Stat 230: Probability

Lecture 26

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Review

Example

Suppose we know $X \sim N(3, 16)$. Find the following:

(1) $F(5)$

(2) The 45th percentile

Review

Last time we talked about:

(1) Joint probability

For today:

(1) Joint probability
(2) Multinomial distribution

Review

- Monday: Quiz 5
- Review

Conditional Probability

Definition

The **conditional probability function** *of X given $Y = y$ is denoted $f_X(x|y)$, and is defined to be*

$$f_X(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f(x, y)}{f_Y(y)}$$

$f_Y(y|x)$ is similarly defined.

Conditional Probability

Example

Suppose that X and Y have a joint probability function defined in the table below:

	f(x,y)	x = 0	x = 1	x = 2
y	0	.2	.3	.1
	2	.25	.13	.02

Find $f_Y(y|0) = f_Y(y|X=0)$.

Conditional Probability

Example

Justin is a sleepy secretary. He often sleeps at work until is woken up by a call. There are two types of phone calls: emergency calls, and non-emergency calls. Emergency calls arrive according to Poisson distribution with $\lambda = 1$ per 6 hours. Non-emergency calls also arrive according to Poisson distribution with $\lambda = 3$ per 6 hours, independently of emergency calls.

- What is the probability that Justin gets 2 emergency calls and 2 non-emergency calls over 6 hours of sleep.
- Calculate the probability that Justin receives 2 emergency calls given that he received a total of 3 call in 6 hours.

Multinomial Distribution

Definition

Consider an experiment in which:

(1) *Individual trials have k possible outcomes, and the probabilities of each individual outcome are denoted p_i, $1 \leq i \leq k$, so that $p_1 + p_2 + \cdots + p_k = 1$.*

(2) *Trials are independently repeated n times, with X_i denoting the number of times outcome i occurred, so that $X_1 + X_2 + \cdots + X_k = n$.*

In this case we say $X_1, ..., X_k$ have a **Multinomial Distribution** *with parameters n and $p_1, ..., p_k$.*

This is a generalization of the binomial distribution.

Multinomial Distribution

Example

In Roulette, a small ball is spun around a wheel in such a way so that the probability it lands in a black or red box is 18/38 each, and the probability it lands in a green box is 2/38. Suppose 10 games are independently played, and let B, R and G denote the number of times the ball landed on black, red, and green, respectively. Compute the probability that $R = 4, B = 4$ and $G = 2$.

Multinomial Distribution

Definition

If $X_1, ..., X_k$ have a joint multinomial distribution with parameters n and $p_1, ..., p_k$, then their joint probability function is

$$f(x_1, ..., x_n) = \frac{n!}{x_1!x_2! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k},$$

where $x_1, ..., x_k$ satisfy $x_1 + \cdots + x_k = n$, $x_i \geq 0$.

Abbreviation: $(X_1, ..., X_k) \sim Mult(n, p_1, ..., p_k)$

Remark

The terms

$$\frac{n!}{x_1!x_2! \cdots x_k!}, \quad x_1 + \cdots + x_k = n,$$

are called the multinomial coefficients.

Multinomial Distribution

Remark (Properties of the multinomial distribution)

(1) *If* $(X_1, ..., X_k) \sim Mult(n, p_1, ..., p_n)$*, then* $X_i \sim Bin(n, p_i)$

(2) $X_i = n - \sum_{j \neq i} X_j$*, so the* X_i *are dependent.*

(3) *Suppose* $(X_1, ..., X_k) \sim Mult(n, p_1, ..., p_n)$*. Then*

$$(X_i | X_j = k) \sim Bin\left(n - k, \frac{p_i}{1 - p_j}\right)$$

Multinomial Distribution

Example

In the game of Roulette, a small ball is spun around a wheel in such a way so that the probability it lands in a black or red box is $18/38$ each, and the probability it lands in a green box is $2/38$. Suppose 10 games are played, and let B, R and G denote the number of times the ball landed on black, red, and green, respectively. Given that $B = 5$ calculate the probability that $R = 5$.

Example

Consider drawing 5 cards from a standard 52 card deck of playing cards **with replacement**. What is the probability that 2 of the drawn cards are hearts, 2 are spades, and 1 is a diamond.

Functions

Suppose that

$$h : \mathbb{R}^2 \to \mathbb{R}$$

Then, for jointly distributed random variables X and Y, $U = h(X, Y)$ is a random variable. If X and Y have joint p.f. $f(x, y)$, then the probability function of U is given by:

$$f_U(t) = P(U = t) = \sum_{(x,y):\ h(x,y)=t} f(x, y)$$

Functions

Example

Suppose X and Y have joint probability function given by the following table:

	f(x,y)	x = 0	x = 1	x = 2
y	0	.2	.3	.1
	2	.25	.13	.02

Let $U = X + Y$. Compute the probability function of U.