Instructions

1. **All electronic devices are to be turned off and put away. If you haven't already done this, please do so now.**
2. **This is a 50-minute closed-book quiz.**
3. **Answer the questions in the spaces provided.**
4. **Final answers to written questions should be rounded to THREE DECIMAL PLACES or left in fraction form unless otherwise specified.**
5. **Only question pages will be marked.**
6. **You may tear off the last two pages and use them for rough work. All pages will be collected at the end of the quiz.**
7. **Only a non-programmable, non-graphical calculator with a pink-tie or blue-goggle sticker will be allowed. No other aids are allowed.**
8. **A reference sheet has been included with this quiz, and can be found at the end of the quiz. You may CAREFULLY detach the rough work pages and reference sheet if you like.**
9. **DO NOT WRITE ON THIS COVER PAGE.**

Marking Scheme:

Questions	**Out of**
1 – 9	9
10	8
11	9
Total	26

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Part 1: Multiple Choice – Each question is worth 1 mark

Please CLEARLY CIRCLE your answer selection on this page. Do NOT simply write your choice next to the question. If you do, the question will not be graded. Each question has one correct answer. Choose the best answer.

1. Suppose X is normally distributed with mean equal to -2 and variance equal to 4. Which of the following is equal to $P(X \leq -1)$?

A. $P(X \geq -1)$ B. $P(X \geq 1)$ **C.** $\boldsymbol{P(X \geq -3)}$ D. More than one of these

- $P(X \leq -1) = P(X \leq -2+1)$
- By symmetry $P(X \leq -2+1) = P(X \geq -2-1)$
- Thus $P(X \leq -1) = P(X \geq -3)$. Note: The others can be checked and are different

2. Suppose we wish to generate a random observation, x, from a distribution with pdf:

$$f(x) = \frac{4}{x^5}\text{, for } x > 1;\ 0 \text{ for } x \leq 1$$

We generate an observation, u, for a continuous $Uniform(0,1)$ distribution and get 0.6357. What is the value, x, this value, u, will produce?

A. 1.445 **B.** $\mathbf{1.287}$ C. 1.120 D. 1.095

- To generate the value, x, we need to find $F(x) = \int_1^x \frac{4}{s^5} ds = -(s^{-4})|_1^x = 1 - x^{-4}$, for $x > 1$.
- Next, we let $u = 0.6357 = 1 - x^{-4}$, and then we solve for x.
- This gives us $x = (1 - 0.6357)^{-\frac{1}{4}} = 1.287$ (to 3 d.p.)

3. Let $Y \sim N(100, 64)$. What is the 66th percentile of the distribution of Y?

A. 124.659 B. 105.231 **C.** $\mathbf{103.300}$ D. 96.918

- We know that $Y \sim N(100, 64)$ and we need the value, y^*, such that $P(Y \leq y^*) = 0.66$
- Using the bottom of the Normal Table, the corresponding z-score is $z = 0.4125$.
- Now, transform Y into r.v. Z, getting $P(\frac{Y-\mu}{\sigma} \leq \frac{y^*-100}{8}) = 0.66$
- Now set $\frac{y^*-100}{8} = 0.4125$ and solve for y^*, $y^* = 100 + 8(0.4125) = 103.300$
- This answer seems reasonable as we know that our answer must be greater than 100, as that is the 50th percentile of the distribution

4. Suppose X is a standard normal random variable with CDF $\Phi(x)$. What is the value of x such that $\Phi(x) = 0.31$?

A. 0.4399 **B.** $\mathbf{-0.4959}$ C. -0.4677 D. -0.2275

- We see $F(X) = P(X \leq x) = 0.31$, which means $P(X \leq x) = P(X \geq -x) = 1 - P(X \leq -x) = 0.31$. Thus $P(X \leq -x) = 0.69$. Looking at the N(0,1) quantile table, $-x = 0.4959$, therefore $x = -0.4959$.

5. Suppose $X \sim N(3, 4)$, if $Y = X - 2$, what is the $P(-2 \leq Y < 1)$?

A. $\mathbf{0.433}$ B. 0.933 C. 0.567 D. 0.84134

- We know $Y \sim N(1, 2^2)$.
- $P(-2 < Y < 1) = P(Y < 1) - P(Y < -2) = P\left(Z < \frac{1-1}{2}\right) - P\left(Z < \frac{-2-1}{2}\right) = P(Z < 0) - P\left(Z < -\frac{3}{2}\right) = \frac{1}{2} - \left(1 - P(Z < 1.5)\right) = 0.5 - (1 - 0.93319) = 0.4333$

6. Suppose X is a random variable that follows a normal distribution with mean equal to 5 and standard deviation equal to 8. What is $P(\,|\,X\ -\ 5\,|\ >\ 4.4)$?

A. 41.77% B. 29.12% **C.** $\mathbf{58.23}$**%** D. 94.42%

- We know that $X \sim N(5, 64)$.
- We want $P(|X-5| > 4.4) = P(\frac{|X-5|}{8} > \frac{4.4}{8}) = P(|Z| > 0.55)$
- $P(|Z|\ >\ 0.55)\ =\ P(Z\ <\ -0.55)\ +\ P(Z\ >\ 0.55)\ =\ (1 - P(Z\ <\ 0.55))\ +\ (1 - P(Z\ <\ 0.55))$
- $P(|Z|\ >\ 0.55)\ =\ 2*(1 - P(Z\ <\ 0.55))\ =\ 2*(1 - 0.70884)\ =\ 0.58232$

7. Suppose the weight of bouncy balls in a bag is normally distributed with a mean of 70 grams and a variance of 100. Roughly 70 percent of bags will weigh at least how much?

A. 77 B. 60 **C. 65** D. 75

- Let Y be the weight of a bag where $Y \sim N(70{,}100)$.We need y^*, such that $P(Y > y^*) = 0.70$
- This is equivalent to $P(Y \geq y^*) = 0.70$. This is equivalent to $P(Y \leq\ y^*) = 0.3$
- We can make use of the table at the bottom of the Normal Table and look up the z-score for the 70th percentile, and then use the negative of this value (due to symmetry)
- From the Table, the $z - score = -0.5244$.
- Now, transform Y into the r.v. Z to get $P\left(\frac{Y-\mu}{\sigma} > \frac{y^*-70}{10}\right) = 0.7$.
- Set $\frac{y^*-70}{10} = -0.5244$ and solve for y^*, $y^* = 70 - 10*(0.5244) = 64.756$
- So approximately 70 percent of bags will weigh at least 65 grams.

8. If $U \sim Uniform(0{,}1)$, what is the distribution of $1 - U$?

A. $\mathbf{Uniform(0,1)}$ B. Uniform(-1,0)
C. Normal(0,1) D. Cannot be determined

- Imagine any value in (0,1). 1 minus that value will flip to around a 0.5 axis. Thus the answer is $Uniform(0{,}1)$
- We also can compute $F_X(x) = P(X \leq x) = P(1 - U \leq x) = P(U \geq 1 - x) = 1 - P(U \leq 1 - x)$. If we plug this value into the CDF, we will achieve a $Uniform(0{,}1)$ random variable

9. At a large grocery store, the time of employment for a cashier is normally distributed with a mean of 8.3 years and a variance of 7.76. If a cashier is randomly selected, what is the probability the cashier has worked at the store for more than 10 years?

A. 32.51% B. 45.29% C. 36.3% **D. 27.09%**

- Let X represent the length of employment for a cashier. $X \sim N(8.3, 7.76)$
- We want $P(X\ >\ 10) = 1 - P(X < 10) = 1 - P\left(\frac{X-\mu}{\sigma} < \frac{10-8.3}{\sqrt{7.76}}\right)$
- Thus we examine $1 - P(Z < 0.6103) \approx 1 - P(Z < 0.61) =\ 1 - 0.72907 =\ 0.27093$

Part 2: Written Answer / Short Answer Questions

Final answers can be left in fraction form OR rounded to THREE DECIMAL PLACES.

For full credit, you need to show all necessary work. Unjustified answers will not receive full credit.

10. (8 marks) Suppose the height of trees in a local forest is normally distributed with mean 260 cm and standard deviation of 10 cm. Let the height be represented by the random variable X.

a. Give the probability density function (pdf) of X with all necessary parameters. What is $P(X = 260)$? **(2 marks)**

- $X \sim N(260, 10^2)$
- $f(x) = \frac{1}{10\sqrt{2\pi}} e^{\frac{-(x-260)^2}{2(100)}}$ for $x \in \mathbb{R}$
- $P(X = 260) = 0$ since this is a continuous distribution

b. What is the probability a randomly selected tree has height greater than 245 cm? **(2 marks)**

- By symmetry, $P(X > 245) = P(X > 260 - 15) = P(X < 260 + 15) = P(X < 275)$
- Through standardization, $P(X < 275) = P\left(Z < \frac{275-260}{10}\right) = P(Z < 1.5) = 0.93319$

c. What is the 80th percentile of the distribution of X? **(2 marks)**

- $P(X < x) = P\left(Z < \frac{x-260}{10}\right) = 0.8$
- Using the table, $\frac{x-260}{10} = 0.8416$
- $x = 260 + 10\,(0.8416) \approx 268.416$
- Therefore, 80% of the trees are no taller than 268.416 cm (or 80% are shorter than this)

d. What is the probability a tree is taller than 245 cm given we know the tree is no taller than 265 cm tall? **(2 marks)**

- $P(X > 245\ |X < 265) = \frac{P(X>245 \cap X<265)}{P(X<265)} = \frac{P(245<X<265)}{P(X<265)} = \frac{F_X(265)-F_X(245)}{F_X(265)}$
- After standardization $\frac{F_X(265)-F_X(245)}{F_X(265)} = \frac{F_Z\left(\frac{265-260}{10}\right)-F_Z\left(\frac{245-260}{10}\right)}{F_Z\left(\frac{265-260}{10}\right)} = \frac{F_Z(0.5)-F_Z(-1.5)}{F_Z(0.5)}$
- Using symmetry to allow use of table $\frac{F_Z(0.5)-F_Z(-1.5)}{F_Z(0.5)} = \frac{F_Z(0.5)-(1-F_Z(1.5))}{F_Z(0.5)}$
- Using the table $\frac{F_Z(0.5)-(1-F_Z(1.5))}{F_Z(0.5)} = \frac{0.69146-(1-0.93319)}{0.69146} \approx 0.903$

11. (9 marks) Suppose that scores on an exam can be modeled by a random variable X that is normally distributed with a mean of 82 and a variance of 36.

a. Determine the z-score for an exam score of 85. Are there any units associated with this z-score? If yes, state the units. If no, state that there are no units. **(2 marks)**

- Let Z be a standard normal random variable, $Z = \frac{X-82}{6}$
- Therefore, the z-score is $\frac{85-82}{6} = 0.5$
- There are no units as the numerator has units that cancel with the denominator

b. Suppose students that score in the top 1% for an exam will receive a special award. What is the minimum score necessary (rounded to the nearest integer) to receive the award? **(2 marks)**

- $F_Z\left(\frac{x-82}{6}\right) = 1 - 0.01$
- Using the table, we get 2.3263
- Solving, $\frac{x-82}{6} = 2.3263$, $x = 82 + 6(2.3263) = 95.9578 \approx 96$
- The top 1% requires a score of at least 96.

c. In this class, it was determined 5 students scored 70 or less on an exam. Given this information, what would you estimate the size of the class to be? Please round your answer to the nearest integer. **(2 marks)**

- $P\left(Z < \frac{70-82}{6}\right) = P(Z < -2) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.97725 = 0.02275$
- So, we know 2.275% of students scored 70 or less.
- Since we know there were 5 students, $(0.02275)x = 5$ and thus $x = \frac{5}{0.02275} = 219.7802$
- Therefore, the class has approximately 220 students.

d. Suppose quiz scores in this class are also normally distributed with the same mean, but a different variance. If 15% of the students scored at least 88 on a quiz, determine the variance associated with quiz scores. **(3 marks)**

- Let Y be the random variable associated with the previous quiz scores
- $P(Y > 88) = P\left(Z > \frac{88-82}{\sigma}\right) = 0.15$
- $P\left(Z < \frac{6}{\sigma}\right) = 0.85$
- $\frac{6}{\sigma} = 1.0364$
- So $\sigma = 5.789$, and $\sigma^2 \approx 33.516$.

This page may be CAREFULLY removed and used for rough work.

It will not be graded, but MUST be handed in.

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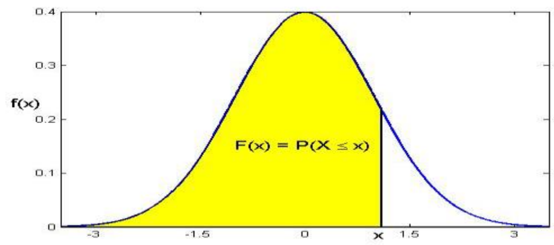
N(0,1) Cumulative Distribution Function



This table gives values of $F(x) = P(X \le x)$ for $X \sim N(0,1)$ and $x \ge 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99979	0.99981	0.99981	0.99982	0.99983	0.99983

N(0,1) Quantiles: This table gives values of $F^{-1}(p)$ for $p \ge 0.5$

p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.075	0.08	0.09	0.095
0.5	0.0000	0.0251	0.0502	0.0753	0.1004	0.1257	0.1510	0.1764	0.1891	0.2019	0.2275	0.2404
0.6	0.2533	0.2793	0.3055	0.3319	0.3585	0.3853	0.4125	0.4399	0.4538	0.4677	0.4959	0.5101
0.7	0.5244	0.5534	0.5828	0.6128	0.6433	0.6745	0.7063	0.7388	0.7554	0.7722	0.8064	0.8239
0.8	0.8416	0.8779	0.9154	0.9542	0.9945	1.0364	1.0803	1.1264	1.1503	1.1750	1.2265	1.2536
0.9	1.2816	1.3408	1.4051	1.4758	1.5548	1.6449	1.7507	1.8808	1.9600	2.0537	2.3263	2.5758

Summary of Discrete Distributions

Notation and Parameters	**Probability Function** $f(x)$	**Mean** $E(X)$	**Variance** $Var(X)$
Discrete Uniform(a, b) $b \geq a$ a, b integers	$\frac{1}{b-a+1}$ $x = a, a+1, \ldots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Hypergeometric(N, r, n) $N = 1,2, \ldots$ $n = 0,1, \ldots, N$ $r = 0,1, \ldots, N$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x = max(0, n-N+r),$ $\ldots, min(r, n)$	$\frac{nr}{N}$	$\frac{nr}{N}\left(1-\frac{r}{N}\right)\frac{N-n}{N-1}$
Binomial(n, p) $0 \leq p \leq 1, q = 1-p$ $n = 1,2, \ldots$	$\binom{n}{x} p^x q^{n-x}$ $x = 0,1, \ldots, n$	np	npq
Bernoulli(p) $0 \leq p \leq 1, q = 1-p$	$p^x q^{1-x}$ $x = 0,1$	p	pq
Negative Binomial(k, p) $0 < p \leq 1, q = 1-p$ $k = 1,2, \ldots$	$\binom{x+k-1}{x} p^k q^x$ $= \binom{-k}{x} p^k (-q)^x$ $x = 0,1, \ldots$	$\frac{kq}{p}$	$\frac{kq}{p^2}$
Geometric(p) $0 < p \leq 1, q = 1-p$	pq^x $x = 0,1, \ldots$	$\frac{q}{p}$	$\frac{q}{p^2}$
Poisson(λ) $\lambda \geq 0$	$\frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1, \ldots$	λ	λ

Summary of Continuous Distributions

Notation and Parameters	**Probability Density Function** $f(x)$	**Mean** $E(X)$	**Variance** $Var(X)$
Uniform(a, b) $b > a$	$\frac{1}{b-a}$ $a \leq x \leq b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential(θ) $\theta > 0$	$\frac{1}{\theta} e^{-x/\theta}$ $x \geq 0$	θ	θ^2
$N(\mu, \sigma^2) = G(\mu, \sigma)$ $\mu \in \mathbb{R}, \sigma^2 > 0$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \in \mathbb{R}$	μ	σ^2