property: Inverse Laplace transform of logarithmic and trigonometric functions:

$\Rightarrow$ $L[t f(t)] = -\frac{d}{ds}[F(s)]$

Therefore $L^{-1}\left[\frac{-d}{ds}(F(s))\right] = t f(t)$

$\Rightarrow$ $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Therefore $L^{-1}\left[(-1)^n \frac{d^n}{ds^n} F(s)\right] = t^n f(t)$

problem Evaluate $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

sol: Given

$$F(s) = \log\left(\frac{s+a}{s+b}\right)$$

$$= \log(s+a) - \log(s+b).$$

Differentiating $F(s)$ w.r.t s,

$$\frac{d}{ds} F(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$(-1)\frac{d}{ds} F(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

Now applying Inv Laplace transformation,

$$L^{-1}\left[-\frac{d}{ds}[F(s)]\right] = L^{-1}\left[\frac{1}{s+b}\right] - L^{-1}\left[\frac{1}{s+a}\right]$$

$$t\,f(t) = e^{-bt} - e^{-at}$$

$$\boxed{f(t) = \frac{e^{-bt} - e^{-at}}{t}}$$

problem Evaluate $L^{-1}\left[\operatorname{Tan}^{-1}\frac{2}{s^2}\right]$

Sol:

Given

$$F(s) = \operatorname{Tan}^{-1}\left(\frac{2}{s^2}\right)$$

$$\frac{d}{ds}\left[\operatorname{Tan}^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= \frac{-4s}{s^4+4}$$

Then

$$\left[-\frac{d}{ds}\operatorname{Tan}^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{4s}{s^4+4}$$

Applying $L^{-1}\left[\frac{4s}{s^4+4}\right] =$

ROUGH WORK

$$y = \operatorname{Tan}^{-1}\left(\frac{2}{x^2}\right)$$

$$\operatorname{Tan} y = \frac{2}{x^2}$$

$$\sec^2 y\,\frac{dy}{dx} = \frac{(2)(-2)}{x^3}$$

$$\frac{dy}{dx} = \frac{-4}{x^3\sec^2 y}$$

$$= \frac{-4}{x^3(1+\tan^2 y)}$$

$$= \frac{-4}{x^3\left(1+\frac{4}{x^4}\right)}$$

$$\frac{4s}{s^4+4} = \frac{4s}{\underbrace{(s^2+2)^2}_{a^2} - \underbrace{(2s)^2}_{-b^2}} \text{ form.}$$

$$= \frac{4s}{\left[\underbrace{s^2+2 - 2s}_{(a-b)}\right]\left[\underbrace{s^2+2 + 2s}_{(a+b)}\right]} \text{ form.}$$

Using partial fraction method,

$$\frac{4s}{(s^2-2s+2)(s^2+2s+2)} = \frac{As+B}{s^2-2s+2} + \frac{Cs+D}{s^2+2s+2}$$

$$4s = (As+B)(s^2+2s+2) + (Cs+D)(s^2-2s+2)$$

Comparing s^3 Terms

$0 = A+C$

Comparing s^2 Terms

$0 = 2A-2C+B+D$

Comparing s Terms

$4 = 2A+2B-2D+2C$

$4 = 2(A+C) + 2(B-D)$

$4 = \underbrace{0}_{} + 2(B-D)$

$\boxed{2 = B-D}$

Comparing Constants

$0 = 2B+2D$

$\boxed{0 = B+D}$

Solving $B-\cancel{D}=2$

$B+\cancel{D}=0$

$2B = 2$

$\boxed{B=1}$

hence $\boxed{D=-1}$

Similarly from $A+C=0$

$2A-2C+B+D=0$

put $B=1$, $D=-1$ $\therefore A=0$, $C=0$

Hence

$$\frac{AS+B}{s^2-2s+2} + \frac{Cs+D}{s^2+2s+2}$$

$$= \frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}$$

Taking L^{-1},

$$\Rightarrow L^{-1}\left[\frac{1}{s^2-2s+2}\right] - L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right].$$

$$\Rightarrow e^{t}.\sin t - e^{-t}\sin t$$

$$\Rightarrow \sin t\,(e^{t} - e^{-t})$$

$$\Rightarrow \sin t\,(2\sinh t)$$

$$\Rightarrow 2\sin t\sinh t$$

problem: Evaluate $L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right]$

Sol: Given

$$F(s) = \log\left[\frac{s^2+1}{s(s+1)}\right]$$

$$F(s) = \log(s^2+1) - \log s - \log(s+1)$$

Then

$$\frac{d}{ds}F(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$-\frac{d}{ds}F(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}$$

Taking L^{-1} operator,

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{2s}{s^2+1}\right]$$

$$\downarrow$$

$$tf(t) = 1 + e^{-t} - 2\cos t$$

$$\boxed{f(t) = \frac{1+e^{-t}-2\cos t}{t}}$$

problem Evaluate $L^{-1}\left[\cot^{-1}\left[\frac{s}{2}\right]\right]$

ROUGH WORK

$\cot^{-1}\left(\frac{s}{a}\right) = \frac{-a}{s^2+a^2}$

Sol: Given

$$F(s) = \cot^{-1}\left[\frac{s}{2}\right]$$

$$\frac{d}{ds}\left[F(s)\right] = \frac{d}{ds}\cot^{-1}\left[\frac{s}{2}\right]$$

$$= \frac{-2}{s^2+2^2}$$

$$= \frac{-2}{s^2+4}$$

$$-\frac{d}{ds}F(s) = \frac{2}{s^2+4}$$

Taking Inverse L.T,

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = 2L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= 2\,\frac{\sin 2t}{2}$$

$$t f(t) = \sin 2t$$

$$\boxed{f(t) = \frac{\sin 2t}{t}}$$

imp problem Find $L^{-1}\left\{ s\log\frac{s-1}{s+1} + 2 \right\}$

Sol:

Consider

$$F(s) = s\log\left(\frac{s-1}{s+1}\right) + 2 = s\log(s-1) - s\log(s+1) + 2$$

$$\therefore \cancel{L^{-1}\left[s\log\left(\frac{s-1}{s+1}\right) + 2\right] = L^{-1}\left[s\log(s-1)\right]}$$

Diff wrt. s,

$$\frac{d}{ds}F(s) = \underbrace{\frac{d}{ds}\left\{ s\log(s-1)\right\}}_{\text{Product Rule of Differention}} - \underbrace{\frac{d}{ds}\left\{ s\log(s+1)\right\}}_{\text{product rule}} + 0$$

$$\frac{d}{ds}F(s) = \left\{ s\cdot\left(\frac{1}{s-1}\right) + \log(s-1)\cdot 1 \right\} - \left\{ s\cdot\left(\frac{1}{s+1}\right) + \log(s+1)\cdot 1 \right\}$$

$$= \frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)$$

$$\frac{d}{ds}F(s) = \frac{s}{s-1} - \frac{s}{s+1} + \log(s-1) - \log(s+1)$$

$$\frac{d}{ds}F(s) = \frac{s}{s-1} - \frac{s}{s+1} + \log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{s(s+1) - s(s-1)}{s^2-1} + \log\left(\frac{s-1}{s+1}\right)$$

Taking L^{-1} on both sides

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = L^{-1}\left[\frac{2s}{s^2-1}\right] + L^{-1}\left[\log\left(\frac{s-1}{s+1}\right)\right]$$

$$-t f(t) = 2\cosh t + L^{-1}\left[\log\left(\frac{s-1}{s+1}\right)\right] \longrightarrow (1)$$

Now we have to compute $L^{-1}\left[\log\left(\frac{s-1}{s+1}\right)\right]$ separately

$$F(s) = \log(s-1) - \log(s+1)$$

Diff wrt s

$$\frac{d}{ds} F(s) = \frac{1}{s-1} - \frac{1}{s+1}$$

Taking L^{-1},

$$L^{-1}\left[\frac{d}{ds} F(s)\right] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$t f(t) = e^{t} - e^{-t}$$

$$\boxed{f(t) = \frac{e^{t} - e^{-t}}{t}}$$

Substitute in equation (1)

$$-t f(t) = 2\cosh t + \frac{e^{t} - e^{-t}}{t}$$

$$\boxed{f(t) = \frac{2\cosh t}{t} + \frac{e^{t} - e^{-t}}{t^2}}$$

Exercise

(i) Find $L^{-1}\left\{ s \log \frac{s+4}{s-4} \right\}$

Ans: $\frac{8}{t}\cosh 4t - \frac{2}{t^2}\sinh 4t$

(ii) Find $L^{-1}\left\{ \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \right\}$

Ans: $\frac{1}{t}\left[\cos at - \cos bt\right]$

(iii) Find $L^{-1}\left\{ \frac{1}{s} \cdot \log\left(\frac{s^2+a^2}{s^2+b^2}\right) \right\}$

(OR)

$$L^{-1}\left\{ \frac{\log\left[\frac{s^2+a^2}{s^2+b^2}\right]}{s} \right\}$$

Ans: $2\int_0^t \frac{\cos bt - \cos at}{t}\,dt.$

CONVOLUTION THEOREM

If $L^{-1}[f(s)] = f(t)$ and $L^{-1}[g(s)] = g(t)$

Then $L^{-1}[f(s).g(s)] = \int_0^t f(u).g(t-u)\,du$

problem Using convolution theorem, evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sol: Let $\frac{s}{(s^2+a^2)^2} = \underbrace{\left(\frac{s}{s^2+a^2}\right)}_{f(s)} \underbrace{\left(\frac{1}{s^2+a^2}\right)}_{g(s)}$

Now $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \text{Cos}at \leftarrow f(t)$

$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\text{Sin}at}{a} \leftarrow g(t)$

By Convolution Theorem,

$L^{-1}[f(s).g(s)] = \int_0^t f(u).g(t-u)\,du$

$= \int_0^t \text{Cos}au \; \frac{\text{Sin}a(t-u)}{a}\,du$

$= \frac{1}{a}\int_0^t \text{Cos}\,au \,.\, \text{Sin}(at-au)\,du$

we know that

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos au \sin(at-au) = \frac{1}{2}\left[\sin at - \sin(2au-at)\right]$$

$$\therefore \frac{1}{a}\int_0^t \cos au \sin(at-au)\,du = \frac{1}{2a}\int_0^t \sin at - \sin(2au-at)\,du$$

$$= \frac{1}{2a}\left\{\sin at\,[u]_{u=0}^{t} + \left[\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left\{[t\sin at - 0] + \left[\frac{\cos at - \cos at}{2a}\right]\right\}$$

$$\boxed{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}(t\sin at)}$$

problem Apply Convolution theorem to evaluate

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] \text{ where } b \neq a$$

Sol:

Let $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \underbrace{\left(\frac{s}{s^2+a^2}\right)}_{f(s)}\underbrace{\left(\frac{s}{s^2+b^2}\right)}_{g(s)}$$

Then $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt$

By Convolution Therem,

$$L^{-1}[f(s).g(s)] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \,.\, \cos b(t-u)\, du$$

By identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\cancel{= \int_0^t \cos(a-b)u\, du + \int_0}$$

$$= \frac{1}{2}\int_0^t \cos[au+bt-bu] + \cos[au-bt+bu]\, du$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}\, dt$$

$$= \frac{1}{2}\left\{\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right\}_{u=0}^{t}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b}\right] - \left[\frac{\sin(0+bt)}{a-b} + \frac{\sin(0-bt)}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{a\sin at - b\sin bt.}{a^2-b^2}\right]$$

problem Apply Convolution Theorem to find $L^{-1}\left\{\frac{s}{(s^2+a^2)^3}\right\}$

Sol:

Consider $\frac{s}{(s^2+a^2)^3} = \underbrace{\left(\frac{1}{s^2+a^2}\right)}_{F(s)}\underbrace{\left(\frac{s}{(s^2+a^2)^2}\right)}_{G(s)}$

we know that

$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a} \leftarrow f(t)$

$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a} \leftarrow g(t)$

Therefore Applying Convolution Theorem,

$$L^{-1}[F(s) * G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

(OR)

$$L^{-1}[F(s) * G(s)] = \int_0^t f(t-u)\, g(u)\, du.$$

we will get,

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^3}\right\} = \int_0^t \left\{\frac{\sin[a(t-u)]}{a}\right\}\left\{\frac{1}{2a} u\sin au\right\} du$$

$$= \frac{1}{2a^2}\int_0^t [u.\sin(at-au).\sin au]\, du$$

Using Sin A Sin B formula

$$= \frac{1}{4a^2}\int_0^t u.\underbrace{[\cos[at-2au) - \cos at]}_{\text{Integration by parts}}\, du$$

$$= \frac{1}{4a^2}\left\{ \left[-\frac{u}{2a}\sin(at-2au)\right]_{u=0}^{u=t} + \frac{1}{2a}\int_0^t \sin(at-2au)\,du - \cos at\left(\frac{u^2}{2}\right)_0^t \right\}$$

$\downarrow$ Simplify

$$= \frac{1}{4a^2}\left\{ \left(\frac{t}{2a}\sin at - 0\right) + \frac{1}{4a^2}\left[\cos(at-2au)\right]_{u=0}^{t} - \frac{t^2}{2}\cos at \right\}$$

$$= \frac{1}{4a^2}\left\{ \frac{t\sin at}{2a} + \frac{1}{4a^2}\underbrace{(\cos at - \cos at)}_{\text{zero}} - \frac{t^2}{2}\cos at \right\}$$

$$= \frac{1}{4a^2}\left\{ \frac{t\sin at}{2a} - \frac{t^2\cos at}{2} \right\} //$$

Exercise problems

(i) Apply Convolution Theorem to find inverse L.T

$$\frac{1}{(s+1)(s^2+1)} \qquad \text{Ans: } \frac{e^{-t} + \sin t - \cos t}{2}$$

(ii) Find $L^{-1}\left\{ \frac{s}{(s+1)^2(s^2+1)} \right\}$

$$\text{Ans: } \frac{\sin t - te^{-t}}{2}$$

property If $L^{-1}[F(s)] = f(t)$ Then

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$$

and also

$$L^{-1}\left[\frac{F(s)}{s^2}\right] = \int_0^t \left[\int_0^t f(t)\,dt\right]dt$$

$$L^{-1}\left[\frac{F(s)}{s^3}\right] = \int_0^t \int_0^t \int_0^t f(t)\,dt\,dt\,dt.$$

problem Find the inverse Laplace transform of

$$\frac{1}{s(s^2+a^2)}.$$

sol: Given function

$$F(s) = \frac{1}{s(s^2+a^2)}$$

$$= \frac{\left[1/s^2+a^2\right]}{s} \quad \text{which is in } \frac{F(s)}{s} \text{ form}$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

Using $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$$L^{-1}\left[\frac{\left(1/s^2+a^2\right)}{s}\right] = \int_0^t \frac{\sin at}{a}\,dt$$

$$= \frac{1}{a}\int_0^t \sin at \, dt$$

$$= \frac{1}{a}\left[\frac{-\cos at}{a}\right]_0^t$$

$$= \frac{-1}{a^2}\left[\cos at - \cos 0\right]$$

$$= \frac{-1}{a^2}\left[\cos at - 1\right]$$

$$\boxed{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-\cos at}{a^2}}$$

problem Find $L^{-1}\left[\frac{1}{s(s+a)^3}\right]$

Sol: Given function $\frac{1}{s(s+a)^3}$ can be written

as $\left[\frac{\frac{1}{(s+a)^3}}{s}\right]$ which is in $\frac{F(s)}{s}$ form.

To find $L^{-1}\left[\frac{1}{(s+a)^3}\right]$, Apply shifting property

$$\Rightarrow L^{-1}\left[\frac{\frac{1}{(s+a)^3}}{s+a-a}\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s+a-a)(s+a)^3}\right]$$

$$\Rightarrow e^{-at} L^{-1}\left[\frac{1}{(s-a)s^3}\right]$$

Now $e^{-at} L^{-1}\left[\frac{(1/s-a)}{s^3}\right] \longrightarrow (1)$

Using property $L^{-1}\left[\frac{F(s)}{s^3}\right] = \int_0^t \int_0^t \int_0^t f(t)\,dt\,dt\,dt$

Since $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

we will get, $L^{-1}\left[\frac{1/s-a}{s^3}\right] = \int_0^t \int_0^t \int_0^t e^{at}\, dt\,dt\,dt$

$$= \int_0^t \int_0^t \left[\frac{e^{at}}{a}\right]_0^t dt\,dt$$

$$= \frac{1}{a}\int_0^t \left(\int_0^t \left[e^{at}-1\right] dt\right) dt$$

$$= \frac{1}{a}\int_0^t \left[\frac{e^{at}}{a} - t\right]_0^t dt$$

$$= \frac{1}{a}\int_0^t \left(\frac{e^{at}}{a} - t\right) - \left(\frac{1}{a} - 0\right)\Big] dt$$

$$= \frac{1}{a^2}\int_0^t \left[e^{at} - at - 1\right] dt$$

$$= \frac{1}{a^2}\left[\frac{e^{at}}{a} - \frac{at^2}{2} - t\right]_0^t$$

$$= \frac{1}{a^2}\left\{\left[\frac{e^{at}}{a} - \frac{at^2}{2} - t\right] - \left[\frac{1}{a} - 0 - 0\right]\right\}$$

$$L^{-1}\left[\frac{1/s-a}{s^3}\right] = \frac{1}{a^3}\left\{e^{at} - \frac{a^2t^2}{2} - at - 1\right\} \longrightarrow (2)$$

Substituting equation ② into equation ①

$$\Rightarrow \bar{e}^{at}\left[\frac{1}{a^3}\left(e^{at}-\frac{a^2t^2}{2}-at-1\right)\right]$$

$$\boxed{\bar{L}^{1}\left[\frac{1}{s(s+a)^3}\right] \Rightarrow \frac{1}{a^3}\left[1-\frac{a^2t^2\bar{e}^{at}}{2}-at\bar{e}^{at}-\bar{e}^{at}\right]}$$

Solving differential equations Using Laplace Transforms:

FORMULAS:

- $L[y'''] = s^3 y(s) - s^2 y(0) - s y'(0) - y''(0).$
- $L[y''] = s^2 y(s) - s y(0) - y'(0).$
- $L[y'] = s y(s) - y(0).$
- $L[y] = y(s).$

problem Solve : $y'' + 5y' + 6y = 5e^{2t}$ with $y(0)=2, y'(0)=1.$

sol: Given

$$y'' + 5y' + 6y = 5e^{2t}$$

Applying Laplace transformation,

$$L[y''] + 5L[y'] + 6L[y] = 5L[e^{2t}]$$

$$= [s^2 y(s) - s y(0) - y'(0)] + 5[s y(s) - y(0)] + 6 y(s) = 5\left(\frac{1}{s-2}\right)$$

~~put~~ Given Condition $y(0)=1$, $y'(0)=1$; substitute

$$= [s^2 y(s) - 2s - 1] + 5[s y(s) - 2] + 6 y(s) = \frac{5}{s-2}$$

= Seperate $y(s)$ terms,

$$[s^2 + 5s + 6] y(s) - 2s - 1 - 10 = \frac{5}{s-2}$$

$$\Rightarrow (s^2+5s+6)\,y(s) = \frac{5}{s-2} + 2s+11$$

$$\Rightarrow \boxed{y(s) = \frac{5}{(s-2)(s^2+5s+6)} + \frac{2s+11}{s^2+5s+6}}$$

Now to get $y(t)$, apply L^{-1}

$$\Rightarrow L^{-1}[y(s)] = \underbrace{L^{-1}\left[\frac{5}{(s-2)(s^2+5s+6)}\right]}_{\text{Term-1}} + \underbrace{L^{-1}\left[\frac{2s+11}{s^2+5s+6}\right]}_{\text{Term-2}}$$

Since $s^2+5s+6 = (s+2)(s+3)$, we will get

$$y(t) = L^{-1}\left\{\frac{5}{(s-2)(s+2)(s+3)}\right\} + L^{-1}\left\{\frac{2s+11}{(s+2)(s+3)}\right\} \quad \text{(1)}$$

Partial fraction method for $L^{-1}\left\{\frac{5}{(s-2)(s+2)(s+3)}\right\}$

Consider

$$\frac{5}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$\Rightarrow \frac{5}{(s-2)\cancel{(s+2)}(s+3)} = \frac{A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)}{(s-2)\cancel{(s+2)}(s+3)}$$

$$5 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$$

Put $s=2$

$$5 = A(2+2)(2+3)$$

$$5 = 20A$$

$$\boxed{A = \frac{1}{4}}$$

Put $s=-3$

$$5 = C(-3-2)(-3+2)$$

$$5 = 5C$$

$$\boxed{C = 1}$$

Put $s = -2$

$$5 = B(-2-2)(-2+3)$$

$$5 = -4B$$

$$\boxed{B = \frac{-5}{4}}$$

hence

$$\frac{5}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{1/4}{s-2} + \frac{-5/4}{s+2} + \frac{1}{s+3}$$

Taking L^{-1},

$$L^{-1}\left\{\frac{5}{(s-2)(s+2)(s+3)}\right\} = \frac{1}{4} L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{5}{4} L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{4} \cdot e^{2t} - \frac{5}{4} \cdot e^{-2t} + e^{-3t} \longrightarrow (2)$$

Similarly Applying partial fractions to $L^{-1}\left\{\frac{2s+11}{(s+2)(s+3)}\right\}$

$$\frac{2s+11}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$\frac{2S+11}{(S+2)(S+3)} = \frac{A}{S+2} \quad \frac{A(S+3)+B(S+2)}{(S+3)(S+2)}$$

$$2S+11 = A(S+3)+B(S+2)$$

put $S=-3$

$$2(-3)+11 = B(-3+2)$$

$$5 = -B$$

$$\boxed{B=-5}$$

Put $S=-2$

$$2(-2)+11 = A(-2+3)$$

$$\boxed{7=A}$$

$$\therefore \quad \frac{2S+11}{(S+2)(S+3)} = \frac{A}{S+2} + \frac{B}{S+3}$$

$$= \frac{7}{S+2} + \frac{(-5)}{S+3}$$

$$= \frac{7}{S+2} - \frac{5}{S+3}$$

Taking L^{-1},

$$L^{-1}\left\{\frac{2S+11}{(S+2)(S+3)}\right\} = L^{-1}\left\{\frac{7}{S+2}\right\} - L^{-1}\left\{\frac{5}{S+3}\right\}$$

$$= 7e^{-2t} - 5e^{-3t} \longrightarrow (3)$$

to get final solution, substitute (2), (3) in eqn (1)

$$y(t) = \left\{\frac{1}{4}e^{2t} - \frac{5}{4}e^{-2t} + e^{-3t}\right\} + \left\{7e^{-2t} - 5e^{-3t}\right\}$$

$$= \left(-\frac{5}{4}+7\right)e^{-2t} - 4e^{-3t} + \frac{1}{4}e^{2t}$$

$$\boxed{y(t) = \frac{23}{4}e^{-2t} - 4e^{-3t} + \frac{1}{4}e^{2t}}$$

<u>problem</u> Using laplace transformation method, solve

$$x'' - 2x' + x = e^{2t} \quad \text{with} \quad x(0) = 0, \quad x'(0) = -1$$

x means x(t)

<u>Sol:</u>

Given that

$$x'' - 2x' + x = e^{2t}$$

Taking Laplace transformation,

$$\Rightarrow L[x''] - 2L[x'] + L[x] = L[e^{2t}]$$

$$\Rightarrow [s^2x(s) - sx(0) - x'(0)] - 2[sx(s) - x(0)] + x(s) = \frac{1}{s-2}$$

Substitute given conditions $x(0) = 0$, $x'(0) = -1$

Then

$$\Rightarrow [s^2x(s) - 0 + 1] - 2[sx(s) - 0] + x(s) = \frac{1}{s-2}$$

writing x(s) terms in L.H.S,

$$\Rightarrow \underbrace{s^2 x(s) - 2s\, x(s) + x(s)} + 1 = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\, x(s) + 1 = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\, x(s) = \frac{1}{s-2} - 1$$

$$\Rightarrow x(s) = \frac{1}{(s-2)(s^2-2s+1)} - \frac{1}{s^2-2s+1}$$

$$\Rightarrow \boxed{x(s) = \frac{1}{(s-2)(s-1)^2} - \frac{1}{(s-1)^2}}$$

To get solution $x(t)$, we need to apply L^{-1},

$$\Rightarrow L^{-1}[x(s)] = L^{-1}\left\{\frac{1}{(s-2)(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$x(t) = L^{-1}\left\{\frac{1}{(s-2)(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} \longrightarrow (1)$$

To find L^{-1}, we will apply partial fractions

Consider

$$\frac{1}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$\frac{1}{\cancel{(s-2)(s-1)^2}} = \frac{A(s-1)^2 + B(s-1)(s-2) + C(s-2)}{\cancel{(s-2)(s-1)^2}}$$

$$\Rightarrow \quad 1 = A(S-1)^2 + B(S-1)(S-2) + C(S-2)$$

<u>put S=1</u>

$1 = C(1-2)$

$1 = -C$

$\boxed{C = -1}$

<u>put S=2</u>

$1 = A(2-1)^2$

$\boxed{1 = A}$

Comparing S^2, on both sides

$0 = A+B$

$0 = 1+B$

$\boxed{B = -1}$

$$\therefore \frac{1}{(S-2)(S-1)^2} = \frac{A}{S-2} + \frac{B}{S-1} + \frac{C}{(S-1)^2}$$

$$= \frac{1}{S-2} + \frac{(-1)}{S-1} + \frac{(-1)}{(S-1)^2}$$

$$\frac{1}{(S-2)(S-1)^2} = \frac{1}{S-2} - \frac{1}{S-1} - \frac{1}{(S-1)^2}$$

Taking L^{-1},

$$L^{-1}\left\{\frac{1}{(S-2)(S-1)^2}\right\} = L^{-1}\left\{\frac{1}{S-2}\right\} - L^{-1}\left\{\frac{1}{S-1}\right\}$$

$$- L^{-1}\left\{\frac{1}{(S-1)^2}\right\}$$

$\Downarrow$ shifting prop

$$= e^{2t} - e^{t} - e^{t} L^{-1}\left\{\frac{1}{S^2}\right\}$$

$$= e^{2t} - e^{t} - e^{t} \cdot \frac{t^1}{1!}$$

$$= e^{2t} - e^{t} - te^{t} \longrightarrow (2)$$

Similarly Considering equation ① Second Term,

$$L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

↓ shifting property

$$= e^{t}\, L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{t}\cdot\frac{t^{1}}{1!}$$

$$= e^{t}t \longrightarrow ③$$

Substitute results of equations ②, ③ in eqn ①

$$x(t) = \left[e^{2t} - e^{t} - te^{t}\right] - \left[e^{t}t\right]$$

$$\boxed{x(t) = e^{2t} - e^{t} - 2te^{t}}$$ //

problem Solve : $y'' + 3y' + 2y = 2t^2 + 2t + 2$ with conditions

$y(0) = 2, \quad y'(0) = 0$

Sol: Given that

$$y'' + 3y' + 2y = 2t^2 + 2t + 2$$

Applying laplace transform,

$$L[y''] + 3L[y'] + 2L[y] = 2L[t^2] + 2L[t] + L[2]$$

$$\Rightarrow \left[s^2 y(s) - s y(0) - y'(0)\right] + 3\left[s y(s) - y(0)\right] + 2 y(s)$$

$$= 2 \cdot \frac{2!}{s^{2+1}} + 2 \cdot \frac{1!}{s^{1+1}} + \frac{2}{s}$$

$$\Rightarrow \left[s^2 y(s) - s y(0) - y'(0)\right] + 3\left[s y(s) - y(0)\right] + 2 y(s) = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s}$$

Substituting given Conditions; $y(0) = 2$
$y'(0) = 0$

$$\Rightarrow \left[s^2 y(s) - 2s - 0\right] + 3\left[s y(s) - 2\right] + 2 y(s) = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s}$$

$\Rightarrow$ Seperating $y(s)$ terms and remaining terms

$$\Rightarrow (s^2 + 3s + 2) y(s) - 2s - 6 = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s}$$

$$\Rightarrow (s^2 + 3s + 2) y(s) = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s} + 2s + 6$$

$$\Rightarrow (s^2 + 3s + 2) y(s) = \frac{4 + 2s + 2s^2 + 2s^4 + 6s^3}{s^3}$$

$$\Rightarrow y(s) = \frac{4 + 2s + 2s^2 + 2s^4 + 6s^3}{s^3 (s^2 + 3s + 2)}$$

(can be written as $(s+1)(s^3+2s^2-s+2)$)

$$= \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3 (s+1)(s+2)}$$

$$\Rightarrow \frac{2\cancel{(s+1)}(s^3 + 2s^2 - s + 2)}{s^3 \cancel{(s+1)}(s+2)}$$

$$\boxed{y(s) \Rightarrow \frac{2(s^3 + 2s^2 - s + 2)}{s^3 (s+2)}}$$

Using partial fraction method

$$\frac{2(s^3+2s^2-s+2)}{s^3(s+2)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+2}$$

$$\Rightarrow \frac{2(s^3+2s^2-s+2)}{\cancel{s^3(s+2)}} = \frac{As^2(s+2)+Bs(s+2)+C(s+2)+Ds^3}{\cancel{s^3(s+2)}}$$

$$\Rightarrow 2(s^3+2s^2-s+2) = As^2(s+2)+Bs(s+2)+C(s+2)+Ds^3$$

put $s=-2$

$2[(-2)^3+2(-2)^2-(-2)+2] = D(-2)^3$

$2[\cancel{-8}+\cancel{8}+2+2] = -8D$

$8=-8D$

$\boxed{D=-1}$

put $s=0$

$4 = C(0+2)$

$4=2C$

$\boxed{C=2}$

Comparing s^3 Terms

$2=A+D$

$2=A+(-1)$

$\boxed{A=3}$

Compare s^2 Terms

$4=2A+B$

$4=2(3)+B$

$\boxed{B=-2}$

hence $\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+2}$ becomes

$$\frac{2(s^3+2s^2-s+2)}{s^3(s+2)} = \frac{3}{s}+\frac{(-2)}{s^2}+\frac{2}{s^3}+\frac{(-1)}{s+2}$$

$y(s)$

Taking L^{-1} on both sides

$$L^{-1}\left\{\frac{2(s^3+2s^2-s+2)}{s^3(s+2)}\right\} = 3L^{-1}\left[\frac{1}{s}\right]-2L^{-1}\left[\frac{1}{s^2}\right]+2L^{-1}\left[\frac{1}{s^3}\right]-L^{-1}\left[\frac{1}{s+2}\right]$$

$$\boxed{y(t) = 3(1)-2(t)+2\left(\frac{t^2}{2}\right)-e^{-2t}}$$

problem Solve : $y'' + 2y' + 5y = e^{-t}\sin t$ with $y(0)=0$
$y'(0)=1$.

Sol: Given D.E is

$$y'' + 2y' + 5y = e^{-t}\sin t$$

Applying Laplace transform,

$$\Rightarrow L[y''] + 2L[y'] + 5L[y] = L[e^{-t}\sin t] \longrightarrow \frac{1}{s^2+1}$$ Then shift

$$\Rightarrow [s^2 y(s) - s y(0) - y'(0)] + 2[s y(s) - y(0)] + 5y(s) = \frac{1}{(s+1)^2+1}$$

Substitute Given conditions $y(0)=0$
$y'(0)=1$.

Then

$$\Rightarrow [s^2 y(s) - 0 - 1] + 2[s y(s) - 0] + 5y(s) = \frac{1}{(s+1)^2+1}$$

Seperating $y(s)$ terms,

$$\Rightarrow (s^2 + 2s + 5)y(s) - 1 = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow (s^2 + 2s + 5)y(s) = \frac{1}{(s+1)^2+1} + 1$$

$$\Rightarrow y(s) = \frac{1}{[(s+1)^2+1][s^2+2s+5]} + \frac{1}{s^2+2s+5}$$

$$\Rightarrow \boxed{y(s) = \frac{1}{[s^2+2s+2][s^2+2s+5]} + \frac{1}{s^2+2s+5}}$$

To get Solution of D.E $y(t)$, Take L^{-1};

$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right] + L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

or $y(t)$

$\downarrow$

Using partial fraction method,

Consider
$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$1 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

Comparing s^3 Terms	Comparing s^2 Terms	Comparing s Terms
$0 = A + C \rightarrow (1)$	$0 = 2A+B+2C+D$	$0 = 5A+2B+2C+2D \rightarrow (3)$
	$0 = 2(A+C)+B+D \rightarrow (2)$	

Comparing Constant Terms
$1 = 5B+2D \rightarrow (4)$

Solving (1), (2) $\Rightarrow$ $B+D=0$

from (4) $\Rightarrow$ $5B+2D=1$

Solving

$\cancel{3B=1}$

$\boxed{\cancel{B=1/3}}$

Therefore $\boxed{\cancel{D=-1/3}}$

$\boxed{B = \frac{1}{3}}$

$\boxed{D = \frac{-1}{3}}$

put $\left.\begin{array}{l} B=\frac{1}{3} \\ D=\frac{-1}{3} \end{array}\right\}$ in equation (3)

$0 = 5A + \frac{2}{3} + 2C - \frac{2}{3}$

$0 = 5A+2C$

solving eqn ① $\Rightarrow A+C=0$
and $\Rightarrow 5A+2C=0$ $\Big\}$ $A=0$, $C=0$

$$\therefore \frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{0+1/3}{s^2+2s+2} + \frac{0-1/3}{s^2+2s+5}$$

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{1}{3}\left[\frac{1}{s^2+2s+2}\right] \frac{-1}{3}\left[\frac{1}{s^2+2s+5}\right]$$

Then

$$L^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right] = \frac{1}{3} L^{-1}\left[\frac{1}{s^2+2s+2}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= \frac{1}{3} L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3}\cdot e^{-t}\cdot \sin t - \frac{1}{3} e^{-t}\left(\frac{\sin 2t}{2}\right) \rightarrow$$

hence

$$y(t) = L^{-1}\left[\frac{1}{(s^2+2s+2)(s^2+2s+5)}\right] + L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= \frac{1}{3} e^{-t} \sin t \underbrace{- \frac{1}{3} e^{-t}\left(\frac{\sin 2t}{2}\right) + e^{-t}\left(\frac{\sin 2t}{2}\right)}$$

$$\boxed{y(t) = \frac{1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t}$$

problem Solve : $[D^3-3D^2+3D-1]y = 2t^2e^t$ where $D=\frac{d}{dt}$

with conditions $y(0)=1$

$y'(0)=0$

$y''(0)=-2$

Sol:

Given

$$y'''-3y''+3y'-y = 2t^2e^t$$

Applying Laplace transformation,

$$L[y''']-3L[y'']+3L[y']-L[y]=2L[t^2e^t]$$

$$\Rightarrow (s^3y(s)-s^2y(0)-sy'(0)-y''(0))-3[s^2y(s)-sy(0)-y'(0)]+3[sy(s)-y(0)]$$

$$-y(s) = \frac{4}{(s-1)^3}$$

Substituting $y(0)=1$, $y'(0)=0$, $y''(0)=-2$ and seperating $y(s)$ terms

$$\Rightarrow (s^3-3s^2+3s-1)y(s)+2+3s-3-s^2 = \frac{4}{(s-1)^3}$$

$$\Rightarrow \underbrace{(s^3-3s^2+3s-1)}y(s) = \frac{4}{(s-1)^3}+s^2-3s+1$$

$$\Rightarrow (s-1)^3\,y(s) = \frac{4}{(s-1)^3}+s^2-3s+1$$

$$\boxed{y(s) = \frac{4}{(s-1)^6}+\frac{s^2-3s+1}{(s-1)^3}}$$

Taking L^{-1} on both sides,

$$L^{-1}[y(s)] = L^{-1}\left[\frac{4}{(s-1)^6}\right] + L^{-1}\left[\frac{s^2-3s+1}{(s-1)^3}\right]$$

Using shifting   Using shifting property

$$= e^t L^{-1}\left[\frac{4}{s^6}\right] + L^{-1}\left[\frac{(s-1+1)^2 - 3(s-1+1)+1}{(s-1)^3}\right]$$

$$= e^t L^{-1}\left(\frac{4}{s^6}\right) + L^{-1}\left[\frac{[(s-1)^2 + 2(s-1)\cdot 1 + 1] - 3(s-1) - 3 + 1}{(s-1)^3}\right]$$

$$= 4e^t\left(\frac{t^5}{5!}\right) + L^{-1}\left[\frac{(s-1)^{\cancel{2}}}{(s-1)^{\cancel{3}}}\right] + L^{-1}\left[\frac{2\cancel{(s-1)}}{(s-1)^{\cancel{3}}}\right] - 3L^{-1}\left[\frac{\cancel{s-1}}{(s-1)^{\cancel{3}}}\right] - L^{-1}\left[\frac{1}{(s-1)}\right]$$

$$= 4e^t\left(\frac{t^5}{5!}\right) + e^t\cdot L^{-1}\left[\frac{1}{s}\right] + 2e^t L^{-1}\left[\frac{1}{s^2}\right] - 3e^t L^{-1}\left[\frac{1}{s^2}\right] - e^t L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{e^t t^5}{30} + e^t + 2e^t\cdot t - 3e^t\cdot t - e^t\frac{t^2}{2}$$

$$= \frac{t^5e^t}{30} + e^t - te^t - \frac{t^2e^t}{2}$$

$$\boxed{y(t) = e^t\left\{\frac{t^5}{30} + 1 - t - \frac{t^2}{2}\right\}}$$

Solving differential equations with Variable Coefficients

<u>problem</u> Solve : $ty'' + 2y' + ty = \sin t$ with $y(0)=1$

<u>Sol:</u>

Given D.E is

$$ty'' + 2y' + ty = \sin t$$

Taking Laplace transformation,

$$L[ty''] + 2L[y'] + L[ty] = L[\sin t]$$

tf(t) type

tf(t) Type

$$= \frac{-d}{ds}\left\{s^2 y(s) - sy(0) - y'(0)\right\} + 2\left\{sy(s) - y(0)\right\} - \frac{d}{ds}y(s) = \frac{1}{s^2+1}$$

product rule

~~Integration by parts~~

$$= -\left\{s^2 \frac{d}{ds}y(s) + y(s).2s - 1 - 0\right\} + 2sy(s) - 2 - \frac{d}{ds}y(s) = \frac{1}{s^2+1}$$

Seperating $\frac{d}{ds}y(s)$ terms,

$$= (-s^2-1)\frac{d}{ds}y(s) - \cancel{2sy(s)} + 1 + \cancel{2sy(s)} - 2 = \frac{1}{s^2+1}$$

$$(-s^2-1)\frac{d}{ds}y(s) - 1 = \frac{1}{s^2+1}$$

$$(-s^2-1)\frac{d}{ds}y(s) = \frac{1}{s^2+1} + 1$$

$$\Rightarrow \quad -(s^2+1)\frac{d}{ds}y(s) = \frac{1}{s^2+1} + 1$$

$$\boxed{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}\left[\sin at - at\cos at\right] \quad \text{formula}}$$

$$\Rightarrow \quad \frac{d}{ds}y(s) = \frac{-1}{(s^2+1)^2} - \frac{1}{s^2+1}$$

$\Rightarrow$ Taking L^{-1},

$$\Rightarrow \quad L^{-1}\left[\frac{d}{ds}\cdot y(s)\right] = -L^{-1}\left[\frac{1}{(s^2+1)^2}\right] - L^{-1}\left[\frac{1}{s^2+1}\right]$$

Using $L[tf(t)] = -\frac{d}{ds}F(s)$

~~Using $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$~~

$$\Rightarrow \quad -t\,y(t) = -\frac{1}{2}(\sin t - t\cos t) - \sin t$$

$$\Rightarrow \quad -t\,y(t) = -\frac{1}{2}[\sin t - t\cos t] - \sin t$$

$$\Rightarrow \quad \boxed{y(t) = \frac{3}{2}\frac{\sin t}{t} - \cos t}$$