couple can be displaced parallel to itself without changing the action of the couple, we conclude that any vector $\overline{CD}$ equal and parallel to the vector $\overline{C'D'}$ represents an equivalent couple. The vectorial representation of a couple is very useful when we have to deal with couples differently oriented in space, and in our further discussions a vector, such as $\overline{CD}$ or $\overline{C'D'}$, will be called a *moment vector* to distinguish it from a *force vector*.

Since the position of a couple in its plane is of no consequence and since the plane of a couple can be displaced parallel to itself without changing the action of the couple, it follows that any system of couples differently oriented in space can be represented by moment vectors that can be taken concurrent at any point in space. Hence it follows at once by analogy to the ease of concurrent forces in space that the resultant couple M of any system of couples $M_1, M_2, \ldots M_n$ may be found by the method of projections.

Consider, for example, the system of couples in space as represented by the moment vectors $M_1, M_2, \ldots, M_n$ in Fig. 5.18, where each couple must be visualized as acting anywhere in a plane normal to the corresponding moment vector. Taking coordinate axes x, y, z, as shown, and denoting by α_i, β_i, γ_i the direction angles of the vector M_i, we see that the projections of this vector are

$$\begin{aligned}(M_x)_i &= M_i \cos \alpha_i \\ (M_y)_i &= M_i \cos \beta_i \\ (M_z)_i &= M_i \cos \gamma_i\end{aligned} \qquad \text{(a)}$$

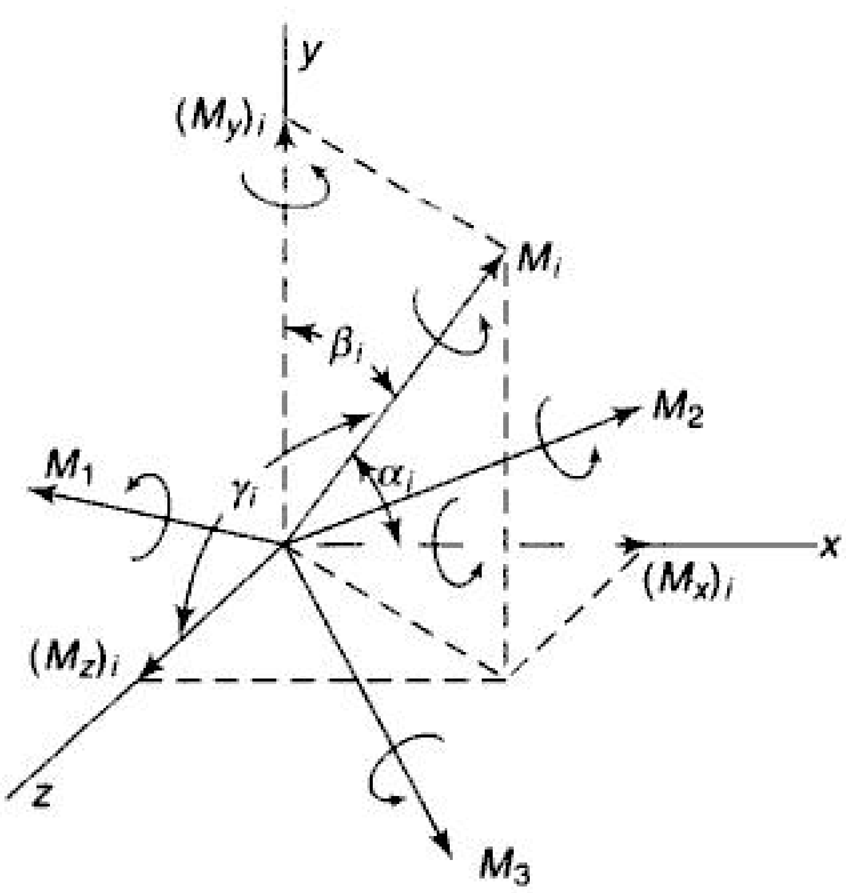

Fig. 5.18

Select the correct statement.
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false and R is true.

[*Ans.* (a)]

2. A couple is completely defined by the following:
   I. The magnitude of its moment.
   II. The aspect of the plane in which it acts.
   III. The direction of rotation in the plane.
   (a) I is correct (b) I and II are correct
   (c) I and III are correct (d) I, II and III are correct

[*Ans.* (d)]

PROBLEM SET 5.3

1. Three circular disks A, B and C of radii $r_a = 375$ mm, $r_b = 250$ mm and $r_c = 125$ mm, respectively, are fastened at right angles to the ends of three rigidly connected arms which all lie in one plane as shown in Fig. A. If couples act on the disks A and B as shown in the figure, find the magnitude of the forces P of the couple that must be applied to the disk C and the angle α that the arm OC must make with the arm OB in order to have equilibrium. (*Ans.* $P = 222.5$ N, $\alpha = 143°08'$)

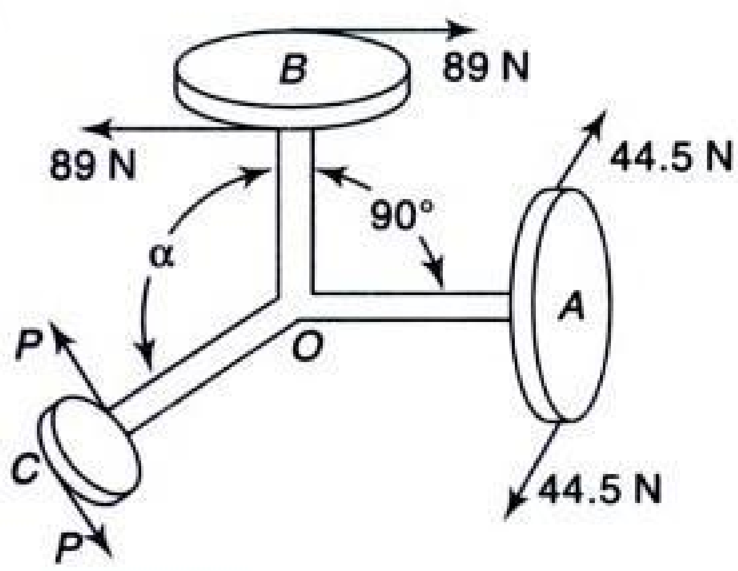

Fig. A

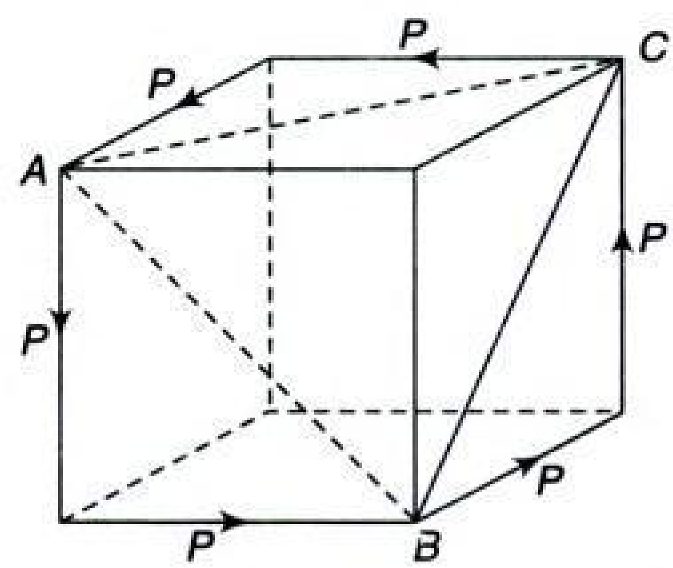

Fig. B

2. Six equal forces $P = 22.25$ N act on a cube with edges of length $a = 125$ mm as shown in Fig. B. Determine the resultant of this system of couples.
(*Ans.* $M = 5.6$ N m in the octahedral plane ABC)

3. Couples of moments M_1, M_2, M_3 act on a prismatic block as shown in Fig. C. What must be the relationship between the magnitudes of moment of these couples if the block is in equilibrium? $\left(Ans. \dfrac{M_1}{\sin\alpha} = \dfrac{M_2}{\sin\beta} = \dfrac{M_3}{\sin\gamma}\right)$

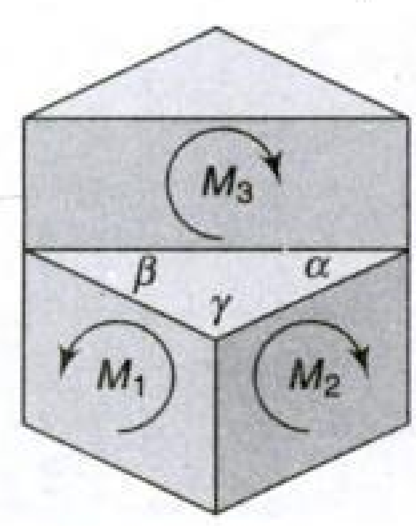

Fig. C

4. A piece of round pipe in the form of a circular quadrant of radius r is attached to two mutually perpendicular vertical wall by flanges A and B as shown in Fig. D. If the pipe is subjected to uniformly distributed twisting moment of intensity m as shown, find the reactions at A and B. Assume that the ends of the pipe can rotate freely inside the flanges so that no twist can be exerted oneither flange.
(*Ans.* $M_a = M_b = mr$)

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Solution: Neglecting friction, the bearing reactions will be vertical and we obtain a system of parallel forces in space that are in equilibrium. With B as origin and coordinate axes directed as shown, Eq. (26) become

$$R_a + R_b - P - S = 0$$
$$-Pc - R_o d + S(c + d) = 0$$
$$Pa - Sb = 0$$

From the last equation, we see that $S = Pa/b$. Substituting this in the other two equations and solving for R_a and R_b, we find

$$R_a = P\frac{a}{b}\left[1 + \frac{c}{d}\left(1 - \frac{b}{a}\right)\right] \qquad \text{(b)}$$

$$R_b = P\frac{a}{b}\left[1 + \frac{c}{d}\left(1 - \frac{a}{b}\right)\right]$$

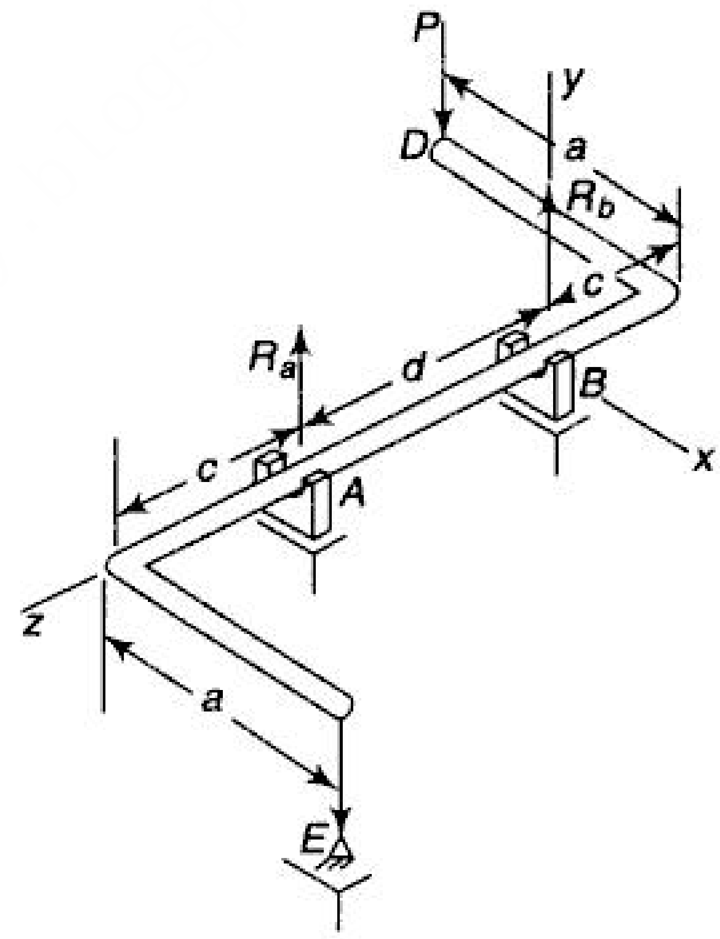

Fig. 5.24

Numerical values of R_a and R_b for given dimensions of the system can now be calculated from these formulas. We note, since the bearings are open, that negative values of R_o and R_b cannot be produced. A study of expression (b) shows that the reactions will both the positive, i.e. up, if the quantities within the brackets are positive. This requires

$$\frac{c + d}{c} > \frac{a}{b} > \frac{c}{c + d} \qquad \text{(c)}$$

Geometrically conditions (c) mean that the line CD joining the two ends of the shaft must cut the central portion of the shaft somewhere between A and B.

2. A movable crane mounted on three wheels, the bearings of which form an equilateral triangle ABC with sides of length a, rests on a horizontal track as shown in Fig. 5.25. The distribution of the weight Q of the crane itself is such that its center of gravity E is vertically above the centroid of the equilateral triangle ABC. For all possible values of the angle α that the vertical plane of the boom can make with the vertical middle plane of the crane, determine the corresponding maximum values of the load P that can be suspended from point D without causing the crane to tip from the track.

Solution: For small values of the angle α, as indicated by the position FD of the boom [Fig. 5.25(b)], the condition of impending tipping of the crane about the axis AB with the wheel C lifting from the track will represent the criterion for the determination of the critical value of the load P, while for larger values of the angle α, as indicated by the position FD' of the boom, the condition of impending tipping of the crane about the axis AC with the wheel B lifting from the track will present the criterion for the determination of the critical value of the load P. Hence, the possibility of tipping of the crane about each of these axes must be investigated separately and the

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Important Formulae

1. The resultant force may be defined analytically by the equations:

$$Z = \Sigma Z_i, \quad \bar{x} = \frac{\Sigma(Z_i x_i)}{\Sigma Z_i}, \quad \bar{y} = \frac{\Sigma(Z_i y_i)}{\Sigma Z_i}$$

2. The conditions that the system reduce to a resultant couple may be expressed analytically as follows:

$$\Sigma Z_i = 0, \quad M_x = \Sigma(Z_i y_i), \quad M_y = \Sigma(Z_i x_i)$$

3. The conditions of equilibrium for any system of parallel forces in space are expressed by the equations

$$\Sigma Z_i = 0, \quad \Sigma(Z_i y_i) = 0, \quad \Sigma(Z_i x_i) = 0$$

PRACTICE SET 5.4

Review Questions

1. What are the possibilities of reducing a parallel force system in space?
2. Write the equilibrium equations for a parallel force system in space.
3. How will you determine the resultant force of the parallel force system, if resultant force exists?
4. What is meant by statically indeterminate?

Objective Questions

1. The simplest resultant of a spatial parallel force system is always
   (a) a wrench
   (b) a resultant force
   (c) a resultant couple
   (d) a resultant force and a resultant couple

[*Ans.* (c)]

PROBLEM SET 5.4

1. A homogenous rectangular plate *ABCD* of width *a*, length *b*, and weight *Q* is supported horizontally by three vertical strings, as shown in Fig. A. Determine the axial forces S_1, S_2 and S_3 in the three supporting strings.

(*Ans.* $S_1 = S_2 = +Q/2$; $S_2 = 0$)

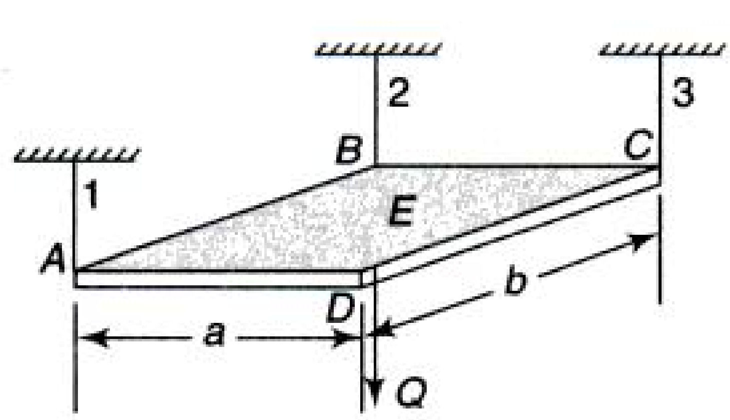

Fig. A

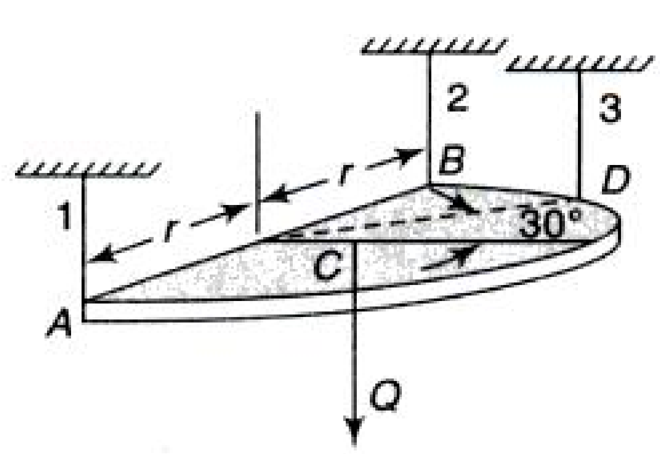

Fig. B

2. A homogenous semicircular plate of weight *Q* and radius *r* is supported in a horizontal plane by three vertical strings as shown in Fig. B. Determine the tensile forces S_1, S_2 and S_3 in these strings. (*Ans.* $S_1 = 0.38Q$; $S_2 = 0.13Q$; $S_3 = 0.49Q$)

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with respect to an axis in this plane that is perpendicular to the direction of the forces and using the above statement regarding moments, we obtain for the perpendicular distance from the yz-plane to the line of action of the resistant (and consequently to the center of parallel forces) the following expression:

$$x_c = \frac{\sum (F_i x_i)}{\sum F_i} \tag{27a}$$

Next, we imagine the parallel lines of action of the given forces to be parallel to the xz-plane. Then proceeding as before, the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$y_c = \frac{\sum (F_i y_i)}{\sum F_i} \tag{27b}$$

Finally, we imagine the given parallel forces to act through their points of application parallel to the xy-plane. Then the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$z_c = \frac{\sum (F_i z_i)}{\sum F_i} \tag{27c}$$

Since we know that the center of parallel forces is independent of the direction of the forces, we conclude that Eq. (27) define the coordinates of this point for any direction of the forces.

Since the center of gravity of a body (see Section 3.3) is the center of parallel gravity forces represented by the weights of the various particles of the body, it follows that the coordinates of the center of gravity of any body can be determined by the use of Eq. (27). In the case of a body of homogeneous material, we conclude that the position of the center of gravity depends only upon the shape of the body and not upon its density. Thus the center of gravity of a body of uniform density is coincident with the centroid of the volume of space occupied by the body.

It follows from the form of Eq. (27) that the center of gravity of a body of uniform density which has a plane of symmetry lies in that plane. If the body has two planes of symmetry, the center of gravity lies on the line of intersection of these planes. If the body has three planes of symmetry, the center of gravity lies at the point of intersection of these planes and is completely determined. Thus the center of gravity of a sphere of uniform density lies at the center of the sphere, the center of gravity of a right circular cylinder of uniform density lies at the midpoint of its geometric axis, the center of gravity of a cone of uniform density lies somewhere on its geometric axis, etc.

If a body may be considered as made up of several finite parts, the centers of gravity of which, individually, are known, then to locate the center of gravity of the composite body, it is only necessary to determine the coordinates of the center of parallel forces represented by the weights of the several parts applied, respectively, at the known centers of gravity of these parts.

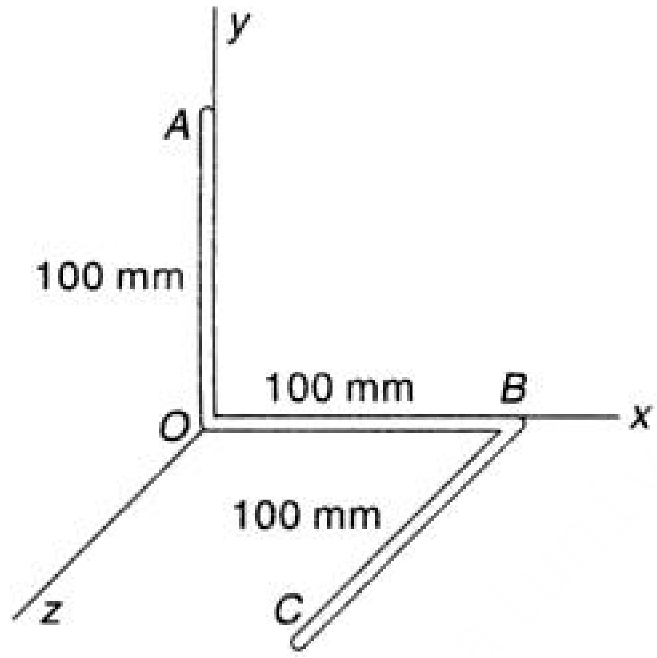

Fig. A

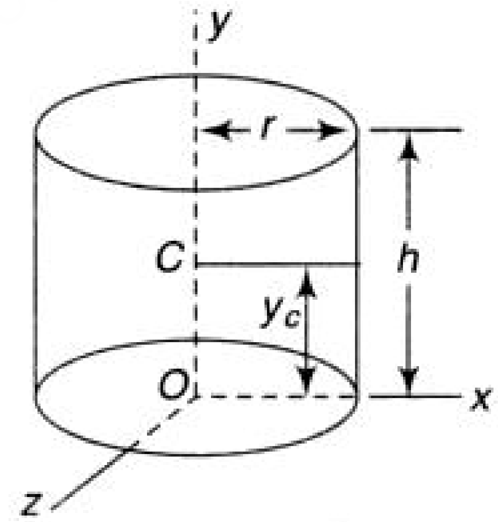

Fig. B

2. Determine the coordinate y_c of the center of gravity C of a right circular cylindrical can of height h and radius of base r if it is made of very thin metal of uniform thickness and density (Fig. B). The can is closed at the bottom and open at the top.

   (*Ans.* $y_c = h^2/(2h + r)$)

3. A steel shaft of circular cross-section has a circular steel hub pressed onto it as shown in Fig. C. For the dimensions shown in the figure, determine the distance x_c from the left end of the shaft to the center of gravity C of the composite body.

   (*Ans.* x_c = 157 mm)

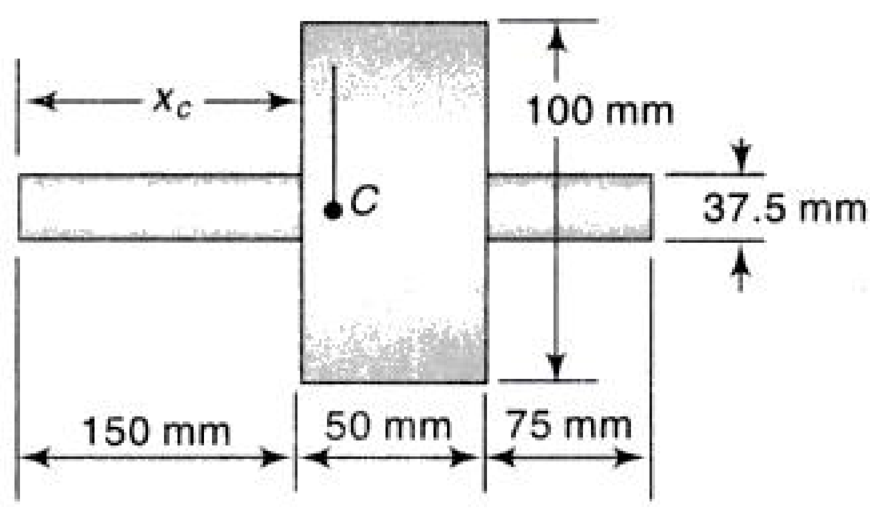

Fig. C

4. The corner of a rectangular box made of this sheet metal of uniform thickness and density is cut-off as shown in Fig. D. Determine the coordinates x_c, y_c and z_c of the center of gravity of the corner.

   (*Ans.* x_c = 31.25 mm, y_c = 31.25 mm; z_c = 56.25 mm)

5. Referring to Fig. E, locate the centroid of the composite area consisting of square in the xy-plane, a triangle in the yz-plane, and a circular quadrant in the xz-plane.

   (*Ans.* $x_c = 0.365a$; $y_c = 0.292a$; $z_c = 0.219a$)

6. Prove that the center of gravity of any homogeneous pyramid with base area A and altitude h lies on the line joining the vertex of the pyramid with the centroid of the area of its base at a distance equal to one-quarter of the altitude from the plane of the base.

7. Determine the height z_c of the center of gravity of a right circular cone above the plane of the base if the density of the material at each point in the cone is proportional to the distance of that point from the plane of the base.

$$\left(Ans.\, x_c = \frac{3}{4} h \right)$$

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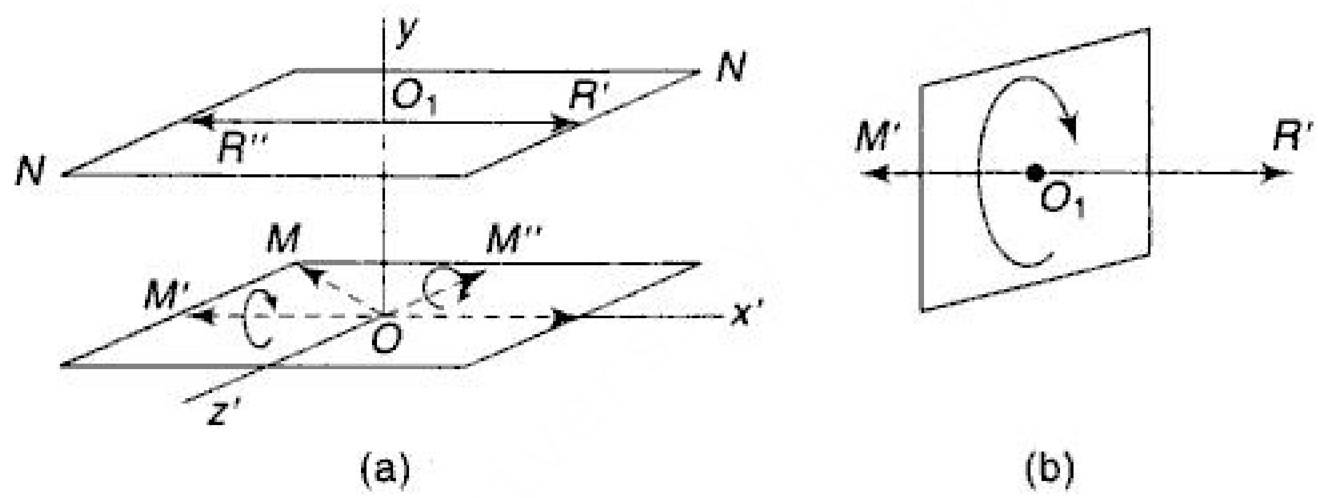

Fig. 5.31

In this plane, we can always resolve the moment vector $\bar{M}$ into two rectangular components: M', coinciding with the line of action of R, and M'' perpendicular there to, as shown in the figure. Now at point O_1 on the y'-axis, we may introduce two oppositely directed collinear forces R' and R'' each equal and parallel to R without altering the action of the system. If we choose the distance $OO_1 = M''/R$, we see that the original force R at O and the force R'' at O_1 constitute a couple which balances the component M'' of the original couple M. Thus we have left only a force R' applied at O_1 and a couple M' in a plane normal to the line of action R'. The moment vector $\bar{M}'$ can now be moved to point O_1, and we have finally the simplified system shown in Fig. 5.31(b), where the resultant couple acts in a plane normal to the line of action of the resultant force. This simplest possible representation of a system of forces in space is sometimes called a *trench*, and *point* O_1 is called the *true center* of the system.

Equations of Equilibrium

From the preceding discussion, we conclude that, in the general case of a system of forces in space, equilibrium can exist only if both the resultant force R and the resultant couple M vanish. Thus the equations of equilibrium, as already obtained in Sections 5.1 and 5.3, are as follows:

$$\Sigma X_i = 0, \qquad \Sigma Y_i = 0, \qquad \Sigma Z_i = 0$$
$$\Sigma(M_x)_i = 0, \qquad \Sigma(M_y)_i = 0, \qquad \Sigma(M_z)_i = 0 \tag{28}$$

These six equations of equilibrium apply to any system of forces, and all cases discussed previously can be obtained from this general case.

If the forces are all parallel and we take the z-axis parallel to them, the first, second, and last equations will always be satisfied and we arrive at Eq. (26) of Section 5.4.

If the forces all intersect in one point and we choose this point as the origin of coordinates, the last three of Eq. (28) will always be satisfied and we arrive at Eq. (20) of Section 5.1.

If the forces are all in one plane which we can take as the xy-plane, the third, fourth, and fifth of Eq. (28) will always be satisfied and we arrive at Eq. (18) of Section 4.2.

In a similar manner, it can be shown that for the cases of parallel and concurrent forces in a plane, Eq. (28) can be reduced to Eq. (11) of Section 3.2 and Eq. (3) of Section 2.4, respectively.

We see from Eq. (28) that there are only six independent conditions of equilibrium for the general case of a system of forces in space. This means that in deal-

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$$S = \frac{Ql}{2\sqrt{l^2 - 4r^2 \sin^2 (\alpha/2)}}, \qquad M = \frac{Qr^2 \sin \alpha}{\sqrt{l^2 - 4r^2 \sin^2 (\alpha/2)}} \tag{e}$$

By way of checking these results it will be noted that the expression for S has the dimension of force while the expression for M has the dimension of force times length. Further, for the particular case where $\alpha = 0$, Eq. (e) reduce to $S = Q/2$ and $M = 0$, as they obviously should. It is interesting to note also that for the case where $\alpha = \pi = 180°$ we obtain, again $M = 0$ but $S = Q/2\sqrt{l^2 - 4r^2}$. In this case the strings are crossed in one plane and a consideration of the equilibrium of the corresponding coplanar system of forces leads to this same result for the tensile force S.

4. A rectangular parallelopiped *ABCD-EFGH* with dimensions a, b and c is supported by six hinged bars arranged as shown in Fig. 5.35. Determine the forces produced in these bars due to the action of a horizontal force P applied at the corner G and acting along the edge GH.

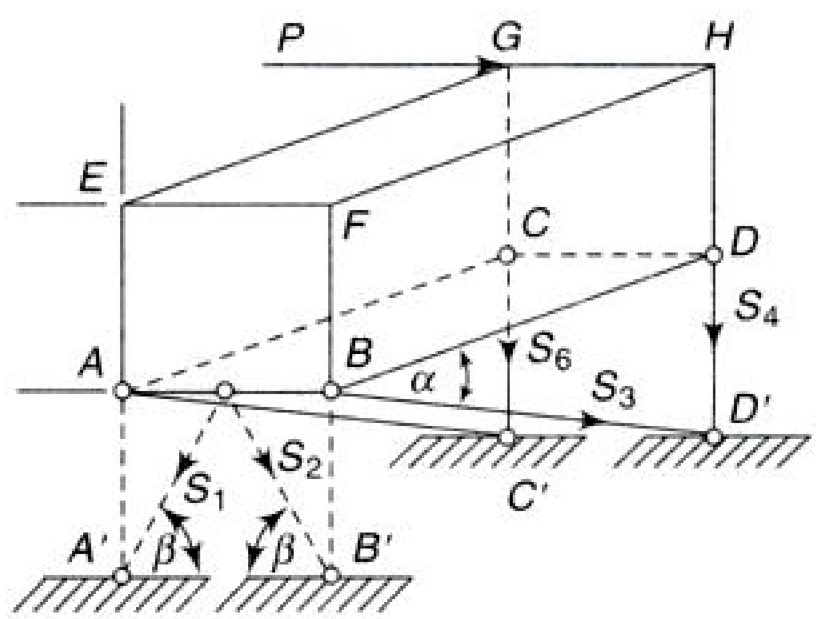

Fig. 5.35

Solution: We begin by considering the parallelopiped as a free body. Replacing the six supporting bars by the reactions that they exert on the body and remembering that these forces must act along the axes of the bars which produce them, we obtain the free-body diagram as shown in the figure. In making this diagram,we assume all bars to be in tension, as indicated by the arrows. If, later, in calculation we find negative values for any of the forces S_i, this will simply indicate that the corresponding bars are in compression.

Writing, first, an equation of moments with respect to the axis BF which is parallel to or intersected by all bars except bar 5, we obtain

$$-(S_5 \cos \alpha)\, a - Pc = 0'$$

from which $S_5 = -Pc/a \cos \alpha$; hence, compression. In the same way, by considering moments of all forces with respect to the axis AE, we conclude that $S_5 = Pc/a \cos \alpha$, tension. Already knowing the value of S_5, this last result may also be obtained directly by equating to zero the algebraic sum of the projections of all forces on the axis AC.

Now by taking moments of all forces, first with respect to the axis AB and again with respect to the axis $C'D'$, we conclude, successively, that

$$S_4 = -S_6 \text{ and } S_1 = -S_2$$

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4. Referring to Fig. C, find the tensions S_1 and S_2 in the two horizontal strings 1 and 2 that support the upper end of the mast AB, assuming a spherical hinge at A. The force P at B is vertical, the one at the mid-point C is parallel to y axis.

(*Ans.* $S_1 = 0.8\,P$; $S_2 = 1.1P$)

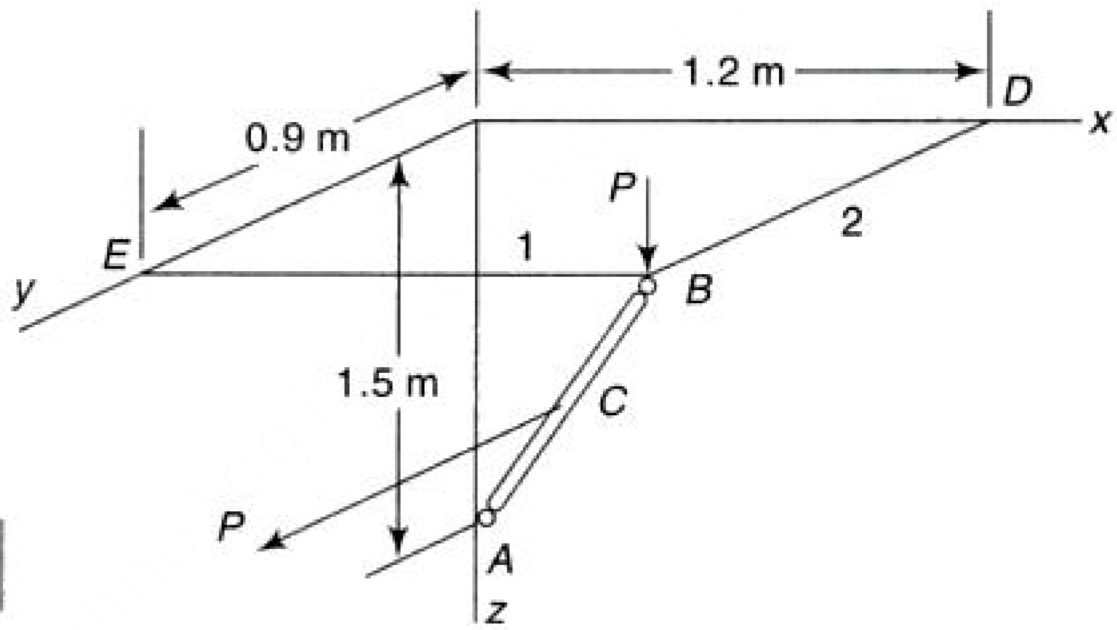

Fig. C

5. Find the tensions S_1 and S_2 in the strings CD and CE attached to the mast AB at its mid-point C, as shown in Fig. D. The load P is vertical. (*Ans.* $S_1 = S_2 = \sqrt{2}P$)

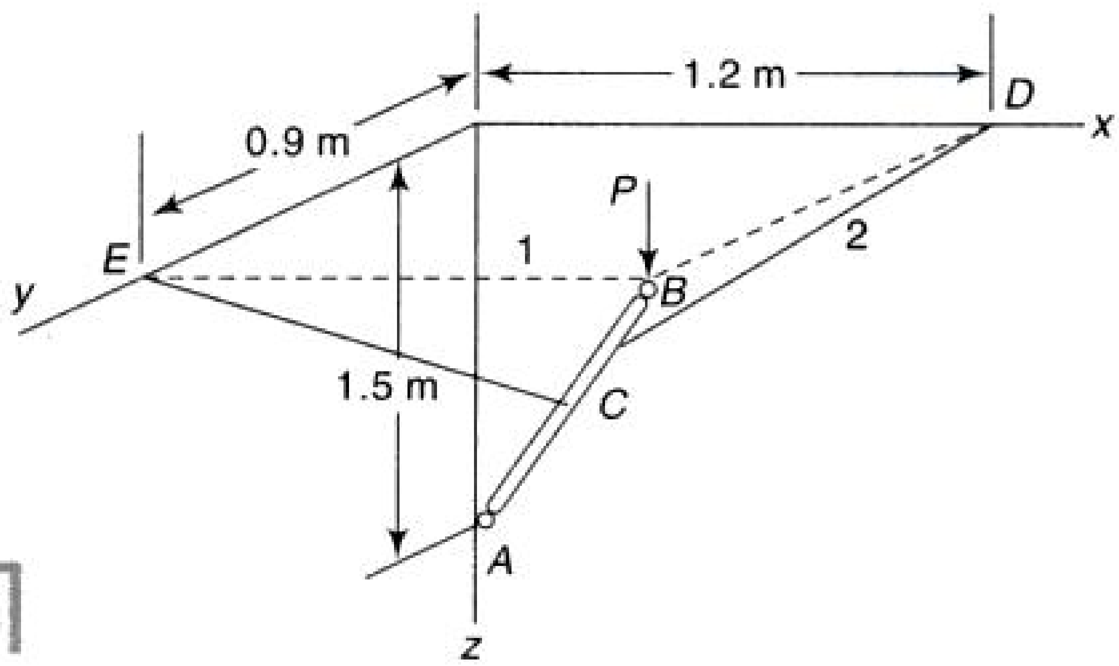

Fig. D

6. To host a load Q, the windlass shown in Fig. E is used. Assuming that the force P applied to the handle of the crank acts at right angles to the arm CD and in a plane

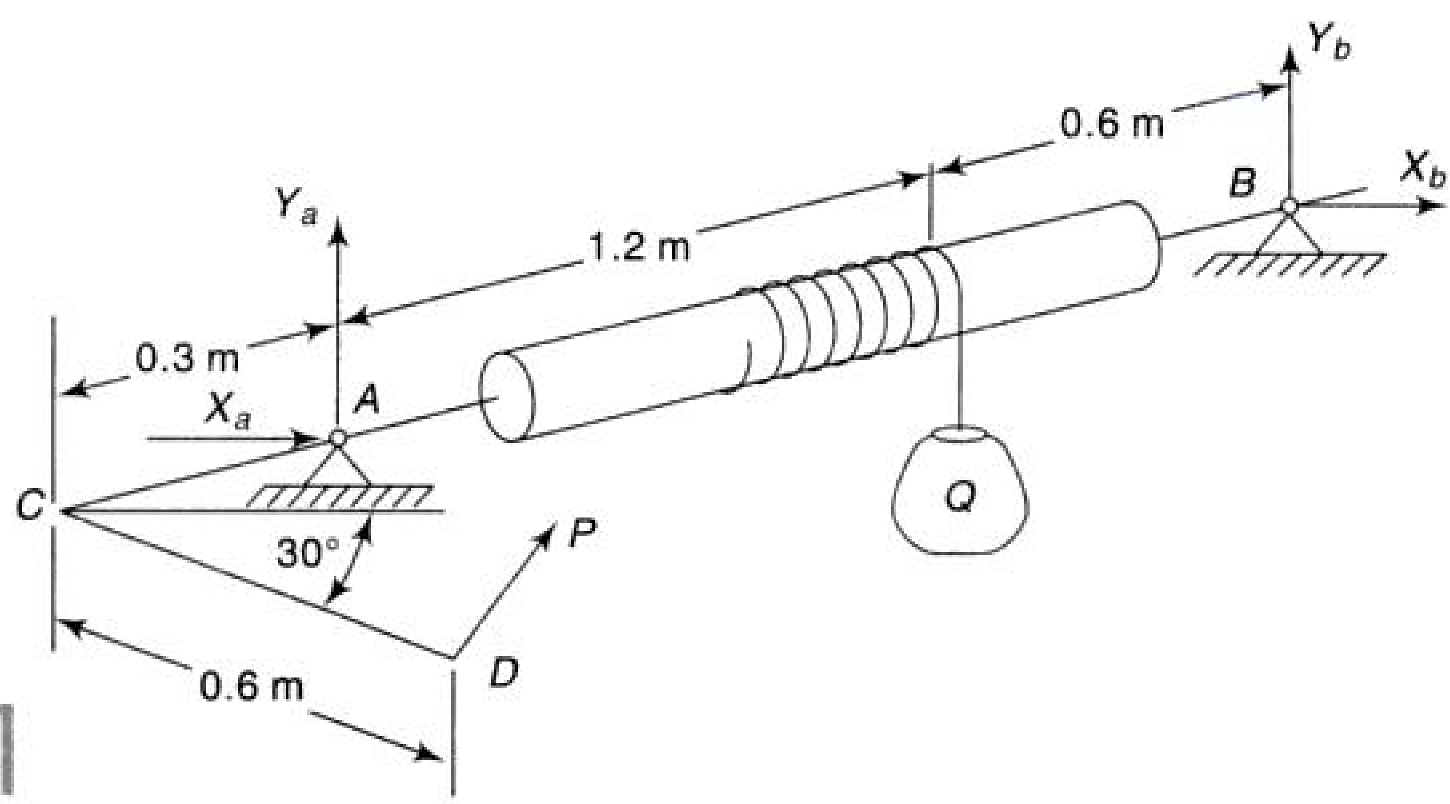

Fig. E

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the virtual displacement of points A and B are $a\delta\theta$ and $b\delta\theta$, perpendicular to the axis of the bar and oppositely directed.

In Fig. 6.2, we have a system of three rigid bodies consisting of crank, connecting rod and piston of an engine. The crank can rotate freely about the z-axis perpendicular to the plane of the figure through O and the configuration of the system is completely defined by the angle θ that OA makes with the fixed x-axis.

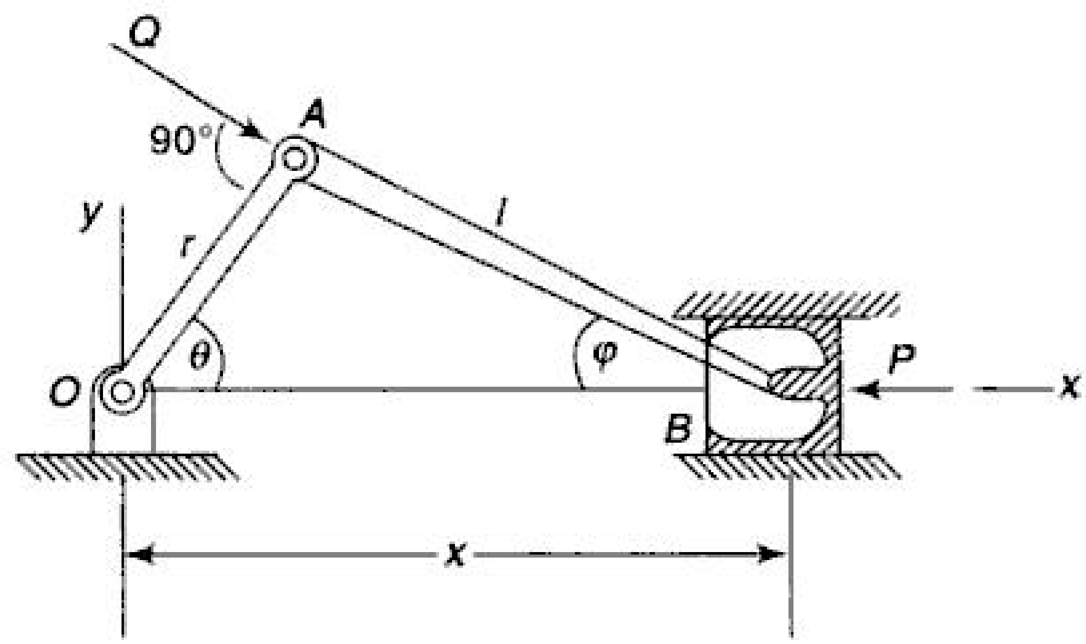

Fig. 6.2

Thus again we have a system with one degree of freedom and the angle θ can be taken as the coordinate of the system. To define virtual displacements of various points in this case, we give to the coordinate θ an infinitesimal increment $\delta\theta$. Then the virtual displacement of the crankpin A is $r\ \delta\theta$ perpendicular to OA. Having the displacement of point A and observing that point B is constrained to move along the x axis, we can now express the corresponding virtual displacement of B as follows: The distance x of the piston from O is

$$x = r\cos\theta + l\cos\varphi$$

Then the change in x due to an increase $\delta\theta$ in the angle θ is

$$\delta x = -r\sin\theta\delta\theta - l\sin\varphi\delta\varphi \tag{a}$$

From ΔAOB (Fig. 6.2), we have

$$\sin\varphi = \frac{r}{l}\sin\theta, \qquad \varphi = \arcsin\left(\frac{r}{l}\sin\theta\right)$$

Substituting these expression into Eq. (a), we obtain for the virtual displacement of the piston

$$\delta x = -r\ \delta\theta\left[1 + \frac{\cos\theta}{\sqrt{(l/r)^2 - \sin^2\theta}}\right]\sin\theta \tag{b}$$

If, for example, we take $\theta = 60°$ and $l/r = 2$, the virtual displacements of A and B will be $r\ \delta\theta$ and $-1.11r\ \delta\theta$, respectively. From these two examples, we see that the problem of defining virtual displacements of various points of a movable system is purely one of geometry.

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4. Two beams AC and CD hinged together at C are supported and loaded as shown in Fig. 6.10(a). Using the principle of virtual work, find the magnitude of the reaction R_b for any position of the load P.

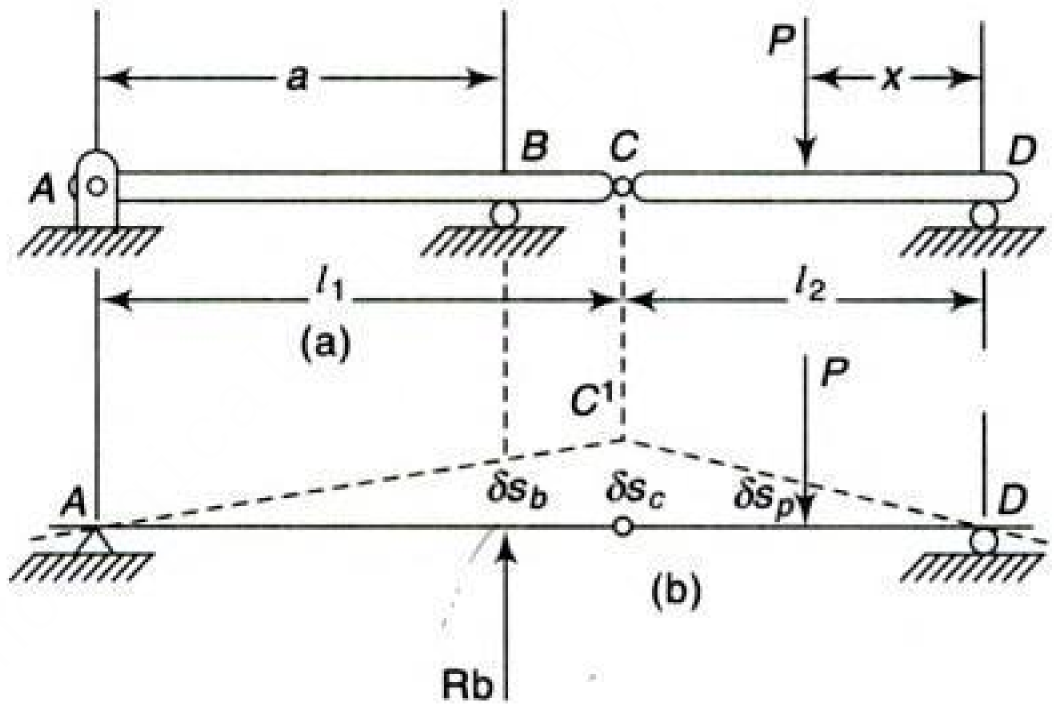

Fig. 6.10

Solution: To obtain a movable system with one degree of freedom, we replace the support at B by a vertical force R_b, as shown in Fig. 6.10(b). A virtual displacement of this system compatible with the remaining constraints is defined by the infinitesimal vertical displacement δs_c of the hinge C as shown. The corresponding vertical displacement of the points of application of R_b and P, respectively, are

$$\delta s_b = \frac{\delta s_c a}{l_1} \quad \text{and} \quad \delta s_p = \frac{\delta s_c x}{l_2}$$

and the equation of virtual work becomes

$$R_b \delta s_b - P\, \delta s_p = 0$$

From this, we obtain

$$R_b = P\frac{\delta s_p}{\delta s_b}$$

or, using the above expression for δs_p and δs_b,

$$R_b = P\frac{x l_1}{l_2 a} \tag{k}$$

Important Terms and Concepts

Completely constrained rigid body
Partially constrained rigid body
Virtual displacements
Degree of freedom
Virtual work
Ideal systems Active forces
Principle of virtual work
Applications of principle of virtual work

SUMMARY

- A virtual displacement is defined to be an imaginary displacement of infinitesimal magnitude.

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from which

$$P = W(\mu \cos \alpha + \sin \alpha) \tag{b}$$

It is seen, from Eq. (a), that the work $P\,\delta s$ produced by the active force P on the virtual displacement δs is used up not only in raising the block W by the amount $\delta s \sin \alpha$ but also in overcoming friction. The work $W\,\delta s$ is called the useful work and its ratio to the expended work $P\delta s$ is called the efficiency of the machine, in this case an inclined plane used for raising the load W. Thus the efficiency of the inclined plane is

$$\frac{\text{Useful Work}}{\text{Expanded Work}} = \frac{\sin \alpha}{\sin \alpha + \mu \cos \alpha} \tag{c}$$

It is seen that the efficiency is unity for $\mu = 0$, that is, for the ideal case without friction, and that it decreases as the coefficient of friction μ increases.

If, instead of raising the load W, we push it downward along the inclined plane by a force P acting parallel to the plane, the magnitude of this force necessary to cause impending motion is obtained from the equation

$$P\,\delta s + W\,\delta s \sin \alpha - F\,\delta s = 0$$

analogous to Eq. (a) above and from which, upon replacing by its value $\mu W \cos \alpha$, we obtain

$$P = W(\mu \cos \alpha - \sin \alpha) \tag{d}$$

When $\mu \cos \alpha - \sin \alpha = 0$, that is, when $\tan \alpha = \mu$ or, in other words, when the angle of inclination of the plane is equal to the angle of friction φ, we see that the force required for impending motion becomes zero. For values of α greater than the angle of friction φ, the load slides down the inclined plane without the application of any pushing force.

In the particular case where $\alpha = \varphi$, the efficiency of the inclined plane as a mechanical device for raising a load W as shown in Fig. 6.11 is obtained by substituting $\tan \alpha$ for μ in expression (c). Then

$$\frac{\text{Useful Work}}{\text{Expanded Work}} = \frac{\sin \alpha}{\sin \alpha + \tan \alpha \cos \alpha} = \frac{1}{2} \tag{e}$$

The method used above for an inclined plane can be applied also in the case of a screw press [Fig. 6.12(a)], the screw of which may be considered as an inclined plane wound around a cylinder [Fig. 6.12(b)].

If an infinitesimal rotation $\delta\theta$ is given to the screw, the work done by the applied couple is $M\,\delta\theta$ and the corresponding work of the compressive force Q is $-Qh\,\delta\theta/2\pi$, where h is the pitch of the thread. In calculating the work done by the friction forces F acting between the threads of the screw and the frame [Fig. 6.12(b)], we assume that the thread is rectangular and that the mean radius of the surface of contact is r. Then the corresponding virtual displacement of the point of application of each friction force F is $\delta s = r\,\delta\theta/\cos\alpha$ and the equation of virtual work becomes

$$M\,\delta\theta - \frac{Qh\delta\theta}{2\pi} - \sum F\,\frac{r\delta\theta}{\cos\alpha} = 0 \tag{f}$$

where F is the friction force acting on one element of the surface of contact of the screw thread and the summation is understood to include all such elements. The

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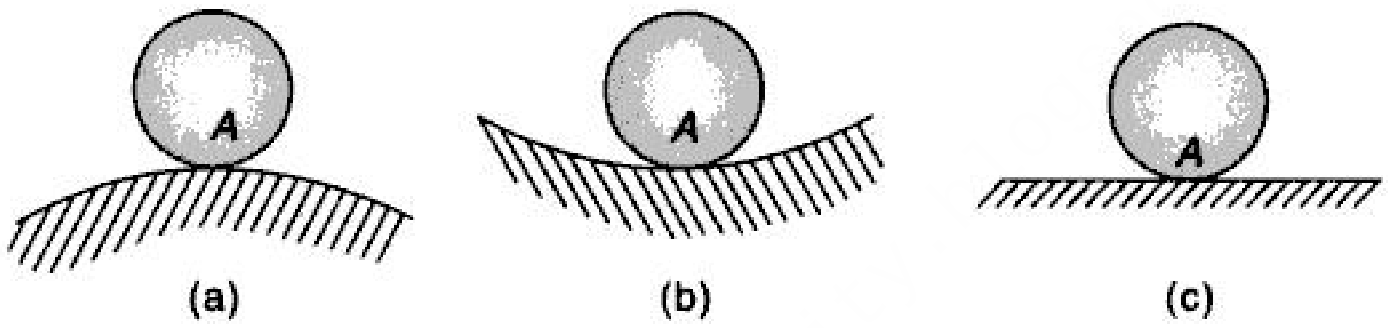

Fig. 6.14

this position, it tends to return back to it as soon as the force is removed. In Fig. 6.14(c), the ball is placed on a smooth horizontal plane, and in this case it is evident that it is in equilibrium in any position and we have *indifferent* equilibrium.

To establish, by using the principle of virtual work, that equilibrium exists in each of the three cases represented in Fig. 6.14, we give to the ball in each case a virtual displacement, i.e. an infinitesimal displacement in which the ball remains in contact with the supporting surface. For such a displacement, we assume that the point of contact of the ball moves horizontally in the plane tangent to the supporting surface at the point of contact. Then the center of gravity of the ball moves also horizontally and the gravity force W does not produce work. Thus we conclude that in all three cases represented in Fig. 6.14 there is a condition of equilibrium.

To decide if each of the above positions of equilibrium is stable or unstable, a more refined calculation of the work done by the active forces on a virtual displacement of the ball is required. Taking the case shown in Fig. 6.14(a) and considering an infinitesimal displacement $AA_1 = \delta s = r\delta\varphi$, we see from Fig. 6.15 that such a displacement involves lowering of the ball by the amount

$$AB = r(1 - \cos \delta\varphi) \approx \frac{r}{2}(\delta\varphi)^2$$

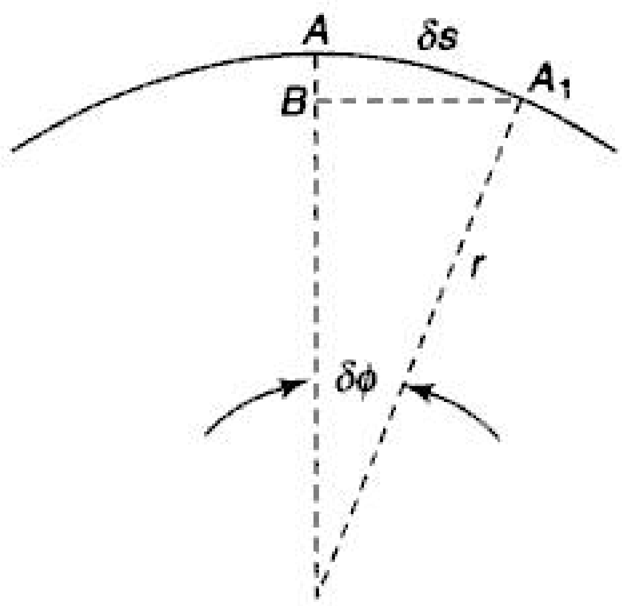

Fig. 6.15

which is a small quantity of the second order if the displacement δs is considered as a small quantity of the first order. It is not necessary to consider this small quantity of higher order in deciding the question of whether or not equilibrium exists. But if the question arises regarding the kind of equilibrium, i.e. whether it is stable or unstable, this second-order quantity must be taken into account. Thus we conclude that, in the case of an unstable equilibrium as represented in Fig. 6.14(a), the gravity force of the ball produces on any virtual displacement a positive work and as a result of the slightest displacement, a reactive force in the direction of motion appears and the ball has the tendency to move away from its position of equilibrium. In the same manner it can be shown that in the case of a stable equilibrium, as represented in Fig. 6.14(b), there will be negative work done by the gravity force on any virtual displacement of the ball. Thus if by some accidental force, the ball is slightly displaced from its position of equilibrium, a reactive force opposing this displacement appears and there is a tendency for the ball to return to its initial position.

as shown in the figure. Neglecting friction and the thickness of the bar AB, establish the criterion of stability of the system in the configuration shown.

(*Ans.* $Wl/2 < Qa\,[(h-a)/a]^2$)

9. A hemispherical cup of radius r and having its center of gravity at C rests on top of a spherical surface of radius R as shown in Fig. I. Assuming that there is sufficient friction to prevent slipping, establish the criterion of stability of the cup in the position shown.

$$\left(Ans.\ R > r\left[\left(\frac{r}{c}\right)-1\right]\right)$$

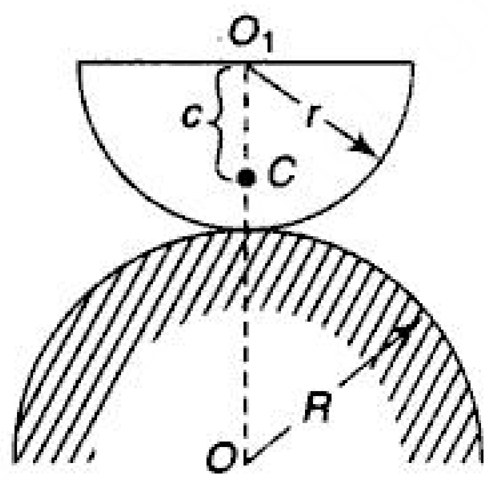

Fig. I

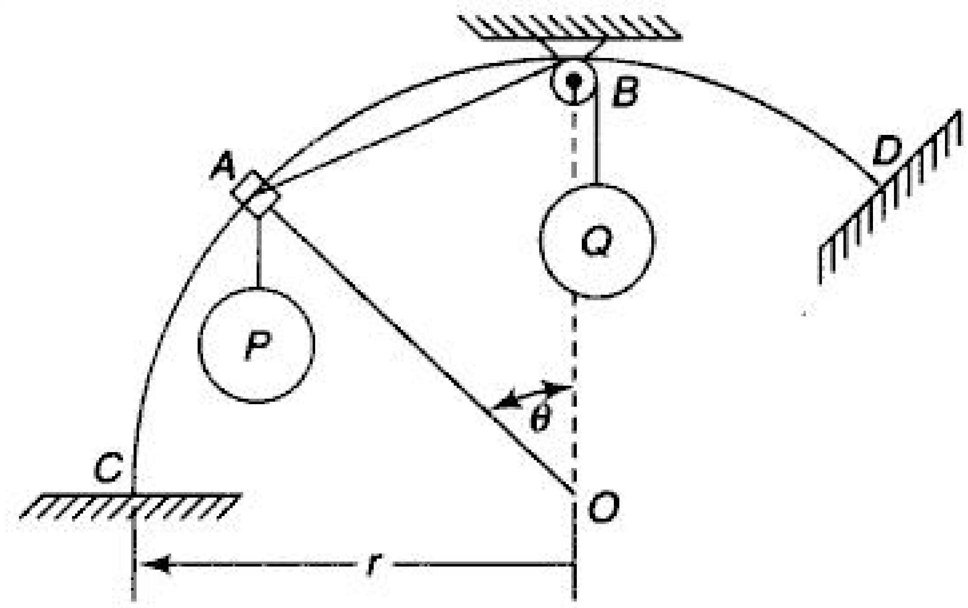

Fig. J

10. Using the principle of virtual work and taking θ coordinate, determine all possible configurations of equilibrium of the system in Fig. J and investigate the stability of each. The bead A can slide freely on the circular wire CD and the pulley at B is negligibly small.

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As a second example of rectilinear motion of a particle, let us consider Eq. (30) in the form

$$x = \frac{1}{2} g t^2 \tag{30b}$$

Here the displacement x is represented as being proportional to the square of the time. Such motion, as is known from elementary physics, we have in the case of freely falling bodies.

A third example of rectilinear motion may be represented by taking Eq. (30) in the following form:

$$x = r e^{-kt} \tag{30c}$$

where r and k are constants and e is the natural logarithmic base. Such motion, in which the displacement decreases exponentially with time, can be encountered in dealing with a particle projected into a highly viscous medium which finally brings it to rest.

Instead of analytical expressions for Eq. (30), it is often useful to represent the displacement–time relationship graphically. Taking time t as abscissa and displacement x as ordinate, the curve represented by Eq. (30) can be plotted for any particular case. This gives us a so-called *displacement–time diagram*. Such diagrams for the particular cases represented by Eqs (30a) to (30c) are shown in Fig. 7.2. These curves represent graphically the same information as given by the analytic expressions.

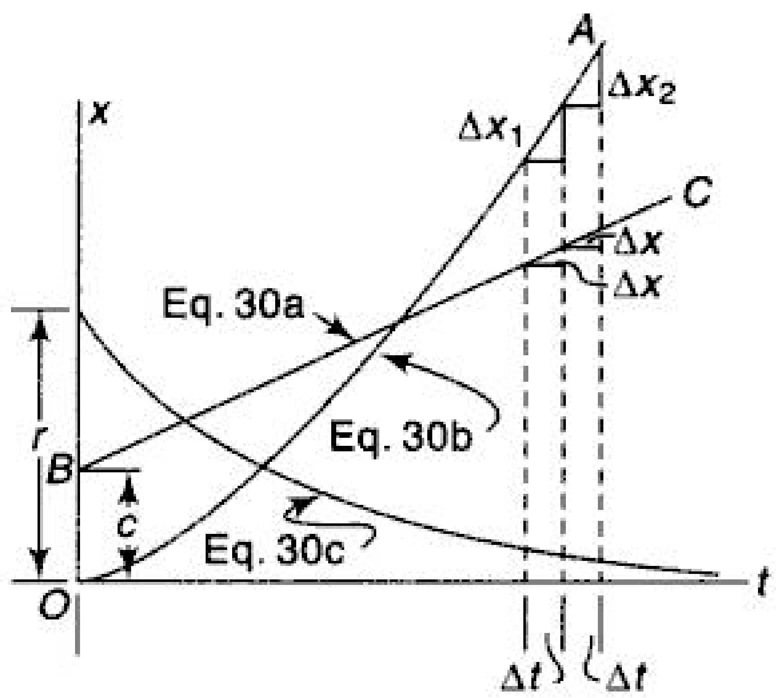

Fig. 7.2

Velocity

In discussing velocity of a particle having rectilinear motion, we begin with the case of uniform rectilinear motion as represented graphically by the straight line *BC* in Fig. 7.2 and we see that for equal intervals of time Δt, the particle receives equal increments of displacement Δx. Thus the velocity v of uniform motion is given by the equation,

$$v = \frac{\Delta x}{\Delta t} \tag{a}$$

This velocity is considered positive if the displacement x is increasing with time and negative if it is decreasing with time. Using the meter as the unit of displacement and the second as the unit of time, the unit of velocity will be the meter per second, usually written m/s.

In the more general case of non-uniform rectilinear motion of a particle as given by Eq. (30b), represented graphically in Fig. 7.2 by the curve *OA*, we see that in equal intervals of time Δt, the particle receives unequal increments of displacement Δx_1 and Δx_2. Since in this case, the increment of displacement increases with time, we have *accelerated motion*. If Δx denotes the increment of

Differentiating expression (g) again with respect to time gives, for the acceleration–time equation,

$$\ddot{x} = -r\,\omega^2 \cos \omega t \tag{h}$$

From Eqs (f), (g), (h), we can now plot displacement–time, velocity–time, and acceleration–time diagrams as shown in Fig. 7.7.

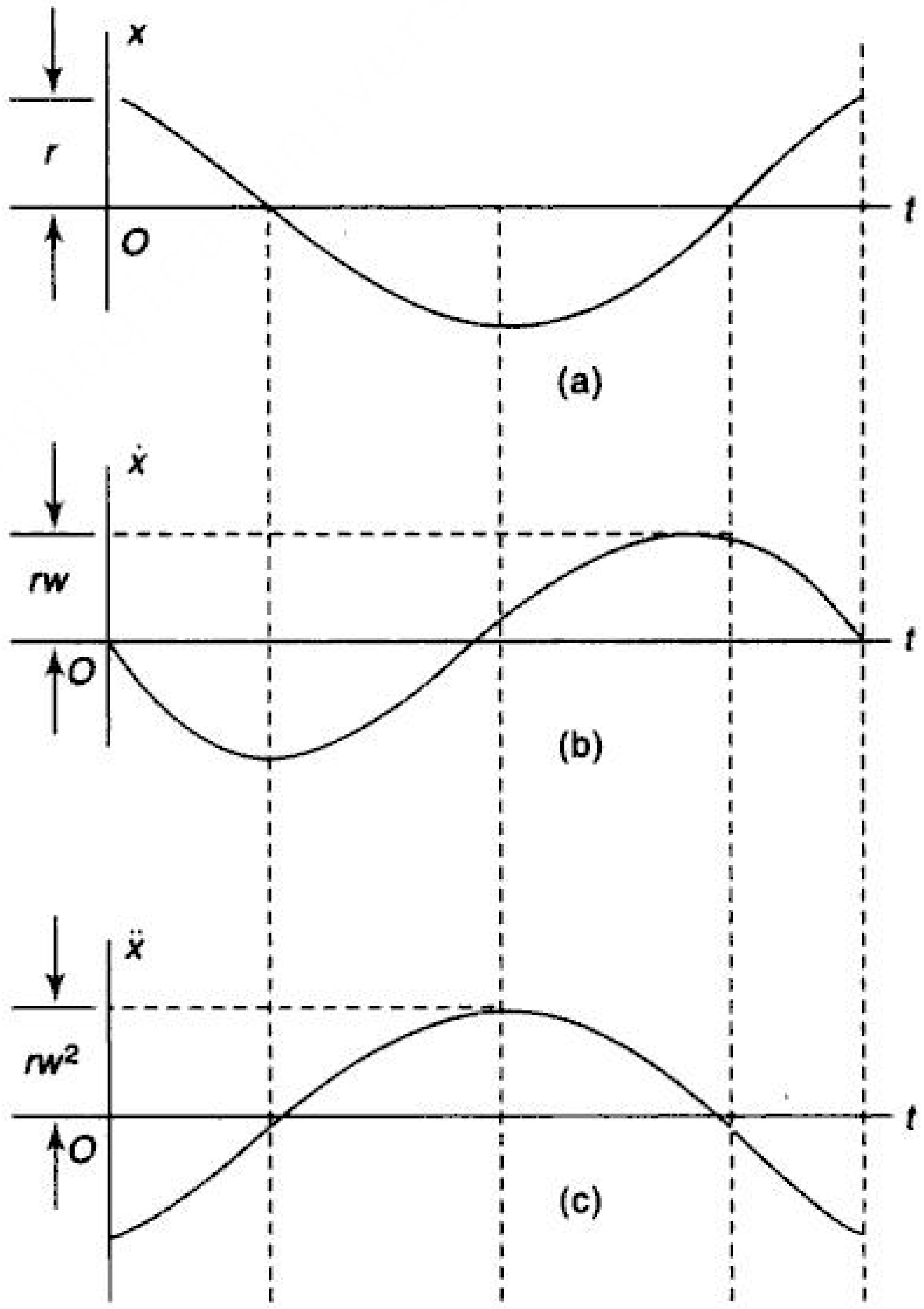

Fig. 7.7

From these diagrams we see that simple harmonic motion is periodic, i.e. after the time interval $\tau = 2\pi/\omega$, the curves repeat themselves. Comparing the three diagrams in Fig. 7.7, we see that the velocity is a maximum or a minimum when the displacement is zero, and vice versa. We also note that the acceleration is proportional to the displacement but always of opposite sign. Simple harmonic motion is a very important type of rectilinear motion and the student should become thoroughly familiar with its various characteristics.

2. A slender bar AB of length l which remains always in the same vertical plane has its ends A and B constrained to remain in contact with a horizontal floor and a vertical wall, respectively, as shown in Fig. 7.8(a). The bar starts from a vertical position, and the end A is moved along the floor with constant velocity v_0 so that the displacement $OA = v_0 t$. Write the displacement–time, velocity–time, and acceleration–time equations for the vertical motion of the end B of the bar.

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Equation (33) is the *general equation of motion* of a particle from which its acceleration at any instant can be obtained provided the acting force F at this instant is known.

It is seen from Eq. (33) that for a given magnitude of the force F, the acceleration produced is inversely proportional to the factor W/g. This factor measures the degree of sluggishness with which the particle yields to the action of an applied force and is a measure of the inertia of the particle. It is called the mass of the particle and is generally denoted by m. Thus using the notation,

$$m = \frac{W}{g} \tag{c}$$

the general equation of motion of a particle becomes

$$ma = F \tag{33'}$$

Taking the kilogram as the unit of mass, the meter as the unit of length, and the second as the unit of time, we find that the unit of force is that force which by acting on 1 kg of mass produces an acceleration of 1 m/s^2. This derived unit of force is called a *newton*.

7.3 DIFFERENTIAL EQUATION OF RECTILINEAR MOTION

The general equation of motion for a particle (Eq. 33) can be applied directly to the case of rectilinear translation of a rigid body, since all particle of the body have the same motion and we consider the entire body as a particle concentrated at its center of gravity. Whenever such a body or particle moves under the action of a force applied at its center of gravity and having a fixed line of action, acceleration of the body is produced in the same direction, and if any initial velocity of the body is also directed along this line, we obtain the case of rectilinear translation. Taking the line of motion as the x-axis and using for acceleration the notation $\ddot{x}$ (see Eq. 32) and for the resultant acting force the notation X, the differential equation of rectilinear motion of a particle becomes

$$\frac{W}{g}\ddot{x} = X \tag{34}$$

This equation can be used for solving two kinds of problems: (1) The motion of the particle is given, i.e. the displacement x is expressed as a certain function of time t and it is required to find the force X necessary to produce such a motion. (2) The force X is given, and it is required to find the motion of the particle, i.e. to find for x such a functions of t that Eq. (34) will be satisfied.

Problems of the first kind are very simple. In each particular case it is only necessary to substitute for x in Eq. (34) the given function of time. Then by successive differentiation, the force X is obtained. The second problem requires the integration of Eq. (34), while such integration can be made will be considered in subsequent articles. For the present, we shall limit our attention to the solution of some problems of the first kind.

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- The differential equation of motion can be used for solving two kinds of problems: (1) The motion of the particle is given, i.e. the displacement x is expressed as a certain function of time t and it is required to find the force X necessary to produce such a motion. (2) The force X is given, and it is required to find the motion of the particle, i.e. to find for x such a functions of t that differential equation of rectilinear motion will be satisfied.

Important Formulae

1. The following equation is the *general equation of motion* of a particle from which its acceleration at any instant can be obtained provided the acting force F at that instant is known.

$$\frac{W}{g}a = F$$

2. Taking the line of motion as the x-axis and using for acceleration the notation $\ddot{x}$ and for the resultant acting force the notation X, the differential equation of rectilinear motion of a particle becomes

$$\frac{W}{g}\ddot{x} = X$$

PRACTICE SET 7.2 AND 7.3

Review Questions

1. State Newton's laws of motion.
2. Write the differential equation of rectilinear motion.

PROBLEM SET 7.2 AND 7.3

1. An elevator of gross weight $W = 4450$ N starts to move upward with constant acceleration and acquires a velocity $v = 18$ m/s, after travelling a distance = 1.8 m. Find the tensile force S in the cable during this accelerated motion. Neglect friction. (*Ans.* $S = 4863.85$ N)
2. The elevator of Prob. 1, when stopping, moves with constant declaration and from the constant velocity $v = 1.8$ m/s, comes to rest in 2 s. Determine the pressure P transmitted during stopping to the floor of the elevator by the feet of a man weighing 765.5 N. (*Ans.* $P = 685.3$ N)
3. A train weighing 1870 kN without the locomotive starts to move with constant acceleration along a straight horizontal track and in the first 60 s acquires a velocity of 56 kmph. Determine the tension S in the draw-bar between the locomotive and train if the total resistance to motion due to friction and air resistance is constant and equal to 0.005 times the weight of the train. (*Ans.* 56.16 N)
4. The driver of an automobile, travelling along a straight level highway, suddenly applies the brakes so that the car slides for 2 s, covering a distance 9.66 m, before coming to a stop. Assuming that during this time the car moved with constant declaration, find the coefficient of friction between the tyres and the pavement. (*Ans.* $\mu = 0.5$)
5. A mine cage of weight $W = 8.9$ kN starts from rest and moves downward with constant acceleration, travelling a distance $s = 30$ m in 10 s. Find the tensile force in the cable during this time. (*Ans.* $S = 8348.2$ N)

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2. From the top of a tower of height $h = 36$ m, a ball is dropped at the same instant that another is projected vertically upward from the ground with an initial velocity $v_0 = 18$ m/s. How far from the top do they pass and with what relative velocity?

Solution: For the ball that is dropped, we choose the top of the tower as the origin and consider downward displacement as positive. Then this ball will have neither initial displacement nor initial velocity and from Eq. (37b), its displacement at any instant will be

$$x_1 = \frac{1}{2} gt^2 \tag{f}$$

For the other ball we choose the ground as the origin and consider upward displacement as positive. Then from Eq. (37a), its displacement at any instant will be

$$x_2 = v_0 t = \frac{1}{2} gt^2 \tag{g}$$

When the ball pass, we must have

$$x_1 + x_2 = h \tag{h}$$

Substituting the values of x_1 and x_2 from Eqs (f) and (g) into Eq. (h), we obtain

$$\frac{1}{2} gt^2 + v_0 t - \frac{1}{2} gt^2 = h$$

from which, using the given numerical data, $t = 2$ s. Using this value of t in Eq. (f), we find

$$(x_1)_{t=2} = 19.32 \text{ m}$$

Differentiating Eqs (f) and (g) once each with respect to time, we find that at the instant $t = 2$ s, the two balls are moving downward with velocities of 19.32 and 1.32 m/s, respectively. Hence the balls pass 19.32 m below the top of the tower with a relative velocity of 18 m/s, 2 s after starting.

3. A particle projected vertically upward is at a height h after t_1 s and again after t_2 s. Find this height h and also the initial velocity v_0 with which the particle was projected.

Solution: Neglecting air resistance, the particle at all times is moving under the action of its own gravity force W which is always directed vertically downward. We take the x-axis along the vertical line of motion, the origin at the starting point, and consider upward displacement as positive. Then, from Eq. (37a), we have for the instant $t = t_1$:

$$h = v_0 t_1 - \frac{1}{2} g t_1^2 \tag{i}$$

Likewise, for the instant $t = t_2$

$$h = v_0 t_2 - \frac{1}{2} g t_2^2 \tag{j}$$

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Second Method

Sometimes the acting force $X = F(t)$ cannot be expressed by an analytic function but is represented graphically by a curve called a force–time diagram, as shown in Fig. 7.19. In such case, we begin with the equation of motion in the form

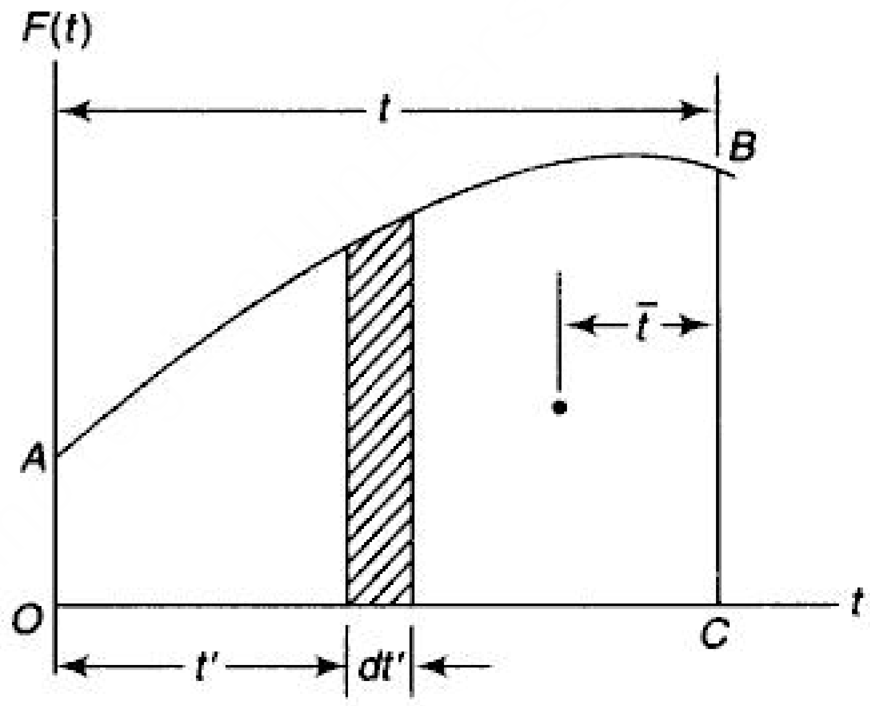

Fig. 7.19

$$m\ddot{x} = F(t')$$

where t' instead of t is used as a temporary symbol for time[6]. Dividing both sides of the equation of motion by the mass m of the particle and making a first separation of variables, we have

$$d\dot{x} = F(t')dt' \tag{h}$$

This equation expresses the differential relation between velocity $\dot{x}$ and time t'. We see that the increment of velocity imparted to the particle during the time interval dt' is proportional to the area $F(t')\,dt'$ of the shaded strip of the force–time diagram in Fig. 7.19. Summing up all such increments of velocity from the initial moment $t' = 0$ to any time $t' = t$, we have

$$\int_{\dot{x}_0}^{\dot{x}} d\dot{x} = \frac{1}{m}\int_0^t F(t')\,dt'$$

where $\dot{x}_0$ is the velocity corresponding to $t' = 0$ and $\dot{x}$ is that corresponding to $t' = t$. Performing the indicated integration on the left side of this equation, we obtain the velocity–time equation in the following form:

$$\dot{x} = \dot{x}_0 + \frac{1}{m}\int_0^t F(t')\,dt' \tag{38a}$$

The first term on the right side of this equation is the initial velocity. The second term represents the additional velocity imparted to the particle by the acting $F(t')$.

We see that the integral in this term simply represents the area of the force–time diagram upto the ordinate t. Hence the velocity–time equation can be expressed in the equivalent form

$$\dot{x} = \dot{x}_0 + \frac{1}{m}\left[A_F\right]_0^t \tag{38b}$$

[6]We do this so that the symbol t can be used as a limit of integration.

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From the definition of the spring constant k, we conclude that the term W/k represents the elongation produced in the spring by the static action of the weight W. Using the notation

$$\delta_{st} = \frac{W}{k} \tag{f}$$

we obtain the third form of Eq. (42), from which we see that to calculate the period of vibration we need to know only the static elongation of the spring under the action of the load W_1.[8] This elongation can often be conveniently obtained by direct experiment, and such an experimentally determined value when substituted into Eq. (42) will give the period τ.

Having the period of vibration, i.e., the time required for one complete cycle, we can easily obtain the number of vibrations per second. This number is called the *frequency of vibration* and will be denoted by f. Then

$$f = \frac{1}{\tau} = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \tag{43}$$

In the preceding discussion we have assumed that at the initial moment the body W was given an initial displacement x_0 and suddenly released without initial velocity. Let us consider now another special case where vibration of the weight is started by giving it, by means of a vertical impact, an initial velocity in its position of static equilibrium. Then at the initial moment we have

$$(x)_{t=0} = 0, \qquad (\dot{x})_{t=0} = \dot{x}_0 \tag{g}$$

Substituting the first of these conditions in Eq. (41) and the second in Eq. (e), we find $C_1 = 0$ and $C_2 = \dot{x}_0/p$, and Eq. (41) becomes

$$x = \frac{\dot{x}_0}{p}\sin pt \tag{41b}$$

The displacement–time, velocity–time and acceleration–time curves for this case are shown in Fig. 7.24.

Figure 7.24 is still a simple harmonic motion. It is seen that the vibration has the same period $\tau = 2\pi/p$ as before, and we observe that the period of vibration does not depend upon either initial displacement or initial velocity but only on the spring constant k and the weight W. Only the amplitude is altered, being in this case equal $\dot{x}_0/p$.

In a more general case we may start vibration of the weight by displacing it from its equilibrium position by an amount x_0 and then giving to it in this displaced position some initial velocity. In this case we have at the initial moment

$$(x)_{t=0} = x_0, \qquad (\dot{x})_{t=0} = \dot{x}_0 \tag{h}$$

and the general solution (41) becomes

[8]It is seen that the period of vibration of the suspended weight is the same as that of a mathematical pendulum, the length of which is equal to the static elongation produced in the spring by the action of the weight W.

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$$S - W - \frac{W}{g}a = 0$$

from which we find for the tensile force in the rope

$$S = W\left(1 + \frac{a}{g}\right) \qquad \text{(d)}$$

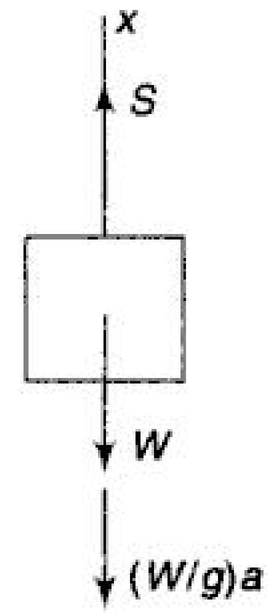

Fig. 7.31

It is seen from Eq. (d) that the tension in the rope depends not only upon the weight W of the body but also upon the acceleration a which is given to it. By giving a great acceleration we can produce in the rope a tension many times greater than the weight W. We are all familiar with the phenomenon of breaking a string to which a small weight is attached, simply by giving the string a quick jerk.

Equation (d) can also be used in the case of deceleration of the weight simply by changing the sign of a. In this case the tensile force becomes less than the weight of the body and we see that when the body is freely falling, that is, when $a = -g$, the tension S becomes zero.

2. A mathematical pendulum of length l and weight W is supported from the ceiling of an elevator (Fig. 7.32). How will its period of oscillation for small amplitudes be affected by a constant upward or downward acceleration a of the elevator?

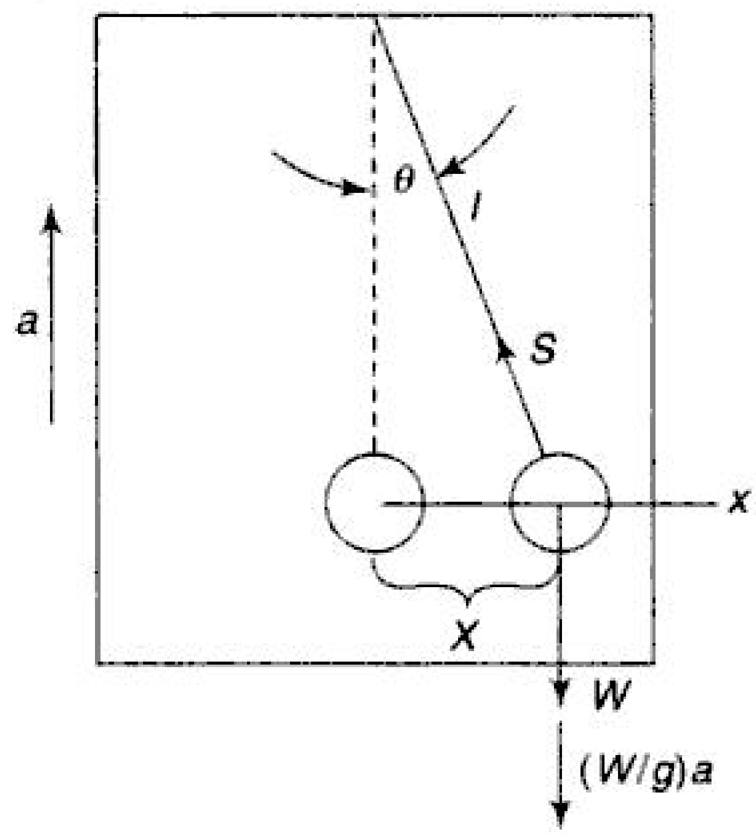

Fig. 7.32

Solution: Let us consider upward acceleration of the elevator as positive. Then the inertia force $-(W/g)a$ of the pendulum bob will add to the gravity force, and we have in effect an increase in the weight of the pendulum without any increase in its mass. Considering now a small displacement of the pendulum from its vertical position of equilibrium and treating the circular arc of radius l along which the weight W moves as coincident with the x-axis, the projection of all forces, including the inertia force $-(W/g)\ddot{x}$, onto the x-axis gives the equation of dynamic equilibrium,

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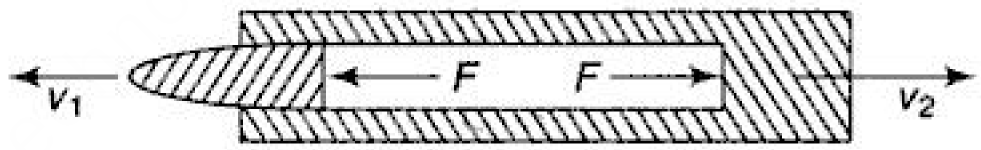
The equation of momentum and impulse is particularly useful when we are dealing with a system of particles, since in such case calculation of the impulse can often be eliminated. As a specific example, let us consider the case of a gun and shell as shown in Fig. 7.36, which may be considered as a system of two particles. During the extremely short interval of explosion, the forces F acting on the shell and gun and representing the gas pressure in the barrel are varying in an unknown manner and a calculation of the impulses of these forces would be extremely difficult. However, the relation between the velocity of the shell and the velocity of recoil of the gun can obtained without calculation of the impulse. Since the forces F are in the nature of action and reaction between the shell and gun, they must at all times be equal and opposite, and hence their impulses for the interval of explosion are equal and opposite, since the forces act exactly the same time t.

Fig. 7.36



Thus if W_1 and W_2 are the weights of the shell and gun, respectively, we find, assuming the initial velocities to be zero and neglecting all external forces that

$$\frac{W_1}{g} v_1 = \int F dt , \qquad \frac{W_2}{g} v_2 = \int F dt$$

Then for the entire system

$$\frac{W_1}{g} v_1 = \frac{W_2}{g} v_2$$

from which we obtain

$$\frac{v_1}{v_2} = \frac{W_2}{W_1}$$

We see that the velocities of the shell and gun after discharge are in opposite directions and inversely proportional to the corresponding weights.

We obtain a great simplification in the above example owing to the fact that no external forces act on the system but only internal forces in the nature of action and reaction. Internal forces in a system of particles always appear as pairs of equal and opposite forces and need not be considered when applying the equation of momentum and impulse. Thus we may state that, in the case of any system of particles to which no external forces are applied, the momentum of the system remains unchanged, since the total impulse is zero. This is sometimes called the principle of *conservation of momentum*.

Examples Examples Examples Examples Examples

1. A flat car can roll without resistance along a horizontal track as shown in Fig. 7.37. Initially, the car together with a man of weight w is moving to the right with speed v_0. What increment of velocity Δv will the car obtain if the

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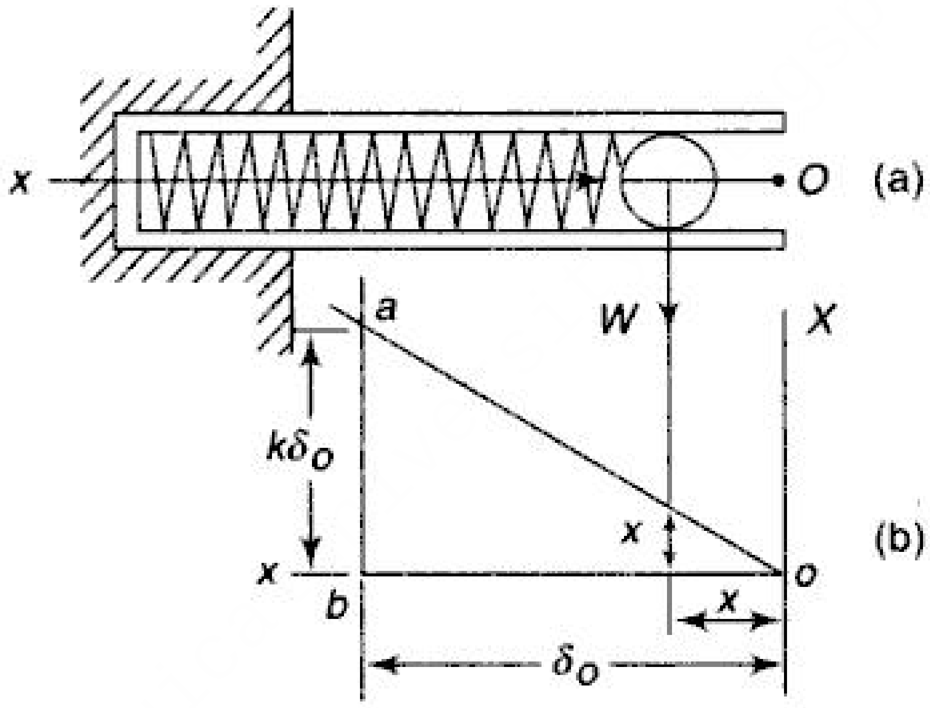

Fig. 7.40

Solution: Denoting by X the force exerted by the spring on the ball after release and during expansion of the spring, we have $X = kx$, where x is measured from the position of the ball corresponding to the unstressed length of the spring. The corresponding force–displacement diagram is shown in Fig. 7.40(b). We see that the initial force on the ball when $x = \delta_0$ is $k\delta_0$ and that it decreases uniformly to zero when $x = 0$. The corresponding work of the force X is represented by the area *oab* of the diagram and is $k\delta_0\ \delta_0/2 = k\delta_0^2/2$. Equation (47) then becomes

$$\frac{W}{g}\frac{\dot{x}^2}{2} = \frac{k\delta_0^2}{2}$$

from which

$$\dot{x} = \delta_0\sqrt{\frac{kg}{w}} \qquad \text{(f)}$$

As would be expected, this is the same result obtained in Section 7.6 for the maximum velocity of a weight W attached to a spring of constant k and set in vibration by an initial displacement δ_0.

2. A weight W that can slide freely up and down a prismatic steel bar BC of length l and cross-sectional area A is allowed a free fall through the distance h (Fig. 7.41). Assuming the mass of the bar to be negligible compared with that of the falling weight, find the resulting dynamic elongation δ of the bar.

Solution: The weight starts from rest, travels a total distance $h + \delta$, and again comes to rest after having stretched the bar an amount δ. Hence the net change in kinetic energy is zero, and we conclude from Eq. (47) that the net work of all forces acting through this displacement

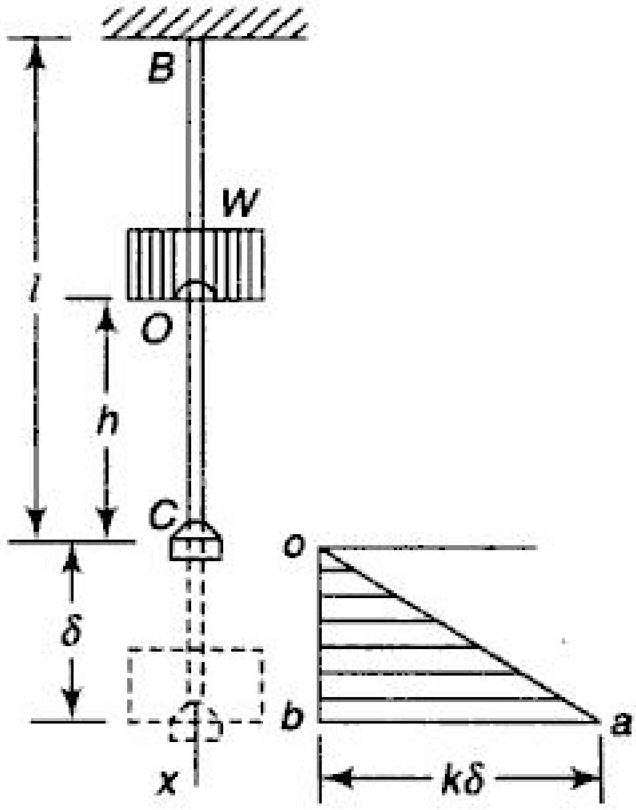

Fig. 7.41

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$$\int_{T_A}^{T_B} dT = \int_{x_A}^{x_B} dU$$

or

$$T_B - T_A = \int_{x_A}^{x_B} dU \tag{48}$$

This is the equation of work and energy for a system of particles. It states that the total change in kinetic energy of the system when it moves from configuration A to configuration B is equal to the corresponding work of all forces acting upon it. In the case of an ideal system, the reactive forces will produce no work and work of all internal forces which occur in equal and opposite pairs will cancel one another. Thus for such systems, only the work of active external forces need be considered on the right side of Eq. (48) and a substantial simplification is realized.

Equation (48) can be written in another from by introducing the concept of potential energy. The potential energy of a system in any configuration A is defined as the work which will be done by the acting forces if the system moves from that configuration back to a certain base or reference configuration O. Thus when the system is in configuration A or B, its potential energy is

$$V_A = \int_A^0 dU \quad \text{or} \quad V_B = \int_B^0 dU \tag{d}$$

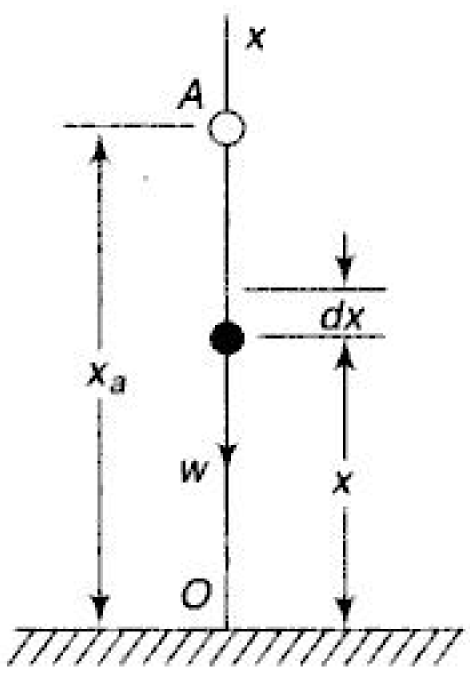

Fig. 7.44

To illustrate, we consider a single particle of weight w at the height x_a above a datum plane O, as shown in Fig. 7.44. The potential energy V_a in position A is defined as the work that the gravity force w will produce if the particle is allowed to fall to the datum plane. Clearly this work is wx_a. To obtain this result by using the definition (d), we consider the particle in any position x and note that $X = -w$. Then the increment of work during a small increase in dx in the displacement is

$$dU = -w\,dx$$

Substituting this into the first of expressions (d), we obtain

$$V_a = -\int_{x_0}^{0} w dx = \int_0^{x_0} w dx = xw_a$$

as anticipated above.

If we have a system of particle of weights, $w_1, w_2, \ldots, w_n$ at the elevation $x_1, x_2 \ldots, x_n$ above the chosen datum plane, the total potential energy of the system is

$$V = w_1x_1 + w_2x_2 + \ldots + w_nx_n = Wx_0 \tag{e}$$

where W is the total weight of the system of particle and x_c is the elevation of their center of gravity.

$$T_B - T_A = \int_A^B dU = \int_0^B dU - \int_0^A dU$$

Then noting from expression (d) that

$$\int_O^B dU = -\int_B^O dU = -V_B$$

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4. A V-tube having a uniform bore of cross-sectional area A stands with its two branches inclined to the vertical by equal angles α as shown in Fig. B. Calculate the period of oscillation of a column of liquid of total length l and specific weight w if initially displaced from its equilibrium position in the tube as shown in the figure.

(*Ans.* $\tau = 2\pi\sqrt{(l \sec\alpha)/2g}$)

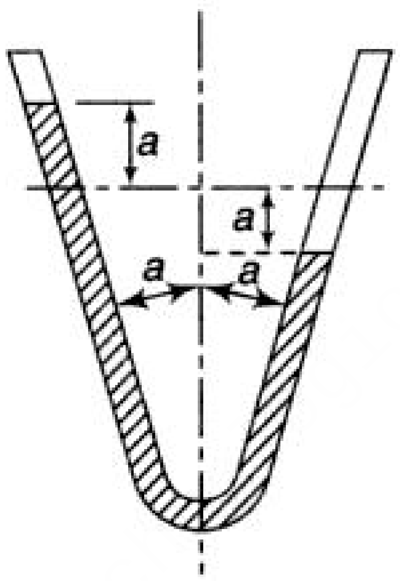

Fig. B

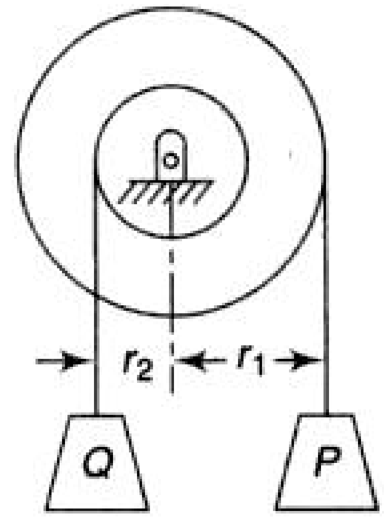

Fig. C

5. If the system in Fig. C is released from rest in the configuration shown, find the velocity v of the falling weight P as a function of its displacement x. Neglect friction and inertia of the pulleys and assume the following numerical data: $P = Q = 44.5$ N, $r_1 = 150$ mm, $r_2 = 100$ mm, $x = 3$ m. (*Ans.* $v = 3.66$ m/s)
6. The two blocks in Fig. D have weights $P = 44.5$ N, $Q = 22.5$ N and the coefficient of friction between the block P and the horizontal plane is $\mu = 0.25$. If the system is released from rest and the block Q falls a vertical distance $h = 0.6$ m, what velocity v will it acquire? Neglect friction in the pulley and extensibility of the string.

(*Ans.* $v = 1.392$ m/s)

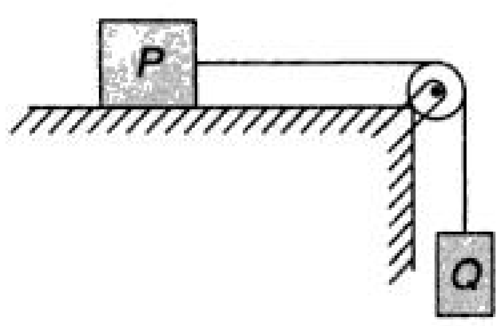

Fig. D

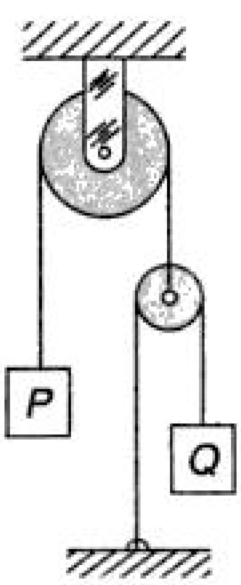

Fig. E

7. If the system in Fig. E is released from rest in the configuration shown, find the velocity v of the block Q after it falls a distance $h = 3$ m. Neglect friction and inertia of the pulleys and assume that $P = Q = 44.5$ N. (*Ans.* $v = 4.82$ m/s)
8. A length l of smooth straight pipe held with its axis inclined to the horizontal by an angle 30° contains a flexible chain also of length l. Neglecting friction and assuming that, after release, the chain falls vertically as it emerges from the open end of the pipe, find the velocity v with which it leaves the pipe. (*Ans.* $v = \sqrt{(3g/2}$)

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we divide the second equation by the first and obtain

$$v_1 + v_1' = v_2 + v_2'$$

or

$$v_1' - v_2' = -(v_1 - v_2) \tag{c}$$

This equation represents a combination of the law of conservation of momentum and conservation of energy. It states that for an elastic impact the relative velocity after impact has the same magnitude as that before impact but with reversed sign. Using this idea in conjunction with that of conservation of momentum, we have for the case of elastic impact.

$$W_1 v_1 + W_2 v_2 = W_1 v_1' + W_2 v_2'$$

$$v_1' - v_2' = -(v_1 - v_2) \tag{51}$$

To illustrate the use of Eq. (51), let us consider several particulars cases. If, in Fig. 7.49, we have $W_1 = W_2$, Eq. (51) become

$$v_1 + v_2 + v_1' + v_2' \tag{d}$$

Subtracting and adding these equations, we find

$$v_1' = v_2 \qquad \text{and} \qquad v_2' = v_1$$

This shows that after an elastic impact, equal weights simply exchange velocities. If the weight W_2 was at rest before impact ($v_2 = 0$), Eq. (d) give

$$v_1' = 0 \qquad \text{and} \qquad v_2' = v_1$$

In this case, the striking ball simply stops after having imparted its velocity to the other ball. This phenomenon can be observed in the case of a moving billiard ball which squarely strikes one that was at rest.

Again, if the two balls were moving toward each other with equal speeds v before impact, an exchange of velocities will simply mean that they rebound from one another with the same speed with which they collided.

As another special case, we assume that $W_2 = \infty$ while W_1 remains finite and further $v_2 = 0$. This will represent the case of an elastic impact of a ball against a flat immovable obstruction, such as dropping a ball on a cement floor. Dividing the first of Eqs. (51) by W_2 we obtain $v_2' = 0$, as would be expected if W_2 is immovable. Then from the second equation, we find $v_1' = -v_1$. This shows that the striking ball rebounds with the same speed with which it hits the obstruction. It must be remembered that each of the examples discussed here assumes perfect elasticity so that no energy is lost during impact.

Semielastic Impact

Under actual conditions we must expect some deviation from perfect elasticity, and owing to this fact there always will be some loss in energy of the system during impact so that the relative velocity after impact is smaller than before and instead of Eq. (c) we must take

$$v_1' - v_2' = -e(v_1 - v_2) \tag{e}$$

where e is a numerical factor less than unity and is called the coefficient of restitution for the materials. Using this, we have for the general case of semi-elastic direct central impact, the following equations:

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8

Curvilinear Translation

8.1 KINEMATICS OF CURVILINEAR MOTION

When a moving particle describes a curved path, it is said to have *curvilinear motion*. We shall now discuss the kinematics of such motion, assuming that the path of the particle is a plane curve.

Displacement

To define the position of a particle P in a plane (Fig. 8.1), we need two coordinates, x and y.

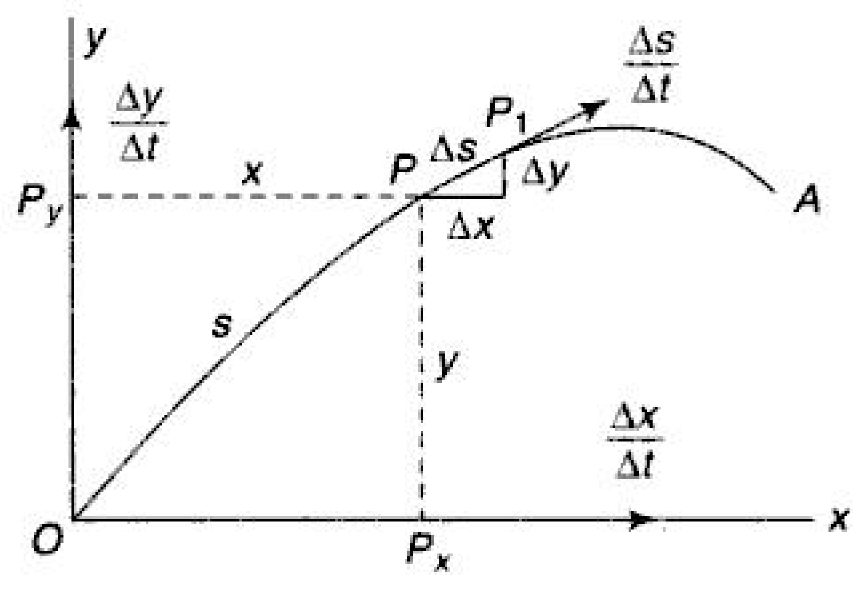

Fig. 8.1

As the particle moves, these coordinates change with time and we have the displacement–time equations

$$x = f_1(t), \qquad y = f_2(t) \tag{53}$$

When these two expressions are given, the motion of the particle in its plane is completely defined.

Instead of Eq. (53), we can also define the motion of a particle in a plane by the equations

$$y = f(x), \qquad s = f_1(t) \tag{53'}$$

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inset diagram of free vectors. Then the corresponding components of the average acceleration will be $\overline{\Delta v}/\Delta t$ will be

$$(a_t)_{av} = \frac{\Delta v_t}{\Delta t}, \qquad (a_n)_{nv} = \frac{\Delta v_n}{\Delta t} \tag{k}$$

We now note from Fig. 8.4 that the angle between the normals at P and P_1 is $\Delta\theta \approx \Delta s/\rho$, where ρ is the radius of curvature of the path at P. Also from the inset diagram of free vectors, we see that $\Delta v_n \approx v\Delta\theta$, while $\Delta v_t \approx v_1 - v = \Delta v$, where Δv without a bar represents only the change in speed of the particle as it moves from P to P_1. Using these relationships, Eq. (k) can be written in the form

$$(a_t)_{av} = \frac{\Delta v_t}{\Delta t} \approx \frac{\Delta v}{\Delta t}, \qquad (a_n)_{av} = \frac{\Delta v_n}{\Delta t} \approx \frac{v}{\rho}\frac{\Delta s}{\Delta t} \tag{k'}$$

As Δt approaches zero, the corresponding limits of these ratios give us the instantaneous tangential and normal accelerations of the particle at the particle point P on its path corresponding to the instant t. Thus we have

$$a_t = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

$$a_n = \lim_{\Delta t \to 0} \frac{v}{\rho} = \frac{\Delta s}{\Delta t} = \frac{v}{\rho}\frac{ds}{dt} = \frac{v^2}{\rho} \tag{56}$$

A study of Eq. (56) shows that the tangential acceleration dv/dt depends only upon the rate of change of speed of the particle, while the normal acceleration, which is always directed toward the center of curvature of the path, depends both upon the square of the speed and the curvature of the path. If the path is a straight line, the curvature is zero and the normal acceleration vanishes. Then we have simply $a = dv/dt$, as obtained in Section 7.1 for the case of rectilinear motion. If the speed of a particle along a curved path is constant, the tangential acceleration vanishes, and we have only normal acceleration v^2/ρ directed toward the center of curvature. We see that high speed around a sharp curve always means very large acceleration.

In summary, we note that total acceleration $\bar{a}$ of a particle in curvilinear motion is a vector. This acceleration may arise as a result of change in magnitude of velocity or change of direction of velocity or both. When we resolve this total acceleration into normal and tangential components, we separate these two effects. The change in speed is accounted for by the tangential acceleration alone, while change of direction of motion is accounted for by the normal acceleration alone.

Examples Examples Examples Examples Examples

1. Small oscillations of the simple pendulum shown in Fig. 8.5 are represented by the displacement–time equation.

$$s = s_o \cos pt \tag{1}$$

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(b) Out of these statements only I and II are true.
(c) Out of these statements only I and III are true.
(d) Out of these statements only I, II and III are true.

[*Ans.* (d)]

PROBLEM SET 8.1

1. Construct the acceleration–time diagram for the motion of a particle that travels with constant speed $v_0 = 26.4$ m along the path *ABCD* in Fig. A, passing points *A*, *B*, *C*, *D* at the instants $t = 0, 10, 20, 30$ sec, respectively.

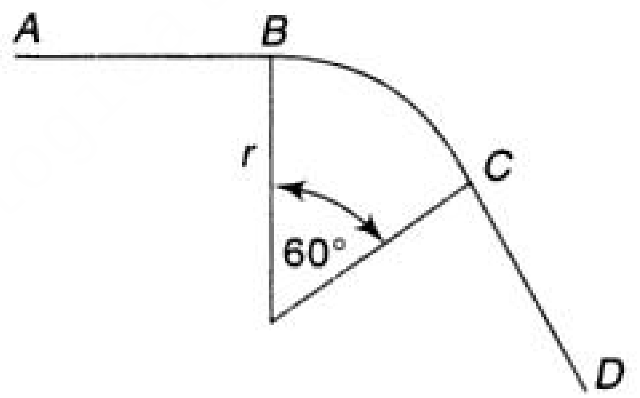

Fig. A

2. At the instant $t = 0$, a locomotive starts to move with uniformly accelerated speed along a circular curve of radius $r = 600$ m and acquires by the end of the first 60 s of motion a speed equal to 24 kmph. Find the tangential and normal acceleration at the instant $t = 30$ s. (*Ans.* $a_t = 110$ mm/s^2; $(a_n)_{t=30} = 18.15$ mm/s^2)
3. Considering only rotation of the earth, determine the resultant acceleration of a point on its surface at the latitude 40°N. Assume the radius of the earth $r = 6336$ km. (*Ans.* $a = 25.5$ mm/s^2)
4. Prove that, if the ends *A* and *B* of a bar *AB* of length $2l$ (Fig. B) are constrained to move along the *y*- and *x*-axes, respectively its mid-point *C* describes a circle of radius l with center at *O* while any intermediate point *D* describes an ellipse with major and minor semiaxes $l + b$ and $l - b$, respectively.

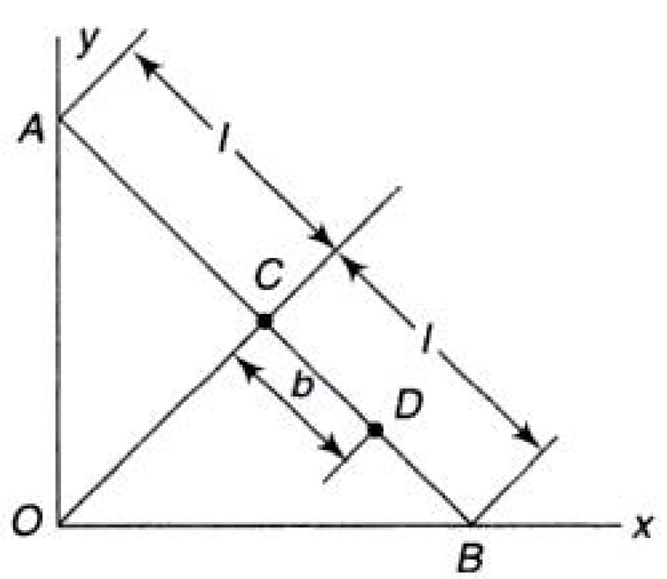

Fig. B

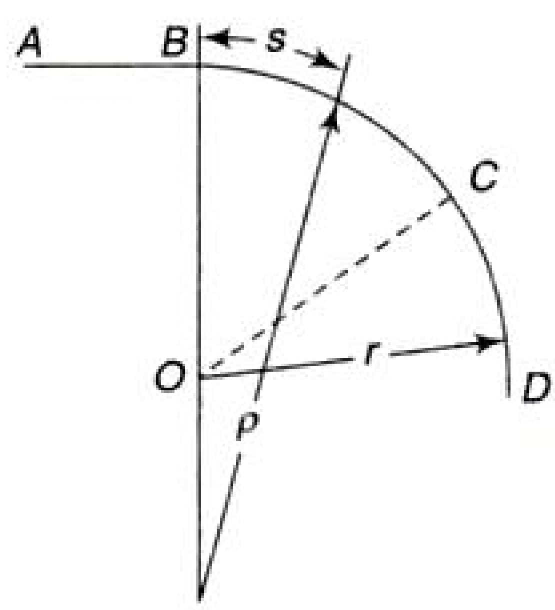

Fig. C

5. In Fig. C, the portion *AB* of a railroad track is straight, the portion *BC* is a spiral the radius of curvature of which is $\rho = r^2/s$, and the portion *CD* is a circle of radius *r*. Construct the acceleration–time diagram for a locomotive moving with constant speed $v_0 = 26.4$ m/s from *A* to *D* if $r = 300$ m.

from which

$$r_m = 2h + \frac{r^2}{8h}$$

2. An airplane is moving with a horizontal velocity v at a height h above a level plane (Fig. 8.11).
If a projectile is fired from a gun at the instant when the plane is vertically above the gun, what must be the angle of elevation α and what is the minimum initial velocity v_o of the projectile in order to hit the airplane?
Solution: It is evident that we must have

$$v_0 \cos \alpha = v \qquad (l)$$

from which

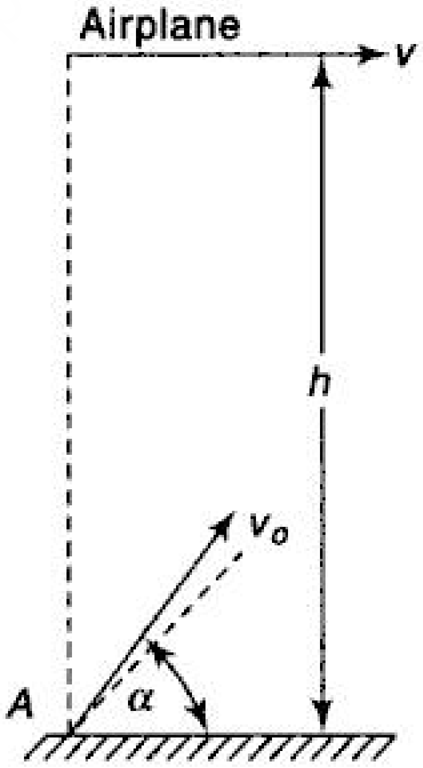

Fig. 8.11

$$\alpha = \text{arc cos}\,\frac{v}{v_0}$$

Also, if the projectile is ever to attain the height h, we must have

$$v_o \sin \alpha \geq \sqrt{2gh} \qquad (m)$$

Eliminating α between Eqs. (l) and (m), we find

$$v_o \geq \sqrt{v^2 + 2gh}$$

3. Referring to Fig. 8.12, determine the range r on an inclined plane for a projectile fired from point O with initial velocity v_o and angle of elevation α. Find also, the maximum range on the inclined plane.

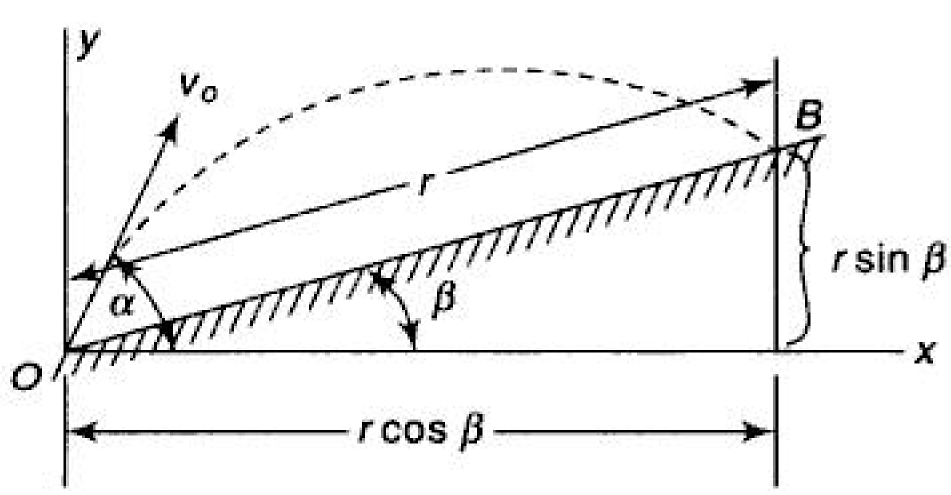

Fig. 8.12

Solution: Taking coordinate axes x, y, as shown, and using Eq. (e), we write for the coordinates of point B where the projectile hits the inclined plane, the following expressions:

$$x = v_o \cos \alpha t = r \cos \beta$$

$$y = v_o \sin \alpha\, t - \frac{1}{2} g t^2 = r \sin \beta \qquad (n)$$

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3. A particle of weight W attached to a fixed point O by a string of length l whirls in horizontal circular path radius r with uniform speed v so that the string generates a cone of height $h = \sqrt{l^2 - r^2}$ [Fig. 8.15(a)].

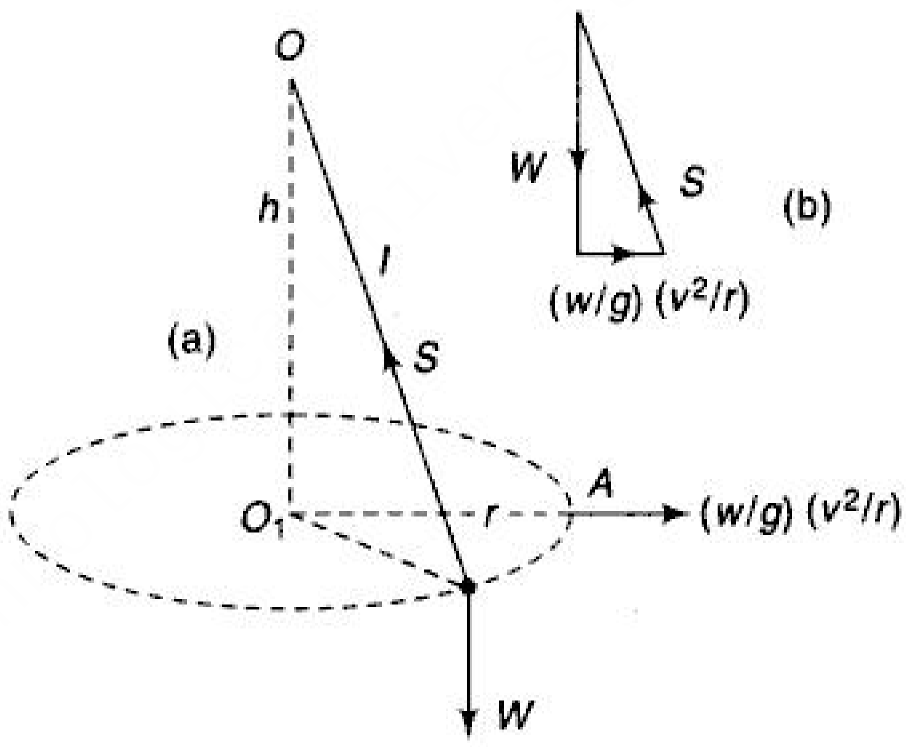

Fig. 8.15

Determine the relation between v, r and h and also the tensile force S in the string during such motion.

Solution: Since the particle moves with uniform speed in a circular path, it has only normal acceleration v^2/r directed toward the center O_1 of the circle and the corresponding inertia force is Wv^2/gr directed outward as shown in the figure. This inertia force together with the weight W and the tensile force S are in equilibrium and must build a closed triangle of forces as shown in Fig. 8.15(b). Form the similarity between the triangle of forces and ΔOAO_1, we may write

$$r : h = \frac{Wv^2}{gr} : W$$

From which

$$v = r\sqrt{\frac{g}{h}} \tag{e}$$

Equation (e) expresses the required relation between v, r and h. Also, from the triangle of forces, we obtain

$$S = W\sqrt{1 + \frac{v^4}{g^2 r^2}} \tag{f}$$

Substituting for v, the value given by Eq. (e), this may also be written in the form

$$S = W\sqrt{1 + \left(\frac{r}{h}\right)^2} \tag{f'}$$

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10. What is the minimum uniform speed that a man and motorcycle of weight W can have in going around the inside of a vertical circular drum of radius r (Fig. G) in order to prevent slipping down the wall if the coefficient of friction between the tyres and the wall is μ? When the motorcycle is running at this speed, what angle α must its middle plane make with the horizontal in order to prevent tipping down?

(*Ans.* $v = \sqrt{(g/m)(r-c)}$; α arc tan μ)

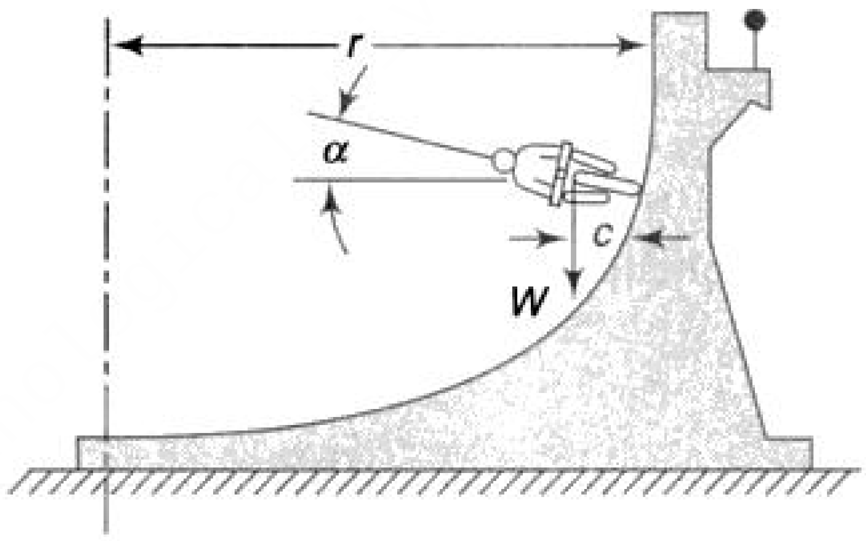

Fig. G

8.5 MOMENT OF MOMENTUM

If a particle A of weight W moves along the curvilinear path CD in the xy plane in Fig. 8.18 and has at any instant the velocity v, its momentum of the particle at any instant can be represented by the vector $\overline{AE}$, the direction of which coincides with that of the velocity, i.e. tangent to the path at the point defining the instantaneous position of the particle. The *moment of momentum* of the particle with respect to the origin O is defined as the product of the momentum and the perpendicular distance OB (Fig. 8.18).

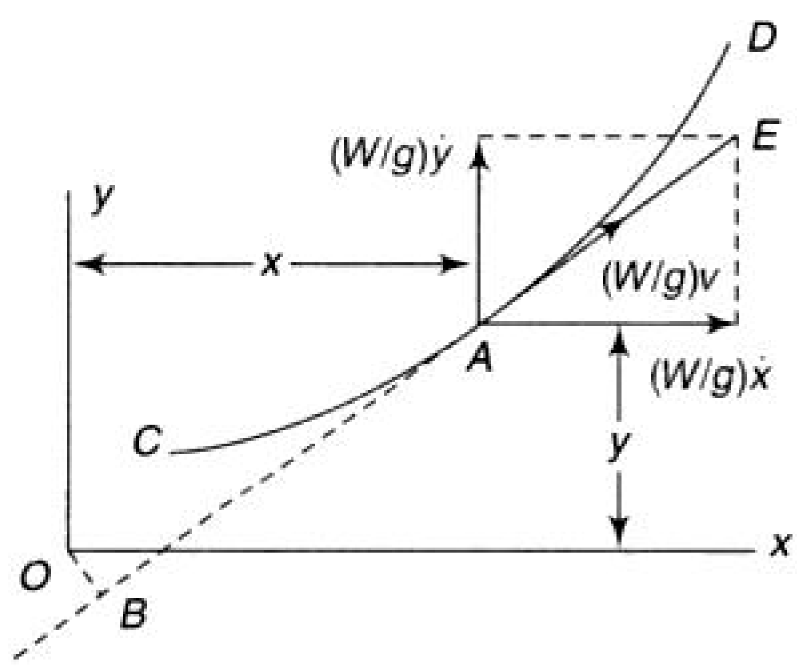

Fig. 8.18

In expressing this moment of momentum of the particle, it is convenient to resolve the momentum vector $\overline{AE}$ into two rectangular components,

$$\frac{W}{g}\dot{x} \quad \text{and} \quad \frac{W}{g}\dot{y}$$

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Solution: Considering the particle in any position along its path as shown, we see that the only forces acting on its are the constant vertical gravity force W and the reaction R exerted by the track. In the absence of friction, the reaction R is always normal to the direction of motion and does not produce work; hence this unknown force can be eliminated from consideration. Projecting the gravity force W onto the tangential direction, we obtain

$$S = -W \frac{dy}{ds}$$

and the energy Eq. (61a) becomes

$$\frac{W}{g}\frac{v_2^2}{2} = \int_{x_1}^{x_2} S ds = -W \int_{y_1}^{y_2} dy = W(y_1 - y_2)$$

from which

$$v_2 = \sqrt{2g(y_1 - y_2)} \tag{e}$$

We see from this example that, provided there is no friction between the particle and the track, the speed gained between B_1 and B_2 is the same as would have resulted from a free fall through the height $(y_1 - y_2)$. This conclusion will be very useful in the solution of further problems.

2. A projectile is fired from point O with initial velocity v_o and angle of elevation α as shown in Fig. 8.24. Find the velocity v of the projectile as a function of its altitude y above the ground.

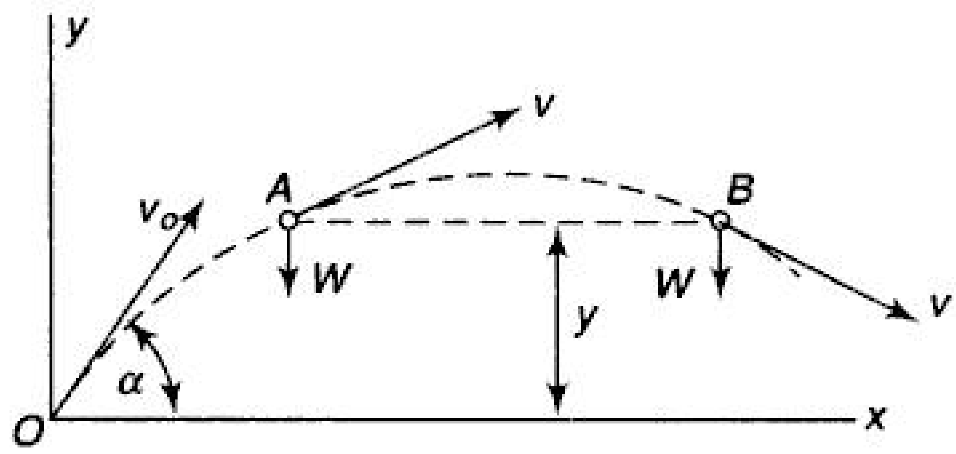

Fig. 8.24

Solution: Neglecting air resistance, we have to consider only the constant vertical gravity force so that $X = 0$ and $Y = -W$, and the energy equation (61b) becomes

$$\frac{W}{g}\frac{v^2}{2} - \frac{W}{g}\frac{v_0^2}{2} = -\int_0^y W dy = -Wy$$

from which

$$v = \sqrt{v_0^2 - 2gy} \tag{f}$$

We note that the speed of the projectile at any given elevation y is independent of the angle of elevation at which it was fired. However, in using Eq. (f), we must select only values of $y \le v_0^2 \sin^2\alpha/2g$, which is the maximum height to which the projectile can rise (Section 8.3).

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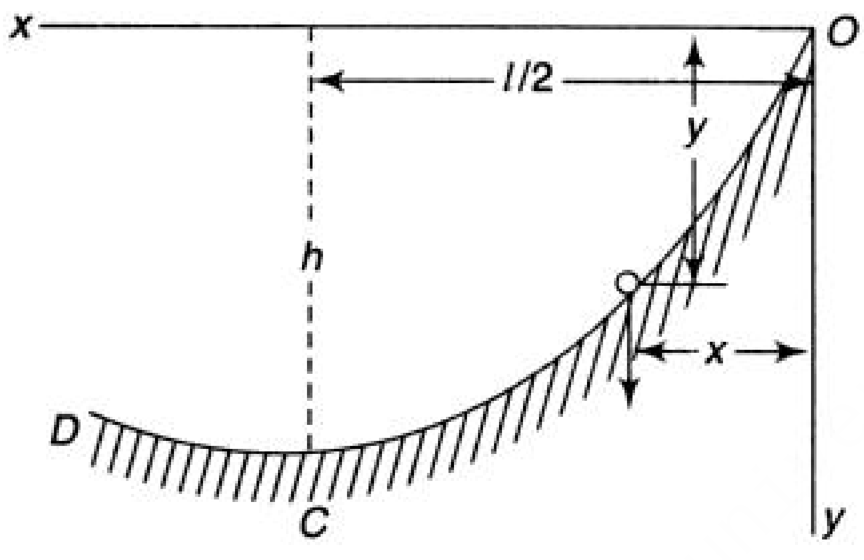

Fig. G

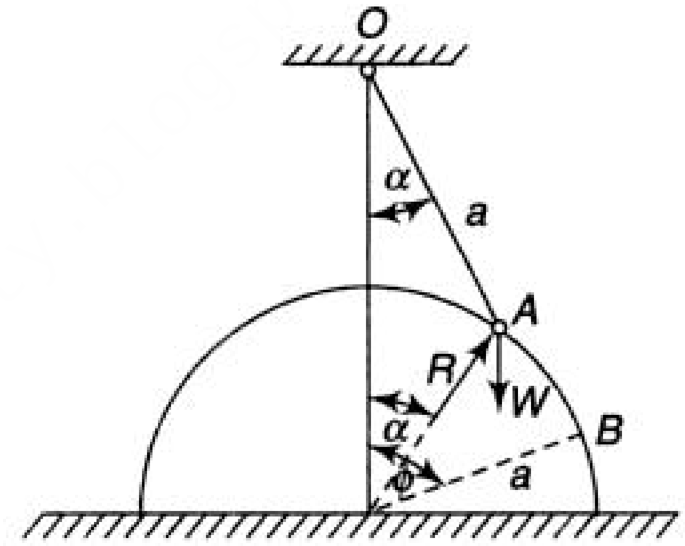

Fig. H

11. A block of weight W starts from rest at A and slides in a vertical plane along the arc AB of smooth circular cylinder of radius r (Fig. I). At point B it leaves the cylinder and travels along the dotted trajectory BC. Neglecting friction, determine the distance b defining the position of the point C at which the block strikes the horizontal plane CD. (*Ans.* $b = 1.46\ r$)

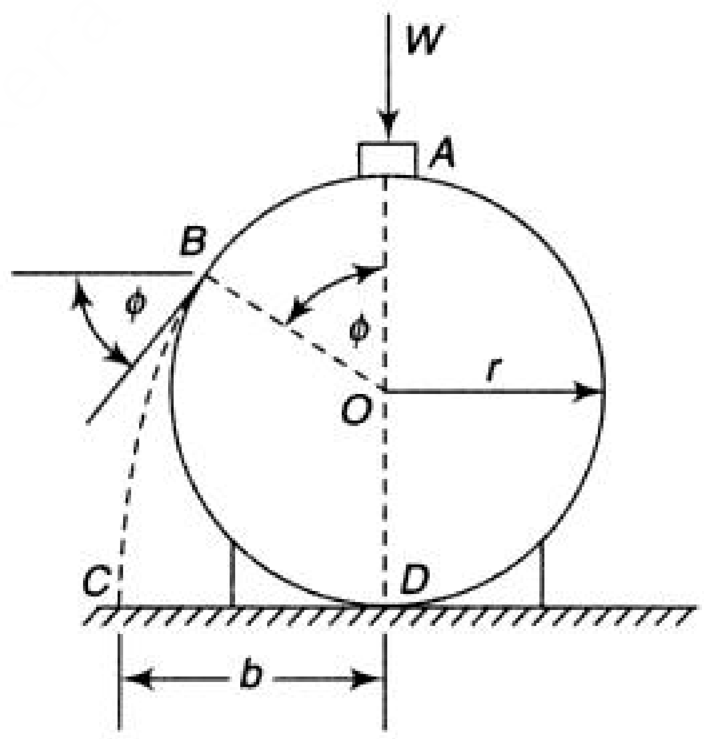

Fig. I

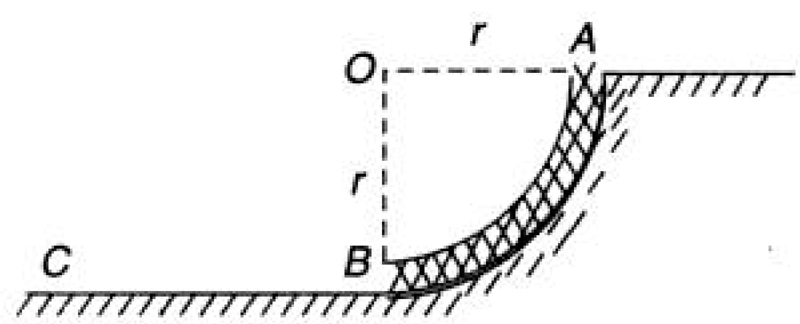

Fig. J

12. A smooth tube AB in the form of a quarter circle of mean radius r is fixed in a vertical plane and contains a flexible chain of length $\pi r/2$ and weight $w\pi r/2$ as shown in Fig. J. If released from rest in the configuration shown, find the velocity v with which the chain will move along the smooth horizontal plane BC after it emerges from the tube. (*Ans.* $v = \sqrt{0.728\,gr}$)

13. In Fig. K, the pendulum A released from rest in the horizontal position O_1A_1 swings down and strikes the pendulum B initially at rest in the vertical position O_2B_1. If the impact is perfectly elastic, find the value of the angle φ where the axial force in OB changes from compression to tension as the ball of the lower pendulum falls along the path $B_1B_2B_3$. The weights of the bobs are $W_a = 4.45$ N, $W_b = 8.9$ N, and the lengths $l = 300$ mm. (*Ans.* $\varphi = 15°38'$)

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The corresponding time for one revolution of the crank is obviously

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}} \tag{k}$$

We shall see later that this is the period of free oscillations of the bar OA, vertically suspended at O. Thus, if the frequency of forced oscillations produced by the rotating crankshaft coincides with the natural frequency of the bar OA as a pendulum, no action from the bar AB is required to maintain these oscillations.

When ω becomes greater than the value given by Eq. (j), expression (i) changes sign and we have compression in the bar AB for the conditions as shown in the figure.

3. Find the pulsating torque acting in a shaft if the rotation of a flywheel of weight $W = 8900$ N at the end of the shaft is defined by the equation

$$\theta = \omega t + \alpha \sin \omega t$$

The average angular speed of the flywheel is 180 rpm, and the maximum deviation from this average angular speed is 0.01ω. The radius of gyration of the flywheel with respect to the axis of rotation is $i = 1.25$ m.

Solution: From the numerical data given above, we have

$$\omega = 180\frac{2\pi}{60} = 6\pi \text{ rad/s}, \qquad \alpha = 0.01$$

$$I = \frac{8900}{9.81} \times 1.25^2 = 1417.6 \text{ kg m}^2$$

Substituting these values in Eq. (67), we obtain

$$M = I\ddot{\theta} = -I\alpha\omega^2 \sin \omega t = 5036.8 \sin \omega t \text{ N m.}$$

Important Terms and Concepts

Equation of motion of a rigid body about a fixed axis
D'Alembert's principle
Moment of inertia of the body with respect to the axis of rotation
Inertia force

SUMMARY

- In case of a rigid body rotating about a fixed axis, the algebraic sum of the moments of external forces and inertia forces, with respect to the axis of rotation, is zero.
- Two kinds of problems can be solved:

  In the first case, the motion is known, i.e. the angle of rotation θ is given as a certain function of time, t, and it is required to find the resultant moment M that produces this motion.

  In the second case, the moment M of external forces with respect to the axis of the rotation is given and it is required to find the motion that this moment produces, i.e. to find an expression for the angle of rotation θ as a function of time.

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The corresponding number of complete revolutions of the rotor is

$$\frac{\theta}{2\pi} = \frac{W\omega^2 ra}{8\pi\mu gPl} \tag{h'}$$

Taking, for example, $W = 1717.7$ N, $n = 1{,}800$ rpm, $r = 250$ mm, $a = 375$ mm, $l = 900$ mm, $\mu = \frac{1}{3}$, and $P = 667.5$N, Eq. (h′) gives 116 revolutions required to stop the rotor.

3. Two rotors having weights W_1 and W_2 and moments of inertia I_1 and I_2 are mounted in bearings and geared together as shown in Fig. 9.8.

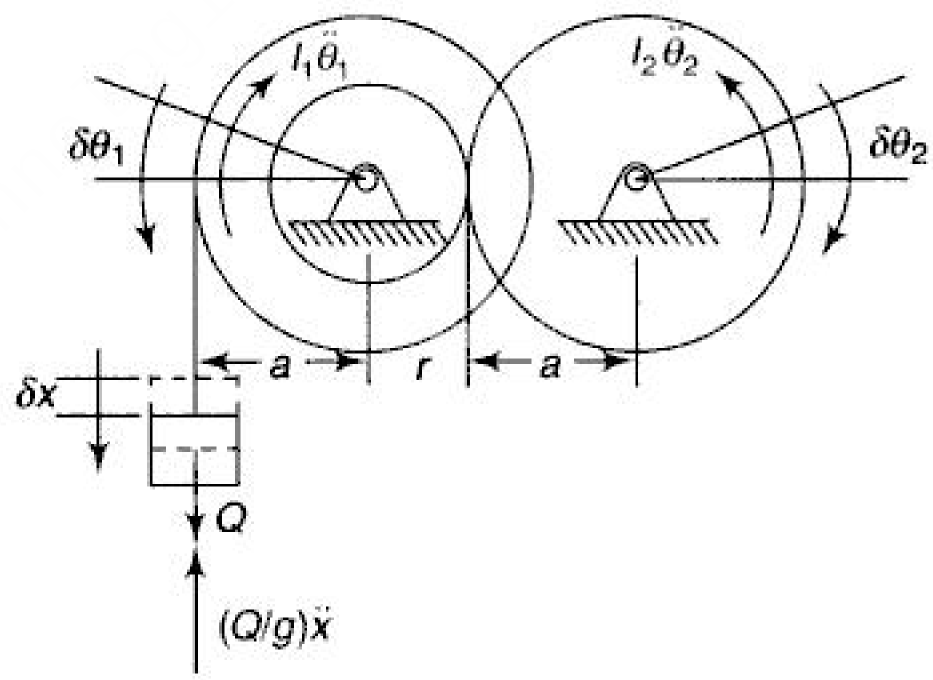

Fig. 9.8

A flexible cord wound around the circumference of one of the rotors carries a weight Q at its free end. Neglecting friction, find the acceleration of the weight Q if it is allowed to fall vertically downward.

Solution: Let $\ddot{x}$ be the acceleration of the falling weight; $\ddot{\theta}_1$ and $\ddot{\theta}_2$ the corresponding angular accelerations of the two rotors. Then introducing the inertia force $-(Q/g)\ddot{x}$ and the inertia couples $-I_1\ddot{\theta}_1$ and $-I_1\ddot{\theta}_2$ as shown in the figure, we put the system in dynamic equilibrium.

We now give a virtual displacement δx to the weight Q and let $\delta\theta_1$ and $\delta\theta_2$ be the corresponding angular displacement of the rotors directed as shown. Then the equation of virtual work for the system becomes

$$\left(Q - \frac{Q}{g}\ddot{x}\right)\delta x - I_1\ddot{\theta}_1\delta\theta_1 - I_2\ddot{\theta}_2\delta\theta_2 \tag{i}$$

Introducing the notations,

$$I_1 = \frac{W_1}{g}i_1^2 \qquad \text{and} \qquad I_2 = \frac{W_2}{g}i_2^2$$

and noting that

$$\ddot{\theta}_1 = \frac{\ddot{x}}{a}, \qquad \ddot{\theta}_2 = \frac{\ddot{x}}{a}\frac{r}{a}$$

while

$$\delta\theta_1 = \frac{\delta x}{a}, \qquad \delta\theta_2 = \frac{\delta x}{a}\frac{r}{a}$$

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From Eq. (71) we see that the rotation of the disk is periodical and that a complete cycle requires the time

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{Il}{GJ}} \tag{72}$$

This expression represents the *period* of torsional vibration.

If all dimensions of the torsional pendulum (Fig. 9.9) are increased n times, the moment of inertia I of the disk is increased in the proportion n^5 and the polar moment of inertia J, in the proportion n^4. Hence the period τ, from Eq. (72), increase in the same ratio n as the linear dimensions of the system. Thus by increasing the size of the torsional pendulum but keeping it geometrically similar, we make the period of torsional vibration larger and the vibrations slower.

The *frequency* of torsional vibration of the pendulum from Eq. 72 is

$$f = \frac{1}{\tau}\frac{1}{2\pi}\sqrt{\frac{GJ}{Il}} \tag{73}$$

It should be noted that frequency is increased by increasing the torsional rigidity of the shaft and decreased by increasing the moment of inertia of the disk.

Examples Examples Examples Examples Examples

1. Determine the frequency of torsional vibration of a horizontal bar AB of weight $W = 17.8$ N and length $a = 0.6$ m (Fig. 9.10), which is suspended at its mid-point on a vertical steel wire of length $l = 0.6$ m and diameter $d = 3.125$ mm. The modulus of elasticity in shear for the wire is $G = 0.86 \times 10^5$ MPa. Neglect the cross-sectional dimensions of the bar and the mass of the wire.

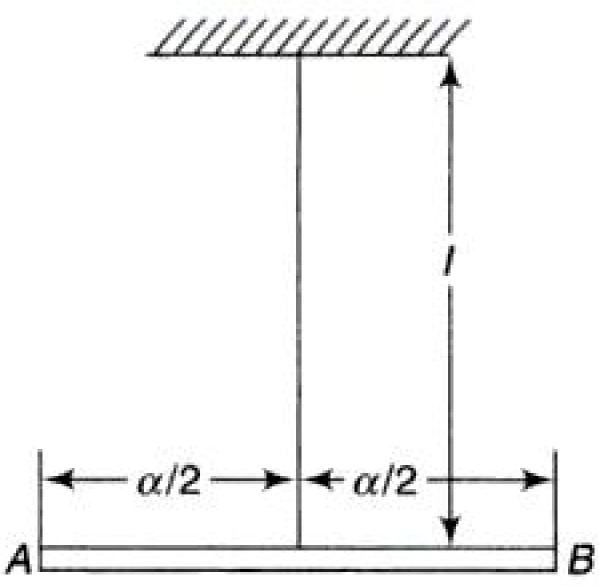

Fig. 9.10

Solution: The moment of inertia of the bar with respect to the axis of the wire is

$$I = \frac{W}{g}\frac{a^2}{12} = \frac{17.8}{9.81}\frac{0.6^2}{12} = 0.544 \text{ kg m}^2 = 5.44 \times 10^4 \text{ kg mm}^2$$

and the polar moment of inertia of the circular cross-section of the wire is

$$J = \frac{\pi d^2}{32} = \frac{\pi}{32} \times (3.125)^4 \text{ mm}^4 = 9.36 \text{ mm}^4$$

Substituting in Eq. (73), we find

$$f = \frac{1}{2\pi}\sqrt{\frac{GJ}{Il}} - \frac{1}{2\pi}\sqrt{\frac{0.86 \times 10^5 \times 9.36}{5.44 \times 10^4 \times 0.6}} = 0.79 \text{ oscillations/sec}$$

$$= 0.79 \text{ Hz}$$

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8. Using the same data as given in Prob. 6 above, calculate the frequency of vibration of the system in Fig. D, if the diameter of the shaft is increased along one-third of its length from 100–125 mm. (*Ans.* f = 8.66 Hz)
9. In Fig. E, a shaft of uniform diameter d carries three right circular disks symmetrically arranged with respect to the mid-point of the shaft. The moments of inertia of the end disks are each I and that of the middle disk is $3I$. If the system performs free torsional vibrations such that the middle disk always rotates opposite to the ones at the ends of the shaft, what are the distances b_1 and b_2 to the nodal cross-sections in the two portions of the shaft? (*Ans.* $b_1 = 2\ l/5$; $b_2 = 2\ l/5$)

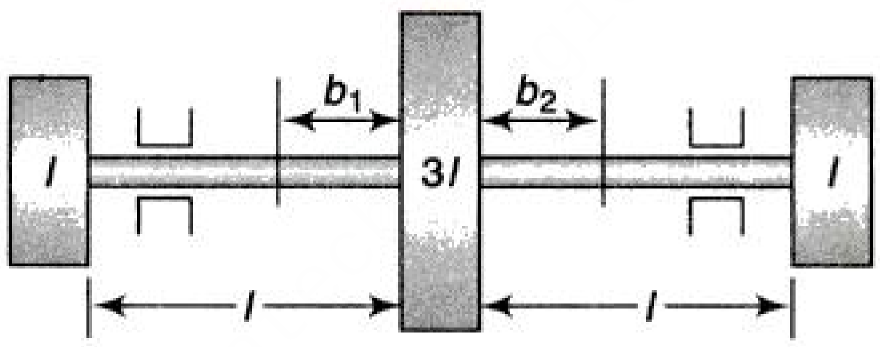

Fig. E

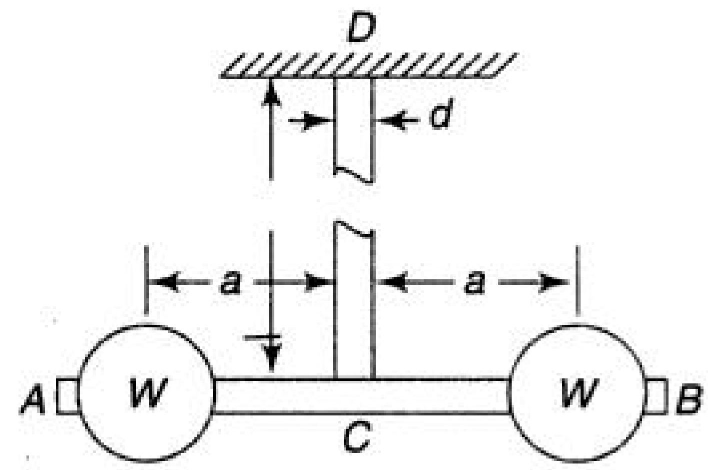

Fig. F

10. A torsion pendulum consists of a rigid bar AB attached at its mid-point C to a shaft DC as shown in Fig. F. Without the two balls, each of weight W = 4.45 N, the observed period of free oscillations is τ_1 = 0.70 s. When the two balls are set at the distance a = 250 mm of the shaft as shown, the observed period of free oscillation τ_2 = 1.00 s. If the shaft DC has length l = 2.75 m and diameter d = 6.25 mm, what is the modulus of elasticity in shear for the material of the shaft?
(*Ans.* $G = 0.82 \times 10^5$ N/mm^2)

9.5 THE COMPOUND PENDULUM

Any rigid body so suspended that it is free to rotate about a fixed horizontal axis through any point O and normal to the vertical xy plane as shown in Fig. 9.13 is called a *compound pendulum*. In its equilibrium position the pendulum hangs with its center of gravity C vertically below the point of support O, and we have the reaction at the support equal and opposite to the gravity force W. If the pendulum is rotated from this position of equilibrium and then released, it performs an oscillatory rotation about the fixed axis through point O. In investigating this motion, we denote by θ the angle of rotation of the body from its equilibrium position at any instant t and apply Eq. (67).

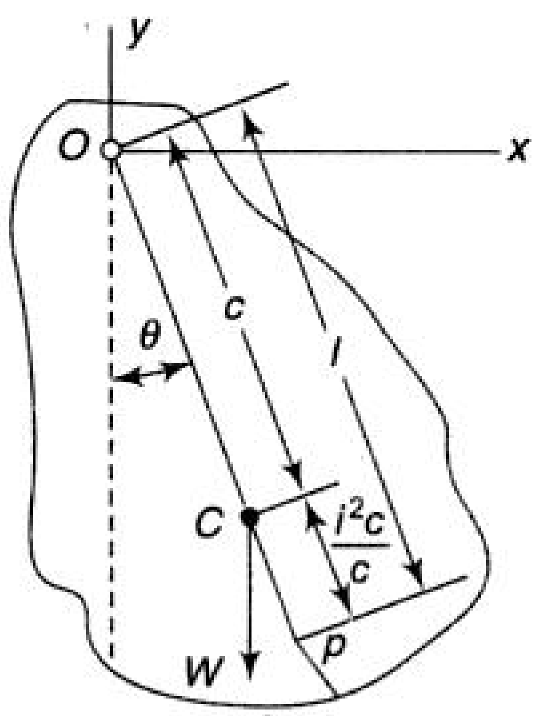

Fig. 9.13

Denoting by c the distance from the axis of rotation to the center of gravity C, the moment of all forces acting on the body, neglecting friction at the axis, is

$$M = -Wc \sin\theta$$

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Solution: From the previous discussion, [Eq. (e)] we know that the minimum period will be obtained by making the distance

$$c = a - x = i_c$$

Also for small oscillations about A, we have by using Eq. (74b) for equivalent length.

$$\tau = 2\pi\sqrt{\frac{a^2 + i_c^2}{ag}}$$

Substituting $a - x$ for i_c in this expression, and squaring both sides, we obtain the following quadratic in x:

$$x^2 + 2ax - \left(2a^2 - \frac{ag\tau^2}{4\pi^2}\right) = 0$$

The roots of this equation are

$$x = a\left(-1 \pm \sqrt{3 - \frac{g\tau^2}{4\pi^2 a}}\right)$$

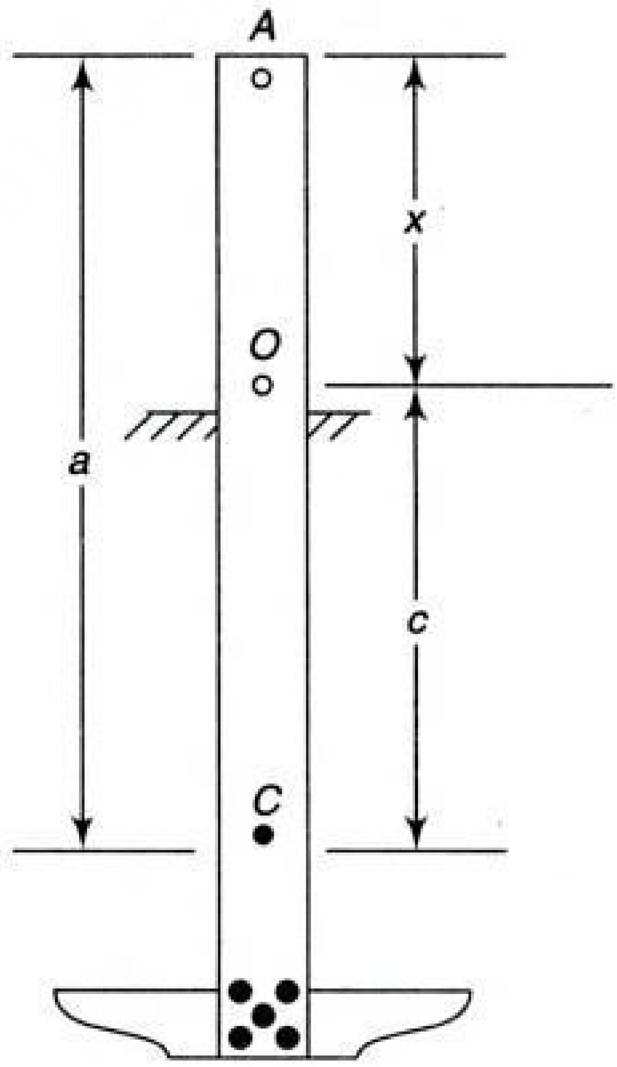

Fig. 9.16

Substituting the given numerical data, we obtain $x = 267.5$ mm. The negative root is of no physical significance.

Important Terms and Concepts

Compound pendulum
Frequency of compound pendulum
Centre of oscillation
Time period of compound pendulum
Equivalent length of compound pendulum
Differential equation for compound pendulum

SUMMARY

- Any rigid body so suspended that it is free to rotate about a fixed horizontal axis through any point O and normal to the vertical xy plane is called a *compound pendulum*.
- In its equilibrium position, the pendulum hangs with its center of gravity vertically below the point of support, and we have the reaction as the support equal and opposite to the gravity force. If the pendulum is rotated from this position of equilibrium and then released, it performs an oscillatory rotation about the fixed axis through point of support.
- The equivalent length of the compound pendulum is larger than the distance from the axis of rotation to the center of gravity.
- The theory of the compound pendulum is often useful in making an experimental determination of the moment of inertia of a body with respect to a given central axis.
- The maximum frequency of oscillation of compound pendulum will be obtained when the center of suspension is at the distance equal to the radius of gyration from the center of gravity of the body.

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Solution: Measuring from the position of equilibrium, a small angle of rotation θ of the bar AB in the horizontal plane, the corresponding angle of inclination α of each string with the vertical, as we see from Fig. 9.20(b), is

$$\alpha = \frac{l_1\theta}{2l} \tag{g}$$

Resolving the load $W/2$ transmitted to each string into two components as shown in Fig. 9.20(b), we obtain a couple of forces $W\alpha/2$ that are approximately horizontal and tend to bring the bar back to its position of stable equilibrium. The moment of this couple is $l_1 W\alpha/2$, which, using expression (g), becomes $l_1^2\theta W/4l$. Equation (67) for rotation of the bar about a vertical axis through its center of gravity then becomes

$$\frac{W}{g}\frac{l_1^2}{12}\ddot{\theta} = -\frac{l_1^2 W}{4l}\theta$$

which reduces to

$$\ddot{\theta} + \frac{3g}{l}\theta = 0$$

and we have for the period of oscillation

$$\tau = 2\pi\sqrt{\frac{l}{3g}} \tag{h}$$

i.e. the same as for a mathematical pendulum of the length $l/3$.

Important Terms and Concepts

General case of moment proportional to angle of rotation
Rotational oscillations of simple harmonic nature

SUMMARY

- In many practical problems we encounter the case of a rigid body which can rotate about a fixed axis but that in so doing brings into play certain reactive forces which create a moment about that axis proportional to the angle of rotation of the body from its equilibrium position and tending always to restore it to that position. In any such case, if the body is initially disturbed from its position of equilibrium, it will perform rotational oscillations of a simple harmonic nature about the fixed axis and we usually are most interested in determining the period or frequency of these oscillations.

Important Formula

For a harmonic oscillatory rotation of the bar, the equation for the period is given by

$$\tau = 2\pi\sqrt{\frac{W}{3kg}} = 2\pi\sqrt{\frac{m}{3k}}$$

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3. Referring to Fig. 9.24, find the bending couple M exerted on the rotating shaft AB by the slender prismatic bar CD of length $2l$ and weight $W = 2ql$ which is rigidly attached to the shaft at its mid-point. The axis of the bar CD makes the angle α with the axis of the shaft AB.

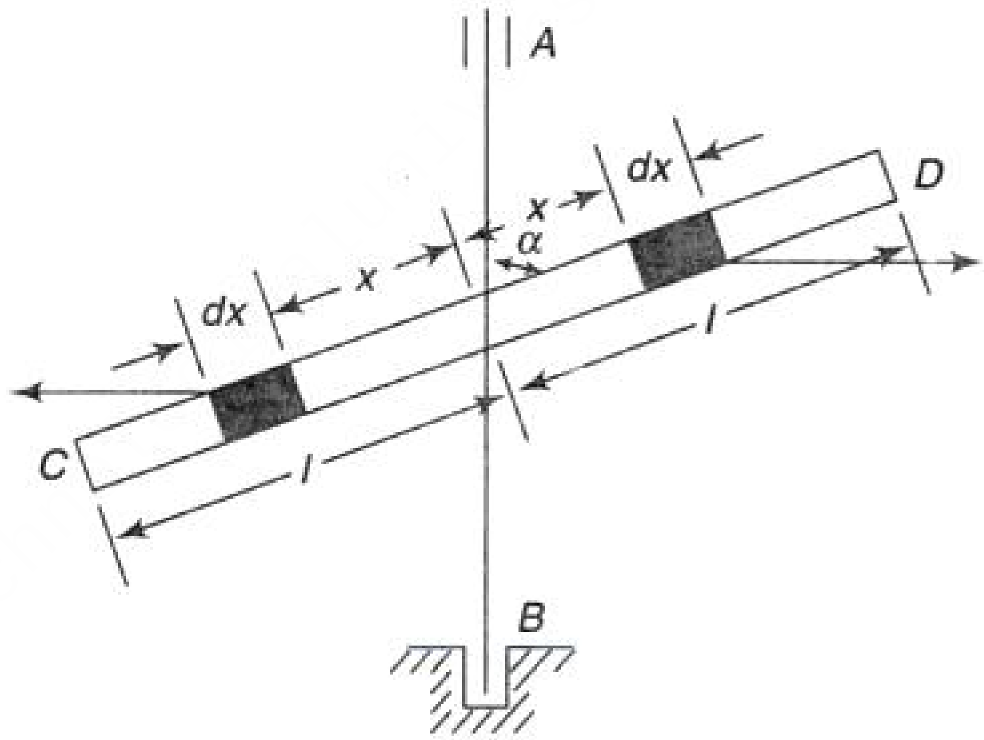

Fig. 9.24

Solution: Considering two infinitesimal elements of length dx, as shown in the figure, we find that the corresponding inertia forces form a couple of the magnitude

$$\frac{qdx}{g}\omega^2 x \sin\alpha \; 2x\cos\alpha = \frac{q\omega^2 \sin 2\alpha}{g}$$

where ω is the angular velocity of rotation. The resultant couple transmitted to the shaft by the inclined bar is

$$M = \int_0^l \frac{q\omega^2 \sin 2\alpha}{g} x^2\,dx = \frac{q\omega^2 l^3}{3g}\sin 2\alpha = \frac{W\omega^2 l^2}{6g}\sin 2\alpha \qquad \text{(c)}$$

which is zero when $\alpha = 0$ or when $\alpha = 90°$ and maximum when $a = 45°$.

4. Calculate the bending couple transmitted to the horizontal shaft AB by the hub of a flywheel, the plane of which makes a small angle α with the plane normal to the shaft (Fig. 9.25). Consider only the mass of the rim of the

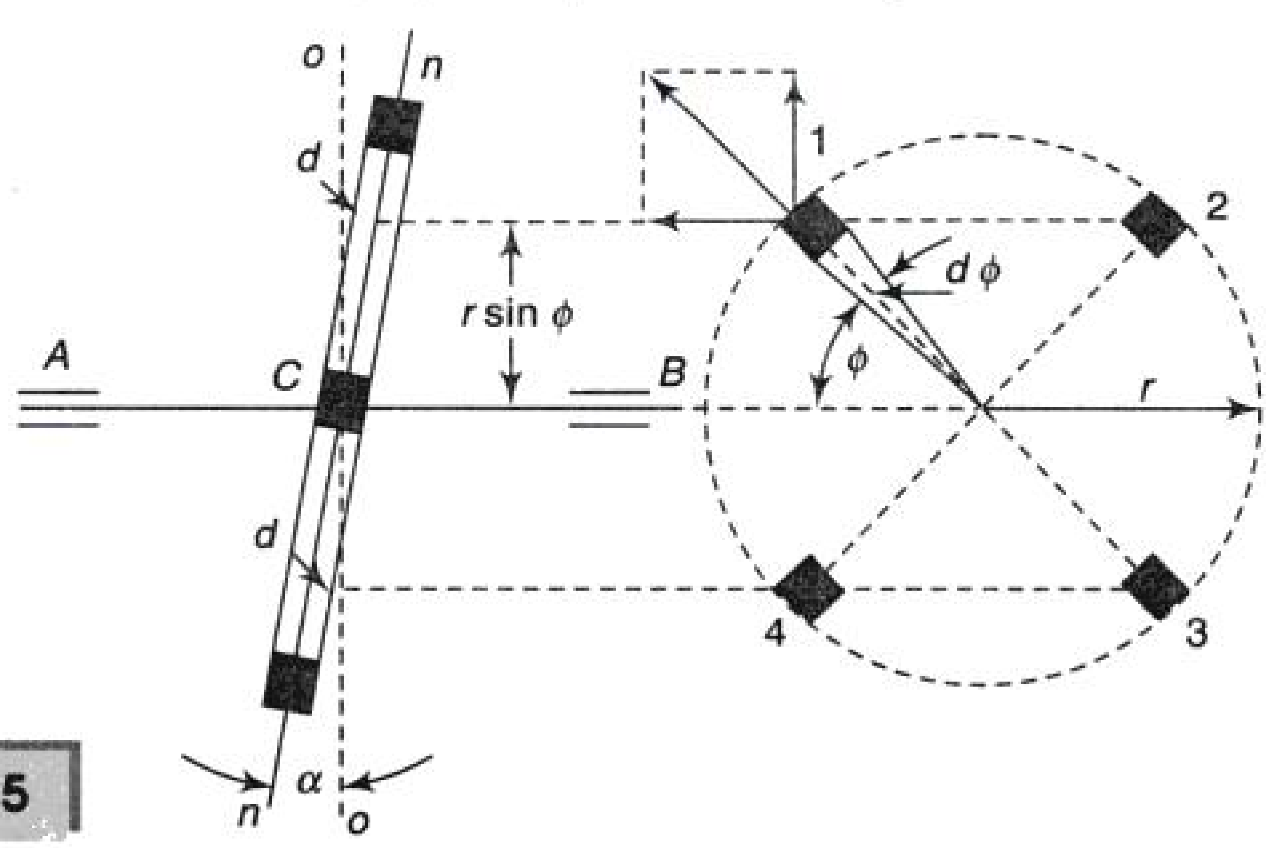

Fig. 9.25

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Referring to Fig. 9.27, we consider a rigid body rotating about a fixed axis through O normal to the plane of the figure which is a plane of symmetry of the body. In this plane, we take coordinate axes x and y through point O as shown. The x-axis is taken through the center of gravity C of the body and both x and y axes are attached to the body so that they rotate with it.

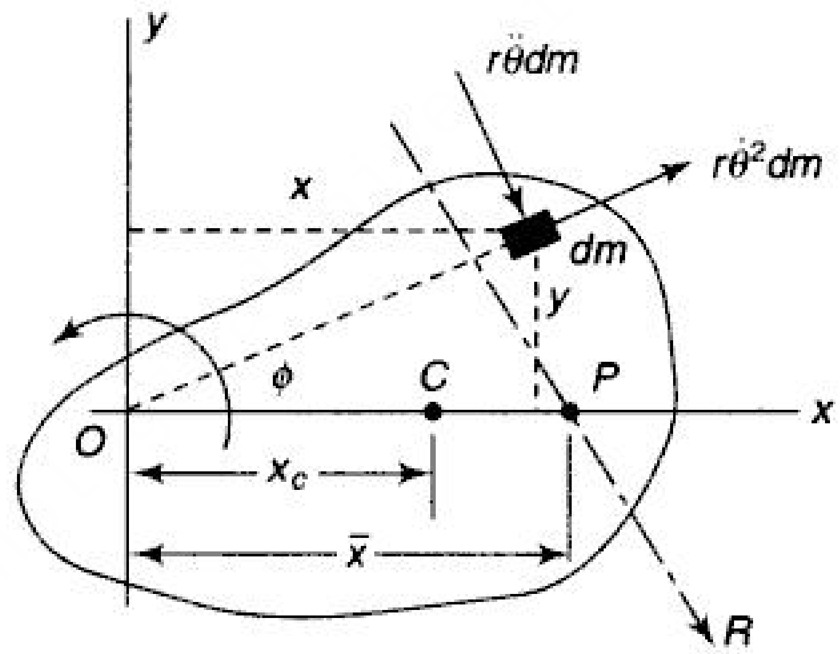

Fig. 9.27

Considering a particle of mass dm at the distance r from the axis of rotation and having coordinates x and y, the normal and tangential inertia forces $2\dot{\theta}^2\,dm$ and $r\ddot{\theta}\,dm$ will be directed as shown in Fig. 9.27. We wish to determine the magnitude and line of their resultant R which, of course, lies in the rotating xy plane. To do this, we denote by X and Y the rectangular components of the force R and by $\bar{x}$ the distance from the axis of rotation to the point P where its line of action cuts the x-axis. Then from statics

$$X = \int r\dot{\theta}^2 dm \cos\varphi + \int r\ddot{\theta} dm \sin\varphi$$

$$Y = \int r\dot{\theta}^2 dm \cos\varphi + \int r\ddot{\theta} dm \sin\varphi$$

Noting from the figure that

$$x = r\cos\varphi \qquad \text{and} \qquad y = r\sin\varphi$$

and remembering that $\dot{\theta}$ and $\ddot{\theta}$ are the same for all particles, these expressions become

$$X = \dot{\theta}^2 \int xdm + \ddot{\theta}\int ydm \tag{a}$$

$$Y = \dot{\theta}^2 \int ydm + \ddot{\theta}\int xdm$$

Observing that

$$\int xdm = \frac{W}{g}x_c \quad \text{and} \quad \int ydm = \frac{W}{g}y_c = 0$$

Eq. (a) reduce to

$$X = \frac{W}{g}\dot{\theta}^2 x_c = m\dot{\theta}^2 x_c \quad \text{and} \quad Y = -\frac{W}{g}\ddot{\theta}x_c = -m\ddot{\theta}x_c \tag{75}$$

We see that these components of the resultant inertia force R represent individually the resultant normal inertia force and the resultant tangential inertia

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θ_0. Find the rectangular components N_0 and S_0 of the reaction at O when the pendulum is in an extreme position. Assume in calculation that $\sin\theta_0 \approx \theta_0$ and $\cos\theta_0 \approx 1$.
(*Ans.* $N_0 \approx W$; $S_0 \approx 0.206W\theta_0$)

5. Referring again to Fig. D, calculate the reaction components N_0 and S_0 as the pendulum swings through the middle position. Use the same data as given in Prob. 4.
(*Ans.* $N_0 = W(1+0.795\theta_0^2)$; $S_0 = 0$)

6. A locomotive wheel 1.8 m in diameter is out of balance to the extent of 890 N at a radius of 0.3 m. The load on the wheel including its own weight is 70 kN. Calculate the maximum and minimum force transmitted to the rail when the speed of the locomotive is 96 km/h.
Hint. Uniform translatory motion of the locomotive does not affect accelerations of the particles of the wheel in its rotory motion; thus inertia forces are the same as in the case of rotation of the wheel about a fixed axis.
(*Ans.* $R_{max} = 86.06$ kN; $R_{min} = 38.45$ kN)

9.9 THE PRINCIPLE OF ANGULAR MOMENTUM IN ROTATION

In discussing motion of a particle in a plane, it was shown (Section 8.5, Eq. (60′)) that the rate of change of *moment of momentum* of the particle with respect to any fixed point in its plane of motion is equal to the moment of all forces acting on the particle with respect to the same point. This principle of moment of momentum can be readily extended to the case of a rigid body rotating about a fixed axis.

Conservation of Angular Momentum

We begin with a consideration of moment of momentum about the axis of rotation (Fig. 9.32).

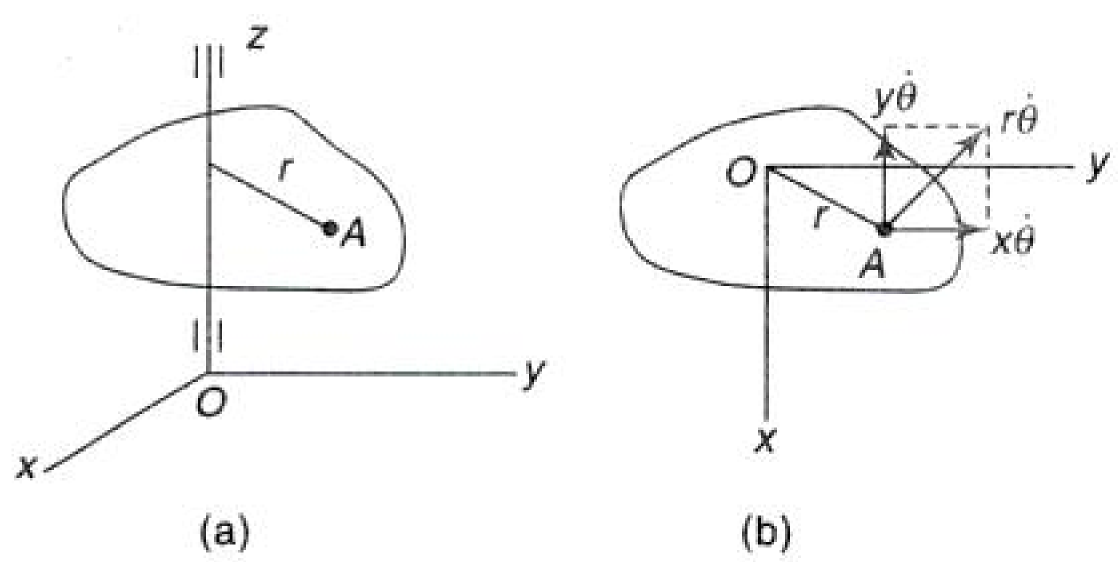

Fig. 9.32

Taking this axis as the z-axis and considering an element dm of the body at a point A, distance r from there; we see that, during rotation, the element describes a circle in the plane perpendicular to the z-axis and having a radius r. If $\dot{\theta}$ is the angular velocity of the body, the velocity of the point A is $r\dot{\theta}$ and the moment of momentum of the element dm with respect to the axis of rotation is

$$r^2\dot{\theta}\ dm \tag{a}$$

The rate of change of this moment of momentum is equal to the moment, about the z-axis, of all forces acting on the element. Summing up the moments of momenta (a) together with the corresponding moments of forces for all elements of the rotating body, we find that the rate of change of this sum must be equal to

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Solution: The initial kinetic energy of the system is zero. After the weight Q has fallen a distance x and acquired a velocity v and the cylinder has acquired an angular velocity $\dot{\theta} = v/r$, the total kinetic energy of the system is

$$\frac{Q}{g}\frac{v^2}{2} + \frac{W}{g}\frac{v^2}{r^2}\frac{v^2}{2r^2} = \frac{v^2}{2g}\left(Q + \frac{W}{2}\right) \qquad \text{(g)}$$

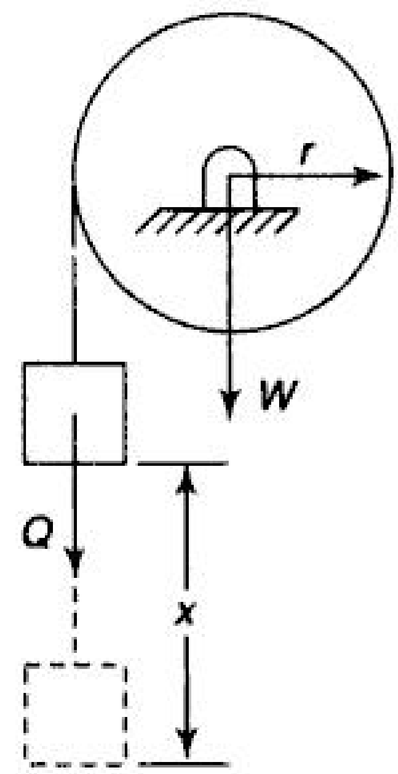

Fig. 9.39

which, in this case, also represents the change in kinetic energy. Neglecting friction at the bearings, the corresponding work of all external forces acting on the system is

$$Qx \qquad \text{(h)}$$

Equating change in kinetic energy (g) to work (h), we obtain

$$\frac{v^2}{2g}\left(Q + \frac{W}{2}\right) = Qx$$

from which

$$v = \sqrt{2gx\frac{Q}{Q + \frac{1}{2}W}}$$

3. If, owing to the impact of a bullet, the ballistic pendulum shown in Fig. 9.35 swings up through an angle θ_m before coming to rest, find the muzzle velocity v of the bullet. Assume that the properties of the pendulum and the weight w of the bullet are given.

Solution: Denoting by $\dot{\theta}$ the initial angular velocity imparted to the pendulum due to impact of the bullet, the initial kinetic energy of the pendulum is $I_0\dot{\theta}_0^2/2$. When the pendulum reaches its extreme position, indicated in the figure by dotted lines, its kinetic energy is zero. Thus the left side of Eq. (82) becomes $I_0\dot{\theta}_0^2/2$. In calculating the work done by external forces, which in this case are the gravity force and the reaction at the axis of rotation, we note that the moment of these forces with respect to the axis of rotation, for any angle of rotation θ, is θ, where the minus sign indicates that the moment acts in the direction opposite to the direction of increasing θ. Thus the total work done during the angle of rotation θ_m is

$$-\int_0^{\theta_m} Wc \sin\theta d\theta = -Wc(1 - \cos\theta_m)$$

Substituting this in Eq. (82), we obtain

$$\frac{I_0\dot{\theta}_0^2}{2} = Wc(1 - \cos\theta_m)$$

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velocity Ω as indicated in the figure? Assume that the angular velocity Ω is small compared with ω.

Solution: In calculating the additional pressures at the bearings due to rotation of the frame, we use the principle of angular momentum. Taking fixed rectangular coordinate axes x, y, z, with origin O at the center of gravity of the disk as shown and the z-axis coinciding with an instantaneous position of the axis of spin, we see that the angular momentum of the disk with respect to the y-axis is small compared with its angular momentum with respect to the z-axis, since ω is large compared with Ω. Thus we are justified in assuming that the angular momentum $I\omega$ with respect to the axis of spin is the resultant angular momentum. With good accuracy, the resultant momentum can be represented by the vector $\overline{OM}$ equal to $I\omega$ and directed along the z-axis in accordance with the right-hand-screw rule. Then, owing to rotation of the frame around the vertical y-axis, the end M of this vector describes a horizontal circle of radius $OM = I\omega$. The velocity of the end M of this vector, perpendicular to the yz plane and equal to $I\omega\Omega$, represents the rate of change of the angular momentum $I\omega$. If the frame has the direction of rotation indicated in the figure, this velocity has the positive direction of the x-axis. Then the vector representing the resultant moment of external forces, which in this case are additional reactions from the bearings due to rotation about the y-axis, must have the same direction. Hence we conclude that the reactions represent a couple in the yz plane as shown in the figure and of such magnitude that

$$Rl = I\omega\Omega \tag{c}$$

from which

$$R = \frac{I\omega\Omega}{l} \tag{d}$$

By putting the bearings A and B on pistons of hydraulic cylinders, the additional forces R, due to the angular velocity Ω, can be determined by measuring the change of oil pressure in the supporting cylinders. In this way formula (d) can be found to be in good agreement with actual conditions.

2. The armature of an electric motor has its axis of spin perpendicular to the longitudinal axis of a rolling ship (Fig. 9.43). The angular velocity of spin of the rotor is ω, and the rolling motion of the ship is assumed to be represented by the equation $\theta = \theta_0 \cos pt$. Find the maximum magnitude of the additional pressures on the bearings of the rotor due to rolling of the ship.

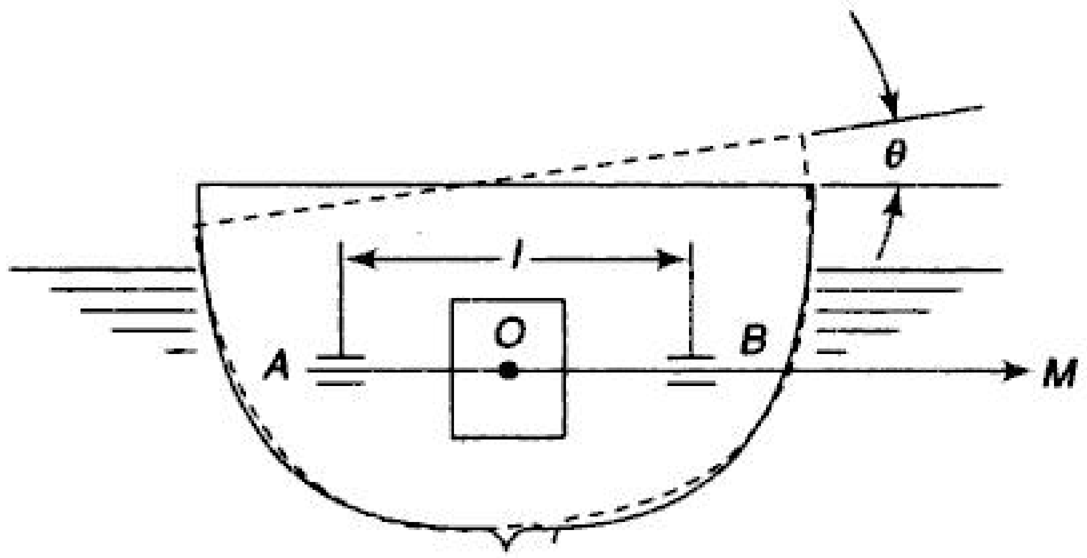

Fig. 9.43

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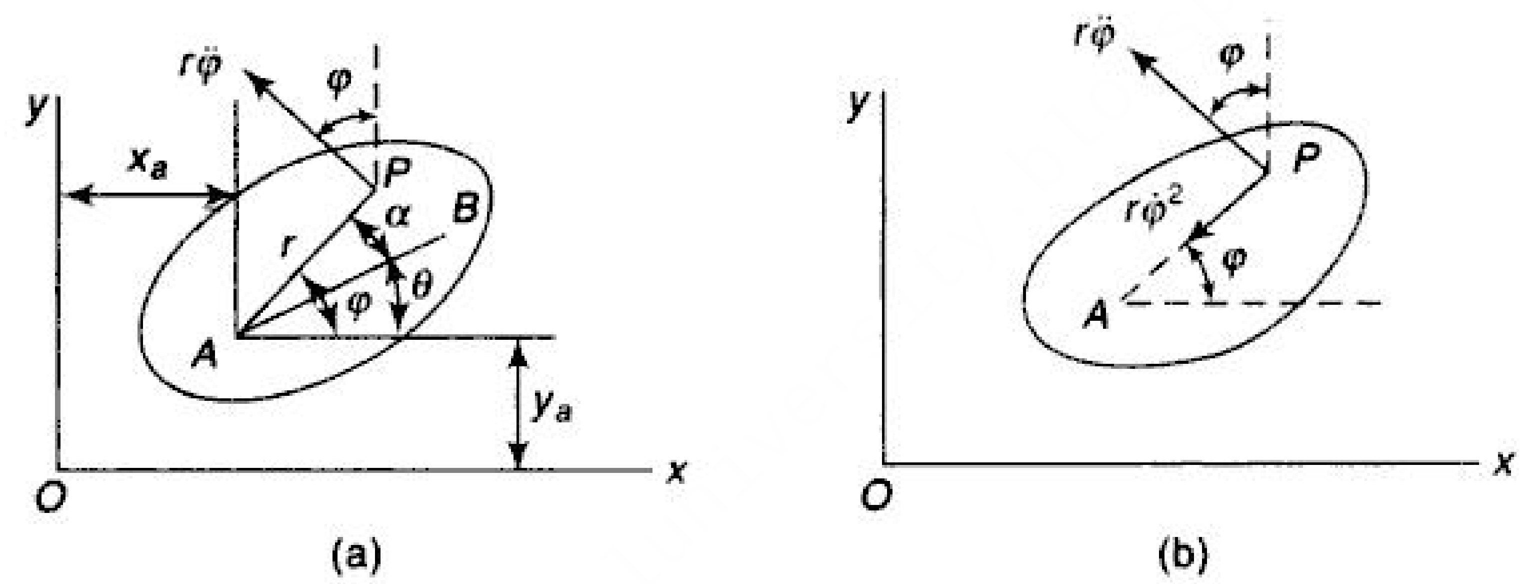

Fig. 10.1

Differentiating each of these expressions once with respect to time and remembering that r is constant while x_a, y_a and φ are changing with time, we find

$$\dot{x} = \dot{x}_a - r\dot{\varphi}\ \sin\varphi, \qquad \dot{y} = \dot{y}_a + r\dot{\varphi}\ \cos\varphi \tag{c}$$

From these equations we see that the $\dot{x}$ and $\dot{y}$ projections of the velocity of P are obtained by adding algebraically to the corresponding projections of the velocity of the pole A the projections of the velocity $r\dot{\varphi}$ due to rotation of the body around point A. Thus, if Eq. (a) defining the motion of a rigid body in a plane are given, the velocity of any point P in the body is obtained as the geometric sum of the velocity of the chosen pole A and the relative rotational velocity of P with respect to A. This may be expressed by the vector equation

$$\bar{v}_p = \bar{v}_a \rightarrow \bar{v}_{p/a} \tag{84}$$

or

$$\bar{v}_p = \bar{v}_a + \bar{v}_{p/a}$$

where the sign $\mapsto$ means that the vectors $\bar{v}_a$ and $\bar{v}_{p/a}$ are to be added geometrically.

Let us consider now the acceleration of the point P of the body shown in Fig. 10.1. Differentiating Eq. (c) with respect to time, we obtain

$$\dot{x} = \ddot{x}_a - r\dot{\varphi}^2 \cos\varphi - r\ddot{\varphi} \sin\varphi \tag{d}$$
$$\dot{y} = \ddot{y}_a - r\dot{\varphi}^2 \sin\varphi - r\ddot{\varphi} \cos\varphi$$

The first term on the right side of each of these equations represents a corresponding projection of the acceleration of the arbitrarily chosen pole A. The second term in each case is seen to represent the projection on the corresponding coordinate axis of the normal component $r\dot{\varphi}^2$ of the relative acceleration of P with respect to A, as indicated in Fig. 10.1(b). Likewise the last term in each equation represents the corresponding projection of the tangential component $r\ddot{\varphi}$ of the relative acceleration of P with respect to A. Thus, we conclude from Eq. (d) that, if Eq. (a), defining the plane motion of a rigid body, are given, a acceleration of any point P of the body is obtained as the geometric sum of the acceleration of the pole A and the relative acceleration of P with respect to A. Expressed in equation form, we have

$$\bar{a}_p = \bar{a}_a \mapsto \bar{a}_{p/a} \quad \text{or} \quad \bar{a}_p = \bar{a}_a + \bar{a}_{p/a} \tag{85}$$

Applications of Eqs (84) and (85) to various problems of plane motion will now be illustrated by the following examples.

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the position B_1A_1' and then rotate it about B_1 through the same angle θ, as before, to come to the position A_1B_1. It is seen that the displacement of the pole depends upon what point in the body is chosen as a pole but that the angle of rotation θ of the figure is independent of the choice of a pole. We can, as a matter of fact, always find in the plane of the figure such a point C that, for displacement of the figure from the position AB to the position A_1B_1, only rotation about C will be required. It is evident from Fig. 10.5 that this center is obtained as the intersection of the perpendicular bisectors CM and CN of the lines AA_1 and BB_1, respectively. For, from the construction, it follows that the two shaded triangles ABC and A_1B_1C are equal. Thus they can be brought into coincidence by rotation θ about point C. If the line AB coincides with the line A_1B_1, the entire figure AB coincides with the figure A_1B_1.

In studying the motion of a plane figure, representing a cross-section of a body performing plane motion, we may consider the continuous motion of the figure as made up of successive infinitesimal displacements from one instantaneous position to the next. Each such infinitesimal displacement may be considered as a rotation of the figure about a certain point in its plane. This point is called the *instantaneous center of rotation* for that particular position of the figure.

Considering, for instance, the cylinder shown in Fig. 10.2(a), which we assume rolls without slipping, it is seen that at any instant the point of contact with the plane has zero velocity. Hence this point is the instantaneous center of the rolling cylinder, and we conclude that the instantaneous velocity of any point on the rim is directed perpendicular to the chord joining it with the point of contact and is equal to the product of the length of this chord and the angular velocity of the cylinder. It should be noted that, as the cylinder moves, the point of contact, i.e. the instantaneous center of rotation, is continually changing its position,[1] in this case it is evidently moving along the line AB.

If we know the directions of the velocities of two points of a moving plane figure for any instantaneous position of the figure, the location of the instantaneous center can always be determined. Consider, for example, a bar AB of length l, the ends of which are constrained to move along the x and y axes (Fig. 10.6). Then the velocity of point A at any instant must be horizontal, and we conclude

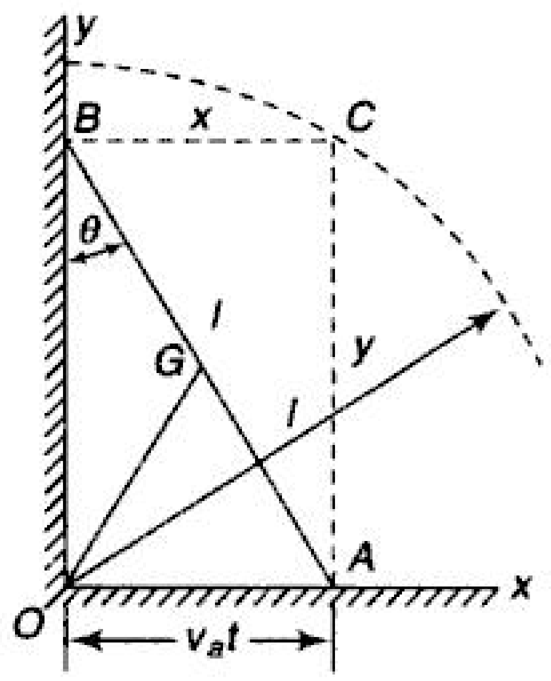

Fig. 10.6

[1]The motion of instantaneous center of rotation should not be confused with the notion of a pole as discussed Section in 10.1. While a pole is always a point fixed in the body, the instantaneous center is not fixed in the body.

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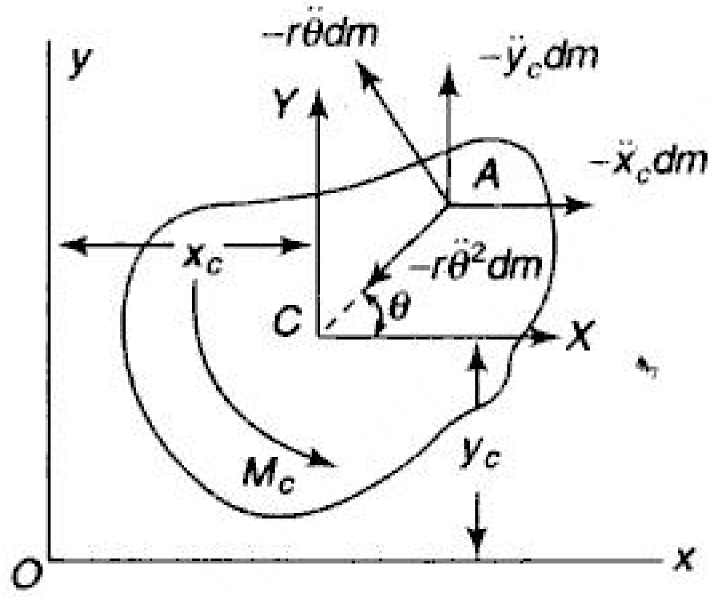

Fig. 10.10

with the pole C and (2) rotation about the axis through C normal to the plane of motion.

As a result of the translation, the particle at A has two components of inertia force $-\ddot{x}_c\, dm$ and $-\ddot{y}_c\, dm$. As a result of rotation about the normal axis through C, it has tangential and normal components of inertia force $-r\ddot{\theta}dm$ and $-r\dot{\theta}^2\, dm$. These four inertia forces are now applied to the particle as shown in Fig. 10.10. Then in accordance with D'Alembert's principle, this system of inertia forces for all particles in the body is in equilibrium with the external applied forces and we may write the following equations of dynamic equilibrium:

$$X - \ddot{x}_c \int dm + \ddot{\theta}\int r \sin\theta\, dm + \dot{\theta}^2 \int r\cos\theta\, dm = 0$$

$$Y - \ddot{y}_c \int dm + \ddot{\theta}\int r \sin\theta\, dm + \dot{\theta}^2 \int r\cos\theta\, dm = 0 \qquad \text{(a)}$$

$$M_c - \ddot{y}_c \int r\cos\theta\, dm + \ddot{x}_c \int r\sin\theta\, dm + \ddot{\theta}\int r^2\, dm = 0$$

where X, Y and M_c represent the applied external forces. Noting that the statical moments

$$\int r\sin\theta\, dm = 0, \qquad \int r\cos\theta\, dm = 0 \qquad \text{(b)}$$

since C is the center of gravity of the body, Eq. (a) become greatly simplified and we have

$$m\ddot{x}_c = \frac{W}{g}\ddot{x}_c = X, \quad m\ddot{y}_c = \frac{W}{g}\ddot{y}_c = Y, \quad I_c\ddot{\theta} = M_c \qquad (86)$$

where $m = W/g = \int dm$ is the total mass of the body and $I_c = \int r^2\, dm$ is its moment of inertia about the centroidal axis normal to the plane of motion.

From the above system of equations, we make a very important observation: in the case of a rigid body performing plane motion under the action of applied external forces defined by X, Y, M_c, the center of gravity of the body moves exactly as if the entire mass were concentrated there and acted upon by forces X, Y. The moment M_c of external forces with respect to the center of gravity has no effect on the motion of that point. At the same time, the body rotates about the axis through the moving center of gravity exactly as if this were a fixed axis. This is sometimes referred to as the principle of independence of translation and rotation in the case of plane motion. It must be observed, however, that this is true

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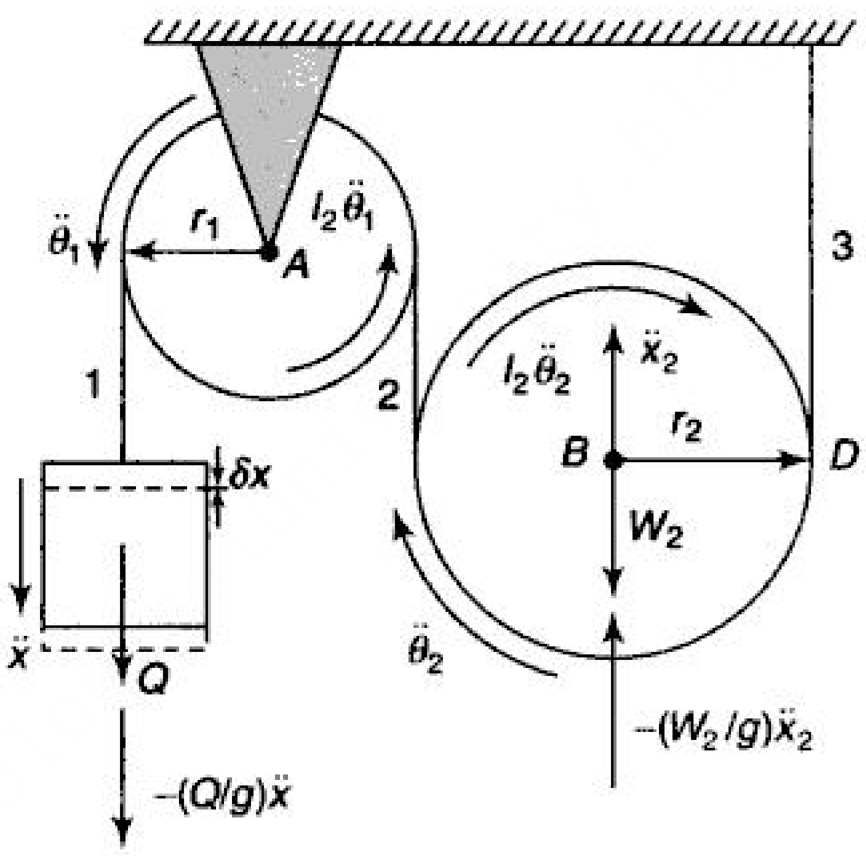

Fig. 10.14

equilibrium. Then making a small virtual displacement δx of the weight Q and denoting by, $\delta\theta_1$, $\delta\theta_2$ and δx_2 the corresponding virtual displacements of the pulleys, we may write one equation of virtual work as follows:

$$\left(Q - \frac{Q}{g}\ddot{x}\right)\delta x - I_1\ddot{\theta}_1\delta\theta_1 - \left(W_2 + \frac{W_2}{g}\ddot{x}_2\right)\delta x_2 - I_2\ddot{\theta}_2\delta\theta_2 = 0 \qquad \text{(a)}$$

Assuming that such quantities as the weights and dimensions of the pulleys and block are given, Eq. (a) contains as unknowns the following four quantities: x, θ_1, θ_2, x_2, as well as their time derivatives. Thus to obtain a solution of the problem, we need three more equations relating these four quantities. These will be obtained from the geometrical conditions of constraint of the system which has only one degree of freedom. This means that when a displacement x of the weight Q is specified, then the angular displacements θ_1, θ_2 of the two pulleys and the displacement x_2 of point B are also known. In this case, we see that if there is no slip between the string and the pulleys, point D is the center of rotation for the pulley B and we must have

$$\theta_1 = \frac{x}{r_1}, \quad x_2 = \frac{x}{2}, \quad \theta_2 = \frac{x}{2r_2} \qquad \text{(b)}$$

Equations (b) relating the coordinates of the several parts of the system are sometimes called *equations of constraint*. Differentiating each of Eq. (b) twice with respect to time, we obtain also

$$\ddot{\theta}_1 = \frac{\ddot{x}}{r_1}, \quad \ddot{x}_2 = \frac{\ddot{x}}{2}, \quad \ddot{\theta}_2 = \frac{\ddot{x}}{2r_2} \qquad \text{(c)}$$

Substituting the values of θ_1, θ_2, x_2 and their time derivatives from Eq. (b) and (c) into Eq. (a) and solving for $\ddot{x}$, we obtain

$$\ddot{x} = g\frac{Q - \frac{1}{2}W_2}{Q + W_1\dfrac{i_1^2}{r_1^2} + \dfrac{1}{4}W_2\left(1 + \dfrac{i_2^2}{r_2^2}\right)} \qquad \text{(d)}$$

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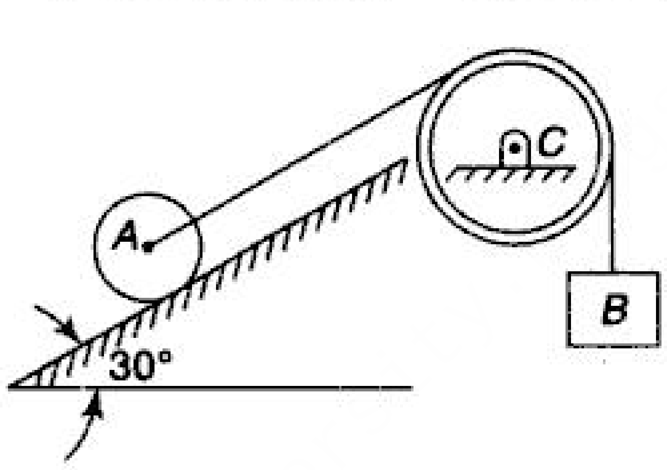

Fig. H

10.5 THE PRINCIPLE OF ANGULAR MOMENTUM IN PLANE MOTION

Let us consider a body moving in the xy plane as shown in Fig. 10.17. Choosing the center of gravity C as a pole, we define the position of this point at any instant by the coordinates x_c, y_c. Through the moving center of gravity, we take also coordinate axes ξ, η, parallel, respectively, to x and y.[2] Then the coordinates of any point A of the body will be

$$x = x_c + \xi,\ y = y_c + \eta \tag{a}$$

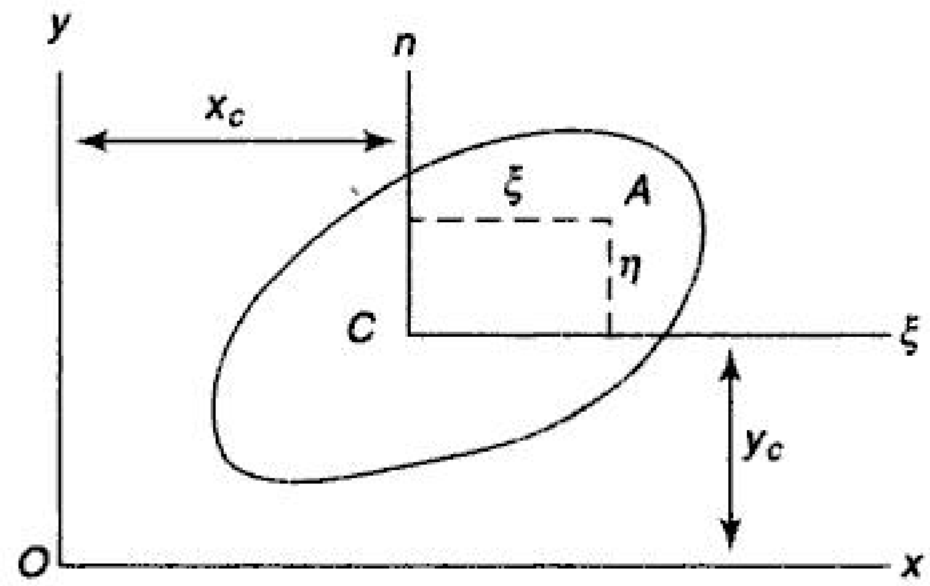

Fig. 10.17

In these expressions, both sets of coordinates x_c, y_c and ξ, η are changing with time; x, y are called absolute coordinates, of point A; ξ, η the relative coordinates.

Differentiating Eq. (a) with respect to time, we obtain

$$\dot{x} = \dot{x}_c + \dot{\xi}, \qquad \dot{y} = \dot{y}_c + \dot{\eta}, \tag{b}$$

With these expressions for the velocity components of any point A in the body, we can easily calculate the total angular momentum of the body with respect to the fixed point O in its plane of motion. By definition, the moment of momentum, with respect to point O, of a particle of mass dm at A is

$$dm\,(\dot{y}x - \dot{x}y) \tag{c}$$

which, by using Eqs. (a) and (b) above, becomes

$$dm[(\dot{y}_c + \dot{\eta})(x_c + \xi) - (\dot{x}_c + \dot{\xi})(y_c + \eta)] \tag{d}$$

Summing up such expressions for all particles of the body, we obtain for the resultant angular momentum with respect to the z-axis the following expression.

[2]These axes move with point C, but they do not rotate with the body.

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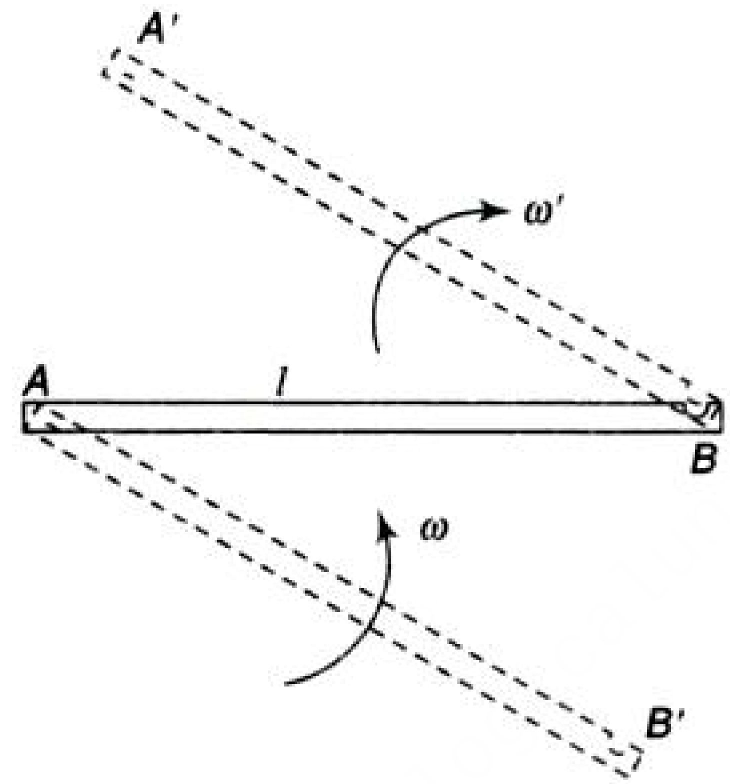

Fig. G

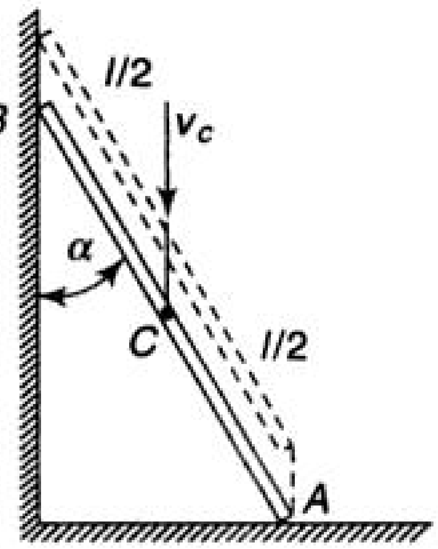

Fig. H

9. Initially, a slender prismatic bar AB of weight W and length l is falling in a vertical plane with velocity v_c and without rotation (Fig. H). The end B of the bar slides freely along a smooth vertical wall and the bar makes an angle a with the wall. Suddenly the end A of the bar comes into contact with a smooth horizontal floor and there is an impact. Assuming no tendency to rebound and that the ends A and B of the bar are frictionless, find the angular velocity ω' with which the bar begins its new motion after impact, if $\alpha = 30°$. (*Ans.* $\omega' = 3v_c/4l$)

*10. A slender prismatic bar AB of weight W and length l rotates in a horizontal plane with constant angular velocity w about a fixed vertical axis through one end A (Fig. I). Suddenly, due to centrifugal tension, the bar snaps in two at the middle. What angular velocity w will the remaining portion AC have? How will the free portion CB move after fracture?

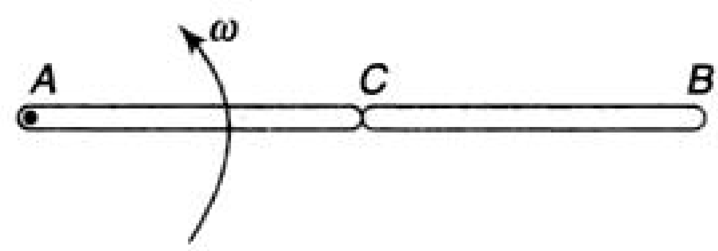

Fig. I

10.6 ENERGY EQUATION FOR PLANE MOTION

Considering motion of a body parallel to the xy plane (Fig. 10.21), the kinetic energy of a particle of mass dm at point A is

$$\frac{dm}{2}(\dot{x}^2 + \dot{y}^2) \qquad \text{(a)}$$

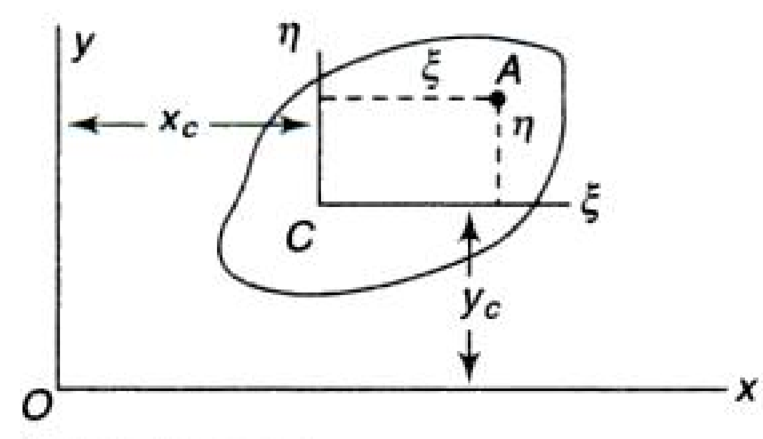

Fig. 10.21

Summing up such expressions for all particles in the body and using, for the components of the velocity, their expressions

$$\dot{x} = \dot{x}_c + \dot{\xi}, \qquad \dot{y} = \dot{y}_c + \dot{\eta}$$

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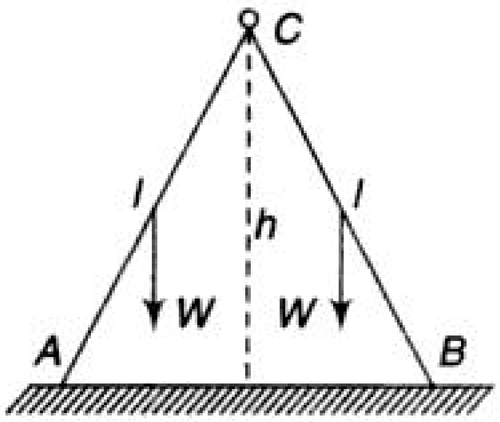

Fig. C

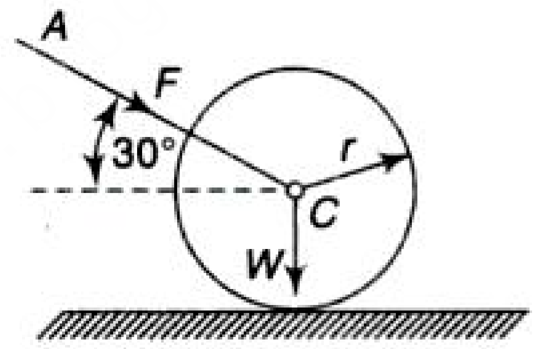

Fig. D

ment $x = 3.6$ m, the roller has a velocity $v = 1.2$ m/s. Assume that the cylinder rolls without sliding. (*Ans.* $F = 179.5$ N)

6. A prismatic bar AB of weight W and length $l = \sqrt{2}\ r$ starts from rest in the position shown in Fig. E and under the action of gravity slides without friction along the constraining vertical plane curve ABD, the portion AB of which is a quadrant of a circle of radius r and the portion BD of which is a horizontal tangent to this circle. With what uniform velocity v will the bar move along the horizontal portion BD ?

(*Ans.* $v = \sqrt{gr}$)

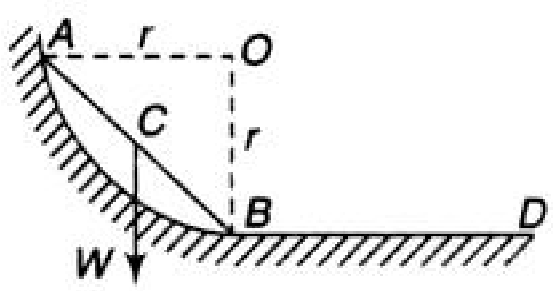

Fig. E

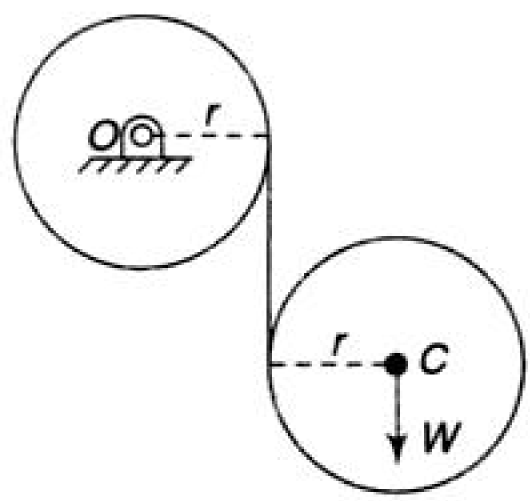

Fig. F

7. Using the method of work and energy, find the velocity v_c of the falling disk in Fig. F as a function of the height h through which it has fallen from rest.

(*Ans.* $v_c = \sqrt{8gh/5}$)

8. (a) Find the period of oscillation of a homogeneous right semicircular cylinder of radius $r = 0.3$ m for small amplitudes of rolling on a horizontal plane. (b) Repeat for a homogeneous hemisphere of radius $r = 0.3$ m.

(*Ans.* $\tau_a = 1.37$ sec; $\tau_b = 1.46$ s)

9. For the one-cylinder gas engine shown in Fig. 10.7, the following numerical data are given: weight of piston and piston rod 161 N; weight of connecting rod, 107.5 N; length of connecting rod 0.9 m. The center of gravity of the connecting rod is 0.3 m from A, and its radius of gyration with respect to the centroidal axis normal to the plane of motion is $i_c = 0.26$ m. The crank radius $r = 0.3$ m, and the engine runs uniformly at 600 rpm. Calculate the kinetic energy T_1 of the piston and piston rod and the kinetic energy T_2 of the connecting rod for the configuration shown in the figure where the angle $wt = 60°$. (*Ans.* $T_1 = 3052.5$ J, $T_2 = 1920$ J)

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Let $\overline{OB}$ (Fig. 11.2) be a vector representing the relative velocity $\bar{v}_r$ of the particle, the projections of which as shown in the figure are represented by the last two terms in each of Eq. (b). Now as the system shown in Fig. 11.1 moves in the xy plane, imagine that the vector $\overline{OB}$ rotates around the origin O with angular velocity $\dot{\theta}$. Thus the end B of this vector has the velocity $v_r\dot{\theta}$ perpendicular to OB as shown. This velocity of the end B of the relative velocity vector $\bar{v}_r$ has the dimension of acceleration. To calculate its projection on the fixed x axis, we note from Fig. 11.2 that the cosine of the angle that it makes with the x axis is $-(\dot{\xi}\sin\theta + \dot{\eta}\cos\theta)/\dot{v}_r$. Thus the required projection is

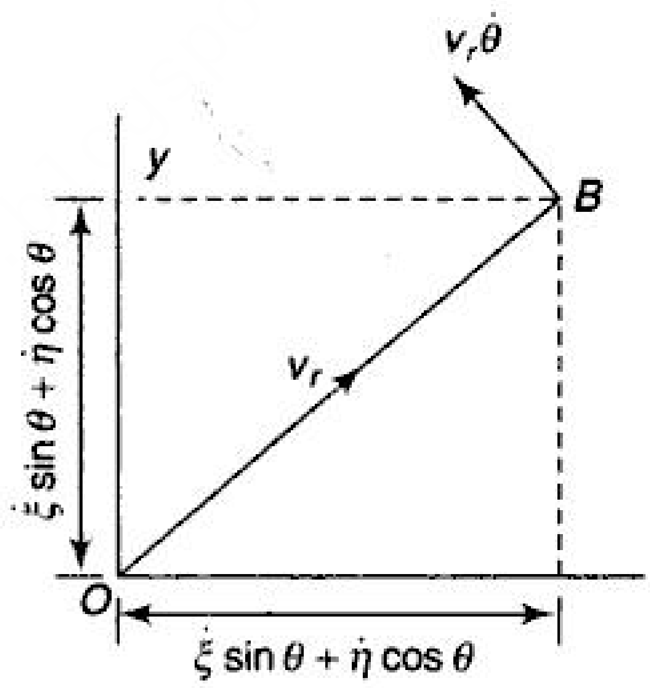

Fig. 11.2

$$-v_r\dot{\theta}\,\frac{(\dot{\xi}\sin\theta + \dot{\eta}\cos\theta)}{v_r} = -\dot{\theta}(\dot{\xi}\sin\theta + \dot{\eta}\cos\theta) \tag{d}$$

In the same manner we find the projection on the y-axis to be

$$-v_r\dot{\theta}\,\frac{(\dot{\xi}\cos\theta + \dot{\eta}\sin\theta)}{v_r} = -\dot{\theta}(\dot{\xi}\cos\theta + \dot{\eta}\sin\theta) \tag{e}$$

Comparing expressions (d) and (e) with the components of the supplementary acceleration in expressions (c), we conclude that this acceleration is represented by the doubled velocity of the end of the vector representing the relative velocity of the particle and rotating with angular velocity $\dot{\theta}$.

Denoting the base acceleration by a_b, the relative acceleration by a_r, and a_s by the supplementary acceleration, we conclude from Eq. (c) that the absolute acceleration can be represented vectorially as follows:

$$\bar{a} = \bar{a}_b \mathbin{+\!\!\!\rightarrow} \bar{a}_r \mathbin{+\!\!\!\rightarrow} \bar{a}_s \qquad \text{or} \qquad \bar{a} = \bar{a}_b + \bar{a}_r + \bar{a}_s \tag{92}$$

In the foregoing discussions it has been assumed that the particle P moves in the $\xi\eta$ plane. If the particle has also some motion in the direction perpendicular to the $\xi\eta$ plane, this motion will not be affected by the motion of the body parallel to the xy plane and that component of the absolute acceleration of P perpendicular to the xy plane is equal to its relative acceleration in the same direction. Thus Eq. (92) holds also in this case, provided a_r is taken as the complete acceleration in relative motion of the particle P.

Examples Examples Examples Examples Examples

1. A particle P moves with constant relative velocity v_r along the circumference of a circular disk of radius r (Fig. 11.3), while the disk rotates with uniform angular velocity ω in the opposite direction. Find the absolute acceleration a of the particle.

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shown, find the absolute acceleration a of the block D which slides freely in a slot cut in the face of the disk and also along the axis of the bar OA. Assume that $h = \frac{1}{2} r$. (*Ans.* $a = 1.5\ \omega^2 r$, outward)

*9. Referring again to the system in Fig. B, find the magnitude of the absolute acceleration of the block D when it reaches the end of the slot in the disk. (*Ans.* $a = 27.9\ \omega^2 r$)

11.2 EQUATIONS OF RELATIVE MOTION

In the formulation of the equations of relative motion of a particle, we observe that in accordance with Newton's second law of motion (Section 7.2) the resultant force F acting on the particle is equal to the product of its mass and its absolute acceleration a and acts in the direction of this acceleration. Multiplying both sides of Eq. (92) by the mass m of the particle, we obtain

$$\bar{F} = m(\bar{a}_b + \bar{a}_r + \bar{a}_s)$$

from this we get

$$ma_r = \bar{F} - m\bar{a}_b + m\bar{a}_s \tag{93}$$

The second and third terms on the right side of Eq. (93) can be regarded as forces of inertia corresponding to the accelerations a_b and a_s. Thus the equations of relative motion of a particle has the same form as the equation of absolute motion of a particle, provided we consider, in addition to the resultant force F, the inertia forces due to the accelerations a_b and a_s.

Equation (93) is given in vectorial form. Projecting the force F and the accelerations a_r, a_b and a_s on rectangular coordinate axes ξ, η, fixed in the body with respect to which the particle is moving, we can obtain two equations of relative motion of a particle in the same form as Eq. (57) of Section 8.2 for absolute motion of a particle. In writing these equations for any particular case of relative motion, we must consider not only the projections of all real forces on the coordinate axes ξ, η, but also the projections on these axes of the two inertia forces due to the base and supplementary accelerations a_b and a_s.

If the body with respect to which we are investigating the relative motion of a particle has a uniform translatory motion, the accelerations a_b and a_s vanish, as may be seen from the discussion of the previous article, and Eq. (93) reduces to

$$m\bar{a} = \bar{F}$$

which coincides with Eq. (33) for absolute motion of a particle. Thus the relative motion in such case will be the same as if the body on which the particle is moving were at rest. Dynamical experiments made in a car moving uniformly along a straight level track will give the same results as in a stationary laboratory. However, if the car enters a curve, the effect of the accelerations a_b and a_s will at once be perceived.

From the equation of relative motion Eq. (93) an equation of *relative equilibrium* of a particle can be obtained. For such equilibrium we need only to assume that

Important Formulae

1. The equations of relative motion of a particle is:

$$\bar{F} = \frac{W}{g}(\bar{a}_b + \bar{a}_r + \bar{a}_s) \quad \text{or} \quad \bar{F} = m(\bar{a}_b + \bar{a}_r + \bar{a}_s)$$

$$\Rightarrow \qquad \frac{W}{g}\bar{a}_r = \bar{F} - \frac{W}{g}\bar{a}_b - \frac{W}{g}\bar{a}_s \quad \text{or} \quad m\bar{a}_r = \bar{F} - m\bar{a}_b - m\bar{a}_s$$

2. The equation of relative equilibrium of a particle is

$$\bar{F} - \frac{W}{g}\bar{a}_b = 0 \quad \text{or} \quad F - m\bar{a}_b = 0$$

PRACTICE SET 11.2

Review Questions

1. Write the equation of relative motion of a particle.
2. Write the equation of relative equilibrium of a particle.

PROBLEM SET 11.2

1. Find the equation of the generator of the vessel shown in Fig. 11.7, and show that it must be a paraboloid of revolution in order that the particle P will be in a condition of relative equilibrium for any position along the meridian AOB.

   (*Ans.* $r^2 = 2gz/\omega^2$)

2. The axes of two ordinary chemist's test tubes make equal angles α with the vertical axis Oz about which they rotate with angular velocity ω (Fig. A). Each tube is completely filled with water and sealed at the end. In one is a steel ball P_1, and in the other a hollow celluloid ball P_2. Prove that the condition of relative equilibrium of the celluloid ball is stable while that of the steel ball is unstable.

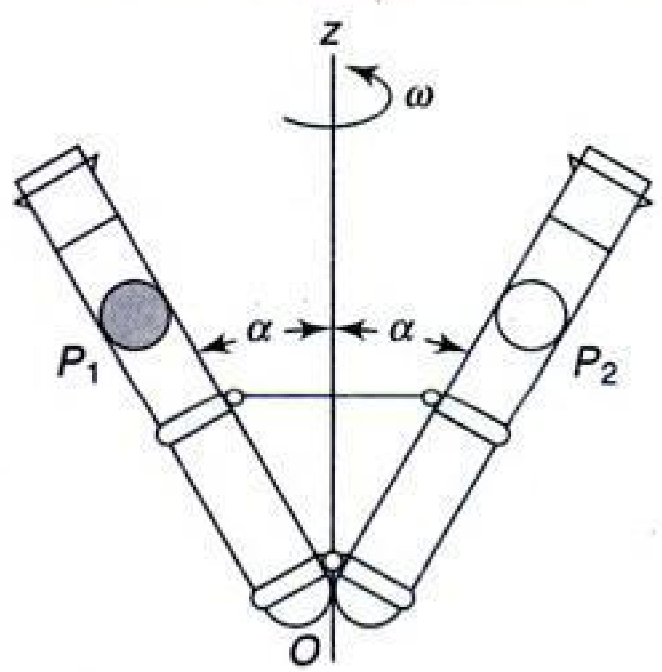

Fig. A

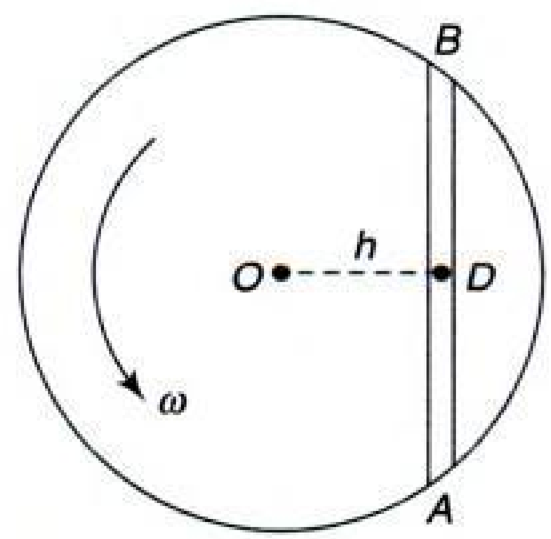

Fig. B

3. In Fig. B, a small ball of weight W rides in a groove cut in the top face of a disk rotating with constant angular velocity ω about its vertical geometric axis. The position shown in the figure is one of unstable relative equilibrium for the ball so that, owing to a slight disturbance, it begins to move along the axis of the groove.

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where dm is the mass of the particle. Denoting by W the total weight of the rim of the wheel and assuming that dm represents the mass of an element of the rim corresponding to the central angle $d\varphi$, we have

$$dm = \frac{W d\varphi}{2\pi g}$$

Substituting this in expression (a) and summing up such expressions for all elements of the rim, we find that the inertia forces due to supplementary acceleration have a resultant moment about the horizontal diameter of the wheel of magnitude.

$$M = 4\frac{W}{2\pi g}\,2\omega\Omega r^2 \int_0^{\pi/2} \sin^2\phi\,\delta\phi = \frac{W}{g} r^2 \omega\Omega = I\omega\Omega \tag{b}$$

where I is the moment of inertia of the wheel with respect to its geometric axis AB.

From Eq. (b) we conclude that all external forces acting on the wheel must constitute an equal and opposite moment with respect to the horizontal diameter. Denoting by R the magnitudes of the reactions on the axle at A and B, we have then

$$Rl = I\omega\Omega$$

which is the same result obtained previously in Section 9.11, by using the principle of angular momentum. For the direction of rotation indicated in Fig. 11.10, the reaction at A will be down and that at B, upwards.

Important Concept

D'Alembert's principle in relative motion

SUMMARY

- In case of relative motion of a particle, we have three components of the inertia force, the relative, base, and supplementary accelerations of the particle. In case of relative motion of a rigid body, we apply to each of its particles the above-mentioned three components of inertia force. Then the system of forces consisting of all such inertia forces together with the external forces acting on the body constitute a system of forces in equilibrium, and we write the equations of relative motion of the body as equations of statics.

PRACTICE SET 11.3

Review Question

1. Write the D'Alembert's equations for relative motion of a particle.

PROBLEM SET 11.3

1. A prismatic bar DE of weight W and length $2r$ is attached at right angles to a horizontal axis AB of length l supported in bearings at A and B (Fig. A). The bar DE rotates about AB with uniform angular velocity ω while the frame to which the

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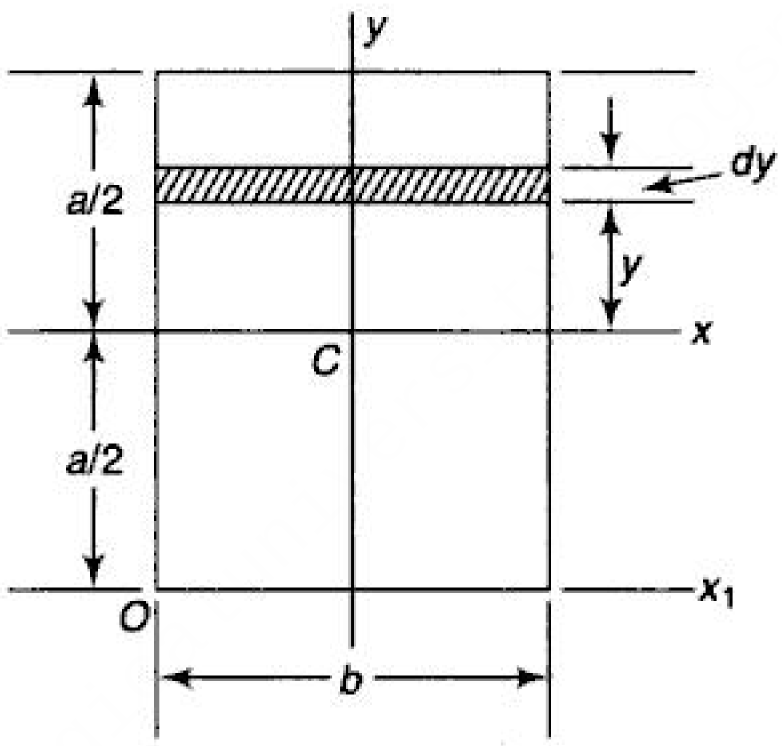

Fig. AI.3

In the same manner, taking the moment of inertia with respect to the y-axis, we find

$$I_y = \frac{ab^3}{12} \tag{d}$$

Formula (c) can also be used for calculating the moment of inertia I_x of the parallelogram shown in Fig. AI.4. This parallelogram may be considered as obtained from the rectangle shown by doted lines by a certain displacement parallel to the x axis of each elemental strip, as shown in the figure. Since this transformation changes neither the area of the element nor its distance from the x axis, we conclude that the value of I_x calculated for the rectangle remains unchanged for the parallelogram.

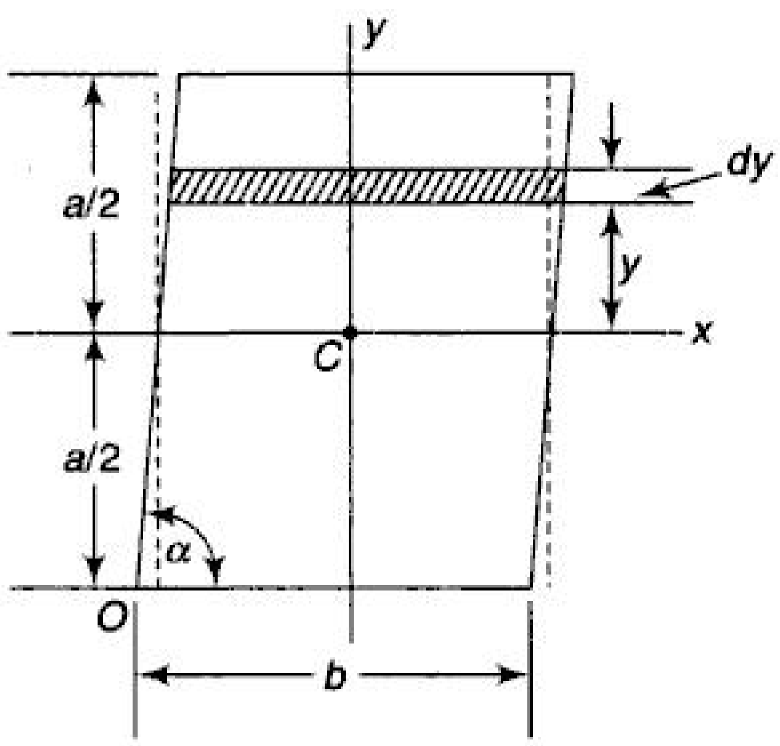

Fig. AI.4

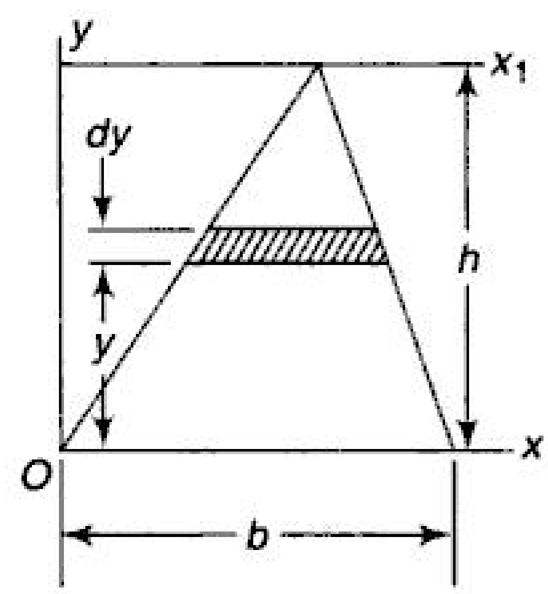

Fig. AI.5

In calculating, the moment of inertia of a triangle with respect to its base (Fig. AI.5), we divide the area into elemental strips, as shown in the figure. Then, for any element at the distance y from the base,

$$dA = \frac{b(h - y)dy}{h}$$

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Likewise, the moment of inertia of the circle in Fig. AI.9 with respect to a tangent is

$$I_T = \frac{\pi d^4}{64} + \frac{p^2}{4}\left(\frac{d}{2}\right)^2 = \frac{5\pi d^4}{64} \tag{s}$$

For the case of the triangle in Fig. AI.5, we already know the moment of inertia about the base [Eq. (c)]. Then by the parallel-axis theorem, we find the centroidal moment of inertia to be

$$\bar{I}_x = \frac{bh^3}{12} - \frac{bh}{2}\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \tag{t}$$

The parallel-axis theorem is especially useful in the calculation of moments of inertia of *composite areas* like the one shown in Fig. AI.13.

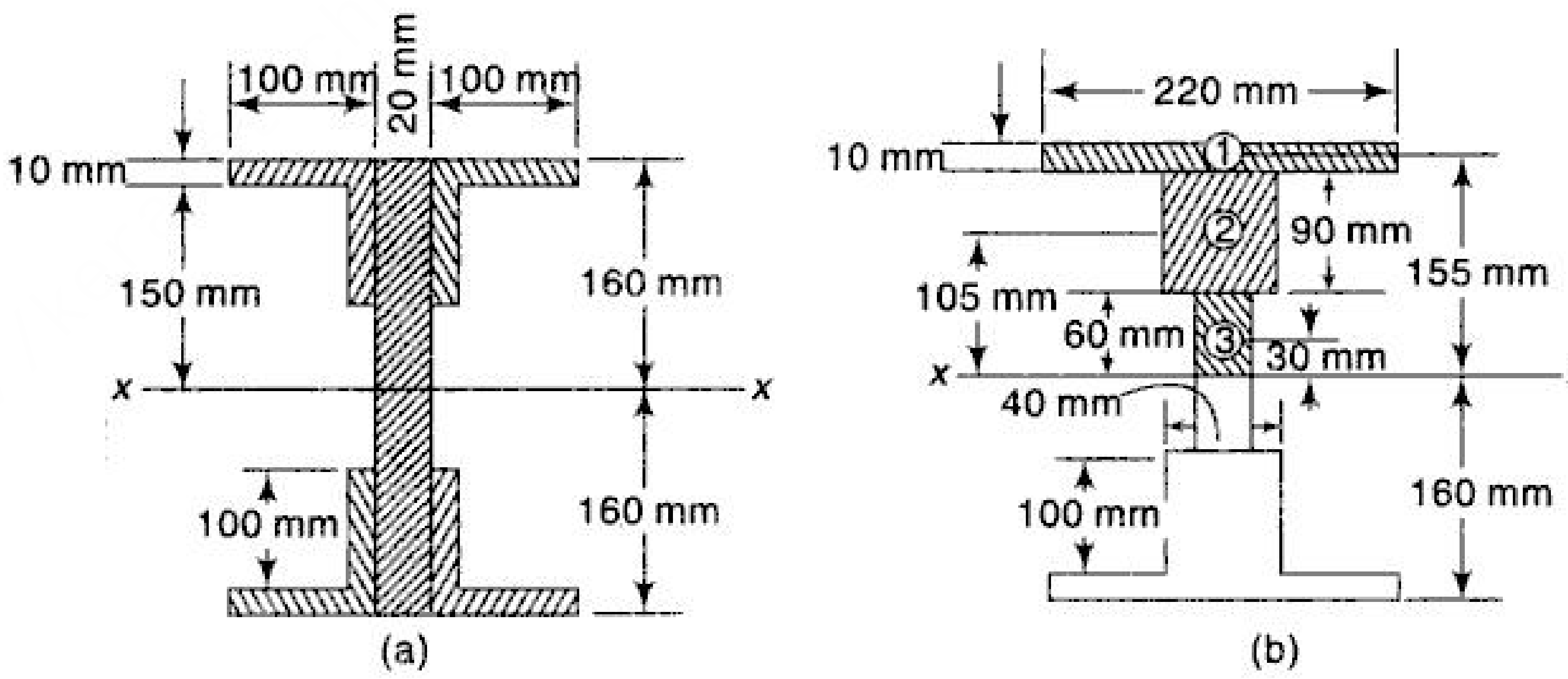

Fig. AI.13

To illustrate, let us calculate the moment of inertia of this composite area with respect to the x axis through the centroid C, which is an axis of symmetry. The area consists of a narrow rectangle 20 mm by 320 mm and four identical angles 100 mm by 100 mm by 10 mm as shown in Fig. AI.13(a). For purposes of calculation, we subdivide the area into rectangles as shown in Fig. AI.13(b). Then denoting by A_1, A_2, A_3 the areas of these rectangles and by y_1, y_2, y_3 the y coordinates of their respective centroids and using Eq. (a) for centroidal moment of inertia of a rectangle, we have, by reference to Fig. AI.13(b) for dimensions.

$A_1 = 2200 \text{ mm}^2$	$A_2 = 3600 \text{ mm}^2$	$A_3 = 1200 \text{ mm}^2$
$y_1 = 155 \text{ mm}^2$	$y_2 = 105 \text{ mm}^2$	$y_3 = 30 \text{ mm}$
$\bar{I}_1 = 18333.3 \text{ mm}^4$	$\bar{I}_2 = 243000 \text{ mm}^4$	$\bar{I}_3 = 40000 \text{ mm}^4$

The required moment of inertia of the entire area with respect to the x-axis is

$$I_x = 2(\bar{I}_1 + A_1 y_1^2 + \bar{I}_2 + A_2 y_2^2 + \bar{I}_3 + A_3 y_3^2) \tag{u}$$

Substituting the numerical values in expression (u), we find $I_x = 1.92 \times 10^8 \text{ mm}^4$.

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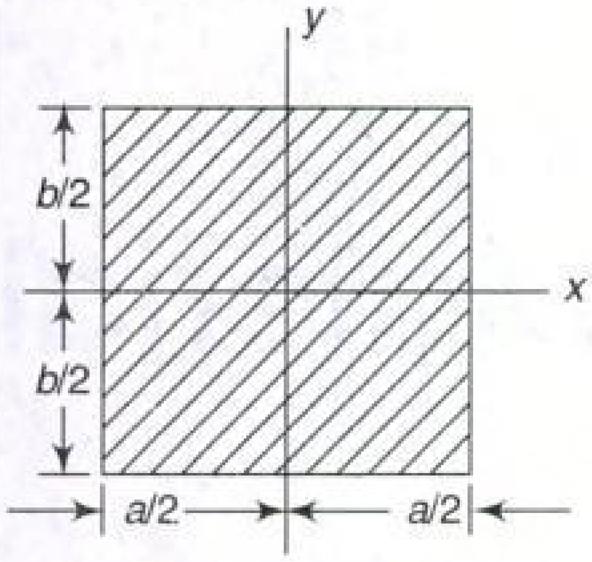
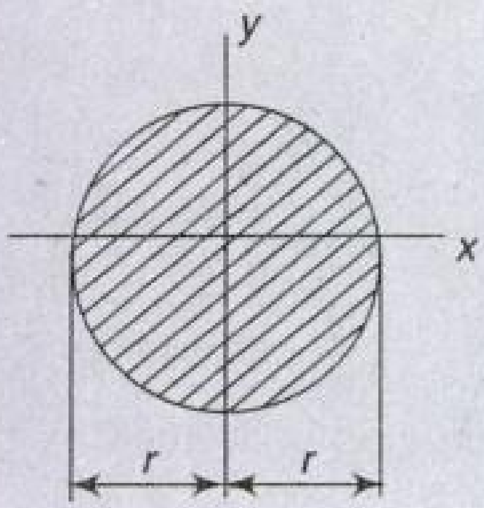
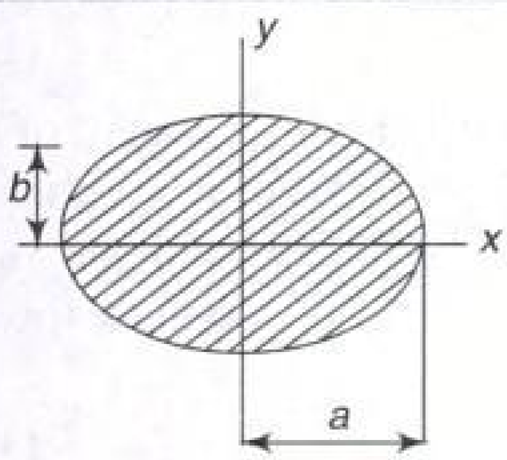
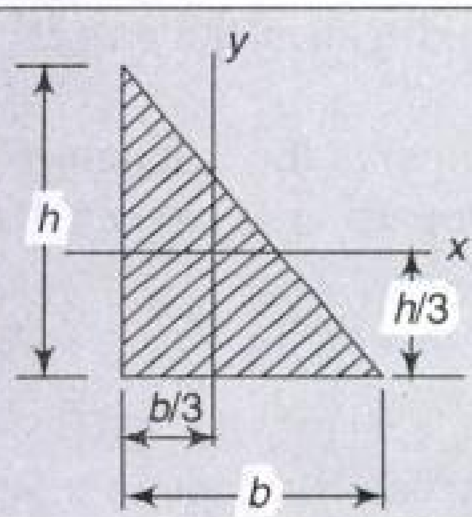
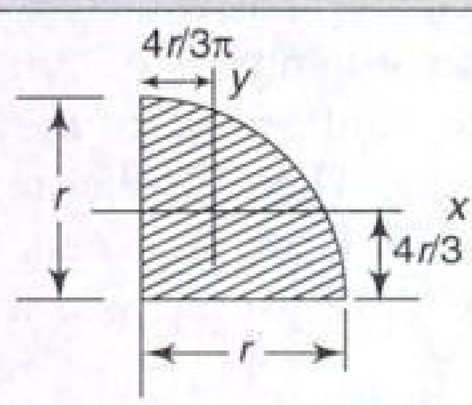
Various Moments and Products of Inertia of Plane Figures.

Figure	$\bar{I}_x$	$\bar{I}_y$	$\bar{I}_{xy}$
	$\dfrac{ab^3}{12}$	$\dfrac{ba^3}{12}$	0
	$\dfrac{\pi r^4}{4}$	$\dfrac{\pi r^4}{4}$	0
	$\dfrac{\pi ab^3}{4}$	$\dfrac{\pi ba^3}{4}$	0
	$\dfrac{bh^3}{36}$	$\dfrac{hb^3}{36}$	$\dfrac{-b^2h^2}{72}$
	$0.0549r^4$	$0.0549r^4$	$-0.0163r^4$

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AII.8 MOMENTS OF INERTIA OF THREE-DIMENSIONAL BODIES

In discussing the moments of inertia of bodies, all three dimensions of which are of the same order of magnitude, it is advantageous to use the parallel-axis theorem. Let x, y, z (Fig. AII.22) be a system of rectangular coordinate axes through the center of gravity C of a body and X, Y, Z a system of corresponding parallel axes with an origin O, a, b, c being the coordinates of the center of gravity C with respect to the axes X, Y, Z.

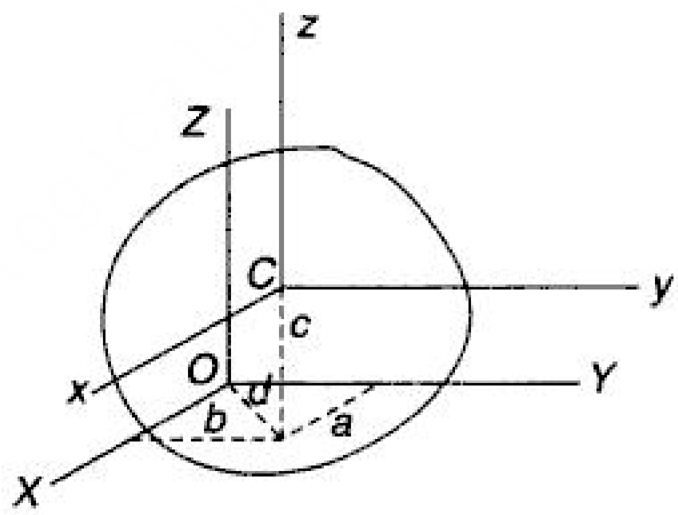

Fig. AII.22

Considering a mass element dm having coordinates $X = a + x$ and $Y = b + y$, we have for the moment of inertia of the body with respect to the z-axis:

$$I_z = \int [(a+x)^2 + (b+y)^2]dm = (a^2+b^2)\int dm$$

$$+2a\int x\,dm + 2b\int y\,dm + \int (x^2+y^2)dm \tag{f}$$

Since the axes x, y, z pass through the center of gravity C of the body, the second and the third integrals in expression (f) vanish. Then, using the notations

$$d^2 = a^2 + b^2 \qquad \text{and} \qquad r^2 = x^2 + y^2$$

we obtain from expression (f)

$$I_t = \int r^2\,dm + d^2\int dm = \bar{I}_z + md^2 \tag{18}$$

and we conclude that the moment of inertia of a body with respect to any axis is obtained by adding to its moment of inertia with respect to a parallel centroidal axis, the product of the mass of the body and the square of the distance between the two axes. This represents the parallel-axis theorem for the mass moment of inertia of any material body.

Writing Eq. (18) in the form

$$mi_t^2 = m\bar{i}_z^2 + md^2$$

we see that

$$i_z^2 = \bar{i}_z^2 + d^2 \tag{19}$$

which states the relationship between radii of gyration for parallel axes.

As an example of the application of the parallel-axis theorem to the calculation of moments of inertia of three-dimensional bodies, let us calculate the moments of inertia of a solid right circular cylinder of uniform density with respect to the centroidal coordsinate axes x, y, z, as shown in Fig. AII.23.

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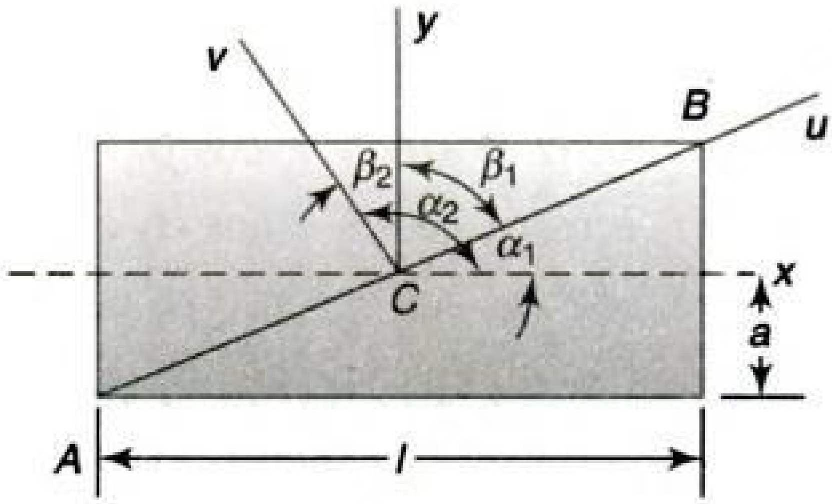

Fig. 25

Substituting the quantities (q) and (r) into Eq. (22a), we obtain

$$I_u = \frac{ma^2(5l^2 + 6a^2)}{6(l^2 + 4a^2)} \tag{s}$$

To calculate the product of inertia I_{uv} of the cylinder, we use Eq. (23a). Then in addition to the quantities (q) and (r) above we have, by reference to Fig. 25,

$$\cos\alpha_2 = -\frac{2a}{\sqrt{l^2 + 4a^2}}, \quad \cos\beta_2 = -\frac{l}{\sqrt{l^2 + 4a^2}}, \quad \cos\gamma_2 = 0 \tag{t}$$

Substituting the quantities (q), (r) and (t) into Eq. (23a), we obtain

$$I_{uv} = \frac{mal(3a^2 - l2)}{6(4a^2 + l^2)} \tag{u}$$

Attention is called to the fact that this product of inertia vanishes when $l = \sqrt{3}a$.

This means that for such proportions of the cylinder, u, v, w, as shown in Fig. AII.25, are also principal axes.

PROBLEM SET AII.10

1. Calculate the moment of inertia of a homogeneous rectangular lamina of dimensions a and b with respect to a diagonal of the rectangle. The mass of the lamina is m.

$$\left(Ans.\ I_d = \frac{1}{6} m \frac{a^2 b^2}{a^2 + b^2}\right)$$

2. Determine the moment of inertia of a homogeneous rectangular parallelepiped of dimensions a, b, c and mass m with respect to a diagonal.

$$\left(Ans.\ I_d = \frac{1}{6} m \frac{a^2 b^2 + b^2 c^2 + a^2 c^2}{a^2 + b^2 + c^2}\right)$$

3. Calculate the moment of inertia of a homogeneous right circular cone of mass m, altitude h, and radius of base a with respect to a generator.

$$\left(Ans.\ I_g = \frac{3ma^2}{20} \frac{a^2 + 6h^2}{a^2 + h^2}\right)$$

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$$\frac{W}{g}\ddot{x} = W - [W + k(x - x_1)] \quad \text{or} \quad m\ddot{x} = W - [W + k(x - x_1)]$$

which, by using expression (d) for x_1, may be written

$$\frac{W}{g}\ddot{x} + kx = ak \cos \omega t \quad \text{or} \quad m\ddot{x} + kx = ak \cos \omega t \tag{e}$$

Dividing both sides of Eq. (e) by m and using the notations

$$\frac{k}{m} = \frac{kg}{W} = p^2 \qquad \frac{ak}{m} = \frac{akg}{W} = q_0 \tag{f}$$

we see that Eq. (e) takes the same form as Eq. (24).
The only difference between this and the preceding case is that here the disturbing force is transmitted to the suspended weight through the spring instead of being applied directly to it, as in Fig. AIII.26.

General Solution

We seek now the solution of Eq. (24). An examination of this equation shows that it can be satisfied by taking

$$x = C \cos \omega t \tag{g}$$

where C is a constant. Substituting expression (g) for x in Eq. (24), we obtain

$$-C\omega^2 + Cp^2 = p_0 \tag{h}$$

from which

$$C = \frac{q_o}{p^2 - w^2} \tag{i}$$

and the solution (g) becomes

$$x = \frac{q_o}{p^2} \frac{1}{1 - w^2/p^2} \cos \omega t \tag{25}$$

This is only a *particular solution* of Eq. (24). To obtain the *general solution* which can be adapted to any initial conditions of motion, we add to this particular solution the general solution for free vibrations, as given by Eq. (41), Section 7.6. Then

$$x = C_1 \cos pt + C_2 \sin pt + \frac{q_o \cos \omega t}{p^2(1 - \omega^2/p^2)} \tag{26}$$

By substitution it can be proved that this expression satisfies Eq. (24), and since it has two constants of integration, we can always adjust these constants so as to satisfy any initial conditions regarding displacement and velocity. The first two terms of Eq. (26) represent *free vibrations* having the period

$$\tau = \frac{2\pi}{p}$$

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Frahm's Tachometer

We come now to a consideration of the condition of resonance. Although our solution in which the effect of friction has been neglected, does not give an exact picture of what happens at resonance, it will be sufficient, for the present, to observe that this condition is characterized chiefly by very large amplitudes of forced vibration. Owing to this fact, the condition of resonance is often a dangerous one because of the corresponding large stresses produced in the spring or springs of the system. However, certain instruments, used for measuring the frequency of vibrations, are designed to operate at or very near resonance.

Figure AIII.34 represents an instrument called Frahm's tachometer, which is used for determining the frequency of any vibrating body. It consists essentially of a number of thin steel strips, each built into a base at one end and carrying a weight at the other. Thus each unit is a small cantilever beam carrying a load at one end and having a definite natural frequency of lateral vibration. The weights are so adjusted that their frequencies represent a series of numbers over some definite range. Now when the base is attached to a vibrating body, each weight undergoes forced vibrations having the frequency of the body to which the base is fastened. If this frequency is within the range of the tachometer, the weight that has most nearly the same natural frequency will vibrate with a larger amplitude than any of the others (see Fig. AIII.34) and thus the frequency is determined.

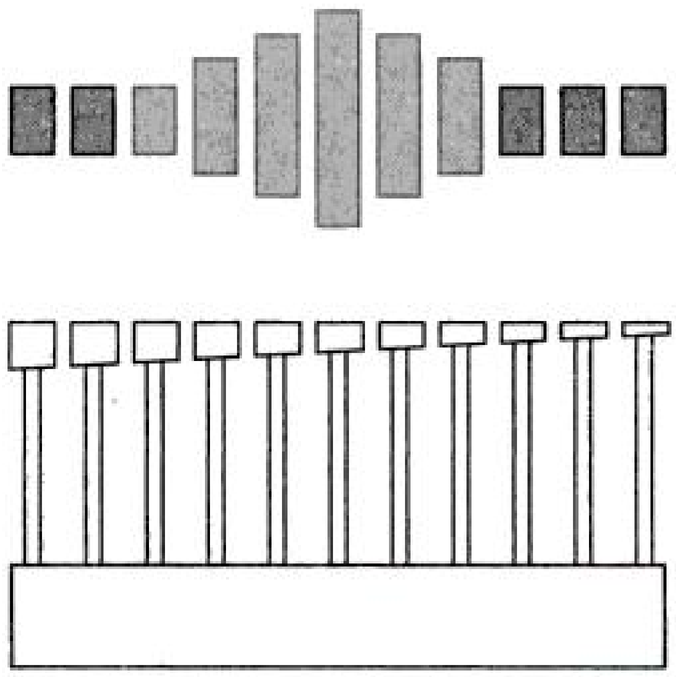

Fig. AIII.34

Vibrator

Another example of a practical application of resonance is the so-called *vibrator* used for finding the natural frequency of vibration of bridges, buildings, and other large structures. This instrument (Fig. AIII.35) consists essentially of two similar rotating masses purposely unbalanced so that when they are rotated in opposite directions, they set up two equal centrifugal forces Q_0. Further, they are so adjusted that these forces always make the same angle with the vertical, thus causing their horizontal projections to balance each other while their vertical projections add together to give a vertical pulsating force, the frequency of which can be controlled by varying the speed of rotation. Suitably mounted in a rigid frame, the vibrator can be stationed on the bridge and gradually speeded up until the resonance condition is reached. This speed will easily be recognised owing to

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$$Y_b = \frac{-\omega^2}{l} I_{yz}$$

$$X_b = \frac{-\omega^2}{l} I_{xz} \tag{26}$$

From these equations the four components of the reactions can be calculated provided the coordinates of the center of gravity and the products of inertia of the body with respect to the x, z and y, z axes are known.

In the particular case where the center of gravity of the body is on the axis of rotation, $x_c = y_c = 0$ and from the first two of Eq. (26) we conclude that

$$X_a = -X_b \qquad \text{and} \qquad Y_a = -Y_b$$

Thus the components X_a, X_b and Y_a, Y_b of the bearing reactions represent a pair of couples in the xz and yz planes, respectively, and we conclude that there is a resultant couple in some intermediate axial plane of the rotating body. During rotation of the body, the plane of this resultant couple rotates also and the bearings are submitted to the action of uniformly rotating forces. Such rotating forces, as we have seen in Appendix III, can produce forced vibrations of the bearing pedestals and are, generally speaking, very undesirable. To eliminate them it is necessary, as is seen from the last two of Eq. (26), to make the products of inertia of the body with respect to the y, z and x, z axes zero. Then the reactions at the bearings due to inertia forces completely vanish. This means that the axis of rotation must go through the center of gravity of the body and must coincide with one of its principal axes of inertia. In short, a body which rotates about a principal central axis will not produce fluctuating pressures on the bearings.

As an example of calculating bearing reactions for a rigid body rotating about a fixed axis AB which is not a principal axis, let us consider the case shown in Fig. AIV.37.

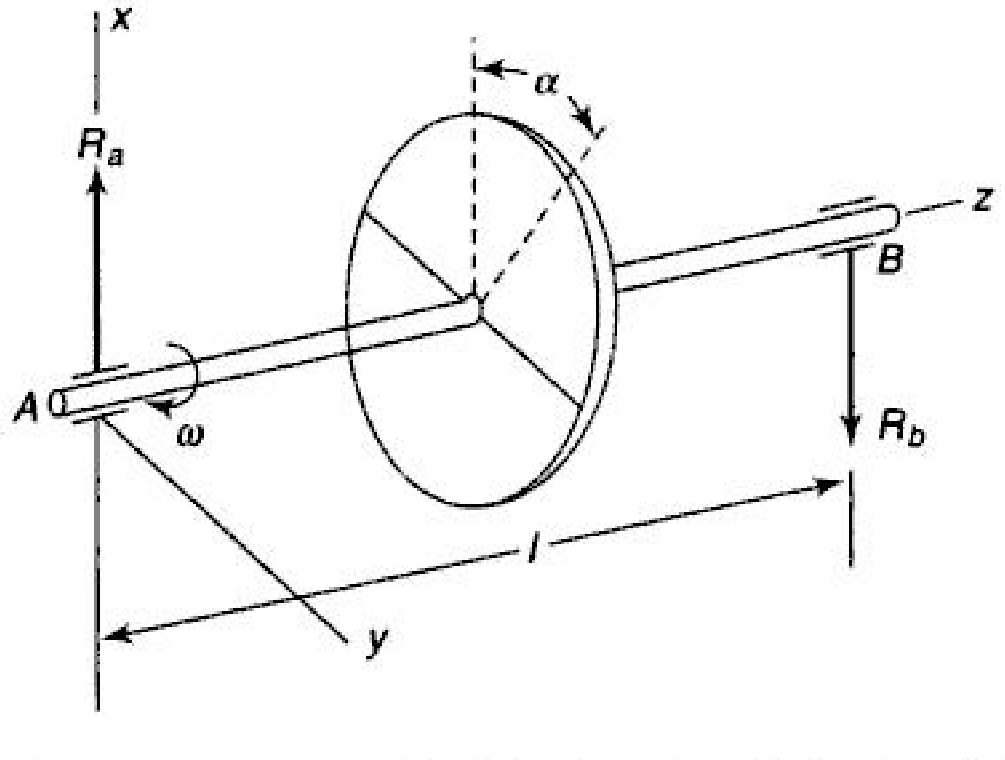

Fig. AIV.37

Here we have a homogeneous solid right circular disk of weight W or mass m and radius r obliquely mounted on a shaft so that the middle plane of the disk makes an angle a with a plane normal to the axis of the shaft. Taking coordinate axes x, y, z, as shown, we see that $x_c = y_c = 0$ and also, since xz is a plane of symmetry, we have $I_{yz} = 0$. Then for this case, Eq. (26) reduce to

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vibrations of the bed. This indicates that the proper correction in plane 2 has been made. That is, the vertical projection of the rotating inertia force R_2 set up by the chosen correction weight is exactly counteracting the effect of the system of inertia forces set up by the rotor properly. This done, the rotor is turned end for end and the same procedure repeated until the proper correction mass m_1/weight W_1 is found. The fact that the previously placed correction mass m_2/weight W_2 is now exerting its inertia force R_2 in the plane containing the fulcrum prevents it from having any influence on the selection of the weight W_1/mass m_1.

In the foregoing discussion it was always assumed that the body rotates with a constant angular velocity ω. It can be shown, however, that, if a rotor is perfectly balanced at constant speed, there will be no dynamic pressures on the bearings if the speed is non-uniform. Considering an element dm of a rotating body, the radial and tangential components of the inertia force in the case of nonuniform rotation are $\omega^2 r$ dm and $\dot{\omega} r$ dm, respectively. Since ω and $\dot{\omega}$ are the same for all elements of the rotating body, it can be concluded that the system of inertia forces due to any angular acceleration $\dot{\omega}$ is obtained from the system of inertia forces due to angular velocity ω simply by multiplying the latter system of forces by a constant factor $\dot{\omega}/\omega^2$ and by rotating this system through $90°$ around the axis of rotation. Now, owing to the balancing described above, the system of forces $\omega^2 r\, dm$ is in equilibrium; hence the system of forces $\dot{\omega} r\, dm$ is also in equilibrium and does not produce pressures on the bearings.

PROBLEM SET A.14

1. A homogeneous circular steel disk rotates about its geometric axis and has two holes drilled through it at A and B, as shown in Fig.AIV. A. Determine the diameter d and the angular position φ of a hole which should be drilled at C in order to balance the disk. (*Ans.* $d = 137.5$ mm; $\varphi = 118°$)

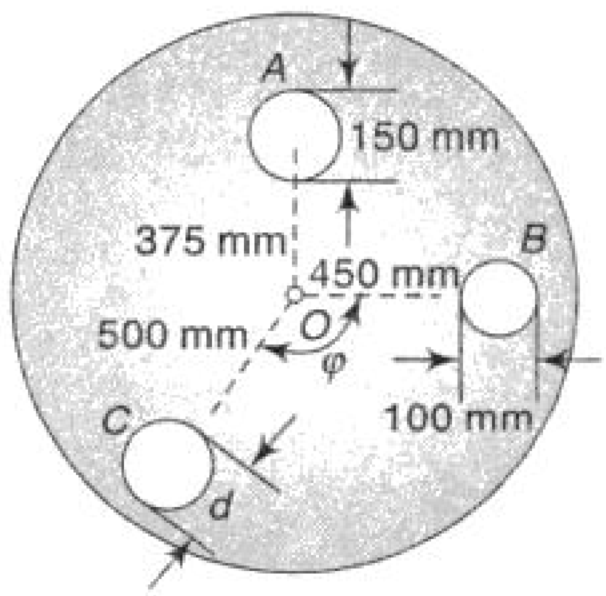

Fig. AIV.A

2. A wheel weighing 10.5 kN has its center of gravity 10 mm from its geometric axis. The wheel is mounted on a shaft which rotates in two bearings A and B, 1.5 apart and on opposites sides of the wheel, bearing A being 0.6 m from the plane of revolution of the wheel. Find the bearing reactions due to the centrifugal force of the unbalanced wheel when the latter has an angular speed $n = 200$ rpm. What correction weight W placed at a radius of 1.05 m, in the plane of revolution of the wheel will balance it? (*Ans.* $R_a = 2.87$ kN; $R_b = 1.91$ kN; $W = 100$ N)

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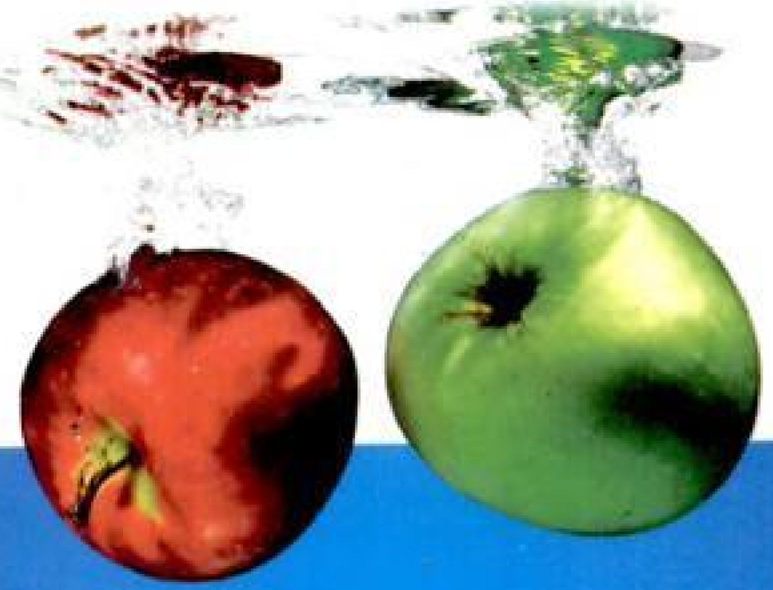

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