Instructions

1. **All electronic devices are to be turned off and put away.**
2. **This is a closed-book quiz.**
3. **When only considering a single random variable, X, $f_X(x) = f(x)$ and $F_X(x) = F(x)$.**
4. **Answer the questions in the spaces provided.**
5. **Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form.**
6. **Only question pages will be marked.**
7. **You may tear off the last two pages and use it for rough work. All pages will be collected at the end of the quiz.**
8. **Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed. No other aids are allowed.**
9. **A formula sheet has been provided at the end of this quiz.**
10. **DO NOT WRITE ON THIS COVER PAGE.**

Marking Scheme:

Questions	Out of
1 – 9	9
10	9
11	8
Total	26

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Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Suppose your friend makes the following 2 statements about any random variable X with probability function $f(x)$ and cumulative distribution function $F(x)$:

I	If $P(X \leq a) \leq P(X \leq b)$ then $a < b$
II	$F_X(x) \geq P(X = x)$

Which statements must be true?

A. Only I **B. Only II** C. I and II D. None

- We can see **I does not hold** in the case a=b.
- We know $P(X = x) = f_X(x)$ and $P(X \leq x) = F_X(x)$. Thus **II holds** as $P(X \leq x) = P(X = x) + P(X < x) \geq P(X = x)$.

2. You are given the following probability function in table form:

x	-2	0	2	3	4
$f_X(x)$	$1.5w$	w	$2w$	$1.5w$	$3w$

If w is a constant, what is the value of w?

A. $\frac{1}{7}$ **B.** $\mathbf{\frac{1}{9}}$ C. $\frac{1}{6}$ D. 9

- Since $\sum_x f_X(x) = 1$, we can compute $\sum_{i\in\{-2,0,2,3,4\}} f_X(x)$ which gets $1.5w + w + 2w + 1.5w + 3w = 1$. Thus $\boldsymbol{p} = \frac{\mathbf{1}}{\mathbf{9}}$.

3. Yvonne is newcomer to Waterloo and decides to pet the cute baby geese. The probability that a mother goose attacks her remains constant and is 0.7 every time Yvonne attempts to pet a baby. If the attacks are independent each time Yvonne tries to pet a cute baby goose, what is the probability Yvonne is attacked exactly 5 times if she attempts to pet a baby 8 times?

A. 0.186 B. 0.055 C. 0.289 **D. 0.254**

- We are counting the number of times Yvonne is attacked out of 8 independent trials. This is fit using a binomial model where $n = 8$ and $p = 0.7$, defining a success as an attack. Stating $X \sim Binomial(8{,}0.7)$, we see $f_X(5) = \binom{8}{5}0.7^5(1-0.7)^3 \approx \mathbf{0.254}$.

4. Suppose that a discrete random variable X has a CDF given by $F(x) = \frac{x^2}{9}$ for $x = 1, 2, 3$. What is the pf of X (on relevant values of x)?

A. $\frac{2x}{9}$ B. $\frac{1-2x}{9}$ **C.** $\mathbf{\frac{2x-1}{9}}$ D. $\frac{x^2+x+1}{9}$

- We know $f_X(x) = F_X(x) - F_X(x-1)$. Therefore $f_X(x) = \frac{x^2}{9} - \frac{(x-1)^2}{9}$ after simplification we get $f_X(x) = \frac{\mathbf{2x-1}}{\mathbf{9}}$ for $x \in \{1,2,3\}$.

5. Raphael decides to ride a hot air balloon to enjoy the sunset while eating grapes. However, he gets bored while the balloon is rising and starts dropping grapes from his huge pile. Since he is over a parking lot, he hits a car according to a Poisson process with an average of 11 hits for every 15 dropped grapes. What is the probability Raphael hits exactly 18 cars on the next 30 drops?

A. $\mathbf{0.064}$ B. 0.068 C. 0.109 D. 0.085

- We are given the chance of hitting a car is modeled by a Poisson distribution. If we say X is the number of hits, then $X \sim Poisson(\lambda)$. We are given $\lambda = 11 \frac{hits}{15-drops}$; however, the question asks about the next 30 drops. We convert λ, such that we get $\lambda = 11 \frac{hits}{15-throws}\left(2 \frac{15-throws}{30-throws}\right) = 22 \frac{hits}{30-throws}$.
- Therefore, $f_X(x) = \frac{e^{-22}22^x}{x!}$ for $x = 1,2,3,\ldots$. So, the probability of hitting 18 cars in the next 30 drops is $f_X(18) = \frac{e^{-22}22^{18}}{18!} \approx \mathbf{0.064}$.

6. The number of wolves sighted over the course of a day in a remote area of Alberta is described by the probability function:

x	f(x)
2	w
3	0.11
4	0.27
5	0.35
6	0.11

What is the probability there are less than 4 wolf sightings in the upcoming week?

A. 0.33 B. 0.29

C. $\mathbf{0.27}$ **D. Cannot be determined from the given information**

- Note, this is written as a fair, but tricky question. Due to the way this question is worded, the answer would most correctly be **D** since we are looking at a **week** and given information on a **day**. Moreover, the days are not necessarily independent based on the question. However, many people overlooked this slight word change so we also accepted answer **C** which is the probability of seeing less than 4 wolves in a day. Be very careful when reading questions and pay extra attention to units.
- If (c), we can first find the value of p. We know $\sum_x f(x) = 1$. Therefore, $w + 0.11 + 0.27 + 0.35 + 0.11 = 1$ and we find $w = 0.16$.
- Now we want $P(X < 4)$. With this distribution, this is the same as $P(X = 2) + P(X = 3)$. Therefore, the answer is $\mathbf{0.27}$.
- Likewise, we can compute $P(X < 4) = 1 - P(X \geq 4)$ for the same solution.

7. Holly buys a bag of balloons of assorted colors for an upcoming birthday. After popping some, she looks inside the bag and sees 11 orange, 16 pink, and 25 green balloons remaining. Which of the following probability functions would be used to determine the exact probability that Holly obtains exactly x green balloons if she randomly selects 5 balloons from the bag to be blown up?

A. $\binom{5}{x}\left(\frac{25}{52}\right)^x\left(1-\frac{25}{52}\right)^{5-x}$ **B.** $\frac{\binom{25}{x}\binom{27}{5-x}}{\binom{52}{5}}$ C. $\frac{\binom{27}{x}\binom{25}{5-x}}{\binom{52}{5}}$ D. $\binom{x+5}{x}\left(\frac{25}{52}\right)^x\left(1-\frac{25}{52}\right)^x$

- We know there are 52 total remaining balloons $(11 + 16 + 25)$, and a success is defined as selecting a green balloon, which there are 25. Since Holly selects 5 and blows up the balloons, the selection must be made without replacement. Therefore, a hypergeometric distribution is appropriate. If X is the number of green balloons Holly selects, the appropriate distribution is $f_X(x) = \frac{\binom{25}{x}\binom{27}{5-x}}{\binom{52}{5}}$ for $0 \leq x \leq 5$ and $x \in \mathbb{Z}$.
- Note, using a binomial distribution may be an appropriate <u>approximation</u>, but the hypergeometric distribution gives the exact probability.

8. Suppose a talking parrot claims the function $f(x) = k\left(\frac{1}{2}\right)^{x+1}$ for $x = 0{,}1{,}2{,}3$ and $f(x) = 0$ otherwise, is a probability function for some constant k. What is $P(X > 0)$?

A. $\frac{8}{15}$

B. $\mathbf{\frac{7}{15}}$

C. $\frac{16}{15}$

D. Cannot be determined from the given information

- Since $\sum_x f_X(x) = 1$, we see $k\left(\frac{1}{2}\right)^{0+1} + k\left(\frac{1}{2}\right)^{1+1} + k\left(\frac{1}{2}\right)^{2+1} + k\left(\frac{1}{2}\right)^{3+1} = 1$. After simplifying, we find $k = \frac{16}{15}$. Thus $f_X(x) = \frac{16}{15}\left(\frac{1}{2}\right)^{x+1}$.
- We can compute $P(X > 0) = 1 - f_X(0) = 1 - \frac{16}{15}\left(\frac{1}{2}\right)^{1} = 1 - \frac{8}{15} = \mathbf{\frac{7}{15}}$

9. A random variable X takes on values in the countably infinite set $\{1,2,3, \dots\}$ and has the property $P(X \geq x) = 0.6\, P(X \geq x - 1)$ for all $x = 2, 3, 4, \dots$. What is $P(X = 4)$?

A. $\mathbf{0.086}$

B. 0.052

C. 0.6

D. Cannot be determined from the given information

- We are given $P(X \geq x) = 0.6\, P(X \geq x - 1)$. We also know since this is a discrete distribution taking integer values, $P(X = 4) = P(X \geq 4) - P(X \geq 4 + 1)$. We can substitute to see $P(X = 4) = P(X \geq 4) - 0.6P(X \geq 4)$. This means $P(X = 4) = (1 - 0.6)P(X \geq 4)$.
- We can continue using the relation to see $P(X = 4) = (1 - 0.6)(0.6\, P(X \geq 4 - 1))$, $P(X = 4) = (1 - 0.6)(0.6^2\, P(X \geq 4 - 2))$, and so on until we reach $P(X = 4) = (1 - 0.6)(0.6^{4-1}\, P(X \geq 1))$.
- We know $\mathrm{P}(X \geq 1) = 1$ since that is all possible values. Thus the solution is $P(X \geq 4) = (1 - 0.6)(0.6^{4-1}) \approx \mathbf{0.086}$.

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work. Unjustified answers will not receive full credit.

10. Yejun goes shopping once a week. When he gets home, he tries to carry everything inside in a single trip. Unfortunately, this means he always drops exactly one item. In fact, he is very consistent and each trip is independent such that he always drops the eggs with probability 0.15, the milk with probability 0.35, the ice cream with probability 0.25, and flour with probability 0.25.

a. What is the probability Yejun drops the eggs at least once in the next two weeks? **(2 marks)**

- We know the probability of dropping the eggs is 0.15. If we define X as the number of times Yejun drops the eggs then we need to look at $P(X \geq 1)$.
- This can be modeled using a Binomial such that $P(X \geq 1) = 1 - F_X(0) = 1 - \binom{2}{0}(0.15)^0(1-0.15)^2 = 0.278$.
- Alternatively, using previous tricks, $P(X = 1) = 2 \times P(eggs\ 1\ day)P(other\ 1\ day) + P(eggs\ day\ 1)P(eggs\ day\ 2) = 2(0.15)(1-0.15) + (0.15)(0.15) = 0.278$.

b. Suppose X represents the number of trips Yejun drops some other kind of food, given above, before dropping the flour for the first time. What type of distribution does X have? For full credit, please include the appropriate probability function and the range of X. **(2 marks)**

- Since we do not know in advance how many trips Yejun will have before dropping the flour and we are only looking at the first time of dropping the flour, this is best modeled by a geometric distribution (though starting with a negative binomial distribution will result in the same probability function and range).
- If X counts the number of trips Yejun takes before dropping the flour then $f_X(x) = (0.25)(1-0.25)^x$ for $x = 0,1,2,\ldots$ (note, $f(x) = 0$ otherwise).

c. What is the probability Yejun drops another type of food exactly 3 times before dropping the flour? **(2 marks)**

- From part (b) we know $f_X(x) = (0.25)(1-0.25)^x$ for $x = 0,1,2,\ldots$, where X counts the number of trips before Yejun drops the flour for the first time.
- Therefore, we can compute $f_X(3) = (0.25)(1-0.25)^3 \approx 0.105$.

d. What is the probability Yejun will drop the milk on at least 2 of the next 4 shopping trips? For full credit, please clearly define the random variable and give the appropriate probability function (with the range of the random variable). **(3 marks)**

- We know this is a 4-week period ($n = 4$), the probability of dropping the milk is 0.35 ($p = 0.35$). This is a perfect application of the Binomial distribution.
- Let X denote the number of trips Yejun drops the milk in 4 trips. Thus $X \sim Binomial(4, 0.35)$. We see $f(x) = \binom{4}{x}(0.35)^x(1-0.35)^{4-x}$ for $x = 0,1,2,3,4$ ($f(x) = 0$ otherwise).
- Therefore, the probability Yejun drops the milk at least 2 times in this 4-week period can be computed $P(X \geq 2) = f(2) + f(3) + f(4)$, or likewise $P(X \geq 2) = 1 - P(X \leq 1) = 1 - \big(f(0) + f(1)\big)$. Either way, we find $P(X \geq 2) \approx 0.437$.

11. Davy loves hunting for caterpillars! Suppose that on a given trip, the number of caterpillars he finds is best modeled with a Poisson distribution, finding an average of 15 caterpillars an hour.

a. Suppose X represents the number of caterpillars Davy finds in t minutes. Please give the appropriate probability function for X. For full credit, please include the range of X. **(2 marks)**

- We are given this is best modeled using a Poisson distribution, and we define X to be the number of caterpillars Davy finds. We are given $\lambda = 15\frac{caterpillars}{hour}$, but since we are looking at minutes, we must convert to $\lambda = \frac{15}{60}\frac{caterpillars}{minutes} = 0.25\frac{caterpillars}{minutes}$.
- Plugging in, $f_X(x) = \frac{e^{-0.25t}(0.25t)^x}{x!}$ for $x = 0,1,2,\ldots$ ($f(x) = 0$ otherwise).

b. What is the probability Davy finds exactly 14 caterpillars in the next hour? For full credit, please clearly define the random variable used. **(2 marks)**

- From part (a), we now just set $t = 60$. Note $\lambda t = 15$.
- X is the number of caterpillars Davy finds in the next hour.
- We find $P(X = 14) = f(14) = \frac{e^{-15}15^{14}}{14!} \approx 0.102$.

c. What is the probability Davy finds at most 2 caterpillars in the next 16 minutes? For full credit, please clearly define the random variable used. **(2 marks)**

- From part (a), we now just set $t = 16$. Note $\lambda t = 4$.
- X is the number of caterpillars Davy finds in the next 16 minutes.
- Since we want 'at most', we need $P(X \leq 2) = F(2) = f(0) + f(1) + f(2)$. Plugging in the values, $P(X \leq 2) \approx 0.0183 + 0.0733 + 0.1465 \approx 0.238$.

d. Consider the Poisson model describing the number of caterpillars Davy finds in t minutes, from part (a). What is the minimum amount time necessary (in minutes, to 3 decimal places) for Davy to have at least an 80% chance of finding at least one caterpillar? **(2 marks)**

- From part (a), $f_X(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ for $x = 0,1,2,\ldots$.
- We want $P(X \geq 1) \geq 0.80$. This can be written $1 - P(X = 0) \geq 0.80$. We can simplify and write $1 - e^{-\lambda t} \geq 0.80$.
- After manipulation, we see $t > \frac{\log(0.2)}{-\lambda} \approx 6.438$ minutes.

Notation and Parameters	**Probability Function** $f(x)$
Discrete Uniform(a, b) $b \geq a$ a, b integers	$\dfrac{1}{b-a+1}$ $x = a, a+1, \ldots, b$
Hypergeometric(N, r, n) $N = 1, 2, \ldots$ $n = 0, 1, \ldots, N$ $r = 0, 1, \ldots, N$	$\dfrac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = max(0, n - N + r), \ldots, min(r, n)$
Binomial(n, p) $0 \leq p \leq 1, q = 1 - p$ $n = 1,2, \ldots$	$\binom{n}{x} p^x q^{n-x}$ $x = 0,1, \ldots, n$
Bernoulli(p) $0 \leq p \leq 1, q = 1 - p$	$p^x q^{1-x}$ $x = 0,1$
Negative Binomial(k, p) $0 < p \leq 1, q = 1 - p$ $k = 1,2, \ldots$	$\binom{x+k-1}{x} p^k q^x = \binom{-k}{x} p^k (-q)^x$ $x = 0,1, \ldots$
Geometric(p) $0 < p \leq 1 , q = 1 - p$	pq^x $x = 0,1, \ldots$
Poisson(λ) $\lambda \geq 0$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1, \ldots$

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