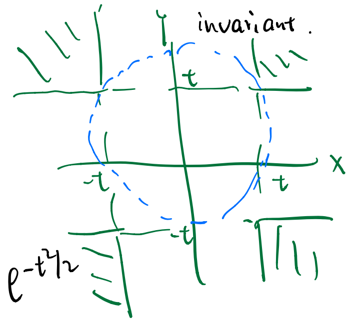
Last time

X is standard Gaussian if X has density: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$E(X) = 0$, $Var(X) = 1$.

If X, Y are indep standard Gaussian, then $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-(x^2+y^2)/2}$ is rotationally invariant.

$Z = \frac{X^2+Y^2}{2}$ is standard exponential, $f_Z(z) = \begin{cases} e^{-z} & z > 0 \\ 0 & z \le 0 \end{cases}$



Useful tail estimate

If X is standard Gaussian, then for $t \ge 0$, $P(|X| \ge t) \le e^{-t^2/2}$

Why? let X, Y be indep standard Gaussian, then:

$$P(|X| \ge t) = \sqrt{P(|X| \ge t)\, P(|Y| \ge t)} \underset{\text{indep}}{=} \sqrt{P(|X| \ge t, |Y| \ge t)} \le \sqrt{P\left(\frac{X^2+Y^2}{2} \ge t^2\right)}$$

$$= \sqrt{e^{-t^2}} = e^{-\frac{t^2}{2}}$$

$=$

$$\int_{t^2}^{\infty} e^{-z}\, dz$$

$$= -e^{-z}\Big|_{t^2}^{\infty}$$

$$= 0 + e^{-t^2}$$

$$\sqrt{P(|X| \ge t)} = \sqrt{P(|X| \ge t) P(|Y| \ge t)}$$

$$= \sqrt{P(|X| \ge t, |Y| \ge t)}$$

$$\le \sqrt{P\left(\frac{X^2+Y^2}{2} \ge t^2\right)}$$

$$= \sqrt{e^{-t^2}} = e^{\frac{-t^2}{2}}$$

$$\int_{t^2}^{+\infty} e^{-z}\, dz$$

$$= -e^{-z}\Big|_{t^2}^{\infty}$$

$$= 0 + e^{-t^2}$$

Useful

If $f_{X,Y}(x,y)$ is rotationally invariant, and $\frac{X^2+Y^2}{2}$ is standard exponential, then (X,Y) are indep standard Gaussian.

If U, V are indep uniform on $[0,1]$,

$$f_U(t) = f_V(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1 \\ 0, & \text{o.w.} \end{cases}$$

then:

$X = \sqrt{-2\ln U}\cos(2\pi V)$, $Y = \sqrt{-2\ln U}\sin(2\pi V)$, are indep standard Gaussian.

Characteristic ftn of the standard Guassian

$$\varphi_X(t) = Ee^{itX} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{itx} e^{-x^2/2}\,dx$$

for real a,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ax} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{a^2/2}\int_{-\infty}^{+\infty} e^{-(x-a)^2/2}\,dx$$

$$= e^{a^2/2}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = e^{a^2/2}$$

$$\varphi_X(a) = Ee^{iaX} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{itx} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ax} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}}\int_{-\infty}^{+\infty} e^{-(x-a)^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}}\int_{-\infty}^{+\infty} e^{-u^2/2}\,du$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}}\sqrt{2\pi} = e^{\frac{a^2}{2}}$$

φ_X $\varphi_X(t) = e^{-t^2/2}$

The integral is an analytic in a, so the formula holds for complex a as well.

For $a = it$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{itx} e^{-x^2/2}\,dx = e^{(it)^2/2} = e^{-t^2/2}$$

$\underline{\varphi_X(t) = Ee^{itX}}$

when $a = it$, $\varphi_X(t) = e^{(it)^2/2}$

$= e^{-t^2/2}$

★ $\varphi_X(t) = e^{-t^2/2}$ Characteristic ftn for standard normal Gaussian

Gaussian Random Variable

Y is Gaussian (normal), if $Y = \sigma X + a$ (σ, a are real #), X standard normal

If $\sigma = 0$, $Y = a \in$ constant

If $\sigma \neq 0$, then the densities of $\sigma X + a$, $(-\sigma)X + a$ are the same. (b/c densities for X and $-X$ are the same)

Find the density of Y, assuming $\sigma > 0$.

$$F_Y(y) = P(Y \le y) = P(\sigma X + a \le y)$$
$$= P(\sigma X \le y - a)$$
$$\overset{\sigma>0}{=} P\left(X \le \frac{y-a}{\sigma}\right)$$
$$= F_X\left(\frac{y-a}{\sigma}\right)$$

density: $f_Y(y) = F_Y'(y) = \frac{1}{\sigma} f_X\left(\frac{y-a}{\sigma}\right)$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-a)^2}{2\sigma^2}}$$

since f_X is the pdf for standard normal - distn

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-a)^2}{2\sigma^2}}, \quad \sigma > 0$$

$E(Y) = \sigma E(X) + a = a$, $\quad Var(Y) = \sigma^2 Var(X) = \sigma^2$

$$\varphi_Y(t) = E e^{itY} = E e^{it(\sigma X + a)}$$
$$= e^{ita} E e^{it\sigma X}$$
$$= e^{ita} \varphi_X(\sigma t)$$
$$= e^{ita} \times e^{-\sigma^2 t^2/2} = e^{-\sigma^2 t^2/2 + ita}$$

X stand normal. $\varphi_X(t) = e^{-t^2/2}$

$\frac{x-u}{\sigma} = z$

$x = \sigma z + u$

$y = \sigma x + u$

$f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

$$F_Y(y) = P(Y \le y)$$
$$= P(\sigma x + a \le y)$$
$$= P(\sigma x \le y - a)$$
$$\overset{\sigma>0}{=} P\left(x \le \frac{y-a}{\sigma}\right)$$
$$= F_X\left(\frac{y-a}{\sigma}\right)$$
$$f_Y(y) = F_Y'(y) = \frac{1}{\sigma} f_X\left(\frac{y-a}{\sigma}\right)$$
$$= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-}$$

$\frac{y-u}{\sigma} = x \rightarrow y = \sigma x + u$

$$\varphi_Y(t) = E e^{itY} = E e^{it(\sigma x + u)}$$
$$= e^{itu} E e^{it\sigma x}$$
$$= e^{itu} \varphi_X(t\sigma)$$
$$= e^{itu} \cdot e^{-t^2\sigma^2/2}$$

Important If Y_1, Y_2 are indep Gaussian, then $Y = Y_1 + Y_2$ is also Gaussian.

$\varphi_{Y_1}(t) = e^{-t^2\sigma_1^2/2 + ita_1}$

$\varphi_{Y_2}(t) = e^{-t^2\sigma_2^2/2 + ita_2}$

$\varphi_{Y_1+Y_2}(t) = e^{-t^2(\sigma_1^2+\sigma_2^2)/2 + it(a_1+a_2)}$

$Y_1 + Y_2 \sim N(a_1+a_2, \sigma_1^2+\sigma_2^2)$

Why? $EY_1 = a_1$, $\operatorname{Var} Y_1 = \sigma_1^2$

$EY_2 = a_2$, $\operatorname{Var} Y_1 = \sigma_2^2$

then $\varphi_{Y_1}(t) = e^{-t^2\sigma_1^2/2 + ita_1}$

$\varphi_{Y_2}(t) = e^{-t^2\sigma_2^2/2 + ita_2}$

$$\varphi_{Y_1+Y_2}(t) \overset{\text{indep}}{=} \varphi_{Y_1}(t)\varphi_{Y_2}(t) = e^{-t^2\sigma_1^2/2 + ita_1} e^{-t^2\sigma_2^2/2 + ita_2}$$

$$= e^{-\frac{t^2}{2}(\sigma_1^2+\sigma_2^2) + it(a_1+a_2)}$$

$\leftarrow$ Characteristic fcn of Gaussian with mean a_1+a_2, variance $\sigma_1^2+\sigma_2^2$.

$\Rightarrow$ Y must be Gaussian with var $\sigma_1^2+\sigma_2^2$, and expectation a_1+a_2.

Self-test exercise

Prove directly that $Y = Y_1 + Y_2$ is Gaussian. ($F_Y'(y) = f_Y(y)$)

Very Typical / Important Example

Suppose X and Y are indep, the density of X is: $\frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$

the density of y is: $\frac{1}{\sqrt{4\pi}} e^{-(y-2)^2/4}$

And $Z = X - 2Y + 1$.

$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Q: Density of Z?

X, Y are indep Gaussian, $\Rightarrow$ $Z = X - 2Y + 1$ must be Gaussian

$E(X) = 1$, $E(Y) = 2$ $\rightarrow$ $E(Z) = E(X) - 2E(Y) + 1 = 1 - 2\times 2 + 1 = -2$

$Var(X) = 1$, $Var(Y) = 2 \to Var(Z) = Var(X) + 4Var(y) = 1 + 8 = 9$

Hence, $f_Z(z) = \frac{1}{3\sqrt{2\pi}} e^{-(x+2)^2/18}$

$f_X(x) = \frac{1}{\sqrt{2\pi}\,6} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi}\,3} e^{-\frac{(x+2)^2}{18}}$

Some interesting / important properties of the Gaussian distn

fact: If X and Y are i.i.d (same cumulative distn ftn)

And $X+Y$ and $X-Y$ are also indep.

then X and Y have to be Gaussian.

We'll show it assuming $E(X)$, $Var(X)$ exist.

The distn of X is symmetric, which means the distns of X and $-X$ are the same.

Let $\varphi_X(t) = \varphi_Y(t) = Ee^{itX} = Ee^{itY} = \varphi(t)$

X, Y have the same distn

$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = \varphi^2(t)$

$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = \varphi^2(t)$

$\varphi_{X-Y}(t) = \varphi_X(t)\varphi_{-Y}(t) = \varphi_X(t)\varphi_Y(t) = \varphi^2(t)$

$\varphi_{X-Y}(t) = \varphi_X(t)\varphi_{-Y}(t) = \varphi_X(t)\varphi_Y(t) = \varphi^2(t)$

$2X = (X+Y) - (X-Y) \Rightarrow \varphi_{2X}(t) = \varphi_{X+Y}(t)\varphi_{X-Y}(t)$

$= \varphi^2(t)\varphi^2(t) = \varphi^4(t)$

$2X = (X+Y) + (X-Y) \to \varphi_{2X}(t) = \varphi_{X+Y}(t)\varphi_{X-Y}(t) = \varphi^2(t)\varphi^2(t) = \varphi^4(t)$

$\parallel$

$Ee^{it2X} = \varphi_X(2t) = \varphi(2t)$

$Ee^{it2X} = \varphi_X(2t) = \varphi(2t)$

$\to \varphi(2t) = \varphi^4(t)$

$\varphi(2t) = \varphi^4(t)$

Returning to $\varphi(t) = \varphi^4(\frac{t}{2}) = \varphi^{16}(\frac{t}{4}) = \cdots \varphi^{4^n}(\frac{t}{2^n})$ for every $n > 0$

Last class: $\varphi(0) = 1$, $\varphi'(0) = iE(X) = 0$

by symmetry

$\varphi(t) = \cdots$

$\varphi''(0) = -E(X^2) \leftarrow$ call it $-\sigma^2$

$a_t = o(t)$

$\frac{a_t}{t} \to 0$ as $t \to \infty$

By Taylor's formula, for $t \approx 0$,

$$\varphi(t) = \underbrace{\varphi(0)}_{=1} + \underbrace{\varphi'(0)}_{=0}\, t + \underbrace{\frac{\varphi''(0)}{2}}_{=-\frac{\sigma^2}{2}} t^2 + o(t)t^2$$

$o(t) \to 0$ as $t \to 0$

$$\varphi(t) = \lim_{n\to\infty} \varphi^{4^n}\left(\frac{t}{2^n}\right)$$

$$\varphi(t) = \varphi(0) + \varphi'(0)\, t + \frac{\varphi''(0)}{2} t^2 + O(t)t^2$$

$$= \lim_{n\to\infty} \left(1 - \frac{\sigma^2}{2\times 4^n} t^2 + o\left(\frac{t}{2^n}\right)\frac{t}{4^n}\right)^{4^n}$$

$e^{-t^2/2n}$

$(e^{-t^2/2})$

$$= \lim_{n\to\infty} \left(1 - \frac{\sigma^2}{2}\frac{t^2}{4^n}\right)^{4^n} = e^{-\frac{\sigma^2 t^2}{2}} \sim N(0, \sigma^2)$$

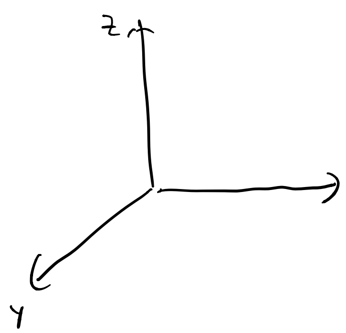
$\uparrow$ characteristic ftn of Gaussian

$\Rightarrow$ X, Y should be Gaussian

$\varphi_X(t) = e^{-t^2/2}$

Application: $\leftarrow \sqrt{X^2+Y^2+Z^2}$, (X, Y, Z) are indep Standard Gaussian

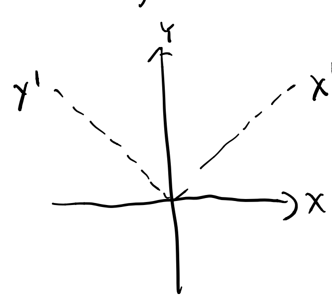
Pick a random molecule of air in the room. Look at its velocity?



Let X, Y, Z be the components of the velocity

$X, Y, Z \leftarrow$ indep distributed

Change the coordinate system: Rotate (X, Y) by $45°$ counterclockwise



For the new coordinate system,

$$\left.\begin{aligned} x' &= \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ y' &= \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \end{aligned}\right\} \text{still indep} \Rightarrow x, y, z \text{ are Gaussian}$$