Stat 230: Probability

Lecture 21

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Review

Example

Suppose X is a random variable such that

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

If we are given $Y \sim -2X + 3$. Determine $E(Y)$.

Review

Last time we talked about

(1) Summary statistics

(2) Transformations

For today:

(1) Transformations

(2) Uniform Distribution

(3) Exponential Distribution

Reading: Middle of chapter 8

Review

- Monday: Quiz 4
- Start Review

Uniform Distribution

Definition

We say that X has a uniform distribution on (a, b) if X has pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b), \\ 0 & \textit{otherwise} \end{cases}$$

This is abbreviated $X \sim U(a, b)$.

Uniform Distribution

Theorem

If $X \sim U(a, b)$ *then*

(1) $E(X) = \frac{a+b}{2}$

(2) $Var(X) = \frac{(b-a)^2}{12}$

Example

Suppose that the angle measured from the principal axis to the point of a spinner is uniformly distributed on $[0, 2\pi]$. You win the prize you want if the point lands in $[\frac{3\pi}{4}, \frac{3\pi}{2}]$. Given that the point will stop in the top half of the circle, what is the probability that you win the prize you want.

Simulation

Remark (One method to simulate $U(0,1)$)

If $(b_1, b_2, b_3, ..., b_L)$ is a random bit string (sequence of 0's and 1's), then

$$\sum_{i=1}^{L} \frac{b_i}{2^i}$$

is approximately $U(0,1)$ for large L.

Remark (Another method to simulate $U(0,1)$)

Use a function such as

$$x_{i+1} = ax_i \ mod \ m$$

With proper selection of a and m, a $U(0,1)$ can be generated using $\frac{x_i}{m}$.

Simulation

Remark (Probability integral transformation)

Suppose we wish to simulate a continuous random variable X with CDF $F(x)$.

Let $F^{-1}(x)$ denote the inverse function of $F(x)$ defined on $(0,1)$. If $Y \sim U(0,1)$, then $X = F^{-1}(Y)$ has CDF $F(x)$.

Simulation

Example

Consider the CDF

$$F(x) = \begin{cases} 0 & x \leq 0, \\ 1 - e^{-x} & x > 0 \end{cases}$$

Devise a transformation h so that if $Y \sim U(0,1)$, $X = h(Y)$ has CDF F.

Exponential Distribution

Definition

We say that X has an exponential distribution with parameter θ ($X \sim exp(\theta)$), where $\theta > 0$, if the density of X is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x > 0, \\ 0 & x \leq 0 \end{cases}$$

Exponential Distribution

Example

Suppose X has an exponential distribution with parameter θ. Compute the CDF of X.

Exponential Distribution

Remark (Relationship between the exponential distribution and the Poisson process)

Consider a Poisson process with rate parameter λ. Then the distribution of amount of time between the occurrence of successive events is exponential.

Theorem

If X is the time to the first event of Poisson process with parameter λ, then $X \sim exp(\frac{1}{\lambda})$.

Exponential Distribution

Theorem

Suppose that $X \sim exp(\theta)$. Then $E(X) = \theta$, and $Var(X) = \theta^2$.