Stat 230: Probability

Lecture 33

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Tuesday, July 26th

Overview

This final presentation is made to add a few more sources for your final preparation. Please reach out if you want any questions tailored to a specific area.

After some intro text, a list of main topics from the course syllabus is included. While this list does not explicitly state everything we learned–for example, approximations–it may help spark your memory. Of course, the best source is the syllabus as it gives more details and the section of the book as a reference.

Finally many examples are included with their solutions on the following slide. I didn't explicitly write out the full solutions so please reach out if you have any questions (or are concerned about an answer, it's possible there are typos I missed).

Review Disclaimer

Attached are many examples (and solutions). However, this may miss concepts discussed in the course (it took us 12 weeks to cover it after all). All material outlined in the course syllabus is fair game, including section 3.6 that we didn't spend much time on in class.

Remember, you know this stuff! If something is asked in a new way, consider drawing pictures, making tables, or listing possible outcomes until you understand it.

Studies Aids

If you are looking for more practice, consider:

- These and other course slides (including those of other sections)
- TA run tutorial reviews
- Review and end-of-chapter questions in the book
- Practice final in the solutions book
- Midterms and quizzes
- Writing and trading questions with each other

Coverage

Exam 1

- Definition of Probability
- Experiment, Sample Space, Events
- Probability (Distribution, Event)
- Addition/Multiplication Rules
- Counting (Permutations, Combinations, Repeats)
- Unions/Intersections (Independence, Mutually Exclusive)
- Conditional Probability (Product Rule, Total probability, Bayes Theorem)

Coverage

Exam 2

- Random Variables (pf, pdf, cdf)
- Named Discrete Distributions
- Summary Statistics (Expected Value + Properties, Variance + Properties, Standard Deviation)
- (Some) Named Continuous Distributions

Coverage

Untested

- Transforming Random Variables (Computer generation)
- (Some) Named Continuous Distributions
- Multivariate (Joint) Distributions
- Named Multivariate distribution
- Summary Statistics (Expected Value, Variance, Covariance, Correlation)
- Linear Combinations of Random Variables (General/Normal Distributions)
- Indicators
- Central Limit Theorem
- Moment Generating Functions

Coverage

Discussed material that is NOT TESTABLE

- R Code (But very useful to know and check your answers when studying or using this in later courses)
- Our post-test days – Casino and Monte Carlo Days (But concepts discussed relate to the course and may help)

Examples

Example

Suppose you flip a coin 3 times.

(1) What is a possible sample space? What is a possible event?

(2) Suppose we want to count the number of heads we see. What is the probability we see 2 heads?

Examples

Solution and Comments

(1) *There are many possible sample spaces (What is required for a sample space?) and events. One sample space is* $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

(2) $P(2\ Heads) = 3/8$ *(Can you solve this by listing solutions and using a distribution?)*

Examples

Example

Callie is buying a new phone. She can buy an Android or an iPhone. Among these, there are 4 varieties of Androids, each with 3 colors, and 3 iPhones, each with 2 colors, worth considering.

(1) How many total options does Callie have?

(2) If she decides to narrow her search to 2 Androids and 2 iPhones, how many ways can she do this? What is the probability she would get this if 4 phones are randomly selected? (Note, she would only want to consider unique options, no duplicates)

Examples

Solution and Comments

(1) $4(3) + 3(2) = 18$

(2) $\binom{12}{2}\binom{6}{2}$, $p = \frac{\binom{12}{2}\binom{6}{2}}{\binom{18}{4}}$

Examples

Example

Terrence loves alphabet soup. At one point while eating he notices the letters arranged to spell his name. Assume these are the only letters left in the soup, and somehow, no matter the arrangement, the letters always organize into some word.

(1) What are the odds in favor of his name being spelled?

(2) How many ways could the letters be arranged to start with "Te"?

Examples

Solution and Comments

(1) $P(\text{'}terrence') = \frac{1}{\frac{8!}{3!2!}} = \frac{1}{3360}$, *but the odds in favor are* $\frac{P(\text{'}terrence')}{1-P(\text{'}terrence')} = \frac{1}{3359}$

(2) $\frac{6!}{2!2!} = 180$

Example

Helga was never one for probability, but she finds drawing a picture is really helpful. Suppose she knows $P(A) = 0.3$, $P(B) = 0.6$, and $P(A \cup B) = 0.8$.

(1) Visualize and compute $P(A \cap B)$

(2) Are A and B independent? Mutually exclusive?

Examples

Solution and Comments

(1) *Try drawing a Venn Diagram.*
$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.1$

(2) *Not independent or mutually exclusive (How do we know this? What are the two definitions of independence?)*

Examples

Example

Walter is scared of the disease known as the hokey pokey being played at a party. 23% of the time, it is played if the chicken dance has been played. It is played 68% of the time when the chicken dance has not been played. The chicken dance is played 34% of the time.

(1) What is the probability the hokey pokey will be played?

(2) Suppose his pal Jerry is instead frightened of the chicken dance and is trying to decide if he should leave when Walter leaves due to the hokey pokey. What is the probability the chicken dance will be played given the hokey pokey was played?

Examples

Solution and Comments

(1) $P(HP) = P(HP|CD)P(CD) + P(HP|\overline{CD})P(\overline{CD}) = 0.527$ *(What properties do we know about conditional probabilities?)*

(2) $P(CD|HP) = \frac{P(HP|CD)P(CD)}{P(HP)} = 0.148$

Examples

Example

Jesse missed all of class and is frightened for the final. In fear, they reach out and ask about the distributions we named in the course.

(1) Describe the reasons for selecting each of the named distributions.

(2) What are three ways to describe the 'average'?

(3) What are two values we used to describe spread?

Examples

Solution and Comments

(1) *This is a personal check. But what is the difference between the distributions? Why would I pick one over another?*

(2) *We discussed expected value, median, and mode. How are these defined? When are they preferred?*

(3) *We discussed variance and standard deviation. How are these defined? When are they preferred?*

Example

Give the appropriate distributions along with the answers to the following

(1) Kathy will work until she makes 4 sells. If she sells to 5% of contacts, and each takes 10 minutes, how long do you expect she will she work?

(2) Oliver has a cookie jar of 12 cookies–3 chocolate chip, 2 lemon, 5 snicker doodles, and 2 sugar cookies. If he selects 4 cookies, eating them immediately, what is the expected number of lemons eaten?

(3) Paul is tone-deaf. When singing he is equally likely to hit any of 5 possible notes (centered on the correct one). What is the probability he is within 1 note of the correct one? Variance of his singing?

(4) Fran has a bag of candy, with 15 remaining pieces - 7 sweet, 5 sour, and 3 weird ones. If Fran selects 6 pieces and puts them back, what is the probability Fran selects 3 sweet and 2 weird?

Examples

Solution and Comments

Note, sometimes different distributions can be used to the same effect.

(1) *I would use a Negative Binomial to find the number of failed contacts, X. Note, $E(X) = 76$. Now at 10 minutes a contact, plus we need to include the successes, $Y \sim 10(X+4)$, so $E(Y) = 10E(X) + 40 = 800$ mins. Also consider using binomial.*

(2) *We only care about lemon so I would use a Hypergeometric, X. We see $E(X) = \frac{8}{12}$.*

(3) *This can be modeled by a discrete uniform. Say it is from 1-5, then $P(2 \leq X \leq 4) = \frac{3}{5}$ and $Var(X) = 2$.*

(4) *Since we care about multiple selections, try the multinomial distribution. Calling sweet S, sour T and weird W, we get $f(s,t,w) = \frac{6!}{s!t!w!}\left(\frac{7}{15}\right)^s\left(\frac{5}{15}\right)^t\left(\frac{3}{15}\right)^w$. Now to get 3 sweet and 2 weird, we must also get 1 sour, thus $f(3,1,2) = 0.081$.*

Examples

Example

Dr. Smith has patients coming in according to a Poisson process with 5 patients per 8-hr day.

(1) How many patients are expected per hour?

(2) What is the probability that on 2 of 5 independent days in a week, less than 3 patients come in?

Examples

Solution and Comments

(1) *We just need to change* λ *to one hour to see* $X \sim Pois(\frac{5}{8})$, *and thus* $E(X) = \frac{5}{8}$.

(2) *The probability less than 3 people come in on a given day (8 hours) is* $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = 0.12465$, *call this* p. *With this, we use a Binomial distribution* $Y \sim Bin(5, p)$ *and thus* $f_Y(2) = \binom{5}{2} p^2 (1 - p)^3 = 0.10421$.

Examples

Example

Suppose we have a random variable X, where $E(X) = 5$ and $Var(X) = 9$. If we also see random variable Y where $Y \sim 2X - 4$

(1) What is $E(Y)$?

(2) What is $E(Y^2)$?

Examples

Solution and Comments

(1) $E(Y) = E(2X - 4) = 2E(X) - 4 = 6$

(2) *Since* $Var(Y) = Var(2X - 4) = 36$ *and* $Var(Y) = E(Y^2) - E(Y)^2$, $E(Y^2) = 72$.

Examples

Example

Suppose we have random variable X with pdf $f(x) = \frac{kx^2}{12} + 8$ for $2 \leq x \leq 4$, and 0 otherwise

(1) What is the median of X?

Examples

Solution and Comments

(1) *The median if the value x where $F(x) = 0.5$. We first find k by* $\int_2^4 \frac{kx^2}{12} + 8dx = 1$, $k = \frac{-135}{14}$. *Then* $\int_2^x \frac{-135}{14}\frac{x^2}{12} + 8dx = 0.5$. *This gives several roots but the only valid one is* 2.1085

Examples

Example

Solve the following scenarios, along with giving the proper distributions

(1) Jose is terrible at cooking. For example, when recently making a cake, he randomly put in 1-2 cups of flour, with no preference to any amount. What is the probability he puts in more than the necessary 1.25 cups?

(2) Carolyn heard the scores on the last exam were normally distributed with mean 86 and standard deviation of 2. What percentile is her 84 point score?

(3) Gary realizes tears on his previous exam came in a Poisson distribution at a rate of 26 tears a minute. What was the variance of the time between his tears? Given it was 15 seconds since his last tear, what is the probability the next tear will be after 0.1 minutes?

Examples

Solution and Comments

Again, potentially other options exist

(1) *I would use a continuous uniform,* $X \sim U(1,2)$.
$P(X > 1.25) = \int_{1.25}^{2} \frac{1}{2-1} dx = 0.75$

(2) $X \sim N(86, 2^2)$. $P(X \leq 84) = P(Z \leq \frac{84-86}{2}) = 0.15866$.

(3) *We are given* $\lambda = 26$. *So* $Y \sim \exp(\frac{1}{26})$. *so* $Var(Y) = \left(\frac{1}{26}\right)^2$. *And*
$P(Y > 0.25 + 0.1 | Y = 0.25) = P(Y > 0.1) = e^{-2.6}$.

Examples

Example

Suppose we wish to generate an observation x from distribution $f(x) = 1/x^2$ for $x > 1$, 0 for $x \leq 1$.

(1) We generate a u from a $Unif(0,1)$ with value 0.42. What value of x does this correspond to?

Solution and Comments

(1) *We find* $F(x) = 1 - \frac{1}{x}$ *for* $x > 1$, 0 *otherwise. Then* $0.42 = 1 - \frac{1}{x}$ *leads to* $x = 1.724$.

Note, we also saw this problem another way. Suppose instead we were given $X = \frac{1}{1-U}$, *what is the CDF of* X*?*

(1) $F_X(x) = P(X \leq x) = P(\frac{1}{1-U} \leq x) = \cdots = P(X \leq 1 - \frac{1}{x}) = F_U(1 - \frac{1}{x})$, *thus the CDF is found*

$$F_X(x) = \begin{cases} 0 & 1 - \frac{1}{x} < 0 \\ 1 - \frac{1}{x} & 0 \leq 1 - \frac{1}{x} \leq 1 \\ 1 & 1 - \frac{1}{x} > 1 \end{cases} = \begin{cases} 0 & x \leq 1 \\ 1 - \frac{1}{x} & x > 1 \end{cases}$$

which is our previously found cdf.

Examples

Example

Suppose we have a X and Y jointly related in the following pf

	$f_{X,Y}(x,y)$	x = 0	x = 1	x = 3
y	0	0.1	0.2	0.2
y	1	0.1	0.1	0.05
y	2	0	0.1	0.15

Find:

(1) $P(X > Y)$

(2) The marginal distribution of X

(3) The conditional distribution $f(y|X = 1)$

(4) Determine if X and Y are independent

Examples

Solution and Comments

(1) $P(X > Y) = 0.6$

(2) *We can write this as the following table*

x	0	1	3
$f_X(x)$	0.2	0.4	0.4

(3) *We can write this as the following table*

y	0	1	2
$f_Y(y)$	0.5	0.25	0.25

(4) *Note all points need to be independent so although* $f(0,0) = f_X(0)f_Y(0)$, *a point such as* $f(0,1) \neq f_X(0)f_Y(1)$, *so* X *and* Y *are dependent.*

Examples

Example

Suppose we know $E(X) = 2$, $Var(X) = 4$, $E(Y) = 1$, $Var(Y) = 9$, and $Cov(X, Y) = -1$

(1) What is ρ, the correlation coefficient, between X and Y?

(2) What is $E(3X - 2Y + 4)$ and $Var(3X - 2Y + 4)$?

(3) What is $Cov(X, 3X - 2Y + 4)$?

Examples

Solution and Comments

(1) $\rho = Corr(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = -0.167$

(2) $E(3X - 2Y + 4) = 8$ *and* $Var(3X - 2Y + 4) = 84$

(3) $Cov(X, 3X - 2Y + 4) = 14$

Examples

Example

Suppose $X \sim G(0,1)$, $Z \sim G(0,1)$ and $Y \sim N(4,9)$, mutually independent.

(1) What is the distribution of W where $W = X + Z + Y$, including all parameters?

(2) Compute $P(W > 5)$.

(3) What is the 41st percentile of W?

Examples

Solution and Comments

(1) $W \sim N(4, 11)$ *(What is we have multiplied or added constants?)*

(2) $P(W > 5) = 1 - P(Z < \frac{5-4}{\sqrt{11}}) \approx 0.38209$ *via the Normal table.*

(3) $F(w) = 0.41$ *leads to* $P(Z < \frac{w-4}{\sqrt{11}}) = 0.41$. *Using the Normal table and simplifying, we see* $w = 3.245$

Examples

Example

The length of springs produced by a company is assumed to be normally distributed with mean $10cm$ and standard deviation $0.5cm$. Since the ends of the spring do not have the same properties as the middle, a machine cuts off a section at each end of the spring. Suppose the length of the cut-off section is normally distributed with mean $1cm$ and standard deviation $0.2cm$, and is independent of the original length of the spring and the amount cut from the opposite end.

(1) Compute the distribution of the length of the remaining middle section of the spring.

(2) Compute the probability that the remaining middle section of the spring is longer than $7.5cm$.

Examples

Solution and Comments

Note the length of the spring $L \sim N(10, 0.5^2)$, *the amount cutoff from each end is* $C_L, C_R \sim N(1, 0.2^2)$, *and all are independent.*

(1) *The length of the middle section* M $L - C_R - C_L$. *Since all are normals, we get* $M \sim N(8, 0.33)$

(2) $P(M > 7.5) = 1 - P(Z < \frac{7.5-8}{\sqrt{0.33}}) = 1 - (1 - \Phi(0.87)) = 0.80785$

Examples

Example

In a circuit containing n switches, the ith switch is closed with probability $1/i^2$, $i = 1, \cdots, n$, independently of the rest. Let X be the total number of switches that are closed.

(1) What is $E(X)$?

(2) Suppose this circuit is infinitely large, what is $E(X)$?

Examples

Solution and Comments

Tip, define an indicator $\mathbb{I}_i$ *where* $P(\mathbb{I}_i = 1) = \frac{1}{i^2}$. *In other words, this indicator is 1 is switch* i *is closed, and* 0 *otherwise. Then the total number of closed switches* X *is such* $X = \sum_{i=1}^{n} \mathbb{I}_i$

(1) $E(X) = E(\sum_{i=1}^{n} \mathbb{I}_i) = \sum_{i=1}^{n} \frac{1}{i^2}$

(2) *If we let* $n \to \infty$ *then* $E(X) = \frac{\pi^2}{6} \approx 1.645$

Examples

Example

A building has 11 floors. Suppose 12 people get into an elevator at floor 1, and each chooses a floor at random to get out (excluding the floor they started on), independently of the others.

(1) At how many floors do you expect the elevator to stop to let out one or more of these 12 people?

Examples

Solution and Comments

Define an indicator $\mathbb{I}_i$ *where it is 1 if at least one person wants out on the floor. This means we will have 10 indicators (one for each possible floor). Note, finding the probabilities the indicators are 1 is easier to find looking at the opposite, i.e. who doesn't want out. We see* $P(\mathbb{I}_i = 1) = 1 - \left(\frac{9}{10}\right)^{12}$. X *is the number of floors we stop at and* $X = \sum_{i=1}^{10} \mathbb{I}_i$.

(1) $E(X) = E(\sum_{i=1}^{10} \mathbb{I}_i) = \sum_{i=1}^{10} \left(1 - \left(\frac{9}{10}\right)^{12}\right) = 7.176$

Examples

Example

Suppose $X_1, \cdots X_{100}$ are independent and identically distributed (iid) continuous random variables such that $X_i \sim F(\mu = 5, \sigma^2 = 9)$–in other words, it is an unknown distribution with $E(X) = 5$ and $Var(X) = \sigma^2$. We define $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$.

(1) What is the approximate distribution of $\bar{X}$?

(2) Approximate $P(\bar{X} > 5.5)$

Examples

Solution and Comments

(1) $\bar{X} \overset{approx}{\sim} N(5, \frac{9}{100})$ *(Why do we know this?)*

(2) $P(\bar{X} > 5.5) = 1 - \Phi\left(\frac{5.5-5}{3/10}\right) = 0.04746$

Examples

Example

Suppose $Y \sim Pois(30)$.

(1) Give the transformation of this that gets an approximate $N(0, 1)$ random variable

(2) Approximate $P(Y = 30)$ using such a transformation

Examples

Solution and Comments

Suppose $Y \sim Pois(30)$*.*

(1) *According to the theorem,* $\frac{Y-30}{\sqrt{30}} \sim N(0,1)$

(2) *Note we need a continuity correction (why?),*
$P(Y=30) = P(29.5 < Y < 30.5) \approx 2\Phi\left(\frac{0.5}{\sqrt{30}}\right) - 1 \approx 0.07172$

and puts them back

Examples

Example

In this course we discussed many approximations. For example, how/when/why can we approximate:

(1) A Hypergeometric distribution with a Binomial distribution

(2) A Binomial distribution with a Poisson distribution

(3) A distribution (both discrete and continuous) with a Normal distribution. (Are there special cases?)

What other relations did we see between distributions?

Examples

Solution and Comments

This is a personal checkup. I will note a few things

(1) *As n gets really big replacement is not such a big issue*

(2) *As n goes to infinity*

(3) *When can we use CLT? Any differences between discrete and continuous random variables? What about cases for Poisson/Binomial?*

What other relations did we see between distributions? For example, why are there two definitions for the exponential distribution?

Examples

Example

Suppose we have random variable X with pf

x	-1	0	1	3
f(x)	0.3	0.2	0.1	0.4

(1) Find the MGF of X

Examples

Solution and Comments

(1) $E(e^{tX}) = \sum_x e^{tx} f(x) = 0.3e^{-t} + 0.2 + 0.1e^{t} + 0.4e^{3t}$ *(Do we know anything about MGFs for independent or joint random variables? How about for adding or multiplying by a constant?)*

Conclusion

Thank you for the fantastic term! Hopefully you learned something this term, you certainly taught me a lot. We covered some useful concepts and I enjoyed talking probability together. You are all very impressive and I know you can do great on this final (and all your other finals).

Thank you again and I hope to see you in the future!