Probability Rules and Conditional Probability: Chapter 4

Chapter Outcomes:

- Rules for unions of events
- Intersections of events and independence
- Conditional probability
- Product rules, Law of total probability, and Bayes' Theorem

Recall: A probability model consists of a **set of simple events**, contained in the sample space:

$$S = \{a_1, a_2, \ldots, a_n\}$$

By defining the event $A = \{a_1, a_2, \ldots, a_r\}$ for $r \leq n$, this leads to

$$P(A) = P(a_1) + P(a_2) + \cdots + P(a_r).$$

These probabilities satisfy the following rules:

1. $P(S) = 1$.

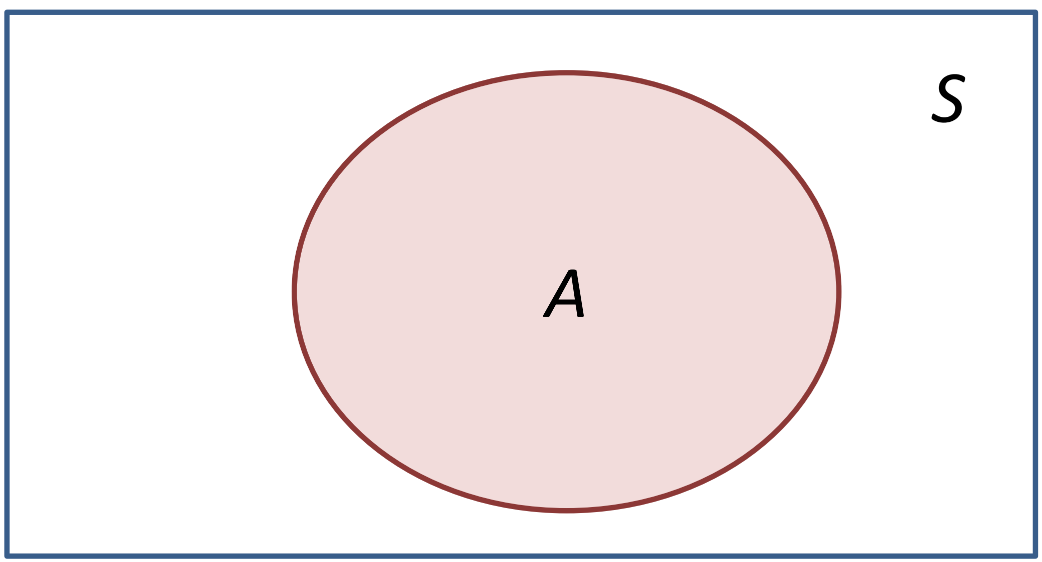
2. For any event A, $0 \leq P(A) \leq 1$.

3. If A and B are two events with $A \subseteq B$, then $P(A) \leq P(B)$.

Venn Diagrams: These are used to illustrate the relationship among sets (or events).

They are made up of:

1. A rectangle which represents *S*
2. Circles within the rectangle which represent the events in *S*



Illustrating the **union** of 2 events - $A \cup B$

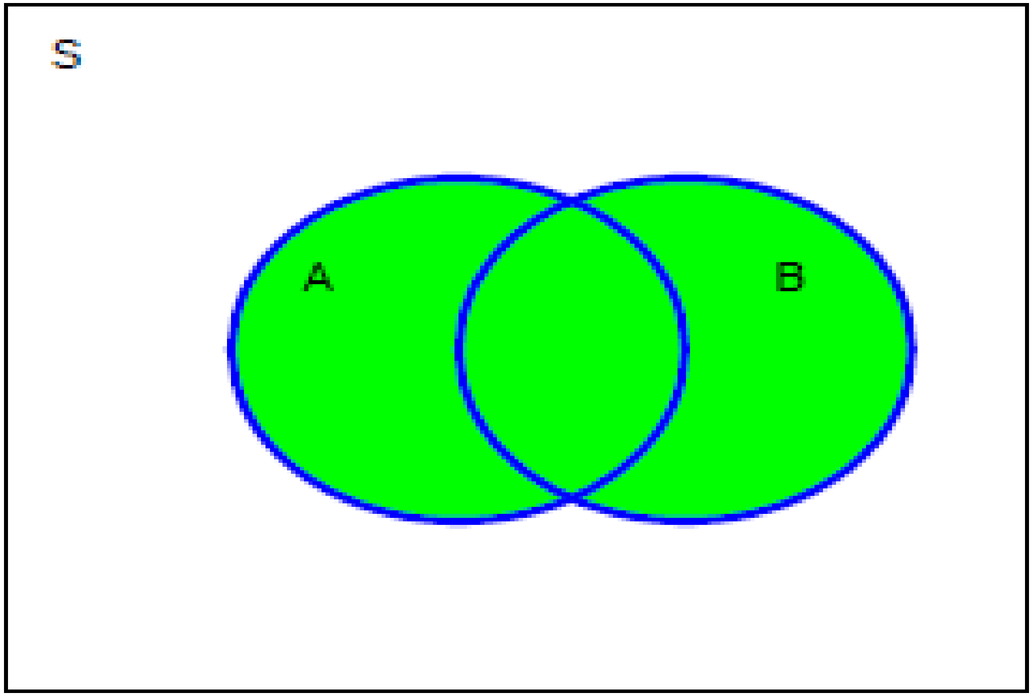


Figure 4.2: The union of two events $A \cup B$

Illustrating the **intersection** of 2 events - $A \cap B$

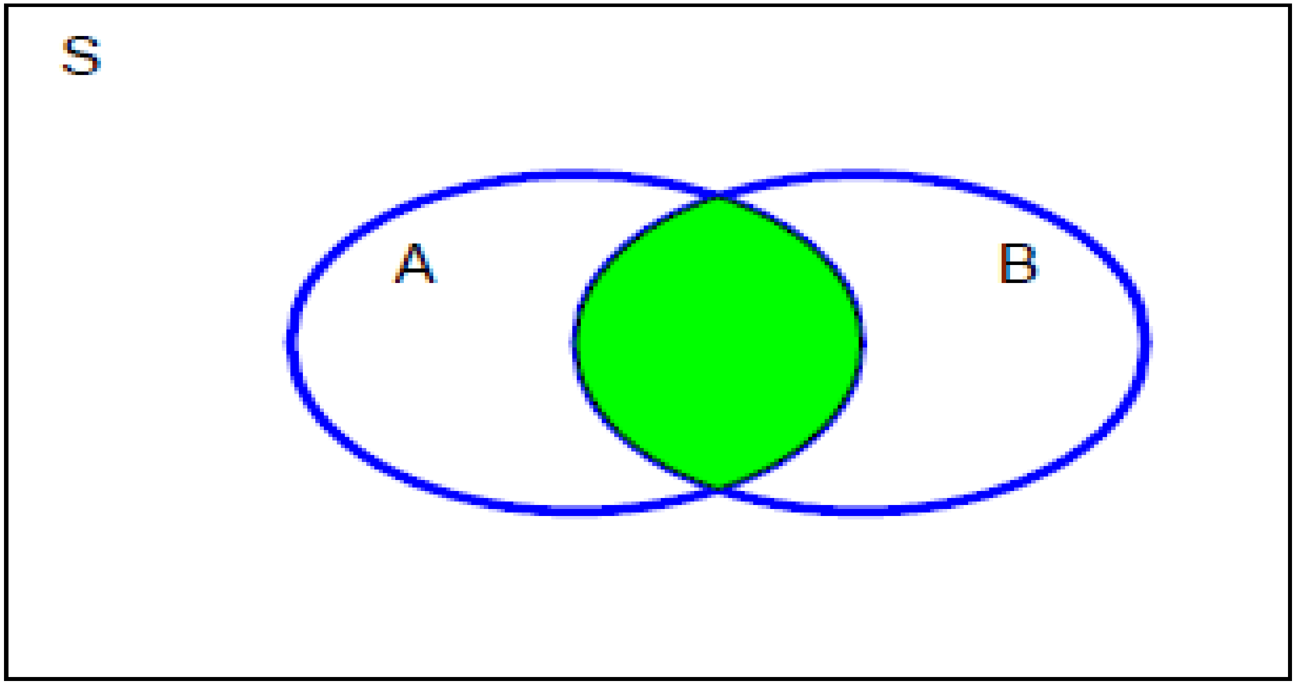


Figure 4.3: The intersection of two events $A \cap B$

Illustrating the **complement** of an event - $\bar{A}$

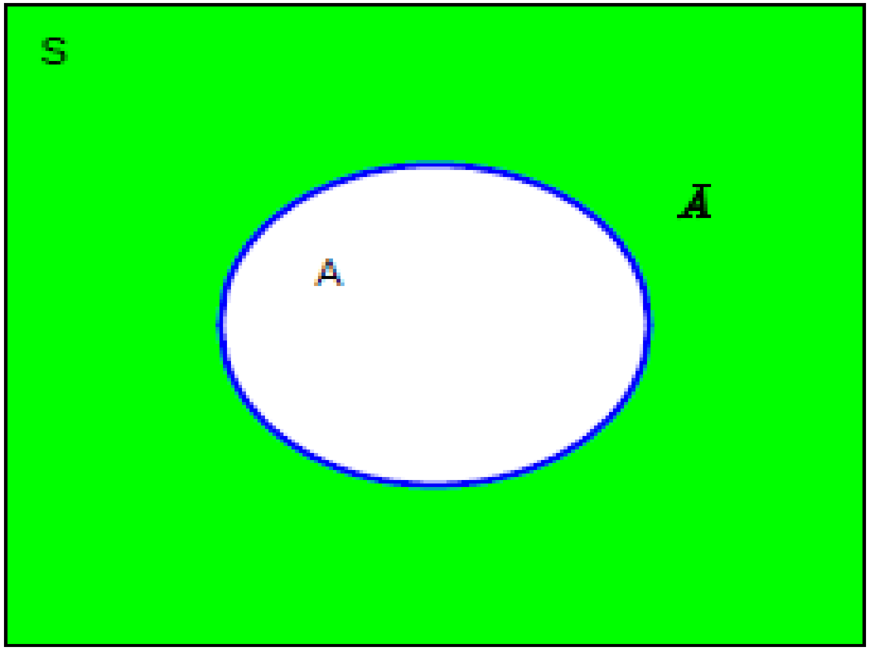


Figure 4.4: $\bar{A}$ = the complement of the event A

Two special events:

1. The entire *sample space* (i.e. universal set), S, where $P(S) = 1$

2. The *null event* (i.e. empty set), ϕ, where $P(\phi) = 0$

De Morgan's Laws:

1. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

2. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Example: Let $S = \{1,2,3,4,5,6,7,8,9,10\}$.

Let $A = \{2,3,5,7,8,9\}$ and $B = \{2,4,5,8,10\}$.

Represent the given events using a Venn diagram.

With A = {2,3,5,7,8,9} and B = {2,4,5,8,10}, prove that De Morgan's Law holds:

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

You try:

With A = {2,3,5,7,8,9} and B = {2,4,5,8,10}, show that $\overline{A \cup B} = \bar{A} \cap \bar{B}$ holds true.

Previously we found:

$\boldsymbol{A \cap B = \{2, 5, 8\}}$

$\boldsymbol{A \cap \bar{B} = \{3, 7, 9\}}$

$\boldsymbol{\bar{A} \cap B = \{4, 10\}}$

This leaves {1, 6} outside, representing $\boldsymbol{\bar{A} \cap \bar{B}}$.

$\boldsymbol{A \cup B = A \cap \bar{B} + A \cap B + \bar{A} \cap B}$ **= {2, 3, 4, 5, 7, 8, 9, 10}.**

So, $\boldsymbol{\overline{A \cup B} = S - (A \cup B)}$ = {1, 6}

which is equivalent to $\boldsymbol{\bar{A} \cap \bar{B}}$.

Rules for Unions of Events:

$$\boldsymbol{1.\ \ P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

This is the "standard" formula.

Is there another way to determine $P(A \cup B)$ using only intersection probabilities?

Union of 3 events:

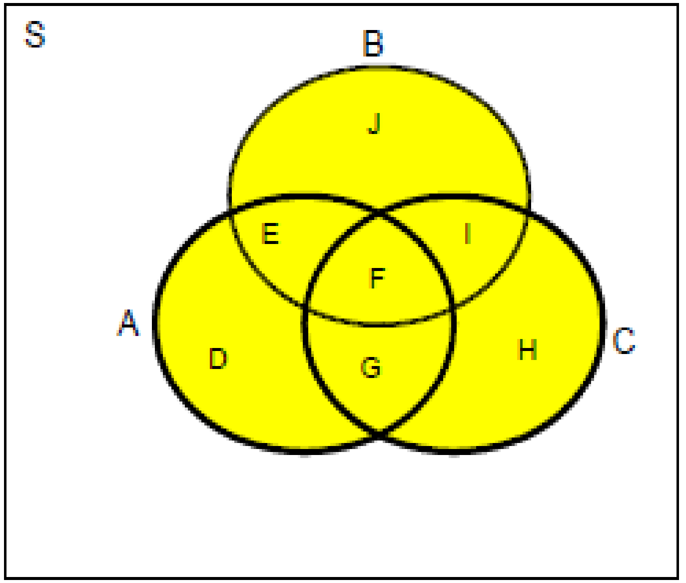


Figure 4.7: The union of three events $A \cup B \cup C$

Is there a "standard" formula?

Is there a formula only using intersection probabilities?

2. Rule 1 can be generalized to n events A_1, $A_2, \ldots, A_n$:

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) \\ &+ \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) - \ldots \\ &+(-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \end{aligned}$$

Definition: Mutually exclusive (ME) events

Events A and B are said to be mutually exclusive if: $A \cap B = \phi$, $\Rightarrow P(A \cap B) = 0$

What is this telling us?

The events A and B have no simple events in common.

In general:

Events $A_1, A_2, \ldots, A_n, \ldots$ *are* **ME if and only if:**

$$A_i \cap A_j = \phi \text{ for all } i \neq j$$

Union of ME events:

$$P(A \cup B) = P(A) + P(B)$$

In general:

If A_1, A_2,…, A_n ,… is a (finite or countably infinite) sequence of ME events, then

$P(A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$

or

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Probability of the complement:

$$P(A) = 1 - P(\bar{A})$$

Proof:

Example: Two fair dice are rolled. Find the probability that at least one of them turns up a six.

Solution 1: Defining the appropriate events.

Solution 2: Using the complement.

Intersections of Events and Independence (Section 4.3):

Events *A* and *B* are said to be **independent** if and only if:

$$P(A \cap B) = P(A) * P(B)$$

If they are NOT independent, they are said to be **dependent**.

- Independence is generally viewed as a mathematically defined concept.
- The above definition works in both directions.
- Independence ≠ Mutually Exclusive

Independence of Complements:

Suppose events A and B are independent.
Then $\bar{A}$ **and** $\bar{B}$ **are also independent.**

Proof:

In fact, if $\bar{A}$ and B are independent, and so are A and $\bar{B}$.

Question: What does independence actually mean?

Whether or not event A happens, this will not affect whether event B happens or not.

Independence can be generalized to a sequence of *n* events.

We say the events $A_1, A_2, \ldots, A_n$ are mutually ***independent*** if and only if:

$$\boldsymbol{P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = P\left(A_{i_1}\right) * P\left(A_{i_2}\right) * \cdots * P\left(A_{i_k}\right)}$$

for all sets $(i_1, i_2, \ldots, i_k)$ of distinct subscripts chosen from (1,2,…,*n*).

We will drop the word mutually to reduce confusion with "mutually exclusive."

Example: Consider the sample space S={1,2,3,…,8}, along with the following events:

$$A = \{1,2,3,4\} \qquad B = \{3,4,5,6\} \qquad C = \{1,2,5,6\}$$

Assuming outcomes in this sample space are equally likely, are the events A, B, and C independent?

Exercise 4.3.1 (*from the Course Notes*):

A weighted die with 6 faces is such that $P(1)=P(2)=P(3)=0.1$, $P(4)=P(5)=0.2$, and $P(6)=0.3$. Assume that events determined by different throws of the die are independent.

a) If the die is thrown twice, what is the probability that the total is 9?

b) If a die is thrown twice, and this process is repeated 4 times, what is the probability that the total will be exactly 9 on 1 of the 4 repetitions?

Conditional Probability (Section 4.4)

The **conditional probability** of event *A*, **GIVEN** event *B*, is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

(provided $P(B) > 0$).

Here, we want to determine the probability of some event *A*, while knowing that some other event *B* has *already occurred*.

In a similar fashion, $P(B|A) = \frac{P(A \cap B)}{P(A)}$ is the **conditional probability** of event *B*, **GIVEN** event *A* (provided $P(A) > 0$).

Example: From five computer chips, of which one is defective, two chips are randomly selected for use (without replacement).

Let A be the event that the second chip chosen is non-defective and B the event that the first chip chosen is non-defective.

What is $P(A|B)$?

Question: What happens if the events A and B are ***independent***?

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A) * P(B)}{P(B)} \\ &= P(A) \end{aligned}$$

In other words, the conditional probability of A given B simply becomes the *ordinary* (i.e.unconditional) probability of A.

Similarly, $P(B|A) = P(B)$ when A and B are independent events.

Product Rules, Law of Total Probability and Bayes' Theorem (Section 4.5)

Product Rule: Let A, B, C, and D be arbitrary events in S. Assume that $P(A) > 0$, $P(A \cap B) > 0$, and $P(A \cap B \cap C) > 0$. Then:

$$P(A \cap B) = P(A)P(B|A)$$
$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$
$$P(A \cap B \cap C \cap D) = P(A)P(B|A)P(C|A \cap B)P(D|A \cap B \cap C)$$

Special Case: If A, B, C, D are independent, then:

$$P(A \cap B \cap C \cap D) = P(A) * P(B) * P(C) * P(D)$$

Law of Total Probability (Partition rule)

Let $A_1, A_2, \ldots, A_k$ be a **partition** of S into disjoint (i.e., mutually exclusive) events, that is:

$$A_1 \cup A_2 \cup \cdots \cup A_k = S \;\; and \;\; A_i \cap A_j = \emptyset \; \forall \; i \neq j.$$

Now, let B be an arbitrary event in S. Then:

$$P(B) = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Special Case (k = 2): $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$

From a Venn diagram perspective:

Example: The probability a randomly selected male is colour blind is 0.05, whereas the probability that a randomly selected female is colour blind is 0.0025.

If the population is 50% male, what proportion of the population is colour blind?

Tree Diagrams

Tree diagrams are useful in giving a visual representation of conditional probabilities.

A tree diagram consists of ***nodes*** and ***branches***, where each branch represents a particular path that could be followed.

Example: Suppose you ask your roommate to water a sickly plant while you are on vacation. While you are away, without water, the plant will die with probability 0.8 and with water, it will die with probability 0.1.

Your roommate will remember to water the plant with probability 0.85.

Represent the given information using a tree diagram.

Example: In a typical year, 20% of the days have a high temperature > $22^{o}C$. On 40% of these days, there is no rain. During the rest of the year, when the temperature is $\leq 22^{o}C$, 70% of the days have no rain.

Represent the information using a tree diagram and solve for the proportion of days in the year which have rain and a temperature $\leq 22^{o}C$.

Bayes' Theorem: This allows us to expand conditional probabilities in terms of similar conditional probabilities:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|\bar{A})P(\bar{A}) + P(B|A)P(A)}$$

This result can also be generalized in a very natural way:

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}$$

Consider the sickly plant example again…

You ask your roommate to water a sickly plant while you are on vacation. Without water, the plant will die with probability 0.8 and with water it will die with probability 0.1. Your roommate will remember to water the plant with probability 0.85.

If the plant is alive when you return, what is the probability that your roommate remembered to water it?

Example: Three methods, A, B, and C, are available for teaching a certain industrial skill. The failure rates are 20%, 10%, and 5% for each of methods A, B, and C, respectively. C is a more expensive method and is only used 20% of the time. The other two methods are used equally often.

Suppose a worker is taught the skill by one of the methods but fails to learn it correctly. What is the probability that he/she was taught by method A?

You Try:

Electric motors coming off two assembly lines are pooled for storage in a common stockroom, and the room contains an equal number of motors from each line. Motors are periodically sampled from that room and tested. It is known that 10% of the motors from line 1 are defective and 15% of the motors from line 2 are defective.

a) If a motor is randomly selected from the stockroom, what is the probability it is defective?

Let's start by defining events.
Let A = motor is from line 1.
Let B = motor is from line 2.
Let D = motor is defective.
Given:
P(A) = 0.5, P(B) = 0.5
P(D|A) = 0.1 => $P(\overline{D}|A) = 0.9$. P(D|B) = 0.15 => $P(\overline{D}|B) = 0.85$.
We want P(D).
Using the Law of Total Probability, P(D) = $P(A \cap D) + P(B \cap D)$.
We can use the product rule to rewrite the intersected probabilities above.
So, P(D) = P(D|A)*P(A) + P(D|B)*P(B) = 0.5(0.1) + 0.5(0.15) = 1/8 = 0.125.

b) If the randomly selected motor is defective, what is the probability it came from line 1?
We want P(A|D).

By definition, P(A|D) = $\frac{P(A \cap D)}{P(D)}$. We can use the result from part a) to solve. Using the product rule, we can rewrite the intersected probability.

This gives P(A|D) = $\frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D)} = \frac{(0.1)(0.5)}{0.125} = \frac{1/20}{1/8} = \frac{2}{5} = 0.4$