# Last time: Counted Trajectories

A very typical example:

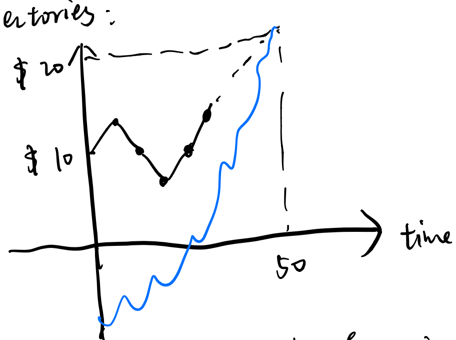
A gambler plays 50 times, each time winning \$2 or losing \$2 with P $\frac{1}{2}$ each.

He starts with \$10.

Q: Probability that he ends up with \$20.

And never goes broke in the process.

Trajectories:




We came above the # of trajectories that started at (0, 10), end at (50, 20), and never visits the x-axis in the process.

Prob of each such a trajectory is: $(\frac{1}{2})^{50}$

The total # of trajectories from (0,10) to (50, 20) including those that visits the x-axis.

$$\left.\begin{aligned} up + down &= 50 \\ up - down &= 20 - 10 = 10 \end{aligned}\right\} \Rightarrow \begin{aligned} up &= 30 \\ down &= 20 \end{aligned}$$

$\binom{50}{30} = \binom{50}{20}$ → Choose which 30 steps should go up, and 20 steps go down

# trajectories:

From $(0,10)$ to $(50,20)$ without a visit to x-axis.

Reflection Principle:

$\parallel$

# all trajectories from $(0,-10)$ to $(50,20)$.

$$\left.\begin{aligned} up + down &= 50 \\ up - down &= 30 \end{aligned}\right\} \Rightarrow \begin{aligned} up &= 40 \\ down &= 10 \end{aligned}$$

$$\binom{50}{40} = \binom{50}{10}$$

A: $\left(\frac{1}{2}\right)^{50}\left(\binom{50}{20} - \binom{50}{10}\right) \approx 0.04$

What if the gambler wins with prob 0.6 and loses with p 0.4 each time.

A: $(0.6)^{30}(0.4)^{20}\left(\binom{50}{20} - \binom{50}{10}\right) \approx 0.11$

Reason: the prob of each trajectory from $(0,10)$ to $(50,20)$ is $(0.6)^{30}(0.4)^{20}$

A tricky example (exam X)

Alice and Bob have \$100 each, they play 100 times and after each play, one of them gives \$1 to the ot with prob $\frac{1}{2}$ each way.

Q: What is more likely?

(a) After 100 plays, they stay even with \$100 each.

(b) One of them is leading all the way after the first game until 100 plays.

(a)

We count trajectories that start at $(0,0)$ and end at $(2n, 0)$, $n=50$.

$$\left.\begin{aligned} up + down &= 2n \\ up - down &= 0 \end{aligned}\right\} \to \begin{aligned} up &= n = 50 \\ down &= n = 50 \end{aligned}$$

# of trajectories: $\binom{2n}{n}$

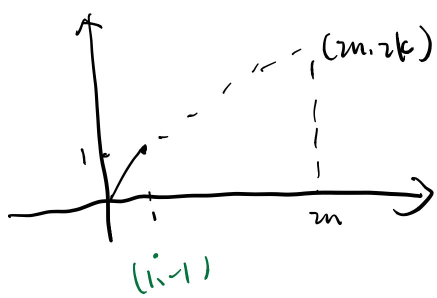
prob of each. $(\frac{1}{2})^{2n} = 2^{-2n}$

Prob of all such trajectories is: $\binom{2n}{n} 2^{-2n} = \frac{(2n)!}{n!\,n!} 2^{-2n}$

(b)

We count trajectories that starts at $(0,0)$ and stay all the way up or all the way down after the first step (And make $2n$ steps together)

Count those that stay all the way up.




$1 \le k \le n$

from 1 to 2k.

$$\left.\begin{aligned} up + down &= 2n-1 \\ up - down &= 2k-1 \end{aligned}\right\} \begin{aligned} up &= n+k-1 \\ down &= n-k \end{aligned}$$

Count the total # of trajectories from $(1,1)$ to $(2n,2k)$, including those that visits the x-axis.

Reflection principle:

# of trajectories from $(1,1)$ to $(2n,2k)$ is the same as # from $(1,-1)$ to $(2n,2k)$

$$\left.\begin{aligned} up + down &= 2n-1 \\ up - down &= 2k+1 \end{aligned}\right\} \to \begin{aligned} up &= n+k \\ down &= n-k-1 \end{aligned}$$

$$\begin{cases} up + down = 2n-1 \\ up - down = 2k+1 \end{cases}$$

So that # is: $\binom{2n-1}{n+k}$

up = $n+k$

down = $n-k-1$

So the # of trajectories starts from $(1,1)$, end at $(2n, 2k)$, and stay all the way up is: $\binom{2n-1}{n+k-1} - \binom{2n-1}{n+k}$

total # of trajectories is:

$$\underbrace{\sum_{k=1}^{n}}\left(\binom{2n-1}{n+k-1} - \binom{2n-1}{n+k}\right) = \binom{2n-1}{n} - \binom{2n-1}{n+1} + \binom{2n-1}{n+1} - \binom{2n-1}{n+2} + \cdots$$

$$\binom{2n-1}{2n-1} - \binom{2n-1}{2n} = \left[\binom{2n-1}{n} + \cancel{\binom{2n-1}{n+1} + \cdots} \binom{2n-1}{2n-1}\right] - \left[\cancel{\binom{2n-1}{n+1} + \binom{2n-1}{n+2}} + \cdots \underbrace{\binom{2n-1}{2n}}_{=0}\right]$$

$k_{max} = n$

$$= \binom{2n-1}{n} - \cancel{\binom{2n-1}{n+1}} + \cancel{\binom{2n-1}{n+1}} - \cancel{\binom{2n-1}{n+2}} + \cancel{\binom{2n-1}{n+2}} - \cancel{\binom{2n-1}{n+3}} + \cdots - \cdots \underbrace{\binom{2n-1}{2n}}_{=0} = \binom{2n-1}{n}$$

$$= \underbrace{\binom{2n-1}{n}}$$ ✓

# of all the way down: also $\binom{2n-1}{n}$

∴ total P of all the way up or down is: $\underbrace{2}_{\text{symmetric}} \times \binom{2n-1}{n} \times \left(\frac{1}{2}\right)^{50}$

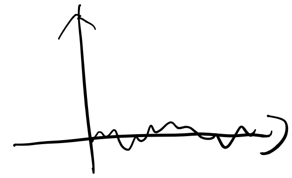
(a) The probability to start at $(0,0)$ and end at $(2n,0)$ is:

$$\binom{2n}{n} 2^{-2n} = \frac{(2n)!}{n!\,n!} 2^{-2n}$$

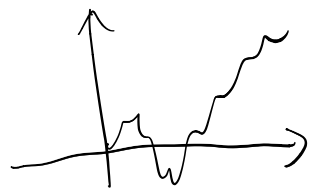
(b) The prob to start at $(0,0)$ and stay all the way up or down in $2n$ steps:

$$2 \times \binom{2n-1}{n} \times 2^{-2n} = \frac{2(2n-1)!}{n!(n-1)!} 2^{-2n} = \frac{2n(2n-1)!}{n!\,n(n-1)!} = \frac{(2n)!}{n!\,n!} \times 2^{-2n}$$

A: The prob are equal.

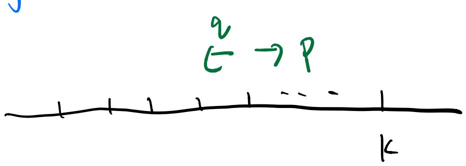


very unlikely



likely

## Infinite Random Walks




Start at some integer $k$,

Move left with prob $q$

Move right with prob $p$

$p+q=1$

if $p=q=\frac{1}{2}$, the walk is called symmetric steps independent.

Probability space $\Omega$:

LLRRLLRL...

Infinite sequences: "L" left and "R" Right.

"Cylindrical Events"

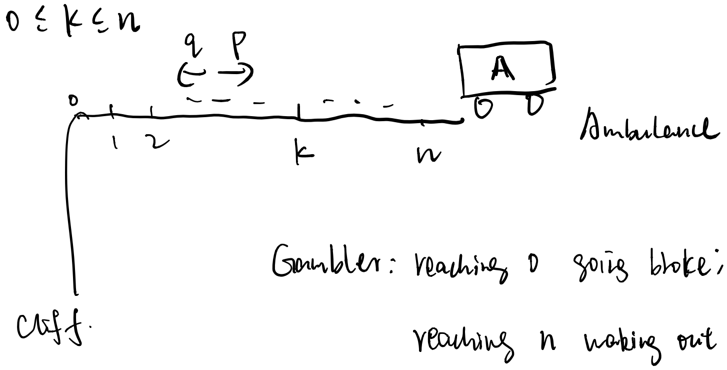
prescribe finitely many moves

3 ... 10 ... 100

L   R   L

+ All countable intersections, unions, etc, ... it works!

$0 \le k \le n$




Gambler: reaching 0 going broke;

reaching n walking out with \$n

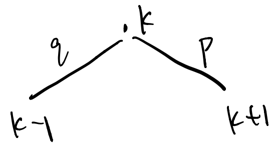
$p+q=1$

survival

$S_k$ = prob of reaching n starting from $0 \le k \le n$

$S_0 = 0$, $\quad S_n = 1$

Condition on the first step:




$$S_k = p S_{k+1} + q S_{k-1}, \quad k = 1, 2, \cdots n-1$$

Symmetric Case: $p = q = \frac{1}{2}$

$$S_k = \frac{1}{2} S_{k+1} + \frac{1}{2} S_{k-1}$$

$$\Rightarrow \frac{1}{2} S_k + \frac{1}{2} S_k = \frac{1}{2} S_{k+1} + \frac{1}{2} S_{k-1}$$

$$\to \frac{1}{2} S_{k+1} - \frac{1}{2} S_k = \frac{1}{2} S_k - \frac{1}{2} S_{k-1}$$

$$\to S_{k+1} - S_k = S_k - S_{k-1}$$

$$S_1 - S_0 = S_2 - S_1 = S_3 - S_2 = \cdots x$$

$$S_k = \underbrace{(S_k - S_{k-1})}_{x} + (S_{k-1} - S_{k-2}) + \cdots \underbrace{(S_1 - S_0)}_{x} + S_0$$

$S_k = kx \qquad S_n = nx = 1 \Rightarrow x = \frac{1}{n}$

$$S_k = q S_{k-1} + p S_{k+1}$$

$$p = q = \frac{1}{2}$$

$$S_k = \frac{1}{2} S_{k-1} + \frac{1}{2} S_{k+1}$$

$$\frac{1}{2} S_k - \frac{1}{2} S_{k-1} = \frac{1}{2} S_{k+1} - \frac{1}{2} S_k$$

$$\underline{S_{k+1} - S_k = S_k - S_{k-1}}$$

let $S_1 - S_0 = S_2 - S_1 = \cdots = x$

$$S_k = (S_k - S_{k-1}) + (S_{k-1} - S_{k-2}) + \cdots (S_1 - S_0) + \underset{0}{S_0}$$

$$= kx$$

$$S_n = nx = 1$$

$\cancel{x \Rightarrow}$, $x = \frac{1}{n}$