STAT 230: Probability (Sec 02)
Spring 2022

Erik Hintz
Department of Statistics and Actuarial Science
erik.hintz@uwaterloo.ca

Lecture 9

Today's Agenda

Last time:

- Independence
- Conditional Probability

Today (Lec 9, 05/20):

- Law of total probability
- Bayes rule

Independent events

Definition

Two events A and B are said to be **independent** if

$$P(A \cap B) = P(A)P(B).$$

Definition

The conditional probability of A *given* B is, so long as $P(B) > 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition (Equivalent definition of independence)

Two events A and B are independent, if

$$P(A|B) = P(A),$$

provided $P(B) > 0$.

Product Rule

Theorem

For events A and B,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

This is known as the **product rule** and follows directly from the definition of conditional probability.

Example

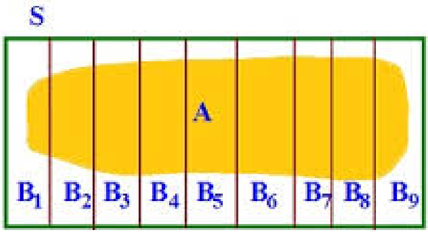
Erik has 12 red pants and 7 green pants. On Monday, Erik picks two pairs of pants for his Monday and Wednesday lectures. He draws one pair of pants at random and then, without replacement, another pair.

a) What is the probability that both pairs of pants are red?

b) What is the probability that the second pair of pants is red?

Definition

A sequence of sets $B_1, B_2, ..., B_k$ are said to **partition** the sample space S if $B_i \cap B_j = \emptyset$ for all $i \neq j$, and $\cup_{j=1}^{k} B_j = S$.



Partitioning the sample space

Theorem (Law of total probability)

Suppose that $B_1, B_2, ..., B_k$ partition S. Then for any event A,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_k)P(B_k).$$

Remark: A common way to use this is

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}),$$

since B and $\bar{B}$ partition S.

Proof

Theorem (Bayes Theorem)

Suppose that $B_1, B_2, ..., B_k$ partition S. Then for any event A,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}.$$

For the partition B and $\bar{B}$:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}.$$

Proof

Example

If I pick a pair of red pants, there is a 20% chance somebody will laugh. If I pick a green pair, there is a 10% chance somebody will laughed. 70% of the time I choose red pants, and 30% of the time green pants. Given that somebody laughed, what is the probability that I picked red pants?

Example

In an insurance portfolio 10% of the policy holders are in Class A1 (high risk), 40% are in Class A2 (medium risk), and 50% are in Class A3 (low risk). The probability there is a claim on a Class A1 policy in a given year is 0.10; similar probabilities for Classes A2 and A3 are 0.05 and 0.02. Find the probability that if a claim is made, it is made on a Class A1 policy.