STAT 230: Probability (Sec 02)
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Lecture 5

Today's Agenda

Last time:

- Counting techniques (factorials, binomial coefficient)

Today (Lec 5, 05/11):

- More on counting techniques

Definition

Given n distinct objects, a **permutation** of size k is an *ordered* subset of k of the individuals. The number of permutations of size k taken from n objects is denoted $n^{(k)}$ and

$$n^{(k)} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

Definition

Given n distinct objects, a **combination** of size k is an *unordered* subset of k of the individuals. The number of combinations of size k taken from n objects is denoted $\binom{n}{k}$ and

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{(n-k)!k!}.$$

The number $\binom{n}{k}$ is called **Binomial coefficient**.

Example

Suppose you have 20 distinct books, 7 of which are written by Mark Twain.

a) How many ways can you arrange 12 books on a shelf if the order they are on the shelf matters?

b) How many ways can you arrange 12 books on a shelf if exactly 3 of them must be Mark Twain books?

c) A monkey picks books at random from the 20 books and puts them on the shelf until it contains 12 books. What is the probability that at least 3 of the books on the shelf are written by Mark Twain?

There are some useful/important results about permutation and combination.

a) $n^{(k)} = n(n-1)^{(k-1)}$ for $k \geq 1$

b) $\binom{n}{k} = \frac{n^{(k)}}{k!}$

c) $\binom{n}{k} = \binom{n}{n-k}$ for $k \geq 0$

d) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

e) Binomial theorem: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

f) $\binom{n}{k}$ is equal to the kth entry in the nth row of Pascal's triangle.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

Question: What relation does identity 4 have with Pascal's triangle?

Let x denote the number of ways that 3 numbers can be randomly selected from the numbers $\{0, 1, 2, 3, 4, 5\}$ WITH replacement, and let y denote the number of ways that 3 numbers can be drawn from the same set of numbers WITHOUT replacement. Which of the following is true:

a) $x > y$

b) $y > x$

c) $x = y$

Example (The Birthday Problem)

Suppose a room contains n people. What is the probability at least two people in the room share a birthday?

Assumption: Suppose that each of the n people is equally likely to have any of the 365 days of the year as their birthday, so that all possible combinations of birthdays are equally likely.

Let A be the event that at least two people share a birthday. Then

$$P(A) = 1 - P(\bar{A}),$$

where $\bar{A}$ is the event that nobody shares birthdays with each other.

For n people to have unique birthdays, we need to arrange them among 365 days w/o replacement. Thus,

$$|\bar{A}| = 365^{(n)}.$$

For the size of the sample space, we see that each person has 365 possibilities for their birthday. Thus,

$$|S| = 365^n.$$

Since we are assuming that all possible combinations of birthdays are equally likely, our desired probability becomes

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{365^{(n)}}{365^n} = 1 - \frac{n!\binom{365}{n}}{365^n}.$$

For $n \in \{100, 30, 23\}$ we find

$$P(A_{100}) = .9999997, \quad P(A_{30}) = .7063 \quad P(A_{23}) = .5073$$

Example

Consider rearranging the letters at random in the word "HELLOKITTY" to form a single word.

a) How many ways can this be done?
b) What is the probability that all of the letters appear in alphabetic order?
c) What is the probability that the word begins and ends with "T"?

Definition

Consider n objects which consist of k types. Suppose that there are n_1 objects which are of type 1, n_2 which are of type 2, and in general n_i objects of type i. Then there are

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

distinguishable arrangements of the n objects. This quantity is known as a **multinomial coefficient** and denoted by

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!n_2!\dots n_k!}.$$

Example

Suppose we make a random arrangement of length three of the letters $\{a, b, c, d, e, f, g, h, i, j\}$. What is the probability of the event $B =$ "letters are in alphabetic order" if

a) letters are selected without replacement?

b) letters are selected with replacement?