STAT 230: Probability (Sec 02) Spring 2022

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Lecture 5

Today's Agenda

Last time:

Counting techniques (factorials, binomial coefficient)

Today (Lec 5, 05/11):

More on counting techniques

Definition

Given n distinct objects, a **permutation** of size k is an *ordered* subset of k of the individuals. The number of permutations of size k taken from n objects is denoted $n^{(k)}$ and

$$n^{(k)} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

Definition

Given n distinct objects, a **combination** of size k is an *unordered* subset of k of the individuals. The number of combinations of size k taken from n objects is denoted $\binom{n}{k}$ and

$$\binom{n}{k} = \frac{n!}{k!} = \frac{n!}{(n-k)!k!}.$$

The number $\binom{n}{k}$ is called **Binomial coefficient**.

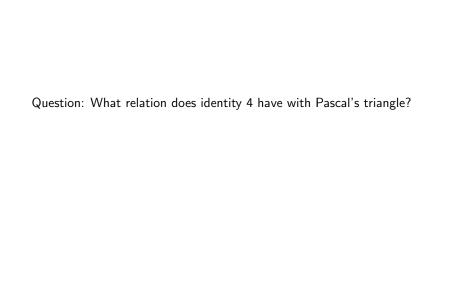
Example

Suppose you have 20 distinct books, 7 of which are written by Mark Twain.

- a) How many ways can you arrange 12 books on a shelf if the order they are on the shelf matters?
- b) How many ways can you arrange 12 books on a shelf if exactly 3 of them must be Mark Twain books?
- c) A monkey picks books at random from the 20 books and puts them on the shelf until it contains 12 books. What is the probability that at least 3 of the books on the shelf are written by Mark Twain?

There are some useful/important results about permutation and combination.

- a) $n^{(k)} = n(n-1)^{(k-1)}$ for k > 1
- b) $\binom{n}{k} = \frac{n^{(k)}}{k!}$
- c) $\binom{n}{k} = \binom{n}{n-k}$ for $k \ge 0$
- d) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- e) Binomial theorem: $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$
- f) $\binom{n}{k}$ is equal to the kth entry in the nth row of Pascal's triangle.



Let x denote the number of ways that 3 numbers can be randomly selected from the numbers $\{0,1,2,3,4,5\}$ WITH replacement, and let y denote the number of ways that 3 numbers can be drawn from the same set of numbers WITHOUT replacement. Which of the following is true:

- a) x > y
- b) y > x
- c) x = y

Example (The Birthday Problem)

Suppose a room contains n people. What is the probability at least two people in the room share a birthday?

Assumption: Suppose that each of the n people is equally likely to have any of the 365 days of the year as their birthday, so that all possible combinations of birthdays are equally likely.

Let A be the event that at least two people share a birthday. Then

$$P(A) = 1 - P(\bar{A}),$$

where \bar{A} is the event that nobody shares birthdays with each other.

For n people to have unique birthdays, we need to arrange them among 365 days w/o replacement. Thus,

$$|\bar{A}| = 365^{(n)}$$
.

For the size of the sample space, we see that each person has 365 possibilities for their birthday. Thus,

$$|S| = 365^n$$
.

Since we are assuming that all possible combinations of birthdays are equally likely, our desired probability becomes

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{365^{(n)}}{365^n} = 1 - \frac{n!\binom{365}{n}}{365^n}.$$

For $n \in \{100, 30, 23\}$ we find

$$P(A_{100}) = .99999997, P(A_{30}) = .7063 P(A_{23}) = .5073$$

Example

Consider rearranging the letters at random in the word "HELLOKITTY" to form a single word.

- a) How many ways can this be done?
- b) What is the probability that all of the letters appear in alphabetic order?
- c) What is the probability that the word begins and ends with "T"?

Definition

Consider n objects which consist of k types. Suppose that there are n_1 objects which are of type 1, n_2 which are of type 2, and in general n_i objects of type i. Then there are

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

distinguishable arrangements of the n objects. This quantity is known as a **multinomial coefficient** and denoted by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}.$$

Example

Suppose we make a random arrangement of length three of the letters $\{a, b, c, d, e, f, g, h, i, j\}$. What is the probability of the event B = "letters are in alphabetic order" if

- a) letters are selected without replacement?
- b) letters are selected with replacement?