Stat 230: Probability

Lecture 32

Jeremy VanderDoes

University of Waterloo

Friday, July 22nd

Review

Example

Suppose you own a store and try to minimize wait time for customers. Your store manager claims the average wait time for customers is 2 minutes. Just as a rough check, you observed a random sample of 36 customers, noting the average wait time for these is 3.1 minutes. Should you believe the manager?

Review

Last time we talked about:

(1) Central Limit Theorem

For today:

(1) Moment generating function

Review

- Monday: TA Review

Moment Generating Function

Definition

The **Moment generating function** *or MGF of a random variable X is given by*

$$M_X(t) = E(e^{tX}), \quad t \in \mathbb{R}$$

In particular, if X is discrete with p.f. $f(x)$ then

$$M_X(t) = \sum_{x \in X(S)} e^{tx} f(x), \quad t \in \mathbb{R}$$

If X is continuous with density $f(x)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx, \quad t \in \mathbb{R}$$

Moment Generating Function

Remark (Properties of the MGF)

The following properties hold for the MGF:

(1)

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j E(X^j)}{j!}$$

(2) *So long as $M_X(t)$ is defined in a neighborhood of the origin*

$$\frac{d}{dt^k} M_X(0) = E(X^k)$$

Moment Generating Function

Theorem (Continuity Theorem)

If X and Y have MGF's $M_X(t)$ and $M_Y(t)$ defined in neighborhoods of the origin, and satisfying $M_X(t) = M_Y(t)$ for all $t \in (-\varepsilon, \varepsilon)$, then

$$X \stackrel{D}{=} Y$$

Moment Generating Function

Remark (Famous Distributions)

The following are the MGF's of some famous distributions:

(1) *If* $X \sim N(\mu, \sigma^2)$, *then* $M_X(t) = e^{t\mu + t^2\sigma^2/2}$

(2) *If* $X \sim Pois(\lambda)$, *then* $M_X(t) = e^{\lambda(e^t - 1)}$

Moment Generating Function

Example

Suppose X has moment generating function given by $M(t) = 0.2 + 0.4e^t + 0.1e^{2t} + 0.3e^{3t}$. What is the probability $P(X < 2)$?

Moment Generating Function

Remark (Properties of MGF)

Other properties of the MGF include:

(1) *If* $Y = aX + b$, *then* $M_Y(t) = e^{bt} M_X(at)$

(2) *If* X *and* Y *are independent random variables, then*

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

Moment Generating Function

Example

If $X \sim Pois(\mu)$ and $Y \sim Pois(\lambda)$, and X and Y are independent, then prove $X + Y \sim Pois(\mu + \lambda)$ using moment generating functions.