Deep Inelastic Scattering

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Rutherford Scattering and Mott Formula - Elastic e-p Scattering

- Treating the electromagnetic field as fixed classical background given by potential $A_\mu(x)$ the interaction Hamiltonian is:

$$H_I = \int \mathrm{d}^3 x e \bar{\psi} \gamma^\mu \psi A_\mu$$

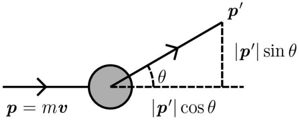

Figure: Rutherford Scattering

- The incident wave packet $|\psi\rangle$ is defined to be (with impact parameter b):

$$|\psi\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{e^{-i\mathrm{b}\cdot\mathrm{k}}}{\sqrt{2E_\mathrm{k}}} \psi(\mathrm{k}) |\mathrm{k}\rangle$$

- The probability that a scattered electron will be found within an infinitesimal element $\mathrm{d}^3 p$ centered at p is:

$$\frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_\mathrm{p}} \int \frac{\mathrm{d}^3 k \, \mathrm{d}^3 k'}{(2\pi)^6 \sqrt{2E_\mathrm{k} 2E_{\mathrm{k}'}}} \psi(\mathrm{k}) \psi^* \left(\mathrm{k}'\right) |i\mathcal{M}|^2 (2\pi)^2 \delta\left(E_\mathrm{p} - E_\mathrm{k}\right) \delta\left(E_\mathrm{p} - E_{\mathrm{k}'}\right) e^{-i\mathrm{b}\cdot(\mathrm{k}-\mathrm{k}')}$$

- The cross section $\mathrm{d}\sigma$ is given by:

$$\mathrm{d}\sigma = \int \mathrm{d}^2 b P(\mathrm{b})$$

- Using the fact that the momentum of the wave packet should be localized around its central value:

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_\mathrm{p}} \frac{1}{2E_\mathrm{k}} \frac{1}{v} (2\pi) |\mathcal{M}|^2 \delta\left(E_\mathrm{p} - E_\mathrm{k}\right) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \psi(\mathrm{k}) \psi^*(\mathrm{k}).$$

- We can further integrate over $|\mathrm{p}|$ to get the differential cross section:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \int \frac{\mathrm{d}pp^2}{(2\pi)^3}\frac{1}{2E_\mathrm{p}}\frac{1}{2E_\mathrm{k}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\delta\left(E_\mathrm{p}-E_\mathrm{k}\right)\\ &= \int \frac{\mathrm{d}pp^2}{(2\pi)^3}\frac{1}{2E_\mathrm{p}}\frac{1}{2E_\mathrm{k}}\frac{1}{v}|\mathcal{M}(k\to p)|^2(2\pi)\frac{E_\mathrm{k}}{k}\delta(p-k)\\ &= \frac{1}{(4\pi)^2}|\mathcal{M}(k,\theta)|^2\end{aligned}$$

- We can work out the amplitude to be:

$$i\mathcal{M}(k,\theta) \ = \mathrm{i}e\bar{u}(p)\gamma^\mu\tilde{A}_\mu(\mathrm{q})u(p) \qquad \text{with } \mathrm{q}=\mathrm{p}-\mathrm{k}$$

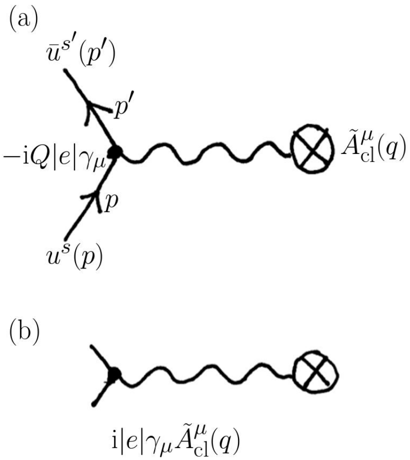

Figure: Feynman Diagrams for Rutherford Scattering

- The squared amplitude with initial spin averaged and final spin summed is simplified to be:

$$\frac{1}{2}\sum_{\text{spin}} |i\mathcal{M}(k,\theta)|^2 = \frac{Z^2 e^4 \left(1 - v^2 \sin^2 \frac{\theta}{2}\right)}{4|\mathrm{k}|^4 v^2 \sin^4(\theta/2)}$$

- In relativistic case, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2 \alpha^2 \left(1 - v^2 \sin^2 \frac{\theta}{2}\right)}{4|\mathrm{k}|^2 v^2 \sin^4(\theta/2)}$$

And in non-relativistic regime, this becomes:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2 \alpha^2}{4m^2 v^4 \sin^4(\theta/2)}$$

Electron-Fermion(Quark) Scattering

- Amplitude for Moller (Electron-Electron) scattering in high energy limit $(m_\mu, m_e << \sqrt{s})$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 2e^4\left[\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}+\frac{2s^2}{tu}\right]$$

- For electron-fermion scattering, u-channel and s-channel forbidden as electron can't change into a Fermion by emitting photon and vice versa

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^4 Q_i^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right) \;\; ; \;\; Q_i - \text{Fermion Charge}$$

- Mandelstram Variables:
  - s, t, u - For Proton-Electron Collision
  - $\hat{s}, \hat{t}, \hat{u}$ - For Quark-Electron Collision
- Cross-Section

$$\frac{d\sigma}{d\cos\theta_{\mathrm{CM}}} = \frac{1}{2\hat{s}} \frac{1}{16\pi} \frac{8e^4 Q_i^2}{\hat{t}^2} \left(\frac{\hat{s}^2 + \hat{u}^2}{4} \right)$$

$$\frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{\hat{s}^2} \left(\frac{\hat{s}^2 + (\hat{s} + \hat{t})^2}{\hat{t}^2} \right)$$

where we have used $\hat{s} + \hat{t} + \hat{u} = 0$ and $\hat{t} = -\hat{s}\left(1 - \cos\theta_{\mathrm{CM}}\right)/2$

Bjorken Scaling

- Parton Model: Proton as a loosely bound collection of 'partons'; For each Parton type i, the probability that it is present in the proton carrying a fraction of the total momentum ξ is $f_i(\xi)$

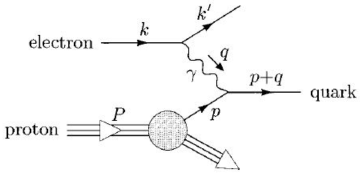

Figure: Deep Inelastic Scattering

- In High Energy Limit, $s >> P^2 \approx 1\ GeV^2$, so

$$\hat{s} = (p+k)^2 = 2p \cdot k = 2\xi P \cdot k = \xi (P+k)^2 = \xi s$$

- Notation:
  - $\hat{t} = q^2 = -\hat{s}\left(1 - \cos\theta_{\rm CM}\right)/2 < 0$
  - $Q^2 \equiv -q^2 > 0$

- Mass of scattered parton $= (p+q)^2 << s, Q^2$ and can be taken to be ≈ 0

$$\Rightarrow 0 \approx (p+q)^2 = 2p\cdot q + q^2 = 2\xi P \cdot q - Q^2$$

$$\Rightarrow \frac{\hat{t}}{\hat{s}} = -Q^2/\xi s$$

- Substituting in Cross Section:

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_i^2}{Q^4}\left(1 + (1 - \frac{Q^2}{\xi s})^2\right)$$

- For Multiple Partons:

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \sum_i f_i(\xi)\frac{2\pi\alpha^2 Q_i^2}{Q^4}\left(1 + (1 - \frac{Q^2}{\xi s})^2\right)$$

- We also need to integrate over all ξ over its range of values. Replacing $|\hat{t}|$ with Q^2

$$\frac{d\sigma}{dQ^2} = \int_0^1 d\xi \sum_i f_i(\xi) Q_i^2 \frac{2\pi\alpha^2}{Q^4}\left[1 + \left(1 - \frac{Q^2}{\xi s}\right)^2\right]\theta\left(\xi s - Q^2\right)$$

- Using the following change of variables:$x = \xi$ and $y = \frac{Q^2}{\xi s}$

$$\Rightarrow \hat{t} = -Q^2 = -xys; \; dxdQ^2 = \frac{dQ^2}{dy}dxdy = xsdxdy$$

$$\frac{d^2\sigma}{dxdy}\left(e^- p \to e^- X\right) = \left(\sum_i xf_i(x)Q_i^2\right)\frac{2\pi\alpha^2 s}{Q^4}\left[1 + (1-y)^2\right]$$

- **Bjorken Scaling: Beyond a certain energy needed to excite partons, the structure of proton visible by probing it electromagnetically is independent of energy.**
  Confirmed by experiments apart from scaling violations which can be accounted for by QCD calculations.
- So we can claim that$f_i(\xi, Q^2) = f_i(\xi)$ for Q^2 >>1 GeV

Weak Interaction Mediated by W Bosons (Low Energy)

- Lagrangian corresponding to W-Boson exchange

$$\mathcal{L}_W = \frac{g}{\sqrt{2}}\left(W_+^\mu J_\mu^+ + W_-^\mu J_\mu^-\right)$$

- Amplitude for W-Boson exchange

$$\left(\frac{ig}{\sqrt{2}}\right)^2 \int d^4x d^4y J^{+\mu}(x) J^{-\nu}(y) W_\mu^+(x) W_\nu^-(y)$$

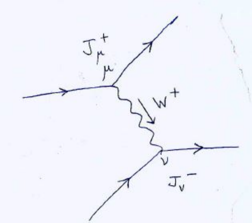

Figure: W Boson Exchange

- Propagator of massive vector boson

$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \int d^4x d^4y \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} J^{+\mu}(x) J^{-\nu}(y) \left(\frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2}\right)}{p^2 - m_W^2} \right)$$

- Further Simplification:
  - $p_\mu p_\nu$ terms is removed using Feynman-t'Hooft Gauge
  - For a low energy effective theory, $p^2 << m_W^2$. So

$$\frac{-g^2}{p^2 - m_W^2} = \frac{g^2}{m_W^2}\left(1 + \frac{p^2}{m_W^2} + \left(\frac{p^2}{m_W^2}\right)^2 + \cdots\right)$$

  - Ignore $\mathcal{O}(\frac{p^2}{m_W^2})$ terms
  - $\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} = \delta^{(4)}(x-y)$
- We get

$$= \frac{-ig^2}{2m_W^2} \int d^4x d^4y \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} J^{+\mu}(x) J^-_\mu(y)$$

$$= \frac{-ig^2}{2m_W^2} \int d^4x J^{+\mu}(x) J^-_\mu(x)$$

- At low energies, range of W-Boson is too small to notice. So effective interaction is a single point

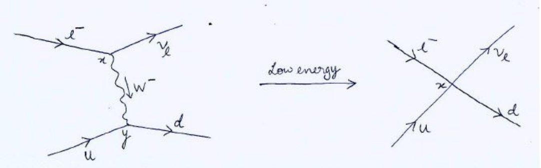

Figure: Effective Interaction is at a Single Point

- Low Energy Effective Lagrangian for weak interaction:

$$\mathcal{L}_{W,eff} = \frac{g^2}{2m_W^2} J^{+\mu} J^{-}_{\mu}$$

- For DIS with Neutrinos

$$\mathcal{L}_{DIS\ \ Neutrino} = \frac{g^2}{2m_W^2} \left[\bar{\ell} \gamma^{\mu} \left(\frac{1-\gamma^5}{2} \right) \nu \right] \left[\bar{u} \gamma_{\mu} \left(\frac{1-\gamma^5}{2} \right) d \right] + \mathrm{h.c.}$$

DIS with Neutrinos

- Calculations shown in Report
- We have
  - $\frac{d\sigma}{d\hat{t}}(\nu d \to \mu^- u) = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2}\left[\frac{\hat{s}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}$
  - $\frac{d\sigma}{d\hat{t}}(\bar{\nu} u \to \mu^+ d) = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2}\left[\frac{\hat{u}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}(1-y)^2$
- Assuming proton is made of u and d quarks and their anti-quarks

$$\frac{d^2\sigma}{dxdy}(\nu p - \mu^- X) = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x f_{\bar{u}}(x) \cdot (1-y)^2\right]$$

$$\frac{d^2\sigma}{dxdy}(\bar{\nu} p \to \mu^+ X) = \frac{G_F^2 s}{\pi}\left[x f_u(x) \cdot (1-y)^2 + x f_{\bar{d}}(x)\right]$$

- The constant nature of curve for neutrino(ν) curve and $(1-y)^2$ nature of anti-neutrino($\bar{\nu}$) curve imply that anti-quark component is largely absent.
- So DIS with neutrinos helps us extract information about proton structure in this manner.

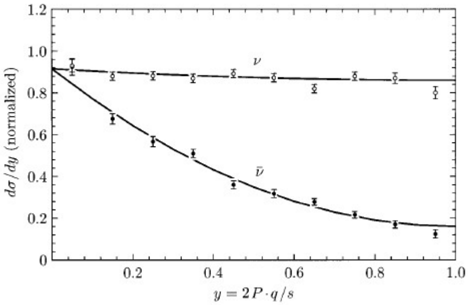

Figure: Experimental observation of neutrino and anti-neutrino scattering from iron target. Fits are to $A + B(1-y)^2$

Deep Inelastic Scattering

- In DIS, spin-averaging leads to a leptonic tensor and a hadronic tensor with two indices each contracted with each other.

$$\frac{d\sigma}{d\Omega} \propto L^{\mu\nu} W_{\mu\nu}$$

- The Leptonic Tensor

$$L_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left(\not{k}\gamma_\mu \not{k}'\gamma_\nu\right) = 2\left(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}\left(k' \cdot k\right)\right)$$

- The Hadronic Tensor: The hadronic tensor is the modulus squared of an expression which if multiplied by polarization vector gives us the amplitude.

$$e^2 \epsilon_\mu \epsilon_\nu^\star W^{\mu\nu} = \frac{1}{2} \sum_{X,\ \text{spins}} \int d\Pi_X (2\pi)^4 \delta^4 \left(q + P - p_X\right) \left|\mathcal{M}\left(\gamma^\star p \to X\right)\right|^2$$

Note: $\gamma^\star$ measn that the photon is off-shell

- Ward Identity: $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$ where

$$W_{\mu\nu} = a_1 \eta_{\mu\nu} + a_2 q_\mu q_\nu + a_3 p_\mu p_\nu + a_4 p_\mu q_\nu + a_5 p_\nu q_\mu$$

- $q^\mu W_{\mu\nu} = 0 \Rightarrow a_1 q_\nu + a_2 q^2 q_\nu + a_3 (p \cdot q) p_\nu + a_4 (p \cdot q) q_\nu + a_5 q^2 p_\nu = 0$
  Hence we have $a_1 + a_2 q^2 + a_4 (p \cdot q) = 0$ and $a_3 (p \cdot q) + a_5 q^2 = 0$.
- $q^\nu W_{\mu\nu} = 0 \Rightarrow a_1 q_\mu + a_2 q^2 q_\mu + a_3 (p \cdot q) p_\mu + a_4 q^2 p_\mu + a_5 (p \cdot q) q_\mu = 0$
  Hence we have $a_1 + a_2 q^2 + a_5 (p \cdot q) = 0$ and $a_3 (p \cdot q) + a_4 q^2 = 0$.
- $\Rightarrow a_4 = a_5 = -\frac{a_1 + a_2 q^2}{p \cdot q} = -\frac{a_3 (p \cdot q)}{q^2}$

- Using this we can write the most general way to write the hadronic tensor as

$$W^{\mu\nu} = W_1 \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + W_2 \left(P^\mu - \frac{P \cdot q}{q^2} q^\mu \right) \left(P^\nu - \frac{P \cdot q}{q^2} q^\nu \right)$$

- $\frac{d\sigma}{d\Omega} \propto \frac{m_p}{2} W_2(x, Q) \cos^2 \frac{\theta}{2} + \frac{1}{m_p} W_1(x, Q) \sin^2 \frac{\theta}{2}$
- Exact Expression:

$$\left(\frac{d\sigma}{d\Omega dE'} \right)_{\text{lab}} = \frac{\alpha_e^2}{8\pi E^2 \sin^4 \frac{\theta}{2}} \left[\frac{m_p}{2} W_2(x, Q) \cos^2 \frac{\theta}{2} + \frac{1}{m_p} W_1(x, Q) \sin^2 \frac{\theta}{2} \right]$$

- Parton Model:
  - Momentum exchanges among partons at time scales $\sim \Lambda_{\mathrm{QCD}}^{-1} \sim m_p^{-1}$, which are much slower than the time scales $\sim Q^{-1}$ that the photon probes. $Q \gg \Lambda_{\mathrm{QCD}}$ justifies using parton model.
  - $f_i(\xi)d\xi$ - Classical Probabilities of photon hitting the parton species
  - $\sigma\left(e^- P \rightarrow e^- X\right) = \sum_i \int_0^1 d\xi f_i(\xi)\hat{\sigma}\left(e^- p_i \rightarrow e^- X\right)$
- Rosenbluth Formula: Derivation in Report

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha_e^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left\{ \left(F_1^2 - \frac{q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} \left(F_1 + F_2\right)^2 \sin^2 \frac{\theta}{2} \right\}$$

- Partons (i.e quarks) interact electromagnetically with the electron we can take $F_1 = 1$ and $F_2 = 0$

$$\left(\frac{d\hat{\sigma}\left(e^- q \rightarrow e^- q\right)}{d\Omega dE'}\right)_{\text{lab}} = \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\cos^2 \frac{\theta}{2} + \frac{Q^2}{2m_q^2} \sin^2 \frac{\theta}{2}\right] \delta\left(E - E' - \frac{Q^2}{2m_q}\right)$$

- $E - E' = \frac{Q^2}{2m_p x}$
  $\Rightarrow \delta\left(E - E' - \frac{Q^2}{2m_q}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p}{Q^2} x^2 \delta(\xi - x)$

$$\Rightarrow \left(\frac{d\sigma\left(e^- P \to e^- X\right)}{d\Omega dE'}\right)_{\mathrm{lab}} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[\frac{2m_p}{Q^2} x^2 \cos^2 \frac{\theta}{2} + \frac{1}{m_p} \sin^2 \frac{\theta}{2}\right]$$

- Comparing with general expression obtained at beginning

$$W_1(x, Q) = 2\pi \sum_i Q_i^2 f_i(x)$$

$$W_2(x, Q) = 8\pi \frac{x^2}{Q^2} \sum_i Q_i^2 f_i(x)$$

- $\Rightarrow W_1(x, Q) = \frac{Q^2}{4x^2} W_2(x, Q)$
  Callan-Gross Relation: Quarks are spin-1/2 particles