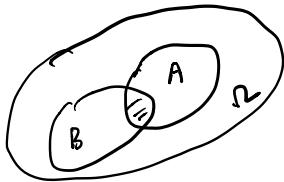


## Conditional Probability



probability space

$B \subset \Omega$  is an event,  $P(B) > 0$

then the probability of A given B is:  $P(A|B) = \frac{P(A \cap B)}{P(B)}$

B strengthens A if  $P(A|B) > P(A)$        $P(A|B) > P(A)$

B rejects A if  $P(A|B) < P(A)$        $P(A|B) < P(A)$

B is independent of A if  $P(A|B) = P(A)$        $P(A|B) = P(A)$

Ex.

(a) pick a random card

A: this is an ace

B: this is a spade black

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = \frac{13}{52} = \frac{1}{4}$$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/52}{1/4} = \frac{1}{13}$$

$P(A|B) = P(A) \Rightarrow$  independent

1b) ex.

Put away Jack of Hearts, pick a random card from remaining 51.

A: this is an Ace

B: this is a spade

$$P(A) = \frac{4}{51}$$

$$P(B) = \frac{13}{51}$$

$$\therefore P(A|B) = \frac{P(AB)}{P(B)} \stackrel{\text{prob of intersection}}{=} \frac{1}{51} = \frac{1}{13} = \frac{4}{52} (\frac{4}{51} = P(A))$$



Self-test exercise:

$$P(A) > 0, P(B) > 0$$

A repels B ( $\Rightarrow$ ) B repels A

-- attracts -- ( $\Rightarrow$ ) -- -- attracts --

-- -- indep -- ( $\Rightarrow$ ) -- -- indep --

$$\frac{1}{2} \times \frac{3}{4}$$

Ex.

My neighbor and I are working independently from our respective houses.  
At some point between 9 Am and 10 Am.



event A: I walked out after 9:30

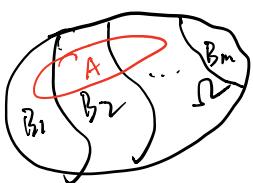
event B: I walked out after my neighbor

$$P(A) = 1/2, P(B) = 1/2$$

$$\therefore P(A|B) = \frac{P(AB)}{P(B)} = \frac{3/8}{1/2} = \frac{3}{4} \Rightarrow P(A)$$

B attracts A

## The Formula of Total Probability



$$\Omega = \bigcup_{i=1}^m B_i$$

$$B_i \cap B_j = \emptyset$$

$$P(B_i) > 0, i=1, 2, \dots, m$$

$$P(A) = \sum_{i=1}^m P(A \cap B_i) = \sum_{i=1}^m P(A|B_i) P(B_i)$$



Same is true if there are Countably many B.

ex ("A Must") → must for exam

there are three boxes in front of me.

1st box: contains 3 red and 2 blue marbles.

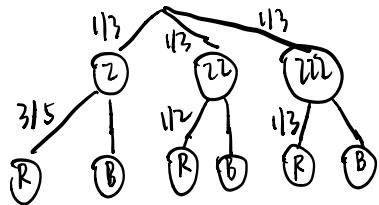
2nd : - - - 3 -- 3 - - - -

3rd : - - 1 - - - 2 - - - -

Q: pick a box at random and a marble from the box.

Q: prob that the marble is red?

Draw A Tree



$$\begin{aligned}P(R) &= 1/3 \times 3/5 + 1/3 \times 1/2 + 1/3 \times 1/3 \\&= \frac{1}{5} + \frac{1}{6} + \frac{1}{9} \\&= \frac{47}{90}\end{aligned}$$

Ex.

Suppose I pick box with eyes closed.

But I see the marble is red.

Q: What's the prob I picked it from the first box.

$$P(Z|R) = \frac{P(Z \cap R)}{P(R)} = \frac{1/3 \times 3/5}{47/90} = \frac{18}{47}$$

Ex "3 CARD SCAM"

there're 3 cards.

one is red on both sides RR

second is blank on both sides BB

third is blank on one and red on the other RB

SAY: I show you a random side of 1 card on card.

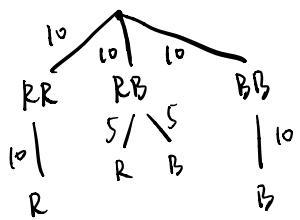
Say: you see red.

$Q$  = prob that the other side is red.

$$P(RR|R) = \frac{P(RR \cap R)}{P(R)} = \frac{P(RR)}{P(R)} = \frac{1/3}{1/2} = \frac{2}{3}$$

frequency interpretation:

Say, play this game 30 times



you'll see red 15 times, of which 10 times from RR card.

A

Self-test Exercise:

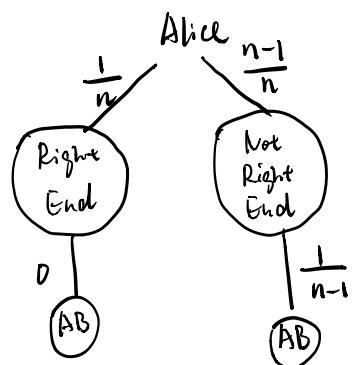
- (a) What strategy should I use, so that I am not losing money against any strategy of yours.
- (b) What strategy should you use, so that you're not losing money against any strategy of mine.

(Ex) -

$n > 2$ . people including Alice and Bob are seated at random in a row.



Q: prob that B<sub>2</sub> is next right to Alice?



$$A: \frac{1}{n} \times 0 + \frac{n-1}{n} \times \frac{1}{n-1} = \frac{1}{n}$$

## Ex / Self-test Exercise

You and I choose a pattern from  $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Starting tossing a fair coin and record the sequence:

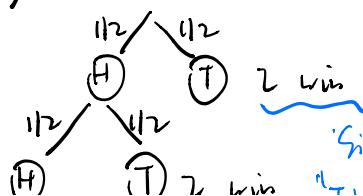
HII HHH III --

Whoever pattern appears first wins.

Show that if I get you choose first, I can always choose a pattern that beats you with prob  $> 2/3$

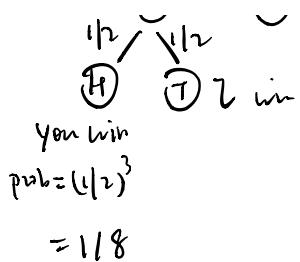
If you choose HHH

I choose THH



I win I win as long as "T" appears.  
Since after "T" appears, if we have  
~~in "T" the T is not~~

718

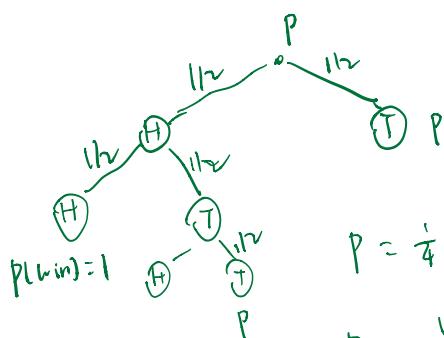


IHH, then L wins; if we have "TT~" or "THT", then we return to the state with just one "T". But prob that "TT~" and "THT" always happen = 0. So "THH" must win. (THH cannot lose in this situation)

If you choose HTH

I choose HHT

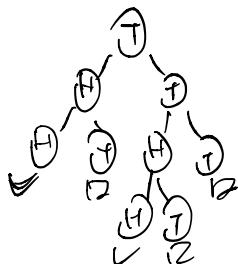
then 2 wins with prob 2/3



$$P = \frac{1}{4}x1 + \frac{1}{8}P + \frac{1}{2}P$$

$$P = \frac{1}{4} + \frac{5}{8} P$$

$$\frac{3}{8}P = \frac{1}{4} \rightarrow P = \frac{2}{3}$$



$$P = f \frac{1}{4} x \frac{8^2}{3}$$