STAT 230
SECTION 2
LECTURE 26

Today's Agenda

Last time:

- Exponential distribution as waiting time in a poisson process
- Mean and variance of the exponential distribution
- Memoryless property

Today (Lec 26, 07/04):

- Brief review exponential distribution
- Percentile
- Generating observations of random variables on a computer

Recall: Exponential distribution

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- λ is the rate parameter from Poisson process, and θ is the waiting time parameter derived from it.

Properties of exponential distribution:

- If $X \sim Exp(\theta)$, then $E[X] = \theta$, $Var(X) = \theta^2$.
- If $X \sim Exp(\lambda)$, then $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$.

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- Exponential distribution is the only memoryless continuous distribution:

$$P(X > s + t | X > s) = P(X > t).$$

Question

Which of the following is true about gamma function?

A $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$

B $\Gamma(\alpha) = (\alpha - 1)!$ for $\alpha \in \mathbb{N}$

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Solution: We know $F(x) = P(X \leq x) = \int_0^x e^{-t}\,\mathrm{d}t = 1 - e^{-x}$ for $x \geq 0$. We want w such that $P(X \leq w) = 0.5$, or

$$F(w) = 0.5 \Leftrightarrow 1 - e^{-w} = 0.5 \Leftrightarrow w = \log(2) \approx 0.693.$$

So with probability 50% Mukhtar won't have to wait longer than $\log(2)$. In other words, $\log(2)$ is the median of the distribution of X.

Definition

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- The **median** of a distribution is its 50% quantile.
- This is different from the **mode**, which is the most likely value, and different from the **mean** which is the expectation (long term "average").

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- Solving $F(c_q) = q$ for x where $q \in (0,1)$ gives

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- For $q = 0.5$ we get $c_{0.5} = -5\log(0.5) \approx 3.466$ as the median.

Question

Suppose X and Y are continuous random variables satisfying $P(X \leq x) < P(Y \leq x)$. Let s_x and s_y denote the 50th percentiles of the distributions of X and Y, respectively. Then

A $s_x < s_y$

B $s_y < s_x$

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Solution: Note that $F_X(s_x) = 0.5 = F_Y(s_y)$. By assumption,

$$0.5 = F_X(s_x) < F_Y(s_x).$$

Since F_Y is increasing, we must have $s_y < s_x$.

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- But how can we sample from any distribution?
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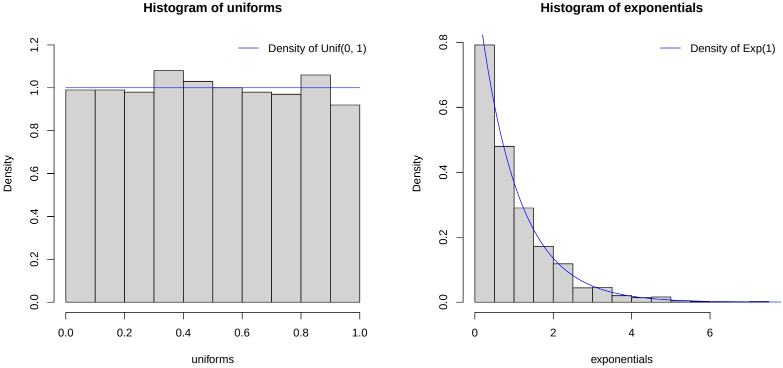
$$F_Y(y) = P(F^{-1}(U) \leq y) = P(U \leq F(y)) = F(y), \quad y \in \mathbb{R}.$$

- So for continuous random variables with strictly increasing cdf, we have $Y = F^{-1} \sim F$. Let look at an example.

Example

We sample `uniforms <- runif(1000)` and then `exponentials <- -log(1-uniforms)`.



Generalization

More generally, one can show the following theorem (see p172).

Theorem

Let F be an arbitrary cdf and $U \sim Unif(0,1)$. The random variable $X = F^{-1}(U)$ has cdf F, where

$$F^{-1}(y) = \inf_{x \in \mathbb{R}} \{F(x) \geq y\}.$$

Example

A random variable X has a $\text{Burr}(c, k)$ distribution, if X has pdf

$$f(x) = ckx^{c-1}(1 + x^c)^{-k-1}, \quad x > 0,$$

where c, k are parameters.

a) Show that f is a valid pdf.
b) Derive a function g such that $g(U)$ has pdf f if $U \sim Unif(0, 1)$.

a) Note that $f(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^{\infty} ckx^{c-1}(1+x^c)^{-k-1}\,\mathrm{d}x = \left[-(1+x^c)^{-k}\right]_0^{\infty} = 1.$$

b) We first compute the cdf as

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_0^{x} ckt^{c-1}(1+t^c)^{-k-1}\,\mathrm{d}t = 1-(1+x^c)^{-k}$$

for $x > 0$ and 0 otherwise. From the theorem, we know we have to choose $g(u) = F^{-1}(u)$. Solving $F(x) = u$ for x gives

$$g(u) = F^{-1}(u) = \left((1-u)^{-1/k} - 1\right)^{1/c}.$$