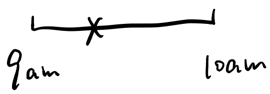
① Sample Space Ω

We have an experiment, list all possible outcomes

Example toss a coin 3 times

HHH, HHT, HTH, HTT, THH, THT, TTT, TTH

Example: I walk out from my house at some point between 9~10.



Example

Roll two dice, look at the numbers

a) number the dice

first shows: 1, 2, 3, 4, 5, 6

second shows: 1, 2, 3, 4, 5, 6

b) Dice look indistinguishable, how many outcomes?

$(36-6)/2+6=21$

Both are legal, but (a) is more convenient, why?

② Events

Some sets of outcomes are called events.

$A \subset \Omega$

Example: {HTT, THT, TTH} is an event

Axioms of event: event is a σ-algebra.

(1) Ω is an event

(2) If $A \subset \Omega$ is an event, then $A^c = \{w \in \Omega;\ w \notin A\}$ is an event.

(3) If $A_1, A_2, \cdots A_n \cdots$ are events, then $\bigcup_{n=1}^{\infty} A_n = \{w,\ w \in A_n \text{ for some } n\}$ is an event.

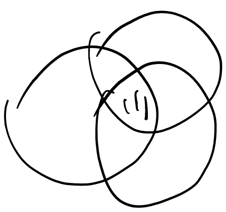
Remarks.

If Ω is countable, there is no harm in calling any $A \subset \Omega$ an event.

If Ω is uncountable, there can be trouble.

Self-test exercise:

If $A_1, \cdots, A_n, \cdots$ are events, then $\bigcap_{n=1}^{\infty} A_n = \{w,\ w \in A_n\}$ is an event



Proof: $\underline{(\bigcap_{n=1}^{\infty} A_n)^c = \bigcup_{n=1}^{\infty} A_n^c}$

$(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c$

A_1, A_2 event

$\downarrow$

$A_1 \cap A_2$ event?

$A_1 \cup A_2$ ✓

A_2 ✓

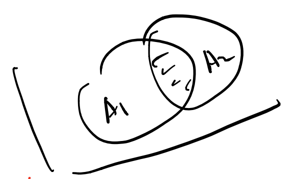
$(A_1 \cup A_2) \setminus A_2$ ✓

$(A_1 \cup A_2) \setminus A_1$ ✓

$(A_1 \cup A_2) \setminus A_2$ ✓

$\therefore A_1 \cup A_2 - (A_1 \cup A_2) \setminus A_2$

$- (A_1 \cup A_2) \setminus A_1$ ✓



$(\bigcap_{n=1}^{\infty} A_n)^c = \bigcup_{n=1}^{\infty} A_n^c$

$\because A_n$ ✓, $\therefore A_n^c$ ✓

$\therefore \bigcup_{n=1}^{\infty} A_n^c$ ✓ $\Rightarrow (\bigcap_{n=1}^{\infty} A_n)^c$ ✓ $\Rightarrow \bigcap_{n=1}^{\infty} A_n$ ✓

③ Probability

To every event $A \subset \Omega$, assign a # $P(A) \geq 0$, called probability

Axioms: ① $P(\Omega) = 1$

② If $A_1, \cdots A_n, \cdots$ are pairwise disjoint, then:

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

$(\bigcap_{n=1}^{\infty} A_i)^c = \bigcup_{n=1}^{\infty} A_i^c$ ✓

Self-test exercise: $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2 \cdots$

$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) = \sum_{i=1}^{\infty} P(A_1) + P(A_2) - P(A_1) + P(A_3) - P(A_2) + \cdots$

If $A_1 \subset A_2 \subset A_3 \subset \cdots A_n \cdots \subset \cdots$

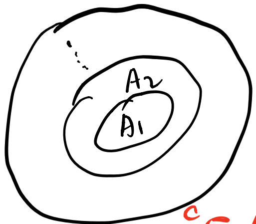
$A_0 = \emptyset$

$B_1 = A_1$ $= \lim_{n\to\infty} P(A_n)$

$B_2 = A_2 \setminus A_1$

$B_k = A_k \setminus A_{k-1}$, $k \geq 2$

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} B_k$



Show that:

① $P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

$\mathbb{P}(\bigcup_{k=1}^{\infty} B_k)$

$= \sum_{k=1}^{\infty} P(B_k)$

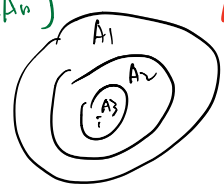
$= \sum_{k=1}^{\infty} [\mathbb{P}(A_k) - \mathbb{P}(A_{k-1})]$

$= \lim_{k\to\infty} \mathbb{P}(A_k)$

$A_1^c \subset A_2^c \subset A_3^c \cdots \subset A_n^c \subset \cdots$

If $A_1 \supset A_2 \supset A_3 \supset \cdots A_n \supset \cdots$

$A_1^c \subset A_2^c \subset \cdots A_n^c \subset \cdots$ $P(\bigcup_{n=1}^{\infty} A_n^c) = \lim_{n\to\infty} P(A_n^c)$



② then $\lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n)$

$P((\bigcap_{n=1}^{\infty} A_n)^c) = \lim_{n\to\infty} P(A_n^c)$

$P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Proof: Prove ① first:

Denote $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus A_2 \cdots$

Then we know $P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n) = P(A_1) + P(A_2) - P(A_1)$

$+ P(A_3) - P(A_2) + \cdots$

$= \lim_{n\to\infty} P(A_n)$ ✓

Then prove ②:

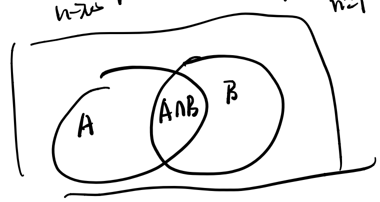
$\because A_1 \supset A_2 \supset A_3 \cdots$, $\therefore A_1^c \subset A_2^c \subset A_3^c \cdots$

$\therefore P(\bigcup_{n=1}^{\infty} A_n^c) = \lim_{n\to\infty} P(A_n^c) \to P(\bigcap_{n=1}^{\infty} A_n)^c = \lim_{n\to\infty} P(A_n^c) \to P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$

Operating with events

Example:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Deduce from axioms:

$P(A \cup B) = P(A \cup (B \cap A^c))$

$= P(A) + P(B \cap A^c)$

And $B = (B \cap A) \cup (B \cap A^c)$

$\therefore P(B) = P(B \cap A) + P(B \cap A^c)$

$P(B) = P(A \cup B) + P(A \cap B) - P(A)$

$\downarrow$

proof: $P(B \cap A^c) = P(A \cap B) - P(A)$

Inclusion:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots + \cdots + (-1)^{k-1} \sum_{1\le i_1 \le i_2 \le \cdots n} P(A_{i_1} \cap \cdots \cap A_{i_n})$$

Attension: $A \cap B$ is denoted AB

Example:

J and M are taking a class with grades A, B, C.

The prob that J gets a B is 0.3,

- - - - - - - M gets - - B - - 0.4

The prob that neither get an A but at least one gets a B is 0.1.

Q: Prob that at least one get a B, but neither C.

John, Mary

Sample space:

AA, AB, AC, BA, BB, BC, CA, CB, CC => nine outcomes

events:

$P(\{BA, BB, BC\}) = 0.3 \Rightarrow P(BA) + P(BB) + P(\cancel{BC})$

$+$

$P(\{AB, BB, CB\}) = 0.4 \Rightarrow P(AB) + P(\cancel{BB}) + P(\cancel{CB})$

$-$

$P(\{BB, BC, CB\}) = 0.1 \Rightarrow P(\cancel{BB}) + P(\cancel{BC}) + P(\cancel{CB})$

Q: $P(\{AB, BA, BB\}) \Rightarrow 0.3 + 0.4 - 0.1 = 0.6$

$\{BA, BB, BC\}$ $P(BA) + P(BB) + P(\cancel{BC}) = 0.3$

$+$

$\{AB, BB, CB\}$ $P(AB) + P(\cancel{BB}) + P(\cancel{CB}) = 0.4$

$-$

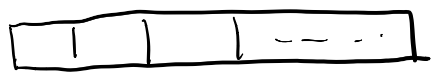
$\{BB, BC, CB\}$ $P(\cancel{BB}) + P(\cancel{BC}) + P(\cancel{CB}) = 0.1$

Q: $\{BA, AB, BB\}$ $P(BA) + P(AB) + P(BB) = 0.3 + 0.4 - 0.1 = 0.6$

Using $\boxed{\text{Symmetry}}$ to reduce the size of the prob space.

Example:

$n \geq 100$ people, including Alice and Bob are arranged in a row at random.



Q: What's the prob of Bob is to the right of Alice?

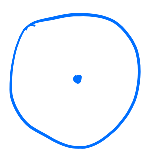
1/2 ($\because$ Bob is either to the left/right of Alice)

Example:

Shuffle a deck of cards, pick the 5 th card from the top.

Q: Prob it is an Ace?

Ans: $\frac{4}{52} = \frac{1}{13}$



Example:

Alice and Bob are playing darts, who ever gets the closest shot wins.

But Alice make 2 and Bob 1.

Q: $P(\text{Bob wins}) = \frac{1}{3}$ (the prob of draw is 0)