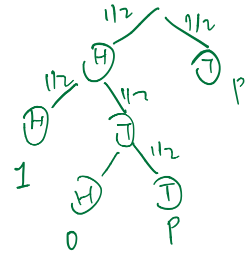
Last time

☆ You and I select a pattern from {HHH, HTH, HHT, HTT, THH, TTH, THT, TTT}

We toss a fair coin and record the sequence.

Whoever's pattern comes up first win the game.




$p = \frac{1}{4}\times 1 + \frac{1}{8}\times 0 + \frac{1}{8}\times p + \frac{1}{2}p$

$p = \frac{1}{4} + \frac{5}{8}p$

$\frac{3}{8}p = \frac{1}{4} \to p = \frac{1}{4}\times\frac{8}{3} = \frac{2}{3}$

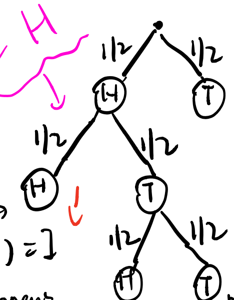
Q: Say, you choose HTH, I choose HHT.

Probability that I win? Continuous

Prob that I win $= x$

$p(w) = x$

Show 2 H




$p(w|T) = x$

$p(w|HH) = 1$

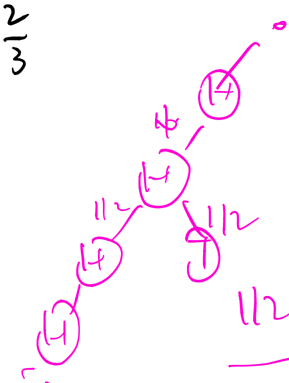
$\because$ T happens sooner or later

$p(w|HTH) = 0$

$p(w|HTT) = x$

$x = \frac{1}{2}x + \frac{1}{4} + \frac{1}{8}x$

$x = \frac{2}{3}$




$1/2 + 1/4 + \cdots = 1$

Probability Space: $\Omega$

HTH⋯TH⋯, which is the infinite sequences of H and T

Events: "Cylindrical Events"

Choose k positions and prescribe outcomes there "H" or "T"

⋯ $\underset{H}{5}$ ⋯ $\underset{T}{10}$ ⋯ $\underset{T}{100}$ prob: $(\frac{1}{2})^k$

Add to the mix all events obtained by repeated countable unions and taking components.

It works.

$P(HT\cdots HT\cdots) = 0$

## Independent Events

$\Omega$ $\varepsilon$ probability space

$\{A_i : i \in I\}$ are events in $\Omega$.

We say $A_i$ are independent $\boxed{\text{if}}$ for every finite $J \subset I$,

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

ex | caution:

I roll two fair dice.

A: The first die shows 3

B: The second die shows 4

C: The sum is 7.

$P(A) = \frac{1}{6}$, $P(B) = \frac{1}{6}$, $P(C) = \frac{6}{36} = \frac{1}{6}$

| | |
|---|---|
| 1 | 6 |
| 2 | 5 |
| 3 | 4 ✓ |
| 4 | 3 |
| 5 | 2 |
| 6 | 1 |

$P(AB) = \frac{1}{36}$, $P(AC) = \frac{1}{36}$, $P(BC) = \frac{1}{36}$

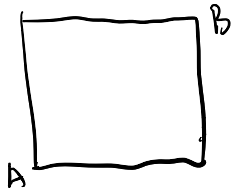
$P(ABC) = \frac{1}{36} \neq P(A)P(B)P(C) = \frac{1}{216}$

$\{A, B\}$ independent. $\{A, C\}$ independent. $\{B, C\}$ independent.

But $\{A, B, C\}$ × independent

Typical Example

Wires arranged like this: Calculate the prob of whole network working




① diagonal x work

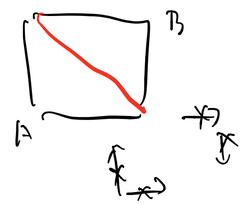
斜边 x ┌ ┘ :

$$P(\uparrow\rightarrow \cup \rightarrow\uparrow) = P(\uparrow\rightarrow) + P(\rightarrow\uparrow) - P(\uparrow\rightarrow\rightarrow\uparrow)$$
$$= 2(1-p)^2 - (1-p)^4$$

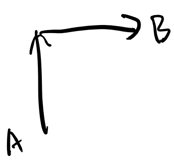
Each fails with prob $0<p<1$

Independly of others

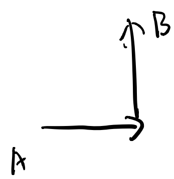
② diagonal work x else:




Q: Prob that there is a connection between A and B?




$(1-p)^2$




$(1-p)^2$

$$P(\uparrow\rightarrow \cup \rightarrow\uparrow) = P(\uparrow\rightarrow) + P(\rightarrow\uparrow) - P(\uparrow\rightarrow\rightarrow\uparrow)$$
$$= 2(1-p)^2 - (1-p)^4$$

$$P(\uparrow\rightarrow \cup \rightarrow\uparrow) = P(\uparrow\rightarrow) + P(\rightarrow\uparrow) - P(\uparrow\rightarrow\rightarrow\uparrow)$$

$$P(\uparrow\rightarrow\rightarrow\uparrow) = (1-p)^4$$

$A: 2(1-p)^2 - (1-p)^4$

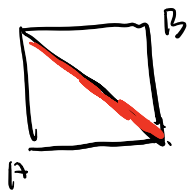
$$P(C) = P(C \mid x\backslash)\cdot P(x\backslash) + P(C\mid \backslash)\cdot P(\backslash)(1-p)$$
$$= p \times [2(1-p)^2 - (1-p)^4]$$

No connection if either

$p^2$ $p^2$ $P(\cup)$ $= P() + P() - P()$ $= 2p^2 - p^4$

$P(C\mid\backslash) = 1 - 2p^2 + p^4$

通电




Each fails with prob $p$ independently of others.

Q: There is a connection between A and B.

Condition on that extra.

斜边 x with P ← If fails with prob $p$, there is connection with prob: $2(1-p)^2 - (1-p)^4$ ✓

斜边 ✓ ← If stays with prob $(1-p)$ ✓

No connection if either:

$p^2$ $p^2$ $P(\text{x clue})$ $= P(\cup)$

$p^2$ ↑ $\quad$ L ✗ $p^2$

$\therefore P(\text{x connection}) = 2p^2 - p^4$

$\therefore P(\text{there is connection}) = 1 - 2p^2 + p^4$

$= P(\dots) + P(\dots) - P(\dots)$

$= 2p^2 - p^4$

$\therefore P(\text{conne}) = 1 - 2p^2 + p^4$

$A: p(2(1-p)^2 - (1-p)^4) + (1-p)(1-2p^2+p^4)$

$= -2p^5 + 5p^4 - 2p^3 - 2p^2 + 1$

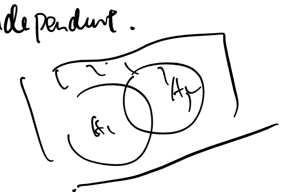
Self-test exercise:

$A_1, A_2, \dots A_n$ independent ⇒ replace some by some intersection still independent

⇒ replace some by the union still independent.

$P((A_1 \cup A_2) \cap (A_3 \cup A_4 \cup A_5))$

$= P(A_1 \cup A_2) \times P(A_3 \cup A_4 \cup A_5)$




eg. $A_1, A_2, A_3, A_4, A_5$ independent, then $A_1 \cup A_2$, $A_3 \cup A_4 \cup A_5$ are independent.

Proof: To prove $A_1 \cup A_2$, $A_3 \cup A_4 \cup A_5$ are independent, trick: $(A_1 \cup A_2)^c = (A_1^c \cap A_2^c)$

we need to prove $\underbrace{P(A_1 \cup A_2)}_{A} \times \underbrace{P(A_3 \cup A_4 \cup A_5)}_{B} = P((A_1 \cup A_2) \cap (A_3 \cup A_4 \cup A_5))$

$P(A^c \cap B^c)$ $\qquad A_1^c \cap A_2^c$

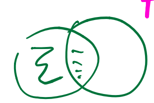
$A_3^c \cap A_4^c \cap A_5^c$

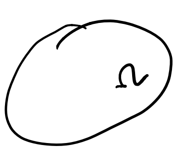
To prove $A, B$ indep → $A^c, B^c$ indep → $(A_1^c \cap A_2^c)$ and $(A_3^c \cap A_4^c \cap A_5^c)$ indep ✓ $\quad A^c, B^c$ indep

$A$ $B$ indep

## Finite Probability Space with Equally Likely Outcomes

proveable




If $\Omega$ is finite,

every $A \subset \Omega$ is an event.

$P(A_1 \cap A_2) = P(A_1) P(A_2)$

$P(A_1 \cap A_2^c) = P(A_1) P(A_2^c)$

$[1 - P(A_2)]$

Equally likely outcomes: $P(A) = \dfrac{|A| \leftarrow \# \text{ outcomes in } A}{|\Omega| \leftarrow \# \text{ outcomes in } \Omega}$

Multiplication Principle:

clubs
diamonds
hearts
spades

If there are n ways to accomplish the task I.

After each of which, there are n ways to finish task II.

How many ways to finish task I and II?

$n \times n$

ex ("Birthday Paradox")

Pick 50 people at random.

Q: What is the prob that some two of them will have birthdays on the same month and day?

think of these 50 people walking in the room one-by-one, and proclaiming there birthday.

1st has 366 possibilities.

2nd has 366 possibilities.

$\vdots$

50th has 366 possibilities.

$|\Omega| = 366^{50}$

$|\Omega| = 366^{50}$

$|A| = 366 \times 365 \times \cdots 317$

$= \frac{366!}{316!}$

$\therefore P = \frac{|A|}{|\Omega|} = \frac{366!/316!}{366^{50}}$, $1-P = $

Complementary Event: No two have birthdays on the same month/day

1st: 366

2nd: 365

3rd: 364

$\vdots$

50th: $366-50+1 = 317$, $\therefore |A| = 366 \times 365 \times \cdots 317 = \frac{366!}{316!}$

366

$1 - \frac{366!}{316!\, 366^{50}}$

$\frac{366 \times 317}{366!}$

Answer: $1-\dfrac{366!}{316!\ 366^{50}} \approx 0.97$

## "Combinations"

A group of $n$ people

Q: In how many ways can we choose a committee of $k$ from the group of $n$ people?

Say, we choose a committee And also rank its members by importance.

We can choose the most important member in $n$ ways.

We - - - - - - - next - - - - - - - $(n-1)$ - -

⋮

We - - - - - - - least - - - - - - - $(n-k+1)$ - -

# ways to choose a committee and to rank its members is:

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

$(n-k)$ factorial

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

If x ranking, then we have: $\dfrac{n!}{k!\,(n-k)!}$ combinations

Now, forget about ranking. Each committee of $k$ can be ranked in $k!$ ways.

A: The total # of (unranked) committees is:

$$\frac{n!}{(n-k)!\ k!} = \binom{n}{k}, \text{ "n choose k"}$$