Multivariate Distributions: Chapter 9

Chapter Outcomes:

- Basic terminology and techniques
- Multinomial distribution
- Covariance and correlation
- Mean and variance of linear combinations of random variables
- Linear combinations of independent normal random variables
- Indicator random variables

So far, we have focused our attention on **one** random variable at a time.

However, sometimes there are multiple random variables associated with a given experiment.

For example, to calculate one's BMI (body mass index), we need both the **height** and **weight** of an individual.

As another example, to calculate your final mark, you would need your **tutorial test marks**, your **term test marks**, and your **final exam mark**.

As a result, we now need to extend the ideas we learned in the previous chapters to deal with **multivariate** problems.

Notation:

Multiple random variables can be denoted by

$$X, Y, Z, \ldots$$

<u>*or*</u>

$$X_1, X_2, X_3, \ldots$$

We will only consider **discrete multivariate** problems.

Definition: For two random variables X and Y, we define

$$\begin{aligned} f(x, y) &= P(\{X = x\} \cap \{Y = y\}) \\ &= P(X = x, Y = y) \end{aligned}$$

as the **joint probability function** of (X, Y).

In general, the joint p.f. of $(X_1, X_2, \dots, X_n)$ is

$$f(x_1, x_2, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$$

Properties:

1. $f(x, y) \geq 0$ for all (x, y)

2. $\sum_{all\,(x,y)} f(x, y) = \sum_{all\,x} \sum_{all\,y} f(x, y) = 1$

Example: Consider the following joint pf given in table form for the r.v.'s X and Y

			x	
	$f(x,y)$	1	2	3
	1	0.09	0.12	0.13
y	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

Then, $f(1,1) = 0.09$**,** $f(1,2) = 0.12$**, ..., $f(3,1) = 0.13$, $f(3,2) = 0.10$, and finally $f(3,3) = 0.09$.**

What if we are given the joint p.f., but our actual interest lies in only **one** of the random variables?

Continuing with the previous example, if we are only interested in the r.v. *X*. *W*e see that:

			x	
	$f(x,y)$	1	2	3
	1	0.09	0.12	0.13
y	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

Similarly, if we are only interested in *Y*.

We have:

			x	
	$f(x,y)$	1	2	3
	1	0.09	0.12	0.13
y	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

What we just calculated is called the **marginal distribution.**

Definition: Given the joint p.f. of X and Y, the **marginal distributions** can be calculated via

$$f_X(x) = \sum_{\substack{all\ y\ with \\ x\ fixed}} f(x, y)$$

and

$$f_Y(y) = \sum_{\substack{all\ x\ with \\ y\ fixed}} f(x, y)$$

This idea can naturally be extended to more than two random variables.

For example, for X_1, X_2, X_3, we might have:

$$f_1(x_1) = P(X_1 = x_1) =$$

$$\sum_{\substack{all (x_2, x_3) \\ with\ x_1 fixed}} f(x_1, x_2, x_3)$$

or

$$f_{1,3}(x_1, x_3) = P(X_1 = x_1, X_3 = x_3)$$

$$= \sum_{\substack{all\ x_2 with \\ x_1 and\ x_3 fixed}} f(x_1, x_2, x_3)$$

Independent Random Variables

Recall: Two events, A and B, are said to be independent if and only if $P(A \cap B)=P(A)P(B)$.

We can extend this definition to random variables.

Definition: Two random variables, X and Y, are **independent if and only if**

$$f(x, y) = f_X(x) f_Y(y) \text{ for all pairs of values } (x, y)$$

or equivalently

$$P(X = x, Y = y) = P(X = x)P(Y = y) \text{ for all pairs of values } (x, y).$$

You can only conclude that X and Y are independent **after checking <u>ALL</u> $(\boldsymbol{x}, \boldsymbol{y})$ combinations.**

Even just a single case in which $f(x, y) \neq f_X(x) f_Y(y)$ would result in X and Y being **dependent** random variables.

In general, $X_1, X_2, \ldots, X_n$ are independent random variables **if and only if**

$$f(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n)$$

for all $x_1, x_2, \ldots, x_n$.

Continuing with our previous example…

		x			
	$f(x,y)$	1	2	3	$f_Y(y)$
y	1	0.09	0.12	0.13	0.34
	2	0.12	0.11	0.11	0.34
	3	0.13	0.10	0.09	0.32
	$f_X(x)$	0.34	0.33	0.33	1

Are X and Y independent random variables?

Conditional Probability Functions

Recall: For events A and B, $P(A|B) = \frac{P(AB)}{P(B)}$.

Definition: The conditional probability function of X given $Y = y$ is

$$f(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f(x, y)}{f_Y(y)}$$

provided that $P(Y = y) > 0$.

Note that this conditional p.f. is defined over all x in the range of the r.v. X with y <u>held fixed</u>.

Also, $f(x|y) \geq 0$ and $\sum_{all\ x} f(x|y) = 1$.

Continuing with our previous example…

		x			
	f(*x*,*y*)	1	2	3	$f_Y(y)$
y	1	0.09	0.12	0.13	0.34
	2	0.12	0.11	0.11	0.34
	3	0.13	0.10	0.09	0.32
	$f_X(x)$	0.34	0.33	0.33	1

Find the conditional p.f. of X given Y=1.

Recall earlier that two random variables X and Y are independent **if and only if**

$f(x, y) = f_X(x) f_Y(y)$ **for all pairs of values** (x, y)**.**

Under this assumption, note that

$$f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x).$$

In other words, the conditional p.f. simply becomes the marginal p.f. in the case of independence.

Functions of Random Variables

Example: Let U = X – Y where *X* and *Y* have the joint p.f. given below. We might now be interested in finding the p.f. of *U*, which is a function of the random variables *X* and *Y*.

		x		
	$f(x,y)$	1	2	3
y	1	0.09	0.12	0.13
	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

		x		
	$u=x-y$	1	2	3
y	1	0	1	2
	2	-1	0	1
	3	-2	-1	0

You Try:

Using the previous example, now find the p.f. of the r.v. T = X + Y.

			x	
	$f(x,y)$	1	2	3
	1	0.09	0.12	0.13
y	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

		x		
	$t = x + y$	1	2	3
y	1	2	3	4
	2	3	4	5
	3	4	5	6

You Try Solution

Following description of the table in the previous slide, we see that T = 2, 3, 4, 5, 6.

The probabilities associated with each value of the r.v. T are as follows:

$f_T(2) = 0.09$,

$f_T(3) = 0.12 + 0.12 = 0.24$,

$f_T(4) = 0.13 + 0.11 + 0.13 = 0.37$,

$f_T(5) = 0.1 + 0.11 = 0.21$,

and $f_T(6) = 0.09$ respectively.

We can organize distribution of T in a table as follows:

t	2	3	4	5	6
$f_T(t) = P(T = t)$	0.09	0.24	0.37	0.21	0.09

Note that to find the p.f. $f_U(u) = P(U = u)$, we are simply adding the probabilities for all (*x*,*y*) combinations such that u = *x*–*y*.

Likewise, to find the p.f. $f_T(t) = P(T = t)$, we are simply adding the probabilities for all (*x*,*y*) combinations such that t = *x*+*y*.

This particular p.f. could be written as

$$f_T(t) = \sum_{\substack{all\ (x,y) \\ with\ x+y=t}} f(x, y)$$

Notice, however, that if $t = x + y$, then this implies that $y = t - x$ and this leads to:

$$f_T(t) = P(T = t) = \sum_{all\ x} f(x, t - x)$$

$$= \sum_{all\ x} P(X = x, Y = t - x)$$

Note: Verify the result for $f_T(3) = P(T = 3)$ in the previous **You Try** exercise using the approach above!

In general, to find the p.f. for a function $U = g(X, Y)$ of two random variables X and Y, we have:

$$f_U(u) = P(U = u) = \sum_{\substack{all\ (x,y) \\ with\ g(x,y)=u}} f(x, y)$$

This can be extended to n random variables, where if $U = g(X_1, X_2, \dots, X_n)$, then:

$$f_U(u) = P(U = u) = \sum_{\substack{(x_1,\dots,x_n) \\ with\ g(x_1,\dots,x_n)=u}} f(x_1, \dots, x_n)$$

Example: Let X and Y be independent random variables having Poisson distributions with expected values μ_1 and μ_2, respectively.

Let $T = X + Y$. Find the p.f. of T, $f_T(t)$.

In a similar fashion, it can be shown that

if $X \sim Bin(m, p)$ and $Y \sim Bin(n, p)$

and

X and Y are independent random variables,

then

$$\boldsymbol{T = X + Y \sim Bin(m + n, p)}.$$