Stat 230: Probability

Lecture 1

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About Me

Background

- Born and raised in Utah, USA
- Enjoy video games, sports, coding

Academic

- **Undergraduate**: Utah State University
- **Graduate**: University of Utah and University of Waterloo
  - Research: Time series, change points, functional data

Industry

- Worked at a startup and government contractor

Today's Schedule

Course Overview

What is Probability?

Probability Basics

Course Overview

Syllabus

Tutorials

- Reminder: First tutorial quiz next Monday

General Advice for doing well in this course:

(1) Do the problems in the course notes

(2) Take all assessments seriously

Piazza

Probability, Statistics, and a Coin Flip

What is Probability?

Figure: Real Money!

Probability, Statistics, and a Coin Flip

Coin Flip!!

Question: What is the probability that, when I flip this coin, it will come up heads?

A 50%

B 50.4%

C 100%

D Something else

E No idea. Please let me sleep in peace

Probability, Statistics, and a Coin Flip

Existential question: Does “randomness” really exist?

- Probability: A subdiscipline of mathematics concerned with describing and modeling uncertain experiments.
- Statistics: The study of the collection and analysis of data.

$$\text{Statistics} \iff \text{Data} \iff \text{Uncertainty} \iff \text{Probability}$$

Probability, Statistics, and a Coin Flip

Figure: Persi Diaconis, Probability of a coin flip landing Heads (standard US quarter) is approximately 50.4% under "typical" coin flip conditions.

Probability Definitions

- **Classical Definition**: The probability of some event is

$$\frac{\text{Number of ways event can occur}}{\text{Total number of possible outcomes}}$$

  - **Example**: Probability of rolling 6 is $1/6$ on a die

- **Relative Frequency**: (Limiting) proportion (or fraction) of times event occurs in a long (infinite) series of (independent) repetitions
  - **Example**: After many rolls, probability rolling 6 is $\sim 1/6$ $\left(\frac{986}{6000}\right)$

- **Subjective Probability**: Best guess at the chances of an event
  - **Example**: Weather forecast – 30% chance of rain

Probability Definitions

To apply definitions:

- Know event of interest
- Know all possible events (sample space)
- Know possibility of each event (probability)

Probability Definitions

Definition

A **sample space** *S is a set of distinct outcomes of an experiment with the property that in a single trial of the experiment only one of these outcomes occurs.*

Example

Experiment	S
Flip a Coin	$\{Heads, Tails\}$
Count LOL games before promotion to platinum	$\{1, 2, 3, 4, ...\}$
Calculate the run time of a program	$\{t \in \mathbb{R}, \quad t > 0\}$

Probability Definitions

Definition

A sample space S is said to be **discrete** *if it is finite, or "countably infinite" (countable), namely it can be put into one-to-one correspondence with the natural numbers. Otherwise a sample space is said to be* **non-discrete**.

Example

S	**Discrete/Non-Discrete**
$\{Heads, Tails\}$	discrete
$\{1, 2, 3, 4, ...\}$	discrete
$\{t, \quad t > 0\}$	non-discrete

Probability Definitions

Definition

An **event** *is a subset of a sample space S.*

Notation: We say A is an event if $A \subset S$ or $A \subseteq S$. Said "A is a subset of S", or "A is contained in S."

Example

Event	**Subset**
The coin flip is "Heads"	$A = \{Heads\}$
You make plat in 20 games or less	$A = \{1, 2, ..., 20\}$
Program takes longer than 30 seconds to run	$A = \{t \ \ , \ t > 30\}$

Probability Definitions

Definition

A simple event contains only one sample point

Definition

A compound event contains two or more sample points. It occurs if any of the simple events occurs

Probability Definitions

Other set notation: Suppose $A, B, A_1, A_2, \ldots$ are events.

- Element of: $x \in A$ if the outcome x is in the event A.
- Union: $A \cup B = \{x \mid x \in A \;\; or \;\; x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \;\; and \;\; x \in B\}$
- Complement: $\bar{A} = \{x \mid x \in S, \;\; x \notin A\}$
- Empty event/set: The empty set is denoted $\emptyset$
- Disjointedness: Two events A and B are said to be disjoint of $A \cap B = \emptyset$. This is also referred to in our textbook as **mutual exclusivity.**

Infinite set versions:

$$\cap_{i=1}^{\infty} A_i \qquad \cup_{i=1}^{\infty} A_i$$

Probability Definitions

Example

Suppose two, distinct, six sided die are rolled, and the number of dots face up on each die are recorded:

(1) Compute S

(2) Compute the event $A =$ "The sum of the dots is 7"

(3) Compute the event $\bar{B}$, where
$B =$ "The sum of the numbers is at least 4"

(4) Compute $A \cap \bar{B}$, and $A \cup \bar{B}$.

Probabilities are defined for such events.

Probability Axioms

Figure: A.N. Kolmogorov, 1963. Developed the Axioms of Probability (1933)

He is my Great, Great, Great, Great (Math) Grandpa: Jeremy VanderDoes – Greg Rice – Lajos Horváth – Sándor Csörgö – Anatoli Skorokhod – Eugene Dynkin – Andrei Kolmogorov

Probability Axioms

Definition

Let $\mathcal{S}$ denote the set of all events of a sample space S.
Example: $S = \{Heads, Tails\}$, $\mathcal{S} = \{\emptyset, \{Heads\}, \{Tails\}, S\}$.
A **Probability** *defined on $\mathcal{S}$ is a real valued function*

$$P : \mathcal{S} \to \mathbb{R},$$

that satisfies the following three conditions (Axioms):

(1) *Scale: If A is an event,* $0 \leq P(A) \leq 1$

(2) *Something happens:* $P(S) = 1$

(3) *Additivity (Infinite) : If $A_1, A_2, \ldots$ is a sequence of disjoint events* $P(A_i \cap A_j) = \emptyset, \quad i \neq j$, *then*

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Probability Axioms

If S is discrete, then $S = \{a_1, a_2, a_3, \dots\}$. In this case, (**probabilities)** $P(a_i)$, $i = 1, 2, 3, \dots$ may be "assigned" to each of the outcomes a_i so that

(1) $0 \leq P(a_i) \leq 1$

(2) $\sum_{\text{all } i} P(a_i) = 1$

then the set of probabilities $\{P(a_i),\ i = 1, 2, \dots\}$ is called a **probability distribution on** S, and will satisfy the axioms of probability so long as additivity holds.

Probability Axioms

Example

Three components of a map labelled A, B, and C have been scrambled up at random and pasted together in sequence to form a (possibly incoherent) map.

(1) What is the sample space for this experiment?

(2) What are some consistent assignments of probability to this sample space?

Probability Axioms

A few elementary facts about the probability function:

(1) $P(\emptyset) = 0$

(2) (Finite Additivity) If $A_1, A_2, ..., A_n$ are disjoint events (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) , then

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

(3) $P(\bar{A}) = 1 - P(A)$.

Probability Definitions

Definition

A **probability model** *contains a sample space, set of events, and a way of assigning probabilities.*

A **probability model** *models a random experiment*

Definition

A **random experiment** *is a trial under controlled conditions of some phenomena*

(1) *Possible Outcomes*

(1) *Repeatable*