Instructions

Marking Scheme:

Questions	Out of
1 – 9	9
10	8
11	9
Total	26

1. **All electronic devices are to be turned off and put away.**
2. **This is a 50-minute closed-book quiz.**
3. **Answer the questions in the spaces provided.**
4. **Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form unless otherwise specified**
5. **Only question pages will be marked.**
6. **You may tear off the last two pages and use them for rough work. All pages will be collected at the end of the quiz.**
7. **Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed. No other aids are allowed.**
8. **A reference sheet has been included with this quiz, and can be found at the end of the quiz. You may CAREFULLY detach the rough work pages and reference sheet if you like.**

DO NOT WRITE ON THIS COVER PAGE.

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Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Let X and Y be discrete random variables with a joint probability function given by:

$$f(x, y) = \frac{x+y}{k}, \text{ for } x = 0, 1, 2;\ \ y = 5, 6\ \ ; 0, \text{ otherwise}$$

What is the value of k that makes this a valid joint pf?

A) 28 B) 40 C) 35 **D) 39**

- **In order for this to be a valid joint pf, the probabilities must sum to equal 1.**
- **So, we have:**
- $\frac{(0+5)+(0+6)+(1+5)+(1+6)+(2+5)+(2+6)}{k} = 1 \Rightarrow \frac{5+6+6+7+7+8}{k} = 1.$
- **So** $\frac{39}{k} = 1$. **So, k = 39**

2. You are given the following joint probability function, f(x, y), for the random variables X and Y in the table below:

y

x

f(x,y)	**1**	**2**	**3**
0	0.1	0.2	0.1
2	0.1	0.1	0.1
4	0.2	0	0.1

What is P(X < Y)?

A) 0.3 B) 0.4 **C) 0.5** D) 0.6

- **To determine this probability, we need to consider all instances.**
- **P(X < Y) = f(0, 1) + f(0, 2) + f(0, 3) + f(2, 3) = 0.1+0.2+0.1+0.1 = 0.5**
- **Of course, you could have also found this probability by looking at the complement.**
- **P(X < Y) = 1 – P(X >= Y) = 1 – [0.1 + 0.1 + 0.2 + 0 + 0.1] = 1 – 0.5 = 0.5**
- **Same answer, as expected.**

3. You are given the following joint probability function, f(x, y), for the random variables X and Y in the table below:

y

x

f(x,y)	**1**	**2**	**3**
1	0	0	0.2
2	0.1	0.2	0.1
3	0.2	0.1	0.1

What is $f_X(2)$?

A) 0.2 B) 0.3 **C) 0.4** D) 0.5

- **To determine this probability, we need to sum across all values of y when x = 2.**
- **This gives us** $f_X(2) = f(2,1) + f(2,2) + f(3,3) = 0.1 + 0.2 + 0.1 = 0.4$

4. You are given the following joint probability function, f(x, y), for the random variables X and Y in the table below:

y

x

f(x,y)	**1**	**2**	**3**
1	0.1	0.2	0.1
2	0.2	0	0.1
3	0.2	0	0.1

Calculate P(X = 2| Y = 1).

A) 1/2 B) 2/3 C) 1/3 **D) 2/5**

- **To determine this probability, we need to set up the appropriate calculation.**
- **By definition, P(X = 2 | Y = 1) =** $\frac{P(X=2,Y=1)}{P(Y=1)} = \frac{f(2,1)}{f(1,1)+f(2,1)+f(3,1)} = \frac{0.2}{0.1+0.2+0.2} = \frac{2}{5}$

5. Suppose that a box contains 6 red, 2 green, and 4 white balls. Two balls are randomly selected without replacement. Let X represent the number of red balls selected, and let Y represent the number of green balls selected. What is P(X = 1, Y = 1).

A) 2/11 B) 1/6 C) 1/12 D) 1/11

- **The joint pf for X and Y is given by:**
- **P(X = x, Y = y) = f(x, y) =** $\frac{\binom{6}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{8+4}{2}}$ **, for x ≥ 0, y ≥ 0, x + y ≤ 2 ; 0, otherwise**
- **We want P(X = 1, Y = 1) =** $\frac{6*2*1}{\frac{(8+4)(7+4)}{2}} = \frac{24}{(8+4)(7+4)} = \frac{2}{11}$

6. Suppose that two fair dice are rolled at once. Let X represent the sum on the two dice. Let Y represent the number of prime numbers rolled on the two dice. The joint probability function for X and Y is denoted by f(x, y). Determine f(7,2).

A) 1/36 B) 1/6 C) 1/3 **D) 1/18**

- **If X = 7, the possibilities for the two dice are:**
- **{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}.**
- **In two of these rolls, Y = 2. So, f(7,2) = 2/36 = 1/18.**

7. A message sent by a transmitter has probability equal to 0.95 of successful transmission on any attempt. Assume that transmission attempts are independent, and that a message will be repeatedly sent until transmission is successful.

Suppose that 18 messages have been successfully transmitted. Which of the following represents the probability that, of the 18 messages, 10 were transmitted successfully on the first attempt, 5 required two attempts, and 3 required 3 or more attempts?

A) $\mathbf{\frac{18!}{10!5!3!}(0.95)^{10}(0.0475)^5(0.0025)^3}$

B) $\frac{18!}{10!5!3!}\left(\frac{5}{9}\right)^{10}\left(\frac{5}{18}\right)^5\left(\frac{1}{6}\right)^3$

C) $\frac{10!5!3!}{18!}\left(\frac{5}{9}\right)^{10}\left(\frac{5}{18}\right)^5\left(\frac{1}{6}\right)^3$

D) $\frac{10!5!3!}{18!}(0.95)^{10}(0.0475)^5(0.0025)^3$

- **If x attempts are needed to obtain a successful transmission, then using the Geometric distribution gives us:**

- **P(X=x) = f(x) = P(x attempts are needed) = $(0.05)^{x-1}(0.95)$, for x = 1, 2, 3, ….. ; 0, otherwise**
- **This leads to f(1) = 0.95, f(2) = 0.0475, and f(3). So, P(X ≥ 3) = 1 – [f(1) + f(2)] = 0.0025.**
- **With these probabilities, we then apply the multinomial theorem and get:**
- $\frac{18!}{10!5!3!}(0.95)^{10}(0.0475)^{5}(0.0025)^{3}$

8. Eleven voters are selected randomly from a very large population consisting of 55% Tories, 15% Whigs, and 30% Liberal-Democrats. Let X represent the number of Tories selected in the sample. Let Y represent the number of Whigs selected in the sample. What is Var(11 – X - Y)?
Hint: Let Z = 11 – X – Y.

A) 2.7225 **B) 2.31** C) 1.4025 D) 2.75

- **Let Z represent the number of Liberal-Democrats in the sample. Again, Z = 11 – X – Y.**
- **Z has a Binomial distribution with n = 11 and p = 0.3.**
- **So, Var(Z) = Var(11 – X – Y) = n*p*(1 – p) = 11*0.3*0.7 = 2.31.**

9. A group of 50 people from a large city were selected randomly and asked about their smoking habits. They were classified as: *regular smoker*, *ex-smoker*, or someone who *never smoked*. The counts are given in the table below:

Smoker Type	*Regular Smoker*	*Ex-Smoker*	*Never Smoked*
Count	5	25	20

You can assume that one person's smoking type is independent of another. Suppose that 7 people are selected randomly (without replacement). We will now define our random variables. Let X_1 represent the number of regular smokers. Let X_2 represent the number of ex-smokers, and let X_3 represent the number of those who never smoked in the sample.

What is $P(X_1 = 1, X_2 = 4, X_3 = 2)$?

A) 12.03% B) 0.1% C) 0.32% D) 10.5%

- **We are sampling without replacement.**
- **There are $\binom{50}{7}$ ways to select the sample of 7 from the 50 people.**
- **The number of samples with $X_1 = 1, X_2 = 4$, and $X_3 = 2$ is $\binom{5}{1}\binom{25}{4}\binom{20}{2}$.**
- **So, $P(X_1 = 1, X_2 = 4, X_3 = 2) = \frac{\binom{5}{1}\binom{25}{4}\binom{20}{2}}{\binom{50}{7}} = 0.1203141$**

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work. Unjustified answers will not receive full credit.

10. Suppose that the discrete random variables X and Y have joint probability function f(x,y) given in the following table, where k is a positive constant. ***(8 marks)***

y

	f(x,y)	**0**	**1**	**2**	**3**
x	**1**	0.2	0.15	k	0.3
	2	0.05	0.1	0	0.05

a. What is the value of k so that f(x, y) is a valid joint probability function? ***(2 marks)***
- The sum over the entire table must be 1, so
  1 = 0.85 + k
- Thus, k = 0.15 makes f(x,y) a valid joint probability function

b. Compute P(X > Y). ***(2 marks)***

We need to sum over all f(x,y) over all (x,y) that satisfy x>y.
P(X>Y) = f(1, 0) + f(2, 0) + f(2, 1) = 0.2 + 0.05 + 0.1=0.35

c. Determine the conditional probability function of X given that Y = 1. Please organize your results in table form. ***(2 marks)***

- By definition, f(x | Y=1) = f(x, 1) / P(Y=1)
- We find P(Y=1) by summing the corresponding column as
  P(Y=1) = f(1, 1) + f(2, 1) = 0.15 + 0.1 = 0.25
- Thus,
  f(x | Y = 1) = 4 f(x, 1) for x=1,2

x	1	2
f(x | Y = 1)	0.6	0.4

d. Let D = Y/X. Determine the probability function of D. Please organize your results in table form. ***(2 marks)***

The random variable Y/X can take the values 0, 0.5, 1, 1.5, 2 and 3.
By summing f(x,y) we find

d	0	0.5	1	1.5	2	3
P(D=d)	0.25	0.1	0.15	0.05	0.15	0.3

11. Suppose three fair coins are tossed and the total number of heads obtained is recorded. This experiment is repeated ten times independently. ***(9 marks)***

a. The results of the experiment have been organized in the frequency table below:

Number of heads	0	1	2	3	**Total**
Frequency observed	1	3	4	2	10

What is the probability of observing the results in the table (to 3 d.p.)? Please clearly define the random variables. When answering this question, please ensure that you clearly define the probability function that will be used to answer this question, along with all associated parameter values, and the range of the random variables, along with any constraints / restrictions. ***(5 marks)***

- Let A_i be the event that i heads occur when tossing three coins for $i = 0,1,2,3$ and $p_i = P(A_i)$ where clearly $\sum_{i=0}^{3} p_i = 1$
- Let X_i be the number of times A_i occurs in $n = 10$ independent trials for $i = 0,1,2,3$
- Then $(X_0, X_1, X_2, X_3) \sim \text{Multinomial}(n, p_0, p_1, p_2, p_3)$ and we want to compute $P(X_0 = 1, X_1 = 3, X_2 = 4, X_3 = 2)$
- Using first principles, $p_0 = (1/2)^3 = 1/8, p_1 = 3(1/2)^3 = 3/8, p_2 = 3(1/2)^3 = 3/8$ and $p_3 = (1/2)^3 = 1/8$
- Thus, (X_0, X_1, X_2, X_3) has joint probability function

$$f(x_0, x_1, x_2, x_3) = \frac{10!}{x_0!\,x_1!\,x_2!\,x_3!}\left(\frac{1}{8}\right)^{x_0}\left(\frac{3}{8}\right)^{x_1}\left(\frac{3}{8}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}$$

where $x_i = 0,1,\dots,10$ for $i = 0,1,2,3$ and $\sum_{i=0}^{3} x_i = 10$

- Plugging in $x_0 = 1, x_1 = 3, x_2 = 4, x_3 = 2$ gives

$$P(X_0 = 1, X_1 = 3, X_2 = 4, X_3 = 2) = f(1,3,4,2) = 0.0257$$

b. Let Y represent the number of times you observed 1 or 3 heads. Clearly state the distribution of Y, with all necessary parameters, and determine the probability function of Y, including the range of values of Y. ***(2 marks)***

- Since we are independently repeating an experiment with constant success probability, the random variable Y satisfies $Y \sim Binomial(10, p)$
- In this case, p is the probability of "1 or 3 heads", i.e., p= $1/8 + 3/8 = 1/2$
- The probability function is then

$$f_Y(y) = \binom{10}{y}\left(\frac{1}{2}\right)^{10}, \quad y = 0,1,\dots,10$$

c. What is Var(Y)? ***(2 marks)***

- From b. $Y \sim Binomial(n = 10, p = 0.5)$ *so that* $Var(Y) = 10 \cdot 1/2 \cdot 1/2 = 2.5$

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It will not be graded, but MUST be handed in.

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