Stat 230: Probability

Lecture 23

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Review

Example

Suppose during a storm little Timmy sees lightning strike on average once every 2 minutes (with an exponential distribution). Naturally, he yells for his mommy. What is the probability of no lightning strikes for 3 minutes after she arrives? What if she already waited 2 minutes?

Review

Last time we talked about:

(1) Exponential Distribution

For today:

(1) Normal Distribution

Reading: Chapter 8

Review

- Today: Quiz 4
- Review timing

Normal Distribution

Definition

X is said to have a **normal distribution** *(or Gaussian distribution) with mean μ and variance σ^2 if the density of X is*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Shorthand, $X \sim N(\mu, \sigma^2)$

Normal Distribution

Figure: Carl Friedrich Gauss, 1777-1855.

Normal Distribution

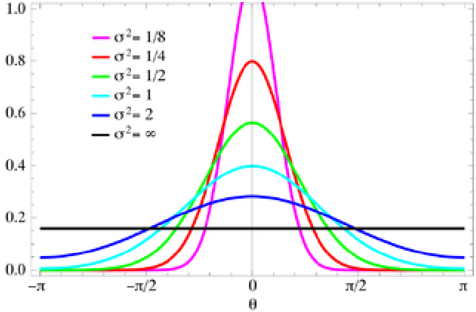

Figure: Normal densities with mean zero and increasing standard deviation.

Normal Distribution

Remark (Properties of the Normal Distribution)

(1) *Symmetric about its mean: If $X \sim N(\mu, \sigma^2)$*

$$P(X \leq \mu + t) = P(X \geq \mu - t)$$

(2) *Density is unimodal: Peak is at μ.*

(3) *Mean and Variance are the parameters: $E(X) = \mu$, and $Var(X) = \sigma^2$.*

Normal Distribution

Theorem

If $X \sim N(\mu, \sigma^2)$*, then*

$$E(X) = \mu, \qquad Var(X) = \sigma^2$$

Normal Distribution

Many probabilists and statisticians would argue (myself included) that the normal distribution is the most important/useful distribution.

(1) The main reason for this is the Central Limit Theorem, which we will discuss by the end of this course.

(2) As described by the Central Limit Theorem, the normal distribution models many different phenomena: Test scores on exams, measurements of heights and weights, error measurements in experiments, etc..

Normal Distribution

Major problem with the normal random variable: If $X \sim N(\mu, \sigma^2)$, then

$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = ???$$

functions of the form e^{-x^2} do not have elementary antiderivatives... Drat!

This is a famous theorem of Liouville in the mid 1800's.

In practice these integrals must be evaluated numerically.

Normal Distribution

Definition

We say that X is a **standard normal** *random variable if $X \sim N(0,1)$.*

Frequently in probability and statistics literature, the density of the standard normal random variable is denoted

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

and the CDF of a standard normal random variable is denoted

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \, dy$$

Example

Suppose that $Z \sim N(0, 1)$. Compute

(1) $P(Z \leq 2.15)$

(2) $P(Z < -1.29)$

Normal Distribution

Theorem

If $Y \sim N(\mu, \sigma^2)$, then if

$$X = \frac{Y - \mu}{\sigma}, \quad X \sim N(0, 1)$$

The operation of centering a random variable and scaling it by the reciprocal of the standard deviation is called "standardization".

Normal Distribution

Example

Suppose the score of a randomly selected student on Midterm 2 follows a normal distribution with mean 81 and variance 6^2.

(1) If a student is selected at random, compute the probability that the students score exceeds 75.

(2) If a student is selected at random, compute the probability that the students score is less than 10.