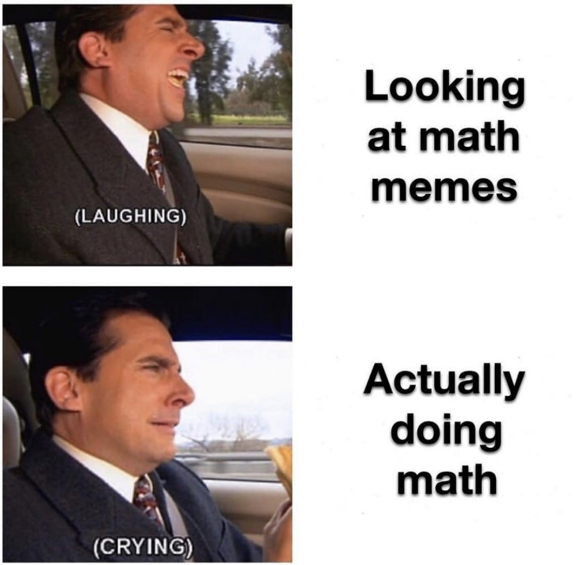
(LAUGHING)
Looking at math memes
(CRYING)
Actually doing math

Today's Agenda

Last time:

- Continuous random variables:
  - definition
  - cdf and pdf
  - expected value and variance of continuous random variables

Today (Lec 23, 06/24):

- More practice
- Distribution of functions of random variables.
- Uniform distribution.

Definition (CDF of continuous random variable)

The cumulative distribution function of a continuous random variable X, $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$, is a function such that

1. $F(x)$ is defined for all $x \in \mathbb{R}$,
2. $F(x)$ is a non-decreasing function of x for all $x \in \mathbb{R}$,
3. $\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to \infty} F(x) = 1$,
4. $P(a < X \leq b) = F(b) - F(a)$,

Note: $P(X = x) = 0$ for all $x \in \mathbb{R}$!

Strict inequalities don't matter...

If X is a continuous random variable, then

$$P(a < X \leq b) = F(b) - F(a)$$

$$P(a \leq X \leq b) = P(a < X \leq b) + P(X = a) = F(b) - F(a) + 0$$

$$P(a < X < b) = P(a < X \leq b) - P(X = b) = F(b) - F(a) - 0$$

$$P(a \leq X < b) = P(a < X \leq b) + P(X = a) - P(X = b) = F(b) - F(a)$$

so if X is continuous, all these probabilities coincide!

If X was discrete, these 4 probabilities could all be different.

Definition (Probability density function)

The *probability density function (pdf)* of a continuous random variable X is the derivative of the cdf

$$f(x) = \frac{d}{dx} F(x),$$

where $f(x)$ is continuous.

Definition (Support of pdf)

The support of a pdf $f(x)$ is defined as

$$supp(f) = \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Integrals of a pdf f over a domain D may be computed on $D \cap supp(f)$.

Properties of the pdf

a)

$$f(x) \geq 0 \quad \forall x \in \mathbb{R}$$

b)

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

c)

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx \quad \text{for } a < b \in \mathbb{R}$$

Definition (Expectation of continuous RV)

If X is a continuous random variable with pdf $f(x)$, and $g : \mathbb{R} \to \mathbb{R}$, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

It follows that

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx.$$

$$\begin{aligned} Var(X) = E[(X - E[X])^2] &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x)dx \\ &= E[X^2] - E[X]^2. \end{aligned}$$

Discrete versus continuous random variables

- If X is **discrete**, then
  - $f(x) = P(X = x)$ is the probability mass function
  - $P(X \in A) = \sum_{x \in X(S) \cap A} f(x)$
  - $E(X) = \sum_{x \in X(S)} xf(x)$
- If X is **continuous**, then
  - $P(X = x) = 0$ for all $x \in \mathbb{R}$.
  - $f(x) = F'(x)$ is the probability density function
  - $P(X \in A) = \int_A f(x)\,\mathrm{d}x$
  - $E(X) = \int_{\mathbb{R}} xf(x)\,\mathrm{d}x$
  - The pdf $f(x)$ is not $P(X = x)$, but for $\delta > 0$ small,

$$\begin{aligned} P(X \in (x - \delta/2, x + \delta/2)) &= P(x - \delta/2 \le X \le x + \delta/2) \\ &= F(x + \delta/2) - F(x - \delta/2) \\ &\approx f(x)\delta. \end{aligned}$$

Example

Suppose X has pdf

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

Compute $E[X]$ and $Var(X)$

Question

Suppose X has pdf $f(x)$, and f is an even function about the origin on $\mathbb{R}$ (i.e. $f(x) = f(-x)$). If $E[X]$ is well defined, which of the following is true:

A $E[X] < 0$

B $E[X] = 0$

C $E[X] > 0$

Example

Suppose X has cdf

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{x^2}{2} & 0 \leq x < 1/2 \\ \frac{7}{4}x - \frac{3}{4} & 1/2 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Compute $E(X)$ and $Var(X)$.

Distribution of $g(X)$

So far, we have a way of dealing with the expectation of a function of random variable $g(X)$ (Law of Unconscious Statistician).

However, we may be interested in the distribution of $g(X)$ itself.

Example: $X =$ time until a light bulb breaks, $g(X) =$ the amount of work done before the light bulb breaks

Distribution of $g(X)$

If the function g has an inverse over the range of X, then we have a fairly easy way of obtaining the distribution of $Y = g(X)$.

In short, the method is as follows.

a) Write the cdf of Y as a function of X, i.e., start by $P(Y \leq y)$ and try to express it using X.
b) Use the cdf of X to find the cdf of Y.
c) If you want the PDF of Y, then you can take the derivative of the cdf.
d) Find the range of Y.

Example

Let X be a continuous random variable with the following pdf and cdf:

$$f(x) = \begin{cases} \frac{1}{4} & 0 < x \leq 4, \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{4} & 0 < x < 4, \\ 1 & x \geq 4. \end{cases}$$

Find the pdf of $Y = X^{-1}$.

Continuous uniform distribution

We now introduce the first continuous distribution.

Definition

We say that X has a continuous uniform distribution on (a, b) if X has pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b), \\ 0 & \text{otherwise} \end{cases}$$

This is abbreviated $X \sim U(a, b)$.

Examples

- Cutting a stick of length 2 at a random position (motivating example!)
- Spinning a wheel in a game show
- ...

Example

Let $X \sim U(a, b)$. Show the following.

a) $E(X) = \frac{a+b}{2}$

b) $Var(X) = \frac{(b-a)^2}{12}$

Question

Suppose $X \sim U(0,1)$, and that $Y = \frac{2}{X} - 1$. What is the range of Y?

A $Y(S) = [0, \infty)$

B $Y(S) = [1, 3]$

C $Y(S) = [0, \infty)$

D $Y(S) = [0, 2]$

E $Y(S) = [1, \infty)$

Example

Let X be a continuous random variable with pdf

$$f(x) = ce^{-\lambda x}, \quad x > 0,$$

and 0 otherwise, where $\lambda > 0$ is a parameter and $c > 0$ is a constant to be determined.

a) Determine c so that f is a valid pdf.
b) Determine the cdf of X.
c) What distribution does the random variable $Y = e^{-\lambda X}$ have?