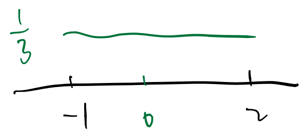
One more exercise

Let X be a r.v. on $[-1,2]$

$$f_X(t) = \begin{cases} \frac{1}{3} & -1 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$



$Y = |X|$, Q: Density with Y.

Start with $F_Y(a)$

$F_Y(a) = P\{Y \le a\} = P\{|X| \le a\}$

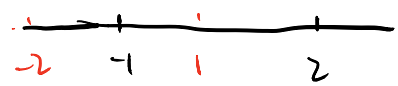
If $a < 0$, then $F_Y(a) = 0$

If $a \ge 0$, then $P\{|X| \le a\} = P\{-a \le X \le a\}$

$= F_X(a) - F_X(-a)$

$f_Y(a) = F_Y'(a) = f_X(a) + f_X(-a)$

$$= \begin{cases} 0 & \text{o.w} \\ 1/3 & \text{if } 1 < a \le 2 \\ 2/3 & \text{if } 0 < a \le 1 \end{cases}$$



$t \le 0$; 0

$F_Y(t) = P(Y \le t) = P(|X| \le t)$

$= P(-t \le X \le t)$

When $0 < t \le 1$,

$P(-t \le X \le t) = 2t \times \frac{1}{3} = \frac{2}{3}t$

When $1 < t \le 2$,

$P(-t \le X \le t) = (1+t) \times \frac{1}{3} = \frac{1}{3} + \frac{t}{3}$

When $t > 2$,

$P(-t \le X \le t) = 1$

$$f_Y(t) = \begin{cases} \frac{2}{3} & 0 < t \le 1 \\ \frac{1}{3} & 1 < t \le 2 \\ 0 & \text{o.w.} \end{cases}$$

Expectation of a random variable

☆ (exam)

Example "Chuck-A-Luck"

You get a # between 1-6. You roll 3 fair dice.

If your # appears 3 times, you get $3.

$\binom{3}{2}\left(\frac{1}{6}\right)^2 \times \frac{5}{6}$

_ _ _ _ - 2, - - - 1 - - - 2,

- - ——— - 1 - - 1 - - - 1.

- - — — ——- 0 - - - , you pay \$1.

Q: Should you play it?

Let X be the payoff.

$\left(\frac{1}{6}\right)^3$, $\binom{3}{2}\left(\frac{1}{6}\right)^2 \times \frac{5}{6}$

$\binom{3}{1}\frac{1}{6}\times\left(\frac{5}{6}\right)^2$

X	3	2	1	-1
Probability	$\left(\frac{1}{6}\right)^3$	$\binom{3}{2}\times\left(\frac{1}{6}\right)^2\frac{5}{6}=\frac{15}{216}$	$\binom{3}{2}\left(\frac{5}{6}\right)^2\times\frac{1}{6}=\frac{75}{216}$	$\left(\frac{5}{6}\right)^3=\frac{125}{216}$

Expected Payoff: $3\times\left(\frac{1}{6}\right)^3+2\times\frac{15}{216}+1\times\frac{75}{216}+(-1)\times\frac{125}{216}=\frac{-17}{216}\approx-0.079$

Frequency Interpretation:

Say we play 216 times, we expect to win \$3 1 time, win \$2 15 times, win \$1 75 times, pay \$1 125 times. All together, we expect to win $1\times3+2\times15+1\times75+(-1)\times125=-17$ dollars.

It's roughly lousing 8 cents per roll.

Definition:

$X:\Omega\to\mathbb{R}$ is called simple if it takes finitely many values

If X is simple, with values $a_1, a_2, \dots a_n$, then we have:

$$E(X)=\sum_{i=1}^{n}a_iP(X=a_i)$$

Properties: Clearly, if $X\geq0$, then $E(X)\geq0$

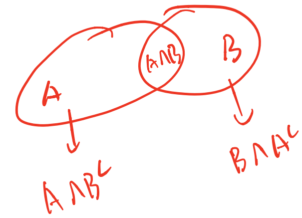
If b is a # then $E(bX)=bE(X)$

Self-test exercise: If $X, Y:\Omega\to\mathbb{R}$ are simple r.v., then $E(X+Y)=E(X)+E(Y)$

$$E(X+Y)=\sum_{x_i}\sum_{y_i}(x_i+y_i)\,p(x_i,y_i)$$
$$=\sum_{x_i}\sum_{y_i}x_i\,p(x_i,y_i)+\sum_{x_i}\sum_{y_i}y_i\,p(x_i,y_i)$$
$$=\sum_{x_i}x_ip(x_i)+\sum_{y_i}y_i\sum_{x_i}p(x_i,y_i)$$
$$=\sum_{x_i}x_ip(x_i)+\sum_{y_i}y_ip(y_i)=E(X)+E(Y)$$

For ex, $X = I_A$, $Y = I_B$

$$X(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases} \qquad Y(w) = \begin{cases} 1 & \text{if } w \in B \\ 0 & \text{if } w \notin B \end{cases}$$



$$(X+Y)(w) = \begin{cases} 2 & w \in A\cap B \\ 1 & w \in A \cap B^c \text{ or } w \in B \cap A^c \\ 0 & \text{if } w \notin (A\cup B) \end{cases}$$

$$(X+Y)(w) = \begin{cases} 2 & w \in A\cap B \\ 1 & w \in A\cap B^c \text{ or } A^c \cap B \\ 0 & w \notin (A\cup B) \end{cases}$$

$$E(X+Y) = 2P(A\cap B) + 1\times P(A\cap B^c) + P(B\cap A^c)$$

$$= P(A\cap B) + P(A\cap B^c) + P(A\cap B) + P(B\cap A^c)$$

$$= P(A) + P(B)$$

$$= E(X) + E(Y)$$

$E(X+Y)$

$$= 2\times P(A\cap B) + 1\times[P(A\cap B^c) + P(A^c\cap B)]$$

$$= P(A\cap B) + P(A\cap B^c) + P(B\cap A) + P(B\cap A^c)$$

$$= P(A) + P(B) = E(X) + E(Y)$$

If $X: \Omega \to R$ is a r.v., we say that X is integrable provided there is a sequence of simple r.v. $X_n : \Omega \to R$ $(n = 1, 2, \dots)$ s.t.: $P\{w \in \Omega : \lim_{n\to\infty} X_n(w) = X(w)\} = 1$

For any $\varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t. if $m, n \geq N(\varepsilon)$, then $E|X_m - X_n| \leq \varepsilon$

Hard self-test exercise

In this case, $\lim_{n\to\infty} E(X_n)$ exist and does not depend on $\{X_n\}$ and

we call $E(X) = \lim_{n\to\infty} E(X_n)$

Properties: If $X \geq 0$, then $E(X) \geq 0$

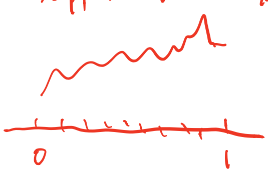
$E(tX) = tE(X)$, $t \in R$

$E(X+Y) = E(X) + E(Y)$ if X, Y are integrable, and $X+Y$ is also integrable

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

We won't prove it, but will use it here is the intuition.

Suppose that $f_X(t) = 0$ for $t \notin [0,1]$ and continuous on $[0,1]$.



We define simple r.v. X_n as follows.

Chop $[0,1]$ into n integrals of length $1/n$ each.

Let I_i be the i th interval. Let $a_i \in I_i$ s.t. $\int_{I_i} f_X(t)\,dt = f_X(a_i)\cdot\frac{1}{n}$

Mean-value thm

Note that: $\frac{1}{n}\sum_{i=1}^{n} f_X(a_i) = \sum \int_{I} f_X(t)\,dt$

$= \int_0^1 f_X(t)\,dt = 1$

If $X(w)$ lies on I_i, define $X_n(w) = a_i$

$$E(X_n) = \sum_{i=1}^{n} a_i P(X_n = a_i) = \sum_{i=1}^{n} a_i \int_{I_i} f(t)\,dt = \frac{1}{n}\sum_{i=1}^{n} a_i f(a_i) \to \int_0^1 x f(x)\,dy$$

ex. If X is uniform on $[0,1]$, then $f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0 & \text{elsewhere} \end{cases}$

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$= \int_0^1 x \cdot 1\,dx$$

$$= \frac{1}{2}$$

$$f_X(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1 \\ 0 & \text{if not} \end{cases}$$

$$\frac{2}{3} t^3$$

$$E(X) = \int_{-\infty}^{+\infty} t \, f_X(t) \, dt$$

$$= \int_0^1 t \cdot 2t \, dt$$

$$= \frac{2}{3}$$