Last time: (exam)

Define the expectation of a R.V. $X: \Omega \to R$.

① If X is simple with values $a_1, a_2, \cdots a_n$

$$E(X) = \sum_{i=1}^{n} a_i P(X=a_i)$$

② In general, use a limit argument.

③ If X is continuous with density $f_X(x)$, then:

$$E(X) = \int_{-\infty}^{+\infty} t f_X(t)\, dt$$

$\hookrightarrow$ integral converges absolutely

Proof: $E g(X)$

$= g(x_1) f(X=x_1) + g(x_2) f(X=x_2) + \cdots g(x_n) f(X=x_n)$

$= \sum_{i=1}^{n} g(x_i) f(X=x_i)$

A Couple of Examples:

If X is simple and if $g: R \to R$ with values $a_1, a_2, \cdots a_n$.

is a fcn then $g(X)$ is a simple r.v.

And $E g(X) = \sum_{i=1}^{n} g(a_i) f(X=a_i)$

self-test exercise

ex. If X takes values $-1, 1, 2$ with prob $1/3$ each.

And $g(x) = x^2$

$E g(X) = \sum_{i=1}^{n} g(a_i) f(X=a_i) = 1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 4 \times \frac{1}{3} = 2$

$= a_1^2 f(X=a_1) + a_2^2 f(X=a_2) + a_3^2 f(X=a_3)$

$g(X)$ takes value 1 with prob $2/3$, and 4 with prob $1/3$.

$E g(X)$ by def $= 1/3 \times 4 + 2/3 \times 1 = 2$

By the formula: $(-1)^2 \times \frac{1}{3} + 1^2 \times \frac{1}{3} + 2^2 \times \frac{1}{3} = 2$

So it works

If X is continuous "Borel Measurable".

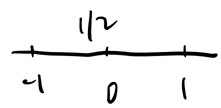
A r.v. and $E\,g(X) = \int_{-\infty}^{+\infty} g(t) f_X(t)\,dt$ provided the integral converges absolutely.

density of X

$f_X(t)$, $f_Y(t)$

density of variable X

ex. If X has density $f_X(t) = \begin{cases} 1/2 & \text{for } -1 \le t \le 1 \\ 0 & \text{elsewhere.} \end{cases}$



$g(t) = t^2$

$E\,g(X) = E(X^2) = \int_{-\infty}^{+\infty} t^2 f_X(t)\,dt$

$= \int_{-1}^{1} \frac{1}{2} t^2\,dt = \frac{1}{3}$

$\frac{1}{6} t^3 \Big|_{-1}^{1}$

$\frac{1}{6} - \frac{-1}{6}$

$E\,g(x) = E\,x^2 = \int_{-1}^{1} t^2 f_X(t)\,dt = \int_{-1}^{1} t^2 \times \frac{1}{2}\,dt = \frac{1}{3}$

Self-test exercise: Compute the density of X^2 compute $E(X^2)$ from the density, check that it is the same.

It is an importance skill to be able to compute the expec of a r.v. by writing it as a linear combination of indicators.

ex. Problems of letters.

Someone wrote n personalized letters to n people, addressed n envelopes, put the letters into the envelops at random. So that every envelops contains exactly one letter and mailed. Let X be the # of people who got the intended letter.

Q: $E(X)$

n letters $\quad$ n people

$X_k = \begin{cases} 1, & \text{if the } k\text{th person gets the right letter} \\ 0 \end{cases}$

$\Omega = \{ \underbrace{n!}_{n \text{ factorial}} \text{ to mail the letter} \}$

$X = X_1 + X_2 + \cdots X_k$

$E(X) = \sum_{i=1}^{n} E(X_i)$

for X_k, $P(X_k = 1) = \frac{1}{n}$, $P(X_k = 0) = \frac{n-1}{n}$

$E(X_k) = \frac{1}{n} \times 1 + \frac{n-1}{n} \times 0 = \frac{1}{n}$

$\therefore E(X) = \sum_{i=1}^{k} E(X_i) = n \times \frac{1}{n} = 1$

For $k = 1, 2, \cdots n$, let:

$X_k = \begin{cases} 1, & \text{if the } k\text{th person gets the right letter} \\ 0, & \text{if not} \end{cases}$

$X = X_1 + X_2 + \cdots X_n \Rightarrow E(X) = E(X_1) + E(X_2) + \cdots E(X_n)$

$P(X_k = 1) = \frac{1}{n}$, $\therefore E(X_k) = \frac{1}{n}$

$\therefore E(X) = \frac{1}{n} \times n = 1$

ex. (Counting record)

Let $a_1, a_2, \cdots a_n$ be a permutation of # $1, 2, \cdots n$

We say that a_k is a record.

If it is greater than all a_i with $i < k$ (a_1 is always a record).

ex. 2, 1, 4, 3, 5

records

$X = X_1 + \cdots X_n$ $\quad$ $X_i = \begin{cases} 1, & \text{record} \\ 0, & \text{o.w.} \end{cases}$

$P(X_1 = 1) = 1$ $\quad$ $E(X_1) = 1$

$P(X_2 = 1) = \frac{1}{2}$ $\quad$ $E(X_2) = \frac{1}{2}$

$P(X_3 = 1) = \frac{1}{3}$ $\quad$ $E(X_3) = \frac{1}{3}$.

$P(X_n = 1) = \frac{1}{n}$ $\quad$ $E(X_n) = \frac{1}{n}$

$X_k = \begin{cases} 1, & \text{if } k \text{ is record} \\ 0, & \text{else} \end{cases}$ $\quad$ $E(X) = \sum_{k=1}^{n} E(X_k)$

$X = X_1 + X_2 + \cdots X_n$, for $P(X_1 = 1) = 1$, $E(X_1) = 1 \times 1 = 1$

$E(X) = \sum_{i=1}^{n} E(X_i)$

$= 1 + \frac{1}{2} + \cdots \frac{1}{n}$

$P(X_2=1)=\frac{1}{2}, E(X_2)=1\times\frac{1}{2}=\frac{1}{2}$

Q: let X be the # of records in a random permutation of $\{1,2,\dots n\}$,

What is $E(X)$?

$P(X_3=1)=\frac{1}{3}, E(X_3)=1\times\frac{1}{3}=\frac{1}{3}$

$\vdots$

let $X_k=\begin{cases}1 & \text{if } a_k \text{ is a record}\\ 0 & \text{if not}\end{cases}$

$P(X_n=1)=\frac{1}{n}, E(X_n)=\frac{1}{n}$

the $E(X)$

$=1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n}$

$\approx \gamma+\ln n$

$X=X_1+X_2+\dots X_n \Rightarrow E(X)=E(X_1)+\dots+E(X_n)$

$P(X_1=1)=1 \Rightarrow E(X_1)=1$

$P(X_2=1)=\frac{1}{2}$ (symmetric) $\Rightarrow E(X_2)=\frac{1}{2}$

$P(X_3=1)=\frac{1}{3} \rightarrow E(X_3)=\frac{1}{3}$

$\vdots$

So $E(X_k)=\frac{1}{k}$

Hence, $E(X)=1+\frac{1}{2}+\dots\frac{1}{n}\approx \underset{\substack{\uparrow\\ \text{Euler constant} \approx 0.577}}{\gamma}+\ln n$ for large n.

Self-test exercise:

Prove the inclusion-exclusion formula along the following lines.

$P=P(A)$.

let $A_1, A_2, \dots A_n$ be events, let $A=\bigcup_{i=1}^{n} A_i$,

$I_A(w)=\begin{cases}1, & w\in A\\ 0, & \text{oth}\end{cases}$

$I_A(w)=\begin{cases}1, & \text{if } w\in A\\ 0, & \text{if } w\notin A\end{cases}$

$I_A(w)=\begin{cases}1, & w\in A\\ 0, & \text{otherwise}\end{cases}$

$I_{A_i}(w)=\begin{cases}1, & \text{if } w\in A_i\\ 0, & \text{if } w\notin A_i\end{cases}$

$I_{A_i}(w)=\begin{cases}1, & w\in A_i\\ 0, & \text{otherwise.}\end{cases}$

$I_A=1-\prod_{i=1}^{n}(1-I_{A_i}) \Rightarrow$

$I_A(w)=1-\prod_{i=1}^{n}(1-I_{A_i}(w))$

$$\underbrace{E\, I_A = E\Big(1-\prod_{i=1}^{n}(1-I_{A_i})\Big)}_{\text{Inclusion-Exclusion formula}}$$

$$E\, I_A = P(A)$$

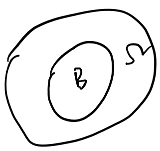
$$E\, I_A(\omega) = P(A) = E\Big[1-\prod_{i=1}^{n}(1-I_{A_i}(\omega))\Big]$$

$$I_A(\omega) = 1-\prod_{i=1}^{n}(1-I_{A_i}(\omega))$$

Conditioned Expectation

Let $X: \Omega \to \mathbb{R}$ be a simple variable, taking values $a_1, a_2, \cdots a_n$.

Let $B \subset \Omega$ be an event with $P(B) > 0$. $E(X|B) = \sum_{i=1}^{n} P(X=a_i|B)\cdot a_i$

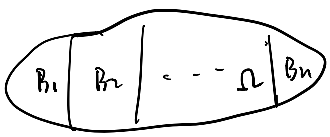


Define $E(X|B) = \sum_{i=1}^{n} a_i \underbrace{P(X=a_i|B)}$

$E(X|B) = \sum_{i=1}^{n} P(X=a_i|B)\cdot a_i$

you can extend this definition to general X by tracking limits.

The formula for the total prob:



$E(X) = \sum$

If $\Omega = \bigcup_{i=1}^{n} B_i$, $B_i \cap B_j = \phi$, $P(B_i) > 0$

then $\boxed{E(X) = \sum_{i=1}^{m} E(X|B_i)\, P(B_i)}$

why? $E(X) = \sum_{i=1}^{n} a_i P(X=a_i)$

$$E(X) = \sum_{i=1}^{n} a_i P(X=a_i)$$
$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} P(X=a_i|b_j)\, P(b_j)$$
$$= \sum_{j=1}^{m} \sum_{i=1}^{n} a_i P(X=a_i|b_j)\, P(b_j)$$

$$E(X) = \sum_{i=1}^{m} E(X|B_i)\, P(B_i) = \sum_{i=1}^{n} a_i \Big(\sum_{j=1}^{m} P(X=a_i|B_j)\, P(B_j)\Big)$$

$$E(X) = \sum_{i=1}^{n} a_i P(X=a_i) = \sum_{j=1}^{m} P(B_j) \sum_{i=1}^{n} a_i P(X=a_i|B_j) = \sum_{j=1}^{m} P(b_j)\, E(X|b_j)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} P(X=a_i|B_j)\cdot P(B_j) = \sum_{j=1}^{m} P(B_j)\, E(X|B_j)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} a_i P(X=a_i|B_j)\, P(B_j)$$

$$= \sum_{j}^{m} P(B_j) \sum_{i}^{n} a_i P(X=a_i|B_j)$$

$$E(X) = \sum_{i=1}^{n} E(X|B_i)\, P(B_i)$$

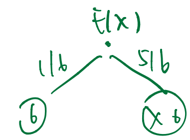
$$E(X|B_i) = \sum_{i=1}^{n} a_i P(X=a_i|B_i)$$

$E(X|B_j)$

ex. I roll a fair die till I get six.

let X be the # of rolls.

Q: $E(X) =$



$E(X|6)$

$= 1$

$E(X|\times 6)$

$= E(X+1)$

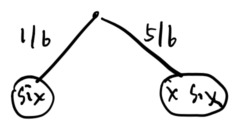
$= E(X)+1$

$E(X) = \frac{1}{6}\times 1 + \frac{5}{6}\times E(X) + \frac{5}{6}$

$\frac{1}{6}E(X) = 1$

$E(X) = 6$

Conditioning on the first roll



$E(X|\text{six}) = 1$ $\qquad$ $E(X|\text{not six}) = E(X)+1$

So $E(X) = \frac{1}{6}E(X|\text{six}) + \frac{5}{6}E(X|\text{not six})$

$= \frac{1}{6}\times 1 + \frac{5}{6}(E(X)+1)$

$E(X) = 6$

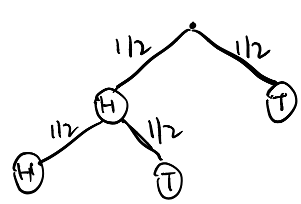
ex. I toss a fair coin till I get two heads in a row.

...HH...

let X be the # of tosses.

Q: $E(X) = ?$

Condition on the first couple of tosses?



$E(X|T) = E(X)+1$

$E(X|HH) = 2$ $\qquad$ $E(X|HT) = E(X)+2$

$\because$ we have to start a new

$E(X) = \frac{1}{2}E(X|T) + \frac{1}{4}E(X|HH) + \frac{1}{4}(E(X)+2)$

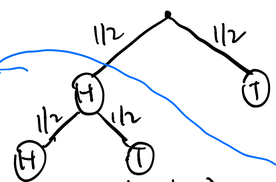
$= \frac{1}{2}(E(X)+1) + \frac{1}{4}\times 2 + \frac{1}{4}E(X) + \frac{1}{2}$

$\therefore E(X) = 6$

Q: I toss a coin till I get ... HT.

let X be the number of tosses.

$E(X) = ?$



$E(x|T) = E(x)+1$

$E(x|HH) = E(x|H)+1$

$E(x|HT) = 2$

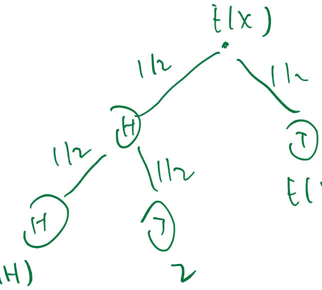
$\therefore \; E(X|H) = \frac{1}{2}\times 2 + \frac{1}{2}\left(E(x|H)+1\right)$

$\Rightarrow \; E(x|H) = 3$

$\therefore \; E(x) = \frac{1}{2}\times 3 + \frac{1}{2}\left(E(x)+1\right)$

$= \frac{1}{2}E(x) + 2$

$E(x) = 4$



$E(x|T) = E(x)+1$

$E(x|HH)$
$= E(x+1|H)$

$E(x) = \frac{1}{4}E(x+1|H) + \frac{1}{4}\times 2 + \frac{1}{2}E(x) + \frac{1}{2}$ ①

$E($ $= \frac{1}{4}E(x|H) + \frac{1}{4} + \frac{2}{4} + \frac{1}{2}E(x) + \frac{1}{2}$

$= \frac{1}{4}E(x|H) + \frac{1}{2}E(x) + \frac{5}{4}$

$E(x) = \frac{1}{2}E(x|H) + \frac{1}{2}E(x|T)$

$= \frac{1}{2}E(x|H) + \frac{1}{2}E(x) + \frac{1}{2}$ ②

$\frac{1}{2}E(x|H) = \frac{1}{2}E(x) - \frac{1}{2}$

$E(x|H) = E(x) - 1$ ③

③ → ①: $\cancel{E(x) = \frac{1}{2}E(x) - \frac{1}{2}}$

$E(x) = \frac{1}{4}E(x) - \frac{1}{4} + \frac{1}{2}E(x) + \frac{5}{4}$

$\frac{1}{4}E(x) = 1 \rightarrow E(x) = 4$