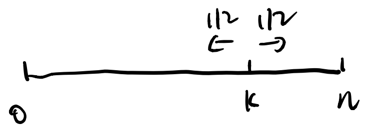
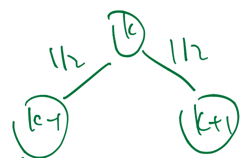
Example (Symmetric Random Walk) ☆



let X_k be the duration (the # of steps of the walk)

Q: $E(X_k) = ?$

$e_k = E(X_k)$. $E(X_k)$

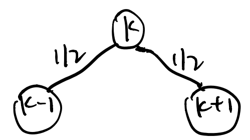


$E(X_k | k \to k-1)$ $e_{k+1}+1$

$= e_{k-1}+1$

let $e_k = E(X_k)$, $k = 0, 1, 2, \cdots n$

$e_0 = 0$, $e_n = 0$



$E(X_k | k \to k-1)$ $E(X_{k+1} | k \to k+1)$

$= \underset{\sim}{e_{k-1}+1}$ $= \underset{\sim}{e_{k+1}+1}$

$e_k = \frac{1}{2}e_{k-1} + \frac{1}{2} + \frac{1}{2}e_{k+1} + \frac{1}{2}$

$e_k = \frac{1}{2}e_{k-1} + \frac{1}{2}e_{k+1} + 1$

$e_k = E(X_k) = \frac{1}{2}E(X_k | k \to k-1) + \frac{1}{2}E(X_{k+1} | k \to k+1) + 1$

$\frac{1}{2}(e_k - e_{k-1}) = \frac{1}{2}(e_{k+1} - e_k) + 1$

$= \frac{1}{2}(e_{k-1}+1) + \frac{1}{2}(e_{k+1}+1)$

$= \frac{1}{2}e_{k-1} + \frac{1}{2}e_{k+1} + 1$

$\frac{1}{2}(e_{k+1} - e_k) = \frac{1}{2}(e_k - e_{k-1}) - 1$

$\underline{e_{k+1} - e_k = (e_k - e_{k-1}) - 2}$

$\therefore \frac{1}{2}e_k + \frac{1}{2}e_k = \frac{1}{2}e_{k-1} + \frac{1}{2}e_{k+1} + 1 \Rightarrow e_k + e_k = e_{k-1} + e_{k+1} + 2$

$\therefore e_{k+1} - e_k = (e_k - e_{k-1}) - 2$

$e_k = \cancel{e_{k+1}}\ (e_k - e_{k-1}) + \cdots (\underbrace{e_1 - e_0}_{x}) + e_0$ $e_0 = 0$

$e_k = (e_k - e_{k-1}) + (e_{k-1} - e_{k-2}) + \cdots (e_1 - e_0) + e_0$

$= x + (x-2) + \cdots (x - 2(k-1)) + 0$ $e_n = 0$

let $e_1 - e_0 = x$,

$= \frac{(2x - 2(k-1))k}{2}$

$= [x - (k-1)]k$

$e_2 - e_1 = x - 2$

$e_3 - e_2 = x - 4$

$= \underline{(x - k + 1)k}$

$\vdots$

$\to e_k = (x - 2(k-1)) + (x - 2(k-2)) + \cdots + x$

$e_n = 0$

$e_k - e_{k-1} = x - 2(k-1)$

$= kx - 2(1 + 2 + \cdots k-1) = kx - k(k-1)$

Since $e_n = 0 \Rightarrow nx - n(n-1) = 0 \Rightarrow x = n-1$

$\therefore e_k = k(n-1) - k(k-1)$

$= k(n-k)$

$e_n = (x-n+1)n = 0$

$\underbrace{x = n-1}$

$e_k = (n-1-k+1)k$

$= \underbrace{(n-k)k}$

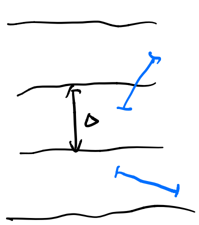
One Interesting Observation

If $k=1$, and $n \to \infty$, then $E(X_k) = n-1 \to \infty$

So a gambler playing against an infinitely rich adversary is going to lose all his money with prob 1. But the expectation duration of the game is ∞.

ex. (Buffon's Needle)

Consider a sheet of paper with horizontal lines one unit apart.



Drop a needle of length 1.

Q: What the prob that the needle intersects a line?

Hard way:

Model: d is the distance from the needle middle point to the nearest line.

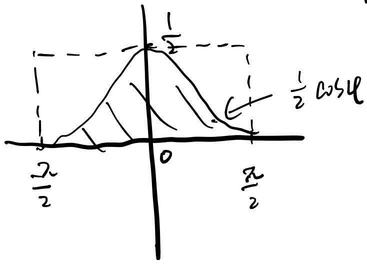
Angle φ between the vertical direction and the needle.

$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

The position of the needle is described by a point (φ, d).

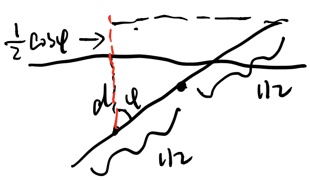
In the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \frac{1}{2}]$

Assume that we pick the point uniformly at random.



Event:

Needle intersect the (nearest to midpoint) line



$\frac{1}{2}\sin\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$d = \frac{1}{2}\cos\varphi$

prob: $\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos\varphi\right) / \left(\pi \times \frac{1}{2}\right)$

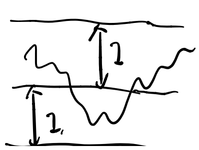
$= \frac{1}{\pi/2} = \frac{2}{\pi}$

A: $\frac{2}{\pi}$

"Easy" Conceptual Way

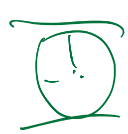
Buffon's noodle

Drop a curve onto the ruled paper. And let X be the # of intersections with lines.



$E(X) = nE(X_1)$

$E(X) = cL$



$L = 2\pi R = 2\pi \times \frac{1}{2} = \pi$

$E(X) = 2 = C\pi, \quad C = \frac{2}{\pi} \qquad E(X) = \frac{2}{\pi} \cdot L$

Q: $E(X)$

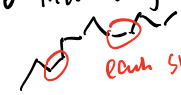
If we drop a needle of length 1, then $E(X)$ = prob of intersection.

Claim: $E(X)$ should be proportional to the length of the curve.

$E(X) = E(X_1) + \cdots F(X_n)$
$= nE(X_1)$
$= CL$

Why?

Chop up the curve into tiny little pieces of the same length, which all look like intervals.



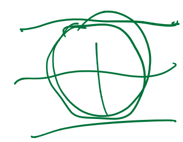
each small section is equivalent to each other. So $E(X_1) = E(X_2) = \cdots E(X_n)$

Let X_k be the # of intersections of the k-th piece.

since each X_i is equivalent

$X = X_1 + X_2 + \cdots X_n \Rightarrow E(X) = E(X_1) + E(X_2) + \cdots E(X_n) = nE(X_1)$

X rely on dependence



If the length of the curve is L, $E(X) = CL$.

$E(X) = C \times$ length of the curve

$\uparrow$
Constant

The longer the curve is, the larger the n is. Then $E(X)$ grows bigger proportionally. So $E(X)$ should be proportional to the length of the curve.

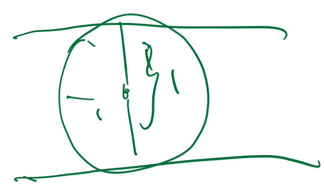
What curve should we drop to get the constant instantly?

Drop a circle of radius $\frac{1}{2}$. $\qquad 2\pi R = 2\pi \times \frac{1}{2} = \pi$

$X = 2 \Rightarrow E(X) = 2$,

$2 = C \cdot \pi \Rightarrow C = \frac{2}{\pi}$ ✓

$E(X) = 2 = C \cdot \pi$
$\Downarrow$
$C = \frac{2}{\pi}$



$E(X) = 2 = C \cdot 2\pi \times \frac{1}{2} = C \cdot \pi$

$\boxed{C = \frac{2}{\pi}}$