STAT 230: Probability (Sec 02)
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Lecture 4

Today's Agenda

Last time:

- Simple/compound events
- Probability distribution
- Counting techniques

Today (Lec 4, 05/09):

- More on counting techniques

3. COUNTING TECHNIQUES

Addition Rule

- If A and B are disjoint events (i.e., $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

- From the addition rule, we can see that "or" means "$+$".

Multiplication rule

- An ordered k-tuple is an ordered set of k values: $(a_1, a_2, ..., a_k)$. If the outcomes in A can be written as an ordered k-tuple where there are n_1 choices for a_1, n_2 choices for a_2,..., and in general n_i choices for a_i, then
$$|A| = n_1 n_2 \cdots n_k = \prod_{i=1}^{k} n_i.$$
- The multiplication rule indicates that "and" means "$\times$".

With versus without replacement

Suppose you have an urn with n distinct balls, and you select $k \leq n$ balls in order. You can do that

- "**with** replacement": Every time an object is selected, it is put back into the pool of possible objects.

- "**without** replacement": Once an object is selected, it stays out of the pool of possible objects.

Sampling without replacement affects the probabilities thereafter!

Factorials!!!!!!

Definition

Given n distinct objects, n **factorial** (denoted $n!$) is the number of ordered arrangements of length n that can be made. Mathematically,

$$n! = n \times (n-1) \times \ldots 2 \times 1.$$

By definition, $0! = 1$.

A useful property of $n!$ is $n! = n \times (n-1)!$.

Example

10 people are standing next to each other for a group picture. How many arrangements are there?

Example

Suppose that the 5 members of of the “STATS IS COOL” club must select a president and a secretary. How many ways can they do this?

Definition

Given n distinct objects, a **permutation** of size k is an *ordered* subset of k of the individuals. The number of permutations of size k taken from n objects is denoted $n^{(k)}$ and

$$n^{(k)} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

Stirling's formula

$n!$ grows really fast as n increases, so sometimes we need to approximate its value for computational reasons.

Stirling's formula provides one such method, and it is given by

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

and their ratio approaches 1 as n goes to infinity.

Definition

Given n distinct objects, a **combination** of size k is an *unordered* subset of k of the individuals. The number of combinations of size k taken from n objects is denoted $\binom{n}{k}$ and

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!}.$$

The number $\binom{n}{k}$ is called **Binomial coefficient**.

Example

Melissa participates in a lottery in which she selects 7 numbers between 1 and 50, and then a computer randomly picks 7 numbers between 1 and 50 (without replacement). She wins if her selected numbers match 5 or more of the randomly selected numbers, in any order. What is the probability that Melissa wins?

Example

Suppose you have 20 distinct books, 7 of which are written by Mark Twain.

a) How many ways can you arrange 12 books on a shelf if the order they are on the shelf matters?

b) How many ways can you arrange 12 books on a shelf if exactly 3 of them must be Mark Twain books?

c) A monkey picks books at random from the 20 books and puts them on the shelf until it contains 12 books. What is the probability that at least 3 of the books on the shelf are written by Mark Twain?

There are some useful/important results about permutation and combination.

a) $n^{(k)} = n(n-1)^{(k-1)}$ for $k \geq 1$

b) $\binom{n}{k} = \frac{n^{(k)}}{k!}$

c) $\binom{n}{k} = \binom{n}{n-k}$ for $k \geq 0$

d) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

e) Binomial theorem: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

f) $\binom{n}{k}$ is equal to the kth entry in the nth row of Pascal's triangle.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

Question: What relation does identity 4 have with Pascal's triangle?

Example (The Birthday Problem)

Suppose a room contains n people. What is the probability at least two people in the room share a birthday?

Assumption: Suppose that each of the n people is equally likely to have any of the 365 days of the year as their birthday, so that all possible combinations of birthdays are equally likely.