STAT 230: Probability (Sec 02)
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Lecture 8

Today's Agenda

Last time:

- Venn Diagram
- Fundamental laws of set algebra
- De Morgan Rules
- Inclusion-Exclusion Principle

Today (Lec 8, 05/18):

- Independence
- Conditional Probability

Independent events

Definition

Two events A and B are said to be **independent** if

$$P(A \cap B) = P(A)P(B).$$

A sequence of events $A_1, A_2, ...$ are said to be independent if

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}),$$

for all possible subsets of size k for any $k \geq 2$ and all possible $i_1, i_2, \ldots, i_k$. Events that are not independent are called **dependent**.

This means that, for a sequence of events to be independent, all of their subsets must be independent as well.

Example

Consider rolling two fair six sided dice, and let

$A = \{\text{the sum is } 10\}$, $B = \{\text{the first die is a } 6\}$ $C = \{\text{the sum is } 7\}$.
Determine:

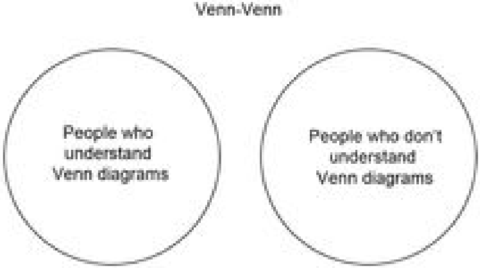
a) Are A and B independent?

b) Are B and C independent?

c) Are A and C independent?

Definition

Two events A and B are **mutually exclusive** if A and B are disjoint.



A common misconception is that if A and B are mutually exclusive, then A and B are independent.

Proposition

Suppose that not both A and B are trivial events (empty sets).
If A and B are independent and mutually exclusive (disjoint), then either $P(A) = 0$ or $P(B) = 0$.

Proposition: If A and B are independent, then $\bar{A}$ and $\bar{B}$ are independent, A and $\bar{B}$ are independent, and $\bar{A}$ and B are independent.

Proof is an exercise.

Our ideas/calculation of probabilities may change upon the attainment of some information.

Example

You roll a die and your friend looks at what number you rolled.

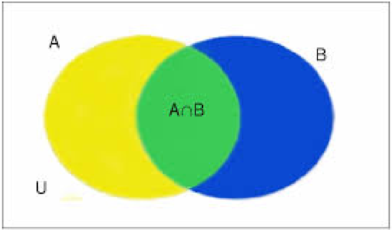
- What's the probability that it's a 6?
- What's the probability that it's a 6 given that your friend tells you that it's an even number?
- What's the probability that it's a 6 given that your friend tells you that it's an odd number?

Conditional Probability of Events

Definition

The conditional probability of A *given* B is, so long as $P(B) > 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$



Example

Consider again rolling two fair six sided dice, and let

$A = \{\text{the sum is } 10\}$, $B = \{\text{the first die is a } 6\}$, $C = \{\text{the sum is } 7\}$.

a) Compute $P(A|B)$
b) Compute $P(B|A)$
c) Compute $P(A|C)$
d) Compute $P(C|B)$

Equivalent definition of independence

Definition (Equivalent definition of independence)

Two events A and B are independent, if

$$P(A|B) = P(A),$$

provided $P(B) > 0$.

Intuition: The probability that A happens is not affected by knowing that B happened.

Some rules for conditional probabilities

Conditional probability behaves the same way as usual probability:

a) $0 \leq P(A|B) \leq 1$

b) $P(\bar{A}|B) = 1 - P(A|B)$

c) If A_1 and A_2 are disjoint: $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B)$

d) $P(S|B) = 1 = P(B|B)$

Product Rule

Theorem

For events A and B,

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

This is known as the **product rule** and follows directly from the definition of conditional probability.

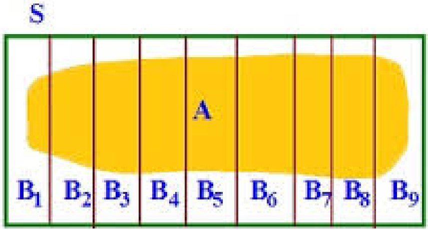
Example

Erik has 12 red pants and 7 green pants. On Monday, Erik picks two pairs of pants for his Monday and Wednesday lectures. He draws one pair of pants at random and then, without replacement, another pair.

a) What is the probability that both pairs of pants are red?
b) What is the probability that the second pair of pants is red?

Definition

A sequence of sets $B_1, B_2, ..., B_k$ are said to **partition** the sample space S if $B_i \cap B_j = \emptyset$ for all $i \neq j$, and $\cup_{j=1}^{k} B_j = S$.



Partitioning the sample space

Theorem (Law of total probability)

Suppose that $B_1, B_2, ..., B_k$ partition S. Then for any event A,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_k)P(B_k).$$

Remark: A common way to use this is

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}),$$

since B and $\bar{B}$ partition S.

Proof