Counting Techniques: Chapter 3

Chapter Outcomes:

- Addition and multiplication rules
- Counting arrangements and permutations
- Counting subsets and combinations
- Number of arrangements when symbols are repeated

- Recall that if we specify a sample space

$S = \{a_1, a_2, \ldots, a_n\}$, where each simple event has probability $1/n$ (i.e. are equally likely), then:

- defining the compound event A which contains r points leads to $P(A) = r/n$.

- To define more complicated probabilities, we first need to review some basic ways to count outcomes from “experiments”.

Basic counting arguments:

1. Addition Rule:

Suppose Job 1 can be done in p ways and Job 2 can be done in q ways.

Then we can do EITHER Job 1 **OR** Job 2 in $p+q$ ways.

- "OR": interpreted as **addition**

2. The Multiplication Rule:

Suppose Job 1 can be done in *p* ways and, **for each of these ways**, Job 2 can be done *q* ways.

Then we can do BOTH Job 1 **AND** Job 2 in *pxq* ways.

- "AND": interpreted as **multiplication**

Sampling “With” and “Without” Replacement

- “With” replacement:

This means that every time an object is selected, it is put back into the pool of possible objects.

Example: Rolling a die, flipping a coin,……

- This method of selection implies that *what we get on the first selection* ***DOES NOT affect*** *what we get on the second selection*. More on this idea later in the course.

- “Without” replacement:

This means that every time an object is selected, it is NOT put back into the pool of possible objects.

- This implies that the same object cannot be selected more than once.

- Hence, what we get on the *first selection* ***WILL affect*** *what we get on the second, and subsequent selections.*

Example: A bag contains 3 blue and 5 red marbles.

a) What is the probability of selecting two blue marbles if the selection process is done ***with*** replacement?

b) What is the probability of selecting two blue marbles if the selection process is done ***without*** replacement?

Example: Suppose a fair die is tossed 3 times. What is the probability that only one of the tosses produced a number greater than 4?

- In many problems, the sample space of interest consists of a set of unique arrangements or sequences (order matters and selection is done without replacement). Often these are called ***permutations***.

Example: How many **different** ordered arrangements of the letters a, b, and c are possible if letters are randomly selected without replacement?

You try:

What if the letters were randomly selected with replacement?

In this case, there would be a total of $3^3 = 27$ arrangements.

Do you see how with / without replacement matters!

One can employ the following counting principles:

Suppose we have *n* **distinct** symbols, so that we can make:

- $n \times (n-1) \times \cdots \times 1 = \boldsymbol{n}!$ (read "*n* factorial") arrangements of **length *n***, where each symbol is used only once (i.e. the number of permutations of *n* distinct objects).

Note: $n! = n \times (n-1)!$ for $n \geq 1$ where $0! = 1$

Examples:

4! = 4 x 3 x 2 x 1 = 24

5! = 5 x 4 x 3 x 2 x 1 = 120

Note: We can write 5! as 5*4!, or n! as n*(n-1)!

- $n \times (n-1) \times \cdots \times (n-k+1) = \boldsymbol{n^{(k)}}$ for $k \leq n$ arrangements of **length** $\boldsymbol{k}$**,** where each symbol is used once (i.e. the number of permutations of *n* distinct objects taken *k* at a time).

$n^{(k)} = \frac{n!}{(n-k)!} = {}_nP_k$ (read "*n* to *k* factors")

where $n^{(k)} = \frac{n!}{(n-k)!} = n(n-1)^{(k-1)}$ for $k \geq 1$

- $n \times n \times \cdots \times n = \boldsymbol{n^k}$

arrangements of length k using each symbol more than once.

Note: The arrangements presented, especially *n*!, tend to grow at an extremely fast rate as *n* gets larger and so approximations can be used.

n	0	1	2	3	4	5	6	7	8	9	10
n!	1	1	2	6	24	120	720	5040	40320	362880	3628800

Stirling's Approximation:

- For large *n*, this is an approximation to *n*! and it states:

$$\boldsymbol{n!} \textbf{ is } \textbf{\textit{asymptotically equivalent}} \textbf{ to } \left(\frac{\boldsymbol{n}}{\boldsymbol{e}}\right)^{\boldsymbol{n}} \sqrt{\boldsymbol{2\pi n}}$$

- What do we mean by *asymptotically equivalent*?

This implies that as $n \to \infty$, the approximation becomes better and better (i.e. $\lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{n!} = 1$).

In other words, the *percentage error* of the approximation approaches zero as *n* increases.

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$	0.9	1.9	5.8	23.5	118.0	710.1	4980.4	39902.4	359536.9	3598695.6

- **Example:** A PIN number of length 4 is formed by randomly selecting 4 digits from the set of digits {0, 1, 2, …, 9}. If selection is done with replacement, find the probability that:

a) the PIN number is even?

b) the PIN number contains at least one 1?

You Try:

- What happens if selection is done without replacement? Find the probability that:

a) the PIN number is even?

There are 5 even numbers to place in the last slot. Then the other numbers can be placed, so P(even)

$$= \frac{5*9*8*7}{10*9*8*7} = \frac{5*9^{(3)}}{10^{(4)}} = \frac{1}{2}$$

b) the PIN number contains **at least one 1**?

P(at least one 1) = 1 – P(no 1's)

P(at least one 1) = 1 - $\left[\frac{9*8*7*6}{10*9*8*7}\right]$**=1 -** $\frac{9^{(4)}}{10^{(4)}} = \frac{2}{5}$

- **Example:** Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a pool of size 30. How many different outcomes are possible if:

a) A student can receive any number of awards?

b) each student can receive at most one award?

c) Suppose that I have 2 nephews and 1 niece in this pool of 30 students. What is the probability that each of my 3 relatives wins exactly one award under assumption (a)?

d) What is this same probability under assumption (b)?

You try:

a) In how many ways can 3 boys and 3 girls sit in a row?

With no restrictions, we know that this is 6! = 720

b) In how many ways can 3 boys and 3 girls be seated if the boys and the girls are each to sit together?

Now, the boys and girls need to be seated together (i.e. BBBGGG or GGGBBB). The # of ways = 2!*3!*3! = 72

c) In how many ways can 3 boys and 3 girls be seated if only the boys must sit together?

Now, the B's are together. Treat them like their own block, and move them "down the line". So, there are 4 slots, and there are 3! ways to arrange the boys. So, the # of ways = 4!*3! = 144

d) In how many ways can 3 boys and 3 girls be seated if no boys and no girls are allowed to sit together?

This is the same as alternating them, so we have: BGBGBG or GBGBGB. So, the # of ways is 2!*3!*3! = 72

- Sometimes the outcomes in the sample space are subsets of a **fixed size**:
- **Example:** Suppose we randomly select a subset of 3 digits from {0,1,2,…,9}, giving rise to

$$S=\{\{0,1,2\},\{0,1,3\},\ldots,\{7,8,9\}\}.$$

- We are now interested in finding the **number of subsets of size *k*** that can be formed from a **set of *n* distinct objects** when **order does not matter** and selection is done without replacement.

- Let us start with a slightly simpler example:
- Suppose we have **three books**: b_1, b_2, b_3.
- We choose **two** of the books to read.
- In how many ways can the two books be read if:

1. The order the books are read matters:

2. The order the books are read does not matter:

- Now, let's return to our earlier example and adopt a similar strategy to find the total number of outcomes in *S*.

- **Recall:** Suppose we randomly select a subset of 3 digits from {0,1,2,…,9} when order does not matter.

- The previous results can be generalized as follows:
- The number of subsets of size *k* that can be chosen from a set of *n* distinct objects is given by:

$$ {}_{n}C_{k} = \binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{k!(n-k)!} $$

- $\binom{n}{k}$ is read as "*n* choose *k*" and represents the *combinatorial symbol*.

- Now, let’s use this idea to calculate some probabilities. Continuing with the previous example:
- Recall that we randomly select a subset of 3 digits from the set {0,1,2,…,9}.

- We found that 120 subsets of 3 digits can be formed. Therefore, if we randomly select a subset, then each subset has the same probability of selection, namely 1/120.

Example:

Find the probability that:

a) All the digits in the selected subset are even.

b) At least one of the digits in the selected subset is less than or equal to 5.

Example: A forest contains 30 moose of which six were captured, tagged, and released. A certain time later, five of the 30 moose are randomly captured.

a) How many samples of size five are possible?

b) How many samples of size five, which include two of the six originally tagged moose, are possible?

c) If the five captured moose represent a simple random sample drawn from the 30 moose (six of which are tagged), find the probability that (i) two of the five captured moose are tagged and (ii) none of the five captured moose is tagged.

- Properties of $\binom{n}{k}$:

1. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{(k)}}{k!}$

2. **Symmetry property:** $\binom{n}{k} = \binom{n}{n-k}$ for all $k = 0,1,\dots,n$ (i.e. $\binom{n}{k}$ is equal to the k^{th} entry in the n^{th} row of *Pascal's Triangle*.)

Pascal's Triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

3. When n is not a non-negative integer $\geq k$, $\binom{n}{k}$ loses its physical meaning. If n is a real number and k is a non-negative integer, then we use

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

Note also that when n and k are non-negative integers and $k > n$, then

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n(n-1)\ldots(1)(0)\ldots(n-k+1)}{k!} = 0$$

4. Recalling that 0! = 1, then

$$\binom{n}{0} = \binom{n}{n} = 1$$

Mathematically: $\binom{n}{0} = \binom{n}{n} = \frac{n!}{0!n!} = 1$

Logically: There is only one way to choose either none of the objects or all of the objects.

5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

6. $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,\ x\ \varepsilon\ \mathbb{R}$

(This is the well-known ***Binomial Theorem***)

Next... How do we determine the number of *ordered* arrangements when symbols are repeated (i.e. they are <u>NOT</u> all distinct)?

Example: Suppose the letters of the word STATISTICS are arranged at random. Find the probability of the event G that the arrangement begins and ends with S.

The results of this STATISTICS example can be generalized in the following way:

- In general, if we have n_i symbols of type i, for $i = 1, 2, \dots, k$ with $n_1 + n_2 + \cdots + n_k = n$, then the number of distinct (i.e., ordered) arrangements using ALL of the symbols is:

$$\binom{\boldsymbol{n}}{\boldsymbol{n_1}} \times \binom{\boldsymbol{n - n_1}}{\boldsymbol{n_2}} \times \binom{\boldsymbol{n - n_1 - n_2}}{\boldsymbol{n_3}} \times \cdots \times \binom{\boldsymbol{n_k}}{\boldsymbol{n_k}}$$

$$= \frac{\boldsymbol{n}!}{\boldsymbol{n_1}!\,\boldsymbol{n_2}!\,\boldsymbol{n_3}! \dots \boldsymbol{n_k}!} = \binom{\boldsymbol{n}}{\boldsymbol{n_1}\ \ \boldsymbol{n_2}\ \dots\ \boldsymbol{n_k}}$$

Example: Find the probability a bridge hand (13 cards randomly dealt from a standard deck of 52 cards without replacement) has:

a) at least 1 ace.

b) 6 spades, 4 hearts, 2 diamonds, and 1 club.

c) a 6-4-2-1 split between the 4 suits.

d) a 4-3-3-3 split between the 4 suits.

You Try:

- If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

From a group of 12 people, we need to form 3 committees of sizes 3, 4, and 5.

The # of committees is:

$$\binom{12}{3}\binom{9}{4}\binom{5}{5} = \frac{12!}{3!4!5!} = 27,720$$

Example: A person has 8 friends, of whom 5 will be invited to a party.

a) How many choices are there if 2 friends are feuding and will not attend together?

b) How many choices are there if 2 of the friends will only attend together?

Example: There are 5 blue beads and 4 green beads to be arranged in a row on a string. The two ends of the string are not connected. Beads with the same colour are indistinguishable. Find the probability of the following events:

a) A = "All 5 blue beads are adjacent to each other"

b) B = “None of the green beads is adjacent to any other green beads”

A Fun Permutation Application:

The Birthday Game

What is the probability that there will be at least one matched birthday in this class?

A birthday is matched if at least two people share the same birthday.

Let’s assume that there are 160 people enrolled in this class.

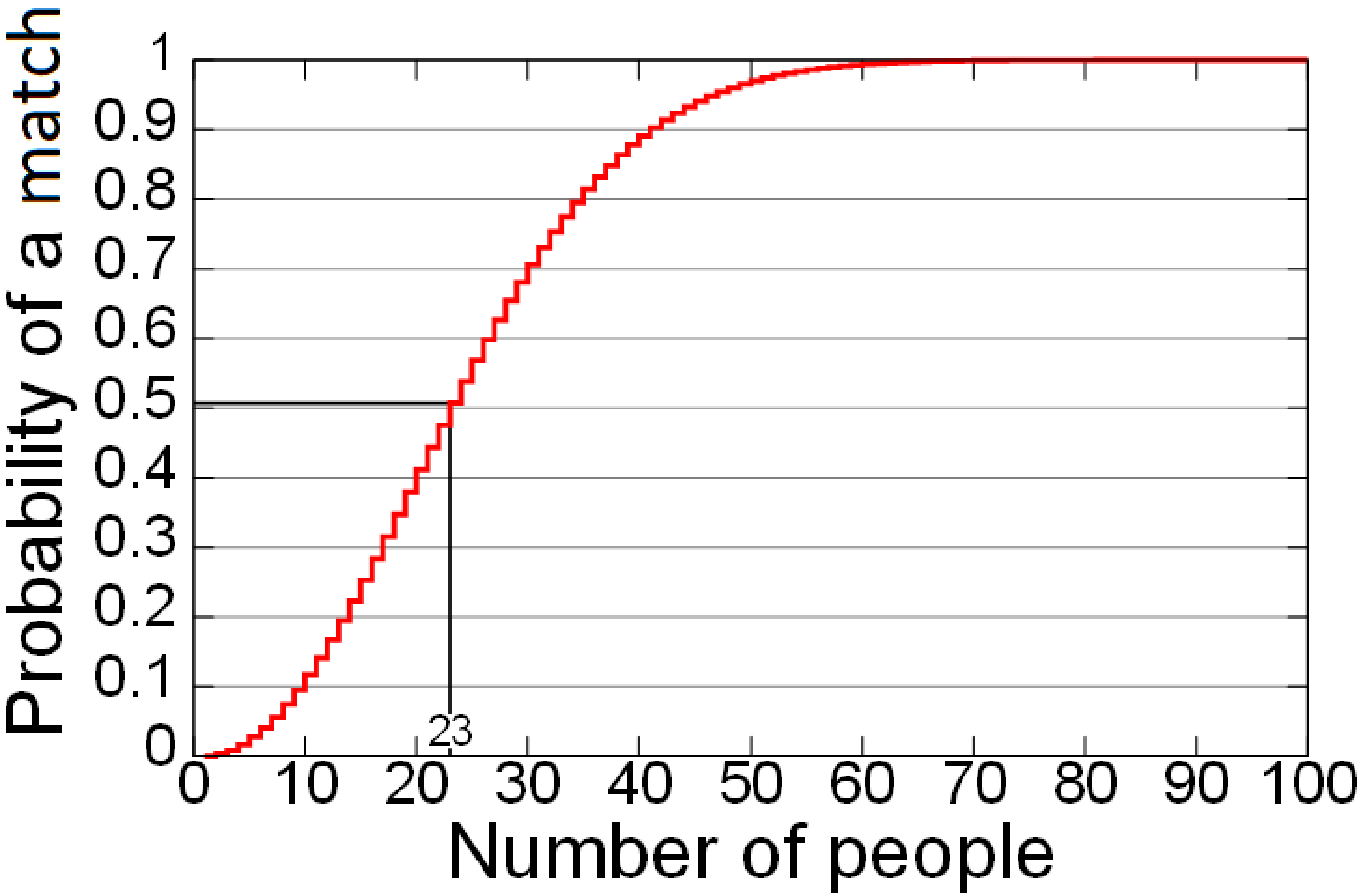
What do you think the probability will be? Let’s hear some guesses.

A Fun Permutation Application:

The Birthday Game

How many participants are needed for the probability of at least one matched birthday to be about 50%?

Guesses?


Probability of a match
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
23
0
10
20
30
40
50
60
70
80
90
100
Number of people