Stat 230: Probability

Lecture 24

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Review

Example

Suppose the score of a randomly selected student on Midterm 2 follows a normal distribution with mean 81 and variance 6^2.

(1) If a student is selected at random, compute the probability that the students score exceeds 75.

(2) If a student is selected at random, compute the probability that the students score is less than 10.

Review

Last time we talked about:

(1) Normal Distribution

For today:

(1) Normal Distribution

(2) Multiple Random Variables: Joint Distributions

Reading: Chapter 8

Review

- Friday: No class!
- Monday: In-class Review
- Monday: TA Tutorial Practice
- Tuesday: Test 2

Percentile

Definition

The α **percentile** *of the distribution of a random variable* X *is the number* χ_α *satisfying*

$$P(X \leq \chi_\alpha) = \alpha$$

The α **critical value** *of the distribution of a random variable* X *is the number* z_α *satisfying*

$$P(X > z_\alpha) = \alpha$$

Percentile

Example

Compute the

(1) 75th percentile of the standard normal distribution

(2) 58th percentile of the $N(5, 9)$ distribution

Percentile

Definition

The **68-95-99.7 Rule** *states that if* $X \sim N(\mu, \sigma^2)$,

$$P(\mu - \sigma \leq X \leq \mu + \sigma) \approx .68$$

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx .95$$

$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx .997$$

Multivariate Distributions

In many cases, we are interested in more than one random variable pertaining to an experiment.

Example

- Suppose we roll two fair six sided die,
  $X =$ outcome of the first die roll, and
  $Y =$ outcome of the second die roll.
- On a given day, let $X =$ temperature in Toronto at noon and
  $Y =$ temperature in San Diego at noon.
- Let $X_1, ..., X_{150}$ denote the heights of people in this room.

In statistical applications, repeated measurements of the same quantity, or measurements of many different quantities can be framed as observations of multiple random variables.

Multivariate Distributions

What is the goal of probability theory for more than one random variable?

- Computing probabilities relating to each random variable and their "joint" behavior. (What is the analog of the probability function/ probability density)
- Compute summary quantities describing the random variables (How to calculate expected values and variances).
- Determine the distributional properties of transformations of the random variables, e.g. the sum/average

Joint Probability Function

Definition

Suppose that X and Y are discrete random variables defined on the same sample space (in general, when we consider two or more random variables it is assumed they are defined on the same sample space.)

The **joint probability function** *of X and Y is*

$$f(x, y) = P(\{X = x\} \cap \{Y = y\}) \;\; x \in X(S), y \in Y(S)$$

A shorthand for this is

$$f(x, y) = P(X = x, Y = y)$$

Joint Probability Function

Example

Suppose two fair six sided die are rolled. Let X denote the outcome of the first die roll, and let Y denote the outcome of the second die roll. Compute the joint probability function of X and Y.

Joint Probability Function

Definition

For a collection of n discrete random variables, $X_1, ..., X_n$, the joint probability function is defined as

$$f(x_1, x_2, ..., x_n) = P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n)$$

Joint Probability Function

Remark (Properties of the joint probability function)

(1) $0 \leq f(x, y) \leq 1$

(2) $\sum_{x,y} f(x, y) = 1.$

Joint Probability Function

Computing probabilities from the joint probability function:

Suppose that A is a collection of ordered pairs, e.g. $A = \{(x, y) : x \in A_x, y \in A_y\}$. Then

$$P((X, Y) \in A) = \sum_{(x,y) \in A} f(x, y)$$

For example,

$$P(X > Y) = \sum_{(x,y):x>y} f(x, y)$$

Joint Probability Function

Example

Suppose that X and Y have a joint probability function defined in the table below:

	f(x,y)	x = 0	x = 1	x = 2
y	0	.2	.3	.1
	2	.25	.13	.02

Compute

(1) $P(X + Y \geq 3)$

(2) $P(XY = 0)$

Marginal Probability Function

Definition

Suppose that X and Y are discrete random variables with joint probability function $f(x, y)$. The **marginal probability function** *of X is*

$$f_X(x) = P(X = x) = \sum_{y \in Y(S)} f(x, y)$$

Similarly, the marginal distribution of Y is

$$f_Y(y) = P(Y = y) = \sum_{x \in X(S)} f(x, y)$$

A common mistake is to think that there is a difference between the marginal distribution of X and the probability function of X. They are the same!

Marginal Probability Function

Example

Suppose that X and Y have a joint probability function defined by:

		x		
	f(x,y)	0	1	2
y	0	.2	.3	.1
	2	.25	.13	.02

(1) Compute the marginal probability functions of X and Y.

(2) Compute $E(X)$.

Marginal Probability Function

Example

Suppose X and Y have joint probability function

$$f(x,y) = \frac{1}{6}\left(\frac{1}{2}\right)^{x}\left(\frac{2}{3}\right)^{y}, \;\; x,y = 0,1,2...$$

(1) Compute the marginal probability functions $f_X(x)$ and $f_Y(y)$.

(2) Compute $P(X < Y)$.