STAT 230: Probability (Sec 02)
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Lecture 7

Today's Agenda

Last time:

- Finished Chapter 3
- Practiced for the quiz

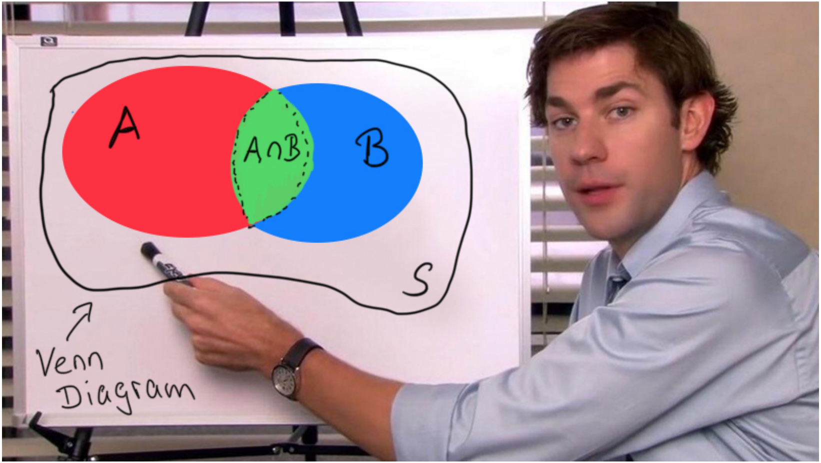
Today (Lec 7, 05/16):

- Chapter 4!!!

4. PROBABILITY RULES AND CONDITIONAL PROBABILITY

- Recall that we define events as sets . . .
- . . . so we need some set theory!

We can visualize sets and set theoretic operations using a Venn Diagram.



Example

Visualize the following set theoretic operations using a venn diagram:

a) $\bar{A}$

b) $A \cap B$

c) $A \cap \bar{B}$

d) $(A \cap B \cap C) \cup (\overline{(B \cup C)} \cap A)$

Fundamental laws of set algebra

Similar to arithmetic (ex: $+$, $\times$), we have the **fundamental laws of set algebra**.

a) Commutativity
$A \cup B = B \cup A, \quad A \cap B = B \cap A$

b) Associativity
$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$

c) Distributivity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Question: Let A, B, and C be events. Which of these sets is necessarily the smallest, i.e. it is contained in all of the others? (assume $\bar{B} = A \cup C$)

A $A \cup B$

B C

C $A \cap C$

D $(A \cap B \cap C) \cup \bar{B}$

E $(\bar{B} \cap A \cap C) \cup (A \cap B)$

DeMorgan's Laws:

a) $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$
"Complement of union is intersection of complements"

b) $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$
"Complement of intersection is union of complements"

Inclusion Exclusion Rules:

For arbitrary events A,B, and C

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

c)

$$P(\cup_{i=1}^{n} A_i) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) \\ - \sum_{i<j<k<l} P(A_i A_j A_k A_l) + \cdots$$

Example

Suppose that two fair six sided die are rolled. What is the probability that at least one of the dice shows a 6?

Independent events

Definition

Two events A and B are said to be **independent** if

$$P(A \cap B) = P(A)P(B).$$

A sequence of events $A_1, A_2, ...$ are said to be independent if

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}),$$

for all possible subsets of size k for any $k \geq 2$ and all possible $i_1, i_2, \ldots, i_k$. Events that are not independent are called **dependent**.

This means that, for a sequence of events to be independent, all of their subsets must be independent as well.

Example

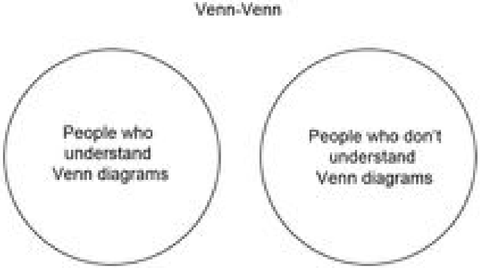
Consider rolling two fair six sided dice, and let

$A = \{\text{the sum is } 10\}$, $B = \{\text{the first die is a } 6\}$ $C = \{\text{the sum is } 7\}$.
Determine:

a) Are A and B independent?
b) Are B and C independent?
c) Are A and C independent?

Definition

Two events A and B are **mutually exclusive** if A and B are disjoint.



A common misconception is that if A and B are mutually exclusive, then A and B are independent.

Proposition

Suppose that not both A and B are trivial events (empty sets).
If A and B are independent and mutually exclusive (disjoint), then either $P(A) = 0$ or $P(B) = 0$.