S-matrix Bootstrap and Bounds on Wilson Coeff.

Vinay Vikramaditya

under
Prof. Aninda Sinha
CHEP, IISc Bangalore

July 6, 2021

Contents

Introduction

- Bootstrap is the use of minimal set of principles to constrain physical observables.
- Its a non-pertubative method using properties like unitarity, crossing symmetry in general and poles due to prescence of bound states, resonances etc. dependending on which theory is in consideration.
- Bootstrap was popular in the 1960s but fell out of favour after strong forces was succesfully described by non-abelian gauge theories. It was later revived for use in CFTs as conformal bootstrap in the 1970s and very recently (2016) the S-matrix bootstrap for QFTs.
- Here we shall be looking at numerical bootstrap methods elucidated in [1],[2] using Semi-Definite Program Solver (SDPB) [3].

Pion Bootstrap

Invariant Tensors in $O(3)$

- The $2 \to 2$ scattering amplitude will have four indices and can be written in terms of 3 invariant tensors of $O(3)$ vector representation.

$$\mathcal{T}_{ab}^{cd} = A(s|t,u)\delta_{ab}\delta^{cd} + A(t|s,u)\delta_a^c\delta_b^d + A(u|s,t)\delta_a^d\delta_b^c$$

$\mathbb{K}_{ij}^{kl} = \delta_{ij}\delta^{kl} =$ 

, $\mathbb{I}_{ij}^{kl} = \delta_i^l\delta_j^k =$ 

, $\mathbb{P}_{ij}^{kl} = \delta_i^k\delta_j^l =$ 

- $\mathbb{P}_{\text{sing}} = \mathbb{P}_0 = \frac{1}{3}\delta_{ab}\delta^{cd}, \mathbb{P}_{\text{anti}} = \mathbb{P}_1 = \frac{1}{2}\left(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c\right), \mathbb{P}_{\text{sym}} = \mathbb{P}_2 = \frac{1}{2}\left(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c - \frac{2}{3}\delta_{ab}\delta^{cd}\right)$ following $\mathbb{P}_I\mathbb{P}_J = \delta_{IJ}\mathbb{P}_I$
- The scattering amplitude in terms of projection operators

$$\begin{aligned}\mathcal{T} =& (3A(s|t,u) + A(t|s,u) + A(u|s,t))\mathbb{P}_0 \\ & + (A(t|s,u) - A(u|s,t))\mathbb{P}_1 + (A(t|s,u) + A(u|s,t))\mathbb{P}_2 \\ =& \mathcal{T}^{(0)}\mathbb{P}_0 + \mathcal{T}^{(1)}\mathbb{P}_1 + \mathcal{T}^{(2)}\mathbb{P}_2\end{aligned}$$

Pion Bootstrap

Partial Wave Unitarity

- $S_\ell^{(I)}(s) = 1 + i\frac{\sqrt{s-4}}{\sqrt{s}} \int_{-1}^{1} dx \frac{P_\ell(x)}{32\pi} \mathcal{T}^{(I)}(s,t)\Big|_{t\to\frac{1}{2}(s-4)(x-1)} = 1 + 2i\frac{\sqrt{s-4}}{\sqrt{s}}\mathcal{T}_l^{(I)}$
- Optical theorem

$$\mathrm{Im}\,\mathcal{T}(\pi\pi\to\pi\pi) \geq 2E_{CM}\,|\vec{p}_i|\,\frac{1}{64\pi E_{CM}^2}\int d(\cos\theta)|\mathcal{T}(\pi\pi\to\pi\pi)|^2$$

- Since $\mathbb{P}_I\mathbb{P}_J = \delta_{IJ}\mathbb{P}_I$,

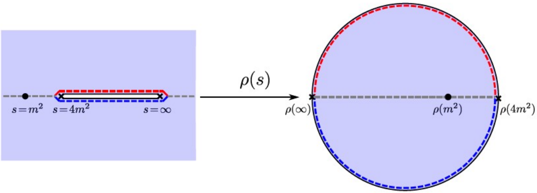
$$\mathrm{Im}\,\mathcal{T}^{(I)}(\pi\pi\to\pi\pi) \geq 2E_{CM}\,|\vec{p}_i|\,\frac{1}{64\pi E_{CM}^2}\int d(\cos\theta)|\mathcal{T}^{(I)}(\pi\pi\to\pi\pi)|^2$$

- Expanding using $\mathcal{T}^{(I)} = 32\pi\sum_\ell \mathcal{T}_\ell^{(I)}(2\ell+1)P_\ell(\cos(\theta))$ gives $\mathrm{Im}\left(\mathcal{T}_l^{(I)}\right) \geq \frac{\sqrt{s-4}}{\sqrt{s}}\left|\mathcal{T}_l^{(I)}\right|^2$ which using $\mathcal{T}_l^{(I)} = \frac{S_l^{(I)}(s)-1}{2i}\frac{\sqrt{s}}{\sqrt{s-4}}$ implies the partial wave unitarity condition

$$\left|S_l^{(I)}(s)\right|^2 \leq 1$$

We use a transformation to map the cut s-plane to a unit disk and similar transformation for t and u:

$$s \mapsto \rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}, \quad s = \frac{s_0\left(1-\rho_s\right)^2 + 16m^2\rho_s}{\left(1+\rho_s\right)^2}$$



This helps since the unit disk isk is a finite region, and the branch cut maps to boundary of this unit disk.

[1] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees and P. Vieira, "The S-matrix Bootstrap III: Higher Dimensional Amplitudes," arXiv:1708.06765

Pion Bootstrap [1][2]

Ansatz

- Crossing symmetry for a process $a\,(p_a) + b\,(p_b) \to c\,(p_c) + d\,(p_d)$ with amplitude $\mathcal{T}_{ab}^{cd} = A(s|t,u)\delta_{ab}\delta^{cd} + A(t|s,u)\delta_a^c\delta_b^d + A(u|s,t)\delta_a^d\delta_b^c$ can be found by interchanging $c \leftrightarrow d$ which interchanges $t \leftrightarrow u$ and comparing coeff. of the invariant tensors to get $A(s|t,u) = A(s|u,t)$ and by crossing in other ways, we can show that $A(x|y,z) = A(x|z,y)$ for any Mandelstram variables.

s

- So a crossing symmetric analytic (in the disk) ansatz for $A(s|t,u)$ in terms of the new variables in absence of poles/bound states is

$$A(s|t,u) = \sum_{n\leq m}^{\infty} a_{nm}\left(\rho_t^n\rho_u^m + \rho_t^m\rho_u^n\right) + \sum_{n,m}^{\infty} b_{nm}\left(\rho_t^n + \rho_u^n\right)\rho_s^m$$

with $\rho_z = \frac{\sqrt{\frac{8}{3}}-\sqrt{4-z}}{\sqrt{\frac{8}{3}}+\sqrt{4-z}}$.

[2]A. L. Guerrieri, J. Penedones, P. Vieira, "Bootstrapping QCD: the Lake, the Peninsula and the Kink" arxiv:1810.12849

- SDPB solves the following problem

$$\text{maximize } a\cdot z \text{ over } z\in\mathbb{R}^{N+1},$$

$$\text{such that } \sum_{n=0}^{N} z_n W_j^n(x)\geq 0 \quad \text{for all } x\geq 0 \text{ and } 1\leq j\leq J$$

$$\text{with normalization } n\cdot z=1$$

- To use this to impose unitarity, we first begin by writing $\mathcal{T}(s,t,4-s-t)=\vec{\eta}\cdot\vec{\mathcal{T}}(s,t)$ where $\vec{\eta}$ is a vector containing all parameters in the ansatz (like all a_{nm}, b_{nm} in case of pion bootstrap).
- Suppresing isospin index, unitarity $|S_I(s)|^2\leq 1$ with definitions $\vec{R}=\mathrm{Re}\left[\vec{\mathcal{T}_\ell}(s)\right]$ and $\vec{I}=\mathrm{Im}\left[\vec{\mathcal{T}_\ell}(s)\right]$, is equivalent to semidefiniteness of

$$M := \begin{pmatrix} 1+\vec{\eta}\cdot\vec{R} & 1-\vec{\eta}\cdot\vec{I} \\ 1-\vec{\eta}\cdot\vec{I} & 1-\vec{\eta}\cdot\vec{R} \end{pmatrix}$$

[3] D. Simmons-Duffin, "A Semi-definite Program Solver for the Conformal Bootstrap" arXiv:1502.02033

Using crossing symmetry, isospin conservation and Bose statistics, [4]

$$\langle ld, qb|M|pc, ka\rangle = \delta_{ab}\delta_{cd}[A + B(s+u) + Ct + \cdots] + \delta_{ad}\delta_{cb}[A + B(s+t) + Cu + \cdots] \\ + \delta_{ac}\delta_{bd}[A + B(u+t) + Cs + \cdots]$$

where $s = (p+k)^2, \quad t = (k-q)^2, \quad u = (p-q)^2$.
Scattering lengths are defined as $a_\ell^I = \lim_{s\to 4m^2} \frac{T_\ell^I(s)}{\left(\frac{s}{4}-m^2\right)^\ell}$,

$$a_0 \cong -(1/32\pi m_\pi)\left[5A + 8{m_n}^2 B + 12{m_\pi}^2 C\right] \quad a_2 \cong -(1/32\pi m_\pi)\left[2A + 8{m_\pi}^2 B\right]$$

$$B - C = 4\left(\frac{g_V}{F_\pi}\right)^2 \qquad 2a_0 - 5a_2 = 6L = 0.69 m_\pi^{-1}$$

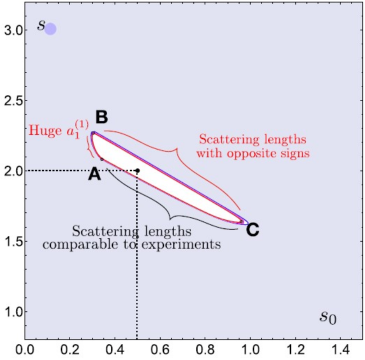
$$A = -m_\pi^2(2B + C) \qquad A = -m_\pi^2(B + C) \qquad \Rightarrow B = 0, \quad A = -{m_\pi}^2 C$$

$$a_0 = (7/4)L = 0.20 m_\pi^{-1}, a_2 = -\frac{1}{2}L = -0.06 m_\pi^{-1}$$

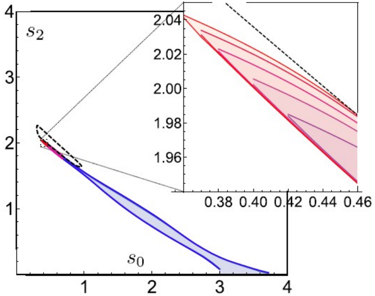
[4] Weinberg, Pion Scattering Lengths, Phys. Rev. Lett. 17, 616

- Adler zeroes $\mathcal{T}_0^{(0)}(s_0) = 0$ and $\mathcal{T}_0^{(2)}(s_2) = 0$ occur in unphysical region and hence can't be experimentally probed. And we would like to find S-matrices that have two Adler zeroes, a ρ-resonance and satisfy unitarity for all spins $l = 0, 1, \ldots, L_{max}$ and isospins $I = 0, 1, 2$ over a grid of ρ_s values (we used 300 values on unit disk).
- To do this we first fix s_0 and impose the following
  1. Unitarity
  2. One Adler Zero $\mathcal{T}_0^{(0)}(s_0) = 0$
  3. ρ resonance at $S_1^{(1)}(m_\rho^2) = 0$ at $m_\rho = 5.5 + 0.5i$
- Now we use SDPB to maximize and minimize $\mathcal{T}_0^{(2)}(s)$ (minimization is done by maximizing $-\mathcal{T}_0^{(2)}(s)$) at $s = s_2$. If both max and min are of the same sign, it will not be possible to impose the second Adler zero as $\mathcal{T}_0^{(2)}(s) = 0$ would not be consistent with the other inputs and such (s_0, s_2) will be disallowed point.

Doing this for many (s_0, s_2) with $0 < s_0, s_2 < 4$ we obtain a region where twoAdler zeroes can't be imposed and this is called the **Pion Lake**.



- At low energies the partial wave amplitude's real part can be expanded in its COM momentum $k = \sqrt{\frac{s-4}{4}}$ as $\text{Re}\left[\mathcal{T}_\ell^{(I)}\right] = k^{2\ell}\left[a_\ell^{(I)} + b_\ell^{(I)}k^2 + \mathcal{O}\left(k^4\right)\right]$ where $a_\ell^{(I)}$'s are called scattering lengths and $b_\ell^{(I)}$'s effective ranges.
- We now impose, apart from **all the conditions imposed in Lake**, the experimental values of the scattering lengths i.e $\left|a_0^{(0)} - 0.2196\right| < 0.034$, $\left|a_1^{(1)} - 0.038\right| < 0.002$ and $\left|a_0^{(2)} - (-0.0444)\right| < 0.0012$ and repeating the same process, we obtain the **Pion Peninsula**.

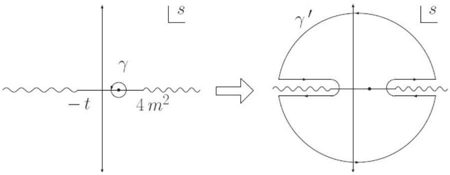


Pion Bootstrap

Analyticity in Mandelstram Plane [5]

- $T^*(s+i\epsilon) = T(s-i\epsilon) \Rightarrow T(s+i\epsilon) - T(s-i\epsilon) = 2i\,\mathrm{Im}\,T(s+i\epsilon) \neq 0$. Non-analyticity for $s \geq 4m^2$ will translate into crossed channels to give **Analyticity for** $s, t \leq 4m^2, s+t \geq 0$
- In complex s-plane with neighbourhood (in s) being analytic and $t < 4m^2$ has branch cuts are at $s > 4m^2$ and $s < -t$ and the following contour can be used to write (if contour at infinity vanishes which is the case for pion scattering with $n = 2$)

$$\frac{d^n}{ds^n} T^I(s,t) = \frac{n!}{\pi} \int_{4m^2}^{\infty} dx \left[\frac{\delta^{II'}}{(x-s)^{n+1}} + (-1)^n \frac{C_u^{II'}}{(x-u)^{n+1}} \right] \mathrm{Im}\, T^{I'}(x+i\epsilon, t)$$



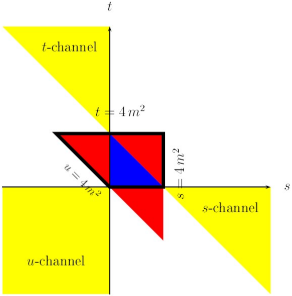
with $T^I(s,t) = C_u^{II'} T^{I'}(u,t)$

[5] A.V.Manohar and V.Mateu,"Dispersion Relation Bounds for $\pi\pi$ Scattering" Phys. Rev. D 77, 094019 (2008)

Pion Bootstrap

Fixed t dispersion relation

- It can be shown that Im $T^I(s,t) > 0$ **for** $t > 0$ and $t > 0$ along with $s, t < 4m^2, s + t > 0$ defines a region $\mathcal{A}$



- For the 3 scattering amplitudes $T = \sum a_I T^I$ has $a_I \geq 0$ and $b_J = \sum_I a_I C_u^{IJ} \geq 0$ and

$$\frac{d^2}{ds^2} T\left(\pi^0\pi^0 \to \pi^0\pi^0\right)[(s,t) \in \mathcal{A}] \geq 0 \qquad \frac{d^2}{ds^2} T\left(\pi^+\pi^0 \to \pi^+\pi^0\right)[(s,t) \in \mathcal{A}] \geq 0$$

$$\frac{d^2}{ds^2} T\left(\pi^+\pi^+ \to \pi^+\pi^+\right)[(s,t) \in \mathcal{A}] \geq 0$$

- Now using $a_\ell^I = \lim_{s\to 4m^2} \frac{T_\ell^I(s)}{\left(\frac{s}{4}-m^2\right)^\ell}$ and going through few steps,

$$\left.\frac{d^2 T^I\left(s,4m^2\right)}{ds^2}\right|_{s=0} = \frac{120}{32} C_t^{IJ} a_2^J \geq 0 \qquad I = 0,1,2 \quad \left(0,4m^2\right) \in \mathcal{A}$$

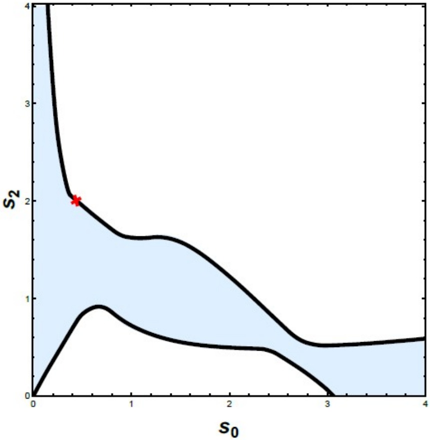
  where $T^I(s,t) = C_t^{II'} T^{I'}(t,s)$

- **D-wave Inequalities:** $a_2^0 + 2a_2^2 \geq 0, \quad a_2^0 - a_2^2 \geq 0.$ Additionally choosing $a_2^{(2)} \geq 0$ makes phenomenological values lie in allowed region.
- Leading order χPT values give [9] **S-wave Inequalitites:** $a_0^{(0)} + 2a_0^{(2)} \geq 0, \quad 2a_0^{(0)} + a_0^{(2)} \geq 0, \quad a_0^{(0)} - a_0^{(2)} \geq 0, \quad a_0^{(2)} \leq 0.$

Pion Bootstrap [9][6]

River

- Constraints in **Lake** plus **S and D-wave Inequalities** gives the **River**



[6] A. Bose, P. Haldar, A. Sinha, Pritish Sinha, S. S. Tiwari, "Relative entropy in scattering and the S-matrix bootstrap" arXiv:2006.12213

Crossing Symmetric Dispersion Relation [6][7]

- Dispersion relations used so far with fixed t in complex s-plane is not crossing symmetric.
- Shifted variables $s_1 = s - \frac{\mu}{3}, s_2 = t - \frac{\mu}{3}, s_3 = u - \frac{\mu}{3}$ and $s_1 + s_2 + s_3 = 0$.
- $s_k = a - \dfrac{a\left(z - z_k\right)^3}{z^3 - 1}, \quad k = 1, 2, 3$ where z_k are cube roots of unity.
- Amplitude can be written as $\overline{\mathcal{M}}(z, a) = \mathcal{M}\left(s_1, s_2\right)$ and for $-\frac{2\mu}{9} < a < 0$, the branch cuts for $s, t, u \geq 4m^2 = \mu$ become $s_1, s_2, s_3 \geq \frac{8m^2}{3} = \frac{2\mu}{3}$ map to arcs in the unit circle in z-plane for fixed a as shown

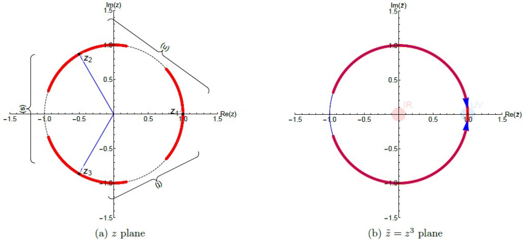


Figure 1: Cuts in z-plane and $\tilde{z}(= z^3)$-plane

[7] A. Sinha, A. Zahed, "Crossing Symmetric Dispersion Relations in QFTs"

Crossing Symmetric Dispersion Relation

- Defining $\tilde{z}(=z^3)$, we can write the following completely symmetric dispersion relation for fixed a

$$\mathcal{M}(\tilde{z},a)=\alpha_0+\frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)H\left(s_1',\tilde{z}\right)$$

where we have s-channel discontinuity as
$\mathcal{A}_1\left(s_1,s_2\right)\equiv\lim_{\epsilon\to 0}\frac{1}{2i}\left[\mathcal{M}\left(s_1+i\epsilon,s_2\right)-\mathcal{M}\left(s_1-i\epsilon,s_2\right)\right],\quad s\geqslant 2\mu/3$, $\alpha_0=\mathcal{M}(z=0,a)$ and

$$H\left(s_1',\tilde{z}\right)=\frac{27a^2\tilde{z}\left(2s_1'-3a\right)}{27a^3\tilde{z}-27a^2\tilde{z}s_1'-(1-\tilde{z})^2\left(s_1'\right)^3}$$

$$s_2^{(+)}\left(s_1',a\right)=-\frac{s_1'}{2}\left[1-\left(\frac{s_1'+3a}{s_1'-a}\right)^{1/2}\right]$$

Univalence and de Branges' theorem [7][8]

- We can write the kernel in the form

$$H\left(s_1', \tilde{z}\right) = \frac{27a^2 \frac{\tilde{z}}{(1-\tilde{z})^2}\left(2s_1' - 3a\right)}{27a^3 \frac{\tilde{z}}{(1-\tilde{z})^2} - 27a^2 \frac{\tilde{z}}{(1-\tilde{z})^2} s_1' - \left(s_1'\right)^3}$$

- We identify the repeatedly occuring factor as the **Koebe function** $k(z) = \frac{z}{(1-z)^2} = z + \sum_{p=2}^{\infty} pz^p$ which has a property called univalence.
- Univalence of a function f implies $f(z) = f(w) \Rightarrow z = w$ in a domain $\mathbb{D}$.
- Consider univalence of $g(z) = az + bz^2 + c$ in unit disk $\mathbb{D} = \{z \,||z| < 1\}$. $g(z) - g(w) = az + bz^2 - aw - bw^2 = a(z-w)(1 + \frac{b}{a}(z+w))$. For $g(z) = g(w)$ to imply necessarily that $z = w$, $|1 + \frac{a}{b}(z+w)| \neq 0 \quad \forall |z| < 1 \Rightarrow \left|\frac{a}{b}\right| < \frac{1}{2}$.

[8] P. Haldar, A. Sinha, A. Zahed, "Quantum field theory and the Bieberbach conjecture"

Univalence and de Branges' theorem

- We shall take $a \in \left(-\frac{2\mu}{9}, 0\right) \cup \left(0, \frac{4\mu}{9}\right)$ and $s_1 \in \left[\frac{2\mu}{3}, \infty\right)$ and abbreviate this condition as $\blacklozenge$
- For $\blacklozenge$, it can be shown that $\mathcal{A}\left(s_1; s_2^{(+)}(s_1, a)\right) > 0$ if $Im\left(\mathcal{T}_l^{(I)}\right) \geq 0$ which is a weaker conditon than the non-linear $Im\left(\mathcal{T}_l^{(I)}\right) \geq \frac{\sqrt{s-4}}{\sqrt{s}} \left|\mathcal{T}_l^{(I)}\right|^2$
- Expanding $H\left(s_1', \tilde{z}\right) = \sum_{n=0}^{\infty} \beta_n\left(a, s_1'\right) \tilde{z}^n$, it can be deduced that $\beta_0 = 0, \;\; \beta_1 < 0$ in $\blacklozenge$
- $F\left(\tilde{z}; s_1, a\right) = \dfrac{H\left(\tilde{z}; s_1, a\right)}{\beta_1\left(a, s_1\right)} = \tilde{z} + \sum_{n=2}^{\infty} \dfrac{\beta_n\left(a, s_1\right)}{\beta_1\left(a, s_1\right)} \tilde{z}^n$
- F has no singularities in the unit disk for $\blacklozenge$ and F can be written as Mobius tranformation of Koebe function as

$$F\left(\tilde{z}; s_1, a\right) = k(\tilde{z})\left(1 - \frac{27a^2\left(a - s_1\right)}{s_1^3} k(\tilde{z})\right)^{-1}$$

- **Mobius transformations of Koebe functions are univalent**. Additionally, univalence of F implies univalence of the kernel H.

Univalence and de Branges' theorem

- **de Branges' Theorem:-** If f is a univalent function with $f(z) = z + \sum_{p=2}^{\infty} b_p z^p, \quad |z| < 1$, then its coefficients satisfy

$$|b_n| \leq n, \quad \forall\, n \geq 2$$

  with equality iff f is a Koebe function.

- Using de Brange's Theorem on F, $\left|\dfrac{\beta_n(a, s_1)}{\beta_1(a, s_1)}\right| \leq n, \quad n \geq 2$

- $\mathcal{M}$, when written in terms of s_1, s_2, can be expanded in $x = -(s_1 s_2 + s_2 s_3 + s_3 s_1) = s_1^2 + s_2^2 + s_1 s_2$ and $y = -s_1 s_2 s_3 = s_1 s_2 (s_1 + s_2)$ as

$$\mathcal{M}(s_1, s_2) = \sum_{p,q=0}^{\infty} W_{p,q} x^p y^q$$

- $\mathcal{M}(\tilde{z}, a) = \sum_{n=0}^{\infty} \alpha_n(a) a^{2n} \tilde{z}^n$ and using the transformation

$$\alpha_p(a) a^{2p} = \sum_{n=0}^{p} \sum_{m=0}^{n} W_{n-m,m} a^m (-1)^{p-n} (-27)^n a^{2n} \begin{pmatrix} -2n \\ p-n \end{pmatrix}$$

Bounds on Wilson Coefficients

Since both $\mathcal{M}$ and H are expanded in $\tilde{z}$, we can write

$$\begin{aligned}
\left|\alpha_n(a)a^{2n}\right| &= \frac{1}{\pi}\left|\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)\beta_n\left(a,s_1'\right)\right| \\
&\leq \frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\left|\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)\beta_n\left(a,s_1'\right)\right| \\
&\leq \frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)n\left|\beta_1\left(a,s_1'\right)\right| \\
&= n\left(-\alpha_1(a)a^2\right) = n\left|\alpha_1(a)a^2\right|
\end{aligned}$$

$\left|\dfrac{\alpha_n(a)a^{2n}}{\alpha_1(a)a^2}\right| \leq n$ for $n \geq 2$ for ⧫

For $n=2$ gives $-2 \leq 2 - \dfrac{27a^2\left(a\left(aW_{0,2}+W_{1,1}\right)+W_{2,0}\right)}{aW_{0,1}+W_{1,0}} \leq 2$

By not allowing denominator to be 0, for $-\dfrac{2\mu}{9} < a < \dfrac{4\mu}{9}$, we get the first bound $-\dfrac{9}{16} < \dfrac{W_{0,1}}{W_{1,0}} < \dfrac{9}{8}$. The other bounds are obtained in [8][9]

[9] A. Sinha, Prashanth Raman, "QFT,EFT and GFT" (Upcoming)

Results

SDPB outputs the parameters and tha can be used to write down the S-matrices/scattering ampltudes at the boundaries of Lake, Peninsula, River.

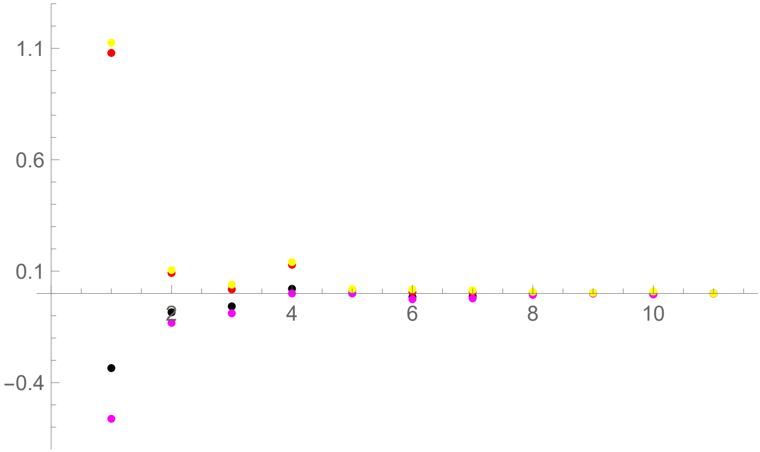
From those, the ratios $\frac{W_{p,q}}{W_{1,0}}$ were extracted (the denominator has been dropped in the table shown below). The River provided all but two of the overall min/max. So its plots for $\frac{W_{p,q}}{W_{1,0}}$ vs s_0 for points on its upper and lower boundaries are shown later.

Theoretical	Min	Max
W01	-0.5625	1.125
W11	-0.13186	0.105
W02	-0.089	0.0396
W20	0	0.140625
W30	0	0.0197
W21	-0.025955	0.0183
W12	-0.0223	0.013735
W03	-0.00664	0.00712
W40	0	0.00278
W31	-0.0047	0.00884
W50	0	0.000391

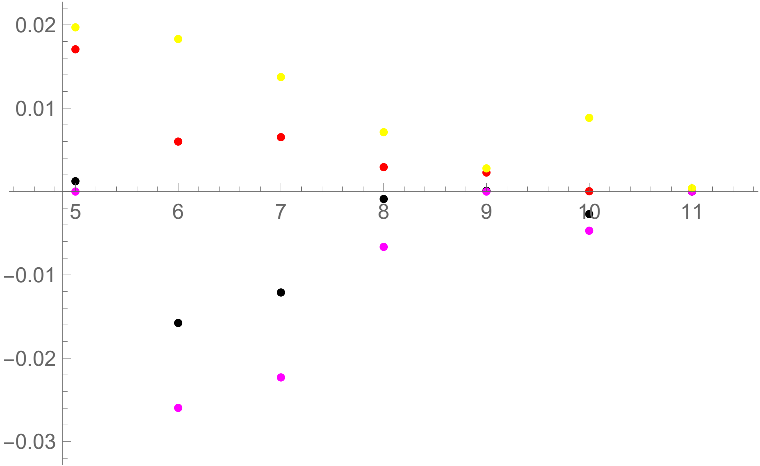
Figure 2: Theoretical Bounds. From [8][10]

Results

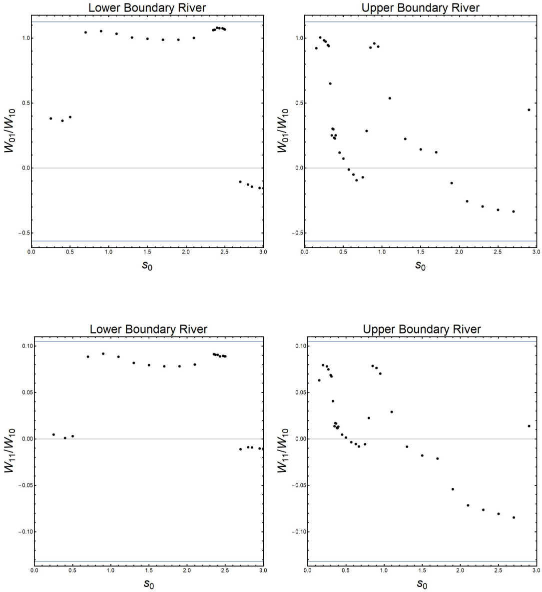
The overall minima and maxima for each ratio are plotted with the theoretical bounds.



Results

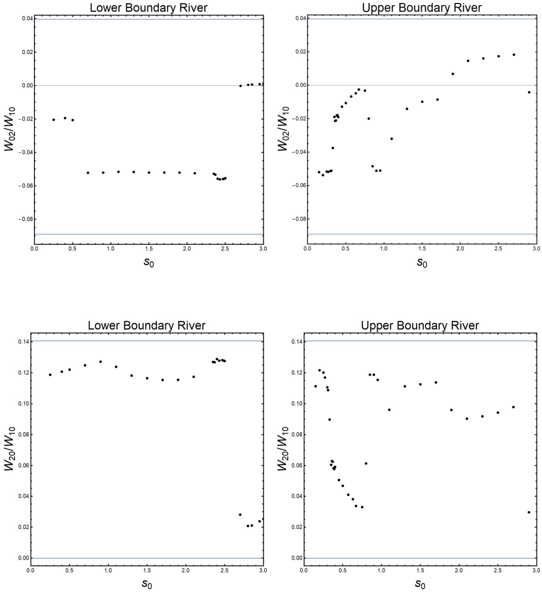


Results



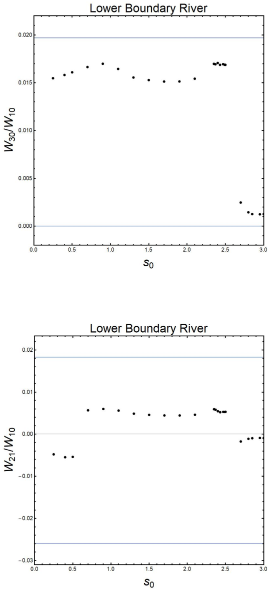
Lower Boundary River
Upper Boundary River
W_{01}/W_{10}
s_0
Lower Boundary River
Upper Boundary River
W_{11}/W_{10}
s_0

Results



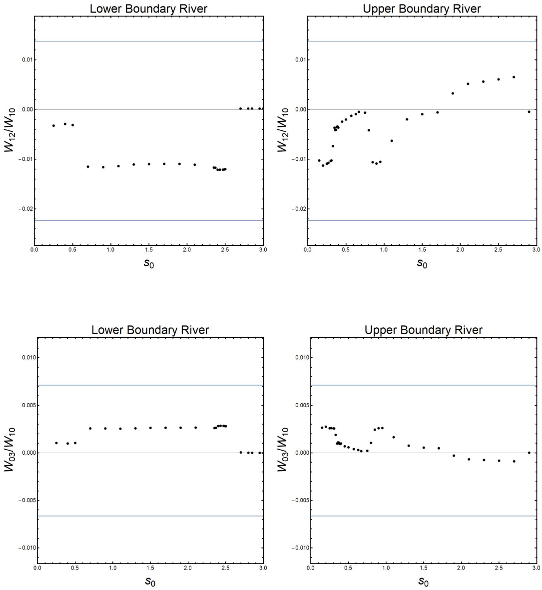
Lower Boundary River
Upper Boundary River
W_{02}/W_{10}
s_0
Lower Boundary River
Upper Boundary River
W_{20}/W_{10}
s_0

Results

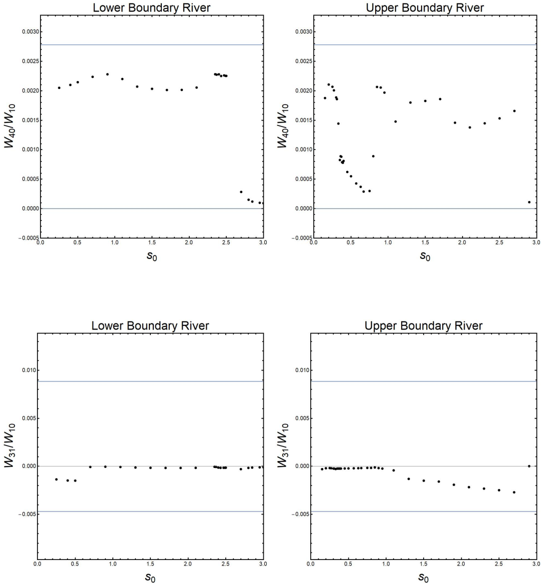


Lower Boundary River
Upper Boundary River
W_{30}/W_{10}
s_0
Lower Boundary River
Upper Boundary River
W_{21}/W_{10}
s_0

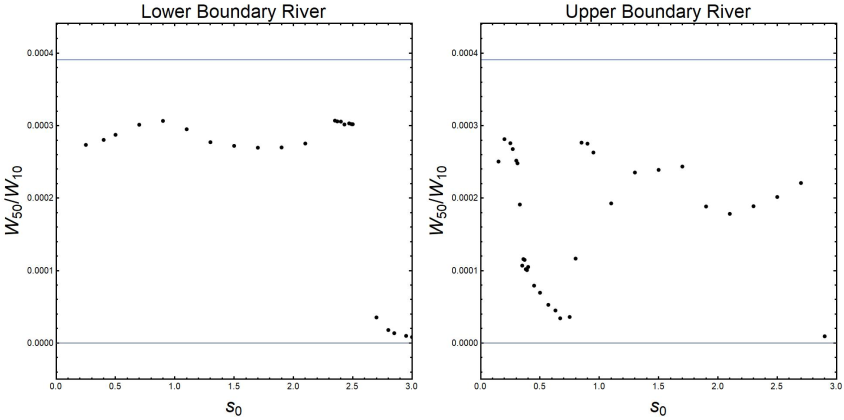
Results


Lower Boundary River
Upper Boundary River
W_{12}/W_{10}
s_0
Lower Boundary River
Upper Boundary River
W_{03}/W_{10}
s_0

Results


Lower Boundary River
Upper Boundary River
W_{40}/W_{10}
s_0
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Lower Boundary River
Upper Boundary River
W_{31}/W_{10}
s_0
0.010
0.005
0.000
−0.005

Results


Lower Boundary River
Upper Boundary River
W_{50}/W_{10}
s_0
0.0004
0.0003
0.0002
0.0001
0.0000
0.0
0.5
1.0
1.5
2.0
2.5
3.0

- Constrain Wilson coeff. at $\mathcal{O}(s^0)$, α in $\frac{T(s,t,u)}{8\pi G_N} = s^4\left(\frac{1}{stu} + \alpha \ell_P^6 + O(s)\right)$.
- Using a transformation to map the cut Mandelstram plane to a unit disk $\rho_s = \frac{\sqrt{s_0}-\sqrt{-s}}{\sqrt{s_0}+\sqrt{-s}}$, we write a crossing symmetric ansatz symmetric in all Mandelstram variables

$$\frac{T}{8\pi G_N} = s^4\left(\frac{1}{stu} + \prod_{A=s,t,u}(\rho_A+1)^2 \sideset{}{'}\sum_{a+b+c\leq N} \alpha_{(abc)}\rho_s^a\rho_t^b\rho_u^c\right)$$

The factor $\prod_{A=s,t,u}(\rho_A+1)^2 \sim \frac{1}{stu}$ keeps that part of the ansatz in control at high energies.

- In 9+1 dim $P_l^{(9)}(x) = \frac{1}{3\cdot 2^{18}\cdot\pi^4}\frac{C_l^{7/2}(x)}{C_l^{7/2}(1)}$ can be used to write for massless case

$$S_\ell(s) = 1 + \frac{8\pi i G_N}{3\cdot 2^{18}\pi^4}s^7\int_{-1}^{1}dx\left(1-x^2\right)^3\frac{C_\ell^{7/2}(x)}{C_\ell^{7/2}(1)}$$
$$\left(\frac{4}{s^3(1-x^2)} + \prod_{A=s,t,u}(\rho_A+1)^2 \sideset{}{'}\sum_{a+b+c\leq N}\alpha_{(abc)}\rho_s^a\rho_t^b\rho_u^c\right)$$

[11] A. Guerrieri, J. Penedones, P. Vieira, "Where is String Theory?", arxiv.org:2102.02847

- Using SDPB, we impose unitarity at grid points but since there is a factor of s^7 is sitting in front and so we need to make sure unitarity holds even at high energies.
- We need to consider integrals of the form

$$I_\ell^{abc}(s) = \rho^a(s) \int_{-1}^{1} \mu_\ell^{(10)}(x)\rho(t(s,x))^b \rho(u(s,x))^c dx$$

where $\mu_\ell^{(10)}(x) = \left(1 - x^2\right)^3 \frac{C_\ell^{(7/2)}(x)}{C_\ell^{(7/2)}(1)}$

- For even ℓ we can use $x \to -x$ from -1 to 0 and expand $\mu_\ell^{(10)}(x) = \sum_{n=3}^{6+\ell} \mu_n^\ell (1-x)^n$ so that now problem boils down to calculating

$$J_n^{bc}(s) = \int_0^1 \rho(t)^b \rho(u)^c (1-x)^n dx$$

- Taking $s_0 = 1$ for simplicity (can easily with few modifications be done for $s_0 = 0.7$), $\rho_t = -1 + \ldots$.series in $\frac{1}{\sqrt{s(1-x)}}$ can cause $(1-x)^n$ with $n > \frac{1}{2}$ appear in the denominator and cause divergences.

- So we split the integral as $J_n^{bc}(s) = \left(\int_0^{1-\delta} + \int_{1-\delta}^1\right) \rho^b_{t(s,x)} \rho^c_{u(s,x)} (1-x)^n dx$
- First integral can be done directly and for second integral we make the tranformation $x = 1 - \frac{2\epsilon^2}{s}$ to get

$$\int_{1-\delta}^1 \rho_t^b \rho_u^c (1-x)^n dx = \frac{2^{n+2}}{s^{n+1}} \int_0^{\Delta} \left(\rho_{-\epsilon^2}\right)^b \left(\rho_{-s+\epsilon^2}\right)^c \epsilon^{2n+1} d\epsilon$$

- After evaluating (explained in Thesis) and expanding for large $\Delta = \sqrt{\frac{\delta s}{2}}$ (large s) and taking $\delta \to 0$ after cancellations of problematic terms, the integral can be written as

$$J_n^{bc}(s) = \sum_{j=0}^{14} \frac{e_j^n(b,c) + \log(s) f_j^n(b,c)}{s^{j/2}} + \mathcal{O}\left(s^{-15/2}\right)$$

- In the end the integral can be written as

$$I_\ell^{abc} = \sum_{i=0}^{14} g_i^\ell(a,b,c)\frac{1}{s^{i/2}} + \sum_{i=8}^{14} h_i^\ell(a,b,c)\frac{\log(s)}{s^{i/2}} + \mathcal{O}\left(s^{-15/2}\right)$$

- And large energies unitarity can be imposed by imposing **for all** ℓ
  1. **All** $\frac{\log(s)}{s^{i/2}}$ **h-terms to vanish**
  2. **All** $s^{-i/2}$ **g-terms upto** $i = 13$ **to vanish**
  3. **14th term** g_{14} **which goes as** s^{-7} **goes to a constant at** ∞ **by exactly cancelling the** s^7 **and this needs to be bounded to respect unitarity**

- To minimize α, we maximize $-\alpha$. For $N_{max} = 13$ and $L_{max} = 28$, the value obtained was $\alpha_{min} = 3.62$ while the authors of obtained $\alpha_{min} = 4.87$.
- Unlike 3D bootstrap which converges for a fixed N_{max} at $L_{max} = N_{max} + 1$, in string bootstrap the values of L_{max} needed are much higher for example for $N = 13$, $L \simeq 70$ and for $N = 24$, $L \simeq 200$.
- Also the convergence as N increases is exponential decay to $\alpha_{min} = 0.13 \pm 0.02$ and hence is very sensitive.
- Moreover the authors used $s_0 = 0.713..$ and at an (N, L) far from convergence, this might be the most probable reason for the mismatch.
- Higher (N, L) computations are very time consuming and we will be working on it in the future.

- Obtain convergent values for string bootstrap by doing higher N_{max} and L_{max} computations.
- It would be interesting to see how the Wilson coeff. ratios corresponding to various boundaries and curves changes with dimensions.
- Since experimental value of ρ-resonance isn't available, this would be a good opportunity to see how changing real and imaginary parts of $m_\rho^2 \in \mathbb{C}$ changes the boundaries of the Lake. In higher dimensions, because of the extra factors of s in S_ℓ, unitarity at large energies need to be imposed separately and the method used in string bootstrap case can be adapted for this purpose.

Conclusion

- We also want to look at coeff. corresponding to symmetric isospin channel

$$\mathcal{T}^{(2)} = A(t|s,u) + A(u|s,t) = \sum_{p,q=0}^{\infty} \widetilde{W}_{p,q}(t+u)^p(tu)^q$$

- 2nd channel amplitude also is a physical amplitude corresponding to $\pi^+\pi^+ \to \pi^+\pi^+$ and hence we expect partial wave's imaginary part to follow positivity and consequently the corresponding Wilson coefficient ratios will have two sided bounds by using Bieberbach conjecture.
- But numerical checks found violations of these expected conditions! Needs further investigation.

References

1. A. L. Guerrieri, J. Penedones, P. Vieira, "Bootstrapping QCD: the Lake, the Peninsula and the Kink" arxiv:1810.12849
2. M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees and P. Vieira, "The S-matrix Bootstrap III: Higher Dimensional Amplitudes," arXiv:1708.06765
3. D. Simmons-Duffin, "A Semi-definite Program Solver for the Conformal Bootstrap" arXiv:1502.02033
4. A.V.Manohar and V.Mateu,"Dispersion Relation Bounds for $\pi\pi$ Scattering" Phys. Rev. D 77, 094019 (2008)
5. A. Guerrieri, J. Penedones, P. Vieira, "Where is String Theory?" arxiv.org:2102.02847
6. A. Sinha, A. Zahed, "Crossing Symmetric Dispersion Relations in QFTs"
7. P. Haldar, A. Sinha, A. Zahed, "Quantum field theory and the Bieberbach conjecture"
8. A. Sinha, Prashanth Raman, "QFT,EFT and GFT"
9. A. Bose, P. Haldar, A. Sinha, Pritish Sinha, S. S. Tiwari, "Relative entropy in scattering and the S-matrix bootstrap" arXiv:2006.12213