expectation of r.v.

$E(X_1X_2\cdots X_n) = E(X_1)\cdots E(X_n)$

$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$

$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\varphi_X(t)\,dt$

Probability Generating Function

Let $X: \Omega \to \mathbb{R}$ be a r.v. taking non-negative integer values (like Poisson).

$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k$ converges for $|s| \le 1$ $\quad s^0 = 1$

$\frac{1}{2}\int_{-\infty}^{+\infty} e^{-itx}\frac{1}{1+x^2}\,dt = e^{-|x|}$

$G_X(1) = \sum_{k=0}^{\infty} P(X=k) = 1$

$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\cdot s^k$

$G_X(0) = P(X=0)$

$G_X(0) = P(X=0)$

$G'_X(s) = \sum_{k=1}^{\infty} k\,P(X=k)\, s^{k-1}$

$G_X(1) = 1$

$G'_X(1) = \sum_{k=1}^{\infty} k\,P(X=k) = E(X)$

$G'_X(s) = \sum_{k=0}^{\infty} P(X=k)\cdot k\, s^{k-1}$

Important: $G_X(s) = E(s^X)$

$G'_X(1) = \sum_{k=0}^{\infty}$

If $X_1, X_2, \dots, X_n$ are indep, then for $X = X_1+X_2+\cdots X_n$, $G_X(s) = E(s^X) = E(s^{X_1+\cdots+X_n})$

$\overset{\text{indep}}{=} E(s^{X_1})E(s^{X_2})\cdots E(s^{X_n})$

$= G_{X_1}(s)\cdots G_{X_n}(s)$

ex.

Let X be Poisson with $E(X) = \lambda$, $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$, $k = 0, 1, \dots$

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\cdot s^k = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda}\cdot e^{\lambda s} = e^{\lambda(s-1)}$$

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\cdot s^k = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\cdot s^k = \sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!}e^{-\lambda} = e^{-\lambda}\cdot e^{\lambda s} = e^{\lambda(s-1)}$$

Check:

$G_X(1) = e^{\lambda(1-1)} = e^0 = 1$

$G_X(0) = e^{\lambda(0-1)} = e^{-\lambda} = P(X=0)$

$G'_X(s) = \lambda e^{\lambda(s-1)}$, $G'_X(1) = \lambda e^{\lambda\cdot 0} = \lambda = E(X)$

ex. let X be the # of success in n indep experiments, where each is successful with prob $0<p<1$ (And failure with $q=1-p$)

$G_X(s) = ?$

let $X_i = \begin{cases} 1, & \text{if the } i\text{th experiment successes (with prob } p) \\ 0, & \text{o.w } (q=1-p) \end{cases}$

$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k$

$i = 1, 2, \dots n$. then $X = X_1 + \dots X_n$

$= E(s^X)$

Then $G_X(s) = G_{X_1}(s) \cdots G_{X_n}(s)$

$G_X(s) = E(s^{X_1}) \times \dots E(s^{X_n})$

$G_{X_i}(s) = P(X_i = 0) + s P(X_i = 1) = q + ps$

Binomial Theorem: $(q+ps)^n = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k s^k$ $= G_{X_1}(s) \times \dots G_{X_n}(s)$

$P(X=k) = \binom{n}{k} q^{n-k} p^k$

$G_{X_i}(s) = E(s^{X_i})$ $= (q+ps)^n$.

$= q + ps$

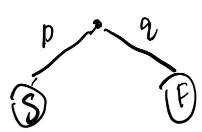
ex. let Y be the time (experiment) of the first success of independent experiments, where each is successful with prob $0<p<1$, and failure with prob $q = 1-p$.

Q: $G_Y(s)$? $= E(s^Y)$

$\boxed{G_X(s) = E(s^X)}$

Condition on the outcome of the first experiment.

$= \sum_{k=0}^{\infty} P(X=k) \cdot s^k$



$\to s$ is a constant, y is r.v.

$E(s^Y | \text{success}) = s$.

$E(s^Y | \text{failure})$

$G_Y(s) = sp + q\, s G_Y(s)$

$= E(s^{Y+1})$

$= s E(s^Y)$

$\frac{sp}{1-qs}$

$= s G_Y(s)$

$G_Y(s) = ps + qs G_Y(s)$

$\Rightarrow (1-qs) G_Y(s) = ps \Rightarrow G_Y(s) = \frac{ps}{1-qs}$

$G_Y(s) = ps \cdot \frac{1}{1-qs}$

$= ps \sum_{k=0}^{\infty} q^k s^k$

$P(Y=k) = pq^{k-1}, \quad k=1,2,\ldots$

$G_Y(s) = ps \frac{1}{1-qs}$

$= ps \sum_{k=0}^{\infty} (qs)^k = \sum_{k=0}^{\infty} ps(qs)^k$

$= \sum_{k=0}^{\infty} pq^k s^{k+1}$

$= \sum_{k=0}^{\infty} P(Y=k) \cdot s^k$

$P(Y=k) = pq^{k-1}$ ~~for~~ $k=1,2\ldots$

$\otimes \quad \otimes \quad \otimes \quad \otimes \quad \bigcirc$

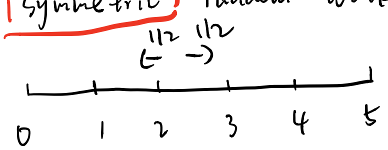
F F F F S

$G_Y'(s) = \frac{p(1-qs)+ps \cdot q}{(1-qs)^2} = \frac{p}{(1-qs)^2}, \quad G_Y'(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$

$G_Y'(s) = \frac{p(1-qs)-ps(-q)}{(1-qs)^2} = \frac{p-pqs+pqs}{(1-qs)^2} = \frac{p}{(1-qs)^2}, \quad G_Y'(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$

Long Example

A $\boxed{\text{symmetric}}$ random walk with absorbing barriers at $x=0$ and $x=5$.



Let X_k be the duration (# of steps) of walk if we start at $k=0,1,2,3,4,5$.

Q: What is $G_{X_k}(s) := G_k(s)$

Symmetric $\Rightarrow$ $G_1(s) = G_4(s)$

$G_2(s) = G_3(s)$

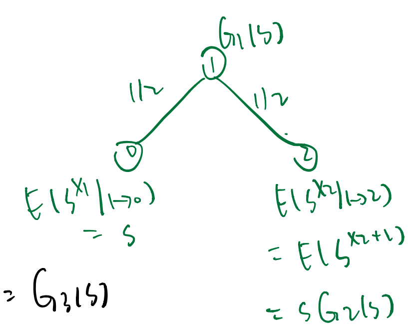
$k=0, \quad X_0 = 0$

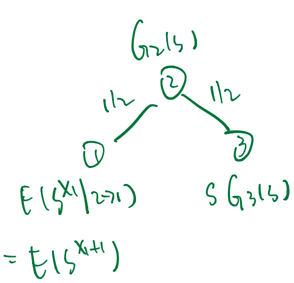
$G_0(s) = E(s^{X_0}) = 1$

$k=5, \quad X_5 = 0$

$G_5(s) = E(s^{X_5}) = 1$

$G_1(s) = \frac{1}{2}s + \frac{1}{2}s G_2(s)$ ①



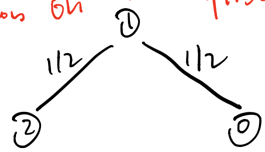


$G_1(s) = G_4(s), \quad G_2(s) = G_3(s)$

symmetry

X_1, X_4 have the same distn. X_2, X_3 has the same distn.

Condition on the first step

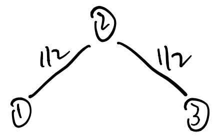


$E(s^{X_1} | 1 \to 2)$ $\qquad$ $E(s^{X_1} | 1 \to 0) = E(s^1) = s$

$= E(s^{X_2+1})$

$= s E(s^{X_2})$

$$\boxed{G_1(s) = \frac{1}{2} s G_2(s) + \frac{1}{2} s}$$ ①



$E(s^{X_2} | 2 \to 1)$ $\qquad$ $E(s^{X_2} | 2 \to 3) = E(s^{X_3+1})$

$= E(s^{X_1+1})$ $\qquad$ $= s G_3(s)$

$= s G_1(s)$

$$\boxed{\begin{aligned} G_2(s) &= \frac{1}{2} s G_1(s) + \frac{1}{2} s G_3(s) \\ &= \frac{1}{2} s G_1(s) + \frac{1}{2} s G_2(s) \end{aligned}}$$ ②

From ① and ②, we know:

$$G_2(s) = \frac{1}{2} s G_1(s) + \frac{1}{2} s G_2(s)$$

$$= \frac{1}{2} s \left(\frac{1}{2} s G_2(s) + \frac{1}{2} s \right) + \frac{1}{2} s G_2(s)$$

$$= \frac{1}{4} s^2 G_2(s) + \frac{1}{2} s G_2(s) + \frac{1}{4} s^2$$

$$G_2(s) \left(1 - \frac{1}{4} s^2 - \frac{1}{2} s \right) = \frac{1}{4} s^2$$

$$\to G_2(s) = \frac{\frac{1}{4} s^2}{1 - \frac{1}{4} s^2 - \frac{1}{2} s} = \frac{s^2}{4 - s^2 - 2s}$$

$G_2(s) = \frac{1}{2} s G_1(s) + \frac{1}{2} s G_3(s)$ ②

$G_2(s) = \frac{1}{2} s G_1(s) + \frac{1}{2} s G_2(s)$ ②

$(1 - \frac{1}{2} s) G_2(s) = \frac{1}{2} s [\frac{1}{2} s + \frac{1}{2} s G_2(s)]$

$(1 - \frac{1}{2} s - \frac{1}{4} s^2) G_2(s) = \frac{1}{4} s^2$

$$G_2(s) = \frac{\frac{1}{4} s^2}{1 - \frac{1}{2} s - \frac{1}{4} s^2} = \frac{s^2}{4 - 2s - s^2}$$

Q: $P(X_2 = m) = ?$ (want a formula)

partial fractions

$$\frac{s^2}{4-2s-s^2} = \sum_{m=0}^{\infty} P(X_2=m)\cdot s^m$$

$$\frac{s^2}{4-2s-s^2} = -1 + \frac{2s-4}{s^2+2s-4}$$

Find the roots of s^2+2s-4.

$$s = \frac{-2\pm\sqrt{4+16}}{2} = \frac{-2\pm\sqrt{20}}{2} = -1\pm\sqrt{5}$$

Call $\alpha = -1+\sqrt{5}$, $\beta = -1-\sqrt{5}$

So $s^2+2s-4 = (s-\alpha)(s-\beta)$

look for a, b,

$$\frac{2s-4}{s^2+2s-4} = \frac{a}{s-\alpha} + \frac{b}{s-\beta}$$

$$= \frac{a(s-\beta)+b(s-\alpha)}{(s-\alpha)(s-\beta)}$$

We have $a+b=2$, $-a\beta - b\alpha = -4$ $\Rightarrow$ $a+b=2$, $a\beta + b\alpha = 4$

$a(\beta-\alpha) = 4-2\alpha \Rightarrow a = \frac{4-2\alpha}{\beta-2\alpha} = \frac{6-2\sqrt{5}}{2\sqrt{5}} = 1-\frac{3}{\sqrt{5}}$

$b(\alpha-\beta) = 4-2\beta \Rightarrow b = \frac{4-2\beta}{\alpha-\beta} = \frac{6+2\sqrt{5}}{2\sqrt{5}} = 1+\frac{3}{\sqrt{5}}$

$$G_{X_2}(s) = -1 + \frac{a}{s-\alpha} + \frac{b}{s-\beta}$$

where $\alpha = -1+\sqrt{5}$, $\beta = -1-\sqrt{5}$; $a = 1-\frac{3}{\sqrt{5}}$, $b = 1+\frac{3}{\sqrt{5}}$

$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$ (Geometric Theories)

$$G_{X_2}(s) = \sum_{k=0}^{\infty} P(X_2=k)\cdot s^k$$

$$= \frac{s^2}{4-2s-s^2}$$

$$= -1 + \frac{-2s+4}{4-2s-s^2}$$

$$= -1 + \frac{2s-4}{s^2+2s-4}$$

$s^2+2s-4=0 \Rightarrow \Delta = 4-4\times(-4) = 20$

$$s = \frac{-2\pm\sqrt{20}}{2} = \frac{-2\pm2\sqrt{5}}{2} = -1\pm\sqrt{5}$$

$\alpha = -1+\sqrt{5}$, $\beta = -1-\sqrt{5}$

$$G_{X_2}(s) = -1 + \frac{2s-4}{(s-\alpha)(s-\beta)}$$

So $G_{X_2}(s) = -1 + \frac{a}{s-\alpha} + \frac{b}{s-\beta}$

$$= -1 + \frac{a}{s-\alpha} + \frac{b}{s-\beta}$$

$$= -1 + \frac{as - a\beta + bs - b\alpha}{(s-\alpha)(s-\beta)}$$

$$= -1 + \frac{(a+b)s - (a\beta+b\alpha)}{(s-\alpha)(s-\beta)}$$

$\left.\begin{array}{l} a+b=2 \\ a\beta+b\alpha=4 \end{array}\right\}$

$a = 2-b$

$(2-b)\beta + b\alpha = 4$

$2\beta + b(\alpha-\beta) = 4$

$$b = \frac{4-2\beta}{\alpha-\beta}$$

$$= \frac{4+2(1+\sqrt{5})}{2\sqrt{5}}$$

$$= \frac{6+2\sqrt{5}}{2\sqrt{5}}$$

$$\frac{a}{s-\alpha} = -\frac{a}{\alpha-s} = -\frac{a}{\alpha}\cdot\frac{1}{1-\frac{s}{\alpha}} = -\frac{a}{\alpha}\sum_{m=0}^{\infty}\left(\frac{s}{\alpha}\right)^m$$

$$\frac{b}{s-\beta} = -\frac{b}{\beta-s} = -\frac{b}{\beta}\cdot\frac{1}{(1-\frac{s}{\beta})} = -\frac{b}{\beta}\sum_{m=0}^{\infty}\left(\frac{s}{\beta}\right)^m$$

So $G_2(s) = -1 + \frac{a}{s-\alpha} + \frac{b}{s-\beta} = \sum_{m=0}^{\infty} \underbrace{P(X_2=m)}\, s^m$ ← coeffi of s^m. so just expand it and compare the coefficients

$$a = 2-b = \frac{4\sqrt{5}-6-2\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}-6}{2\sqrt{5}}$$

$$\frac{a}{s-\alpha} = \frac{-a}{\alpha-s} = \frac{-a}{\alpha}\cdot\frac{1}{1-\frac{s}{\alpha}} = \frac{-a}{\alpha}\cdot\sum_{k=0}^{\infty}\left(\frac{s}{\alpha}\right)^k$$

$$\frac{b}{s-\beta} = \frac{-b}{\beta-s} = \frac{-b}{\beta}\cdot\frac{1}{1-\frac{s}{\beta}} = \frac{-b}{\beta}\sum_{k=0}^{\infty}\left(\frac{s}{\beta}\right)^k$$

A: For $m>0$, $P(X_2=m) = -\frac{a}{\alpha}\cdot\frac{1}{\alpha^m} - \frac{b}{\beta}\cdot\frac{1}{\beta^m}$

$$= \frac{5-\sqrt{5}}{10}\,\frac{1}{(\sqrt{5}-1)^m} + (-1)^m\,\frac{5+\sqrt{5}}{10}\left(\frac{1}{1+\sqrt{5}}\right)^m$$

For $m=0$, subtract 1, from the above expression and get 0.