Instructions

1. **All electronic devices are to be turned off and put away.**
2. **This is a closed-book quiz.**
3. **Answer the questions in the spaces provided.**
4. **Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form.**
5. **Only question pages will be marked.**
6. **You may tear off the last page and use it for rough work. All pages will be collected at the end of the quiz.**
7. **Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed. No other aids are allowed.**
8. **DO NOT WRITE ON THIS COVER PAGE.**

Marking Scheme:

Questions	Out of
1 – 9	9
10	9
11	8
Total	26

*** *This page was intentionally left blank.* ***

Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Suppose that two fair dice are rolled. We note the numbers on the top faces. Let M represent the maximum of the numbers showing on the top faces. What is E(M)?

A) 3.5 **B) 4.472** C) 4.7467 D) 4.9583

- **If M ≤ m, then all three dice will show a number less than or equal to m.**
- **So, F(m) = P(M ≤ m) = $\frac{m^2}{6^2} = \frac{m\hat{\ }2}{36}$.**
- **To find E(M), we will need to determine P(M = m) for m = 1, 2, …, 6.**
- **We can obtain P(M = m) using the cdf.**
- **To start, P(M = 1) = F(1) = 1/36. Then, P(M = 2) = F(2) – F(1) = 3/36**
- **Then, P(M = 3) = F(3) – F(2) = 5/36. P(M = 4) = F(4) – F(3) = 7/36.**
- **Then, P(M = 5) = F(5) – F(4) = 9/36. P(M = 6) = F(6) – F(5) = 11/36.**
- **So, E(M) = $\frac{1*1+2*3+3*5+4*7+5*9+6*11}{36} = 4.472$**

2. Suppose that the number of surface flaws on windshields to be installed in brand new cars has a Poisson distribution with a mean of 0.02 flaws per square foot. Assume that each windshield has an area of 14 square feet. Let X represent the number of windshields that will need to be examined in order to find one with no flaws. What is E(X)?

A) 1 B) 1.020 C) 0.28 **D) 1.323**

- **In this case, X has a Geometric distribution with p. We need to solve for p.**
- **Let's let Y represent the number of flaws in a windshield.**
- **Y~Pois(0.02*14 = 0.28) in this case.**
- **P(no flaws) = P(Y = 0) = $e^{-0.28}$ = 0.755783741. This is p.**
- **So, X~Geom(p = $e^{-0.28}$)**

- **The expected number of failures before our first success = = 0.323**
- **So, the expected number of windshields that will need to be examined in order to find one with no flaws is 1.323.**

3. In the following game, three cards are labelled \$2, \$14, and \$*k*, respectively. A player pays \$10 to play the game and selects 2 cards at random <u>with replacement,</u> shuffling the cards after each selection. The player will receive an amount equal to the sum of the values indicated on the 2 cards. If the net winnings are equal to the amount won minus the \$10 fee to play, what is a possible positive value of *k* such that the variance of the net winnings (in dollars squared) is equal to 48?

A) 2 **B) 8** C) 10 D) None of these

- **Let W represent the possible (gross) winnings (before the \$10 to play is applied). With three cards labeled \$2, \$14, and \$k, the possible values of W lie in the set:**
- **{\$4, \$16, \$(k + 2), \$28, \$(k + 14), \$2k}**
- **Thus, E(W) = $4\left(\frac{1}{9}\right) + 16\left(\frac{2}{9}\right) + (k+2)\left(\frac{2}{9}\right) + 28\left(\frac{1}{9}\right) + (k+14)\left(\frac{2}{9}\right) + 2k\left(\frac{1}{9}\right) = \frac{2(16+k)}{3}$**
- **Similarly, E(W²) = $16\left(\frac{1}{9}\right) + 256\left(\frac{2}{9}\right) + (k+2)^2\left(\frac{2}{9}\right) + 784\left(\frac{1}{9}\right) + (k+14)^2\left(\frac{2}{9}\right) + 4k^2\left(\frac{1}{9}\right)$**
- **E(W²) = $\frac{8(214+8k+k^2)}{9}$**
- **Var(W) = E(W²) – [E(W)]² = $\frac{8(214+8k+k^2}{9} - \frac{4(16+k)^2}{9} = \frac{4(172-16k+k^2)}{9}$**

- **Let N represent the net winnings (with the $10 fee to play subtracted).**
- **So, N = W – 10**
- **Thus, Var(N) = Var(W -10) = Var(W).**
- **So, we need to solve for k, such that Var(W) = 48.**
- **So,** $48 = \frac{4(172-16k+k^2)}{9} \Rightarrow 64 - 16k + k^2 = 0 \Rightarrow (k-8)^2 = 0.$
- **The negative root is inadmissible, as we are looking for a possible positive value of k.**
- **Thus, k = 8.**

4. Suppose that a random variable *X* has probability function of the form:

$f(x) = k * 0.9^x$ for x = 1,2, 3, 4 where k is a constant and 0 otherwise.

What is the variance of *X*?

A) 1.5391 **B) 1.2383** C) 0.3231 D) 2.1167

- **In order for our pf to be valid, we know that** $\sum_{x=1}^{4} k(0.9)^x = 1.$
- **We have the sum of a finite geometric series inside the brackets with r = 0.9.**
- **We are able to determine that k =** $\frac{1-0.9}{0.9(1-0.9^4)} = \frac{1/10}{(\frac{9}{10})(1-(\frac{6561}{10,000}))} = \frac{(\frac{1}{10})}{(\frac{30951}{100000})} = \frac{10000}{30951}.$
- **In this case, E(X) =** $\left(\frac{10000}{30951}\right)\sum_{x=1}^{4} x * (0.9)^x = \left(\frac{73314}{30951}\right) = \frac{8146}{3439}$
- **Similarly, E(X²) =** $\left(\frac{10000}{30951}\right)\sum_{x=1}^{4} x^2 * (0.9)^x = \frac{211986}{30951}$
- **So, Var(X) =** $\frac{211986}{30951} - \left(\frac{8146}{3439}\right)^2 = 1.238288280.$

5. Suppose that P(X = x) = kx , for x = 1, 2, 3, 4 ; 0, otherwise, for some constant k.

What is E(X²)?

A) 225 B) 12.25 **C) 10** D) 3.667

- **First, we will find k.**
- **k*(1+2+3+4) = 1 => k = 1/10**
- **E[X²] =** $\sum x^2 f(x) = \frac{1}{10}\left[1^3 + 2^3 + 3^3 + 4^3\right] = 10$

6. Suppose that a discrete random variable X has Var(X)=5. Define Y = 1 – 3X. Then Var(Y) is

A) -15 B) -14 C) 15 **D) 45**

- **We know Var(X) = 5**
- **We also know Var(Y)=Var(-3X+1)**
- **Using the properties of the variance, Var(Y) = (-3)^2 Var(X)**
- **Plugging in, Var(Y) = 9*5 = 45**

<u>Use the following information to answer the next TWO questions</u>:

Toby Moose enters a local casino and agrees to play the following game. Toby will pay $20 to toss a fair coin 20 times. For every head that is flipped, he will receive $2. **Note:** A loss is a negative gain.

7. Toby's expected gain from this game is closest to

A) -\$5.00 B) \$5.00 C) **\$0 (fair game)** D) \$10.00

- **Let X represent the number of heads tossed.**
- **Y~Bin(20,0.5). E(Y) = np = 20(0.5) = 10**
- **We can set up a loss function where L = 2Y – 20**
- **E(L) = E[2Y – 20] = 2E(Y) – 20 = 20 – 20 = 0**

8. The variance of Toby's gain is:

A) -10 B) 0 C) 18.5 **D) 20**

- **Let X represent the number of heads tossed.**
- **X~Bin(20,0.5). E(X) = np = 20 * 0.5 = 10; Var(X) = 20 * 0.5 * 0.5 = 5**
- **From the previous part, we have L = 2Y – 20**
- **Var(L) = Var(2Y – 20) = 4Var(Y) = 4*5 = 20**

9. Mr. Lucky enters a local casino and is thinking about playing the following game: Mr. Lucky will be asked to pay a certain amount of money to toss a fair die 18 times. Every time a "6" is tossed, he will receive \$4. Otherwise, he receives no money.

If the game is considered to be fair, then Mr. Lucky should pay an amount equal to his expected winnings. How much should Mr. Lucky pay to play?

A) \$8 B) \$10 **C) \$12** D) \$18

- **Let X = # of 6's in 18 tosses of a fair die. X ~ Binom(n = 18, p = 1/6)**
- **E(X) = np = 18*(1/6) = 3**
- **Let W = Mr. Lucky's winnings. W = \$4X.**
- **So, Mr. Lucky's expected winnings would be E[W] = \$4*3 = \$12**
- **So, for this game to be considered fair, Mr. Lucky should pay \$12 to play.**
- **Any less would be an advantage to him, and any more would be a disadvantage to him.**
- **Of course, a casino would charge you more than \$8 to play this game. That is one of the ways that they make their money. They don't give you "fair" odds.**

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work. Unjustified answers will not receive full credit.

10. Suppose that the random variable X has the following probability function

x	1	2	3	4	5
f(x)	0.15	0.1	0.15	0.35	0.25

a. Compute E(X) ***(2 marks)***

- We compute E(X) with the definition as

E(X) = 1 * 0.15 + 2 * 0.1 + 3 * 0.15 + 4 * 0.35 + 5 * 0.25 =3.45

b. Compute E(1/X) ***(2 marks)***

- The random variable 1/X has expectation

E(1/X) = 0.15 * 1 + 0.1 * ½ + 0.15 * 1/3 + 0.35 * ¼ + 0.25 * 1/5 = 0.3875

c. Compute the standard deviation of X. ***(3 marks)***

- We compute E(X^2)
  E(X^2)=1*0.15 + 4 * 0.1 + 9 * 0.15 + 16 * 0.35 + 25 * 0.25 =13.75
- Then, by the formula for the variance,

Var(X) = E(X^2) – E(X)^2 = 13.75 – 3.45^2= 1.8475

- Therefore,

SD(X) = sqrt(Var(X))=sqrt(1.8475) = 1.359 (3 dp)

d. Compute E(X(X-1)) ***(2 marks)***

- By linearity

E(X(X-1))= E(X^2) - E(X)

- We have computed E(X^2) and E(X) before. Plugging in,

E(X(X-1))=13.75 - 3.45 =10.3

11. Through previous studies, it is known that 3% of the people in Timbuktu have Covid-19. The government of Timbuktu is using two different kinds of tests: Rapid tests and PCR tests. Rapid tests only cost $5 per person and give a false positive result with probability 0.05 and a false negative result with probability 0.02. PCR tests are much more accurate in that they always show a positive result for people with covid, but at a cost of $100 per test, they are also much more expensive.

To balance the costs, the government of Timbuktu decided on the following testing regime: A person first gets the rapid test. If the rapid test is positive, the person is given a PCR test. If the rapid test is negative, the person is deemed negative.

a) What are the expected costs per person for this testing regime? ***(4 marks)***

- We determine the distribution of the random variable C ("cost"), which can also be done in a tree diagram.
- We have 4 cases:
    - Person has covid, tests positive with rapid test => C = 105 wp 0.03 * 0.98 = 0.0294
    - Person has covid, tests negative with rapid test => C = 5 wp 0.03 * 0.02 = 0.0006
    - Person has no covid, tests positive with rapid test => C = 105 wp 0.97 * 0.05 = 0.0485
    - Person has no covid, tests negative with rapid test => C = 5 wp 0.97 * 0.95=0.9215
- Combining,
  E(C) = 105 * 0.0294 + 5 * 0.0006+ 105 * 0.0485+ 5 * 0.9215=12.79
  (4 marks)

b) Despite this testing regime, the government of Timbuktu is noting a spike in covid cases, so they rethink their strategy and come up with a new one. In the new strategy, everybody is still given the rapid test first. If the rapid test is positive, just like before, the person is given a PCR test. If the rapid test is negative, a second rapid test is given to the person, just to be sure. If that second rapid test is positive, a PCR test is given, otherwise no further testing is done.

Still assuming that 3% of the people in Timbuktu have covid, what are the expected costs per person for this modified testing scheme? ***(4 marks)***

- Again, we determine the distribution of the random variable C ("cost")
- We have 6 cases (well illustrated in a tree diagram)
    - Person has covid, tests positive with rapid test => C = 105 wp 0.03 * 0.98 = 0.0294
    - Person has covid, tests negative with first rapid test, positive with second rapid test => C = 110 wp 0.03 * 0.02 * 0.98= 0.000588
    - Person has covid, tests negative with both rapid tests => C = 10 wp 0.03 * 0.02 * 0.02 = 0.000012
    - Person has no covid, tests positive with 1st rapid test => C = 105 wp 0.97 * 0.05 = 0.0485
    - Person has no covid, tests positive with 2nd rapid test => C = 110 wp 0.97*0.95*0.05=0.046075
    - Person has no covid, tests negative with both rapid tests => C = 10 wp 0.97 * 0.95 * 0.95=0.875425
- Combining,
    - E(C) = 105 * 0.0294 + 110 * 0.000588 + 10 * 0.000012 + 105 * 0.0485 + 110 * 0.046075 + 10 * 0.875425=22.0668

(4 marks)

****** This page was intentionally left blank. ******

This page may be CAREFULLY removed and used for rough work.

It will not be graded, but MUST be handed in.

This page may be CAREFULLY removed and used for rough work.

It will not be graded, but MUST be handed in.

Summary of Distributions

Notation and Parameters	Probability function f(x)	Mean E(X)	Variance Var(X)
Discrete Uniform(a,b) $b \geq a$ a, b integers	$\frac{1}{b-a+1}$ $x = a, a+1, \dots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Hypergeometric(N,r,n) $N = 1,2, \dots$ $n = 0,1, \dots, N$ $r = 0,1, \dots, N$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = \max\{0, n-N+r\},$ $\dots, \min\,(r, n)$	$\frac{nr}{N}$	$\frac{nr}{N}\left(1-\frac{r}{N}\right)\frac{N-n}{N-1}$
Binomial(n, p) $0 \leq p \leq 1, q = 1-p$ $n = 1,2, \dots$	$\binom{n}{x} p^x q^{n-x}$ $x = 0,1, \dots, n$	np	npq
Bernoulli(p) $0 \leq p \leq 1, q = 1-p$	$p^x q^{1-x}$ $x = 0,1$	p	pq
Negative Binomial(k,p) $0 < p \leq 1, q = 1-p$ $k = 1,2, \dots$	$\binom{x+k-1}{x} p^k q^x$ $= \binom{-k}{x} p^k (-q)^x$ $x = 0,1, \dots$	$\frac{kq}{p}$	$\frac{kq}{p^2}$
Geometric(p) $0 < p \leq 1, q = 1-p$	pq^x $x = 0,1, \dots$	$\frac{q}{p}$	$\frac{q}{p^2}$
Poisson(λ) $\lambda \geq 0$	$\frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1, \dots$	λ	λ