STAT 230 **SECTION 2** **LECTURE 24**

Today's Agenda

Last time:

- More examples on continuous random variables
- Determining the distribution of $g(X)$

Today (Lec 24, 06/27):

- Continuous uniform distribution
- Exponential distribution

Discrete versus continuous random variables

- If X is **discrete**, then
  - $f(x) = P(X = x)$ is the probability mass function
  - $P(X \in A) = \sum_{x \in X(S) \cap A} f(x)$
  - $E(X) = \sum_{x \in X(S)} x f(x)$
- If X is **continuous**, then
  - $P(X = x) = 0$ for all $x \in \mathbb{R}$.
  - $f(x) = F'(x)$ is the probability density function
  - $P(X \in A) = \int_A f(x)\,\mathrm{d}x$
  - $E(X) = \int_{\mathbb{R}} x f(x)\,\mathrm{d}x$
  - The pdf $f(x)$ is not $P(X = x)$, but for $\delta > 0$ small,

$$\begin{aligned} P(X \in (x - \delta/2, x + \delta/2)) &= P(x - \delta/2 \le X \le x + \delta/2) \\ &= F(x + \delta/2) - F(x - \delta/2) \\ &\approx f(x)\delta. \end{aligned}$$

Continuous uniform distribution

We now introduce the first continuous distribution.

Definition

We say that X has a continuous uniform distribution on (a, b) if X has pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b), \\ 0 & \text{otherwise} \end{cases}$$

This is abbreviated $X \sim U(a, b)$.

Examples

- Cutting a stick of length 2 at a random position (motivating example!)
- Spinning a wheel in a game show
- ...

Example

Let $X \sim U(a, b)$. Show the following.

a) $E(X) = \frac{a+b}{2}$

b) $Var(X) = \frac{(b-a)^2}{12}$

Question

Suppose $X \sim U(0, 1)$, and that $Y = \frac{2}{X} - 1$. What is the range of Y?

A $Y(S) = [0, \infty)$

B $Y(S) = [1, 3]$

C $Y(S) = [0, \infty)$

D $Y(S) = [0, 2]$

E $Y(S) = [1, \infty)$

A new continuous distribution

Let X be a continuous random variable with pdf

$$f(x) = ce^{-\lambda x}, \quad x > 0,$$

and 0 otherwise, where $\lambda > 0$ is a parameter and $c > 0$ is a constant to be determined.

a) Determine c so that f is a valid pdf.
b) Determine the cdf of X.
c) What distribution does the random variable $Y = e^{-\lambda X}$ have?

The exponential distribution

Definition (λ-parametrization of exponential distribution)

We say that X has an exponential distribution with parameter λ, denoted by $X \sim Exp(\lambda)$, if the density of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & x \leq 0\ . \end{cases}$$

Exponential distribution models waiting times

Here is one way to derive the exponential distribution.

In a Poisson process with respect to time, let $X =$ the length of **time** taken until the first event occurrence.

Note: This is slightly different from the actual Poisson process, as it count the **number** of event occurrences.

Let's consider the cdf of X (length of time until first event occurs)

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= P(\text{time to 1st occurrence} \leq x) \\ &= 1 - P(\text{time to first occurrence} > x) \\ &= 1 - P(\text{no occurrence between } (0, x)) \end{aligned}$$

...and we know how to model the number of event occurrences between time $(0, x)$: it follows $Poi(\lambda x)$.

$$\begin{aligned}
&1 - P(\text{no occurrence between } (0, x)) \\
&= 1 - \frac{\exp(-\lambda x)(\lambda x)^0}{0!} \text{ for } x > 0 \quad (\text{and } 0 \text{ otherwise}). \\
&= 1 - \exp(-\lambda x)
\end{aligned}$$

So we have $F(x) = 1 - \exp(-\lambda x)$ for $x > 0$.

Then, we can take the derivative with respect to x, for $x > 0$, to obtain the pdf.

$$
\begin{aligned}
f(x) &= \frac{d}{dx} F(x) \\
&= \lambda \exp(-\lambda x) \text{ for } x > 0, \quad \text{(and 0 otherwise)}
\end{aligned}
$$

$\Rightarrow$ exponential distribution models the waiting time between each event occurrence in a Poisson distribution.

Different parametrizations

- It is sometimes more convenient to express the parameter as $\frac{1}{\theta} = \lambda$.

Definition (θ-parametrisation of exponential distribution)

We say that X has an exponential distribution with parameter θ ($X \sim Exp(\theta)$) if the density of X is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0, \\ 0 & x \leq 0 \ . \end{cases}$$

If λ denotes the **rate** of event occurrence in a Poisson process, then $\theta = 1/\lambda$ denotes the **waiting time** until the first occurrence

Example

Nupur decided to enjoy a relaxing Summer away from student housing, so he rented a place in Simcoe, Ontario. However, the busses there are far and few between. Suppose busses arrive according to a Poisson process with an average of 3 busses per hour.

a) Find the probability of waiting at least 15 minutes.

b) Find the probability of waiting at least another 15 minutes given that you have already been waiting for 6 minutes.

Moments of Exp(θ)

- When computing $E(X)$ and $Var(X)$, we need to solve integrals

$$E(X) = \int_0^\infty x \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} \, \mathrm{d}x$$

  and

$$E(X^2) = \int_0^\infty x^2 \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} \, \mathrm{d}x$$

  which can be done using integration by parts.
- Alternatively, we can use the gamma function

Definition (Gamma function)

The integral

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy, \ \alpha > 0$$

is called the gamma function of α.

Some useful properties of $\Gamma(\alpha)$ are

- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$
- $\Gamma(\alpha) = (\alpha - 1)!$ for $\alpha \in \mathbb{N}$
- $\Gamma(1/2) = \sqrt{\pi}$
- Gamma function is used to derive the Gamma distribution ($\Rightarrow$ STAT 330), which is extremely important in non-life insurance pricing, and it can be used to model certain brain signals in neuroscience.

Mean and variance of Exp(θ)

With the Gamma function at hand, we can show that if $X \sim Exp(\theta)$, then

$$E(X) = \theta, \quad Var(X) = \theta^2.$$