Convergence

Convergence in distn $X_n: \Omega_n \to \mathbb{R}$ random variables

$X: \Omega \to \mathbb{R}$ r.v.

$\lim_{n\to\infty} F_n(a) = F(a)$

let $F_n(a) = P(X_n \le a) \leftarrow$ CDF of X_n

$F(a) = P(X \le a) \leftarrow$ CDF of X

We say that $X_n \xrightarrow{D} X$ ("in distn")

If for all real a, F is cts at a, we have $\lim_{n\to\infty} F_n(a) = F(a)$

Levy's continuity theorem

① If $X_n \xrightarrow{D} X$ then: for all $t \in \mathbb{R}$ we have: $\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t)$

② let X_n be a sequence of random variable, and suppose there is a ftn: $\varphi: \mathbb{R} \to \mathbb{C}$

s.t. $\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t)$ for all $t \in \mathbb{R}$.

For all $t \in \mathbb{R}$ and φ is cts at $t=0$, then φ is the characteristic ftn of some random variable X and $X_n \xrightarrow{D} X$.

Central Limit Theorems

Generally, they deal with situations $X_n \xrightarrow{D} X$ where X is Gaussian.

Main ex:

let $X_1, X_2, \cdots X_n$ be i.i.d (have the same CDF) r.v. $E(X_n) = 0$, $Var(X_n) = 1$.

let $Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k$, $n = 1, 2, \cdots$

$\sqrt{n}(\bar{Y} - \mu) \Rightarrow N(0, \sigma^2)$

then $Y_n \xrightarrow{D} Y$ where Y is standard normal.

Note $E(Y_n) = 0$, $Var(Y_n) = 1$ sanity check

Proof:

let $\varphi(t) = \varphi_{X_n}(t) = E e^{itX_n}$

$$\varphi_{Y_n}(t) = E e^{itY_n} = E e^{i\frac{t}{\sqrt{n}}\sum_{k=1}^{n} X_k}$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1} e^{i\frac{t}{\sqrt{n}}X_2} \cdots e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$\overset{\text{indep}}{=} E\left[e^{i\frac{t}{\sqrt{n}}X_1}\right] \times \cdots E\left[e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$= \varphi^n\left(\frac{t}{\sqrt{n}}\right)$$

We can see $a_t = O(t)$. Since $\left|\frac{a_t}{t}\right|$ is bounded as $t \to 0$, we know $a_t \to 0$ as t in some neighborhood of 0.

$\underbrace{\frac{O(t)}{t} \to 0 \text{ as } t \to 0}_{M \text{ cst}}$

$\frac{\varphi'''(0)}{3!} t^3$

$O(t) \cdot t^2$

Taylor Expansion: $\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{\varphi''(0)}{2!}t^2 + O(t)t^2$ where $O(t) \to 0$ as $t \to 0$. Δ

$\varphi(0) = 1$, $\varphi'(0) = iE(X_n) = 0$, $\varphi''(0) = -E(X_n^2) = -Var(X_n) = -1$

Hence $\varphi_{X_n}(t) = \left(1 - \frac{t^2}{2n} + \underbrace{O(t)}_{\to 0 \text{ as } t \to 0}\frac{t^2}{2n}\right)^n$

$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{\varphi''(0)}{2!}t^2 + O(t) \cdot t^2$

$O(t) \to 0$ as $t \to 0$

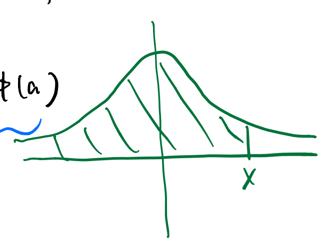
$\to e^{-t^2/2}$, which is the characteristic ftn of standard normal

$\Downarrow$ levy's continuity theorem

$Y_n \xrightarrow{D} \underset{\text{standard normal}}{Y}$

let $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$, which is the CDF of the standard normal.

$Y_n \xrightarrow{D} \underset{\text{standard Gaussian}}{Y} \iff$ for all $a<b$, $P(a \le Y_n \le b) \to \Phi(b) - \Phi(a)$



$\Phi(x)$ is tabulated for $x \ge 0$. by symmetry, $\Phi(x) + \Phi(-x) = 1$

Q: How fast is the convergence?

A: Tricky. Buzzwords: Stein Method, Berry-Esseen Theorem.

CLT under special situations (not always true)

Sometimes, if $X_1, \cdots, X_n \cdots$ are indep r.v. no need for i.i.d

$$Y_n = \frac{\sum_{k=1}^{n} X_k - \left(\sum_{k=1}^{n} E(X_k)\right)}{\sqrt{\sum_{k=1}^{n} Var(X_k)}}, \text{ so that } E(Y_n) = 0, \ Var(Y_n) = 1.$$

$Y_n \xrightarrow{D} \underbrace{Y}_{\text{Standard Gaussian}}$

If that happens, we have central limit theorem.

One More Example (Lyapunov Theorem) X need

If $E(X_n) = 0$, $Var(X_n) = \sigma_n^2$. $E|X_n^3| = \gamma_n$

And $S_n^2 = \sum_{k=1}^{n} \sigma_k^2 = Var(Y_n)$,

and $\lim_{n\to\infty} \frac{1}{S_n^3} \sum_{k=1}^{n} \gamma_k = 0$

then $Y_n \xrightarrow{D} \underbrace{Y}_{\text{standard normal}}$

ex. (Typical)

We roll a fair die n times.

let Y_n be the total sum.

$Y_n = X_1 + \cdots X_n$

X_k: # shown on the k-th roll

$E(Y_n) = \sum_{i=1}^{n} E(X_i) = \frac{7}{2} n$

$D(Y_n) = \sum_{i=1}^{n} Var(X_i) = \frac{35n}{12}$

CLT: $\frac{Y_n - \frac{7}{2}n}{\sqrt{\frac{35n}{12}}} \xrightarrow{D} N(0,1)$

$E(Y_n) = E(X_1) + \cdots E(X_n) = \frac{7}{2}n$

$Var(Y_n) = Var(X_1) + \cdots Var(X_n) = \frac{35}{12}n$
↑
indep

$P\left(a \le \frac{Y_n - \frac{7}{2}n}{\sqrt{\frac{35n}{12}}} \le b\right) \to \Phi(b) - \Phi(a)$
as $n \to \infty$

$P\left(\sqrt{\frac{35n}{12}}a + \frac{7}{2}n \le Y_n \le \sqrt{\frac{35n}{12}}b + \frac{7}{2}n\right)$
$\to \Phi(b) - \Phi(a)$

$\boxed{\text{Central Limit says:}}$

$$\frac{Y_n - \frac{7}{2}n}{\sqrt{\frac{35n}{12}}} \xrightarrow{D} \text{standard normal}$$

For any $a < b$,

$$P\left(a \le \frac{Y_n - \frac{7}{2}n}{\sqrt{\frac{35n}{12}}} \le b\right) \to \Phi(b) - \Phi(a) \quad \text{when } n \to \infty.$$

$$\parallel$$

$$P\left(\frac{7}{2}n + a\sqrt{\frac{35n}{12}} \le Y_n \le \frac{7}{2}n + b\sqrt{\frac{35n}{12}}\right) \xrightarrow[n\to\infty]{} \Phi(b) - \Phi(a)$$

Choose, for ex.

$a = -5\sqrt{\frac{12}{35}}$, $b = 5\sqrt{\frac{12}{35}}$
≈ -2.93 ≈ 2.93

then: $P\left(\frac{7}{2}n - 5\sqrt{n} \le Y_n \le \frac{7}{2}n + 5\sqrt{n}\right) \xrightarrow{n\to\infty} \Phi\left(5\sqrt{\frac{12}{35}}\right) - \Phi\left(-5\sqrt{\frac{12}{35}}\right) = 2\Phi\left(5\sqrt{\frac{12}{35}}\right)$
≈ 0.997

Hence, $\lim_{n\to\infty} P\left(\left|Y_n - \frac{7}{2}n\right| > 5\sqrt{n}\right) \approx 0.003$

rely on n (hard to figure out)

Compare with Chebyshev,

$$P\left(\left|Y_n - \frac{7}{2}n\right| > 5\sqrt{n}\right) \le \frac{Var(Y_n)}{25n} = \frac{\frac{35}{12}n}{25n} = \frac{35}{12 \times 25} = \frac{7}{60} \approx 0.12$$

Compare with Bernstein,

$$P\left(\left|Y_n - \frac{7}{2}n\right| > 5\sqrt{n}\right) = P\left(\left|\frac{2}{5}Y_n - \frac{7}{5}n\right| > 2\sqrt{n}\right) \le 2e^{-2} \approx 0.27$$

true for any n

$$Pr(|X| > a) \leq 2e^{-a^2/2n}$$

$$\leq 2e^{-\varphi n/2n}$$