Black Holes

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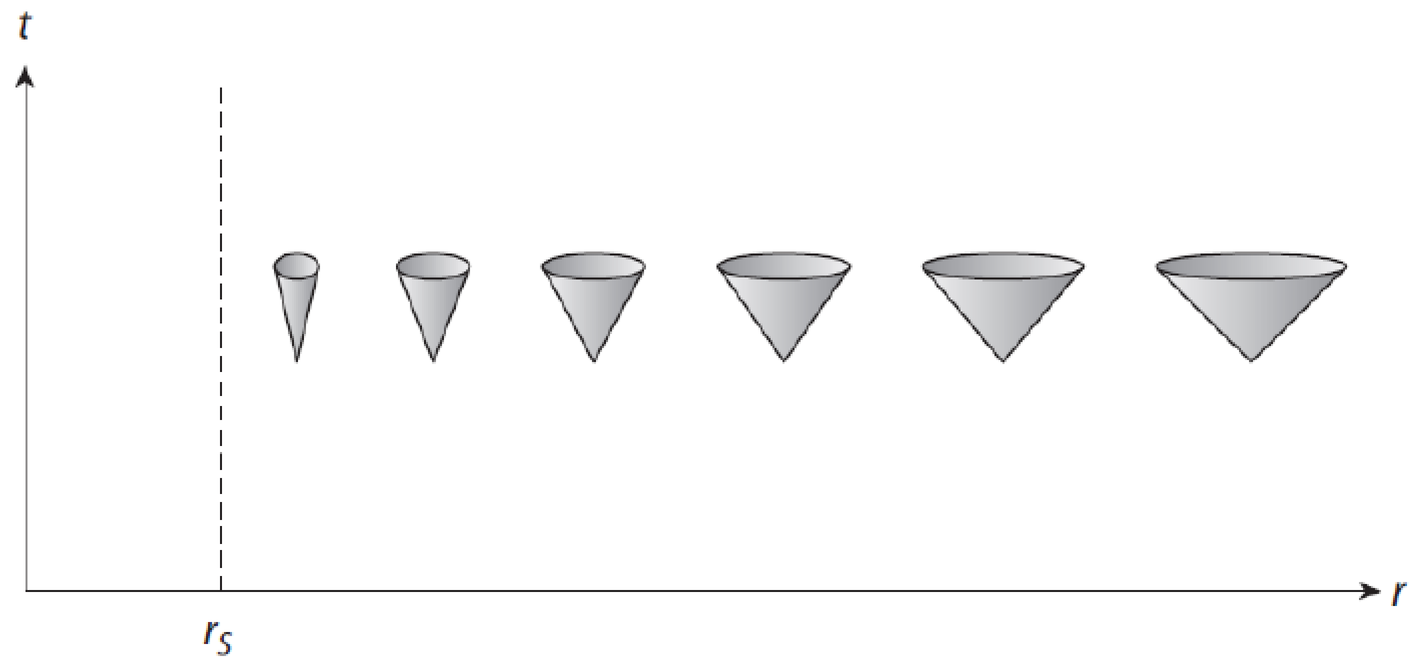
Kruskal-Szekeres Coordinates

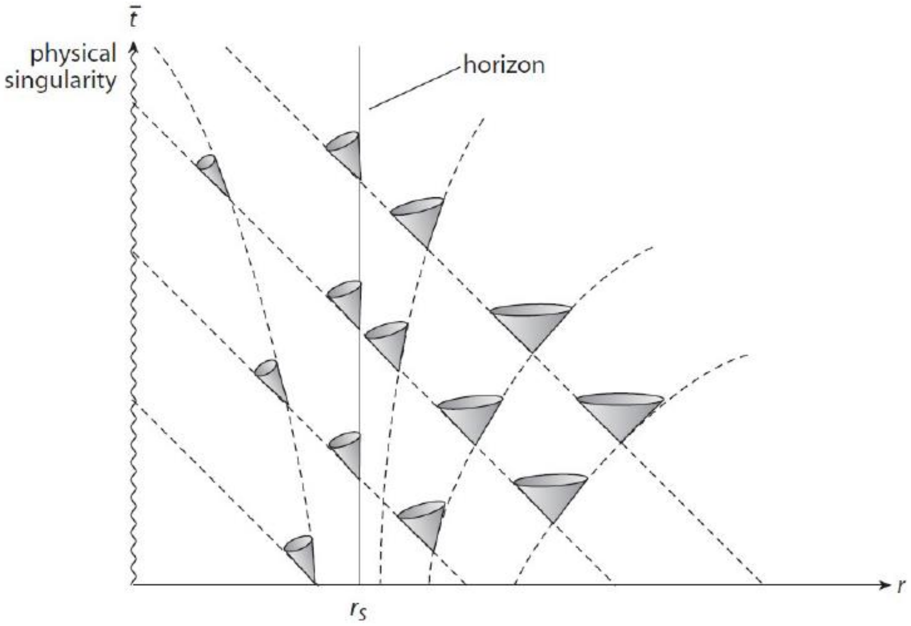
Radial Rays: $dt = \pm \frac{1}{1 - \frac{r_S}{r}} dr = \pm \frac{r}{r - r_S} dr$



$$ds^2 = -\left(1 - \frac{r_S}{r}\right)\left(dt^2 - \frac{1}{\left(1 - \frac{r_S}{r}\right)^2} dr^2\right) + r^2 d\Omega^2$$

$$= -\left(\frac{r - r_S}{r}\right)\left(dt + \frac{r}{r - r_S} dr\right)\left(dt - \frac{r}{r - r_S} dr\right) + r^2 d\Omega^2$$

$$d\bar{t} \equiv dt + \tfrac{r_S}{r-r_S}dr \longrightarrow ds^2 = -\left(\frac{r-r_S}{r}\right)(d\bar{t}+dr)\left(d\bar{t} - \frac{r+r_S}{r-r_S}dr\right) + r^2 d\Omega^2$$



$$dp = dt + \tfrac{r}{r-r_S}dr \qquad dq = dt - \tfrac{r}{r-r_S}dr \longrightarrow ds^2 = -\left(\frac{r-r_S}{r}\right)dp\,dq + r^2 d\Omega^2$$

with $\quad p + q = 2t \qquad p - q = 2r + 2r_S \log \frac{|r-r_S|}{r_S}$

$$P = e^{p/2r_S} \qquad Q = -e^{-q/2r_S} \longrightarrow ds^2 = -\frac{4r_S^3}{r}e^{-r/r_S}\,\mathrm{sign}\,(r-r_S)\,dP\,dQ + r^2 d\Omega^2$$

$$r > r_S \qquad V = \tfrac{1}{2}(P+Q),\, U = \tfrac{1}{2}(P-Q)$$

$$r < r_S \qquad V = \tfrac{1}{2}(P-Q),\, U = \tfrac{1}{2}(P+Q)$$

$$\longrightarrow \quad ds^2 = -\frac{4r_S^3}{r} e^{-r/r_S}\left(dV^2 - dU^2\right) + r^2 d\Omega^2$$

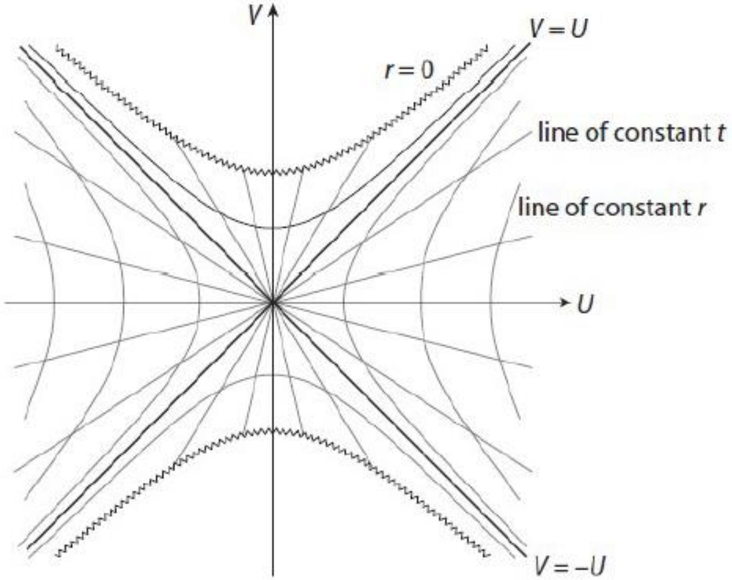
Radial Rays have slope ±1

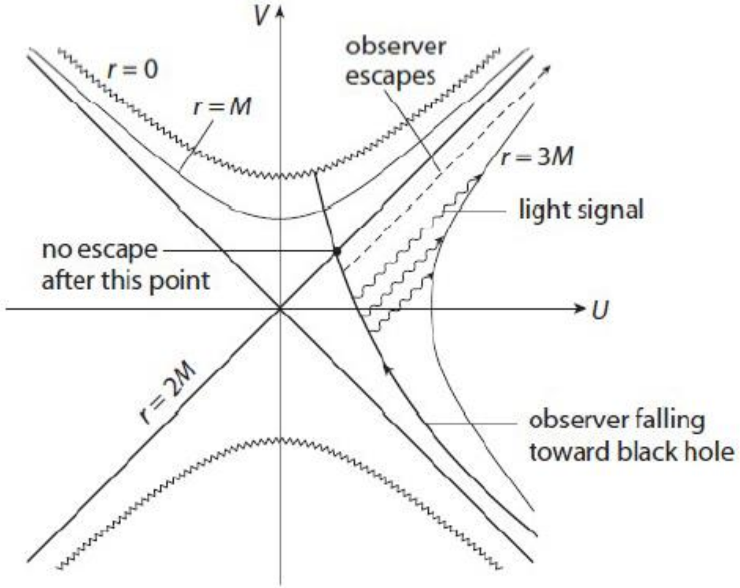
$$\frac{V}{U} = \Theta\left(r - r_S\right)\tanh\frac{t}{2r_S} + \Theta\left(r_S - r\right)\coth\frac{t}{2r_s} \longrightarrow$$

Lines of constant t correspond to straight lines with constant slope passing through the origin.

$$V^2 - U^2 = \left(1 - \frac{r}{r_S}\right) e^{r/r_S} \longrightarrow$$

Lines of constant r are hyperbolae: Vertically oriented for outside horizon and horizontally for inside it.

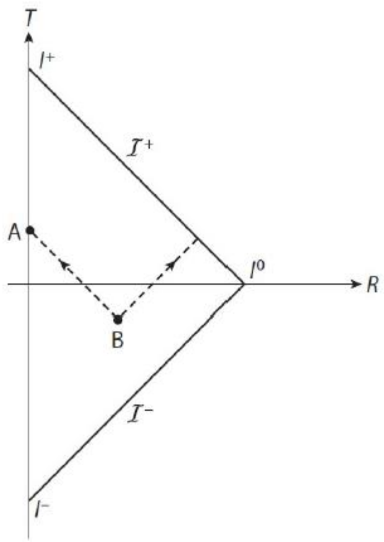




Penrose Diagrams

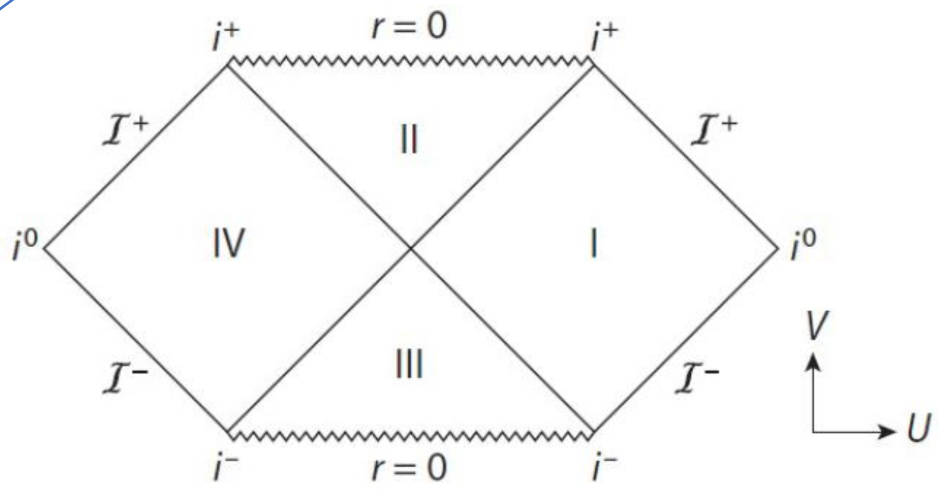
$$
\begin{aligned}
ds^2 &= -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 \\
&= -dpdq + \frac{1}{4}(p-q)^2 d\Omega_{d-2}^2 \\
&= \frac{1}{4\cos^2 P \cos^2 Q}\left(-dT^2 + dR^2 + R^2 d\Omega_{d-2}^2\right) \\
&= \frac{1}{4\cos^2(\frac{T+R}{2})\cos^2(\frac{T-R}{2})}\left(-dT^2 + dR^2 + R^2 d\Omega_{d-2}^2\right)
\end{aligned}
$$

$$t \pm r \to \tan(\tfrac{T \pm R}{2})$$



$$ds^2 = -\frac{4r_S^3}{r} e^{-r/r_S}\left(dV^2 - dU^2\right) + r^2 d\Omega^2 \xrightarrow{V \pm U \to \tan(\frac{T \pm R}{2})} ds^2 = \frac{r_S^3}{r}\frac{e^{\frac{r}{r_S}}\left(-dT^2 + dR^2\right)}{\cos^2(\frac{T+R}{2})\cos^2(\frac{T-R}{2})} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

r = r(T,R) is a solution of $\left(1 - \frac{r}{r_S}\right) e^{\frac{r}{r_S}} = \tan\left(\frac{T+R}{2}\right)\tan\left(\frac{T-R}{2}\right)$



Charged Black Hole: Reissner-Nordstrom Solution

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

Only non-vanishing component: $E = F_{0r} = -F_{r0}$

Spherical symmetry

Maxwell's equation $D_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0 \Rightarrow \partial_r\left(r^2 E/\sqrt{AB}\right) = 0 \Rightarrow E = \frac{Q\sqrt{AB}}{r^2}$

$E(r) \to Q/r^2$ as $r \to \infty$

$$T = g^{\mu\nu}T_{\mu\nu} = F_{\mu\lambda}F^{\mu\lambda} - \frac{1}{4}\delta^\mu_\mu F_{\sigma\rho}F^{\sigma\rho} = 0 \Rightarrow R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$T_{00} = \frac{E^2}{2B} \quad T_{rr} = -\frac{E^2}{2A} \Rightarrow \frac{R_{00}}{A} + \frac{R_{rr}}{B} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 8\pi G\left(\frac{T_{00}}{A} + \frac{T_{rr}}{B}\right) = 0 \Rightarrow$$

Using boundary conditions we have AB=1 as in Schwarzschild Solution

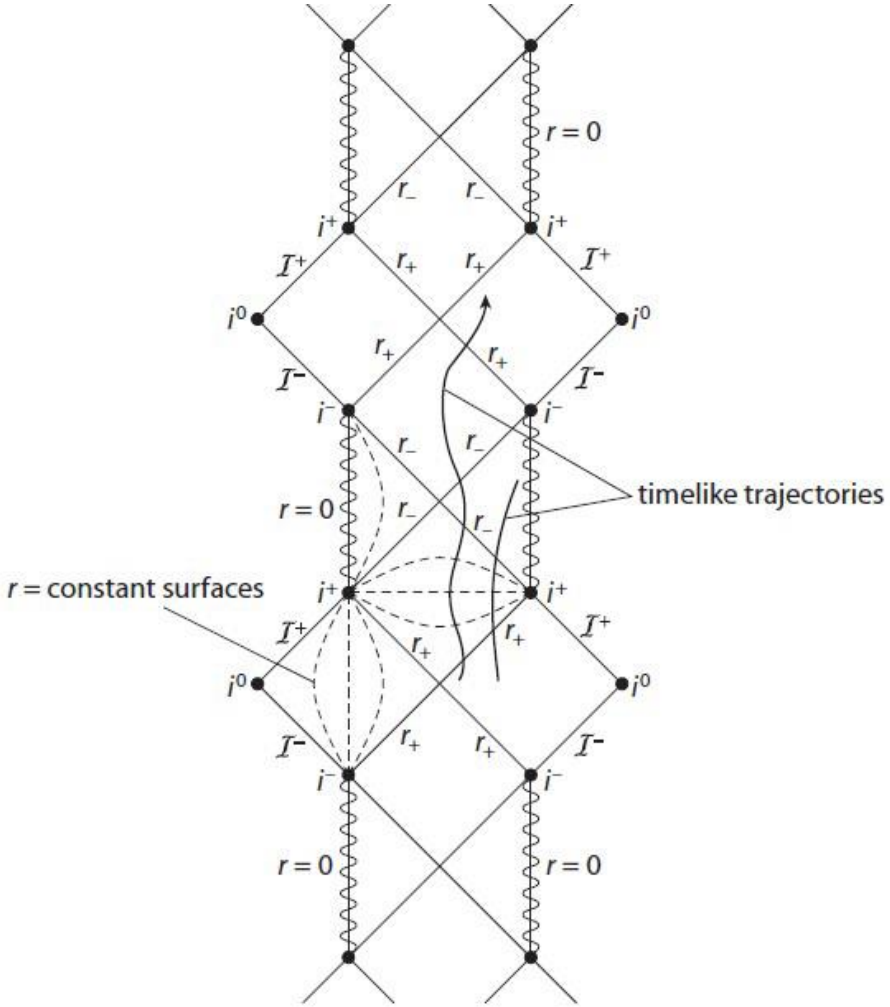
$$T_{\theta\theta} = \frac{Q^2}{2r^2} \Rightarrow R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 1 - A - rA' = 8\pi G T_{\theta\theta} = \frac{4\pi G Q^2}{r^2} \Rightarrow (rA)' = 1 - \frac{4\pi G Q^2}{r^2}$$

Taking $G = 1$ and absorbing factor of 4π by redefining Q

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(\frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}\right) dr^2 + r^2 d\Omega^2$$

$$A(r) = \left(1 - \tfrac{2M}{r} + \tfrac{Q^2}{r^2}\right) = (r - r_+)\,(r - r_-)\,/r^2$$ with $r_\pm = M \pm \sqrt{M^2 - Q^2}$

In subextremal black hole, the result of having two horizons is that if an object crosses the outer horizon from the outside or crosses the inner horizon from inside of it it has no choice but to travel to the other horizon. But an object falling into outer horizon after necessarily falling into the inner horizon can now escape by crossing the inner horizon which is possible since t is now a timelike coordinate. This behaviour can be seen by looking at the Penrose diagram.



Q=M means both horizons coincide and this is called an extremal charged/Reissner-Nordstrom black hole (Extreme RNBH)

Extremal Reissner-Nordstrom Black Hole

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(\frac{1}{1-\frac{M}{r}}\right)^2 dr^2 + r^2 d\Omega^2 \longrightarrow ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(\frac{1}{1-\frac{M}{r}}\right)^2 \left(dr^2 + (r-M)^2 d\Omega^2\right)$$

This suggests we make a change of coordinates as follows

$$\rho = r - M \longrightarrow ds^2 = -f(\rho)^{-2} dt^2 + f(\rho)^2 \left(d\rho^2 + \rho^2 d\Omega^2\right) = -f(\rho)^{-2} dt^2 + f(\rho)^2 \left(dx^2 + dy^2 + dz^2\right) \qquad f(\rho) = 1 + \frac{M}{\rho}$$

Metric Ansatz for Multiple Extremal RNBHs $ds^2 = -f(x,y,z)^{-2} dt^2 + f(x,y,z)^2 \left(dx^2 + dy^2 + dz^2\right)$

$$A_0 = \frac{Q}{\sqrt{4\pi} r} = \frac{M}{\sqrt{4\pi}(\rho + M)} = \frac{1 - f(\rho)^{-1}}{\sqrt{4\pi}}$$

Vector Potential Ansatz for Multiple Extremal RNBHs $A_0 = \frac{1}{\sqrt{4\pi}}\left(1 - f(x,y,z)^{-1}\right)$

NOTE:- This $f(x,y,z)$ is the same function as in the previous ansatz. There is no reason for this to be the case but we will check if doing this allows us to simultaneously solve both Einstein's and Maxwell's equations.

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_i A_0 = \frac{\partial_i f}{\sqrt{4\pi} f^2}, \quad F_{ij} = 0 \qquad \sqrt{-g} = f^2$$

Maxwell's Equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu v}\right) = \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}F^{iv}\right) = 0 \Rightarrow \partial_i\left(f^2 F_{0i}\right) = 0$$

$$\Rightarrow \nabla^2 f = 0$$

$$T_{00} = F_{0i}F_0{}^i - \frac{1}{4}g_{00}F_{\sigma\rho}F^{\sigma\rho} \qquad \Rightarrow 8\pi T_{00} = \frac{(\partial_i f)^2}{f^6}$$

$$T_{11} = F_{10}F_1{}^0 - \frac{1}{4}g_{11}F_{\sigma\rho}F^{\sigma\rho} \qquad \Rightarrow 8\pi T_{11} = \frac{(\partial_i f)^2 - 2(\partial_1 f)^2}{f^2}$$

$$T_{12} = F_{10}F_2{}^0 \qquad \Rightarrow 8\pi T_{12} = -\frac{2(\partial_1 f)(\partial_2 f)}{f^2}$$

$$T_{0i} = 0$$

$$T = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} + 3g^{11}T_{11} = 0 \longrightarrow \text{Einstein's equation is } R_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$R_{00} = \frac{(\partial_i f)^2 - f\nabla^2 f}{f^6} \qquad R_{11} = \frac{(\partial_i f)^2 - 2(\partial_1 f)^2 - f\nabla^2 f}{f^2} \qquad R_{12} = -\frac{2(\partial_1 f)(\partial_2 f)}{f^2}$$

$$R_{\mu\nu} = 8\pi T_{\mu\nu} \longrightarrow \nabla^2 f(x, y, z) = 0$$

Remarkably the same condition as Maxwell's equation!

$$\nabla^2 f(x,y,z) = 0 \Rightarrow f(\vec{x}) = 1 + \sum_{a=1}^{N} \frac{M_a}{|\vec{x} - \vec{x}_a|}$$ By demanding that $A_0 \to 0$ far away to fix additive constant

This is a time independent simultaneous solution of N Extremal RNBHs sitting stationary at various locations.
The physical reason behind this is the fact that any Extremal RNBhs do not either attract or repel each other allows this situation to occur and the reason they don't attract or repel is that their repulsive Coulombic force $= \frac{Q_a Q_b}{4\pi r}$ perfectly cancels the attractive Gravitational force $= \frac{M_a M_b}{4\pi r}$ since Extremality $\Rightarrow Q_{a/b} = \sqrt{4\pi} M_{a/b}$ which in hindsight makes looking for such a solution more hopeful.

Computation of Ricci Tensor

Step 1:- Write down vielbin e from the metric using $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\alpha\beta} e^\alpha e^\beta$.
Step 2:- Use $de + \omega e = 0$ to extract ω.
Step 3:- Use $R = d\omega + \omega^2$ to get Riemann tensor components.
Step 4:- Contract to get Ricci tensor components.

Step 1 $e^0 = f^{-1}(x,y,z)dt,\ e^i = f(x,y,z)dx^i$

Step 2 $de^0 + \left(-\frac{\partial_i f}{f^3}dt + (\ldots)dx^i\right)e^i = 0$ $de^i = (\partial_j f)dx^j dx^i \longrightarrow$ No *dt* term

$$\omega^0{}_i = -\frac{\partial_i f}{f^3}dt = -\frac{\partial_i f}{f^2}e^0$$

$$de^i = (\partial_j f)dx^j dx^i \quad \left\{ \begin{aligned} de^1 &= (\partial_2 f)\, dx^2 dx^1 + (\partial_3 f)\, dx^3 dx^1 \\ de^2 &= (\partial_3 f)\, dx^3 dx^2 + (\partial_1 f)\, dx^1 dx^2 \\ de^3 &= (\partial_1 f)\, dx^1 dx^3 + (\partial_2 f)\, dx^2 dx^3 \end{aligned} \right.$$

$$\omega^1{}_2 = \tfrac{\partial_2 f}{f}dx^1 + (\cdots)dx^2 \qquad \omega^2{}_1 = -\omega^1{}_2 = \tfrac{\partial_1 f}{f}dx^2 + (\cdots)dx^1$$

$$\omega^1{}_2 = \frac{\partial_2 f}{f}dx^1 - \frac{\partial_1 f}{f}dx^2$$

$$\omega^i{}_j = \frac{\partial_j f}{f}dx^i - \frac{\partial_i f}{f}dx^j = \frac{\partial_j f}{f^2}e^i - \frac{\partial_i f}{f^2}e^j$$

Step 3 and 4

$$d\omega^0{}_i = \frac{f\partial_j\partial_i f - 3\left(\partial_i f\right)\left(\partial_j f\right)}{f^4} e^0 e^j \qquad \omega^0{}_\alpha \omega^\alpha{}_i = \omega^0{}_0 \omega^0{}_j + \omega^0{}_k \omega^k{}_i = \omega^0{}_j \omega^j{}_i = \left(-\frac{\partial_j f}{f^2} e^0\right)\left(\frac{\partial_i f}{f^2} e^j - \frac{\partial_j f}{f^2} e^i\right)$$

$$R^0{}_1 = \frac{f\partial_j\partial_1 f - 4\left(\partial_1 f\right)\left(\partial_j f\right)}{f^4} e^0 e^j + \frac{\left(\partial_j f\right)^2}{f^4} e^0 e^1 \qquad R^0{}_{101} = \frac{f\partial_1^2 f - 4\left(\partial_1 f\right)^2 + \left(\partial_i f\right)^2}{f^4}$$

$$R^0{}_0 = \sum_{j=1}^{3} R^0{}_{j0j} = \frac{f\nabla^2 f - 4\left(\partial_i f\right)^2 + 3\left(\partial_i f\right)^2}{f^4} = \frac{f\nabla^2 f - \left(\partial_i f\right)^2}{f^4}$$

$$R_{tt} = e_t^\mu e_t^\nu \eta_{\mu\rho} R^\rho{}_\nu = e_t^0 e_t^0 \eta_{00} R^0{}_0 = -\frac{1}{f^2} R^0{}_0 = \frac{\left(\partial_i f\right)^2 - f\nabla^2 f}{f^6} \qquad R_{tt} = \frac{\left(\partial_i f\right)^2 - f\nabla^2 f}{f^6}$$

$$d\omega^i{}_j = \frac{\left(\partial_j f\right)\left(\partial_k f\right) - f\partial_k\partial_j f}{f^4} e^i e^k + \frac{f\partial_k\partial_i f - \left(\partial_i f\right)\left(\partial_k f\right)}{f^4} e^j e^k$$

$$\omega^i{}_\alpha \omega^\alpha{}_j = \omega^i{}_0 \omega^0{}_j + \omega^i{}_k \omega^k{}_j \qquad \omega^i{}_k \omega^k{}_j = \left(\frac{\partial_k f}{f^2} e^i - \frac{\partial_i f}{f^2} e^k\right)\left(\frac{\partial_j f}{f^2} e^k - \frac{\partial_k f}{f^2} e^j\right)$$

$$R^1{}_2 = d\omega^1{}_2 + \omega^1{}_3\omega^3{}_2$$
$$= d\left[\frac{(\partial_2 f)\,dx^1 - (\partial_1 f)\,dx^2}{f}\right] + \left[\frac{(\partial_3 f)\,dx^1 - (\partial_1 f)\,dx^3}{f}\right]\left[\frac{(\partial_2 f)\,dx^3 - (\partial_3 f)\,dx^2}{f}\right]$$

$$\left.\begin{aligned} R^1{}_{212} = R^2{}_{121} &= \frac{1}{f^4}\left[(\partial_j f)^2 - 2(\partial_3 f)^2 - f\nabla^2 f + f(\partial_3\partial_3 f)\right] \\ R^3{}_{131} &= \frac{1}{f^4}\left[(\partial_j f)^2 - 2(\partial_2 f)^2 - f\nabla^2 f + f(\partial_2\partial_2 f)\right] \\ R^0{}_{101} &= \frac{f\partial_1^2 f - 4(\partial_1 f)^2 + (\partial_i f)^2}{f^4} \end{aligned}\right\}$$

$$R_{11} = \frac{(\partial_j f)^2 - 2(\partial_1 f)^2 - f\nabla^2 f}{f^4}$$

$$R_{xx} = e_x^1 e_x^1 R_{11} = \frac{(\partial_j f)^2 - 2(\partial_1 f)^2 - f\nabla^2 f}{f^2}$$

$$R_{12} = R^0{}_{102} + R^3{}_{132} = R^0{}_{102} + R^1{}_{323}$$

One can replace 2 with 3 in $R^1{}_2$ to get $R^1{}_3$ and extract coefficient of e^2e^3

$$R^1{}_{323} = \frac{2(\partial_1 f)(\partial_2 f) - f\partial_2\partial_1 f}{f^4}$$

$$R^0{}_1 = \frac{f\partial_j\partial_1 f - 4(\partial_1 f)(\partial_j f)}{f^4}e^0e^j + \frac{(\partial_j f)^2}{f^4}e^0e^1 \Rightarrow R^0{}_{102} = \frac{f\partial_2\partial_1 f - 4(\partial_1 f)(\partial_2 f)}{f^4}$$

$$R_{12} = -2\frac{(\partial_1 f)(\partial_2 f)}{f^4}$$

$$R_{xy} = e_x^1 e_y^2 R_{12} = -2\frac{(\partial_1 f)(\partial_2 f)}{f^2}$$