Instructions

1. All electronic devices are to be turned off and put away.
2. This is a closed-book quiz.
3. When only considering a single random variable, X, $f_X(x) = f(x)$ and $F_X(x) = F(x)$.
4. Answer the questions in the spaces provided.
5. Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form.
6. Only question pages will be marked.
7. You may tear off the last two pages and use it for rough work. All pages will be collected at the end of the quiz.
8. Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed. No other aids are allowed.
9. A formula sheet has been provided at the end of this quiz.
10. DO NOT WRITE ON THIS COVER PAGE.

Marking Scheme:

Questions	Out of
1 – 9	9
10	9
11	8
Total	26

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Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Suppose your friend makes the following 2 statements about any random variable X with probability function $f(x)$ and cumulative distribution function $F(x)$:

I	$F_X(x) \geq P(X = x)$
II	If $P(X \leq a) \leq P(X \leq b)$ then $a < b$

Which statements must be true?

A. **Only I** B. Only II C. I and II D. None

- We know $P(X = x) = f_X(x)$ and $P(X \leq x) = F_X(x)$. Thus **I holds** as $P(X \leq x) = P(X = x) + P(X < x) \geq P(X = x)$.
- We can see **II does not hold** in the case $a = b$.

2. You are given the following probability function in table form:

x	-1	0	1	4	5
$f_X(x)$	p	$0.5p$	$3p$	p	$2.5p$

If p is a constant, what is the value of p?

A. $\frac{1}{9}$ B. $\frac{1}{5}$ **C.** $\boldsymbol{\frac{1}{8}}$ D. 8

- Since $\sum_x f_X(x) = 1$, we can compute $\sum_{i \in \{-1,0,1,4,5\}} f_X(x)$ which gets $p + 0.5p + 3p + p + 2.5p = 1$. Thus $\boldsymbol{p = \frac{1}{8}}$.

3. Sue is newcomer to Waterloo and decides to pet the cute baby geese. The probability that a mother goose attacks her remains constant and is 0.8 every time Sue attempts to pet a baby. If the attacks are independent each time Sue tries to pet a cute baby goose, what is the probability Sue is attacked exactly 8 times if she attempts to pet a baby 10 times?

A. 0.168 B. 0.682 **C.** $\mathbf{0.302}$ D. 0.242

- We are counting the number of times Sue is attacked out of 10 independent trials. This is fit using a binomial model where $n = 10$ and $p = 0.8$, defining a success as an attack. Stating $X \sim Binomial(10, 0.8)$, we see $f_X(8) = \binom{10}{8} 0.8^8 (1 - 0.8)^2 \approx \mathbf{0.302}$.

4. Suppose that a discrete random variable X has a CDF given by $F(x) = \frac{x^2}{25}$ for $x = 1, 2, 3, 4, 5$. What is the pf of X (on relevant values of x)?

A. $\frac{2x}{25}$ **B.** $\mathbf{\frac{2x-1}{25}}$ C. $\frac{x^2+x+1}{25}$ D. $\frac{1-2x}{25}$

- We know $f_X(x) = F_X(x) - F_X(x-1)$. Therefore $f_X(x) = \frac{x^2}{25} - \frac{(x-1)^2}{25}$ after simplification we get $f_X(x) = \boldsymbol{\frac{2x-1}{25}}$ for $x \in \{1,2,3,4,5\}$.

5. Halfway through a boring movie at the theater, Agatha starts throwing popcorn at the seats in front of her. However, the air conditioning is running and blows the popcorn. The popcorn randomly hits the people sitting in front her, according to a Poisson process with an average of 3 hits for every 10 pieces thrown. If Agatha has a massive bucket of popcorn, what is the probability Agatha hits exactly 8 people on the next 20 throws?

A. 0.168 B. 0.029 C. 0.122 **D. $\mathbf{0.103}$**

- We are given out chances of hitting a person is modeled by a Poisson distribution. If we say X is the number of hits, then $X \sim Poisson(\lambda)$. We are given $\lambda = 3\frac{hits}{10-throws}$; however, the question asks about the next 20 throws. We convert λ, such that we get $\lambda = 3\frac{hits}{10-throws}\left(2\frac{10-throws}{20-throws}\right) = 6\frac{hits}{20-throws}$.
- Therefore, $f_X(x) = \frac{e^{-6}6^x}{x!}$ for $x = 1,2,3,\dots$. So, the probability of hitting 8 people in the next 20 throws is $f_X(8) = \frac{e^{-6}6^8}{8!} \approx \mathbf{0.103}$.

6. The probability function for the number of moose sightings in a remote area of Ontario over the course of a week is the following.

x	**f(x)**
0	0.02
1	0.22
2	0.16
3	0.31
4	p

What is the probability there are at least 3 moose sightings in the upcoming week?

A. $\mathbf{0.6}$ B. 0.29
C. 0.31 D. Cannot be determined from the given information

- We must first find the value of p. We know $\sum_x f(x) = 1$. Therefore, $0.02 + 0.22 + 0.16 + 0.31 + p = 1$ and we find $p = 0.29$.
- Now we want $P(X \geq 3)$. With this distribution, this is the same as $P(X = 3) + P(X = 4)$. Therefore, the answer is $\mathbf{0.6}$.
- Likewise we could compute $P(X \geq 3) = 1 - P(X \leq 2)$ for the same result.

7. Steve buys a bag of chocolates, individually wrapped in different colored foil. After eating some, he looks inside the bag and sees 20 red, 10 yellow and 15 blue chocolates remaining. Which of the following probability functions would be used to determine the exact probability that Steve obtains exactly x red chocolates if he randomly selects 7 chocolates from the bag, eating each candy before selecting the next one?

A. $\frac{\binom{25}{x}\binom{20}{7-x}}{\binom{45}{7}}$ B. $\binom{x+7}{x}\left(\frac{20}{45}\right)^x\left(1-\frac{20}{45}\right)^x$ C. $\binom{7}{x}\left(\frac{20}{45}\right)^x\left(1-\frac{20}{45}\right)^{7-x}$ **D.** $\frac{\binom{20}{x}\binom{25}{7-x}}{\binom{45}{7}}$

- We know there are 45 total remaining candies $(20 + 10 + 15)$, and a success is defined as selecting a red candy, which there are 20. Since Steve eats the candy after selection, the selection must be made without replacement. Therefore, a hypergeometric distribution is appropriate. If X is the number of red candies Steve selects, the appropriate distribution is $f_X(x) = \frac{\binom{20}{x}\binom{25}{7-x}}{\binom{45}{7}}$ for $0 \leq x \leq 7$ and $x \in \mathbb{Z}$.
- Note, using a binomial distribution may be an appropriate approximation, but the hypergeometric distribution gives the exact probability.

8. Suppose a co-worker claims the function $f(x) = k\left(\frac{1}{3}\right)^x$ for $x = 1,2,3,4$ and $f(x) = 0$ otherwise, is a probability function for some constant k. What is $P(X > 1)$?

A. $\mathbf{\frac{13}{40}}$ B. $\frac{27}{40}$

C. $\frac{81}{40}$ D. Cannot be determined from the given information

- Since $\sum_x f_X(x) = 1$, we see $k\left(\frac{1}{3}\right)^1 + k\left(\frac{1}{3}\right)^2 + k\left(\frac{1}{3}\right)^3 + k\left(\frac{1}{3}\right)^4 = 1$. After simplifying, we find $k = \frac{81}{40}$. Thus $f_X(x) = \frac{81}{40}\left(\frac{1}{3}\right)^x$.
- We can compute $P(X > 1) = 1 - f_X(1) = 1 - \frac{81}{40}\left(\frac{1}{3}\right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}$

9. A random variable X takes on values in the countably infinite set $\{1,2,3,\ldots\}$ and has the property $P(X \geq x) = 0.8\, P(X \geq x - 1)$ for all $x = 2, 3, 4, \ldots$. What is $P(X = 4)$?

A. 0.8 B. 0.082

C. $\mathbf{0.102}$ D. Cannot be determined from the given information

- We are given $P(X \geq x) = 0.8\, P(X \geq x - 1)$. We also know since this is a discrete distribution taking integer values, $P(X = 4) = P(X \geq 4) - P(X \geq 4 + 1)$. We can substitute to see $P(X = 4) = P(X \geq 4) - 0.8P(X \geq 4)$. This means $P(X = 4) = (1 - 0.8)P(X \geq 4)$.
- We can continue using the relation to see $P(X = 4) = (1 - 0.8)(0.8\, P(X \geq 4 - 1))$, $P(X = 4) = (1 - 0.8)(0.8^2\, P(X \geq 4 - 2))$, and so on until we reach $P(X = 4) = (1 - 0.8)(0.8^{4-1}\, P(X \geq 1))$.
- We know $P(X \geq 1) = 1$ since that is all possible values. Thus the solution is $P(X \geq 4) = (1 - 0.8)(0.8^{4-1}) \approx \mathbf{0.102}$.

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work. Unjustified answers will not receive full credit.

10. Doug walks by the cafeteria exactly once every day. Each day he walks by, he ends up stepping in some kind of food. In fact, on any given day, he either steps on a banana peel with probability 0.25, a noodle with probability 0.15, a roll with probability 0.4, or a brownie with probability 0.2. Doug only steps on a single type of food in a day and is very consistent such that these probabilities stay constant day to day and the daily occurrences are also independent.

a. What is the probability Doug steps on a banana peel exactly once in the next two days? **(2 marks)**

- We know the probability of stepping on a banana peel is 0.25. If we define X as the number of times Doug steps on a banana peel then we need to look at $P(X = 1)$, noting that he must step on something else on the other days.
- This can be modeled using a Binomial such that $P(X = 1) = f_X(1) = \binom{2}{1}(0.25)^1(1 - 0.25)^1 = 0.375$.
- Additionally, using previous tricks, $P(X = 1) = P(banana\ day\ 1)P(other\ day\ 2) + P(other\ day\ 1)P(banana\ day\ 2) = (0.25)(1 - 0.25) + (1 - 0.25)(0.25) = 0.375$.

b. Suppose X represents the number of days Doug steps in some other kind of food, given above, before stepping on a brownie for the first time. What type of distribution does X have? For full credit, please include the appropriate probability function and the range of X. **(2 marks)**

- Since we do not know in advance how many times Doug can pass the cafeteria and we are only looking at the first time, this is best modeled by a geometric distribution (though starting with a negative binomial distribution will result in the same probability function and range).
- If X counts the number of days Doug passes the cafeteria without stepping on a brownie then $f_X(x) = (0.2)(1 - 0.2)^x$ for $x = 0,1,2, \ldots$ (note, $f(x) = 0$ otherwise).

c. What is the probability Doug steps in another type of food exactly 2 times before stepping on a brownie? **(2 marks)**

- From part (b) we know $f_X(x) = (0.2)(1 - 0.2)^x$ for $x = 0,1,2, \ldots$, where X counts the number of days before Doug steps in a brownie for the first time.
- Therefore, we can compute $\boldsymbol{f_X(2) = (0.2)(1 - 0.2)^2 \approx 0.128}$.

d. What is the probability Doug will step on a roll on at least 3 of the next 5 days? For full credit, please clearly define the random variable and give the appropriate probability function (with the range of the random variable). **(3 marks)**

- We know this is a 5-day period ($n = 5$), the probability of stepping on a roll is 0.4 ($p = 0.4$). So this is a perfect application of a Binomial distribution.
- Let X denote the number of days Doug steps on a roll in a 5-day period. Thus $X \sim Binomial(5, 0.4)$. We see $f(x) = \binom{5}{x}(0.4)^x(1 - 0.4)^{5-x}$ for $x = 0,1,2,3,4,5$ ($f(x) = 0$ otherwise).
- Therefore, the probability Doug steps on at least 3 rolls in this 5-day period can be computed $P(X \geq 3) = f(3) + f(4) + f(5)$ or likewise $P(X \geq 3) = 1 - P(X \leq 2) = 1 - \left(f(0) + f(1) + f(2)\right)$. Either way, we find $P(X \geq 3) \approx 0.317$.

11. Ava loves watching butterflies! Suppose that on a given trip, the number of butterflies she watches is best modeled with a Poisson distribution, watching an average of 12 butterflies a minute.

a. Suppose X represents the number of butterflies Ava watches in t seconds. Please give the appropriate probability function for X. For full credit, please include the range of X. **(2 marks)**

- We are given this is best modeled using a Poisson distribution, we define X to be the number of butterflies Ava watches. We are given $\lambda = 12\frac{butterflies}{min}$, but since we are looking at seconds, we must convert to $\lambda = \frac{12}{60}\frac{butterflies}{seconds} = 0.2\frac{butterflies}{second}$.
- Plugging in, $f_X(x) = \frac{e^{-0.2t}(0.2t)^x}{x!}$ for $x = 0,1,2,\ldots$ ($f(x) = 0$ otherwise).

b. What is the probability Ava watches exactly 8 butterflies in the next minute? For full credit, please clearly define the random variable used. **(2 marks)**

- From part (a), we now just set $t = 60$. Note $\lambda t = 12$.
- X is the number of butterflies Ava sees in the next minute (60 seconds)
- We find $P(X = 8) = f(8) = \frac{e^{-12}12^8}{8!} \approx 0.0655$.

c. What is the probability Ava watches at most 2 butterflies in the next 30 seconds? For full credit, please clearly define the random variable used. **(2 marks)**

- From part (a), we now just set $t = 30$. Note $\lambda t = 6$.
- X is the number of butterflies Ava sees in the next 30 seconds.
- Since we want 'at most', we need $P(X \leq 2) = F(2) = f(0) + f(1) + f(2)$. Plugging in the values, $P(X \leq 2) \approx 0.00248 + 0.0149 + 0.0446 \approx 0.062$

d. Consider the Poisson model describing the number of butterflies Ava watches in t seconds, from part (a). What is the minimum amount time necessary (in seconds, to 3 decimal places) for Ava to have at least a 50% chance of watching at least one butterfly? **(2 marks)**

- From part (a), $f_X(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ for $x = 0,1,2,\ldots$.
- We want $P(X \geq 1) \geq 0.5$. This can be written $1 - P(X = 0) \geq 0.5$. We can simplify and write $1 - e^{-\lambda t} \geq 0.5$.
- After manipulation, we see $t > \frac{\log\left(\frac{1}{2}\right)}{-\lambda} \approx 3.466\ seconds$.

Notation and Parameters	**Probability Function** $f(x)$
Discrete Uniform(a, b) $b \geq a$ a, b integers	$\dfrac{1}{b-a+1}$ $x = a, a+1, \dots, b$
Hypergeometric(N, r, n) $N = 1, 2, \dots$ $n = 0, 1, \dots, N$ $r = 0, 1, \dots, N$	$\dfrac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = max(0, n-N+r), \dots, min(r, n)$
Binomial(n, p) $0 \leq p \leq 1, q = 1-p$ $n = 1,2, \dots$	$\binom{n}{x} p^x q^{n-x}$ $x = 0,1, \dots, n$
Bernoulli(p) $0 \leq p \leq 1, q = 1-p$	$p^x q^{1-x}$ $x = 0,1$
Negative Binomial(k, p) $0 < p \leq 1, q = 1-p$ $k = 1,2, \dots$	$\binom{x+k-1}{x} p^k q^x = \binom{-k}{x} p^k (-q)^x$ $x = 0,1, \dots$
Geometric(p) $0 < p \leq 1 , q = 1-p$	pq^x $x = 0,1, \dots$
Poisson(λ) $\lambda \geq 0$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1, \dots$

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