STAT 230: Probability (Sec 02) Spring 2022

Erik Hintz
Department of Statistics and Actuarial Science
erik.hintz@uwaterloo.ca

Lecture 4

Today's Agenda

Last time:

- Simple/compound events
- Probability distribution
- Counting techniques

Today (Lec 4, 05/09):

More on counting techniques

3. COUNTING TECHNIQUES

Addition Rule

■ If A and B are disjoint events (i.e., $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

■ From the addition rule, we can see that "or" means "+".

Multiplication rule

■ An ordered k-tuple is an ordered set of k values: $(a_1, a_2, ..., a_k)$. If the outcomes in A can be written as an ordered k-tuple where there are n_1 choices for a_1 , n_2 choices for a_2 ,..., and in general n_i choices for a_j , then

$$|A| = n_1 n_2 \cdots n_k = \prod_{i=1}^k n_i.$$

■ The multiplication rule indicates that "and" means "×".

With versus without replacement

Suppose you have an urn with n distinct balls, and you select $k \le n$ balls in order. You can do that

- "with replacement": Every time an object is selected, it is put back into the pool of possible objects.
- "without replacement": Once an object is selected, it stays out of the pool of possible objects.

Sampling without replacement affects the probabilities thereafter!

Factorials!!!!!!

Definition

Given n distinct objects, n factorial (denoted n!) is the number of ordered arrangements of length n that can be made. Mathematically,

$$n! = n \times (n-1) \times \dots 2 \times 1.$$

By definition, 0! = 1.

A useful property of n! is $n! = n \times (n-1)!$.

 $10\ people$ are standing next to each other for a group picture. How many arrangements are there?

Suppose that the 5 members of of the "STATS IS COOL" club must select a president and a secretary. How many ways can they do this?

Definition

Given n distinct objects, a **permutation** of size k is an *ordered* subset of k of the individuals. The number of permutations of size k taken from n objects is denoted $n^{(k)}$ and

$$n^{(k)} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}.$$

Stirling's formula

n! grows really fast as n increases, so sometimes we need to approximate its value for computational reasons.

Stirling's formula provides one such method, and it is given by

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$
 ,

and their ratio approaches 1 as n goes to infinity.

Definition

Given n distinct objects, a **combination** of size k is an *unordered* subset of k of the individuals. The number of combinations of size k taken from n objects is denoted $\binom{n}{k}$ and

$$\binom{n}{k} = \frac{n!}{k!} = \frac{n!}{(n-k)!k!}.$$

The number $\binom{n}{k}$ is called **Binomial coefficient**.

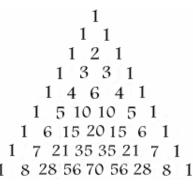
Melissa participates in a lottery in which she selects 7 numbers between 1 and 50, and then a computer randomly picks 7 numbers between 1 and 50 (without replacement). She wins if her selected numbers match 5 or more of the randomly selected numbers, in any order. What is the probability that Melissa wins?

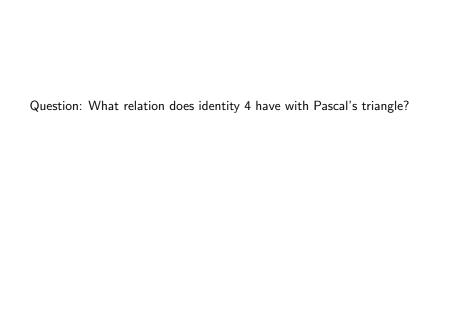
Suppose you have 20 distinct books, 7 of which are written by Mark Twain.

- a) How many ways can you arrange 12 books on a shelf if the order they are on the shelf matters?
- b) How many ways can you arrange 12 books on a shelf if exactly 3 of them must be Mark Twain books?
- c) A monkey picks books at random from the 20 books and puts them on the shelf until it contains 12 books. What is the probability that at least 3 of the books on the shelf are written by Mark Twain?

There are some useful/important results about permutation and combination.

- a) $n^{(k)} = n(n-1)^{(k-1)}$ for k > 1
- b) $\binom{n}{k} = \frac{n^{(k)}}{k!}$
- c) $\binom{n}{k} = \binom{n}{n-k}$ for $k \ge 0$
- d) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- e) Binomial theorem: $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$
- f) $\binom{n}{k}$ is equal to the kth entry in the nth row of Pascal's triangle.





Example (The Birthday Problem)

Suppose a room contains n people. What is the probability at least two people in the room share a birthday?

Assumption: Suppose that each of the n people is equally likely to have any of the 365 days of the year as their birthday, so that all possible combinations of birthdays are equally likely.