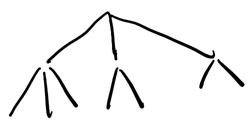
Last time: Branching Process

$X_0 = 1$

X_n: population at level n



$G_n(s) = E(s^{X_n})$

Easy: $E(X_n)$

$E(X_n) = G_n'(1) = p(G_{n-1}(s))' |_{s=1}$

$= p'(G_{n-1}(1))\, G_{n-1}'(1)$

?? G_n $= p'(1)\, G_{n-1}'(1)$

$= E(X_1) E(X_{n-1})$

Conclusion: $E(X_n) = (E(X_1))^n$

$p(s) = s$ $\quad p(s) = \sum_{k=0}^{\infty} P(X_1 = k)\cdot s^k$

$p(s) = \sum_{k=0}^{\infty} P(X_1 = k)\cdot s^k$

$G_n(s) = p(G_{n-1}(s))$

$p(s) = \sum_{k=0}^{\infty} p_k s^k$

$p(1) > 1$

$G_1(1) = 1$

$\boxed{p(1) = 1}$

$E(X_n) = G_n'(1)$

$= p'(G_{n-1}(s)|_{s=1})$

$= p'(G_{n-1}(1))\, G_{n-1}'(1)$

$= p'(1)\, G_{n-1}'(1)$

$= E(X_1) E(X_{n-1})$

$E(X_n) = (E(X_1))^n$

$G_n(s) = p(G_{n-1}(s))$

$= p(p(G_{n-2}(s)))$

$= p(p \cdots G_1(s))$

$\therefore G_n(1) = p(p \cdots \underbrace{G_1(1)}_{=1})$

$= 1 = G_{n-1}(1)$

So $G_1(1) = \sum_{k=0}^{\infty} P(X_1 = k) = 1$

Variance

Let $X: \Omega \to \mathbb{R}$ be a r.v. s.t. $E(X)$ exists.

$Var(X) = E(X - E(X))^2$

① $Var(X) \geq 0$

② If $Var(X) = 0 \iff P\{\omega \in \Omega : X(\omega) = E(X)\} = 1$

③ More computation efficient formula:

$E((X - E(X))^2) = E(X^2 - 2X E(X) + E^2(X))$

$E(X_n) = G_n'(1)$

$= p'(G_{n-1}(1))$

$= p'(G_{n-1}(s) \mid s = 1)$

$= p'(\underbrace{G_{n-1}(1)}_{1})\, G_{n-1}'(1)$

$= \underbrace{p'(1)}_{E(X_1)}\, G_{n-1}'(1)$

$= \underbrace{1}_{} \times E(X_{n-1})$

$p(s) = \sum_{k=0}^{\infty} P(X_1 = k)\cdot s^k = G_1(s)$

$= E(X^2) - 2E^2(X) + E^2(X)$

$= E(X^2) - E^2(X)$

$p'(s) = \sum_{k=0}^{\infty} p(X_i=k)\cdot k s^{k-1}$ $= \cdots \boxed{E^n(X_1)}$

$p'(1) = E(X_1)$

ex. I roll a fair die. let X be the #.

Q: Var(X) = ?

$\text{Var}(X) = E(X^2) - E^2(X)$

$= \dfrac{1^2+2^2+3^2+4^2+5^2+6^2}{6} - \left(\dfrac{1+2+3+4+5+6}{6}\right)^2$

$= \dfrac{35}{12}$

ex. X is cts with density $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x>0 \\ 0 & , x\le 0 \end{cases}$, $\lambda > 0$.

Var(X)?

$\text{Var}(X) = E(X^2) - E^2(X)$

$= \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx - \left(\int_0^\infty x\lambda e^{-\lambda x}\,dx\right)^2$

$= \dfrac{2}{\lambda^2} - \left(\dfrac{1}{\lambda}\right)^2$

$= \dfrac{1}{\lambda^2}$

$\int_0^\infty (-x) e^{-\lambda x}\,dx = \dfrac{-1}{\lambda^2}$

$\int_0^\infty x e^{-\lambda x}\,dx = \dfrac{1}{\lambda^2}$

$\int_0^\infty e^{-\lambda x}\,dx = \dfrac{-1}{\lambda} e^{-\lambda x}\Big|_0^\infty = 0 + \dfrac{1}{\lambda} = \dfrac{1}{\lambda}$

$\int_0^\infty e^{-\lambda x}\,dx = \dfrac{1}{\lambda}$ $\Rightarrow \dfrac{d\int_0^\infty e^{-\lambda x}dx}{d\lambda} = \int_0^\infty \dfrac{d e^{-\lambda x}}{d\lambda}\,dx$

Differentiate it w.r.t λ

once: $\int_0^\infty (-x) e^{-\lambda x}\,dx = \dfrac{-1}{\lambda^2} \Rightarrow \int_0^\infty x e^{-\lambda x}\,dx = \dfrac{1}{\lambda^2} = \lambda^{-2}$

Units: if x is in \$, then x^2 is in $\$^2$.

twice: $\int_0^\infty -x^2 e^{-\lambda x}\,dx = -2\lambda^{-3} \Rightarrow$

$\int_0^\infty x^2 e^{-\lambda x}\,dx = \dfrac{-2}{\lambda^3}$

Variance of the sum of R.V.

Covariance

$X, Y : \Omega \to R$

random variables:

$Cov(X, Y) = E(XY) - E(X)E(Y)$

$Var(X) = Cov(X, X)$

If X, Y are indep $\Rightarrow Cov(X, Y) = 0$

$Var(X_1 + \cdots X_n)$

$= \sum_{i=1}^{n} Var(X_i^2) + \sum_{i \neq j} Cov(X_i, X_j)$

$Var(X_1 + X_2 + \cdots + X_n)$

$= E(X_1 + X_2 + \cdots X_n)^2 - E^2(X_1 + \cdots X_n)$

$= E\left(\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j\right) - \left(\sum_{i=1}^{n} E(X_i)\right)^2$

$= \underbrace{\sum_{i=1}^{n} E(X_i^2)}_{} + \underbrace{\sum_{i \neq j} E(X_i X_j)}_{} - \underbrace{\sum_{i=1}^{n} E^2(X_i)}_{} - \underbrace{\sum_{i \neq j} E(X_i)E(X_j)}_{}$

$= \underbrace{\sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)}_{}$ ← important: we count i, j and j, i as difference

Important:

If $X_1, \cdots X_n$ are pairwise independent, then:

$$\underbrace{Var(X_1 + X_2 + \cdots X_n) = \sum_{i=1}^{n} Var(X_i)}_{}$$

ex. Problem of letters

Someone wrote n letters to n people, put letters into envelopes randomly, so that each envelop contains one letter.

Let X be the # of people who get the correct letter.

Q: What is the variance of X?

For $i = 1, 2, \cdots n$, let

$X_i = \begin{cases} 1, & \text{if the } i\text{th person gets the right letter} \\ 0, & \text{if not} \end{cases}$

$X = X_1 + \cdots X_n$

$Var(X) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$

$Var(X_i) = E(X_i^2) - E^2(X_i)$

$P(X_i = 1) = \frac{1}{n}$, $\therefore E(X_i) = \frac{1}{n}$, $E(X_i^2) = \frac{1}{n} \rightarrow Var(X_i) = \frac{1}{n} - (\frac{1}{n})^2 = \frac{n-1}{n^2}$

$Cov(X_i, X_j) = \underbrace{E(X_i X_j)}_{\frac{1}{n(n-1)}} - \underbrace{E(X_i)}_{= \frac{1}{n}} \underbrace{E(X_j)}_{= \frac{1}{n}}$

$X_i X_j$ takes 2 values.

$X_i X_j = \begin{cases} 1, & \text{if both } i\text{th and } j\text{th get the right letters} \\ 0, & \text{o.w.} \end{cases}$

$P(X_i X_j = 1) = \frac{1}{n(n-1)}$

$\therefore E(X_i X_j) = \frac{1}{n(n-1)}$

$Cov(X_i, X_j) = \frac{1}{n(n-1)} - (\frac{1}{n})^2 = \frac{n-(n-1)}{n^2(n-1)} = \frac{1}{n^2(n-1)}$

Hence, $Var(X) = n \cdot \frac{n-1}{n^2} + n(n-1) \cdot \frac{1}{n^2(n-1)} = \frac{n-1}{n} + \frac{1}{n} = 1$ ✓

$X = X_1 + \cdots X_n$

$n^2 - n$

$X_n = \begin{cases} 1 & \frac{1}{n} \\ 0 & \frac{n-1}{n} \end{cases}$

$Var(X) = \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$

$E(X_i) = \frac{1}{n}$

$E(X_i^2) = \frac{1}{n}$

$Var(X_i) = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$

$Cov(X_i, X_j)$

$= E(X_i X_j) - E(X_i) E(X_j)$

$= \frac{1}{n(n-1)} - \frac{1}{n^2}$

$= \frac{n-(n-1)}{n^2(n-1)} = \frac{1}{n^2(n-1)}$

So $Var(X) = n \times \frac{n-1}{n^2} + n(n-1) \times \frac{1}{n^2(n-1)}$

$= \frac{n-1}{n} + \frac{1}{n} = \boxed{1}$

$Var(X) = n \times (\frac{1}{n} - \frac{1}{n^2}) + \sum_{i \neq j} \frac{1}{n^2(n-1)}$

$= (1 - \frac{1}{n}) + \frac{1}{n^2 \cancel{(n-1)}} \times n(n-1)$

$= (1 - \frac{1}{n}) + \frac{1}{n^2}$

$= 1$

ex. We toss a fair coin n times ($n \geq 5$).

Let X be the # of switches from heads to tails.

$\cdots HT \cdots HT$

Q: $Var(X) = ?$

$X_i = \begin{cases} 0 & \text{o.w.} \\ 1 & \text{if } i\text{th toss is } H \text{, and } (i+1)\text{th toss is } T \end{cases}$

$X = X_1 + \cdots X_{n-1}$

$Var(X) = \sum_{i=1}^{n-1} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$

$P(X_i = 1) = \frac{1}{4}$

$Var(X_i) = E(X_i^2) - E^2(X_i) = \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$

$Cov(X_i, X_j) = 0$, if $\underbrace{|i-j| > 1}_{\text{indep}}$

HT $\cdots$ HT

$Cov(X_i, X_{i+1}) = \overset{=0}{E(X_i X_{i+1})} - E(X_i) E(X_{i+1}) = \frac{-1}{16}$

$X_i X_{i+1} = \begin{cases} 1 & \text{, if ith is H, } \underbrace{\text{(i+1) th is T, (i+1) th is}}_{\text{never happens}} \text{ H, and (i+2) th is T} \\ 0 & \text{o.w} \end{cases}$

$\rightarrow$ (i, j) and (j, i) are diff combi

$Var(X) = (n-1) \times \frac{3}{16} - \frac{2(n-2)}{16}$