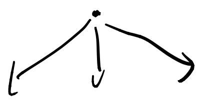
Branching Processes

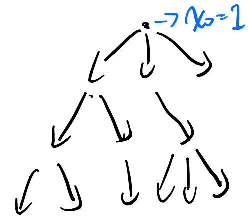
(Galton-Watson Trees)

$$E(X)=\sum_{i=1}^{n} E(X|B_i)\,P(B_i)$$

We start with a single root:

$X_0=1$.





$$E(X)=\sum_{i=1}^{n} a_i\, P(X=a_i)$$

$$=\sum_{i=1}^{n} a_i \sum_{j=1}^{m} P(X=a_i|B_j)\,P(B_j)$$

$$=\sum_{j=1}^{m} P(B_j)\sum_{i=1}^{n} a_i\, P(X=a_i|B_j)$$

$$=\sum_{j=1}^{m} P(B_j)\,E(X|B_j)$$

This root has k descendants with prob p_k, for $k=0,1,2,\ldots$

$$\sum_{k=0}^{\infty} p_k=1$$

Every descendant behaves the same way as the root independent of others.

And so forth.

Q: The prob of eventual extinction?

the process stops (x descents)

Let X_n be the population (# of nodes) at the n-th level; so $X_0=1$.

Q: What is $G_{X_n}(s)=G_n(s)$?

$\to p_k=P(X_1=k)$

$$G_1(s)=\sum_{k=0}^{\infty} P(X_1=k)s^k=\sum_{k=0}^{\infty} p_k s^k = P(s)$$

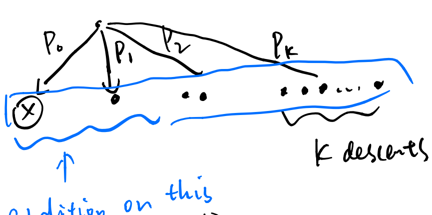
$$G_1(s)=\sum_{k=0}^{\infty} P(X_1=k)\cdot s^k$$

$$=\sum_{k=0}^{\infty} p_k\cdot s^k$$

$$=P(s)$$

Condition on the first step of the process

$n\geq 1$



$$G_1(s)=\sum_{k=0}^{\infty} P(X_1=k)\cdot s^k=\sum_{k=0}^{\infty} \underbrace{p_k}_{P(X_1=k)}\cdot s^k = P(s)$$

$$G_n(s)=E(s^{X_n})=\sum_{k=0}^{\infty} p_k\, E(s^{X_n}\,|\,\text{k descents on the first step of the process})$$

$$=p_0\,E(s^{X_n}\,|\,0\text{ descents})+p_1\,E(s^{X_n}\,|\,1\text{ descent})+p_2\,E(s^{X_n}\,|\,2\text{ descents})+\cdots$$

$$= \underline{P_0 s^0} + P_1 E(s^{X_{n-1}}) + P_2 E(s^{Y_1+Y_2}) + \cdots$$

$$= p_0 + p_1 E(s^{X_{n-1}}) + p_2 (E(s^{X_{n-1}}))^2 + \cdots + p_k (E(s^{X_{n-1}}))^k$$

don't have descants ← $P_0 s^0 = 1$

on the first level, so no descents on the n-th level ($n>1$)

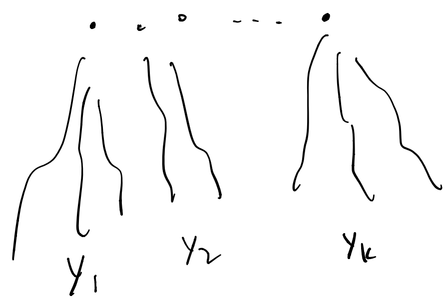
Y_1: descent of the first node
Y_2: descent of the second node
} at the n-th level

$$E(s^{Y_1+Y_2}) = E(s^{Y_1})E(s^{Y_2})$$

$$= E(s^{X_{n-1}})E(s^{X_{n-1}})$$

n-th level →

If k descents



$$X_n = Y_1 + \cdots + Y_k$$

$$E(s^{X_n}) = E(s^{Y_1+\cdots+Y_k})$$

$$= E(s^{Y_1}) \cdots E(s^{Y_k})$$

$G_n(s) = P(G_{n-1}(s))$ $\quad P(s) = G_1(s) = \sum_{k=0} P(X_1=k) \cdot s^k$

$\forall \; G_n(s) = P(G_{n-1}(s))$

$G_n(s) = P(G_{n-1}(s))$ → composition

$\lim_{n\to\infty} P(X_n=0)$

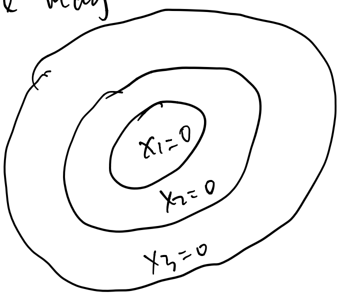
$G_n(s) = \sum_{k=0}^{\infty} P(X_n=k) \cdot s^{X_n}$

$G_n(0) = P(X_n=0)$

$\lim_{n\to\infty} P(X_n=0) = \lim_{n\to\infty} G_n(0)$

Prob of eventual distinction:

We may have $X_1=0$ or $X_2=0$, or $X_3=0$.



$= P(X_1=0$ or $X_2=0$ or $X_3=0 \ldots)$

$= \lim_{n\to\infty} P(X_n=0)$

$= \lim_{n\to\infty} G_n(0)$

nested family of events

What is $\lim_{n\to\infty} G_n(0) = ?$

$\lim_{n\to\infty} G_n(0)$

$\lim_{n\to\infty} P(X_n=0)$

$= \lim_{n\to\infty} G_n(0)$ $\quad P(s)=s$

$G_1(s)$

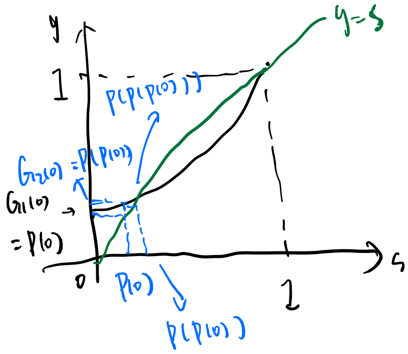
$G_{X_1}(s) = \sum_{k=0}^{\infty} P(X_1=k) \cdot s^k = P(s)$

$\lim_{n\to\infty} G_n(0)$ = smallest non-negative solution of the equation $P(s)=s$.

Self-test exercise: prove it

$G_n(s) = P(G_{n-1}(s))$

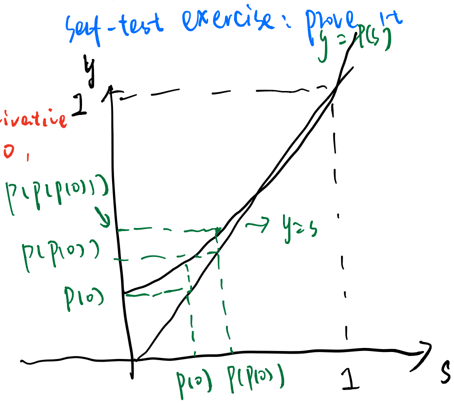
Look at the ftn $P(s) = \sum_{k=0}^{\infty} P_k \cdot s^k$ → first & second derivative ≥ 0, so convex



increasing convex

$P(1) = \sum_{k=0}^{\infty} P_k = 1$

$G_1(0) = P(0)$



We are interested in:

$$\lim_{n\to\infty} \underbrace{P(P(P\cdots(P(0))\cdots)}_{n \text{ times}}$$

ex.

$P_0 = P_1 = P_2 = P_3 = \frac{1}{4}$ ($P_k = 0$ for $k \geq 4$)

$\lim G_n(0) = \lim P(X_n = 0)$

Q: Prob of eventual extinction? ($s=1$ is always a soln)

$\boxed{P(s)=s}$

$P(s) = s$

$\rightarrow \frac{1}{4} + \frac{1}{4}s + \frac{1}{4}s^2 + \frac{1}{4}s^3 = s \Rightarrow$

$\frac{1}{4}s^3 + \frac{1}{4}s^2 - \frac{3}{4}s + \frac{1}{4} = 0 \Rightarrow$

$s^3 + s^2 - 3s + 1 = 0 \Rightarrow (s-1)(s^2+2s-1) = 0$

$s = 1$ or $s = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

So the smallest positive root is $\sqrt{2} - 1 \approx 0.41$.

A: $\sqrt{2} - 1$

$P(s) = \sum_{k=0}^{\infty} P(X_1 = k) \cdot s^k$

$= \frac{1}{4} \times 1 + \frac{1}{4}s + \frac{1}{4}s^2 + \frac{1}{4}s^3 = s$

$\frac{1}{4}s^3 + \frac{1}{4}s^2 - \frac{3}{4}s + \frac{1}{4} = 0$

$s^3 + s^2 - 3s + 1 = 0$

$$\begin{array}{r} s^2+2s-1 \\ s-1 \overline{)\, s^3+s^2-3s+1} \\ \underline{s^3-s^2} \\ 2s^2-3s \\ \underline{2s^2-2s} \\ -s+1 \end{array}$$

$\rightarrow (s-1)(s^2+2s-1) = 0$

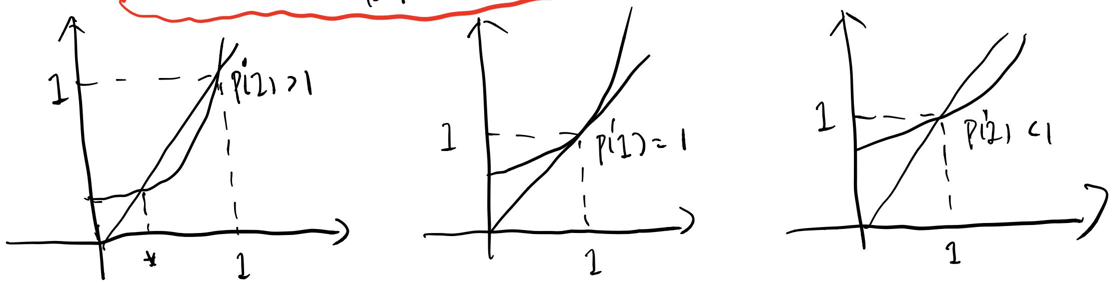
$s_1 = 1$, $s_2 = \frac{-2 + \sqrt{4+4}}{2} = \frac{-2+2\sqrt{2}}{2} = \boxed{-1+\sqrt{2}}$

$s_3 = -2-2\sqrt{2}$

$\frac{\cdot}{2}$

Observation: Suppose there $\underbrace{P_0 > 0}_{\text{prob of extinction is positive}}$

$$\boxed{\text{The prob of eventual distinction is 1 iff: } \quad P'(1) = \sum_{k=1}^{\infty} k\, P(X_1 = k) = E(X_1) \leq 1}$$



$$G_n(s) = P(G_{n-1}(s))$$

Example

suppose

$P_0 = \frac{1}{2}$, $P_1 = P_2 = P_3 = \frac{1}{6}$, $P_k = 0$ for $k \geq 4$

$$P(s) = \sum_{k=0}^{\infty} \underbrace{P_k}_{P(X_1 = k)} \cdot s^k$$

$P(s) = s \Rightarrow$

$\frac{1}{2} + \frac{1}{6}s + \frac{1}{6}s^2 + \frac{1}{6}s^3 = s \Rightarrow$

$\frac{1}{6}s^3 + \frac{1}{6}s^2 - \frac{5}{6}s + \frac{1}{2} = 0 \Rightarrow$

$$\boxed{E(X) = \frac{1+2+3}{6} = 1}$$

$s^3 + s^2 - 5s + 3 = 0 \Rightarrow$

$(s-1)(s^2 + 2s - 3) = 0$

roots are $s = 1$, $s = \frac{-2 \pm \sqrt{4+12}}{2} = \frac{-2 \pm 4}{2} = -1 \pm 2$

$s = 1$, $s = -3$

$\uparrow$

double root

prob of eventual extinction is 1.

So $E(X_1) = \frac{1+2+3}{6} = 1$.

$\underline{P(s) = s}$.

$s \quad -1$

$s \quad 3$

$\sum_{k=0}^{\infty} P(X_1 = k) \cdot s^k = s$

$(s-1)(s+3) = 0$

$\boxed{s = 1}$

$\frac{1}{2} + \frac{1}{6}s + \frac{1}{6}s^2 + \frac{1}{6}s^3 = s$

$\frac{1}{6}s^3 + \frac{1}{6}s^2 - \frac{5}{6}s + \frac{1}{2} = 0$

$s^3 + s^2 - 5s + 3 = 0$

$$\begin{array}{r} s^2 + 2s - 3 \\ s-1 \overline{) s^3 + s^2 - 5s + 3} \\ \underline{s^3 - s^2} \\ 2s^2 - 5s \end{array}$$

$(s-1)(s^2 + 2s - 3) = 0$

$\boxed{s = 1}$ ✓

Usually, it is not easy to get $G_n(s) = G_{X_n}(s)$ in a nice form.

But sometimes it's possible.

ex. $P_0 = \frac{1}{2},\ P_1 = \frac{1}{4}, \dots, P_k = \frac{1}{2^{k+1}},\ k = 0,1,2\dots$

$$P(s) = \sum_{k=0}^{\infty} P_k s^k = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} s^k = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{s}{2}\right)^k$$

$$= \frac{1}{2}\,\frac{1}{1-s/2}$$

$$= \frac{1}{2-s} \checkmark$$

∴ $G_1(s) = \frac{1}{2-s}$,

$$G_2(s) = P(G_1(s)) = \frac{1}{2-\frac{1}{2-s}} = \frac{2-s}{3-2s}$$

$$G_3(s) = P(G_2(s)) = \frac{1}{2-\frac{2-s}{3-2s}} = \frac{3-2s}{4-3s}$$

⋮

$$G_n(s) = \frac{n-(n-1)s}{(n+1)-ns} = \frac{n-1}{n} + \frac{1}{n(n+1)}\,\frac{1}{1-\frac{n}{n+1}s}$$

→ very similar to partial fraction

$$= \frac{n-1}{n} + \frac{1}{n(n+1)}\sum_{k=0}^{\infty}\left(\frac{n}{n+1}\right)^k s^k$$

$$P(X_n = k) = \frac{n^{k-1}}{(n+1)^{k+1}} \quad \text{for } k>0$$

$$P(X_n = 0) = \frac{n}{n+1}$$

$G_n(s) = P(G_{n-1}(s))$

$P(s) = \sum_{k=0}^{\infty} P(X_1 = k)\cdot s^k$

$= \sum_{k=0}^{\infty} P_k \cdot s^k$

$= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}}\cdot s^k$

$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{s}{2}\right)^k$

$= \frac{1}{2}\,\frac{1}{1-\frac{s}{2}}$

$= \frac{1}{2-s}$

$G_1(s) = \frac{1}{2-s}$

$G_2(s) = P(G_1(s))$

$= \frac{1}{2-\frac{1}{2-s}}$

$= \frac{2-s}{4-2s-1} = \frac{2-s}{3-2s}$

$G_3(s) = P(G_2(s))$

$= \frac{1}{2-\frac{2-s}{3-2s}}$

$= \frac{3-2s}{6-4s-2+s} = \frac{3-2s}{4-3s}$

⋮

$G_n(s) = \frac{n-(n-1)s}{(n+1)-ns}$

$\frac{n^2-1}{n} = n - \frac{1}{n}$

$= \frac{(n+1)\times\frac{n-1}{n} - (n-1)s}{(n+1)-ns} + \frac{\frac{1}{n}}{\cdots}$

$= \frac{n-1}{n} + \frac{1}{n[(n+1)-ns]}$

$= \frac{n-1}{n} + \frac{1}{n(n+1)}\left[\frac{1}{1-\frac{n}{n+1}s}\right]$

$= \frac{n-1}{n} + \frac{1}{n(n+1)}\sum_{k=0}^{\infty}\left(\frac{n}{n+1}s\right)^k$

$$\frac{\frac{n-1}{n}(n+1) - (n-1)s}{(n+1)-ns} + \frac{\frac{1}{n}}{(n+1)-ns}$$

$\frac{n^2-1}{n}$ $\quad \frac{n^2}{n}$ $\quad = \frac{n-1}{n} + \frac{1}{n(n+1)-n^2 s}$

$$= \frac{n-1}{n} + \frac{1}{n(n+1)\left[1-\frac{n}{n+1}s\right]}$$

$$= \frac{n-1}{n} + \frac{1}{n(n+1)}\sum_{k=0}^{\infty}\left(\frac{n}{n+1}\right)^k s^k$$