Stat 230: Probability

Lecture 12

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Example

Suppose a tack when flipped has probability 0.7 of landing point up. If the tack is flipped 6 times, what is the probability it lands point up more than twice?

Review

Last time we talked about:

(1) Common distributions
   - Binomial Distribution
   - Negative Binomial Distribution

For today:

(1) Common distributions
   - Negative Binomial Distribution
   - Geometric Distribution
   - Poisson Distribution

Review

- Reminder: Exam Tomorrow
- TA review tutorial today

Negative Binomial

Example (Negative Binomial Random Variables)

(1) Flip a fair coin until 5 heads are observed, and let X denote the number of tails observed. Then

$$X \sim NB(5, 0.5)$$

(2) Roll a fair die until the number 6 has appeared three times. Let Y denote the number of rolls required beyond 3.

$$Y - 3 \sim NB(3, 1/6)$$

Negative Binomial

Example

Compute the probability function of a Negative Binomial random variable with parameters k and p.

Negative Binomial

Example

Suppose names are drawn with replacement out of a hat containing 100 names, of which 12 are the name "Kyle" until the name "Kyle" is drawn three times.

(1) Give the distribution of the number of draws required

Suppose 17 names are drawn with replacement out of a hat containing 100 names, of which 12 are the name "Kyle".

(2) Give the distribution of the number of names drawn that are not Kyle

Geometric

The Geometric distribution is a special case of the Negative Binomial:

Definition

We say the random variable X has a **geometric** *distribution with parameter p if $X \sim NB(1, p)$ (abbr. $X \sim Geo(p)$). In this case, the p.f. of X is*

$$f_X(x) = (1-p)^x p, \quad x = 0, 1, 2, 3, \ldots$$

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} (1-p)^x p = \frac{p}{1-(1-p)} = 1.$$

Geometric

Example

Suppose a fair coin is to be flipped until it comes up heads. What is the probability it will take more than 5 flips?

Poisson

Definition

We say the random variable X has a **Poisson** *distribution with parameter λ if*

$$f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots.$$

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1.$$

Poisson

Figure: Siméon Denis Poisson, 1781-1840

Poisson

Example (Poisson approximation of Binomial)

Let $\lambda = np$. Then if n is large and p is close to zero,

$$\binom{n}{x} p^x (1-p)^{n-x} \approx e^{-\lambda} \frac{\lambda^x}{x!}$$

More precisely, one can show that if $n \to \infty$ and $p = p_n \to 0$ as $n \to \infty$ in such a way that $np_n \to \lambda$, then

$$\binom{n}{x} p^x (1-p)^{n-x} \to e^{-\lambda} \frac{\lambda^x}{x!}, \quad as \quad n \to \infty$$

Poisson

Example

A bit error occurs for a given data transmission method independently in one out of every 1000 bits transferred. Suppose a 64 bit message is sent using the transmission system. What is the probability that there are exactly 2 bit errors? Approximate this using a Poisson approximation.

Example (Poisson process (counting events))

Consider counting the number of occurrences of an event that happens at random points in time (or space). Examples include,

(1) Counting emissions of radioactive particles from a radioactive substance

(2) Hits on a web site during a given time period

(3) Counting how many pokemon you encounter in an hour

Poisson Process

Suppose that the events you are counting satisfy the following assumptions:

(1) **Independence:** the number of occurrences in non-overlapping intervals are independent.

(2) **Individuality:** for sufficiently short time periods of length Δt, the probability of 2 or more events occurring in the interval is close to zero

$$\frac{P\left(2 \text{ or more events in } (t, t+\Delta_t)\right)}{\Delta_t} \to 0, \quad \Delta_t \to 0$$

(3) **Homogeneity or Uniformity:** events occur at a uniform or homogeneous rate λ, i.e.

$$\frac{P\left(\text{one event in } (t, t+\Delta_t)\right) - \lambda\Delta_t}{\Delta_t} \to 0.$$

Poisson Process

Let X denote the number of events observed up to time t. Then, assuming the previous conditions on the occurrence of events,

$$X \sim Pois(\lambda t)$$

Definition

A process that satisfies the prior conditions on the occurrence of events is often called a **Poisson process***. More precisely, if $X_t,\ t \geq 0$ denotes the number of events that have occurred up to time t, then X_t is called a Poisson process.*

Poisson Process

Example

Shiny versions of Pokemon are possible to encounter and catch starting in Generation 2 (Pokemon Gold/Silver). Normal encounters with Pokemon while running in grass occur according to a Poisson process with rate 1 per minute on average. 1 in every 8192 encounters will be a Shiny Pokemon, on average.

(1) If you run around in grass for 15 hours, what is the probability you will encounter at least one Shiny pokemon?

(2) How long would you have to run around in grass so that you have a better than 50 percent chance of encountering at least one Shiny pokemon?