Langrange Interpolation (Quadratic $n=2$)

x_0, x_1, x_2

let $f(x)$ is a conti. f^n on (a,b) & x_0, x_1, x_2 are three destint nodes of (a,b)

Condition $\rightarrow f(x_i) = P(x_i) \quad i = 0,1,2$ —— ①

let $P(x) = a_0 + a_1 x + a_2 x^2$ is a quadratic poly.

then $f(x_0) = P(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$ —— ②

$f(x_1) = P(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$ —— ③

$f(x_2) = P(x_2) = a_0 + a_1 x_2 + a_2 x_2^2$ —— ④

Now a_0, a_1 & a_2 are eliminated by eq. ①, ②, ③, ④ —°o.

$$\begin{vmatrix} P(x) & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{vmatrix} = 0.$$

Now expanding via first column —°o.

$$P(x) \underbrace{\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}}_{\text{let it be } D_0} - f(x_0) \underbrace{\begin{vmatrix} 1 & x & x^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}}_{\text{let it be } D_1}$$

$$+ f(x_1) \underbrace{\begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}}_{D_2} - f(x_2) \underbrace{\begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \end{vmatrix}}_{D_3}$$

$D_0 = (x_0 - x_1)(x_1 - x_2)(x_2 - x_0)$

$D_1 = (x - x_1)(x_1 - x_2)(x_2 - x)$

$D_2 =$

$D_3 =$

$P(x) D_0 - f(x_0) D_1 + f(x_1) D_2 - f(x_2) D_3 = 0$

$$P_x = \frac{D_1}{D_0} f(x_0) - \frac{D_2}{D_0} f(x_1) + \frac{D_2}{D_0} f(x_2)$$

formula

$$\boxed{P_x = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \times f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \times f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \times f(x_2)}$$

$P_2 x = L_0(x) f(x_0) + L_1(x) f_1(x_1) + L_2(x) f(x_2)$

where $L_0(x) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$

Q. Given $f(0) = 1$

$f(1) = 3$

$f(3) = 55$

Find Langrange interpolation.

Sol. Here $x_0 = 0$ $x_1 = 1$ $x_2 = 3$

$f(x_0) = 1$ $f(x_1) = 3$ $f(x_2) = 55$

Now by langrange quadratic interpolation :-

$P_2(x) = L_0(x) f(x_0) - L_1(x) f(x_1) + L_2(x) f(x_2)$

Now, $L_0(x) = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \dfrac{(x-1)(x-3)}{(0-1)(0-3)}$

$= \frac{(x-1)(x-3)}{3}$ or $\frac{1}{3}(x^2-4x+3)$ (verify by putting x_0, x_1 or x_2 must $=0$)

$L_1(x) = \frac{1}{2}(x^2-3x)$

$L_2(x) = \frac{1}{6}(x^2-x)$

Applying formula gets :-

$P_2(x) = \left[\frac{1}{3}(x^2-4x+3) \times (1)\right] + \left[\frac{1}{2}(x^2-3x)(3)\right] + \left[\frac{1}{6}(x^2-x)55\right]$

$P_2(x) = 8x^2 - 6x + 1$

Comparing answer satisfy condition $f(x_i) = P(x_i)$

$f(x_0) = P_2(x_0) = P_2(0) = 1$
$f(x_1) = P_2(x_1) = P_2(1) = 3$
$f(x_2) = P_2(x_2) = P_2(3) = 55$

Q. $f(x) = \sin x + \cos x$

x	10°	20°	30°
$f(x)$	1.1585	1.287	1.3660

Ans $\rightarrow$ Convert $10^\circ = 10 \times \frac{\pi}{180} = \frac{\pi}{18} = \frac{22}{7 \times 18} = \underline{\underline{0.1745}}$.

Solve urself.

Shift operator Denoted by E.

① $E f(x_i) = f(x_i + h)$ $\quad [a, b]$

$h = \frac{b-a}{h}$ $\quad a = x_0 < x_1 < x_2 < x_3 \cdots x_n = b$.

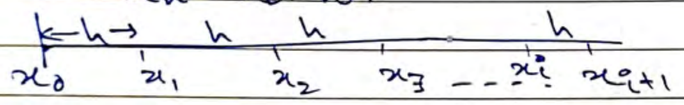
$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_0 + 3h$.

$\vdots$

$x_n = x_0 + nh$



② $E^n f(x_i) = f(x_i + nh)$

Forward difference operator. Denoted by Δ.

$\Delta f(x_i) = f(x_i + h) - f(x_i)$

$= E f(x_i) - f(x_i)$

$\Delta f(x_i) = (E - 1) f(x_i)$

$\Rightarrow \boxed{\Delta = E - 1}$

Backward Difference operator. Denoted by ∇

$$\boxed{\nabla f(x_i) = f(x_i) - f(x_i - h)}$$

$f(x_i) - E^{-1} f(x_i)$

$\nabla f(x_i) = (1 - E^{-1}) f(x_i)$

$\boxed{\nabla = 1 - E^{-1}}$

Central Difference operator. Denoted by δ

Defined by :-

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$$= E^{1/2} f(x_i) - E^{-1/2} f(x_i)$$

$$\delta f(x_i) = (E^{1/2} - E^{-1/2}) f(x_i)$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Average Difference operator. Denoted by μ

$$\mu f(x_i) = \frac{f(x_i + \frac{h}{2}) + f(x_i - \frac{h}{2})}{2}$$

$$= E^{1/2} f(x_i) + E^{-1/2} f(x_i) / 2$$

$$\mu f(x_i) = \frac{(E^{1/2} + E^{-1/2})}{2} f(x_i)$$

$$\boxed{\mu = \frac{E^{1/2} + E^{-1/2}}{2}}$$

x_i — $x/2$ — $x_i + \frac{h}{2}$

Relation between the operator.

	E	Δ	∇	δ (Mostly comes)
E	E	$\Delta+1$	$(1-\nabla)^{-1}$	$1+\delta^2/2+\delta\sqrt{1+\delta^2/4}$
Δ	$E-1$	Δ	$(1-\nabla)^{-1}-1$	$\delta^2/2+\delta\sqrt{1+\delta^2/4}$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\delta^2/2+\delta\sqrt{1+\delta^2/4}$
δ	$E^{1/2}-E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ
μ	$\frac{E^{1/2}+E^{-1/2}}{2}$	$(1+\frac{\Delta}{2})(1+\Delta)^{-1/2}$	$(1-\frac{\nabla}{2})(1-\nabla)^{-1/2}$	$\sqrt{1+\frac{\delta^2}{4}}$

(Relation proof comes Definitely)

① $E = \Delta + 1$

RHS $= \Delta + 1$ — we know, $\Delta = E - 1$

$= E - \cancel{1} + \cancel{1}$

$= E =$ LHS

(2) $\nabla = 1-(1+\Delta)^{-1}$

$RHS = 1-(1+\Delta)^{-1}$ we know, $\Delta = E-1$

$= 1-(\cancel{1}+E-\cancel{1})^{-1}$

$= 1-E^{-1}$

$= \nabla = LHS.$

Imp. (3) $\delta = \Delta(1+\Delta)^{-1/2}$

$RHS = \Delta(1+\Delta)^{-1/2}$ we know, $\Delta = E-1$

$= (E-1)(1+E-1)^{-1/2}$

$= (E-1)E^{-1/2}$

$= EE^{-1/2} - E^{-1/2}$

$= E^{1/2} - E^{-1/2}$

$= \delta = LHS.$

(For relationship proving put the value of E in RHS, solve and convert the ending result in terms of E into that of operator)

(4) $\mu = \left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2}$ we know $\Delta = E-1.$

$RHS = \left(\frac{1+\Delta}{2}\right)(1+\Delta)^{-1/2}$

$= \left(\frac{1+E-1}{2}\right)(\cancel{1}+E-\cancel{1})^{-1/2}$

$= \left(\frac{1+E}{2}\right)(E)^{-1/2} = \frac{E^{-1/2}E + E^{-1/2}}{2} = \frac{E^{1/2}+E^{-1/2}}{2} = \mu$ LHS.

(5) $E = 1+\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$

$RHS = 1+\frac{\delta^2}{2}+\delta\sqrt{1+\frac{\delta^2}{4}}$

$\delta = E^{1/2}-E^{-1/2}$

$= 1+\frac{(E^{1/2}-E^{-1/2})^2}{2} + (E^{1/2}-E^{-1/2})\sqrt{1+\frac{(E^{1/2}-E^{-1/2})^2}{4}}$

$$= 1 + \frac{1}{2}\left(E + E^{-1} - 2\right) + \left[E^{1/2} - E^{-1/2}\right]\sqrt{1 + \frac{(E + E^{-1} + 2)}{4}}$$

$$= 1 + \left(\frac{E+E^{-1}}{2}\right) - 1 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{1}{4}(E + E^{-1} + 2)}$$

$$= \left(\frac{E+E^{-1}}{2}\right) + E^{1/2} - E^{-1/2}\sqrt{\frac{1}{4}\left(E^{+1/2} + E^{-1/2}\right)^2}$$

$$= \left(\frac{E+E^{-1}}{2}\right) + \left(E^{1/2} - E^{-1/2}\right)\left(\frac{E^{1/2} + E^{-1/2}}{2}\right)$$

$$= \frac{E + E^{-1}}{2} + \frac{1}{2}\left[\left(E^{1/2}\right)^2 - \left(E^{-1/2}\right)^2\right]$$

$$= \frac{E+E^{-1}}{2} + \frac{1}{2}\left(E - E^{-1}\right) = \frac{2E}{2} = E = \underline{\underline{LHS}}$$

(6). $\nabla = -\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$

$$RHS = -\frac{\delta^2}{2} + \delta\left(1 + \frac{\delta^2}{4}\right)$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$= -\frac{\left(E^{1/2} - E^{-1/2}\right)^2}{2} + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}}$$

$$= -\left(\frac{E + E^{-1} - 2}{2}\right) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= -\left(\frac{E + E^{-1} - 2}{2}\right) + E^{1/2} - E^{-1/2}\sqrt{\frac{\left(E^{1/2} + E^{-1/2}\right)^2}{4}}$$

$$= -\left(\frac{E + E^{-1} - 2}{2}\right) + \left(E^{1/2} - E^{-1/2}\right)\left(\frac{E^{1/2} + E^{-1/2}}{2}\right)$$

$$= \left(\frac{-E - E^{-1} + 2}{2}\right) + \left(\frac{E - E^{-1}}{2}\right) = \frac{-2E^{-1}}{2} + 1 = 1 - E^{-1}$$

$$= \nabla \ \underline{\underline{LHS}}$$

Newton Divided Difference [with forward]

first divided difference with x_0 & x_1

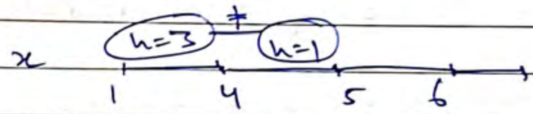
$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \qquad \boxed{x_0 < x_1}$$

we know,

$$\left[\begin{array}{l} f(x_i) = f_i \\ f(x_0) = f_0 \\ [a, b] \\ x_0 < x_1 < x_2 \ldots < x_n \\ x_n = x_0 + nh \\ x_1 = x_0 + h \\ x_2 = x_1 + h = x_0 + 2h \end{array}\right.$$

$$\Rightarrow \frac{f[x_0+h] - f[x_0]}{(x_0+h-x_0)} \Rightarrow \frac{f[x_0+h] - f[x_0]}{h} = \frac{\Delta f(x_0)}{h}$$

$$\Delta f(x_i) = f(x_i+h) - f(x_i)$$



$f(x)$: ↓ ↓ ↓ ↓ (value given)

$$f[x_0, x_1] = \frac{1}{h} \Delta f(x_0) \quad \text{(forward)}$$

from this

Second Divided difference –

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)}{x_0 + 2h - x_0}$$

$$= \frac{\frac{1}{h}\Delta\boxed{(f(x_1) - f(x_0))}}{2h} = \frac{1}{2h^2}\Delta[\Delta f(x_0)]$$

from above

$$= \frac{\Delta^2 f(x_0)}{2h^2}$$

Third Divided difference

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{1}{2h^2}\Delta^2 f(x_1) - \frac{1}{2h^2}\Delta^2 f(x_0)$$
$$\overline{x_0 + 3h - x_0}$$

$$= \frac{1}{2h^2}\Delta^2\left(\frac{f(x_1) - f(x_0)}{3h}\right) = \frac{1}{6h^3}\Delta^2(\Delta f(x_0))$$

$$= \frac{1}{\lfloor 3 h^3}\Delta^3 f(x_0)$$

n^{th} Divided difference

$$f[x_0, x_1, x_2, x_3 \ldots x_n] = \frac{1}{\lfloor n\, h^n}\Delta^n f(x_0)$$

Newton Divided difference (relation with backward)

First Divided difference $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$\nabla f(x_i) = f(x_i) - f(x_i - h) \qquad \because x_1 = x_0 + h$$
$$\boxed{x_0 = x_1 - h}$$

$$= \frac{f(x_1) - f(x_1 - h)}{\cancel{x_0} + h - \cancel{x_0}}$$

$$= \frac{\nabla f(x_1)}{h}$$

Second divided difference

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

$$= \frac{\frac{\nabla}{h} f(x_2) - \frac{\nabla}{h} f(x_1)}{\cancel{x_0} + 2h - \cancel{x_0}}$$

$$= \frac{\frac{\nabla}{h}[f(x_2) - f(x_1)]}{2h}$$

$$= \frac{\nabla}{2h^2}(\nabla f(x_2)) = \frac{\nabla^2}{2h^2} f(x_2)$$

Third Divided difference

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{\frac{\nabla^2}{2h^2} f(x_3) - \frac{1}{2h^2} \nabla^2 f(x_2)}{x_0 + 3h - x_0}$$

$$= \frac{\nabla^2}{2h^2}\left[\frac{f(x_3) - f(x_2)}{3h}\right] = \frac{\nabla^2}{3!\, h^3}\left(\nabla f(x_3)\right) =$$

$$= \frac{\nabla^3}{3!\, h^3} f(x_3)$$

n^{th} Divided Difference

$$f[x_0, x_1, x_2, x_3 \ldots x_n] = \frac{1}{n!\, h^n} \Delta^n f(x_n)$$

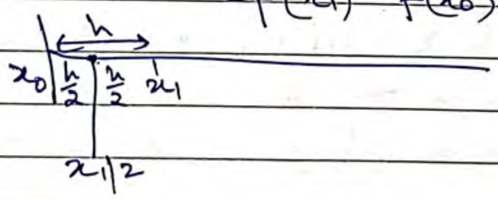
Relation b/w Newton Divided Diff. & Central Diff. operator

First Divided Diff. $f[x_0 x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

we know $\delta f(x_i) = f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2})$

and $\delta f(x_{1/2}) = f(x_{1/2} + \frac{h}{2}) - f(x_{1/2} - \frac{h}{2})$

$= f(x_1) - f(x_0)$



Now, $f[x_0, x_1] = \dfrac{\delta f(x_{1/2})}{x_0 + h - x_0} = \boxed{\dfrac{1}{h} \delta f(x_{1/2})}$

Second Divided Diff.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\frac{1}{h}\delta f_{3/2} - \frac{1}{h}\delta f_{(1/2)}}{x_0 + 2h - x_0} \Rightarrow \frac{1}{2h^2}\delta(f_{3/2} - f_{1/2})$$

$$\boxed{= \frac{1}{\lfloor 2 h^2}\delta f_{1}} \rightarrow \text{Second Diff} \; \frac{(2)}{2} = 1$$

Third Divided Diff.

$$f[x_0, x_1, x_2, x_3] = \frac{1}{\lfloor 3 h^3}\delta_{3/2} \rightarrow \text{Third Diff.} \; \frac{(3)}{2} = \frac{3}{2}$$

N^{th} Divided Diff.

$$f[x_0, x_1, x_2, x_3 \ldots x_n] = \frac{1}{\lfloor n h^n}\delta_{n/2}$$

But odd / even case :-

- $f[x_0, x_1, x_2, \overline{x_3 \ldots x_{2m}}^{\text{even}}] = \frac{1}{\lfloor 2m h^{2m}}\delta f_m$
- $f[x_0, x_1, x_2, x_3 \ldots x_{2m+1}^{\text{odd}}] = \frac{1}{\lfloor 2m+1 \, h^{2m+1}}\delta f_{m+1/2}$

N.D.D with forward.

$$f[x_0, x_1, x_2 \ldots x_n] = \frac{1}{\lfloor n h^n}\Delta^n f(x_0)$$

N.D.D interpolation

$$P(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) \times f(x_0, x_1, x_2) + \ldots + (x-x_0)(x-x_1)(x-x_2) \ldots \ldots (x-x_{n-1}) f(x_0, x_1, x_2 \ldots x_n)$$

~~P(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1)~~

Using both things we get,

$$P(x) = f(x_0) + (x-x_0)\frac{1}{\lfloor 1 h}\Delta f(x_0) + (x-x_0)(x-x_1)\frac{1}{\lfloor 2 h^2}\Delta^2 f(x_0)$$

$$+ \ldots (x-x_0)(x-x_1) \ldots (x-x_{n-1})\frac{1}{\lfloor n h^n}\Delta^n f(x_0)$$

[This value of $P(x)$ is called Newton-forward-Diff. interpolation.]

let $u = \frac{x-x_0}{h}$

$$\boxed{P(x) = f(x_0) + \frac{u}{\lfloor 1}\Delta f(x_0) + \frac{u(u-1)}{\lfloor 2}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 f(x_0) + \dots + u(u-1)(u-2)\dots(u-(n-1))\Delta^n f(x_0)}$$

$$\boxed{u = \frac{x-x_0}{h}}$$

$u = \frac{x-x_0}{h}$ (we suppose)

Now, $uh = x - x_0$

$x = uh + x_0$

$x - x_1 = uh + x_0 - x_1$

$= uh - (x_1 - x_0)$

$= uh - h$

$x - x_1 = (u-1)h$

$\frac{x-x_1}{h} = (u-1)$

Similarly $\frac{x-x_2}{h} = u-2$.

Alternative Proof.

$$f(x) = f\left(x_0 + \underbrace{\left(\frac{x-x_0}{h}\right)}_{u} h\right) \qquad \text{let } u = \frac{x-x_0}{h}.$$

$$= f(x_0 + uh)$$

$$= E^u f(x_0)$$

$$= (1+\Delta)^u f(x_0)$$

$$f(x) = ({}^uC_0 + {}^uC_1\Delta + {}^uC_2\Delta^2 + {}^uC_3\Delta^3 + \dots) f(x_0)$$

$$\boxed{f(x) = 1.f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{\lfloor 2}\Delta^2 f(x_0) + \dots + \frac{u(u-1)(u-2)\dots(u-(n-1))}{\lfloor n}\Delta^n f(x_0)}$$

$\Delta f(x_i) = f(x_i+h) - f(x_i)$
$= f(x_{i+1}) - f(x_i)$

NDD Table

Forward Diff. Table :-

x	$f(x)$		
x_0	$f(x_0)$	$f(x_1)-f(x_0) = \Delta f(x_0)$	
x_1	$f(x_1)$		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$
		$\Delta f(x_1) = f_2 - f_1$	
x_2	$f(x_2)$		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$
		$\Delta f(x_2) = f_3 - f_2$	
x_3	$f(x_3)$		

$$\boxed{\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0}$$

Q.

x	0.1	0.2	0.3	0.4
$f(x)$	1.40	1.56	1.76	2.00

Find ① Interpolation ② Newton Diff. Fwd ③ interpolate at $x = 0.25$, $x = 0.35$

Note: (Our choice of interpolation : Newton or Langrange However for equal interval & value of x till 3 units use Langrange but for unequal (even equal) intervals & value of x greater than 3 units use Newton D.D interpolation or Newton Diff. fwd interpolation. If Ques specifies interpolation then do as directed in ques.)

Sol. Newton Diff forward Table.

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 \to$	0.1	$f_0 = 1.40$	$\Delta f(x_0)$			
			0.16	$\Delta^2 f(x_0)$		
$x_1 \to$	0.2	$f_1 = 1.56$		0.04	$\Delta^3 f(x_0)$	
			0.20		0	
$x_2 \to$	0.3	$f_2 = 1.76$		0.04		$\Delta^4 f(x_0)$ 0
			0.24		0	

$x_3 \to 0.4$	$f_3 = 2.00$		0.04	0
		0.28		
$x_4 \to 0.5$	$f_4 = 2.28$			

Newton forward Diff interpolation.

$$p(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{h} + (x-x_0)(x-x_1)\frac{1}{\lfloor 2 h^2}\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\frac{1}{\lfloor 3 h^3}\Delta^3 f(x_0) = 0.$$

$$p(x) = 1.40 + \frac{(x-0.1)(0.16)}{0.1} + \frac{(x-0.1)(x-0.2)}{2(0.1)^2}(0.04) + 0 + 0$$

$$p(x) = 1.40 + 1.6(x-0.1) + 2(x-0.1)(x-0.2)$$

$$= 1.40 + \frac{1.6x}{2} - 0.16 + 2x^2 - 0.6x + 0.04$$

$$p(x) = 2x^2 + x + 1.28 \text{ Ans.}$$

At $x = 0.25$

$$P(0.25) = 2(0.25)^2 + (0.25) + 1.28$$

$$= \text{ Ans.}$$

Q (1)

x	0	0.1	0.2	0.3	0.4
$f(x)$	2.00	2.11	2.28	2.39	2.56

(2)

x	0.1	0.3	0.5	0.7	0.9	0.11
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Find the Newton Fwd & Bwd Diff. interpolation.

Proofs won't come

Newton Backward Difference interpolation

Proof: $f(x) = f\left[x_n + \frac{x - x_n}{h} \cdot h\right]$ let $u = \frac{x - x_n}{h}$

$= f(x_n + uh)$

$= E^u f(x_n)$

$= (1-\nabla)^{-u} f(x_n)$

$\Delta = E - 1$ (fwd operator)

$\nabla = 1 - E^{-1}$ (Bwd operator)

$E^{-1} = 1 - \nabla$

$E = (1-\nabla)^{-1}$

$= 1 + u\nabla + \frac{u(u+1)}{\lfloor 2}\nabla^2 + \frac{u(u+1)(u+2)}{\lfloor 3}\nabla^3 + \dots f(x_n)$

$$p(x) = f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{\lfloor 2}\nabla^2 f(x_n) + \dots$$

* Newton Bwd and fwd Diff. interpolation only for equal intervals ques.

Q.

x:	0.1	0.2	0.3	0.4	0.5
$f(x)$:	1.40	1.56	1.76	2.00	2.28

① Find the Newton Bwd Diff. interpolation

② interpolate at $x = 0.25$ & 0.35.

Sol. Newton Bwd Diff. Table.

	x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
$x_0 \rightarrow$	0.1	1.40				
			$\nabla f_1 = f_1 - f_0$ $= 0.16$			
$x_1 \rightarrow$	0.2	1.56		$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$		
			$= 0.20$	0.04		
$x_2 \rightarrow$	0.3	1.76			$\nabla^3 f_3 = \nabla^2 f_2 - \nabla^2 f_1$	
			0.24	$= 0.04$	$= 0$	
$x_3 \rightarrow$	0.4	2.00				0
			0.28	$= 0.04$	$= 0$	
$x_4 \rightarrow$	0.5	2.28				

$\nabla f(x_i) = f(x_i) - f(x_i - h)$
$\nabla f(x_1) = f(x_1) - f(x_0)$

Newton Backward Diff interpolation.

$p(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{\lfloor 2}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{\lfloor 3}\nabla^3 f(x_n)$

and $u = \frac{x - x_n}{h}$; $x_n = 0.5$

$= f(0.5) + u\nabla f(0.5) + \frac{u(u+1)}{\lfloor 2}\nabla^2 f(0.5) + 0$

$= 2.28 + u(0.28) + \frac{u(u+1)}{\lfloor 2}(0.04)$

$= 2.28 + \left(\frac{x-0.5}{0.1}\right)(0.28) + \frac{1}{2}\left(\frac{x-0.5}{0.1}\right)\left(\frac{x-0.4}{0.1}\right)(0.04)$

$= 2.28 + 2.8x - 1.4 + 2\left[x^2 - 0.9x + 0.2\right]$

$= 2x^2 + x + 1.28$

* for verification put any value of x in it. If it gives corresp. value of $f(x)$ as answer.

Q. Prove that

① $\nabla - \Delta = -\Delta\nabla$

LHS $= \nabla - \Delta$ $\qquad \Delta = E - 1$

$= (1 - E^{-1}) - (E - 1)$ $\qquad \nabla = 1 - E^{-1}$

$= 1 - E^{-1} - E + 1$

$= 2 - E^{-1} - E$

RHS : $-\Delta\nabla$

$= -(E-1)(1-E^{-1})$

$= -E + EE^{-1} + 1 - E^{-1}$

$= -E + 1 + 1 - E^{-1}$

$= 2 - E^{-1} - E$

LHS = RHS

$\frac{1}{1+a}$ (16) Date 2/5

(2) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$\Delta = E-1$

$\nabla = 1-E^{-1}$

LHS $= \Delta + \nabla$

$= E-1+1-E^{-1}$

$= E-E^{-1}+1-1$

$\Rightarrow \Delta + \nabla$ H.P.

RHS $= \cancel{\frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}}$

$= \cancel{\frac{(E-1)^2 - (1-E^{-1})^2}{(1-E^{-1})(E-1)}} \Rightarrow \cancel{\frac{E^2+1-2E-(1+E^{-2}-2E^{-1})}{E-EE^{-1}-1+E^{-1}}}$

$= \cancel{\frac{E^2-E^{-2}+1-1-2E+2E^{-1}}{E-1-1+E^{-1}}} \Rightarrow$

(3) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

LHS: $\sum_{k=0}^{n-1} \Delta^2 f_k = \sum_{k=0}^{n-1} \Delta[\Delta f(k)]$

$= \sum_{k=0}^{n-1} \Delta(f_{k+1} - f_k)$

$= \sum_{k=0}^{n-1} \Delta f_{k+1} - \Delta f_k$

$= (\Delta f_1 - \Delta f_0) + (\Delta f_2 - \Delta f_1) + (\Delta f_3 - \Delta f_2) + \dots + (\Delta f_n - \Delta f_{n-1})$

$= -\Delta f_0 + \Delta f_n \Rightarrow \Delta f_n - \Delta f_0 =$ RHS

H.P.

$\cancel{\Delta f_k = \Delta f(x_k) = f(x_k) - f(x_{k-1}) = f_k - f(x_{k-1}) = f_k}$

$\Delta f_k = \Delta f(x_k)$
$\Rightarrow f(x_k+h) - f(x_h)$
$\Rightarrow f(x_{k+1}) - f(x_k)$
$\Rightarrow f_{k+1} - f_k$

(4) $\Delta(f_i g_i) = f_i \Delta g_i + g_{i+1} \Delta f_i$

LHS $\Rightarrow \Delta(f_i g_i)$

$= \Delta f(x_i+h) g(x_i+h) - f(x_i) \cdot g(x_i)$
$+ f(x_i) g(x_i+h) - f(x_i) g(x_i+h)$

$= g(x_i+h)\,[f(x_i+h) - f(x_i)] + f(x_i)\,[g(x_i+h)$ divided into

$= g(x_i+h)\,\Delta f(x_i) + f(x_i)\,\Delta g(x_i)$

$\Rightarrow g_{i+1}\,\Delta f_i + f_i\,\Delta g_i$ = RHS H.P.

(5) $\Delta f_i^2 = (f_i + f_{i+1})\,\Delta f_i$

$g_i = f_i$

$\Delta f_i^2 = f_i \cdot \Delta f_i + f_{i+1}\,\Delta f_i$ (from prev. Ques)
$= (f_i + f_{i+1})\,\Delta f_i$

(6) $\Delta\left(\frac{f_i}{g_i}\right) = (g_i\,\Delta f_i - f_i\,\Delta g_i)/g_i \cdot g_{i+1}$

LHS $\Rightarrow$ $\Delta\left(\frac{f_i}{g_i}\right) = \Delta\left(\frac{f\,x_i}{g\,x_i}\right)$ we know, $g_i \cdot \Delta f_i$ $\Rightarrow g_i\,(f_{i+1} - f_i)$

$= \frac{f(x_i+h)}{g(x_i+h)} - \frac{f(x_i)}{g(x_i)}$

$= \frac{g(x_i)\,f(x_i+h) - g(x_i+h)\,f(x_i) + f(x_i)\,g(x_i) - f(x_i)\,g(x_i)}{g(x_i)\,g(x_i+h)}$

$\frac{1}{1+\frac{1}{16}}$

$\frac{1\times 16}{17}$

$= \frac{g(x_i)\,[f(x_i+h) - f(x_i)] - f(x_i)\,[g(x_i+h) - g(x_i)]}{g(x_i)\,g(x_i+h)}$

$=$

$= \frac{g(x_i)\,\Delta f(x_i) - f(x_i)\,\Delta g(x_i)}{g(x_i)\,g(x_i+h)}$

$\Rightarrow \frac{g_i\,\Delta f_i - f_i\,\Delta g_i}{g_i\,(g_{i+1})}$ = RHS

H.P.

7) $\Delta\left(\frac{1}{f_i}\right) = \frac{-\Delta f_i}{f_i \cdot f_{i+1}}$ (See from prev. Ques)

<u>Peice-wise interpolation</u>

The given interval $[a,b]$ into a no. of sub-interval (x_{i-1}, x_i) $i = 1,2 \ldots n$ and approx. fn. by sum lower degree poly. in each sub-interval where $a = x_0 < x_1 < x_2 < \ldots x_n < x_{n+1} = b$.
These sub divided interval $[a,b]$ into no. of long overlapping sub-intervals each containing two or three or four nodal points then we construct corresp. cubic / linear / quadratic interpolation. These poly. define the peice-wise linear / quadratic / cubic interpolation resp.

$$\boxed{\begin{array}{l}\text{Unequal interval} \rightarrow \text{Newton D.D / Lagrange} \\ \text{Equal interval} \rightarrow \text{Newton Back / fwd}\end{array}}$$

$\underbrace{x_0 \quad x_1}_{\text{linear}} \quad \underbrace{\quad x_2}_{\text{linear}} \quad x_3$

$x_0 \le x_1 \le x_2$
$x_1 \le x_2 \le x_3$

$x_0 \le x_1 \le x_2$
$x_2 \le x_3 \le x_4$

$\underbrace{x_0 \quad x_1 \quad x_2}_{\text{Quadratic}} \quad \underbrace{x_3 \quad x_4}_{\text{Quadratic}} \quad x_5$

$x_0 \le x_1 \le x_3$
$x_0 \le x_2 \le x_3$

$\underbrace{x_0 \quad x_1 \quad x_2 \quad x_3}_{\text{cubic}} \quad x_4 \quad x_5$

1.) Obtain peicewise interpolation linear. (Lagrange)

x :	1	2	4	8
$f(x)$:	3	7	21	73

Also find $f(3)$ and $f(7)$

Sol. ~~Now in the interval~~ Now the given data divided into subinterval [1,2] [2,4] & [4,8]

In interval [1,2]

$x_0 = 1 \quad x_1 = 2$

then $P_{1,1}(x) = l_0(x) f_0 + l_1(x) f_1$ $\quad f_0 = 3 \quad f_1 = 7$

where $l_0(x) = x - x_1 / x_0 - x_1 = -x + 2$

$l_1(x) = x - x_0 / x_1 - x_0 = x - 1$

then $P_{1,1}(x) = (-x+2)3 + (x-1)7$

$= -3x + 6 + 7x - 7 = 4x - 1$ where $1 < x < 2$

In interval [2,4] $\quad x_0 = 2 \quad x_1 = 4$

$f_0 = 7 \quad f_1 = 21$

then $P_{2,1}(x) = l_0(x) f_0 + l_1(x) f_1$

$l_0 = \frac{x - x_1}{x_0 - x_1} = \frac{x-4}{2} \qquad l_1 = \frac{x - x_0}{x_1 - x_0} = \left(\frac{x-2}{2}\right)$

$P_{2,1}(x) = \left(\frac{x-4}{2}\right) 7 + \left(\frac{x-2}{2}\right) \times 21 = 7x - 7$

where $2 < x < 4$

In interval [4,8] $\quad x_0 = 4 \quad x_1 = 8$

$f_0 = 21 \quad f_1 = 73$

$P_{3,1}(x) = l_0(x) f_0 + l_1(x) f_1$

$l_0 = \frac{x - x_1}{x_0 - x_1} = \frac{x-8}{4-8} = \frac{x-8}{-4}$

$l_1 = \frac{x - x_0}{x_1 - x_0} = \frac{x-4}{4}$

$P_{3,1}(x) = \frac{x-8}{-4} \times 21 + \frac{x-4}{4} \times 73$

$= 13x - 31 \quad 4 < x < 8$

73
21
52
4
13.

73
21
4
4/5
23.

91
31
60

Piece wise f^n or bracket f^n $P(x) = \begin{cases} 4x-1 & 1 \le x \le 2 \\ 7x-7 & 2 \le x \le 4 \\ 13x-31 & 4 \le x \le 8 \end{cases}$

$f(3) = P(3) = 7*3-7 \quad 3 \in [2,4]$
$= 14$

$f(7) = P(7) = 13(7)-31 \quad 7 \in [4,8]$
$= 60$

Piecewise Quadratic interpolation. (Lagrange)

Q.
x :	-3	-2	-1	1	3	6	7
$f(x)$:	369	222	171	165	207	990	1779

Sol: now the given data is divided into intervals.
$\{-3,-2,-1\}, \{-1,1,3\}, \{3,6,7\}$

In interval $\{\overset{x_0}{-3}, \overset{x_1}{-2}, \overset{x_2}{-1}\}$

$P_{1,2}(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2$

$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x^2+3x+2}{2}$

$\boxed{\begin{matrix} x-x_0 \to \times \text{ for } L_0 \\ x-x_1 \to \times \text{ for } L_1 \\ x-x_2 \to \times \text{ for } L_2 \end{matrix}}$

$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x^2+4x+3}{-1}$

$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x^2+5x+6}{2}$

$P_{1,2}(x) = \left(\frac{x^2+4x+3}{2}\right)369 + \left(\frac{x^2+4x+3}{-1}\right)222 + \frac{x^2+5x+6}{2}(171)$

=

for verification

$L_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$

to (x_0, x_1, x_2) check krne k liye first check f_{x_0} then further.

$$P_{1,2}(x) = \begin{cases} 48x^2+93x+216 & -3 \le x \le -1 \\ 6x^2-3x+162 & -1 \le x \le 3 \\ 132x^2-927x+1800 & 3 \le x \le 7 \end{cases}$$

$f(-1) \Rightarrow$ can be found from first both intervals since $-1 \in (-3,-1)$ as well as $(-1,3)$.

Piecewise cubic interpolation. (N.D.D)

1.

x :	-3	-2	-1	1	3	6	7
$f(x)$:	369	222	171	165	207	990	1779.

Sol) Divide in given data in two intervals.
$\{-3,-2,-1,1\}$ & $\{1,3,6,7\}$

In first interval $\{\overset{x_0}{-3}, \overset{x_1}{-2}, \overset{x_2}{-1}, \overset{x_3}{1}\}$
using N.D.D interpolation we get

$$P(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

Newton D.D. table

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} \qquad f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

x	$f(x)$	$f(x_0,x_1)$	$f(x_0,x_1,x_2)$	$f(x_0,x_1,x_2,x_3)$
-3	369			
		$\Rightarrow \frac{222-369}{-2+3} = -147$		
-2	222		48	
		-51		-8. $\quad \frac{16-48}{1+3}$
-1	171		16 $= \frac{-3+51}{1+2}$.	
		-3		
1	165			
~~3~~	~~207~~			

$P(x) = 369 + (x+3)(-147) + (x+3)(x+2)48 + (x+3)(x+2)(x+1)(-8)$
$= -8x^3 + 5x + 168.$

$$P(x) = \begin{cases} P_{1,3} = -8x^3 + 5x + 168 & -3 \le x \le 1 \\ P_{2,3} = 14x^3 - 92x^2 + 207x + 36 & 1 \le x \le 7 \end{cases}$$

Ordinary diff. eq. (ODE)

100%

1.) Euler Method

$\frac{du}{dt} = f(t,u)$

$\boxed{u(x_0) = u_0}$

then $u(b) = ?$

u is f^n of $t = f(t)$

$\frac{du}{dt} = f(t,u)$ ← dependent var. ; t indep.

let $[x_0, b]$ be divided into sub interval such that

$x_0 = x_0 < x_1 < x_2 \ldots < x_n = b.$

$h = \frac{b - x_0}{n}$

where n means no. of steps in which Ques. will be completed. n is assumed.

$x_0 = x_0$
$u_1 \leftarrow x_1 = x_0 + h$
$u_2 \leftarrow x_2 = x_0 + 2h$
$u_n \leftarrow x_n = x_0 + nh = b.$
$u(b) = ?$

By Euler method :-

$u_{j+1} = u_j + h f(t_j, u_j)$

$j = 0, 1, 2 \ldots n-1$

$u_1 = u_0 + h f(t_0, u_0)$

$u_2 = u_0 + h f(t_1, u_1)$

$u_3 = u_0 + h f(t_2, u_2)$

Ques. has be completed within 2 steps or max 4 steps.

Q. $\frac{du}{dt} = u' = -2tu^2$

$u(0) = 1$ $\Rightarrow t_0 = 0, u_0 = 1$

Find $u(\underline{0.4}) = ?$
$= b$

Sol $\Rightarrow$ $(t_0, b) \sim (0, 0.4)$

$h = \frac{b - x_0}{n} = \frac{0.4 - 0}{\boxed{2}} \Rightarrow h = 0.2$

assumed if $n = 2$, $h = 0.2$

if $n = 4$, $h = 0.1$

if $n = 8$, $h = 0.05$

$u_0 \leftarrow t_0 = 0$

$u_1 \leftarrow t_1 = t_0 + h$

$= 0 + 0.2 = 0.2$

$u_2 \leftarrow t_2 = t_0 + 2h = \underline{0.4}$

By Euler method :-

$u_{j+1} = u_j + h f(t_j, u_j)$ for $j = 0, 1, 2, \ldots n-1$

for $h = 0.2$

if $j = 0$, $u_1 = u_0 + h f(t_0, u_0)$

$f(t, u) = -2tu^2$

$= 1 + 0.2 (-2t_0 u_0^2)$

$f(t_0, u_0) = -2t_0 u_0^2$

$= 1 + 0.2 (-2(0)(1)^2)$

$u_1 = 1$

for $j = 1$, $u_2 = u_1 + h f(t_1, u_1)$

$u(t_1) = u(0.2) = 1$

$= 1 + 0.2 (-2t_1 u_1^2)$

$= 1 + 0.2 (-2(0.2) \times (1)^2)$

$= 1 - 0.08 = 0.92$

$u_2 = u(t_2) = u(0.4) = \underline{\underline{0.92}}$

Q. (1) $y' = t + y$

$y(1) = 0 \Rightarrow \boxed{t_0 = 1, y_0 = 0}$

compute $y(\underset{b}{1.2})$ by Euler Method.

Ans. $[1, 1.2]$

$h = \frac{0.2}{2} = 0.1$

$t_0 = 1$

$t_1 = 1.1$

$t_2 = 1.2$

(2) $y' = -y^2$

$y(1) = 1 \Rightarrow t_0 = 1, y_0 = 1$

Compute $y(\underset{b}{1.2})$ by Euler Method.

Sol. (1). $[1, 1.2] \sim (t_0, y_0)$

$h = \frac{b - t_0}{n} = \frac{(1.2 - 1)}{2} = \underline{0.1}$

.819

$$\begin{array}{r} 0.900 \\ .081 \\ \hline 0.819 \end{array}$$

classmate
Date
Page

$t_n = t_0 + nh$

$y_0 \leftarrow t_0 = t_0 = 1$

$y_1 \leftarrow t_1 = t_0 + h = 1.1$

$y_2 \leftarrow t_2 = t_0 + 2h = \underline{1.2}$

By Euler method.

$y_{j+1} = y_j + h f(t_j, y_j)$ $f(t,y) = t+y$

for $j=0$, $y_1 = y_0 + 0.1\,(1+0)$ $f(t_0,y_0) = t_0 + y_0$

$= 0 + 0.1 = \underline{0.1}$

for $j=1$, $y_2 = y_1 + 0.1\,(1.1 + 0.1)$ $f(t_1,y_1) = t_1 + y_1$

$= 0.1 + 0.1\,(1.2)$

$= 0.1 + 0.12$

$= \underline{0.22}$

$y_2 = y(1.2) = \underline{0.22}$

② $(t_0, y_0) \sim (1, 1.2)$ $t_0 = 1,\ y_0 = 1$

$h = \dfrac{b - t_0}{n} = \dfrac{1.2 - 1}{2} = \underline{0.1}$

$t_n = t_0 + nh$

$y_0 \leftarrow t_0 = t_0 = 1$

$y_1 \leftarrow t_1 = t_0 + h = 1.1$

$y_2 \leftarrow t_2 = t_0 + 2h = 1.2$

By Euler method

$y_{j+1} = y_j + h f(t_j, y_j)$ $f(t,y) = -y^2$

for $j=0$. $y_1 = y_0 + 0.1\,(-y_0)^2$ $f(t_0, y_0) = -y_0^2$

$= 1 + 0.1\,(-1)^2$

$= 1 - 0.1 = \underline{0.9}$

$y_2 = y_1 + 0.1\,(-)(y_1)^2$

$= 0.9 + 0.1(-)(0.9)^2$

$= 0.9 + (0.1)(-)(0.81)$

$= 0.9 - 0.081 = \underline{0.819}$

$0.16+1$

Variable Separation.

$\frac{du}{dt} = -2tu^2$ $u(0)=1$, $u(0.4)=?$.

$\int \frac{du}{u^2} = \int -2t\,dt$ (By method of var. separation)

$\Rightarrow \frac{-1}{u} = -\frac{2t^2}{2} + c.$

$\Rightarrow -\frac{1}{u} = -t^2 + c$

Given $u(0)=1$ i.e., $t=0$, $u=1$

$-\frac{1}{1} = -(0)^2 + c \Rightarrow \underline{c=-1}.$

$\Rightarrow \frac{-1}{u} = -t^2 - 1 \Rightarrow \frac{1}{u} = t^2+1 \Rightarrow u = \frac{1}{t^2+1}$

$u(0.4) = \frac{1}{(0.4)^2+1} = \frac{1}{1.16}.$

$\frac{dy}{dt} = t+y$ $y(1)=0 \Rightarrow y=0, t=1$

$\frac{dy}{dt} - y = t$ (By integrating factor Method.)

$\cancel{\frac{dy}{dt}}$ If $= \boxed{e^{\int -1\,dt}} = e^{-t}$, $e^{-\int P\,dt}$

$y \times \text{If.} = \int Q \times \text{If.}\ dt + c$

$y \times e^{-t} = \int t e^{-t}\,dt + c$

$y e^{-t} = e^{-t}(-t-1) + c$

$\boxed{y = -t-1 + ce^{t}}$

~~y(1.2) = ...~~

$0 = -1-1 + ce^{1}$

$ce^{1} = 0.$

$c = e^{-1}(0)$

$\underline{c=0}$

$y = t-1$

$y(1.2) = 1.2 - 1 = \underline{0.2}$

Runge - Kutta II^{nd} order Method.

R-K II^{nd} order method.

$$\frac{du}{dt} = f(t, u)$$

$u(t_0) = u_0$

then find $u(b) = ?$

Now, interval (t_0, b) divided into sub-interval

$t_0 < t_1 < t_2 \ldots t_n = b$

$h = \frac{b - t_0}{n}$

Now,

$u_0 \leftarrow t_0 = t_0$

$u_1 \leftarrow t_1 = t_0 + h$

$u_2 \leftarrow t_2 = t_0 + 2h$

$u_n \leftarrow t_n = t_0 + nh = b$

$$\boxed{u_n = u(t_n) = u(b)}$$

then by R-K II^{nd} order —o.

$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$

where $K_1 = hf(t_j, u_j)$

$K_2 = hf(t_j + h, u_j + K_1)$

for $j = 0, 1, 2, 3$

1.) $\frac{du}{dt} = u' = -2tu^2$

$u(0) = 1$

Find $u(0.4)$ by R-K II^{nd} order method.

Sol. Now, $(0, 0.4)$ into interval $t_0 < t_1 < t_2 < \ldots t_n = b$.

$h = \frac{b - t_0}{n} = \frac{0.4 - 0}{2} = 0.2$

$u_0 \leftarrow t_0 = t_0 = 0$

$u_1 \leftarrow t_1 = t_0 + h = 0.2$

0.2
-2×0.2
-0.08

$u_2 = t_2 = t_0 + h2 = 0.4$

by R-K IInd order Method —

$$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$$

$$K_1 = hf(t_j, u_j)$$
$$K_2 = hf(t_j + h, u_j + K_1) \qquad j = 0, 1, 2, 3 \ldots$$

if $j = 0$, then

$$u_1 = u_0 + \frac{1}{2}(K_1 + K_2) \qquad f(t,u) = -2tu^2$$
$$K_1 = 0.2\, f(t_0, u_0) \qquad f(t_0, u_0) = -2t_0 u_0^2$$
$$= 0.2\,(-2(0)(1)) \qquad = -2(0)(1)$$
$$= 0$$

$$K_2 = hf(t_0 + h, u_0 + K_1)$$
$$= 0.2\, f(0.2, 1+0)$$
$$= 0.2\, f(0.2, 1)$$
$$= 0.2\,[-2(0.2)(1)] = \underline{-0.08}$$

$$u_1 = 1 + \frac{1}{2}(0 - 0.08) = 1 - 0.04 = \underline{0.96}$$

$$u_1 = u(0.2) = \underline{\underline{0.96}}$$

if $j = 1$ then

$$u_2 = u_1 + \frac{1}{2}(K_1 + K_2)$$

$$K_1 = hf(t_1, u_1)$$
$$= 0.2\,(-2(0.2)(0.96)^2)$$
$$= -0.073728$$

$$K_2 = hf(t_1 + h, u_1 + K_1)$$
$$= hf(0.2 + 0.2, 0.96 - 0.0737)$$
$$= 0.2\, f(0.4, 0.886272)$$
$$= 0.2\,(-2(0.4) \cdot (0.886272)^2$$
$$= -0.102262.$$

$u_2 = 0.96 + \frac{1}{2}(-0.073728 - 0.162262)$

$u(0.4) = u_2 = 0.857738$ Ans.

Range - Kutta IVth order Method.

$$\frac{du}{dt} = f(t,u)$$

$u(t_0) = u_0$

$u(b) = ?$

$h = \frac{b - t_0}{n}$

$u_0 \leftarrow t_0 = t_0$

$u_1 \leftarrow t_1 = t_0 + h$

$u_2 \leftarrow t_2 = t_0 + 2h$

$u_n \leftarrow t_n = t_0 + nh$

By R-K IVth order :-

$$u_{j+1} = u_j + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$K_1 = hf(t_j, u_j)$

$K_2 = h\, f\left(t_j + \frac{h}{2}, u_j + \frac{K_1}{2}\right)$ → coeff (2)

[yaad krna ka method]

$K_3 = hf\left(t_j + \frac{h}{2}, u_j + \frac{K_2}{2}\right)$ → coeff (2)

$K_4 = hf(t_j + h, u_j + K_3)$

$u' = -2tu^2$

$u(0) = 1$

with $h = 0.2$ on interval $(0, 0.4)$. use R-K IVth order compare with exact solution/value.

$u(0.4) = ?$, $b = 0.4$

$u_0 \leftarrow t_0 = 0$

$u_1 \leftarrow t_1 = t_0 + h = 0.2$

$u_2 \leftarrow t_2 = t_0 + 2h = 0.4$

$0.2 \times -2 \times 0.1$

0.04

$$u_{j+1} = u_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

if $j = 0$, $u_1 = u_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = h f(t_0, u_0)$$
$$= 0.2(-2t_0 u_0^2)$$
$$= 0.2(-2 \times 0 \times 1^2)$$
$$= 0$$

$$k_2 = h f\left(t_j + \frac{h}{2}, u_j + \frac{k_1}{2}\right)$$
$$= 0.2 f\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right)$$
$$= 0.2 f(0 + 0.1, 1 + 0) = 0.2 f(0.1, 1)$$
$$= 0.2[-2(0.1)(1)^2] = -0.04$$

$$k_3 = h f\left(t_j + \frac{h}{2}, u_j + \frac{k_2}{2}\right)$$
$$= 0.2 f\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right)$$
$$= 0.2 f(0 + 0.1, 1 + 0.02)$$
$$= 0.2 f(0.1, 0.98) = 0.2[-2(0.1)(0.98)^2]$$
$$= -0.038416$$

$$k_4 = h f(t_j + h, u_j + k_3)$$
$$= h f(t_0 + h, u_0 + k_3) = h f(0 + 0.2, 1 + 0.038416)$$
$$= 0.2 f(0.2, 0.961584)$$
$$= 0.2[-2(0.2)(0.96584)^2] = -0.0739715$$

$$u_1 = u_0 + \frac{1}{6}(0 - 0.08 - 2(0.038416) - 0.0739)$$

$$u(0.2) = u_1 = 0.9615328$$

* Find u_2 in exam or just leave it. It will cost 1.5MM so just leave it.

Some Ques,

① $\frac{dy}{dx} = \sqrt{x+y}$

$y(0.4) = 0.41$

$h = 0.4$, $y(0.8) = ?$

Find by RK IV order

② $\frac{du}{dt} = \frac{t}{u}$ $u(0) = 1$

$h = 0.2$, $u(0.2) = ?$

Find by R-K IV order.

Mid point method

$\frac{du}{dt} = f(t,u)$,

dependent var. always $\leftarrow u(t_0) = u_0 \rightarrow$ independent var. , then find $u(b) = ?$

Now $[t_0, b]$ divide into sub interval

$t_0 = t_0 < t_1 < t_2 \ldots < t_n = b$

i.e. $h = \frac{t_n - t_0}{n} = \frac{b - t_0}{n}$

then

$t_0 = t_0$

$t_1 = t_0 + h$

$t_2 = t_0 + 2h$

$t_n = t_0 + nh$

then by mid-pt method :-

$u_{j+1} = u_{j-1} + 2h f(t_j, u_j)$ for $j = 1, 2, \ldots, n-1$

$u_2 = u_0 + 2h f(t_1, u_1)$

$u_3 = u_1 + 2h f(t_2, u_2)$

$\vdots$

Now, by taylor series :-

$u_1 \approx u(h) = u(0) + h u'(0) + \frac{h^2}{2} u''(0)$

Q. $\frac{du}{dt} = -2tu^2$

$u(0) = 1$, $u(1) = ?$ for $h = 0.2$

Sol:
$t_0 = 0 \rightarrow u_0$
$t_1 = t_0 + h = 0.2 \rightarrow u_1$
$t_2 = t_0 + 2h = 0.4 \rightarrow u_2$
$t_3 = t_0 + 3h = 0.6 \rightarrow u_3$
$t_4 = t_0 + 4h = 0.8 \rightarrow u_4$
$t_5 = t_0 + 5h = 1.0 \rightarrow u_5$

$u_5 = u(t_5) = u(1.0) = ?$ (to be found)

~~Here~~ Now by taylor series :-

$u_1 = u_0 + hu'(0) + \frac{h^2}{2} u''(0)$
$= 1 + h \times 0 + \frac{h^2}{2}(-2)$
$= 1 + \frac{0.2 \times 0.2}{\cancel{2}} \times -\cancel{2}$
$= 1 - 0.04 \Rightarrow \underline{0.96}$

$u_1 = u(t_1) = u(0.2) = \underline{0.96}$

$f(t, u) = -2tu^2$
$f(t_j, y_j) = -2t_j u_j^2$
$t_0 = 0, u_0 = 1$
$u' = -2tu^2$
$u'(0) = 0.$ [put $t = 0$]
$u'' = -4tu - 2u^2$
$u''(0) = -4(0) \times (1) - 2(1)^2$
$= \underline{-2}$

Now by mid pt. method :-

$u_{j+1} = u_{j-1} + 2hf(t_j, u_j)$ for $j = 1, 2 \ldots n-1$

$j=1$ $\quad u_2 = u_0 + 2hf(t_1, u_1)$
$= 1 + 2 \times 0.2[-2(0.2)(0.96)^2]$
$= 1 - 0.16 + (0.96)^2$

$f(t, u) = -2tu^2$
$f(t_1, u_1) = -2t_1 u_1^2$

$u(0.4) = u(t_2) = u_2 = \underline{0.852544}$

$j=2$ $\quad u_3 = u_1 + 2hf(t_2, u_2)$
$= 0.96 + 2 \times 0.2(-2(0.4)(0.8525)^2]^2$
$= 0.7274$

$j=3.$ $\quad u_4 = u_2 + 2hf(t_3, u_3)$
$= 0.8525 + 2 \times 0.2[-2(0.6)(0.7274)^2]$

$= 0.5985$

$j=4$ $\quad u_5 = u_3 + 2h\, f(t_4, u_4)$

$= 0.7274 + 2 \times 0.2 \left[(-2(0.8)(0.5985)^2\right]$

$= 0.49811$

$\boxed{u_5 = u(t_5) = u(1.0) = 0.49811}$ Ans.

Q. $y' = t+y$

$y(1) = 0$ $\quad$ find $y(1.2) = ?$ $\quad h = 0.1$

$y_0 \leftarrow t_0 = 1$

$y_1 \leftarrow t_1 = t_0 + h = 1.1$

$y_2 \leftarrow t_2 = t_0 + 2h = 1.2$

$f(t,y) = t+y$

$y' = t+y$

By taylor series method.

$y_1 = y(t_0) + h\, y'(t_0) + \frac{h^2}{2} y''(t_0)$

$\therefore \left[y(1+0.1) = y(1.1) = y(1) + h\, y'(1) + \frac{h^2}{2} y''(1) \right]$

$y'(1) = t + y(1)$

$= 1+0 = 1$

$y'' = y' + 1$

$y''(1) = y'(1) + 1$

$= 2$

$y_1 = 0.1 + 0.01 = 0.11$

$y_1 = y(t_1) = y(1.1) = 0.11$

By mid pt method —

$y_{j+1} = y_{j-1} + 2h f(t_j, y_j)$ for $j = 1, 2, \ldots n-1$

$j=1$ $\quad y_2 = y_0 + 2h f(t_1, y_1)$

$= 0 + 2 \times 0.1 [t_1 + y_1]$

$= 0 + 0.2 [1.1 + 0.11] = 0.242$

$\boxed{y(1.2) = y(t_2) = y_2 = 0.242}$ Ans

CE Maths (conti.)

Q. $y' = -y^2$

$y(1) = 1$, $h = 0.1$, find $y(1.2) = ?$

[Ans → $y_1 = 0.91$, $y_2 = 0.8344$]

Q. $y' = y$

$y(0) = 1$, $y(0.3) = ?$

[Ans → $h = 0.3/3 = 0.1$

Imp

Thomas Method of Tridiagonal Sys.

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \vdots \\ 0 & 0 & 0 & a_{n-1} & c_{n-1} \\ 0 & 0 & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}$$

then $\alpha_1 = c_1/b_1$

$\gamma_1 = r_1/b_1$

$$\alpha_j = \frac{c_j}{b_j - a_j \alpha_{j-1}} \qquad j = 2, 3, 4, \ldots n$$

$$\gamma_j = \frac{r_j - a_j \gamma_{j-1}}{b_j - a_j \alpha_{j-1}} \qquad j = 2, 3, 4 \ldots n$$

with $c_n = 0$.

then the sol. of tridiagonal sys. is $\boxed{x_n = \gamma_n}$

$x_j = \gamma_j - \alpha_j x_{j+1}$ $j = n-1, n-2 \ldots 3, 2, 1$

2017 Q

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 17 \\ 22 \end{bmatrix}$$

Solve by Thomas Method of Triadiagonal sys.

Sol: Compare with –

$$\begin{bmatrix} b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

then by thomas method –:

$$\alpha_1 = \frac{c_1}{b_1} = \frac{2}{1} = 2.$$

$$\gamma_1 = \frac{r_1}{b_1} = \frac{10}{1} = 10$$

$$\alpha_2 = \frac{c_2}{b_2 - a_2\alpha_1} = \frac{3}{3 - 1\times 2} \Rightarrow \frac{3}{1} = \underline{3}.$$

$$\alpha_3 = \frac{c_3}{b_3 - a_3\alpha_2} = \frac{0}{10 - 3\times 3} = \frac{0}{1} = \underline{0}$$

$$\gamma_2 = \frac{r_2 - a_2\gamma_1}{b_2 - a_2\alpha_1} \Rightarrow \frac{17 - 1\times 10}{3 - 1\times 2} \Rightarrow \frac{7}{1} = \underline{7}$$

$$\gamma_3 = \frac{r_3 - a_3\gamma_2}{b_3 - a_3\alpha_2} \Rightarrow \frac{22 - 3\times 7}{10 - 3\times 3} \Rightarrow \underline{1}.$$

then by sol. of triadiagonal sys. $\underline{n=3}$

$x_n = \gamma_n$

$x_3 = \gamma_3 = 1 \Rightarrow \boxed{x_3 = 1}$

$x_j = \gamma_j - \alpha_j x_{j+1}$, $j = n-1, n-2 \ldots 3, 2, 1$

here $j = 2, 1$

$$x_2 = \gamma_2 - \alpha_2 x_3$$
$$= 7 - 3(1) = \underline{4}$$

$$x_1 = \gamma_1 - \alpha_1 x_2$$

$x_1 = 10 - 2(4) = 2$

Ans $\Rightarrow$ $(2, 4, 1)$

Q. $\begin{bmatrix} 1 & 2 & 0 \\ 3 & -4 & 5 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -9 \end{bmatrix}$

Solve by thomas Method.

Sol: Compare with eq.

$$\begin{bmatrix} b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\alpha_1 = \frac{c_1}{b_1} \Rightarrow \frac{2}{1} = 2.$$

$$\gamma_1 = \frac{r_1}{b_1} = \frac{5}{1} = 5$$

$$\alpha_2 = \frac{c_2}{b_2 - a_2\alpha_1} = \frac{5}{-4-3(2)} = \frac{5}{-4-6} \Rightarrow \frac{5}{-10} = \frac{-1}{2}.$$

$$\alpha_3 = \frac{c_3}{b_3 - a_3\alpha_2} = \frac{0}{-7-6(\frac{1}{2})} = \frac{0}{-7+3} \Rightarrow 0$$

$$\gamma_2 = \frac{r_2 - a_2\gamma_1}{b_2 - a_2\alpha_1} = \frac{10-3\times(5)}{-4-3(2)} \Rightarrow \frac{-5}{-10} \Rightarrow \frac{1}{2}.$$

$$\gamma_3 = \frac{r_3 - a_3\gamma_2}{b_3 - a_3\alpha_2} = \frac{-9-6\times 1/2}{-7-6\times -1/2} \Rightarrow \frac{-12}{-4} = 3.$$

then by tridiagonal sys. $\boxed{n=3}$

$x_n = \gamma_n$

$x_3 = \gamma_3 = 3.$

$x_j = \gamma_j - \alpha_j x_{j+1}$ for $j = n-1, n-2, \ldots 3, 2, 1$

$x_2 = \gamma_2 - \alpha_2 x_3$

$= \frac{1}{2} + \frac{1}{2} \times 3 \Rightarrow \underline{2}$

$x_1 = \gamma_1 - \alpha_1 x_2$

$= 5 - (2)(2) = \underline{1}$

Ans → (1, 2, 3)

Finite Difference Method (for IInd order ODE)

$\frac{d^2u}{dx^2} = u'' = f(x, u)$

$u(x_0) = u_0$

$u(x_n) = u_n$

Now interval $[x_0, x_n]$ divided into n subinterval.

$h = \frac{x_n - x_0}{n}$ ①

then by finite diff. method -

$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f(x_j, u_j)$

$j = 1, 2, 3 \ldots n-1$ ②

Q. $u'' = u + x$ ③

$u(0) = 0$

$u(1) = 0$

with $h = \frac{1}{4}$ find by finite Diff. Method

Sol: $h = \frac{x_n - x_0}{n}$ $f(x_j, u_j) = x_j + u_j$

$u(x_0) = u_0$ $u(x_n) = u(n)$

$u(0) = 0$

$h = \frac{x_n - x_0}{n} = \frac{1-0}{n} = \frac{1}{n}$ $\boxed{\text{if } n = 4}$

then $h = 1/4$

by Finite Diff. method :-

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f(x_j, u_j) \qquad j = 1,2,3 \ldots n-1$$

$x_0 = 0$

$x_1 = x_0 + h = 0.25$

$x_2 = x_0 + 2h = 0.50$

$x_3 = x_0 + 3h = 0.75$

$\Rightarrow \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = x_j + u_j \qquad j = 1,2,3$ [as $n = 4$]

$\Rightarrow (u_{j-1} - 2u_j + u_{j+1}) = h^2 (x_j + u_j)$

$\Rightarrow u_{j-1} - 2u_j + u_{j+1} = \frac{1}{16}(x_j + u_j)$

$\rightarrow 16u_{j-1} - 32u_j + 16u_{j+1} = x_j + u_j$

$\rightarrow \boxed{16u_{j-1} - 33u_j + 16u_{j+1} = x_j} \qquad j = 1,2,3$

① for $j = 1$.

$16u_0 - 33u_1 + 16u_2 = x_1$ Given. $u_0 = 0$

$-33u_1 + 16u_2 = 0.25$

② for $j = 2$

$16u_1 - 33u_2 + 16u_3 = 0.50$

③ for $j = 3$.

$16u_2 - 33u_3 + 16u_4 = x_3 = 0.75$ Given. $u_4 = 0$

$\Rightarrow 16u_2 - 33u_3 = 0.75$

All 3 equations :

① $-33u_1 + 16u_2 = 0.25$

② $16u_1 - 33u_2 + 16u_3 = 0.50$

③ $16u_2 - 33u_3 = 0.75$

$-33 + 16 \times \frac{16}{33}$ $\quad -33 + \frac{256}{33}$.

$$\begin{bmatrix} -33 & 16 & 0 \\ 16 & -33 & 16 \\ 0 & 16 & -33 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.75 \end{bmatrix}$$

Solve the sys. of eq. by thomas method - tridiagonal
Can also use Gauss-Elimination Method.

Sol. Ans $\Rightarrow$ $u_1 = -0.034885$ — $u(0.25)$
$u_2 = -0.056326$ — $u(0.50)$
$u_3 = -0.050037$ — $u(0.75)$

③

④

Q. $u'' = ux$
interval $= 0,1$
$u(0) + u'(0) = 1$
$u(1) = 1$, $h = 1/3$. Solve by finite Diff.

Sol. Now the interval $(0,1)$ divided into subinterval with $h = 1/3$.

$u_0 \leftarrow x_0 = 0$
$u_1 \leftarrow x_1 = x_0 + h = 1/3$
$u_2 \leftarrow x_2 = x_0 + 2h = 2/3$
$u_3 \leftarrow x_3 = x_0 + 3h = 1$

$$\boxed{\begin{aligned} f'(x_0) &= [f(x_0+h) - f(x_0-h)]/2 \\ f'(x_0) &= [f_1 - f_{-1}]/2h \\ f'(0) &= [f_1 - f_{-1}]/2h \end{aligned}}$$

by finite Diff. method —

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} = f(x_j, u_j) \quad j = 0,1,2,3 \ldots n-1.$$

$$9u_{j-1} - 18u_j + 9u_{j+1} = x_j u_j$$

① for $j = 0$

$$9u_{-1} - 18u_0 + 9u_1 = x_0 u_0$$

$x_0 = 0$

$$9u_{-1} - 18u_0 + 9u_1 = 0$$

Since

$$\rightarrow u_{-1} - 2u_0 + u_1 = 0 \quad \text{— ①}$$

② for $j = 1$

$$9u_0 - 18u_1 + 9u_2 = x_1 u_1$$

$\Rightarrow 9u_0 - 18u_1 + 9u_2 = \frac{1}{3}u_1$

$\rightarrow (u_0 - 2u_1 + u_2) = u_1/27$ — (2)

(3) $j=2$:

$9u_1 - 18u_2 + 9u_3 = x_2 u_2$

$9(u_1 - 2u_2 + u_3) = 2/3\, u_2$

$\Rightarrow u_1 - 2u_2 + u_3 = 2/27\, u_2$ — (3)

(4) $j=3$

not relevant.

$9u_2 - 18u_3 + 9\boxed{u_4} = x_3 u_3$

we know that, $u'(0) = \frac{u(x_0+h) - u(x_0-h)}{2h}$

$$\boxed{u'_0 = \frac{u_1 - u_{-1}}{2h}}$$ Remember

Given: $u(0) + u'(0) = 1$

$u_0 + u'_0 = 1$

$u_{(1)} = 1$

$u(x_3) = 1$

$u_3 = 1$

$u_0 + \frac{u_1 - u_{-1}}{2h} = 1$

$u_0 + \frac{3}{2}(u_1 - u_{-1}) = 1$

$u_0 + 3/2\, u_1 - 1 = \frac{3}{2}(u_{-1})$

$u_{-1} = 2/3\,(u_0 + 3/2\, u_1 - 1)$ — (4)

Put u_{-1} in eq. (1) - (0).

$\rightarrow 2/3\,(u_0 + 3/2\, u_1 - 1) - 2u_0 + u_1 = 0$

$\rightarrow -4/3\, u_0 + 2u_1 = 2/3$

$\rightarrow \frac{2}{3}(-2u_0 + 3u_1) = \frac{2}{3} = 1$

$\rightarrow -2u_0 + 3u_1 = 1$ — (5)

$u_0 - 2u_1 + u_2 = u_1/27$ — (2)
$u_0 - 55/27\, u_1 + u_2 = 0$ — (6)

$u_1 - 2u_2 + u_3 = 2/27\, u_2$ — (3)   Given $u_3 = 1$
$u_1 - 56/27\, u_2 = 1$ — (7)

$-2u_0 + 3u_1 = 1$
$u_0 - 55/27\, u_1 + u_2 = 0$
$u_1 - 56/27\, u_2 = -1$

$$\begin{bmatrix} -2 & 3 & 0 \\ 1 & -55/27 & 1 \\ 0 & \cancel{-56/27}\ 1 & -56/27 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Use Gauss-Jordan / Elimination method.

Numerical Differentiation

Linear Interpolation. $[a_0 + a_1 x]$

Langrange (linear) $(n=1)$ $[x_0, x_1]$

$P_1 x = L_0(x) f(x_0) + L_1(x) f(x_1)$

$f(x) = P_1 x = \left(\dfrac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\dfrac{x - x_0}{x_1 - x_0}\right) f(x_1)$

$$f'(x) = P_1'(x) = \frac{1}{x_0 - x_1} f(x_0) + \frac{1}{x_1 - x_0} f(x_1)$$

$$\boxed{f'(x) = P_1'(x) = \frac{-f_0}{x_1 - x_0} + \frac{f_1}{x_1 - x_0} \Rightarrow \frac{f_1 - f_0}{x_1 - x_0}}$$

remember when two points given & slope was found by $\boxed{\dfrac{y_1 - y_0}{x_1 - x_0}}$

Quadratic interpolation (x_0, x_1, x_2) $(n=2)$

$P_2(x) = L_0(x) f(x_0) + L_1(x) f_1 + L_2(x) f_2$

$$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \qquad l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$f'(x) = P_2'(x) = l_0'(x) f_0 + l_1'(x) f_1 + l_2'(x) f_2$$

$$l_0'(x) = \frac{(x-x_2)(1)}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_1)(1)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)} \qquad l_1'(x) = \frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}$$

$$l_2'(x) = \frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}$$

$$\boxed{P_2'(x) = \frac{(2x-x_1-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(2x-x_0-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(2x-x_0-x_1)}{(x_2-x_0)(x_2-x_1)} f_2}$$

Again diff. wrt x

$$\boxed{P_2''(x) = f''(x) = \frac{2f_0}{(x_0-x_1)(x_0-x_2)} + \frac{2f_1}{(x_1-x_0)(x_1-x_2)} + \frac{2f_2}{(x_2-x_0)(x_2-x_1)}}$$

$$P_2(x) = f(x) = a_0 + a_1 x + a_2 x^2$$

$$f'(x) = P_2'(x) = a_1 + 2a_2 x$$

$$f''(x) = P_2''(x) = 2a_2$$

$$f'''(x) = 0$$

Q. $f(x) = \log x$

x	x_0 2.0	x_1 2.2	x_2 2.6
$f(x)$	0.6931S	0.78846	0.95551

① $f'(2.0)$ using linear interpolation

② $f'(2.0)$ using quadratic "

③ $f''(2.0)$ using quadratic interpolation.
④ Compare with exact value.

Sol. ① we know by linear interpolation

$$f'(x_0) = \frac{f_1 - f_0}{x_1 - x_0} = \frac{0.788 - 0.6931}{2.2 - 2.0} = 0.4765$$

$f'(2.0) \in (2.0, 2.2) = 0.47655$ Ans

* (if $f'(2.5) \in (2.2, 2.6)$)

② Quadratic interpolation - First Degree

$$f'(x_0) = \frac{(2x - x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(2x - x_0 - x_2)}{(x_1 - x_0)(x_1 - x_2)} f_2$$

$$+ \frac{(2x - x_0 - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_3$$

$$= \frac{2\times 2 - 2.2 - 2.6}{(2-2.2)(2-2.6)}(0.6931) + \frac{2\times 2 - 2 - 2.6}{(2.2-2)(2.2-2.6)}(0.7884)$$

$$+ \frac{2\times 2 - 2 - 2.2}{(2.6-2)(2.6-2.2)} 0.9555$$

$= \underline{\underline{0.49619}}$

③ Quadratic interpolation - Second degree

$$f''(x_0) = \frac{2f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{2f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{2f_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{2\times 0.69315}{(2-2.2)(2-2.6)} + \frac{2\times 0.78846}{(2.2-2)(2.2-2.6)} + \frac{2\times 0.9555}{(2.6-2)(2.6-2.2)}$$

$= \underline{\underline{-0.19642}}$

④ Exact value

$f(x) = \log x$

$f'(x) = 1/x$

$f'(2) = 1/2 = 0.5$

$f''(x) = -1/x^2$

$f''(2) = -1/4 = -0.25$

Q.2) $f(x) = x^4$

x:	x_0 0.4	x_1 0.6	x_2 0.8
$f(x)$:	0.0256	0.1296	0.4096

$0.2 = h$

① $f'(0.8)$
② $f''(0.8)$] quadratic interpolation.
③ compare with exact values of both above things.

Sol: $h = 0.2$

$x_0 = 0.4$

$x_1 = x_0 + h = 0.6$

$x_2 = x_0 + 2h = 0.8$

Ans ⇒ ① 1.84
② 4.4
③ 2.048, 7.68

$$f'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f_0 + \frac{(\quad)}{h(-h)} + \frac{(\quad)}{(2h)(h)}$$

easing the dues
$$= \frac{2x - x_1 - x_2}{(-h)(-2h)} f_0 + \frac{(\quad)}{h(-h)} + \frac{(\quad)}{(2h)(h)}$$

Solve urself later.

Newton interpolation.

Linear ($n=1$) $f(x) = P_1(x) = f(x_0) + (x - x_0) f(x_0, x_1)$

$f(x)$ $P_1(x) = f(x_0) + (x - x_0)\left(\frac{f_1 - f_0}{x_1 - x_0}\right)$

Diff. wrt x -°.

$$f'(x) = 0 + 1\left(\frac{f_1 - f_0}{x_1 - x_0}\right)$$

for Quadratic, langrange is more Suitable
Use any Newton / langrange Ur choice
for both linear / Quadratic
Use langrange only

Q. $u'' = u - 4xe^{x}$

$u(0) - u'(0) = -1$

$u(1) + u'(1) = -e$

Solve by finite Diff. Method $\quad 0<x<1$

Q. $f(x) = \sin x$

$x_0 = \pi/4 \qquad h = \pi/12$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

(i) $f'(\pi/4)$ } using quadratic interpolation.
(ii) $f''(\pi/4)$
(iii) Compare with exact value.

Q. $f(0) = 1$, $f(1) = 3$, $f(3) = 55$

(i) $f'(0)$ } using quadratic interpolation.
(ii) $f''(0)$

Numerical integration

Newton cotes formula

1.) $I = \int_a^b w(x) f(x)\, dx \approx \sum_{k=0}^{n} \lambda_k f_k$ where $w(x)$ is weight f^n.

$w(x) > 0$ in $[a,b]$ $\qquad h = \frac{b-a}{n}$

$a = x_0 < x_1 < \dots x_n$ in (a,b)

2.) $\int_a^b x\, dx = \frac{x^2}{2}\Big|_a^b = \frac{1}{2}(b^2 - a^2)$ $\qquad \boxed{f_k = f(x_k)}$

Types : Numerical integration.

→ closed. Newton-cotes formula $\quad [a,b]$ includes a & b

→ open Newton-cotes formula $\quad (a,b)$ excludes a, b.

Closed Newton-Cotes formula Methods

1.) Trapezoidal Rule = $w(x) = 1$, $h = \frac{b-a}{n}$, $\boxed{n=1}$

closed

$$\boxed{I = \int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a) + f(b)\right]}$$ → simple trap. rule

- Composite trapezoidal rule

$$I = \int_a^b f(x)\,dx \qquad h = \frac{b-a}{n}$$

$$a = x_0 < x_1 < x_2 \ldots x_n = b.$$

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots \int_{x_{n-1}}^{x_n} f(x)\,dx$$ [broke the formula]

$$\Rightarrow \int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right] + \frac{h}{2}\left[f(x_1)+f(x_2)\right] + \frac{h}{2}\left[f(x_2)+f(x_3)\right]$$

$$\ldots + \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots f(x_n)\right]}$$

Q. $\int_0^1 \frac{dx}{1+x}$ find by ① simple trap. rule
② composite trap. rule.

$N = 2, 4, 8.$ (Given.)

Sol① Trap. rule $\Rightarrow \int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$

$$\Rightarrow \int_0^1 \frac{dx}{1+x} = \frac{b-a}{2}\left[f(a)+f(b)\right] \qquad f(x) = \frac{1}{1+x}$$

$$a=0,\ b=1,\ f(0)=1 \quad f(1) = 1/2$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x} = \frac{1-0}{2}\left[f(0)+f(1)\right]$$

$$= \frac{1}{2}\left[1+\frac{1}{2}\right] \Rightarrow \frac{3}{4} \text{ or } 0.75.$$

② Composite trap. formula.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+2\{f x_1 + f x_2 + \dots\} + f(x_n)\right]$$

$h = \frac{b-a}{n}$ for $n = 2, 4, 8$

i. for $n=2$, $h = \frac{1-0}{2}$ or $1/2$. $b=1, a=0$

$x_0 = 0$
$x_1 = x_0+h = 1/2$
$x_2 = x_0+2h = 1$

$$\int_0^1 \frac{dx}{x+1} = \frac{1}{2}\times\frac{1}{2}\left[f(0)+2f\left(\tfrac{1}{2}\right)+f(1)\right]$$

$$= \frac{1}{4}\left[1+2\times\frac{2}{3}+\frac{1}{2}\right] = \frac{17}{24}$$

ii For $n=4$ $h = 1-0/4 = 1/4$ $b=1, a=0$.

$x_0 = 0$
$x_1 = x_0+h = 1/4$
$x_2 = x_0+2h = 1/2$
$x_3 = x_0+3h = 3/4$
$x_4 = x_0+4h = 1$

$$\int_0^1 \frac{dx}{x+1} = \frac{1}{2}\times\frac{1}{4}\left[f(0)+2f\left(\tfrac{1}{4}\right)+2f\left(\tfrac{1}{2}\right)+2f\left(\tfrac{3}{4}\right)+f(1)\right]$$

$$= \frac{1}{8}\left[1+2\times\frac{4}{5}+2\times\frac{2}{3}+2\times\frac{4}{7}+\frac{1}{2}\right] = 0.69702$$

iii For $n=8$, $h = 1-0/8$ or $1/8$ $b=1, a=0$.

$x_0 = 0$
$x_1 = x_0+h = 1/8$ | $x_5 = x_0+5h = 5/8$
$x_2 = x_0+2h = 1/4$ | $x_6 = x_0+6h = 6/8$
$x_3 = x_0+3h = 3/8$ | $x_7 = x_0+7h = 7/8$
$x_4 = x_0+4h = 1/2$ | $x_8 = x_0+8h = 1$

$$\int_0^1 \frac{dx}{x+1} = \frac{1}{2}\cdot\frac{1}{8}\left[f(0) + 2f(1/8) + 2f(1/4) + 2f(3/8) + 2f(1/2) + 2f(5/8) + 2f(6/8) + 2f(7/8) + f(1) \right]$$

$$= \underline{0.694124}$$

note: Ques. gave values of N, if not given, we can take n any number to ease the ques.
eg − n = 4, 5, 6 --- 8

$$\int_0^1 \frac{1}{1+x}\,dx = \log_e(1+x)\Big|_0^1$$

$$= \log_e(1+1) - \log_e(1+0) \Rightarrow \log_e 2 - 0$$

$$= \underline{0.693155} \quad \text{Exact value}$$

2.) Simpson $1/3^{rd}$ Rule | Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \quad \text{in } [a,b]$$

$$h = \frac{b-a}{n} \qquad \boxed{n=2} \quad \text{so} \quad h = \frac{b-a}{2}.$$

$$\begin{cases} x_0 = a \\ x_1 = x_0 + h = (a+b)/2 \\ x_2 = x_0 + 2h = a + b - a = b. \end{cases}$$

$$\int_{x_0=a}^{x_2=b} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

- Composite Simpson's 1/3 Rule − In this rule, we need three nodes, we divide interval [a,b] into n no. of given subintervals of equal length (given: n has odd no. of nodes)
If we devide [a,b] into two subintervals each of h is equal to $b-a/n$ for n = even.

$$a = x_0 < x_1 \ldots - x_n = b \text{ in } [a,b]$$

Page x_{2n}

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \dots \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx$$

$$= \frac{h}{3}\Big[\big(f(x_0) + 4f(x_1) + f(x_2)\big) + \big(f(x_2) + 4f(x_3) + f(x_4)\big) + \dots \big(f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\big)\Big]$$

$$= \frac{h}{3}\left\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots f(x_{2n})\right\}$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\big[f(x_1) + f(x_3) + f(x_5) + \dots\big] + 2\big[f(x_2) + f(x_4) \dots\big] + f(x_{2n})\Big]}$$

Q. $\int_0^1 \frac{dx}{1+x}$ ① Simpson's 1/3 Rule.
② Composite simpson. $n = 4, 8$.

Sol. ② Simpson 1/3 rule (composite)

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$h = \frac{b-a}{n} \Rightarrow \frac{1-0}{4} \Rightarrow 1/4$. for $n=4$.

$x_0 = 0$.
$x_1 = x_0 + h = 1/4$
$x_2 = x_0 + 2h = 2/4$
$x_3 = x_0 + 3h = 3/4$
$x_4 = x_0 + 4h = 1$

$$\Rightarrow \int_0^{1/2} \frac{dx}{1+x} + \int_{1/2}^{1} \frac{dx}{1+x} = \frac{h}{3}\left[f(0) + 4f(1/4) + f(1/2)\right] + \frac{h}{3}\left[f(1/2) + 4f(3/4) + f(1)\right]$$

$$= \frac{1}{12}\left[1 + 4 \times \frac{4}{5} + \frac{2}{3} + \frac{2}{3} + 4 \times \frac{4}{7} + \frac{1}{2}\right] = 0.69325$$

~~6.64~~

Next put $n=8$. and solve.

(4) Simpson Rule (Simple)

$x_0 = 0$ $\quad n=2 \quad h = 1/2$

$x_1 = a+b/2 \Rightarrow 1/2$

$x_2 = 1$

$$\int_0^1 \frac{dx}{1+x} = \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{1}{6}\left[f(0) + 4f(1/2) + f(1)\right]$$

$$= \frac{1}{6}\left[1 + 4\times\frac{2}{3} + \frac{1}{2}\right] = 0.694444$$

3.) Simpson's 3/8 Rule $\quad (n=3)$

$$h = \frac{b-a}{n} = \frac{b-a}{3}.$$

$x_0 = a$

$x_1 = x_0 + h = (2a+b)/3$

$x_2 = x_0 + 2h = (a+2b)/3$

$x_3 = x_0 + 3h = b$.

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

$$\boxed{\int_a^b f(x)dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]}$$

- Composite Simpson 3/8 rule : we need 4 nodes and divide interval $[a,b]$ into multiple of 3 of subinterval of equal length, then given $3n+1$ nodes such that $x_0, x_1, x_2, x_3 \ldots x_n$ (where n is multiple of 3).

$$h = \frac{b-a}{n} \rightarrow \text{multiple of } 3$$

$$a = x_0 < x_1 < \ldots x_n = b.$$

$\frac{9.6}{6}$ $\quad \frac{2}{6}$ $\quad 1+\frac{8}{3}+\frac{1}{2}$ $\quad \frac{6+16+3}{6}$

classmate
Date ____
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$$\int_{a=x_0}^{b=x_n} f(x)\,dx \Rightarrow \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \int_{x_6}^{x_9} f(x)\,dx + \dots \int_{x_{n-3}}^{x_{3n}} f(x)\,dx$$

$$\Rightarrow \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right] + \frac{3h}{8}\left[f(x_3)+3f(x_4) + 3f(x_5)+f(x_6)+\dots f(x_{3n})\right.$$

$$\Rightarrow \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+2f(x_3)+3f(x_4)+3f(x_5)+\dots\right]$$

$$\boxed{\Rightarrow \frac{3h}{8}\left[f(x_0)+3\left[f(x_1)+f(x_2)+f(x_4)+\dots\right]+2\left[f(x_3)+f(x_6)+\dots\right] + f(x_{3n})\right]}$$

Q. $\int_0^1 \frac{dx}{1+x}$ ① Simpson 3/8.
② Composite Simpson 3/8. $n=6$

Sol: ① Simple Simpson 3/8 $n=3$ $h=\frac{1}{3}$

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

$x_0 = 0 \rightarrow a$
$x_1 = x_0+h = 1/3 \rightarrow \frac{2a+b}{3}$
$x_2 = x_0+2h = 2/3 \rightarrow \frac{a+2b}{3}$
$x_3 = x_0+3h = 1 \rightarrow b$

$$= \frac{3}{8}\times\frac{1}{3}\left[f(0)+3f(1/3)+3f(2/3)+f(1)\right]$$

$$= \frac{1}{8}\left[1+\frac{3}{4}\times 3 + 3\times\frac{3}{5}+\frac{1}{2}\right] =$$

② Composite Simpson 3/8. $n=6$ $h=1/6$

$$\int_0^1 \frac{dx}{1+x} = \int_0^{1/2}\frac{dx}{1+x} + \int_{1/2}^1 \frac{dx}{1+x}$$

$$= \frac{3h}{8}\left[f(0)+3f\left(\frac{1}{6}\right)+3f\left(\frac{2}{6}\right)+f\left(\frac{1}{2}\right)\right] + \frac{3h}{8}\left[f\left(\frac{1}{2}\right)+3f\left(\frac{4}{6}\right)+3f\left(\frac{5}{6}\right)+f(1)\right]$$

$x_0 = 0$
$x_1 = x_0+h = 1/6$
$x_2 = x_0+2h = 2/6$
$x_3 = x_0+3h = 1/2$
$x_4 = x_0+4h = 4/6$
$x_5 = x_0+5h = 5/6$
$x_6 = x_0+6h = 1$

$= \frac{3h}{8} \times \frac{1}{6}\left[1 + 3\times\frac{6}{7} + 3\times\frac{6}{8} + \frac{2}{3} + \frac{2}{3} + \frac{6}{10}\times 3 + 3\times\frac{6}{11} + 1/2\right]$

$= \frac{3 \cdot 1}{16}\left[1 + \frac{18}{7} + \frac{18}{8} + \frac{4}{3} + \frac{18}{10} + \frac{18}{11} + \frac{1}{2}\right] =$

Q1) $\int_0^2 e^x \, dx$ by Simpson rule $h = 1, 1/2$.

Ans: $h = \frac{b-a}{n} = \frac{2-0}{2\times1} = 1$ General. $n=2$.

multiple of 2

and $h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{1}{2}$ thus $n=4$ Composite

Q2) $\int_{0.1}^{0.2} \frac{x^2}{\cos x} dx$ by Trapz. & Simpson 1/3 rule also composite of both.

10% chance Open Newton cotes formula Methods

1) Mid point rule $\rightarrow$ $n=2$ $\quad h = \frac{b-a}{2}$

$\int_a^b f(x)\,dx = 2h\, f(x_0+h)$

$= 2h\, f(x_1)$

$= 2\left(\frac{b-a}{2}\right) + \left(\frac{b-a}{2}\right)$

a $\frac{b-a}{2}$ b

x_0 x_n

$$\boxed{\int_a^b f(x)\,dx = \frac{(b-a)}{2} + (b-a)/2}$$

2) Two point rule $n=3$ $\quad h = \frac{b-a}{3}$

$\int_a^b f(x)\,dx = \frac{3h}{2}\left[f(x_0+h) + f(x_0+2h)\right]$

$$\boxed{= \frac{3h}{2}\left[f(x_1) + f(x_2)\right]}$$

a x_1 x_2 b

x_0 x_n

3) Three point rule $n=4$ $\quad h = \frac{b-a}{4}$

$$\int_a^b f(x)\,dx = \frac{4h}{3}\left[2f(x_0+h) - f(x_0+2h) + 2f(x_0+3h)\right]$$

$$\boxed{= \frac{4h}{3}\left[2f(x_1) - f(x_2) + 2f(x_3)\right]}$$

$a = x_0$, x_1, x_2, x_3, $b = x_n$

Q1. $I = \int_0^1 \frac{\sin x}{x}\,dx$ ① Midpt. ② Two pt. ③ three pt.

Sol: 1) Mid point $\boxed{n=2}$ $h = \frac{b-a}{2} \Rightarrow \frac{1}{2}$.

$$\int_0^1 \frac{\sin x}{x}\,dx = 2h[f(x_1)]$$
$$= 2 \cdot \frac{1}{2} \cdot 2\sin(1/2)$$
$$= 0.95288$$

$x_0 = 0$

$x_0 + h = x_1 = 1/2$.

$f(\frac{1}{2}) = 2\sin(1/2)$

$x_2 = 1$

2) Two point $\boxed{n=3}$ $h = \frac{b-a}{3} = \frac{1}{3}$.

$$\int_0^1 \frac{\sin x}{x}\,dx = \frac{3}{2}h\left[f(x_1) + f(x_2)\right]$$
$$= \frac{3}{2} \cdot \frac{1}{3}\left(3\sin(\tfrac{1}{3}) + \frac{3}{2}\sin\frac{2}{3}\right)$$
$$= 0.9546$$

$x_0 = 0$.

$x_1 = x_0 + h = 1/3$

$x_2 = x_0 + 2h = 2/3$.

$x_3 = x_0 + 3h = 1$

$f(1/3) = 3\sin(1/3)$

$f(2/3) = 3/2 \sin(2/3)$.

3) Three point $\boxed{n=4}$ $h = \frac{b-a}{n} = \frac{1}{4}$.

$$\int_0^1 \frac{\sin x}{x}\,dx = \frac{4}{3}h\left[2f(x_1) - f(x_2) + 2f(x_3)\right]$$

$$= 0.946027$$

$x_0 = 0$

$x_1 = x_0 + h = 1/4$

$x_2 = x_0 + 2h = 2/4$

$x_3 = x_0 + 3h = 3/4$

$x_4 = x_0 + 4h = 1$

$f(1/4) = 4\sin(1/4)$

$f(1/2) = 0.47644$

$f(3/4) = \frac{4}{3}\sin(\frac{3}{4})$

Q. $\int_0^1 \frac{dx}{1+x}$ ① Mid pt. ② Two pt. ③ Three pt.

Gauss Quadrature Methods.

$$\int_{-1}^{1} w(x) f(x) dx \approx \sum_{k=0}^{n} \alpha_k f(x_k)$$

Now finite interval [a,b] convert into [-1,1] using transformation

$$x = \left(\frac{b-a}{2}\right) t + \left(\frac{b+a}{2}\right)$$

solve in this only after converting.

1.) Gauss lengendre integration method $\boxed{w(x) = 1}$

then $\int_{-1}^{1} f(x) dx = \sum_{k=0}^{n} \alpha_k f(x_k)$

(i) One point formula $\boxed{n=0}$

then $\int_{-1}^{1} f(x) dx = 2f(0)$

(ii) Two point formula. $\boxed{n=1}$

$\int_{-1}^{1} f(x) dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

(iii) Three point formula. $\boxed{n=2}$

$\int_{-1}^{1} f(x) dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$

Q. $\int_0^1 \frac{dx}{1+x}$

$a=0$

$b=1$

$x = \left(\frac{b-a}{2}\right) t + \frac{b-a}{2}$

$= \frac{1}{2} t + \left(\frac{1}{2}\right)$

$\boxed{2x = t+1}$

verify $x=0$.

then $0 = t+1$

$t = -1$

$x=1$ gives $t=1$

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or $x = \alpha + \beta t$

If $x = 0$, $t = -1$

$0 = \alpha - \beta = \boxed{\alpha = \beta}$

If $x = 1$, $t = 1$

$1 = \alpha + \beta$ or $2\alpha = 1$

$\alpha = \beta = 1/2$

$\Rightarrow$ $x = (1+t)/2$

$$\int_{-1}^{1} \frac{1}{1+\frac{1+t}{2}} \frac{1}{2} dt = \boxed{\int_{-1}^{1} \frac{1}{3+t} dt}$$

(i) One point $n=0$

$$\int_{-1}^{1} \frac{1}{3+t} dt = 2f(0) \qquad f(0) = \frac{1}{3}$$

$$= \frac{2}{3} = \underline{0.6667}$$

(ii) Two pt $n=1$

$$\int_{-1}^{1} \frac{1}{3+t} dt = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{3 - 1/\sqrt{3}} + \frac{1}{3+1/\sqrt{3}} = \frac{\sqrt{3} \cdot 6\sqrt{3}}{26} = \frac{9}{13}$$

(iii) Three pt. $n=2$

$$\int_{-1}^{1} \frac{dt}{3+t} = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5\left(\frac{1}{3-\sqrt{3}/\sqrt{5}}\right) + \frac{8}{3} + 5\left(\frac{1}{3+\sqrt{3}/\sqrt{5}}\right)\right]$$

$=$

Q. $I = \int_{1}^{2} \frac{2x\,dx}{1+x^4}$ Gauss legendre. 1 pt, 2 pt, 3 pt.

Sol. $x = \alpha + \beta t$

If $x = 1$, $t = -1$.

$1 = \alpha - \beta$

if $x = 2,\ t = 1$

$2 = \alpha + \beta$ $\rightarrow \alpha = 3/2 \quad \beta = 1/2$

and $x = \frac{3}{2}t + \frac{1}{2}$ or $x = \left(\frac{3t+1}{2}\right)$

$$\int_1^2 \frac{2x}{1+x^4}\,dx = \int_{-1}^{1} \frac{2\left(\frac{3+t}{2}\right)}{1+\left(\frac{3+t}{2}\right)^4}\frac{dt}{2} \Rightarrow 8\int_{-1}^{1} \frac{3+t}{16+(3+t)^4}\,dt$$

Then solve further.

Gauss-Chebyshev integration. $\rightarrow$ open-type

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1} w(x)f(x)\,dx = \int_{-1}^{1} \frac{dx\, f(x)}{\sqrt{1-x^2}} = \sum_{k=0}^{n} \lambda_k f(x_k)$$

(i) ~~of~~ One-pt. formula $\boxed{n=0}$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \pi f(x_0)$$

(ii) Two-pt formula $\boxed{n=1}$

$$\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[f\left(\frac{-1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right]$$

(iii) Three-pt. formula $\boxed{n=2}$

$$\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right]$$

[Note: If Gauss-~~legendre~~ chebyshev written in Ques. only then do it by that method else if only Gauss is written do with Gauss-~~cheb~~ legendre]

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Q. $I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x \, dx$

In Gauss-L take whole as $f(x)$
In Gauss-C convert in $\left(\frac{fx}{\sqrt{1-x^2}}\right)$

Sol: $I = \int_{-1}^{1} \frac{(1-x^2)^2 \cos x}{\sqrt{1-x^2}} dx$

$f(x) = (1-x^2)^2 \cos x.$

(i) one point $(n=0)$

$$\int_{-1}^{1} \frac{fx}{\sqrt{1-x^2}} dx = \pi f(x_0) = \pi f(0) = \pi(1) = \pi = 0.314159$$

(ii) Two-pt $(n=1)$

$$\int_{-1}^{1} \frac{fx}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\left[f\left(\frac{-1}{\sqrt{2}}\right) + f\left(\boxed{\frac{1}{\sqrt{2}}}\right)\right]$$

radians value

$= 0.59709$

(iii) Three pt $(n=2)$

$$\int_{-1}^{1} \frac{fx}{\sqrt{1-x^2}} dx = \frac{\pi}{3}\left[f\left(\frac{-\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right]$$

$= 1.13200$

Q. $\int_{0}^{1} \cos 2x \, (1-x^2)^{-1/2} dx$ by Gauss-Chebyshev.

$\Rightarrow \int_{0}^{1} \frac{\cos 2x}{\sqrt{1-x^2}} dx$

we have to go. $\int_{0}^{1} \frac{fx}{\sqrt{1-x^2}} dx$ to $\int_{-1}^{1} \frac{ft}{\sqrt{1-t^2}} dt$

By $x = \left(\frac{b-a}{2}\right)t + \left(\frac{b-a}{2}\right)$ or

$x = \alpha t + \beta$

if $x=0$, $t=-1$.

$\Rightarrow \alpha = \beta$.

if $x=1$, then $t=1$

$\Rightarrow \alpha = \beta = \frac{1}{2}$ and $x = \frac{t+1}{2}.$

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$1 - \frac{1}{2} \cdot 2\left[\left(\frac{1}{2}\right)^2 \cos\left(-1/\sqrt{2}\right)\right] +$
$2 \times \frac{1}{4} \cos\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \int_0^1 \frac{\cos 2x}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos 2\left(\frac{t+1}{2}\right)\frac{1}{2}dt}{\sqrt{1-\left(\frac{t+1}{2}\right)^2}}$ | Put $x = (t+1)/2$; $x \to 0$ then $t = -1$; $x \to 1$ then $t = 1$

$\Rightarrow \int_{-1}^{1} \frac{\cos(t+1) \cdot \cancel{2}}{\sqrt{4-t^2-1-2t}} \cdot \frac{1}{\cancel{2}}\,dt$

$\Rightarrow \int_{-1}^{1} \frac{\cos(t+1)\sqrt{1-t^2}}{\sqrt{3-t^2-2t}\sqrt{1-t^2}}\,dt = \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\,dt$

multiply/divide by $\sqrt{1-t^2}$ — Now solve urself.

Richardson Extrapolation formulae

① $f'(x_0) = \frac{f(x_0+h) - f(x_0)}{2h}$

$f'(0) = \frac{f(x_1) - f(x_0)}{2h} = \frac{f(1) - f(0)}{2h}$

② $g^{(1)}(h) = \frac{4g(h/2) - g(h)}{3}$

③ $g^{(1)}(h/2) = \frac{4g(h/4) - g(h/2)}{3}$

④ $g^{(2)}(h) = \frac{4^2 g^{(1)}(h/2) - g^{(1)}(h)}{4^2 - 1}$

In general,
$g^{(m)}(h) = \frac{4^m g^{(m-1)}(h/2) - g^{(m-1)}(h)}{4^m - 1}$ $m = 1, 2, 3 \ldots$

⑤ $g^{(0)}(h) = g(h)$

⑥ $g(x) = f'(x_0)$

Extrapolation table

order → h ↓	Second	fourth	Sixth	Eighth
h	g(h)	$g^{(1)}(h)$		
h/2	g(h/2)			
$h/2^2$	$g(h/2^2)$	$g^{(1)}(h/2)$	$g^{(2)}h$	$g^{(3)}(h)$ find answer
$h/2^3$	$g(h/2^3)$	$g^{(1)}(h/2^2)$	$g^{(2)}(h/2)$	

By Taylor-series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x)$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots$$

Put $x=0$
$h=x$

Q.

x	-1	1	2	3	4	5	7
$f(x)$	1	1	16	81	256	625	2401

using $f'(x_1) = \frac{f(x_2) - f(x_1)}{2h}$ & Richardson Extrapolation

find $f'(3) = ?$

The ques will always ask value of mid pt.

Correct formula written

$h=4$, when 3 moves towards 7 & -1

$h=2$, when 3 moves towards 5 & 1

$h=1$, when 3 moves towards 4 & 2

$$f'(0) = \frac{f(1) - f(0)}{2h}$$

for I column

$h=2$: $f'(3) = \frac{f(5) - f(1)}{2\times 2}$

$h=4$: $f'(3) = \frac{f(7) - f(-1)}{2\times 4}$

$h=1$: $f'(3) = \frac{f(4) - f(2)}{2\times 1}$

Table

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
4	$\frac{2401-1}{8} = 300$		
		$\frac{4\times156-300}{3} = 108$	
2	$\frac{625-1}{4} = 156$		$\frac{16\times108-108}{15} = \underline{108}$ Ans
		$\frac{4\times120-156}{3} = 108$	
1	$\frac{256-16}{2} = 120$		

for II column

$$\Rightarrow g^{(1)}h = \frac{4g(h/2) - g(h)}{3}$$

$$= \frac{4\times156-300}{3} = 108$$

$$\Rightarrow g^{(1)}(h/2) = \frac{4\,g(h/2^2) - g(h/2)}{4^m-1}$$

$$\Rightarrow \frac{4\times120-156}{3} = 108$$

for III column

$$g^{(2)}h = \frac{4^2\,g^{(1)}(h/2) - g^{(1)}(h)}{4^2-1}$$

$$= \frac{16\,g^{(1)}(h/2) - g^{(1)}(h)}{15}$$

$$= \frac{16\times108-108}{15} = \boxed{108}$$ Ans

Correct formula written ahead

Romberg integration with (Simpson's Rule)

$$I_S^{(m)}(h) = \frac{4^{(m+1)}\,I_S^{(m-1)}(h/2) - I_S^{(m-1)}(h)}{4^{(m+1)}-1} \quad \text{for } m=1,2,3\ldots$$

Differentiation.

① Newton - fwd diff. approximation.

② Newton - Bwd Diff. approximation.

① Newton - fwd Diff. approximation

$$f(x) = P(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{\lfloor 2}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 f(x_0)$$

$$f(x) = P(u) = f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{\left(\frac{x-x_0}{h}\right)\left(\frac{x-x_0}{h}-1\right)}{\lfloor 2}\Delta^2 f(x_0)$$

$$+ \frac{\left(\frac{x-x_0}{h}\right)\left(\frac{x-x_1}{h}-1\right)\left(\frac{x-x_2}{h}-2\right)}{\lfloor 3} \Delta^3 f(x_0) + \ldots$$

$$\Rightarrow f(x) = p(x) = f(x_0) + \left(\frac{x-x_0}{h}\right) \Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{h^2 \lfloor 2} \Delta^2 f(x_0)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{h^3 \lfloor 3} f(x_0) + \ldots$$

Diff wrt x — %.

$$\Rightarrow f'(x) = p'(x) = 0 + \frac{1}{h} \Delta f(x_0) + \frac{(2x-x_0-x_1)}{2h^2} \Delta^2 f(x_0) + \ldots$$

$$\boxed{f'(x) = p'(x) = \frac{\Delta f(x_0)}{h} + \frac{(2x-x_0-x_1)}{2h^2} \Delta^2 f(x_0) + \ldots}$$

Q. $f(x) = \sin x \quad f'(\pi/4) = ?, \; h = \pi/12$
Newton fwd diff interpolation, $f'(\pi/4)$ of $O(h^2)$.
(means cal. till two terms of formula)

Now we know that $f'(x) = \frac{\Delta f(x_0)}{h} + \frac{2x - x_0 - x_1}{2h^2} \Delta f(x_0) + O(h^3)$

$$f'(\pi/4) = \frac{\Delta f(x_0)}{h} + \frac{(2 \times \pi/4 - x_0 - x_1)}{2 \cdot h^2} \Delta^2 f(x_0)$$

$x_0 = \pi/4$
$x_1 = x_0 + h = \pi/3$
$x_2 = x_0 + 2h = 5\pi/12$

we know, $\boxed{\Delta f(x) = f(x+h) - f(x)}$

$$\Delta f(x_0) = f(x_0 + h) - f(x_0)$$
$$= f(x_1) - f(x_0)$$
$$= f(\pi/3) - f(\pi/4)$$
$$= \sin(\pi/3) - \sin(\pi/4)$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = 0.15891$$

$$\Delta^2 f(x_0) = \Delta[\Delta f(x_0)]$$
$$= \Delta[f(x_1) - f(x_0)]$$
$$= \Delta f(x_1) - \Delta f(x_0)$$
$$= [f(x_2) - f(x_1)] - [f(x_1) - f(x_0)]$$
$$= f(x_2) - f(x_1) - f(x_1) + f(x_0)$$
$$= f(x_2) - 2f(x_1) + f(x_0)$$
$$= f\left(\frac{5\pi}{12}\right) - 2 f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{4}\right)$$
$$= \sin(5\pi/12) - 2\sin(\pi/3) + \sin(\pi/4)$$
$$= -0.0590182$$

$$f'(\pi/4) = \frac{0.15891(12)}{\pi} + \frac{\pi/2 - \pi/4 - \pi/3}{2(\pi^2)/144}(-0.05901)$$
$$= \frac{12}{\pi}(0.15891) - \frac{\pi \times 144}{6\times 2 \pi^2}(-0.05901)$$
$$= \frac{12}{\pi}\left(0.15891 + \frac{0.05901}{2}\right)$$
$$= \frac{12\times 7}{22}(0.18842) \Rightarrow \boxed{0.7196}$$

Exact value

$$f(x) = \sin(x)$$
$$f'(x) = \cos x$$
$$f'(\pi/4) = \cos(\pi/4) = \boxed{0.707106}$$

Error $\Rightarrow$ Approx value − exact value

$\Rightarrow$ ~~0.707106~~ $0.7196 - 0.7071 = \boxed{0.0125}$

Q.

	x_0	x_1	x_2	x_3	x_4
x :	6.0	6.1	6.2	6.3	6.4
$f(x)$:	0.1750	−0.1998	−0.2223	−0.2422	−0.2596

$f'(6) = ?$ and $f''(6.3) = ?$

Use Newton fwd Diff interpolation.

$$f'(6) = \frac{\Delta f(x_0)}{h} + \frac{(2x - x_0 - x_1)}{2h^2}\Delta^2 f(x_0)$$

$x_0 = 6$

$x_1 = 6.1$

$h = x_1 - x_0 = 0.1$

$$= \frac{f(x_1) - f(x_0)}{h} + \frac{2x - x_0 - x_1}{2h^2}[f(x_2) - 2f(x_1) + f(x_0)]$$

$$= \frac{-0.1998 - 0.1750}{0.1} + \frac{2\times 6 - 6 - 6.1}{2(0.1)^2}[-0.2223 + 2(0.1998) + (0.1750)]$$

$=$

$f''(6.3) = ?$

$$f'(x) = f'(x) = \frac{\Delta f(x_0)}{h} + \frac{2x - x_0 - x_1}{2h^2}\Delta^2 f(x_0) + \ldots$$

$$f''(x) = 0 + \frac{2}{2h^2}\Delta^2 f(x_0) + \ldots$$

$$f''(x) = \frac{1}{h^2}\Delta^2 f(x_0) + \ldots$$

In this Ques,

$$f''(x) = \frac{1}{h^2}\Delta^2 f(x_0) + O(h^3)$$

$$= \frac{1}{h^2}\Delta^2 f(x_0) + \frac{6x - 2(x_0 - x_1 - x_2)}{\lfloor 3\, h^3}\Delta^3 f(x_0)$$

$$= \frac{1}{h^2}[f(x_2) - 2f(x_1) + f(x_0)] + \frac{6\times 6 - 2(6 - 6.1 - 6.2)}{6.(0.1)^3}[f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)]$$

$$= \frac{1}{(0.1)^2}[-0.2223 + 2(0.1998) + 0.1750] + \frac{8.1}{(0.1)^3}[-0.2422 + 3(0.2223) + 3(0.1998) - 0.1750]$$

$=$

$$\Delta^3 f(x_0) = \Delta(\Delta^2 f(x_0))$$

$$= \Delta(f(x_2) - 2f(x_1) + f(x_0))$$

$$= \Delta f(x_2) - 2\Delta f(x_1) + \Delta f(x_0)$$

$$= f(x_3) - f(x_2) - 2[f(x_2) - f(x_1)] + [f(x_1) - f(x_0)]$$

$$= f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)$$

② Newton Bwd. Diff. Interpolation.

we know

$$p(x) = f(x) = f(x_n) + \frac{u}{1}\nabla f(x_n) + \frac{u(u+1)}{\lfloor 2}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{\lfloor 3}\nabla^3 f(x_n)$$

where $u = \frac{x - x_n}{h}$

$$f(x) = p(x) = f(x_n) + \frac{(x-x_n)}{h}\nabla f(x_n) + \frac{\frac{(x-x_n)}{h}\left(\frac{x-x_n}{h}+1\right)}{\lfloor 2}\nabla^2 f(x_n) + \ldots$$

$$f(x) = p(x) = f(x_n) + \left(\frac{x-x_n}{h}\right)\nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{\lfloor 2\, h^2}\nabla^2 f(x_n) + \ldots$$

Diff. wrt. x —

$$\boxed{f'(x) = p'(x) = 0 + \frac{1}{h}\nabla f(x_n) + \frac{2x - x_n - x_{n-1}}{\lfloor 2\, h^2}\nabla^2 f(x_n) + O(h^2)}$$

Q. $f(x) = y = \sin x$

find $f'(\pi/4)$ using newton Bwd Diff. interpolation with $h = (\pi/12)$, Oh^2

we know,

$$f'(x) = p'(x) = \frac{1}{h}\nabla f(x_n) + \left(\frac{2x - x_n - x_{n-1}}{\lfloor 2\, h^2}\right)\nabla^2 f(x_n) + Oh^2$$

$x_n = \pi/4$

$x_{n-1} = x_n - h = \pi/6$

$x_{n-2} = x_n - 2h = \pi/12$

$$\nabla f(x_n) = f(x_n) - f(x_n - h)$$
$$= f(x_n) - f(x_{n-1})$$
$$= f(\pi/4) - f(\pi/6)$$
$$= \sin \pi/4 - \sin \pi/6$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{2} = \underline{0.2070}$$

$$\nabla^2 f(x_n) = \nabla[\nabla f(x_n)]$$
$$= \nabla[f(x_n) - f(x_{n-1})]$$

$= \nabla f(x_n) - \nabla f(x_{n-1})$

$= f(x_n) - f(x_{n-1}) - [f(x_{n-1}) - f(x_{n-2})]$

$= f(x_n) - 2f(x_{n-1}) + f(x_{n-2})$

$= f(\pi/4) - 2 f(\pi/6) + f(\pi/12)$

$= \sin(\pi/4) - 2\sin(\pi/6) + \sin(\pi/12)$

$= \underline{-0.0341}$

$$f'(\pi/4) = (0.2071)\frac{12}{\pi} + \frac{2\frac{\pi}{4} - \frac{\pi}{6} - \frac{\pi}{4}}{2(\pi/12)^2}(-0.0341)$$

$$= \frac{12}{\pi}(0.2071) + \frac{12}{\pi}\left(\frac{-0.0341}{2}\right)$$

$$= \frac{12}{\pi}\left(0.2071 - \frac{0.0341}{2}\right) = \boxed{0.7259}$$

Exact : $f'(\pi/4) = \boxed{0.70716}$

[Thus, we get more approx value i.e. close to exact value) on Newton fwd Diff.].

Romberg Integration (RI)

$$I_T^{(m)}(h) = \frac{4^m I_T^{(m-1)}(h/2) - I_T^{(m-1)} h}{4^m - 1} \quad m = 1, 2, 3 \ldots$$

which is extrapolation with trapezoidal

$$I_S^{(m)}(h) = \frac{4^{(m+1)} I_S^{(m-1)}(h/2) - I_S^{(m-1)}(h)}{4^{(m+1)} - 1} \quad m = 1, 2, 3 \ldots$$

which is extrapolation with Simpsons

Q) $I = \int_0^1 \frac{dx}{1+x}$ ① composite trap. with nodes 2, 3, 5 & RI
② " Simpson rule " " 3, 5, 9 & RI.

① Trapezoidal

2 nodes $n=1$ $\qquad h = \frac{b-a}{n} = b-a$

3 nodes $n=2$

5 nodes $n=4$

for 2 nodes $n=1$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1-0}{2}\left[f(0)+f(1)\right] = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4} \text{ or } \underline{0.75}$$

for 3 nodes $\boxed{n=2}$ $\boxed{h=1/2}$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+2f(x_1)+f(x_2)\right]$$

$x_0 = 0$

$x_1 = x_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$

$x_2 = x_0 + 2h = 1$

$$= \frac{1}{4}\left[f(0)+2f(1/2)+f(1)\right] = \frac{17}{24} \text{ or } \underline{0.70835}$$

for 5 nodes $n=4$ $\boxed{h=1/4}$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)\right)+f(x_4)\right]$$

$$= \frac{h}{2}\left[f(0)+2\left(f(1/4)+f(1/2)+f(3/4)\right)+f(1)\right]$$

$x_0 = 0$

$x_1 = 0 + 1/4 = 1/4$

$x_2 = 2/4$

$x_3 = 3/4$

$x_4 = 1$

$$= \frac{1}{8}\left[1 + 2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)+\frac{1}{2}\right]$$

$$= \underline{0.697024}$$

Note: If subintervals given ~~as~~ on place of nodes and of nodes is given

$n =$ subinterval

$n =$ nodes -1.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
1	0.7500		
		0.69444	
1/2	0.708333		0.698175
		0.693754	
1/4	0.697024		

(ii) Simpsons

for nodes = 3 $\quad n=2 \quad h = \frac{b-a}{2} = \frac{1-0}{2} = \frac{1}{2}$.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{6}\left[f(0) + 4f(1/2) + f(1)\right]$$

$$= \frac{1}{6}\left(1 + \frac{8}{3} + \frac{1}{2}\right) = \frac{25}{36} \text{ or } 0.694444$$

for nodes = 5 $\quad \boxed{n=4} \quad h = (1/4)$

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\right]$$

$$= \frac{1}{12}\left[1 + \frac{4\times 4}{5} + \frac{4\times 4}{7} + \frac{2\times 2}{3} + \frac{1}{2}\right]$$

$$= 0.693254$$

$x_0 = 0$, $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 1$

for nodes = 9 $\quad \boxed{n=8} \quad \boxed{h = 1/8}$

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\Big[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) + 2(f(x_2) + f(x_4) + f(x_6)) + f(x_8)\Big]$$

$$= \frac{1}{24}\left[1 + 4\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) + 2\left(\frac{8}{10} + \frac{8}{12} + \frac{8}{14}\right) + \frac{1}{2}\right]$$

$$= 0.693154$$

$x_0 = 0$, $x_1 = 1/8$, $x_2 = 2/8$, $x_3 = 3/8$, $x_4 = 4/8$, $x_5 = 5/8$, $x_6 = 6/7$, $x_7 = 7/8$, 1

	$m=1$	$m=2$	
h	$O(h^2)$	$O(h^4)$	$O(h^6)$
$1/2$	0.694444		
		0.693175	
$1/4$	0.693254		0.693146
		0.693148	
$1/8$	0.693155		