Deep Inelastic Scattering

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1 Elastic Proton-Electron Scattering a.k.a Rutherford Scattering and Mott Formula

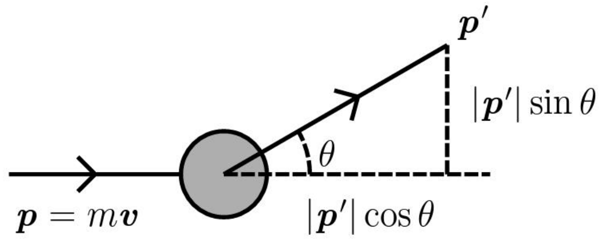


Figure 1: Rutherford Scattering

The Rutherford scattering is the scattering of an electron by the coulomb field of a nucleus. We calculate the cross section by treating the electromagnetic field as fixed classical background given by potential $A_\mu(x)$. Then the interaction Hamiltonian is,

$$H_I = \int \mathrm{d}^3 x e \bar{\psi} \gamma^\mu \psi A_\mu$$

First we calculate the cross section $\mathrm{d}\sigma$ in terms of the matrix elements $i\mathcal{M}$. The incident wave packet $|\psi\rangle$ is defined to be:

$$|\psi\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{e^{-i\mathbf{b}\cdot\mathbf{k}}}{\sqrt{2E_{\mathbf{k}}}} \psi(\mathbf{k}) |\mathbf{k}\rangle$$

where b is the impact parameter. The probability that a scattered electron will be found within an infinitesimal element $\mathrm{d}^3 p$ centered at $\mathbf{p}$ is,

$$\begin{aligned}
P = &\ \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathrm{p}}} \left| {}_{\text{out}} \langle \mathbf{p} \mid \psi \rangle_{\text{in}} \right|^2 \\
&= \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathrm{p}}} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 k'}{(2\pi)^6 \sqrt{2E_{\mathrm{k}} 2E_{\mathbf{k}'}}} \psi(\mathbf{k}) \psi^*(\mathbf{k}') \left({}_{\text{out}} \langle \mathbf{p} \mid \mathbf{k} \rangle_{\text{in}} \right) \left({}_{\text{out}} \langle \mathbf{p} \mid \mathbf{k}' \rangle_{\text{in}} \right)^* e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')} \\
&= \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 k'}{(2\pi)^6 \sqrt{2E_{\mathbf{k}} 2E_{\mathbf{k}'}}} \psi(\mathbf{k}) \psi^*(\mathbf{k}') \left(\langle \mathbf{p} | iT | \mathbf{k} \rangle \right) \left(\langle \mathbf{p} | iT | \mathbf{k}' \rangle \right)^* e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}
\end{aligned}$$

In the last equality we throw away the trivial scattering part from the S -matrix. Note that,

$$\langle p' | iT | p \rangle = i\mathcal{M} (2\pi) \delta \left(E_{\mathbf{p}'} - E_{\mathbf{p}} \right)$$

so we have,

$$P = \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathrm{p}}} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 k'}{(2\pi)^6 \sqrt{2E_{\mathrm{k}} 2E_{\mathrm{k}'}}} \psi(\mathbf{k}) \psi^*(\mathbf{k}') |i\mathcal{M}|^2 (2\pi)^2 \delta\left(E_{\mathbf{p}} - E_{\mathbf{k}} \right) \delta\left(E_{\mathbf{p}} - E_{\mathbf{k}'} \right) e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}$$

The cross section $\mathrm{d}\sigma$ is given by:

$$\mathrm{d}\sigma = \int \mathrm{d}^2 b P(\ \mathrm{b})$$

thus the integration over b gives a delta function:

$$\int \mathrm{d}^2 b e^{-i\mathbf{b}\cdot\left(\mathbf{k}-\mathbf{k}'\right)} = (2\pi)^2 \delta^{(2)}\left(\mathbf{k}_\perp - \mathbf{k}'_\perp\right)$$

The other two delta functions in the integrand can be modified as follows

$$\delta\left(E_{\mathbf{k}} - E_{\mathbf{k}'}\right) = \frac{E_{\mathbf{k}}}{k_\parallel}\delta\left(k_\parallel - k'_\parallel\right) = \frac{1}{v}\delta\left(k_\parallel - k'_\parallel\right)$$

where we have used $|\mathbf{v}| = v = v_\parallel$. Taking all these delta functions into account, we get,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{1}{v}\psi(\mathbf{k})\psi^*(\mathbf{k})|i\mathcal{M}|^2(2\pi)\delta\left(E_{\mathbf{p}} - E_{\mathbf{k}}\right)$$

Since the momentum of the wave packet should be localized around its central value, we can pull out the quantities involving energy $E_{\mathbf{k}}$ outside the integral,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}(2\pi)|\mathcal{M}|^2\delta\left(E_{\mathbf{p}} - E_{\mathbf{k}}\right)\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\psi(\mathbf{k})\psi^*(\mathbf{k}).$$

Recall the normalization of the wave packet,

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\psi(\mathbf{k})^*\psi(\mathbf{k}) = 1$$

then,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\delta\left(E_{\mathbf{p}} - E_{\mathbf{k}}\right)$$

We can further integrate over |p| to get the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \int \frac{\mathrm{d}p p^2}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\frac{1}{2E_{\mathrm{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\delta\left(E_{\mathrm{p}} - E_{\mathrm{k}}\right) \\
&= \int \frac{\mathrm{d}p p^2}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\frac{E_{\mathrm{k}}}{k}\delta(p-k) \\
&= \frac{1}{(4\pi)^2}|\mathcal{M}(k,\theta)|^2
\end{aligned}$$

Above we worked out the integral by virtue of delta function, which constrains the outgoing momentum |p| = |k| but leave the angle θ between p and k arbitrary. Thus the amplitude $\mathcal{M}(k,\theta)$ is a function of momentum $|\mathbf{k}|$ and angle θ. We work directly for the relativistic case. Firstly the Coulomb potential $A^0 = Ze/4\pi r$ in momentum space is

$$A^0(\mathbf{q}) = \frac{Ze}{|\mathbf{q}|^2}$$

This can be easily worked out by Fourier transformation, with a "regulator" e^{-mr} inserted:

$$A^0(\mathbf{q}, m) \equiv \int \mathrm{d}^3 x e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-mr}\frac{Ze}{4\pi r} = \frac{Ze}{|\mathbf{q}|^2 + m^2}$$

This is simply Yukawa potential, and Coulomb potential is a limiting case when $m \to 0$.

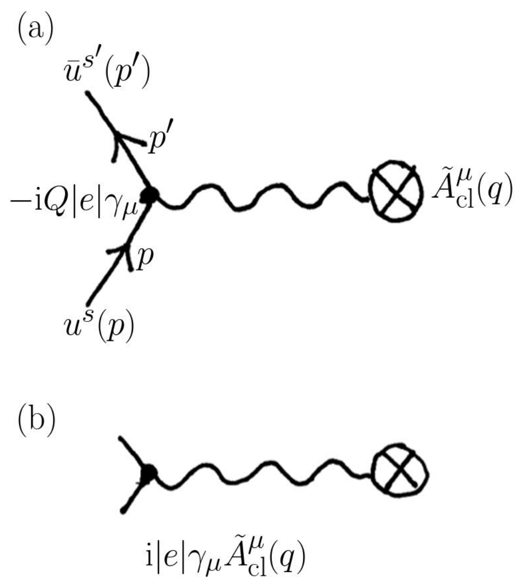


Figure 2: Feynman Diagrams for Rutherford Scattering

We first calculate the T -matrix to lowest order,

$$\begin{aligned}
{}_{\text{out}}\left\langle p' \mid p\right\rangle_{\text{in}} &= \left\langle p' \left| T\exp\left(-i\int \mathrm{d}^4 x H_I\right)\right| p\right\rangle = \left\langle p' \mid p\right\rangle - ie\int \mathrm{d}^4 x A_\mu(x)\left\langle p'\left|\bar{\psi}\gamma^\mu\psi\right| p\right\rangle + O\left(e^2\right) \\
&= \left\langle p' \mid p\right\rangle - ie\int \mathrm{d}^4 x A_\mu(x)\bar{u}\left(p'\right)\gamma^\mu u(p) e^{i\left(p'-p\right)\cdot x} + O\left(e^2\right) \\
&= (2\pi)^4\delta^{(4)}\left(p-p'\right) - ie\bar{u}\left(p'\right)\gamma^\mu u(p)\tilde{A}_\mu\left(p'-p\right) + O\left(e^2\right)
\end{aligned}$$

On the other hand,

$${}_{\text{out}}\left\langle p' \mid p\right\rangle_{\text{in}} = \langle p'|S|p\rangle = \left\langle p' \mid p\right\rangle + \langle p'|iT|p\rangle$$

Thus to the first order of e, we get,

$$\langle p'|iT|p\rangle = -ie\bar{u}\left(p'\right)\gamma^\mu u(p)\tilde{A}_\mu\left(p'-p\right)$$

Therefore the amplitude is given by (replacing p' with k)

$$i\mathcal{M}(k,\theta) = \mathrm{i}e\bar{u}(k)\gamma^\mu\tilde{A}_\mu(\mathbf{q})u(p) \quad \text{with } \mathbf{q}=\mathbf{p}-\mathbf{k}$$

Then we have the squared amplitude with initial spin averaged and final spin summed,

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spin}}\left|\mathcal{M}(k,\theta)\right|^2 &= \frac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\sum_{\text{spin}}\bar{u}(p)\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(p) \\
&= \frac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\operatorname{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\
&= 2e^2\left[2(p\cdot\tilde{A})(k\cdot\tilde{A}) + \left(m^2-(k\cdot p)\right)\tilde{A}^2\right]
\end{aligned}$$

Note that

$$\tilde{A}^0(\mathbf{q}) = \frac{Ze}{|\mathbf{p}-\mathbf{k}|^2} = \frac{Ze}{4|\mathbf{k}|^2\sin^2(\theta/2)}$$

thus

$$\frac{1}{2}\sum_{\text{spin}}|\mathcal{M}(k,\theta)|^2 = \frac{Z^2e^4\left(1-v^2\sin^2\frac{\theta}{2}\right)}{4|\mathbf{k}|^4 v^2\sin^4(\theta/2)}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{Z^2\alpha^2\left(1-v^2\sin^2\frac{\theta}{2}\right)}{4|\mathbf{k}|^2v^2\sin^4(\theta/2)}$$

In non-relativistic case, this formula reduces to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)}$$

2 Electron-Fermion(Quark) Scattering

The amplitude for Moller (Electron-Electron) scattering in high energy limit i.e m_μ,$m_e << \sqrt{s}$ is

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2=2e^4\left[\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}+\frac{2s^2}{tu}\right]$$

This has three parts: the t-channel part, the u-channel part and an interference term. In Electron-Fermion scattering we will have only the t-channel part since u-channel diagram will not be allowed since electron can't change into another Fermion by emitting photon and vice-versa. So the spin averaged squared amplitude will be

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2=\frac{8e^4Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right)$$

where Q_i is charge of Fermion (with Charge of electron = 1).
Note that while $\hat{s},\hat{t},\hat{u}$ are Mandelstram variables of Quark-Electron collision while s,t,u will be reserved for Proton-Electron collision kinematics.
Using $\hat{s}+\hat{t}+\hat{u}=0$ (Masses can be ignored in high-energy limit),the cross-section is

$$\frac{d\sigma}{d\cos\theta_{\mathrm{CM}}}=\frac{1}{2\hat{s}}\frac{1}{16\pi}\frac{8e^4Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right)=\frac{\pi\alpha^2Q_i^2}{\hat{s}}\left(\frac{\hat{s}^2+(\hat{s}+\hat{t})^2}{\hat{t}^2}\right)$$

Using $\hat{t}=-\hat{s}\left(1-\cos\theta_{\mathrm{CM}}\right)/2$ we get

$$\frac{d\sigma}{d\hat{t}}=\frac{2\pi\alpha^2Q_i^2}{\hat{s}^2}\left(\frac{\hat{s}^2+(\hat{s}+\hat{t})^2}{\hat{t}^2}\right)$$

3 Bjorken Scaling

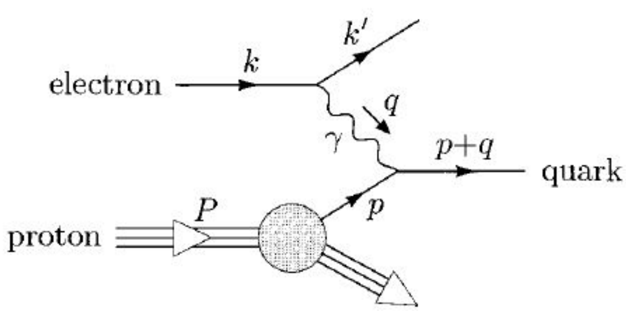


Figure 3: Deep Inelastic Scattering

q^μ is exchange momentum and since that is t-channel: $\hat{t}=q^2=-\hat{s}\left(1-\cos\theta_{\mathrm{CM}}\right)/2<0$.
We define a positive quantity $Q^2\equiv-q^2>0$ for convenience.

We use the parton model here to think of the proton as a loosely bound collection of partons with each parton i has an associated probability that proton contains a parton of type i and that parton carries a fraction ξ of the total momentum of the proton ($p = \xi P$) is $f_i(\xi)$. In high-energy limit we have

$$\hat{s} = (p+k)^2 = 2p \cdot k = 2\xi P \cdot k = \xi (P+k)^2 = \xi s$$

s is Electron-Proton center of mass energy which must be $>> P^2 \approx$1 GeV which we have ignored.
Mass of scattered parton $= (p+q)^2 << s, Q^2$ and can be taken to be ≈ 0.

$$0 \approx (p+q)^2 = 2p \cdot q + q^2 = 2\xi P \cdot q - Q^2$$

This gives a way to calculate ξ from electron momentum measurements alone. We have $\xi = \frac{Q^2}{2P \cdot q}$. Using $\hat{t} = -Q^2$ and $\hat{s} = 2p \cdot k = 2\xi P \cdot k = \xi s$, we have

$$\frac{\hat{t}}{\hat{s}} = -Q^2/\xi s$$

Now we use the above expression in

$$\frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{\hat{s}^2}\left(\frac{\hat{s}^2 + (\hat{s}+\hat{t})^2}{\hat{t}^2}\right)$$

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2 + (\hat{s}+\hat{t})^2}{\hat{s}^2}\right)$$

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{\hat{t}^2}\left(1 + (1 + \frac{\hat{t}}{\hat{s}})^2\right)$$

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{Q^4}\left(1 + (1 - \frac{Q^2}{\xi s})^2\right)$$

For multiple partons,

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \sum_i f_i(\xi)\frac{2\pi\alpha^2 Q_i^2}{Q^4}\left(1 + (1 - \frac{Q^2}{\xi s})^2\right)$$

We also need to integrate over all ξ over its range of values. And noting that scattering cross seections have an implicit modulus sign and hence $\hat{t}$ can be replaced by Q^2.

$$\frac{d\sigma}{dQ^2} = \int_0^1 d\xi \sum_i f_i(\xi) Q_i^2 \frac{2\pi\alpha^2}{Q^4}\left[1 + \left(1 - \frac{Q^2}{\xi s}\right)^2\right]\theta\left(\xi s - Q^2\right)$$

Defining $x = \xi$ and $y = \frac{Q^2}{\xi s}$,

$$\frac{d\sigma}{dx dQ^2} = \sum_i f_i(\xi) Q_i^2 \frac{2\pi\alpha^2}{Q^4}\left[1 + (1-y)^2\right]\theta\left(xs(1-y)\right)$$

By using $\hat{t} = -Q^2 = -xys$ and $dxdQ^2 = \frac{dQ^2}{dy}dxdy = xsdxdy$,

$$\frac{d^2\sigma}{dxdy}\left(e^- p \to e^- X\right) = \left(\sum_i x f_i(x) Q_i^2\right)\frac{2\pi\alpha^2 s}{Q^4}\left[1 + (1-y)^2\right]$$

This expression below

$$\frac{\left(1 + (1 - \frac{Q^2}{\xi s})^2\right)}{Q^4}$$

is in terms of measurable quantities independent of i and can be factored out. The statement of Bjorken Scaling is that the quantity left after dividing this factor out is independent of Q^2. In other words,

$f_i(\xi, Q^2) = f_i(\xi)$. If that were not the case dependence on Q^2 wouldn't go away after factoring out the expression.This is a statement for Q^2 >>1 GeV.

Bjorken Scaling: ***Beyond a certain energy needed to excite partons, the structure of proton visible by probing it electromagnetically is independent of energy.***

This is confirmed by experiments apart from scaling violations which can be accounted for by QCD calculations.

3.1 Bjorken Scaling and Asymptotic Freedom

Bjorken Scaling refers to a scaling of a large class of dimensionless physical quantities in elementary particles; it implies that hadrons have an underlying structure made up of point like constituents when beams of high energy electrons are incident on them. Since a higher energy scale implies a smaller length-one, this phenomenon suggests that no continuous resolution exists which means effectively a point like structure. In QCD,this means that scattering occurs at the level of quarks, rather than at a homogeneous object with the size of a proton. Apparently, at sufficiently high-momentum transfers, quarks behave like free or weakly bound particles. The fact that the strong interaction becomes "weak" at high- energy scales, and vanishes to zero at asymptotically high energies, led to the term "asymptotic freedom".

4 Low Energy Effective Theory for Weak Interaction Mediated by W Bosons

The part of the Standard Model Lagrangian that corresponds to W-Boson exchange is

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \left(W_+^\mu J_\mu^+ + W_-^\mu J_\mu^- \right)$$

The amplitude for the W-Boson exchange can be written as follows

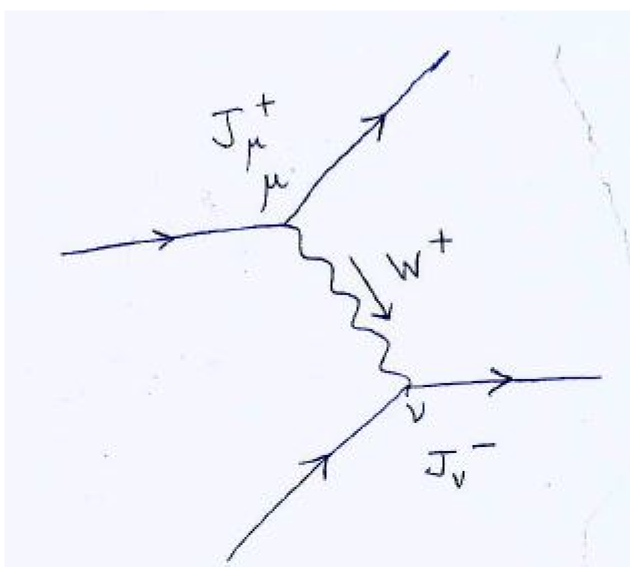


Figure 4: W Boson Exchange

$$\left(\frac{ig}{\sqrt{2}}\right)^2 \int d^4x d^4y J^{+\mu}(x) J^{-\nu}(y) W_\mu^+(x) W_\nu^-(y)$$

Contracting the W's gives propagator of massive vector boson.

$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \int d^4x d^4y \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} J^{+\mu}(x) J^{-\nu}(y) \left(\frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2}\right)}{p^2 - m_W^2}\right)$$

Using Feynman-t'Hooft gauge, we can get rid of $p_\mu p_\nu$ term. Since we are looking for a low energy effective theory, $p^2 << m_W^2$, we can write

$$\frac{-g^2}{p^2 - m_W^2} = \frac{g^2}{m_W^2}\left(1 + \frac{p^2}{m_W^2} + \left(\frac{p^2}{m_W^2}\right)^2 + \cdots\right)$$

Ignoring terms of $\mathcal{O}(\frac{p^2}{m_W^2})$, we get

$$= \frac{-ig^2}{2m_W^2} \int d^4x d^4y \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} J^{+\mu}(x) J^-_\mu(y)$$

Using $\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} = \delta^{(4)}(x-y)$, we get the amplitude to be

$$= \frac{-ig^2}{2m_W^2} \int d^4x J^{+\mu}(x) J^-_\mu(x)$$

Note that the effective interaction at low energies is a single point with the range of W-Boson propagator being too small to notice.

The Lagrangian that gives this amplitude is the low energy effective Lagrangian for weak interactions

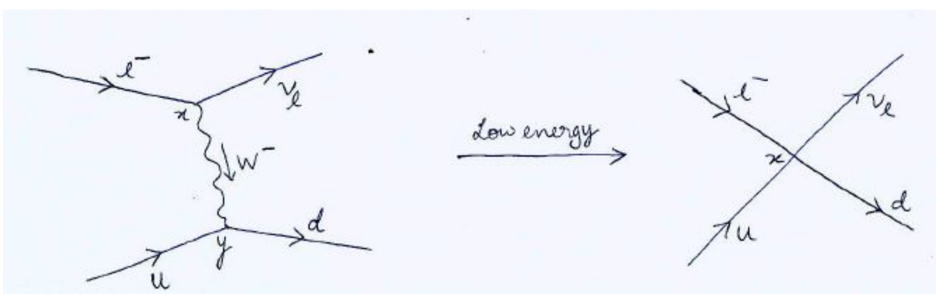


Figure 5: Effective Interaction is at a Single Point

mediated by W bosons given by

$$\mathcal{L}_{W,eff} = \frac{g^2}{2m_W^2} J^{+\mu} J^-_\mu$$

The part of this Lagrangian we are interested in is

$$\mathcal{L}_{DIS\ Neutrino} = \frac{g^2}{2m_W^2}\left[\bar{\ell}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\nu\right]\left[\bar{u}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)d\right] + \text{h.c.}$$

5 DIS with Neutrinos

We have

$$\frac{d\sigma}{d\hat{t}}\left(\nu d \to \mu^- u\right) = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2}\left[\frac{\hat{s}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}$$

and

$$\frac{d\sigma}{d\hat{t}}\left(\bar{\nu} u \to \mu^+ d\right) = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2}\left[\frac{\hat{u}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}(1-y)^2$$

Defining x and y in a similar manner as earlier and assuming proton is made of u and d quarks and their anti-quarks,

$$\frac{d^2\sigma}{dxdy}\left(\nu p - \mu^- X\right) = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x f_{\bar{u}}(x)\cdot(1-y)^2\right]$$
$$\frac{d^2\sigma}{dxdy}\left(\bar{\nu} p \rightarrow \mu^+ X\right) = \frac{G_F^2 s}{\pi}\left[x f_u(x)\cdot(1-y)^2 + x f_{\bar{d}}(x)\right]$$

The constant nature of curve for neutrino(ν) curve and $(1-y)^2$ nature of anti-neutrino($\bar{\nu}$) curve imply that anti-quark component is largely absent. So DIS with neutrinos helps us extract information about proton structure in this manner.

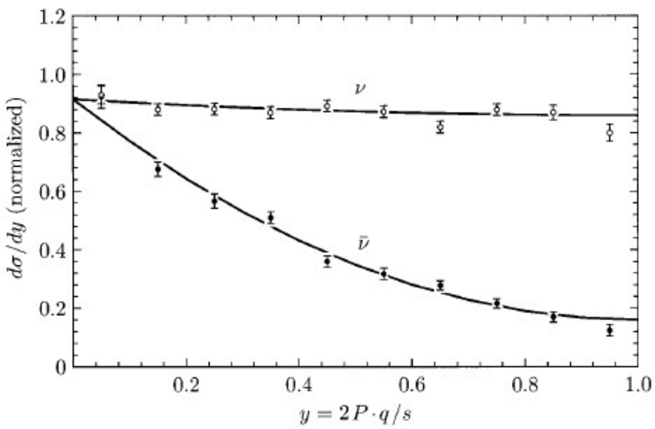


Figure 6: Experimental observation of neutrino and anti-neutrino scattering from iron target. Fits are to $A + B(1-y)^2$

5.1 Calculation of Neutrino DIS Cross-Section

$$\begin{aligned}
\left[\bar{l}(k)\gamma^\mu\left(\frac{1-\gamma^\mu}{2}\right)v(p)\right] &= \left[l^\dagger(k)\gamma^0\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)v(p)\right]^\dagger \\
&= \left[v^\dagger(p)\left(\frac{1-\gamma^5}{2}\right)\gamma^{\mu\dagger}\gamma^0 l(k)\right] \\
&= \left[v^\dagger(p)\left(\frac{1-\gamma^5}{2}\right)\gamma^0\gamma^\mu l(k)\right] \\
&= \left[v^\dagger(p)\gamma^0\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)l(k)\right] \\
&= \left[\overline{v}(p)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)l(k)\right]
\end{aligned}$$

$$\mathcal{M} = \frac{g^2}{2m_W^2}\left[\bar{l}(k)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)v(p)\right]\left[\overline{u}(k')\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)d(p')\right]$$

$$|\mathcal{M}|^2 = \frac{g^4}{4m_W^4}\left[\overline{v_s}(p)_\alpha\left[\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\right]_{\alpha\beta} l_\gamma(k)_\beta\right]\cdot\left[\overline{l_r}(k)_\gamma\left[\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]_{\gamma\delta} v_s(p)_\delta\right]$$
$$\cdot\left[\overline{d}_{s'}(p')_{\alpha'}\left[\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\right]_{\alpha'\beta'} u_{r'}(k')_{\beta'}\right]\cdot\left[\overline{u}_{r'}(k')_{\gamma'}\left[\gamma_\nu(\frac{1-\gamma^5}{2})\right]_{\gamma'\delta'} d_{s'}(p')_{\delta'}\right]$$

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{r,r',s,s'}|\mathcal{M}|^2\\
&= \frac{g^4}{8m_W^4}\,\mathrm{Tr}\left[\not{p}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\not{k}\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\cdot\mathrm{Tr}\left[\not{p}'\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\not{k}'\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]\\
&= \frac{g^4}{8m_W^4}\,\mathrm{Tr}\left[\not{p}\gamma^\mu\not{k}\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\mathrm{Tr}\left[\not{p}'\gamma_\mu\not{k}'\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]\end{aligned}$$

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{g^4}{8m_W^4}\cdot 2\big[p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}p\cdot k + i\epsilon^{\alpha\mu\beta\nu}p_\alpha k_\beta\big]\cdot 2\big[p'_\mu k'_\nu + k'_\mu p_\nu - g^{\mu\nu}p'\cdot k' + i\epsilon_{\rho\mu\sigma\nu}p'^\rho k'^\sigma\big]\\
&= \frac{g^4}{2m_W^4}[2(p\cdot p')(q\cdot q') + 2(p\cdot k')(p'\cdot k) + 0(p\cdot k)(p'\cdot k') - \epsilon^{\alpha\mu\vee\nu}\epsilon_{\rho\mu\sigma\nu}p_\alpha k_\beta p'^\rho k'^\sigma]\end{aligned}$$

Using

$$\begin{aligned}-\epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}p_\alpha k_\beta p'^\rho k'^\rho &= 2\big(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho\big)p_\alpha k_\beta p'^\rho k'^\sigma\\
&= 2[(p\cdot p')(k\cdot k') - (p\cdot k')(k\cdot p')]\end{aligned}$$

Substituting, we get

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{2g^4}{m_W^4}(p\cdot p')(k\cdot k')\\
&= \frac{g^4}{2m_W^4}\hat{s}^2\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{|\mathcal{M}|^2}{64\pi^2E_{CM}^2} = \frac{g^4\hat{s}^2}{128\pi^2 m_W^4\hat{s}}\\
\frac{\mathrm{d}}{\mathrm{d}\Omega} &= \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)} = \frac{\hat{s}}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\hat{t}};\qquad \hat{t} = -\frac{\hat{s}}{2}(1-\cos\theta)\\
\frac{\mathrm{d}\sigma}{\mathrm{d}\hat{t}} &= \frac{g^4\hat{s}^2}{128\pi^2m_W^4\hat{s}}\cdot\frac{4\pi}{\hat{s}} = \frac{g^4}{32\pi m_W^4} = \frac{G_F^2}{\pi}\end{aligned}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\hat{t}}\big(\nu d\to\mu^- u\big) = \frac{G_F^2}{\pi}$$

$$\begin{aligned}q^2 = \hat{t} &= (p-k)^2 = -Q^2\\
&= -2p\cdot k = -2p'\cdot k'\\
\hat{s} &= (p+p')^2 = 2p\cdot p' = 2k\cdot k'\\
\hat{u} &= (p-k')^2 = -2p\cdot k' = -2k\cdot p'\end{aligned}$$

$$\mathcal{M} = \frac{g^2}{2m_W^2}\left[\overline{v}(p)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)l(k)\right]\cdot\left[\overline{d}(k')\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)u(p')\right]$$

$$|\mathcal{M}|^2 = \frac{g^4}{4m_W^4}\left[\bar{l}_r(k)_\alpha\left[\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\right]_{\alpha\beta}v_s(p)\right]\cdot\left[\overline{v}_s(p)_\gamma\left[\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\right]_{\alpha\beta}l_r(k)_\gamma\right]$$
$$\cdot\left[\overline{u}_{s'}(p')_{\alpha'}\left[\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\right]_{\alpha'\beta'}d_{r'}(k')_{\beta'}\right]\cdot\left[\overline{d}_{r'}(k')_{\gamma'}\left[\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]_{\gamma'\delta'}u_{s'}(p')_{\delta'}\right]$$

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{r,r',s,s'}|\mathcal{M}|^2\\ &= \frac{g^4}{8m_W^4}\,\mathrm{Tr}\left[\not{k}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\not{p}\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\cdot\mathrm{Tr}\left[\not{p}'\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)\not{k}'\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]\\ &= \frac{g^4}{8m_W^4}\,\mathrm{Tr}\left[\not{k}\gamma^\mu\not{p}\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)\right]\cdot\mathrm{Tr}\left[\not{p}'\gamma_\mu\not{k}'\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]\end{aligned}$$

Same as for $(\nu d\to\mu^- u)$ with $k\leftrightarrow p$

$$\overline{|\mathcal{M}|^2} = \frac{2g^4}{m_W^4}(k\cdot p')(p\cdot k') = \frac{g^4}{2m_W^4}\hat{u}^2 = \left(\frac{g^4\hat{s}^2}{2m_W^4}\right)\frac{\hat{u}^2}{\hat{s}^2}$$

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{|\mathcal{M}|}{64\pi^2E_{CM}^2}\\ \frac{\mathrm{d}\sigma}{\mathrm{d}\hat{t}} &= \frac{G_F^2}{\pi}\cdot\frac{\hat{u}^2}{\hat{s}^2}\end{aligned}$$

$$\frac{\hat{t}}{\hat{s}} = \frac{-Q^2}{\xi s}\quad\frac{\hat{t}+\hat{u}}{\hat{s}} = -1\quad\frac{\hat{u}}{\hat{s}} = -\left(1-\frac{Q^2}{\xi s}\right)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\hat{t}}(\overline{\nu}\mu\to\mu^+d) = \frac{G_F^2}{\pi}\left(1-\frac{Q^2}{\xi s}\right)^2$$

6 Deep Inelastic Scattering

In DIS, spin-averaging leads to a leptonic tensor and a hadronic tensor with two indices each contracted with each other.

$$\frac{d\sigma}{d\Omega}\propto L^{\mu\nu}W_{\mu\nu}$$

The leptonic tensor is

$$L_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left(\not{k}\gamma_\mu\not{k}'\gamma_\nu\right) = 2\left(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}\left(k'\cdot k\right)\right)$$

The hadronic tensor is the modulus squared of an expression which if multiplied by polarization vector gives us the amplitude. So hadronic tensor contracted with polarization vector and its conjugate gives the rate of process $\gamma^\star p\to$ anything. Note that $\gamma^\star$ measn that the photon is off-shell.

$$e^2\epsilon_\mu\epsilon_\nu^\star W^{\mu\nu} = \frac{1}{2}\sum_{X,\ \text{spins}}\int d\Pi_X(2\pi)^4\delta^4\left(q+P-p_X\right)\left|\mathcal{M}\left(\gamma^\star p\to X\right)\right|^2$$

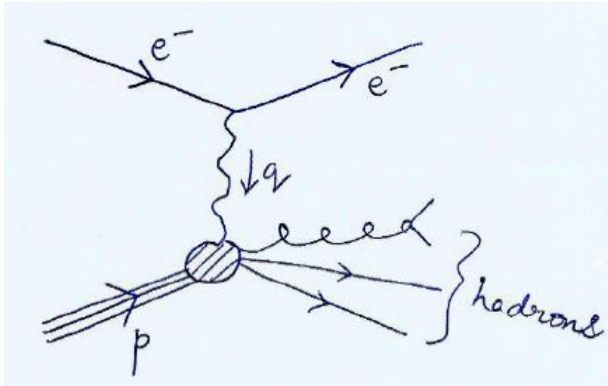


Figure 7: DIS

We have $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$ by the Ward Identity.

$$W_{\mu\nu} = a_1\eta_{\mu v} + a_2 q_\mu q_\nu + a_3 p_\mu p_\nu + a_4 p_\mu q_\nu + a_5 p_\nu q_\mu$$

$$q^\mu W_{\mu\nu} = 0 \Rightarrow a_1 q_\nu + a_2 q^2 q_\nu + a_3 (p\cdot q) p_\nu + a_4 (p\cdot q) q_\nu + a_5 q^2 p_\nu = 0$$

Hence we have $a_1 + a_2 q^2 + a_4(p\cdot q) = 0$ and $a_3(p\cdot q) + a_5 q^2 = 0$.

$$q^\nu W_{\mu\nu} = 0 \Rightarrow a_1 q_\mu + a_2 q^2 q_\mu + a_3 (p\cdot q) p_\mu + a_4 q^2 p_\mu + a_5 (p\cdot q) q_\mu = 0$$

Hence we have $a_1 + a_2 q^2 + a_5(p\cdot q) = 0$ and $a_3(p\cdot q) + a_4 q^2 = 0$.

$$\Rightarrow a_4 = a_5 = -\frac{a_1 + a_2 q^2}{p\cdot q} = -\frac{a_3(p\cdot q)}{q^2}$$

Using this we can write the most general way to write the hadronic tensor as

$$W^{\mu\nu} = W_1\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

Now a simple contraction gives

$$\frac{d\sigma}{d\Omega} \propto \frac{m_p}{2} W_2(x,Q)\cos^2\frac{\theta}{2} + \frac{1}{m_p} W_1(x,Q)\sin^2\frac{\theta}{2}$$

The exact expression turns out to be

$$\left(\frac{d\sigma}{d\Omega dE'}\right)_{\text{lab}} = \frac{\alpha_e^2}{8\pi E^2 \sin^4\frac{\theta}{2}}\left[\frac{m_p}{2} W_2(x,Q)\cos^2\frac{\theta}{2} + \frac{1}{m_p} W_1(x,Q)\sin^2\frac{\theta}{2}\right]$$

Now we turn to the parton model with the classical probabilities $f_i(\xi)d\xi$ of the photon hitting parton species i which has a fraction ξ of the proton momentum P. $f_i(\xi)$ are the parton distribution functions (PDFs). Momentum exchanges among partons at time scales $\sim \Lambda_{\text{QCD}}^{-1} \sim m_p^{-1}$. These time scales are much slower than the time scales $\sim Q^{-1}$ that the photon probes. The separation of scales $Q \gg \Lambda_{\text{QCD}}$ justifies using parton model.

$$\sigma\left(e^- P \to e^- X\right) = \sum_i \int_0^1 d\xi f_i(\xi)\hat{\sigma}\left(e^- p_i \to e^- X\right)$$

Since the quarks (partons) interact electromagnetically with the electron we can take $F_1 = 1$ and $F_2 = 0$ in the Rosenbluth Formula given below

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha_e^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\left\{\left(F_1^2 - \frac{q^2}{4m_p^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right\}$$

to get

$$\left(\frac{d\hat{\sigma}\left(e^- q \to e^- q\right)}{d\Omega dE'}\right)_{\text{lab}} = \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_q^2}\sin^2\frac{\theta}{2}\right] \delta\left(E - E' - \frac{Q^2}{2m_q}\right)$$

Since $E - E' = \frac{Q^2}{2m_p x}$,

$$\delta\left(E - E' - \frac{Q^2}{2m_q}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p}{Q^2} x^2 \delta(\xi - x)$$

and we obtain

$$\left(\frac{d\sigma\left(e^- P \to e^- X\right)}{d\Omega dE'}\right)_{\text{lab}} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p}\sin^2\frac{\theta}{2}\right]$$

Comparing the expressions from general considerations and that of the parton model, we get

$$W_1(x, Q) = 2\pi \sum_i Q_i^2 f_i(x)$$

$$W_2(x, Q) = 8\pi \frac{x^2}{Q^2} \sum_i Q_i^2 f_i(x)$$

We see that Bjorken scaling i.e $f_i(x, Q) = f_i(x)$ implies $W_1(x, Q) = W_1(x)$ is independent of Q^2 and so is $Q^2 W_2(x, Q)$. And we also have

$$W_1(x, Q) = \frac{Q^2}{4x^2} W_2(x, Q)$$

known as the Callan-Gross relation which lets us know that quarks are spin-$\frac{1}{2}$ particles.

6.1 Derivation of Rosenbluth Formula

Consider the elastic scattering of a relativistic electron off of a proton while correcting the vertex function of the photon. The amplitude for this process is:

$$i\mathcal{M} = \bar{u}\left(k'\right)\left(-ie\gamma_\mu\right) u(k) \frac{-i}{q^2} \bar{u}\left(p'\right)\left(-ie\Gamma^\mu\right) u(p)$$

The generalized vertex function is of the form,

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$$

Inserting this into the amplitude and using the Gordon identity, we get

$$\begin{aligned}
i\mathcal{M} &= i\frac{e^2}{q^2} \bar{u}\left(k'\right)\gamma_\mu u(k) \bar{u}\left(p'\right)\Gamma^\mu u(p) \\
&= i\frac{e^2}{q^2} \bar{u}\left(k'\right)\gamma_\mu u(k) \bar{u}\left(p'\right)\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2\right) u(p) \\
&= i\frac{e^2}{q^2} \bar{u}\left(k'\right)\gamma_\mu u(k) \bar{u}\left(p'\right)\left(\gamma^\mu F_1 + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2 + \frac{(p' + p)^\mu}{2m} F_2 - \frac{(p' + p)^\mu}{2m} F_2\right) u(p) \\
&= i\frac{e^2}{q^2} \bar{u}\left(k'\right)\gamma_\mu u(k) \bar{u}\left(p'\right)\left(\gamma^\mu \left(F_1 + F_2\right) - \frac{(p' + p)^\mu}{2m} F_2\right) u(p)
\end{aligned}$$

Spin-averaging the amplitude squared , we see that

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{e^4}{4q^4}\sum_{\text{spin}} \bar{u}\left(k'\right)\gamma_\mu u(k)\bar{u}\left(p'\right)\left(\gamma^\mu\left(F_1+F_2\right)-\frac{\left(p'+p\right)^\mu}{2m}F_2\right)\\
&u(p)\bar{u}(p)\left(\gamma^\nu\left(F_1+F_2\right)-\frac{\left(p'+p\right)^\nu}{2m}F_2\right)u\left(p'\right)\bar{u}(k)\gamma_\nu u\left(k'\right)\\
=&\frac{e^4}{4q^4}\operatorname{Tr}\left[\left(k'+m_e\right)\gamma_\mu\left(k+m_e\right)\gamma_\nu\right]\times\\
&\left\{\left(F_1+F_2\right)^2\operatorname{Tr}\left[\left(\not{p}'+m\right)\gamma^\mu(\not{p}+m)\gamma^\nu\right]-F_2\left(F_1+F_2\right)\frac{\left(p'+p\right)^\nu}{2m}\operatorname{Tr}\left[\left(p'+m\right)\gamma^\mu(p+m)\right]\right.\\
&\quad-F_2\left(F_1+F_2\right)\frac{\left(p'+p\right)^\mu}{2m}\operatorname{Tr}\left[\left(p'+m\right)(\not{p}+m)\gamma^\nu\right]\\
&\quad\left.+F_2^2\frac{\left(p'+p\right)^\mu\left(p'+p\right)^\nu}{4m^2}\operatorname{Tr}\left[\left(p'+m\right)(\not{p}+m)\right]\right\}\\
=&\frac{4e^4}{q^4}\left(k'_\mu k_\nu+k'_\nu k_\mu-g_{\mu\nu}\left(k'\cdot k-m_e^2\right)\right)\times\\
&\left\{\left(F_1+F_2\right)^2\left(p'^\mu p^\nu+p'^\nu p^\mu-g^{\mu\nu}\left(p'\cdot p-m^2\right)\right)-F_2\left(F_1+F_2\right)\left(p'+p\right)^\mu\left(p'+p\right)^\nu\right.\\
&\quad\left.+\frac{F_2^2}{4m^2}\left(p'+p\right)^\mu\left(p'+p\right)^\nu\left(p'\cdot p+m^2\right)\right\}\\
=&\frac{4e^4}{q^4}\left(k'_\mu k_\nu+k'_\nu k_\mu-g_{\mu\nu}\left(k'\cdot k-m_e^2\right)\right)\times\\
&\left\{\left(F_1+F_2\right)^2\left(p'^\mu p^\nu+p'^\nu p^\mu-g^{\mu\nu}\left(p'\cdot p-m^2\right)\right)+\left(p'+p\right)^\mu\left(p'+p\right)^\nu\left(\frac{p'\cdot p+m^2}{4m^2}F_2^2-F_2\left(F_1+F_2\right)\right)\right\}\\
=&\frac{8e^4}{q^4}\left[\left(F_1+F_2\right)^2\left(k'\cdot p'k\cdot p+k'\cdot pk\cdot p-k'\cdot km^2-p'\cdot pm_e^2+2m^2m_e^2\right)\right.\\
&\left.+\left(\frac{p'\cdot p+m^2}{4m^2}F_2^2-F_2\left(F_1+F_2\right)\right)\left(k'\cdot\left(p'+p\right)k\cdot\left(p'+p\right)-\frac{1}{2}\left(k'\cdot k-m_e^2\right)\left(p'+p\right)^2\right)\right]
\end{aligned}$$

In the initial rest frame of the proton: $p=(m,\overrightarrow{0}), k=(E,E\hat{z}), k'=\left(E',\overrightarrow{k'}\right), p'=\left(E-E'+m,-\vec{k}\right)$. With $p'-p=q=k-k'$ and $p\cdot p'=m^2+Em-E'm$, we compute p^2:

$$p'^2=(p+q)^2=p^2+2p\cdot q+q^2=m^2+2p\cdot\left(p'-p\right)+q^2=-m^2+2p'\cdot p+q^2=m^2+2Em-2E'm+q^2=m^2$$
$$\Longrightarrow q^2=2E'm-2Em$$
$$\Longrightarrow E'=E+\tfrac{q^2}{2m}$$

Since $E'=E+\frac{q^2}{2m}$:

$$q^2=-4E^2\sin^2\tfrac{\theta}{2}-\tfrac{q^2}{2m}4E\sin^2\tfrac{\theta}{2}$$
$$\Longrightarrow q^2=-\frac{4E^2\sin^2\frac{\theta}{2}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}$$

So,

$$p'^2=m^2=p^2+2p\cdot q+q^2=-m^2+2p'\cdot p+q^2$$
$$p'^2=m^2=p^2+2p\cdot q+q^2=m^2+2p\cdot k-2p\cdot k'+q^2$$
$$k'^2=0=k^2-2k\cdot q+q^2=2k\cdot k'+q^2=0$$
$$k'^2=0=k^2-2k\cdot q+q^2=-2k\cdot p'+2k\cdot p+q^2$$
$$k^2=0=k'^2+2q\cdot k'+q^2=2p'\cdot k'+q^2$$

Thus, we get identities as follows

$$\begin{array}{lll} k' \cdot k = -\frac{q^2}{2} & p' \cdot p = m^2 - \frac{q^2}{2} & k' \cdot p = Em + \frac{q^2}{2} \\ p' \cdot k = Em + \frac{q^2}{2} & k' \cdot p' = Em & p \cdot k = Em \end{array}$$

$$\Rightarrow k \cdot (p' + p) = 2Em + \frac{q^2}{2}, \quad k' \cdot (p' + p) = 2Em + \frac{q^2}{2}, \quad \text{and} \quad (p + p')^2 = 4m^2 - q^2$$

We had

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4} \left[\begin{array}{c} (F_1 + F_2)^2 \overbrace{\left(k' \cdot p' k \cdot p + k' \cdot pk \cdot p - k' \cdot km^2 - p' \cdot pm_e^2 + 2m^2 m_e^2\right)}^{\text{i}} \\ + \\ \underbrace{\left(\frac{p' \cdot p + m^2}{4m^2} F_2^2 - F_2 (F_1 + F_2)\right)}_{\text{ii}} \underbrace{\left(k' \cdot (p' + p)\, k \cdot (p' + p) - \frac{1}{2}\left(k' \cdot k - m_e^2\right)(p' + p)^2\right)}_{\text{iii}} \end{array} \right]$$

In the approximation $k^2 \approx 0$, we take $m_e \to 0$. Thus computing each term separately:
i.

$$(k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p') - (k' \cdot k)\, m^2 = (Em)^2 + (Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2} m^2$$

ii.

$$\begin{aligned} \frac{p' \cdot p + m^2}{4m^2} F_s^2 - F_1 F_2 - F_2^2 &= \frac{1}{2} F_2^2 - \frac{q^2}{8m^2} F_2^2 - F_1 F_2 - F_2^2 \\ &= -\frac{1}{2}\left[\left(F_2^2 + 2F_1F_2 + F_1^2 - F_1^2 + \frac{q^2}{4m^2} F_2^2\right]\right. \\ &= -\frac{1}{2}\left[\left((F_1 + F_2)^2 - \left(F_1^2 - \frac{q^2}{4m^2} F_2^2\right)\right]\right. \end{aligned}$$

iii.

$$(k' \cdot (p' + p))(k \cdot (p' + p)) - \frac{1}{2}(k' \cdot k)(p' + p)^2 = 4(Em)^2 + 2Emq^2 + \frac{q^4}{4} + q^2 m^2 - \frac{q^4}{4}$$

For the coefficient of $(F_1 + F_2)^2$:

$$2(Em)^2 + Emq^2 + \frac{q^4}{4} + \frac{q^2}{2} m^2 - 2(Em)^2 - Emq^2 - \frac{q^2}{2} m^2 = \frac{q^4}{4} = -\frac{2E^2 m^2}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \frac{q^2}{2m^2}$$

For the coefficient of $\left(F_1^2 - \frac{q^2}{4m^2} F_2^2\right)$:

$$\begin{aligned} 2(Em)^2 + Emq^2 + \frac{q^2}{2} m^2 &= 2E^2 m^2 - \frac{4E^3 m \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} - \frac{2E^2 m^2 \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \\ &= \frac{2E^2 m^2 + 4E^3 m \sin^2\frac{\theta}{2} - 4E^3 m \sin^2\frac{\theta}{2} - 2m^2 E^2 \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \\ &= 2E^2 m^2 \frac{1 - \sin^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} = \frac{2E^2 m^2 \cos^2\frac{\theta}{2}}{1 + \frac{2E}{m}\sin^2\frac{\theta}{2}} \end{aligned}$$

Thus, combining everything we get

$$\overline{|\mathcal{M}|^2} = \frac{16e^4 E^2 m^2}{q^4\left(1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right)} \left[\left(F_1^2 - \frac{q^2}{4m^2} F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}(F_1 + F_2)^2 \sin^2\frac{\theta}{2}\right]$$

Now we shall compute the differential cross section $\frac{d\sigma}{d\cos\theta}\Big|_{\text{lab}}$ by computing the general cross section.

$$\begin{aligned}
d\sigma &= \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}\left|v_{\mathcal{A}}-v_{\mathcal{B}}\right|}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\overline{|\mathcal{M}|^2}(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum p_f\right)\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E_p'}\delta^{(4)}\left(p+k-p'-k'\right)
\end{aligned}$$

Integrating,

$$\begin{aligned}
\sigma = \int d\sigma &= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{d^3p'd^3k'}{(2\pi)^2}\frac{1}{4E'E_p'}\delta^{(4)}\left(p+k-p'-k'\right)\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{d^3k'}{(2\pi)^2}\frac{1}{4E'E_p'}\delta^{(1)}\left(E'-E-m+\sqrt{m^2+E^2+E'^2-2EE'\cos\theta}\right)\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{E'^2dEd\Omega}{(2\pi)^2}\frac{1}{4E'E_p'}\delta^{(1)}\left(E'-E-m+\sqrt{m^2+E^2+E'^2-2EE'\cos\theta}\right)\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{d\Omega}{(2\pi)^2}\frac{E'}{4E_p'}\left(1+\frac{E'-E\cos\theta}{E_p'}\right)^{-1}\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{d\Omega}{(2\pi)^2}\frac{E'}{4E_p'}\left(\frac{E_p'}{E_p'+E'-E\cos\theta}\right)\\
&= \frac{1}{4mE}\overline{|\mathcal{M}|^2}\int \frac{d\cos\theta}{(2\pi)}\frac{E'}{4E_p'}\left(\frac{E_p'}{E_p'+E'-E\cos\theta}\right)\\
&= \frac{1}{32\pi mE}\overline{|\mathcal{M}|^2}\int d\cos\theta\frac{E'}{m+E(1-\cos\theta)}\\
&= \frac{1}{32\pi m^2E}\overline{|\mathcal{M}|^2}\int d\cos\theta\frac{E'}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}\\
&= \frac{1}{32\pi m^2E}\overline{|\mathcal{M}|^2}\int d\cos\theta\frac{E+\frac{q^2}{2m}}{1+\frac{2E}{m}\sin^2\frac{\theta}{2}}\\
&= \frac{1}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\overline{|\mathcal{M}|^2}\cos\theta
\end{aligned}$$

So,

$$\frac{d\sigma}{d\cos\theta}\bigg|_{\text{lab}} = \frac{1}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\overline{|\mathcal{M}|^2}$$

We have the Rosenblunth Formula

$$\begin{aligned}
\frac{d\sigma}{d\cos\theta}\bigg|_{\text{lab}} &= \frac{16e^4E^2m^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right]}{32\pi m^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2q^4\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}\\
&= \frac{e^4E^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right]}{2\pi\frac{16E^4\sin^4\frac{\theta}{2}}{\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)^2\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}\\
&= \frac{e^4\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right]}{32\pi E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}\\
&= \frac{\pi\alpha^2\left[\left(F_1^2-\frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right]}{2E^2\sin^4\frac{\theta}{2}\left(1+\frac{2E}{m}\sin^2\frac{\theta}{2}\right)}
\end{aligned}$$

7 Operator Product Expansion

7.1 A Simple Example

Using the low energy limit expansion for Lagrangian corresponding to Weak interaction we used (replacing p^2 with $\square$ since $\square$ acting on the Fourier exponential gives p^2)

$$g^2 \int d^4x d^4y J^\mu(x) D^{\mu\nu}(x,y) J_\nu(y) \sim \frac{g^2}{2m_W^2} \int d^4x \left[J^\mu J_\mu - J^\mu \frac{\square}{m_W^2} J_\mu + J^\mu \frac{\square^2}{m_W^4} J_\mu + \cdots \right]$$

This is a case of the Operator Product Expansion

$$\lim_{x \to y} \mathcal{O}_1(x) \mathcal{O}_2(y) = \sum_n C_n(x-y) \mathcal{O}_n(x)$$

which in a more practical form looks like

$$\int d^4x e^{iqx} \mathcal{O}(x) \mathcal{O}(0) = \sum_n C_n(q) \mathcal{O}_n(0)$$

This is a telling example since here we have used the concept of effective field theory i.e long-distance physics is independent of short-distance physics. OPE should work for the same reason effective field theories work: physics naturally compartmentalizes itself so that all irrelevant scales can be taken to be either 0 or ∞ without strongly affecting the physics in which we are interested.

7.2 Application to Deep Inelastic Scattering

The conserved current for spin-$\frac{1}{2}$ particle is

$$J^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Writing out in terms of creation and annihilation operators and simplifying, we get

$$\langle p' | J^\mu(x) | p \rangle = \bar{u}_2(p') \gamma^\mu u_1(p) e^{i(p'-p)x}$$

The spinor product $\bar{u}_2(p') \gamma^\mu u_1(p)$ coming out of a Feynman diagram matrix element calculation is just the current matrix element at $x = 0 : \langle p' | J^\mu(0) | p \rangle$.

For the process $\gamma^\star P \to X$, we need the matrix element of this current (since that is what the photon couples to) at $x = 0$ between an initial proton state $|P\rangle$ and an arbitrary hadronic final state $\langle X|$. That is, we need

$$\mathcal{M}\left(\gamma^\star p^+ \to X\right) = e\epsilon^\mu \langle X | J_\mu(0) | P \rangle$$

Defining $\omega = \frac{1}{\xi}$,

$$\begin{aligned} W_{\mu\nu}(\omega, Q) &= \sum_X \int d\Pi_X \langle P | J_\mu(0) | X \rangle \langle X | J_\nu(0) | P \rangle (2\pi)^4 \delta^4\left(q^\mu + P^\mu - p_X^\mu\right) \\ &= \sum_{\mathrm{V}} \int d\Pi_X \int d^4x e^{i(q+P-p_X)x} \langle P | J_\mu(0) | X \rangle \langle X | J_\nu(0) | P \rangle \end{aligned}$$

Now we use translation operator to write $J_\mu(x)$ in terms of $J_\mu(0)$.

$$\langle P | J_\mu(0) | X \rangle = \left\langle P \left| e^{-i\hat{\mathcal{P}}\cdot x} J_\mu(x) e^{i\hat{\mathcal{P}}\cdot x} \right| X \right\rangle = e^{-i(P-p_X)\cdot x} \langle P | J_\mu(x) | X \rangle$$

$$\begin{aligned} \Rightarrow W_{\mu\nu} &= \sum_X \int d\Pi_X \int d^4x e^{iq\cdot x} \langle P | J_\mu(x) | X \rangle \langle X | J_\nu(0) | P \rangle \\ &= \int d^4x e^{iq\cdot x} \langle P | J_\mu(x) J_\nu(0) | P \rangle \end{aligned}$$

We have used completeness of states in the second step.

1. We know how to calculate matrix elements of time-ordered products of fields at different points using Feynman rules but the product of currents is not time-ordered.
2. We would like to Taylor expand in Q^{-2}. $\omega = \frac{2P\cdot q}{Q^2}$ means this implies $\omega \to 0$. However, kinematically $P \cdot q > \frac{1}{2}Q^2$, i.e $\omega > 1$ (i.e. $x < 1$). So a large Q^2 expansion will take us out of the physical region. We need to somehow Taylor expand around $\omega = 0$.

The first problem can be soled using Optical Theorem which gives us

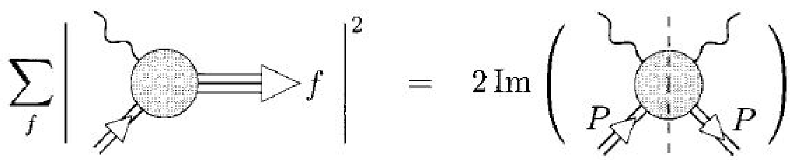


Figure 8: Optical Theorem

$$W^{\mu\nu} = 2\,\mathrm{Im}\,T^{\mu\nu}$$

where $T^{\mu\nu}(\omega, Q)$ is the forward (i.e initial and final states are the same) Compton amplitude and hence can be calculated using Feynman rules.

$$e^2 \epsilon_\mu \epsilon_\nu^\star T^{\mu\nu}(\omega, Q) = \mathcal{M}\left(\gamma^\star p \to \gamma^\star p\right)$$

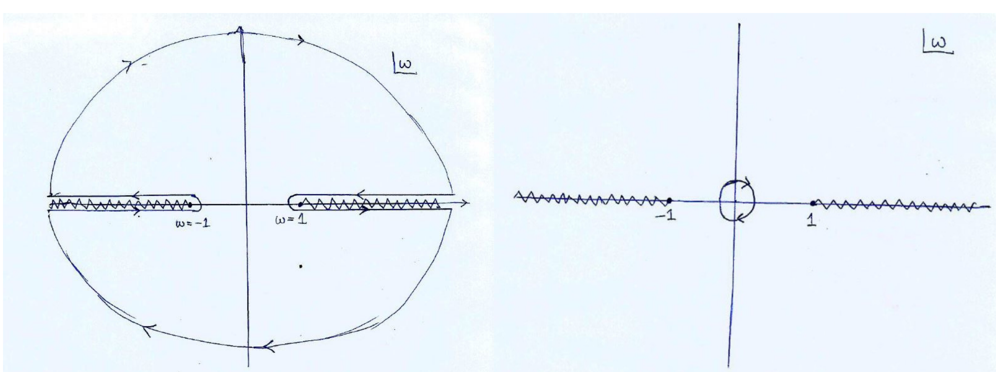
This can be written as a time-ordered product of currents.

$$T_{\mu\nu} = i \int d^4x e^{iq\cdot x} \left\langle P \left| T\left\{J_\mu(x) J_\nu(0)\right\}\right| P\right\rangle$$

Ward Identity again allows us to write

$$T^{\mu\nu} = T_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{T_2}{P\cdot q}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right)$$

and hence we have $W_1 = 2\,\mathrm{Im}\,T_1$ and $W_2 = 4\,\mathrm{Im}\,\frac{1}{\omega Q^2} T_2$ with an extra factor of $\frac{2}{\omega Q^2} = \frac{2\xi}{Q^2} = P \cdot q$ since we have $\frac{T_2}{P\cdot q}$ instead of just T_2.



$$\begin{aligned} W_{\mu\nu}(\omega, Q) = 2\,\mathrm{Im}\,T_{\mu\nu}(\omega, Q) &= \frac{T_{\mu\nu}(\omega + i\epsilon) - T^*_{\mu\nu}(\omega + i\epsilon, Q)}{i} \\ &= -iT_{\mu\nu}(\omega + i\varepsilon, Q) + iT_{\mu\nu}(\omega - i\varepsilon, Q) = \mathrm{Disc}\left(-iT_{\mu\nu}\right) \end{aligned}$$

We need to integrate over $1 \leq \omega < \infty$, which corresponds to integrating x from 0 to 1 . Since $T_{\mu\nu}$ is analytic away from the real axis, the contour can be deformed to be around $\omega = 0$, as shown in Figure. Any stray poles can be accounted for by calculating their residues. Thus, we only need to know $T_{\mu\nu}(\omega, Q)$ near $\omega = 0$ and we can justify Taylor expanding at small ω .And hence we can justify using the OPE of $J^\mu(x)J^\nu(0)$ as $x^\mu \to 0$ to derive results about $W^{\mu\nu}$ as long as we integrate over ω.

We need to find an expansion of the form

$$T\left\{J^\mu(x)J^\nu(y)\right\} = \sum_n C_n(x-y)\mathcal{O}_n^{\mu\nu}(x)$$

Let's write down the forward Compton amplitude for $\gamma^\star q \to \gamma^\star q$

$$i\int d^4x e^{iqx}\left\langle p\left|T\left\{J^\mu(x)J^\nu(0)\right\}\right|p\right\rangle = -\bar{u}(p)\frac{\gamma^\mu(\not{p}+\not{q})\gamma^\nu}{(p+q)^2+i\varepsilon}u(p) - \bar{u}(p)\frac{\gamma^\nu(\not{p}-\not{q})\gamma^\mu}{(p-q)^2+i\varepsilon}u(p)$$

For $Q^2 \gg p^2$, we can expand the denominators as follows

$$\frac{1}{(p+q)^2} = \frac{1}{-Q^2+2q\cdot p+p^2} = -\frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{2p\cdot q+p^2}{Q^2}\right)^n$$

and

$$\frac{1}{(p-q)^2} = \frac{1}{-Q^2-2q\cdot p+p^2} = -\frac{1}{Q^2}\sum_{n=0}^{\infty}\left(\frac{-2p\cdot q+p^2}{Q^2}\right)^n$$

$$\begin{aligned} i\int d^4x e^{iqx}\left\langle p\left|T\left\{J^\mu(x)J^\nu(0)\right\}\right|p\right\rangle &= \frac{1}{Q^2}\bar{u}(p)\gamma^\mu(\not{p}+\not{q})\gamma^\nu u(p)\sum_{n=0}^{\infty}\left(\frac{2p\cdot q+p^2}{Q^2}\right)^n \\ &+ \frac{1}{Q^2}\bar{u}(p)\gamma^\nu(\not{p}-\not{q})\gamma^\mu u(p)\sum_{n=0}^{\infty}\left(\frac{-2p\cdot q+p^2}{Q^2}\right)^n \end{aligned}$$

We can symmetrize $\mu \leftrightarrow \nu$ since the currents are identical.

$$\begin{aligned} \gamma^\mu\not{p}\gamma^\nu + \gamma^\nu\not{p}\gamma^\mu &= \gamma^\mu(2p^\nu - \gamma^\nu p^\mu) + \gamma^\nu(2p^\mu - \gamma^\mu p^\nu) \\ &= 2\gamma^\mu p^\nu + 2\gamma^\nu p^\mu - (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\not{p} \\ &= 2\gamma^\mu p^\nu + 2\gamma^\nu p^\mu - 2g^{\mu\nu}\not{p} \end{aligned}$$

$$\gamma^\mu(\not{p}\pm\not{q})\gamma^\nu + \gamma^\nu(\not{p}\pm\not{q})\gamma^\mu = 2\gamma^\mu\left(p^\nu \pm q^\nu\right) + 2\gamma^\nu\left(p^\mu \pm q^\mu\right) - 2g^{\mu\nu}(\not{p}\pm\not{q})$$

DIS limit is $Q \to \infty$ at fixed ω. $p^\mu = \xi P^\mu, \Rightarrow p^2 = \xi^2 P^2 = \xi^2 m_p^2$. So, terms such as $\Box/Q^2$ in operators give factors of p^2/Q^2 that are small. Terms such as ∂^μ/Q in operators then give factors of $q\cdot p/Q^2 \sim \omega$ that are not small. Also we can use the Dirac equation to get $\not{p}u(p) = m_q u(p) \approx 0$ and $\bar{u}(p)\not{p} = m_q\bar{u}(p) \approx 0$.

A few examples of how to identify OPE operators and their coefficients.

Term	Operator $\mathcal{O}_n$	Wilson Coefficient C_n
p^μ	$i\partial^\mu$	1
p^2	$i^2\Box$	1
$\left(\frac{p^2}{Q^2}\right)^n$	$i^{2n}\bar{\psi}\Box^n\psi$	$\frac{1}{Q^{2n}}$
$\left(\frac{2p\cdot q}{Q^2}\right)^3\frac{p^2}{Q^2}$	$i^5\bar{\psi}\partial^{\mu_1}\partial^{\mu_2}\partial^{\mu_3}\Box\psi$	$\frac{8}{Q^8}q_{\mu_1}q_{\mu_2}q_{\mu_3}$

Since second part of the expansion is related to first part by $q \leftrightarrow -q$, we can drop terms odd in q and double the terms even in q. This gives us the following

$$\begin{aligned} i\int d^4x e^{iqx}\left\langle p\left|T\left\{J^\mu(x)J^\nu(0)\right\}\right|p\right\rangle &= \frac{2}{Q^2}\bar{u}(p)\left(p^\mu\gamma^\nu + \gamma^\mu p^\nu\right)\sum_{n=0,2,\cdots}^{\infty}\left(\frac{2q\cdot p}{Q^2}\right)^n u(p) \\ &+\frac{2}{Q^2}\bar{u}(p)\left(\gamma^\mu q^\nu + q^\mu\gamma^\nu - g^{\mu\nu}\not{q}\right)\sum_{n=1,3,\cdots}^{\infty}\left(\frac{2q\cdot p}{Q^2}\right)^n u(p) \end{aligned}$$

The operators and Wilson coefficients corresponding to the two parts are

Operator	Wilson Coefficient
$\mathcal{O}^{(1)\mu\nu\mu_1\cdots\mu_n}_{n+2} = i^{n+1}\bar{\psi}_q(x)\left(\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu\right)\partial^{\mu_1}\cdots\partial^{\mu_n}\psi_q(x)$	$C^{(1)n+2}_{\mu_1\cdots\mu_n}(q) = \left(\frac{2}{Q^2}\right)^{n+1} q_{\mu_1}\cdots q_{\mu_n}$
$\mathcal{O}^{(2)\mu\nu\rho\mu_1\cdots\mu_n}_{n+3} = i^{n}\bar{\psi}_q(x)\left(\gamma^\mu g^{\nu\rho} + \gamma^\nu g^{\mu\rho} - \gamma^\rho g^{\mu\nu}\right)\partial^{\mu_1}\cdots\partial^{\mu_n}\psi_q(x)$	$C^{(2)n+3}_{\rho\mu_1\cdots\mu_n}(q) = \left(\frac{2}{Q^2}\right)^{n+1} q_\rho q_{\mu_1}\cdots q_{\mu_n}$

It is customary to work in a basis of gauge-invariant operators that transform in irreducible representations of the Lorentz group here the symmetric trace-less (A,A) representation of Lorentz group with $A\otimes A = 2A\oplus\cdots\ldots..\oplus 0$. The spin of an operator is the number of free indices on it and spin-l corresponds to $(\frac{l}{2},\frac{l}{2})$ representation. So operators have to be symmetrized in their indices and their traces be subtracted.An example of spin-2 operator is

$$\widehat{\mathcal{O}}^{\mu\nu}_{2,0} = \bar{\psi}_q\left(i\gamma^\mu\partial^\nu + i\gamma^\nu\partial^\mu - \frac{1}{2}ig^{\mu\nu}\not{\partial}\right)\psi_q$$

with $g_{\mu\nu}\widehat{\mathcal{O}}^{\mu\nu}_{2,0} = 0$. More generally for spin-s

$$\widehat{\mathcal{O}}^{\mu_1\cdots\mu_s}_{s,r} = \bar{\psi}\gamma^{\mu_1}i\partial^{\mu_2}\cdots i\partial^{\mu_s}(-\Box)^r\psi + \text{ symmetrizations of } \mu_i - \text{traces}$$

Doing a bit of dimensional analysis

$$\begin{aligned}
&[m\bar{\psi}\psi] = 4 = [\mathcal{L}]\\
&\Rightarrow [\psi] = [\bar{\psi}] = \tfrac{3}{2}\\
&[\partial_\mu] = [\partial^\mu] = 1\\
&[\Box] = 2
\end{aligned}$$

we get the dimension of the operator $\widehat{\mathcal{O}}^{\mu_1\cdots\mu_s}_{s,r}$ to be $d = 3 + (s-1) + 2r = 2 + s + 2r$ and this fixes twist of the operator defined as $t \equiv 2 + 2r = d - s$. Twist = Dimension - Spin. We will have

$$\left(\frac{2P\cdot q}{Q^2}\right)^s\left(\frac{1}{Q}\right)^{t-2}$$

Operators with extra $\Box$ factors are suppressed, the OPE will be dominated by operators with the lowest twist since power of $\Box$ is $r = \frac{t-2}{2}$. These are operators with $r = 0 \Rightarrow t = 2$ such as $\widehat{\mathcal{O}}^{\mu\nu}_{2,0}$. To also make these gauge invariant we promote the derivatives to covariant derivatives and add a label for the quark flavor to get gauge-invariant twist-2 quark operators

$$\widehat{\mathcal{O}}^{\mu_1\cdots\mu_n}_i(x) = \bar{\psi}_q(x)\gamma^{\mu_1}iD^{\mu_2}\cdots iD^{\mu_n}\psi_q(x) + \text{symmetrizations of } \mu_i - \text{traces.}$$

At the operator level it turns out the following gives the required OPE.

$$\begin{aligned}
i\int d^4x e^{iq\cdot x}T\left\{J_\mu(x)J_\nu(0)\right\} = \sum_i Q_i^2\Bigg\{&\sum_{n=2,4,\cdots}^{\infty}\frac{(2q^{\mu_1})\cdots(2q^{\mu_n})}{Q^{2n}}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\widehat{\mathcal{O}}^{\mu_1\cdots\mu_n}_i\\
&+4\sum_{n=2,4,\cdots}^{\infty}\frac{(2q^{\mu_3})\cdots(2q^{\mu_n})}{Q^{2n-2}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)\widehat{\mathcal{O}}^{\mu_1\cdots\mu_n}_i\Bigg\}
\end{aligned}$$

Since the Operators depend only on P^μ, the only combination with the required indices is

$$\sum_{\text{spins}}\left\langle P\left|\widehat{\mathcal{O}}^{\mu_1\cdots\mu_n}_i\right|P\right\rangle = \mathcal{A}^n_i P^{\mu_1}\cdots P^{\mu_n} - \text{ traces}$$

The traces give terms $\propto m_p^2 \to 0$. And since P and q contract to give $q\cdot P = \frac{1}{2}\omega Q^2$, we have

$$\begin{aligned}
T^{\mu\nu} = \sum_i Q_i^2\Bigg\{&\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\sum_{n=2,4,\cdots}^{\infty}\omega^n\mathcal{A}^n_i\\
&+\frac{4}{Q^2\omega^2}\left(P^\mu-\frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu-\frac{P\cdot q}{q^2}q^\nu\right)\sum_{n=2,4,\cdots}^{\infty}\omega^n\mathcal{A}^n_i\Bigg\}
\end{aligned}$$

where we see that

$$T_1 = \frac{\omega}{2} T_2 = \sum_i Q_i^2 \left[\sum_{n=2,4,\cdots} \omega^n \mathcal{A}_i^n \right]$$

and hence since $W_1(x, Q) = 2\pi \sum Q_i^2 f_i(x)$ and $W_1 = 2 \operatorname{Im} T_1$ and $\xi = x = \frac{1}{w}$, we have the Operator definition of parton distribution function

$$f_i(x) = \frac{1}{\pi} \sum_{n=2,4,\cdots} x^{-n} \operatorname{Im} \mathcal{A}_i^n$$

7.2.1 Mellin Moment

Consider a Mellin moment defined as

$$\int_0^1 dx x^{m-1} f_i(x) = \operatorname{Im} \frac{1}{\pi} \int_1^\infty d\omega \sum_n \omega^{n-m-1} \mathcal{A}_i^n$$

since $x = \frac{1}{\omega}$, $dx = -\frac{d\omega}{\omega^2}$ and minus sign taken care of by reversing limits of integration. Using contour integration to relate this integral to a contour around origin as we did earlier and using residue theorem, we get

$$\sum_n \frac{1}{2\pi i} \int \frac{d\omega}{\omega^{m-1}} \omega^{n-2} \mathcal{A}_i^n = \mathcal{A}_i^m$$

So,

$$\mathcal{A}_i^m = \int_0^1 dx x^{m-1} f_i(x)$$

7.2.2 Sum Rules

Let's look at the case of $m = 1$, where $\left\langle P \left| \bar{\psi}_i(x) \gamma^\mu \psi_i(x) \right| P \right\rangle$ needs to be evaluated.
In general with quantum corrections the vertex factor is in terms of form factors.

$$\Gamma^\mu (p', p) = \gamma^\mu F_1 \left(q^2\right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2 \left(q^2\right)$$

For $p = p'$ we only have the first term. We can use Gordon's Identity and inner product relations, we get

$$\left\langle P \left| \bar{\psi}_i(x) \gamma^\mu \psi_i(x) \right| P \right\rangle = \bar{u}(P) \gamma^\mu u(P) F_{i1}(0) = \bar{u}(P) \frac{2P^\mu}{2m} u(P) F_{i1}(0) = \frac{P^\mu}{m} 2m = 2P^\mu F_{i1}(0)$$

$F_{i1}(0)$ is conserved charge corresponding to proton state which is number of particles minus number of anti-particles which we name N_i.

$$\mathcal{A}_i^1 = N_i$$

For a proton,

$$N_i = \begin{cases} 2 & i = u \\ 1 & i = d \end{cases}$$

$$\Rightarrow \int_0^1 dx \left[f_u(x) - f_{\bar{u}}(x) \right] = 2, \quad \int_0^1 dx \left[f_d(x) - f_{\bar{d}}(x) \right] = 1$$