

Poisson Distribution (Section 5.7)

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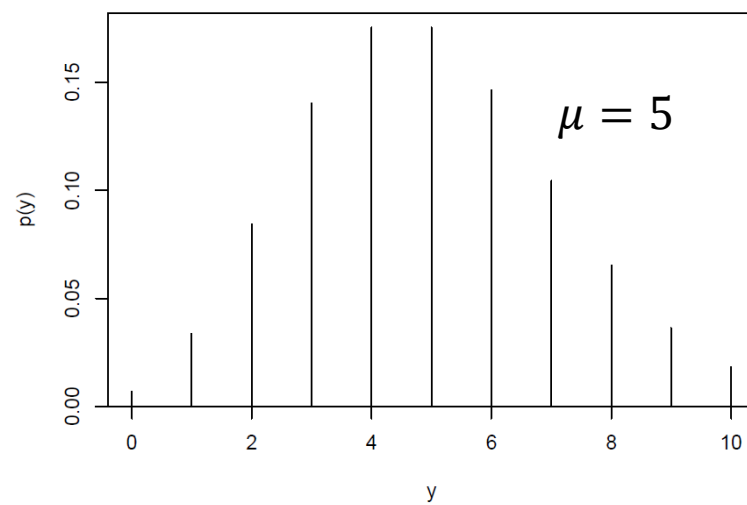
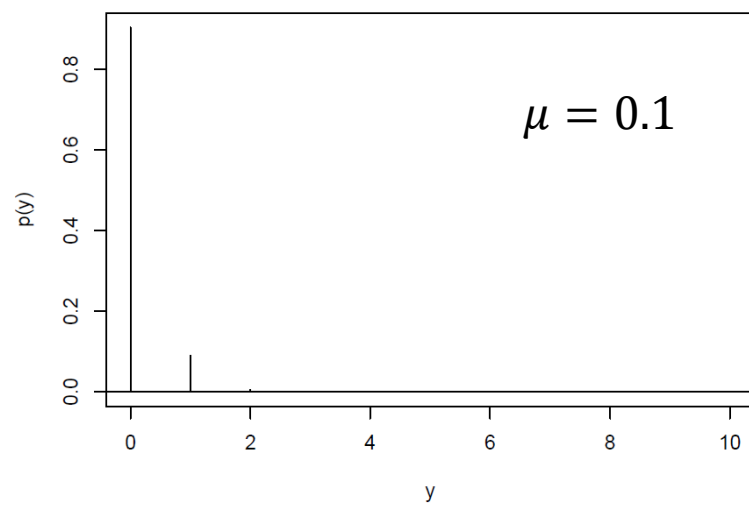
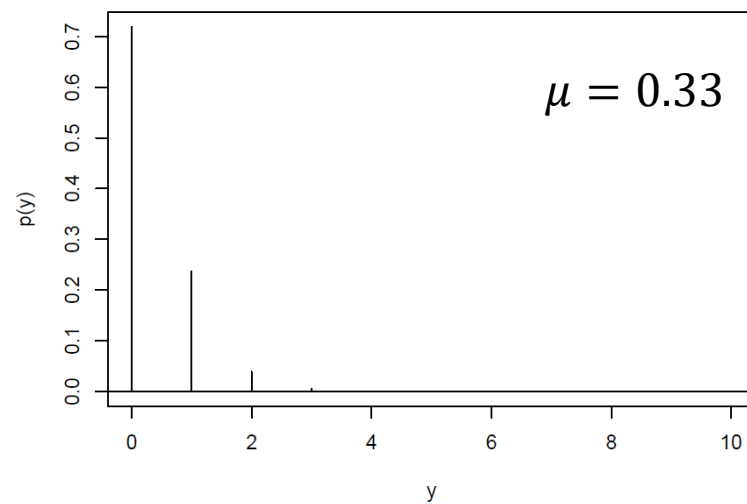
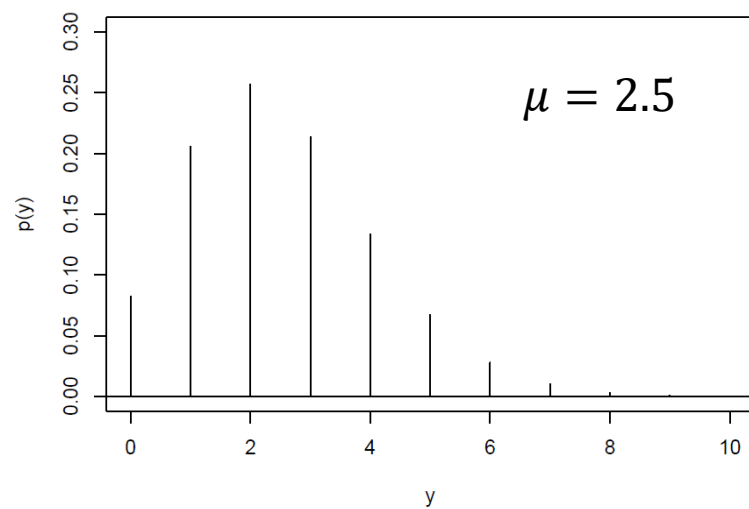
- In this setting, the r.v. X represents the number of events of some type.
- The events occur according to some **rate**, denoted by μ , where $\mu > 0$.
- We write $X \sim \text{Poisson}(\mu)$.
- X has p.f. given by

$$f(x) = \frac{e^{-\mu} \mu^x}{x!} \text{ for } x = 0, 1, 2, \dots$$

We have that $A = \{0, 1, 2, \dots\}$.

Clearly, $f(x) \geq 0 \forall x \in A$.

In addition, $\sum_{x \in A} f(x) = \sum_{x=0}^{\infty} f(x) = 1$.



Some examples of random variables that generally follow a Poisson distribution:

1. The number of misprints on a page (or a group of pages) of a book.
2. The number of people in a community who survive to age 100.
3. The number of wrong telephone numbers that are dialed in a day.
4. The number of customers entering a post office on a given day (or other types of arrivals).
5. The number of α -particles discharged in a fixed period of time from some radioactive material.
6. The number of new potholes in a stretch of highway during the winter months.
7. The number of earthquakes in a region of Canada in a month.
8. The number of lightning strikes in a region of Canada in a month.

Next, we show that the Poisson distribution has connections to a binomial distribution.

Recall that the binomial distribution applies when we have a fixed number of independent trials n , each with a constant probability of success p .

Imagine now that we do not know the number of trials that are needed, but instead we only know the *average number of successes* (per unit of time).

In other words, we know the rate of successes (say, per day), but not the number of trials n , or the probability of success p that led to that rate.

We define an average rate:

$$\mu = np$$

Let this be the rate of success (say, per day). That is the number of trials n —however many there are—multiplied by the chance of success p for each of those trials.

Think of it like this: If the chance of success is p and we run n trials per day, we will observe np successes per day **on average**. That is our average success rate μ .

For example: When we flip a fair coin, we observe a head with probability $p = 0.5$. Hence, if we flip a coin 50 times in a day, our expectation is to see **an average of** $np = 50 \times 0.5 = 25$ heads per day.

Remember that we could be observed anywhere from 0 to 50 heads. 25 is the (long-run) average.

Now, recall that if $X \sim \text{Bin} \left(n, p = \frac{\mu}{n} \right)$,

then we have:

$$\begin{aligned} f(x) &= \binom{n}{x} p^x (1 - p)^{n-x} \\ &= \binom{n}{x} \left(\frac{\mu}{n} \right)^x \left(1 - \frac{\mu}{n} \right)^{n-x} \end{aligned}$$

for $x = 0, 1, 2, \dots, n$.

Under this setting, we then consider the **limiting case** of the binomial distribution.

In other words, we want to know what happens to $f(x)$ as $n \rightarrow \infty$ (so that $p = \frac{\mu}{n} \rightarrow 0$)?

In summary, a Poisson distribution arises from a binomial distribution when $n \rightarrow \infty$ and $p \rightarrow 0$.

In other words, a **Poisson distribution approximates a binomial distribution** when n is **very large** and p (the probability of success) is **small**.

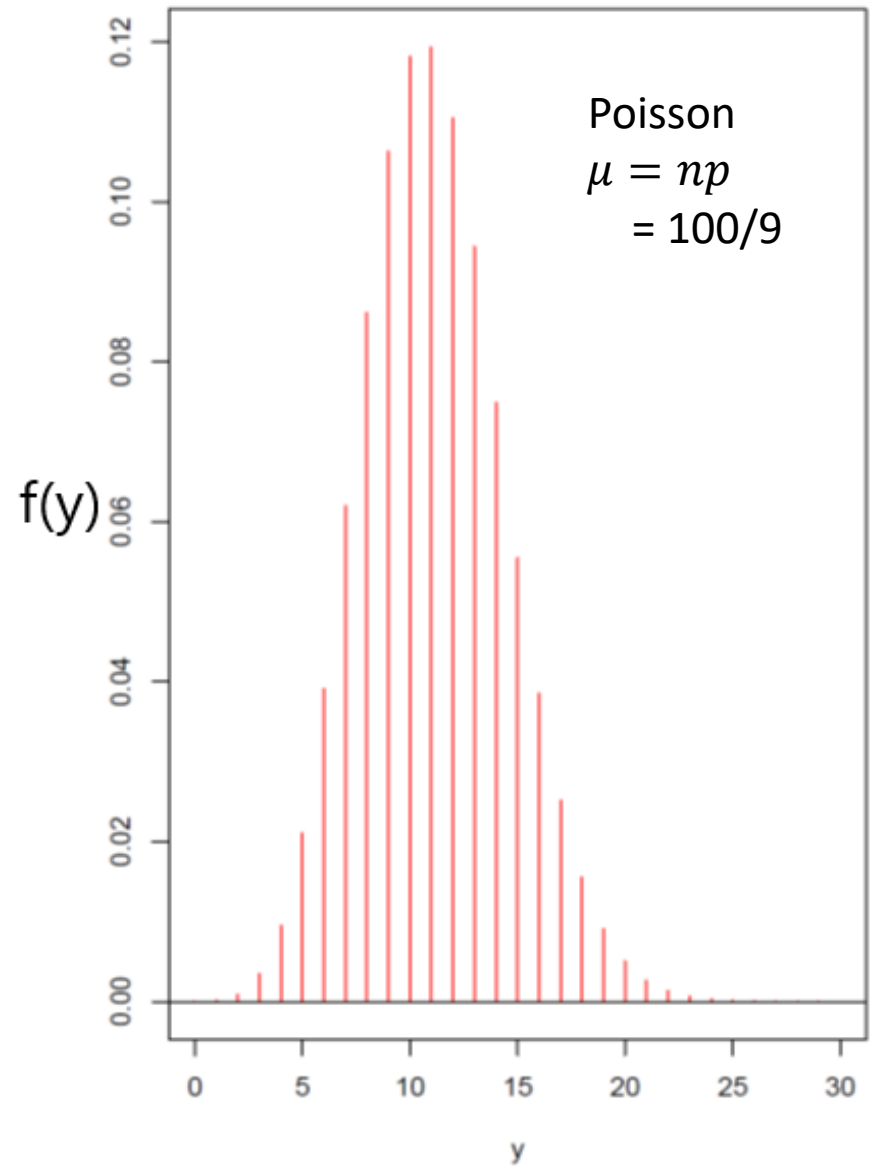
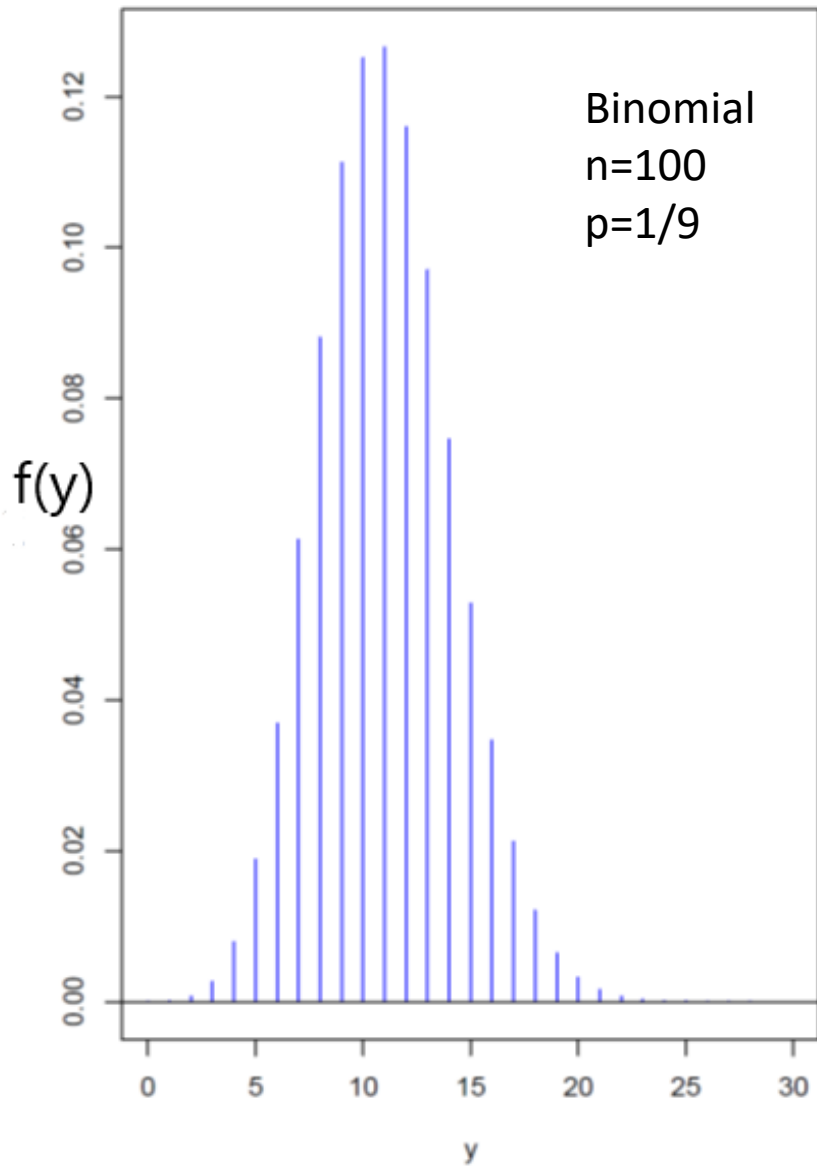
******That is, a Poisson distribution could be used to approximate a binomial probability.

The idea is to keep the product np ***fixed*** at some value, μ , and then let $n \rightarrow \infty$. This automatically makes $p \rightarrow 0$.

Example: A local restaurant is running a contest. A customer receives a ticket each time they purchase a combo (i.e. sandwich, side, and a drink). They claim that 1 in 9 tickets are winners. Say you buy 100 combos!

Assuming that the trials are independent, let's use the **Poisson approximation to the binomial** to solve for the probability that you get no more than 10 winning tickets.

(We will compare the approximate probability with the exact probability to see how well the approximation worked).



You Try:

If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $1/100$, what is the (approximate) probability that you will win a prize:

Solution:

We can define the r.v. X here one time as it will apply to all parts of this question.

Here, we let X represent the number of times a prize is won.

We will use a Poisson approximation to the Binomial to determine the approximate probabilities.

So, X has an approximate Poisson distribution with $\mu = 50 \cdot (1/100) = 0.5$.

$$P(X = x) = \frac{e^{-0.5} 0.5^x}{x!}, x = 0, 1, \dots; 0, \text{ otherwise}$$

a) at least once?

$$\text{We want to approximate } P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.5} = 0.393469$$

b) exactly once?

$$\text{We want to approximate } P(X = 1) = 0.5 * e^{-0.5} = 0.303265$$

c) at least twice?

$$\begin{aligned} \text{We want to approximate } P(X \geq 2) &= 1 - [P(X=0) + P(X = 1)] \\ &= 1 - [e^{-0.5} + 0.5 * e^{-0.5}] = 1 - 1.5e^{-0.5} = 0.090204 \end{aligned}$$

Notes:

1. The same idea can be applied when p is close to 1. So, if p is close to 1, simply interchange the labels of “success” and “failure”. In doing so, now the probability of “success” is close to 0, and you can proceed with the Poisson approximation to the binomial.
2. Historically, the Poisson approximation was a useful tool as it was easier to work with computationally. However, with the advent of better computers these days, calculating exact probabilities is no longer an issue under any circumstances.

Poisson Distribution from the Poisson Process

(Section 5.8)

- In order to discuss the **Poisson process**, we first need to introduce some notation:
- Define the so-called “*order*” notation:

$$g(\Delta t) = o(\Delta t) \text{ as } \Delta t \rightarrow 0.$$

- This means that **the function g approaches 0 faster than Δt does as Δt approaches 0.**

$$\text{i.e. } \frac{g(\Delta t)}{\Delta t} \rightarrow 0 \text{ as } \Delta t \rightarrow 0$$

For example:

1. $g(x) = x^2$

2. $g(x) = \sqrt{x}$

Physical setup:

Assume that a certain type of event occurs at random points in time (or space) and satisfies the following conditions:

1. **Independence:** The number of occurrences in non-overlapping intervals are independent.
2. **Individuality:**

$$P(\text{2 or more events in } (t, t + \Delta t)) = o(\Delta t) \\ \text{as } \Delta t \rightarrow 0.$$

In other words, this is telling us that as $\Delta t \rightarrow 0$, the probability of 2 or more events occurring in the interval $(t, t + \Delta t)$ is close to zero (i.e. the probability of more than one event occurring during such a small interval is negligible.)

3. Homogeneity or Uniformity: Events occur at a homogeneous (uniform) rate λ over time so that

$$P(\text{one event in } (t, t + \Delta t)) = \lambda \Delta t + o(\Delta t)$$

for small Δt for any value of t .

These three conditions define what is known as a **Poisson process**.

If we now let X represent the number of event occurrences in a time period of length t , then it can be shown that X has a Poisson distribution with parameter $\mu = \lambda t$.

In a Poisson process with rate of occurrence λ , the number of event occurrences, X , in a time interval of length t has a Poisson distribution with parameter $\mu = \lambda t$, so that the p.f. of X is

$$f(x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \text{ for } x = 0, 1, 2, \dots$$

Note that μ can be “scaled up” or “scaled down” accordingly.

For example:

1. Visits to a website during a given time period often follow a Poisson process.
2. Occurrences of certain diseases over time can sometimes be modelled by a Poisson process.
3. The number of customers entering a store or bank is commonly modelled by a Poisson process.

How to Interpret μ and λ :

1. λ refers to the **intensity** or **rate of occurrence**.

For example: A student types on average 60 words a minute. Here, $\lambda = 60$.

2. $\mu = \lambda t$ represents the average number of occurrences in **t units** of time.

For example: On average, how many words will the student type in 3 minutes? In 10 seconds?

$$\mu = 60 * (3) = 180.$$

$$\mu = 60 * (1/6) = 10.$$

Example: Suppose earthquakes recorded in Ontario each year follow a Poisson process with an average of 6 per year.

What is the probability that 7 will be recorded in a 2-year period?

You Try:

At a certain location on Highway 401, the number of cars exceeding the speed limit by more than 10 km/hr is a random variable having a Poisson distribution with $\lambda = 8.4$ per 30 minutes.

What is the probability that, in 1 hour, 10 cars exceed the speed limit by more than 10 km/hr?

Solution:

Let X = the number of cars exceeding the speed limit in 1 hour. The rate parameter is given as $\lambda = 8.4$ per 30 minutes, so we have to rescale it. In this case, $\mu = \lambda t = 8.4 * 2 = 16.8$ per hour

X has a Poisson distribution with $\mu = 8.4 * 2 = 16.8$

In this case, $f(x) = P(X = x) = \frac{e^{-16.8} 16.8^x}{x!}$, $x = 0, 1, \dots$; 0, otherwise

We want $P(X = 10) = \frac{e^{-16.8} (16.8)^{10}}{10!} = 0.0249564$

The Poisson process also applies when “events” occur randomly in **space**.

X could represent the number of events in a **space (like volume or area) of size v** . If λ is the average number of events per unit volume (or area), then X has a Poisson distribution with parameter $\mu = \lambda v$. (same idea as previously when $\mu = \lambda t$).

So, this model is valid when we replace “time” by “volume” or “area”.

Example: In the manufacturing process of commercial carpet, small faults occur at random in the carpet according to a Poisson process at an average rate of 0.95 per 20 m². One of the rooms of a new office block has an area of 80 m² and has been carpeted using the same commercial carpet described above.

What is the probability that the carpet in that room contains at least 4 faults?

Problem 5.8.1 (from the Course Notes):

Suppose that emergency calls to 911 follow a Poisson process with an average of 3 calls per minute. Find the probability there will be:

- a) 6 calls in a period of 2.5 minutes.
- b) 2 calls in the first minute of a 2.5 minute period, given that 6 calls occur in the entire period.

Combining Other Models with the Poisson Process (Section 5.9)

Sometimes we will need to use two or more probability distributions in a given application.

Example: Server requests come in according to a Poisson process with a rate of 100 requests per minute. A second is defined as “quiet” if it has no requests.

a) Find the probability that a second is “quiet”.

b) Find the probability that we observe 10 “quiet” seconds in a 60-second (1 minute) period.

c) Find the probability that we have to wait 30 seconds to get 2 “quiet” seconds.

d) If 10 “quiet” seconds occur in 60 seconds, what is the probability that exactly 2 occurred among the first 20 seconds?

Summary:

Discrete Distribution	Probability Function	Range
Discrete Uniform	$f(x) = \frac{1}{b - a + 1}$	$x = a, a + 1, \dots, b$
Hypergeometric	$f(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}$	$x \geq \max\{0, n - (N - r)\} \text{ \& } x \leq \min\{r, n\}$
Binomial	$f(x) = \binom{n}{x} p^x (1 - p)^{n-x}$	$x = 0, 1, 2, \dots, n$

Discrete Distribution	Probability Function	Range
Negative Binomial (# of failures before the k^{th} success)	$f(x) = \binom{x+k-1}{x} p^k (1-p)^x$	$x = 0, 1, 2, \dots$
Geometric (# of failures before the first success)	$f(x) = p(1-p)^x$	$x = 0, 1, 2, \dots$
Poisson	$f(x) = \frac{e^{-\mu} \mu^x}{x!}$	$x = 0, 1, 2, \dots$