STAT 230 Spring 2022 Test #2 Solutions

Instructions

1. **All electronic devices are to be turned off and put away. If you haven't already done this, please do so now.**
2. **This is a 90-minute closed-book test.**
3. **Answer the questions in the spaces provided.**
4. **Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form.**
5. **Only question pages will be marked.**
6. **You may tear off the last page and use it for rough work. All pages will be collected at the end of the test.**
7. **Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed.**
8. **A formula sheet has been included with this test, and can be found at the end of the test. You may CAREFULLY detach the rough work pages and formula sheet if you like.**
9. **The test will be marked out of 51. With the bonus opportunities, it will be possible to score more than 100% on this test.**

10. **DO NOT WRITE ON THIS COVER PAGE.**

Marking Scheme:

Questions	**Out of**
1 – 15	15
16	8
17	8
18	11
19	9
BONUS 1	2
BONUS 2	4
Total	51

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Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Consider the following probability function, P(X = x) = kx^2 for x = 1, 2, 3, 4; 0, otherwise. Let F(x) represent the cdf of X. What is F(2)?

A) 1/6 B) 1/10 C) 1/180 D) 4/30

- **We need to solve for k first.**
- **In this case, $k[1^2 + 2^2 + 3^2 + 4^2]$ => 30k = 1. So k = 1/30.**
- **Thus, F(2) = P(X ≤ 2) = ($1^2 + 2^2$)/30 = 5/30 = 1/6**

<u>Use the following information to answer the next TWO questions</u>:

Consider the following cdf for the random variable X:

x	F(x)
4	0.05
5	0.1
6	0.35
7	0.65
8	0.85
9	1

2. Calculate P(6 ≤ X ≤ 8).

A) 0.7 B) 0.5 **C) 0.75** D) 0.9

- **P(6 ≤ X ≤ 8) = P(X = 6) + P(X = 7) + P(X = 8)**
- **Using the cdf, we see that this equivalent to F(8) – F(5) = 0.85 – 0.1 = 0.75**
- **You might be tempted to calculate this using F(8) – F(6), but this is incorrect, as it excludes P(X = 6).**

3. Calculate sd(X), the standard deviation of X.

A) 7.152 **B) 1.203** C) 7.05 D) 3.1667

- **From the given cdf, we can determine P(X = x) values.**
- **P(X = 4) = 0.05, P(X = 5) = F(5) – F(4) = 0.1 – 0.05 = 0.05, P(X = 6) = F(6) – F(5) = 0.25, P(X = 7) = F(7) – F(6) = 0.3, P(X = 8) = F(8) – F(7) = 0.2, P(X = 9) = F(9) – F(8) = 0.15**
- **Var(X) = E(X^2) – $[E(X)]^2$**
- **E(X) = 4*(0.05) + 5(0.05) + 6(0.25) + 7(0.3) + 8(0.2) + 9(0.15) = 7**
- **E(X^2) = 4^2(0.05) + 5^2(0.05) + 6^2(0.25) + 7^2(0.3) + 8^2(0.2) + 9^2(0.15) = 50.7**
- **So, Var(X) = 50.7 – $(7)^2$ = 1.7, and sd(X) = $\sqrt{Var(X)} = \sqrt{1.7}$.**
- **The best answer is 1.203, given the choices.**

4. Suppose that a fair coin is to be tossed a total of three times. A player will win \$1 if the first toss is a head, but will lose \$1 if the first toss is a tail. Similarly, the player wins \$2 if the second toss is a head, but will lose \$2 if the second toss is a tail. Finally, they will win \$3 if the third toss is a head, but will lose \$3 if the third toss is a tail. Define the random variable X to represent the total winnings after three tosses.

Note: It is possible for the value of X to be negative. This represents a loss after three tosses.

What is the most likely value of the random variable X?

A) -2 **B) 0** C) 2 D) 4

- **We can use first principles to solve. We can list all outcomes in the sample space:**
- **S = {(H,H,H), (H,H,T), (H,T,H), (H,H,T), (H,T,T), (T,H,T), (T,T,H), (T,T,T)}**
- **We can turn the outcomes int a set of winnings.**
- **S={(1,2,3), (1,2,-3), (1,-2,3), (-1,2,3), (-1,-2,3), (-1,2,-3), (1,-2,-3), (-1,-2,-3)}**
- **S={6, 0, 2, 4, 0, -2, -4, -6}. The coin is fair, so all outcomes are equally likely. We see that 0 occurs most often, so 0 is the most likely value of the random variable X (with probability 1/4).**

5. You are given the following pdf of X:

f(x) = kx for, 2 < x < 4; 0, otherwise.

Determine the value of k that makes this a valid pdf.

A) 2/15 B) 1/8 **C) 1/6** D) None of these

- **Given: f(x) = kx, 2 < x < 4; 0, otherwise.**
- **We want to find the value of k that makes f(x) a valid pdf.**
- **In this case, one of the conditions is that $\int_2^4 f(x)dx = 1 => k\int_2^4 xdx = 1$.**
- **So, $k[\frac{x^2}{2}]|_2^4 = 1 => 6k = 1$. Thus, k = 1/6**

6. You are given the following pdf of X:

$$f(x) = \begin{cases} x^2, \text{for } 0 < x \leq 1 \\ \frac{2}{3}, \text{for } 1 < x \leq 2 \\ 0, \text{otherwise} \end{cases}$$

What is the median of X?

A) 7/4 B) 3/2 C) 3/4 **D) 5/4**

- **There are a couple of ways to do this one.**
- **Let's check the total area under the curve from 0 to 1.**
- **The cdf for the first piece is given by: F(x) = $\frac{x^3}{3}$, and the total area from 0 to 1 is 1/3. So, we know that the median must be greater than 1.**
- **The second piece of the pdf is a rectangle with height 2/3. We can find the median by solving for the value, x, such that P(X > x) = 0.5 in this piece.**
- **This suggests that we want the total area under the curve to the right of x to be 0.5. Again, it's a rectangle, with height 2/3. So area = b*h = 1/2 => b(2/3) = 1/2.**
- **So, b = 3/4, but this is the base from x to 2, so the median value of this distribution is given by 2 – 3/4 = 5/4**

7. A student is writing a multiple choice test with 10 questions. Each question has 4 possible answers to choose from. The student has not prepared very well for this test. However, they do know the answer to three questions, for sure, but will need to randomly guess the answer for the other questions. You can also assume that their guesses are independent from question to question. If a minimum grade of 50% is needed to pass the test, what is the probability that the student will pass the test?

A) 0.311 **B) 0.555** C) 0.623 D) 0.445

- **Let X represent the number of correct answers. X~Binomial(7, 0.25)**
- **In order to pass the test, the student needs at least two correct from the group of 7 that they need to guess the answers to.**
- **So, we want P(X ≥ 2) = 1 – [P(X=0) + P(X=1)].**
- **In this case, P(X = x) = $\binom{7}{x}(0.25)^x(0.75)^{7-x}$ for x = 0, 1, 2, ..., 7; 0, otherwise**
- **P(X ≥ 2) = 1 – [$\binom{7}{0}(0.25)^0(0.75)^7 + \binom{7}{1}(0.25)^1(0.75)^6$] = 0.555**

8. Suppose that a certain breed of dog will have 6 pups in a litter. Let's assume that P(male) = P(female) = 0.5, and that the sex of each pup is independent. What is the probability that there are at least two females in the litter given that there is at least one male in the litter?

A) 0.889 B) 0.781 C) 0.111 D) 0.794

- **Let X represent the number of male pups in the litter. In this case, X~Binomial(6,0.5).**
- **We want P(at least 2 females | at least one male)**
- **Note: at least 2 females = no more than 4 males**
- **So, we want P(X ≤ 4 | X ≥ 1) = $\frac{P(1 \le X \le 4)}{P(X \ge 1)}$**
- **Where P(X = x) = $\binom{6}{x}(0.5)^6$ for x = 0, 1,..., 6; 0, otherwise**
- **P(X ≤ 4 | X ≥ 1) = $\frac{P(X=1)+P(X=2)+P(X=3)+P(X=4)}{1-P(X=0)}$ = 0.889**

9. Suppose that the number of surface flaws in plastic panels used in a car's interior has a Poisson distribution with a mean of 0.05 flaws per square foot. Assume that a car's interior contains 10 square feet of plastic panel. Suppose that 10 cars are sold to a rental company. What is the probability that at most one of them has exactly one interior surface flaw?

A) 0.921 B) 0.856 C) 0.117 **D) 0.144**

- **There are two layers to this question.**
- **First, let X represent the number of surface flaws in plastic panels in a car.**
- **In this case, X has a Poisson distribution with μ = 0.05*10 = 0.5**
  and P(X = x) = $\frac{e^{-0.5}0.5^x}{x!}$ for x = 0, 1, ...; 0, otherwise.
- **Now let Y represent the number of cars in a random sample of 10 that have exactly one interior surface flaw.**
- **Y has a Binomial distribution with n = 10 and p = P(X = 1) = 0.5$e^{-0.5}$**
- **So, P(Y = y) = $\binom{10}{y}(0.5e^{-0.5})^y(1-0.5e^{-0.5})^{10-y}$ for y = 0, 1, 2, ..., 10; 0, otherwise**
- **We want P(Y ≤ 1).**
- **P(Y ≤ 1) = $\binom{10}{0}(0.5e^{-0.5})^0(1-0.5e^{-0.5})^{10} + \binom{10}{1}(0.5e^{-0.5})^1(1-0.5e^{-0.5})^9$ = 0.144**

10. DEF Laptop Co. has determined that 1 in every 500 laptops that it produces are defective. A random sample of 750 laptops is taken from a production run and the laptops are tested. Using the Poisson approximation to the Binomial calculate the approximate chance that at least 3 of them are found to be defective.

A) 27.48% B) 18.99% **C) 19.12%** D) 44.22%

- **We want to use the Poisson approximation to the Binomial to solve this.**
- **Let X represent the number of defective laptops.**
- **X has an approximate Poisson distribution with μ = 750/500 = 1.5.**
- **We want $P(X \geq 3) = 1 - P(X \leq 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$**
  where $P(X = x) = \frac{e^{-1.5}1.5^x}{x!}$ for x = 0, 1, ...; 0, otherwise.
- **$P(X \geq 3) = 1 - e^{-1.5}[1 + 1.5 + \frac{1.5^2}{2}] = 0.1912$**

11. In a best of seven game playoff series, two teams play each other until one team wins 4 games. In this series, team A is favoured in such a way that they will beat team B with probability 0.6. We will assume that the probability stays constant from game to game, and that the results of each game are independent. What is the probability that the series will end in 6 games?

A) 0.0922 B) 0.393 C) 0.2074 **D) 0.2995**

- **We want the probability that the 4^{th} win comes in the 6 game (for either team).**
- **Let X_A represent the number of losses by team A before their 4^{th} win.**
- **Let X_B represent the number of losses by team B before their 4^{th} win.**
- **X_A has a negative binomial distribution with k = 0.4 and p = 0.6.**
- **X_B has a negative binomial distribution with k = 0.4 and p = 0.6.**
- **P(series ends in 6 games) = $P(X_A = 2) + P(X_B = 2)$**
- **P(series ends in 6 games) = $\binom{5}{2}(0.6)^4(0.4)^2 + \binom{5}{2}(0.4)^4(0.6)^2 = 0.2995$**

12. Suppose that numbers are to be randomly drawn <u>without replacement</u> from the set:

{1, 2, 3, 4, ……, 20}.

Let X represent the number of draws until the third prime number is drawn.

What is the range of X? **Note:** 1 is not a prime number.

A) {3, 4, 5, 6, …., 15} B) {3, 4, 5, 6, …., ∞}

C) {1, 2, 3, 4, 5, 6, 7, 8} D) {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

- **In this case, we would need to draw at least 3 numbers to see the third prime number. We are drawing without replacement, so we could draw all 12 of the non-primes before we see the third prime. So, the range for X is {3, 4, 5, ..., 15}.**

13. Suppose that machine breakdowns occur according to a Poisson process with λ = 0.2 per day. This also means that the time between machine breakdowns is Exponentially distributed with λ = 0.2 per day. Let Y represent the time between machine breakdowns. What is P(Y > E(Y))?

A) $\frac{e-1}{e}$ **B)** $\mathbf{\frac{1}{e}}$ C) $\frac{1}{e^5}$ D) $\frac{e^5-1}{e^5}$

- **Let Y represent the time between breakdowns. Y has an Exponential distribution with λ = 0.2. F(y) = 1 – $e^{-0.2y}$ and E(Y) = $\frac{1}{\lambda} = 5$.**
- **So, we want P(Y > 5) = 1 – F(5) = $e^{-0.2(5)}$ = e^{-1} = $\frac{1}{e}$.**

14. Suppose that the cdf for a continuous random variable X is given by F(x) = kx^2, 0 ≤ x ≤ 3. What is the value of k?

A) 1/6 B) 1/3 C) 1 **D) 1/9**

- **The cdf of X is given by F(x) = kx^2, 0 ≤ x ≤ 3. We know that the cdf would be 1 for y = 3.**
- **So 9k = 1 => k = 1/9.**

15. The pdf of the magnitude, *X*, of a load on a bridge (in Newtons) is given by:

$$f(x) = \frac{1+3x}{8},\ 0 \le x \le 2;\ 0,\ \text{otherwise}$$

Calculate Var(X).

A) 13/48 B) 25/16 C) 11/6 D) 151/96 D) None of these

- **Var(X) = E(X²) – [E(X)]²**
- **E(X²) = $\int_0^2 x^2 * f(x)dx = \int_0^2 x^2 * \left(\frac{1+3x}{8}\right)dx = \frac{1}{8}\int_0^2 (x^2+3x^3)dx = \frac{1}{8}\left[\frac{x^3}{3}+\frac{3x^4}{4}\right]\Big|_0^2 = \frac{22}{12} = \frac{11}{6}$**
- **E(X) = $\int_0^2 x * f(x)dx = \int_0^2 x * \left(\frac{1+3x}{8}\right)dx = \frac{1}{8}\int_0^2 (x+3x^2)dx = \frac{1}{8}\left[\frac{x^2}{2}+x^3\right]\Big|_0^2 = \frac{10}{8} = \frac{5}{4}$**
- **So, Var(X) = (11/6) – (5/4)² = 26/96 = 13/48**

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work / justify your answer. Unjustified answers will not receive full credit.

16. ***(8 marks)*** Driving across the country, Josie gets bored and starts counting the cows she sees on her never-ending car ride. Suppose that cow sightings follow a Poisson process at an average of 12 cows an hour on this cross-country journey.

a. Determine the probability function of the distribution for the number of cows Josie will see in the next t hours. For full credit, please state the range of the random variable. ***(2 marks)***

- **Let X be the number of cows Josie sees in t hours.**
- **We are given** $\lambda = 12\,\frac{\text{cows}}{\text{hour}}$ **for a Poisson process.**
- **Therefore** $X \sim \text{Pois}(12\,t)$
- $f(x) = \frac{e^{-12t}(12t)^x}{x!}$ **for** $x = 0, 1, 2, 3, 4 \ldots.;\ 0$**, otherwise**

b. Let Y represent the number of cows that Josie will see in the next 30 minutes. Determine the probability function of Y and use the probability function to calculate the probability Josie will see fewer than 3 cows in the next 30 minutes. ***(3 marks)***

- **Let Y be the random variable counting the number of cows Josie sees in the next 30 minutes**
- $f_{Y(y)} = \frac{e^{-6}6^y}{y!}$ **for** $y = 0, 1, 2, 3, 4, \ldots.;\ 0$**, otherwise.**
- $P(Y < 3) = P(Y \le 2) = f_Y(0) + f_Y(1) + f_Y(2) = e^{-6} + 6e^{-6} + \frac{36e^{-6}}{2} = 0.062$ **(to 3 d.p.)**

c. Now suppose her Dad stops every 30 minutes for the next 5 hours, thus creating 10 distinct, non-overlapping 30-minute segments. What is the probability she sees fewer than 3 cows on exactly 2 of these segments? For full credit, please clearly define the random variable of interest and state its probability function with related range. **Note:** If you need to use the probability in part b. to answer this question, please use the probability in part b. to 3 d.p. ***(3 marks)***

- **From part (b), we know the probability of seeing fewer than 3 cows is** 0.062.
- **If we let W represent the number of segments that Josie sees fewer than 3 cows.**
- $W \sim \text{Bin}(10, 0.062)$
- $f_W(w) = \binom{10}{w}(0.062)^w(1-0.062)^{10-w}$ **for** $w = 0, 1, 2, \ldots, 9, 10;\ 0$**, otherwise.**
- $P(W = 2) = f_W(2) = \binom{10}{2}(0.062)^2(1-0.062)^{10-2} = 0.104$ **(to 3 d.p.)**

17. ***(8 marks)*** Jack is playing games with his younger brother Sam. Suppose that no games end in a tie, and that Jack wins a game with probability 0.65. This probability remains the same each time a game is played. The outcomes of each game are also assumed to be independent.

a. Let X represent the number of games Sam wins before Jack wins his first game. State the distribution of X with all appropriate parameters and determine probability function of X. For full credit, state the range of the random variable. Use the probability function to calculate the probability that Sam wins at most 1 game before Jack wins his first game. ***(3 marks)***

- **Let X represent the number of games Sam wins before Jack wins his first game.**
- **X has a Geometric distribution with $p = 0.65$.**
- $f_X(x) = (0.65)(1-0.65)^x$ **for** $x = 0, 1, 2, 3, \ldots;$ **0, otherwise**
- **We want P(Sam wins at most 1 game before Jack wins his first game).**
- **In terms of the r.v. X, we want** $P(X \leq 1) = P(X = 0) + P(X = 1)$
- $P(X \leq 1) = (0.65)(0.35)^0 + (0.65)(0.35)^1 =$ **0.878 (to 3 d.p.)**

b. Let Y represent the number of games that Jack wins before Sam wins exactly three games. State the distribution of Y with all appropriate parameters and determine probability function of Y. For full credit, state the range of the random variable. Use the probability function to calculate P(Y > 2). ***(3 marks)***

- **Let Y represent the number of games Jack wins before Sam wins exactly 3 games.**
- **Y has a Negative Binomial distribution with** $k = 3$ **and** $p = 1 - 0.65 = 0.35$
- $f_Y(y) = \binom{y+2}{y}(0.35)^3(0.65)^y$ **for** $y = 0, 1, 2, 3, \ldots;$ **0, otherwise**
- $P(Y > 2) = 1 - P(Y \leq 2) = 1 - [P(Y = 0) + P(Y = 1) + P(Y = 2)]$
- $P(Y > 2) = 1 - [\binom{2}{0}(0.35)^3(0.65)^0 + \binom{3}{1}(0.35)^3(0.65)^1 + \binom{4}{2}(0.35)^3(0.65)^2]$
- $P(Y > 2)$ **= 0.765 (to 3 d.p.)**

c. Suppose the brothers have 12 board games, 9 video games, 4 outdoor games, and 15 card games to choose from. Suppose the brothers randomly select a total of 8 games to play, without replacement. Let W represent the total number of card games selected. In a sample of 8 games, selected without replacement, how many board games will be selected, on average? For full credit, clearly define the random variable of interest and state the distribution of the random variable with all appropriate parameters. ***(2 marks)***

- **Let B represent the number of board games selected.**
- **B has a Hypergeometric distribution with N = 40, r = 12, and n = 8.**
- **E(B) =** $\frac{8*12}{40}$ **= 2.4**

18. ***(11 marks)*** A University of Waterloo Math Major wishes to study births in the city in which he lives. Suppose it has been determined that there are 6,059 births on average in this city each year, and that the births are assumed to follow a Poisson process. We will also assume that a year has 365 days, and we will ignore any time changes due to daylight savings. We will think of the 365 days in a year as 365 non-overlapping 24-hour intervals.

a. Discuss whether the three assumptions of a Poisson process are satisfied. ***(3 marks)***

- **We discuss the three assumptions.**
- **Independence. It is reasonable to think that the number of babies born in different time intervals are independent, as there is no reason to believe they would affect each other.**
- **Individuality. It is unlikely that in a sufficiently short time interval (say one tenth of a second) there are two or more babies born, so individuality holds.**
- **Homogeneity. Births can be believed to occur uniformly throughout the day proportional to time.**

b. Let X represent the number of births in a day (24-hour period). Provide an expression for the probability that there will be exactly 16 births in a 24-hour period. You do not need to give a numerical answer here, just give the appropriate expression that you would use to calculate the probability.

Note: Do not simply write P(X=16). Write the expression used to determine P(X=16). ***(2 marks)***

- **We know that 6,059 births have been recorded in a year, that makes on average 6,059/365 = 16.6 births a day. The number of births in 24 hours (or one day), X, is then $X \sim \text{Poisson}(16.6)$ and we find that $P(X = 16) = e^{-16.6}\left(\frac{16.6^{16}}{16!}\right)$**

c. Consider that a year consists of 365 non-overlapping 24-hour periods. On average, how many 24-hour periods would you have to wait until there are exactly 16 babies born in one of the 24-hour periods? ***(3 marks)***

- **From b., and using the fact that two non-overlapping days are independent, we found that $P(X = 16) = e^{-16.6}\left(\frac{16.6^{16}}{16!}\right) = p$**
- **Let Y represent number of days (24-hour periods) in which there are not exactly 16 births in a 24-hour period (i.e. failures). In this case, $Y \sim \text{Geometric}(p)$.**
- **From the Summary of Distributions, we find $E(Y) = (1-p)/p = 9.190$.**
- **So, on average, you would have to wait have to wait 10.190 24-hour periods, on average, to find a 24-hour period with exactly 16 births.**

d. Suppose that the Math Major is a "number nerd" and would like to give the first baby that was born on July 6, 2022, in the city that he lives in, a special pacifier with his initials on it. He will get up precisely at midnight and will wait for news of the next birth. What is the probability he could have slept at least 3 more hours, instead of getting up right at midnight to wait for the first baby to be born on July 6, 2022? For full credit, please clearly define the random variable of interest, along with the distribution of the random variable of interest and show all necessary work when calculating this probability. ***(3 marks)***

- **Let Y be the time between midnight and the first baby born on July 6, 2022, measured in hours. On average, there are $6,059/(365 \cdot 24)$ births per hour. Thus, $Y \sim \text{Exp}(\lambda)$ with $\lambda = 6,059/(365 \cdot 24)$ and the probability that the Math Major could have slept at least 3 more hours is given by:**

- $P(Y > 3) = \int_3^\infty \lambda e^{-\lambda t}dt = e^{-3\lambda} = e^{-2.075} = 0.126$ **(to 3 d.p.)**

19. ***(9 marks)*** You have formed a study group with six of your classmates to help with homework. You have all been working remotely working on one of the homework questions and cannot seem to get the answer. Suppose that three of your classmates in the group have solved the question successfully. You decide to call each member of the group randomly until you contact one that has successfully solved the question. Let the random variable X represent the number of classmates you contact.

a. Determine the probability function (pf) of X. Express the pf in table form and leave the probabilities in fraction form. ***(3 marks)***

- **From the question, the r.v. X represents the number of members you contact to find someone who knows how to solve the question correctly.**
- **You may have found it helpful to use a probability tree to visualize this.**
- **P(X = 1) = 3/6 = 1/2**
- **P(X = 2) = (3/6)*(3/5) = 3/10**
- **P(X = 3) = (3/6)*(2/5)*(3/4) = 3/20**
- **P(X = 4) = (3/6)*(2/5)*(1/4)*(3/3) = 1/20**
- **The probability that you will need to contact 5 or 6 classmates, in this case, is 0.**
- **We can easily organize this into a table:**

x	**1**	**2**	**3**	**4**
f(x)	1/2	3/10	3/20	1/20

- **Quick check: 1/2 + 3/10 + 3/20 + 1/20 = 20/20 (=1). This is good news!**

b. What is the expected number of group members you contact? ***(3 marks)***

- **We want E(X) =** $\sum_{x=1}^{4} x * f(x) = 1 * \left(\frac{1}{2}\right) + 2 * \left(\frac{3}{10}\right) + 3 * \left(\frac{3}{20}\right) + 4 * \left(\frac{1}{20}\right) = \frac{35}{20} = \frac{7}{4} = 1.75$

c. Now, let the random variable T represent the total amount of time (in minutes) that you spend on the phone in an effort to get the answer to the homework question. Suppose that T is related to X via the functional relationship: $T = 2X^2 - 6X + 11$.

How many minutes should you expect to spend on the phone? ***(3 marks)***

- **To answer this question, we need to find E(T), we will need E(X^2).**
- **E(X^2)** $= \sum_{x=1}^{4} x^2 * f(x) = 1 * \left(\frac{1}{2}\right) + 4 * \left(\frac{3}{10}\right) + 9 * \left(\frac{3}{20}\right) + 16 * \left(\frac{1}{20}\right) = 3.85$
- **E(T) = E($2X^2$ – 6X + 11) = 2*E(X^2) – 6*E(X) + 11 = 2*(3.85) – 6*(1.75) + 11 = 8.2**
- **Thus, you will spend an average of 8.2 minutes on the phone.**

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work / justify your answer. Unjustified answers will not receive full credit.

BONUS QUESTION 1: (2 marks)

Consider the following game. A fair coin is flipped until the first tail appears. You will win \$2 if it appears on the first toss, \$4 if it appears on the second toss, and in general, $\$2^k$ if it appears on the k^{th} toss. Let X represent your total winnings.

How much money would you have to pay in order for this to be a "fair" game? Would you play this game? Explain why or why not.

Note: For this game to be considered "fair", a player would have to pay an amount equal to their expected winnings, so that the expected gain is equal to 0.

This is referred to as the "St. Petersburg paradox". In this case, X has no finite expected value. That is, E(X) is infinite. So, in order for this game to be "fair", you would need to pay an infinite amount of money to play.

Likewise, your opponent would have to be willing to risk an infinite amount of money if they allowed you to play!

For these reasons, you would not play this game!

BONUS QUESTION 2: ***(4 marks)***

A casino game has a very large board marked out on a grid of squares so that each square has sides with length x cm. A player will toss a one-dollar coin with diameter 2.7 cm from a short distance onto the grid. If the dollar coin lies entirely in any of the squares, the player will get the dollar coin back, plus they will win a dollar! If any part of the coin lands on any grid line, the player will lose their coin, and thus, will lose a dollar.

Note: You can assume that the board is very large so that any dollar coin will land somewhere on the board, and that the lines dividing the squares have an ignorable thickness. You can also assume that the dollar coin is equally likely to land anywhere on the board.

a. Determine the value of x (the length of each side, to 3 d.p.), so that the probability of winning the game is 0.5. ***(2 marks)***

The coin must lie entirely inside a square with sides of length x for the player to win. This also means that the centre of the coin (think of this as a point), must be inside a smaller (x – 2.7) cm square. Picture this smaller square that is 1.35 cm away from the outside of the larger square.

Thus, it is a square with a side that has length (x – 2.7) cm.

The required probability is represented by the area of the smaller square divided by the area of the larger square with sides of length x.

So, we would need to find the length of the side, or x, that satisfies the following:

$$\frac{(x-2.7)^2}{x^2} = 0.5.$$

Solving for x, and ignoring the inadmissible root, of x = 1.582 cm, we see that the sides would need to be of length 9.218 cm for the probability of a player winning to be 0.5, and the game to be considered fair.

Note: You can attempt part b. of this question without solving part a.

b. Assuming the casino would not want to lose money, on average, would this game have squares with sides of length greater than x, or less than x (where x is the solution to part a.)? Explain.

(2 marks)

The casino would not want to lose money, so the squares would have a length less than x. Otherwise, the advantage would go to the player, and the casino would lose money, on average.

Summary of Discrete Distributions

Notation and Parameters	**Probability Function** $f(x)$	**Mean** $E(X)$	**Variance** $Var(X)$
Discrete Uniform(a, b) $b \geq a$ a, b integers	$\frac{1}{b-a+1}$ $x = a, a+1, \ldots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Hypergeometric(N, r, n) $N = 1,2, \ldots$ $n = 0,1, \ldots, N$ $r = 0,1, \ldots, N$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = max(0, n-N+r),$ $\ldots, min(r, n)$	$\frac{nr}{N}$	$\frac{nr}{N}\left(1-\frac{r}{N}\right)\frac{N-n}{N-1}$
Binomial(n, p) $0 \leq p \leq 1, q = 1-p$ $n = 1,2, \ldots$	$\binom{n}{x} p^x q^{n-x}$ $x = 0,1, \ldots, n$	np	npq
Bernoulli(p) $0 \leq p \leq 1, q = 1-p$	$p^x q^{1-x}$ $x = 0,1$	p	pq
Negative Binomial(k, p) $0 < p \leq 1, q = 1-p$ $k = 1,2, \ldots$	$\binom{x+k-1}{x} p^k q^x$ $= \binom{-k}{x} p^k (-q)^x$ $x = 0,1, \ldots$	$\frac{kq}{p}$	$\frac{kq}{p^2}$
Geometric(p) $0 < p \leq 1, q = 1-p$	pq^x $x = 0,1, \ldots$	$\frac{q}{p}$	$\frac{q}{p^2}$
Poisson(λ) $\lambda \geq 0$	$\frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1, \ldots$	λ	λ

Summary of Continuous Distributions

Notation and Parameters	**Probability Density Function** $f(x)$	**Mean** $E(X)$	**Variance** $Var(X)$
Uniform(a, b) $b > a$	$\frac{1}{b-a}$ $a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential(θ) $\theta > 0$	$\frac{1}{\theta} e^{-x/\theta}$ $x \geq 0$	θ	θ^2