Last time:

$X : \Omega \to \mathbb{R}$ r.v. taking values $0, 1, 2, \dots$

$G_X(s) = E(s^X)$

$= \sum_{m=0}^{\infty} P(X=m)\, s^m$

We define:

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k = E(s^k)$$

Did an ex of a symmetric random walk absorbing at $x=0$ and $x=5$.

We let X_2 be the duration of the walk if we start at $x=2$.

$$G_2(s) = G_{X_2}(s) = \frac{s^2}{4-2s-s^2} = -1 + \frac{2s-4}{s^2+2s-4}$$ (By conditioning) used partial fractions.

$$P(X_2 = m) = \frac{5-\sqrt{5}}{10}\left(\frac{1}{\sqrt{5}-1}\right)^m + (-1)^m \frac{5+\sqrt{5}}{10}\left(\frac{1}{1+\sqrt{5}}\right)^m \quad (m>0)$$

For $m=0$, subtract 1 from the above.

$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k$

Remark (tail estimate)

$\limsup_{k\to\infty} \sqrt[k]{P(X=k)} = \frac{1}{\text{radius of convergence}}$

As m grows,

$P(X_2 = m)$ decreases roughly as $\approx (0.81)^m$

Taill Estimates (A bit of complex analysis)

$\limsup_{m\to\infty} a_m = \lim_{m\to\infty} \sup_{k\geq m} a_k$

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k$$

$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k$

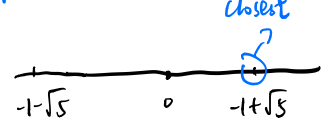
$$\limsup_{k\to\infty} \sqrt[k]{P(X=k)} = \frac{1}{R} \leftarrow \text{radius of convergence}$$

$\limsup_{k\to\infty} \sqrt[k]{P(X=k)} = \frac{1}{\text{radius of conv}}$

$$= \frac{1}{\text{distance from } 0 \text{ to the nearest complex singular point of } G_X(s)}$$

(the fcn loses analyticity at singular points)

$4-2s-s^2 = 0$



For $G_X(s) = \frac{s^2}{4-2s-s^2}$, singular points are: $4-2s-s^2=0 \to s = -1 \pm \sqrt{5}$

So $\lim_{m\to\infty} \sqrt[m]{P(X_2=m)} = \frac{1}{\sqrt{5}-1}$

$\boxed{\limsup_{m\to\infty} a_m = \lim_{m\to\infty} \underbrace{\sup_{k\ge m} a_k}_{\text{decreasing sequence}}}$

② (F_0 ...) ①

RHS is the def of $\limsup_{m\to\infty} a_m$

$\limsup_{m\to\infty} a_m = \lim_{m\to\infty} \sup_{k\ge m} a_k$

calculate ① first, then ②

$X_k = Y_k + Y_{k-1} + \cdots Y_1$ $\quad \boxed{X_1 = Y_1}$

$G_{X_k}(s) = G_{Y_k}(s) \cdots G_{Y_1}(s)$ $\quad E(s^{Y_1})$

$= G^k_{Y_1}(s)$

ex. 1 $\boxed{\text{Asymmetric}}$ Random Walk with absorbing barrier at $x=0$) $\quad Y_k =$ the steps from k to $k-1$

$q \leftarrow \rightarrow p$ $\qquad q = 1-p, \quad 0 < p \le \frac{1}{2}$

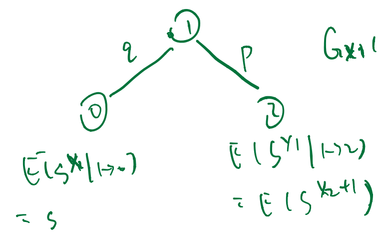
$x=0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$X_k = Y_k + \cdots Y_1$

$G_{X_k}(s) = G_{Y_1}(s) G_{Y_2}(s) \cdots G_{Y_k}(s)$

$X_k =$ Duration of the walk if we start at k ($k = 0, 1, 2, \cdots$) $\quad = G^k_{Y_1}(s)$

Q: $G_k(s) = G_{X_k}(s) = E(s^{X_k}) = ?$



$G_{X_1}(s) = qs + ps\,G_{X_2}(s)$

$= qs + ps\,G^2_{Y_1}(s)$

$= qs + ps\,G^2_{X_1}(s)$

$X_k = Y_k + Y_{k-1} + \cdots + Y_1$ $\;(X_1 = Y_1)$

$Y_1, Y_2, \cdots Y_k, \cdots$

$Y_i =$ the # of steps it takes to move from $x=i$ to $x=i-1$ (one step to the left)

Here $Y_k, Y_{k-1}, \cdots Y_1$ are indep, and have the same distn as X_1.

So $G_k(s) = G_{X_k}(s) = G_{Y_k}(s) \times \cdots G_{Y_1}(s)$ $\qquad X_k = Y_k + Y_{k-1} + \cdots Y_1$

$= G^k_{Y_1}(s)$ $\qquad \underline{X_1 = Y_1}$

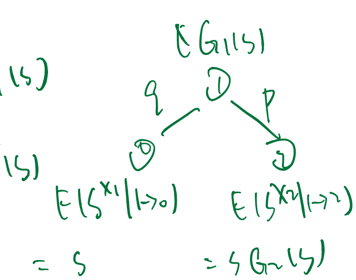
$= G^k_1(s)$ $\qquad G_k(s) = G_{X_k}(s)$

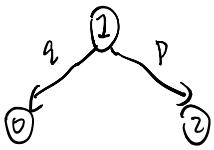
$= G_{Y_k}(s) \cdots G_{Y_1}(s)$

$= G^k_{Y_1}(s) = G^k_1(s)$

Let us find $G_1(s)$?

Condition on the first step:





$E(s^{X_1}|1\to 0)=s$ $\quad E(s^{X_1}|1\to 2)=E(s^{X_2+1})=sE(s^{X_2})$

$G_1(s)=E(s^{X_1})=qs+psE(s^{X_2})=qs+psG_2(s)$

$G_1(s)=qs+psG_1^2(s)$

$psG_1^2(s)-G_1(s)+qs=0$

$\rightarrow G_1(s)=\dfrac{1\pm\sqrt{1-4pqs^2}}{2ps}$ $\quad (G_X(0)=P(X=0))$

Must choose $G_1(s)=\dfrac{1-\sqrt{1-4pqs^2}}{2ps}$ because $G_1(0)=P(X_1=0)=0$

$G_1(s)=\dfrac{1-\sqrt{1-4pqs^2}}{2ps}$

$G_1(s)=qs+psG_2(s)$

$=qs+psG_1^2(s)$

$G_1^2(s)\,ps-G_1(s)+qs=0$

$G_1(s)=\dfrac{1\pm\sqrt{1-4psqs}}{2ps}$

$=\dfrac{1\pm\sqrt{1-4pqs^2}}{2ps}$

$\boxed{G_1(0)=P(X_1=0)=0}$

$G_1(s)=\dfrac{1-\sqrt{1-4pqs^2}}{2ps}$

A: $G_k(s)=\left(\dfrac{1-\sqrt{1-4pqs^2}}{2ps}\right)^k$

Q: Where is the singularity nearest to $s=0$. $(\sim\frac{s^2}{s}=s)$

Where $4pqs^2=1\rightarrow s=\pm\dfrac{1}{\sqrt{4pq}}$

$\limsup_{k\to\infty}\sqrt[k]{P(X_1=k)}=\sqrt{4pq}$

Specifically, if $p=q=\frac{1}{2}$, $\limsup_{k\to\infty}\sqrt[k]{P(X_1=k)}=1$

$\dfrac{4pqs^2}{2ps\cdot(1+\sqrt{1-4pqs^2})}$ $\quad \| \; s\approx 0$

$\|$

for $s\approx 0$, $\dfrac{1-\sqrt{1-4pqs^2}}{2ps}\approx\dfrac{2pqs^2}{2ps}=qs$

$=\dfrac{1-(1-4pqs^2)}{2ps(1+\sqrt{1-4pqs^2})}$

$\approx\dfrac{4pqs^2}{2ps\times 2}=qs$

Self-test Exercise

Show that:

$E(X_1)=G_1'(1)=\dfrac{q}{1-2p}$

$f(x)=f(a)+f'(a)(x-a)+f''(a)(x-a)^2$

$\sqrt{1-x}=1-\frac{1}{2}x-\sum_{k=2}^{\infty}\dfrac{(2k-3)!!\,2!}{2^k\cdot k!}$ $\quad G_1(s)=$

Using Expansion, $\sqrt{1-x}=1-\frac{1}{2}x-\sum_{k=2}^{\infty}\dfrac{(2k-3)!!}{2^k\cdot k!}x^k$ $\quad ((2k-3)!!=1\times3\times5\times\cdots(2k-3))$

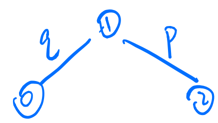
Taylor Expansion

Show that:

$$P(X_1 = 2m-1) = \frac{(2m-3)!!\,(4pq)^m}{2^m \times m!\; 2p} \qquad m \geq 2$$

And check the answer using trajectories.

Interesting Trick



$E(X_1 \mid 1 \to 0)$ $= 1$

$E(X_1 \mid 1 \to 2)$ $= \underbrace{E(X_2)}+1$ $= \underbrace{2E(X_1)}+1$

$E(X_2) = 2E(X_1)$ b/c $X_2 \to 0 \Leftrightarrow X_2 \to X_1,\ X_1 \to 0$. And $X_2 \to X_1$ and $X_1 \to 0$ are equivalent to each other. So $E(X_2) = 2E(X_1)$

So $E(X_1) = q \times 1 + p \times (2E(X_1)+1)$

$E(X_1)(1-2p) = q + p = 1$

$\to E(X_1) = \frac{1}{1-2p}$

Expanding (Remark)

last time, we used partial fractions

ex. $G_X(s) = \frac{1}{2-s^2}$

Use partial fractions: $2-s^2 = 0 \Rightarrow s = \pm\sqrt{2}$

$(2-s^2) = (\sqrt{2}-s)(\sqrt{2}+s)$

So $\frac{1}{2-s^2} = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}-s} + \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}+s}$

$\underbrace{\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}}$ (infinite geometric sequence)

$= \frac{1}{4}\frac{1}{1-\frac{s}{\sqrt{2}}} + \frac{1}{4}\frac{1}{1+\frac{s}{\sqrt{2}}}$

So $P(X=k) = \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^k + (-1)^k \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^k$ ✓

$$G_X(s) = \frac{1}{2-s^2} = \frac{1}{(\sqrt{2}+s)(\sqrt{2}-s)} = \frac{a}{\sqrt{2}+s} + \frac{b}{\sqrt{2}-s}$$

$$G_X(s) = \frac{a}{\sqrt{2}+s} + \frac{b}{\sqrt{2}-s} = \frac{-a}{-\sqrt{2}-s} + \frac{b}{\sqrt{2}-s} = \frac{-a}{-\sqrt{2}}\frac{1}{1+\frac{s}{\sqrt{2}}} + \frac{b}{\sqrt{2}}\frac{1}{\left(1-\frac{s}{\sqrt{2}}\right)}$$

$$= \frac{a\sqrt{2}a - as + \sqrt{2}b + bs}{2-s^2} = \frac{\sqrt{2}(a+b)+(b-a)s}{2-s^2} =$$

$b = a$

$\sqrt{2}(a+b) = 1 \to \sqrt{2}\times 2a = 1$

$a = \frac{1}{2\sqrt{2}}$

$b = \frac{1}{2\sqrt{2}}.$

$$\frac{1}{2-s^2} = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}-s} + \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2}+s} = \frac{1}{4}\frac{1}{1-\frac{s}{\sqrt{2}}} + \frac{1}{4}\frac{1}{1-\left(-\frac{s}{\sqrt{2}}\right)}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{s}{\sqrt{2}}\right)^k + \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{-s}{\sqrt{2}}\right)^k$$

A shorter way:

$$\frac{1}{2-s^2} = \frac{1}{2}\frac{1}{1-\frac{s^2}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{s^2}{2}\right)^k$$

$$P(X=m) = \begin{cases} 0, & \text{if } m \text{ is odd} \\ 2^{-k-1} & \text{if } m \text{ is } m=2k \text{ even} \end{cases}$$

$$\frac{1}{2-s^2} = \frac{1}{2}\frac{1}{1-\frac{s^2}{2}}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{s^2}{2}\right)^k$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cdot s^{2k} = \sum_{n=0}^{\infty} P(X=n)\cdot s^n$$

$2k = n,\ k = \frac{n}{2}\ (n = 2k)$

$p(x = n) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1} = \left(\frac{1}{2}\right)^{\frac{n}{2}+1}$

$= 0 \cdot \boxed{0.u}$