Last time

Define a r.v. as a ftn: $X: \Omega \to \mathbb{R}$ s.t. for any real $a \in \mathbb{R}$,

$\{w: X(w) \le a\}$ is an event. $F_X(a) = P\{w \in \Omega : X(w) \le a\}$

$\boxed{\text{Cumulative distribution ftn:}}$ $F_X(a) = P\{w \in \Omega : X(w) \le a\}$

ex. Toss a fair coin 3 times, count heads

$F_X(a) = P\{w \in \Omega : X(w) \le a\}$

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$X(HHH) = 3$, $X(HHT) = X(HTH) = X(THH) = 2$

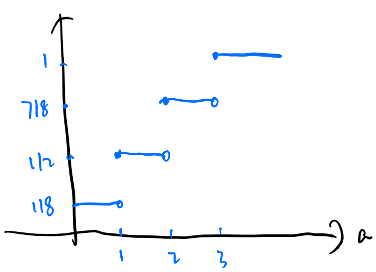
$X(HTT) = X(THT) = X(TTH) = 1$

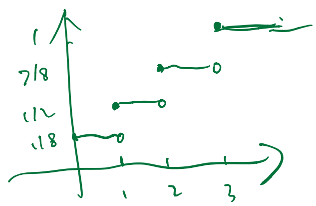
$X(TTT) = 0$

$P(X(w) \le a)$

$= P\{w \in \Omega : X(w) \le a\}$

$$F_X(a) = P\{X \le a\} = \begin{cases} 1 & a \ge 3 \\ 7/8 & 2 \le a < 3 \\ \frac{1}{2} & 1 \le a < 2 \\ \frac{1}{8} & 0 \le a < 1 \\ 0 & a < 0 \end{cases}$$





Properties of F_X

① $0 \le F_X \le 1$

② If $a < b$, then $F_X(a) \le F_X(b)$ (non-decreasing)

③ $\lim_{a \to -\infty} F_X(a) = 0$, $\lim_{a \to \infty} F_X(a) = 1$ ← self-test

④ If $a_n \to a+$

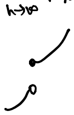
$\lim_{n \to \infty} F_X(a_n)$

$= \lim_{n \to \infty} P(X \le a_n)$

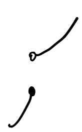
$\lim_{n \to \infty} F_X(a_n) = F_X(a)$

$\lim_{n \to \infty} P(X \le a_n) = P(X \le a)$

$A_n = \{w \in \Omega : X(w) \le a_n\}$



these jumps are allowed



these jumps are not allowed

$A = \{w \in \Omega : X(w) \le a\}$

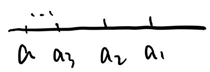
$A_1 \supset A_2 \supset \cdots$

$A = \bigcap_{n=1}^{\infty} A_n$, then $P(A) = \lim_{n \to \infty} P(A_n)$

Proof of ④:

Define Events (A_n is decreasing)

$A_n = \{\omega \in \Omega : X(\omega) \leq a_n\}$



$A = \{\omega \in \Omega : X(\omega) \leq a\}$

$A_n = \{\omega \in \Omega : X(\omega) \leq a_n\}$

$A = \{\omega \in \Omega : X(\omega) \leq a)$

$\therefore A_1 \supset A_2 \supset A_3 \cdots$

$A_1 \supset A_2 \supset \cdots$

$A = \bigcap_{n=1}^{\infty} A_n$ (self-test of first class)

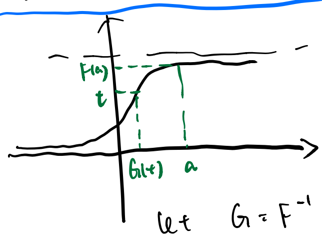
then $P(A) = \lim_{n\to\infty} P(A_n) \Leftrightarrow \lim_{n\to\infty} F_X(a_n) = F_X(a)$

$\therefore \lim_{n\to\infty} P(A_n) = P(\underbrace{\bigcap_{n=1}^{\infty} A_n}_{A})$

$\lim_{n\to\infty} F_X(a_n) = F_X(a)$

Important, we will use it.

$\boxed{\text{If } F \text{ is a ftn } F: \mathbb{R} \to [0,1], \text{ satisfies ①–④, then there is a r.v. } X: \Omega \to \mathbb{R} \text{ s.t. } F = F_X}$



Why?

If F is continuous and strictly increasing

Pick $\Omega = (0,1)$ why??? → just need to prove the existence

let $G = F^{-1}$ be the inverse of F

$G = F^{-1}$ here, so we can pick

$G(t) \leq a$, $G(a) = F^{-1}(a)$ any Ω

$F^{-1}(t) \leq a$ that leads to desirable result.

Define $X: (0,1) \to \mathbb{R}$, $G: (0,1) \to \mathbb{R}$

$= \{c : F(c) = t\}$

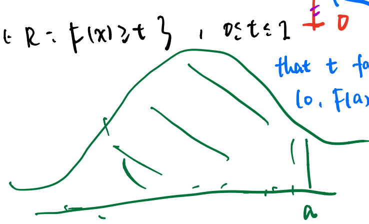
$X(t) = G(t)$

$F_X(a) = P(X \leq a) = P(G(t) \leq a) = P(t \leq G^{-1}(a)) = P(t \leq F(a)) = F(a)$

the prob that t falls within $[0, F(a))$

Hard self-test exercise:

In general, try $G(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\}$, $0 \leq t \leq 1$



Continuous Random Variable

We say that $X: \Omega \to \mathbb{R}$ is continuous if there is an integrable ftn s.t.:

$$P\{\omega \in \Omega : X(\omega) \in A\} = \int_A f_X(t)\,dt$$

$f_X: \mathbb{R} \to \mathbb{R}_+$, called density of X

↑
non-negative reals

Here $A \subset \mathbb{R}$, and A is Borel

$F_X(a) = P(\omega \in \Omega : X(\omega) \le a)$

$= \int_{-\infty}^{a} f_X(t)\,dt \Rightarrow F_X'(a) = f_X(a)$ for almost all a (except measure 0)

$F_X(a) = \int_{-\infty}^{a} f_X(t)\,dt$

$F_X(a+\Delta) = \int_{-\infty}^{a+\Delta} f_X(t)\,dt$

$\frac{F_X(a+\Delta) - F_X(a)}{\Delta} = F_X'(a)$

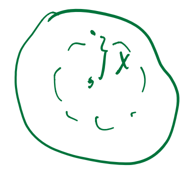
$LHS = \frac{\int_a^{a+\Delta} f_X(t)\,dt}{\Delta} \approx \frac{f_X(a)\cdot\Delta}{\Delta}$

$= f_X(a)$

ex. Dartboard is a circle of radius 1. The dart lands "uniformly".

Probability to hit any region is proportional to the area of the region.

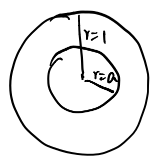
Let X be the distance from dart to the center.



Q: Density of X?

$F_X(a)$: the dart lands at the distance $\le a$ from the center

$$F_X(a) = \begin{cases} 1 & a \ge 1 \\ \frac{\pi a^2}{\pi} = a^2 & 0 \le a < 1 \\ 0 & a < 0 \end{cases}$$



$$f_X(a) = F_X'(a) = \begin{cases} 2a & 0 \le a < 1 \\ 0 & \text{otherwise} \end{cases}$$

$F_X(x) = P(X \le x)$

① when $x \ge 1$,

$F_X(x) = P(X \le x) = 1$

② when $0 \le x < 1$

$F_X(x) = P(X \le x)$

$= \frac{\pi x^2}{\pi \times 1} = x^2$

$\therefore F_X(x) = \begin{cases} 1 & x \ge 1 \\ x^2 & 0 \le x < 1 \end{cases}$

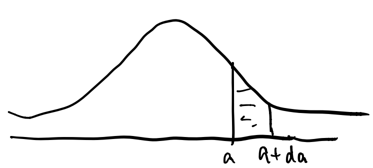
$\therefore f_X(x) = \begin{cases} 0 & x \ge 1 \\ 2x & 0 \le x < 1 \end{cases}$

Good News:

You can change the density on any countable set, or even on a set of measure zero free.

Intuition:

$P\{\omega : a \le X(\omega) \le a + da\} \approx f(a)\,da$

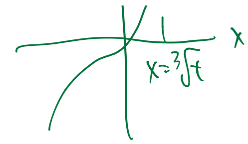


$P\{\omega : a \le X(\omega) \le a+da\} \approx f(a)\,da$

$\int_{-\infty}^{+\infty} f(t)\,dt = 1$ (the prob of landing in $(-\infty, +\infty)$ is 1)

It is an important skill to compute the density of a fcn of r.v.

Always start with the cdf.



ex. let X be a r.v. with density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ (standard normal).

let $Y = X^3$.

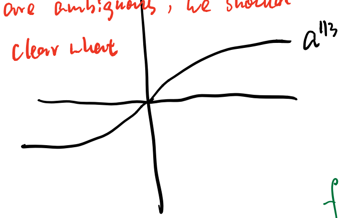
Q: Density of Y?

$F_Y(a) = P\{Y \le a\} = P\{X^3 \le a\}$

$= P\{X \le a^{1/3}\}$

$= F_X(a^{1/3})$

Generally, non-integer powers of negative # are ambiguous, we should always make clear what we mean.

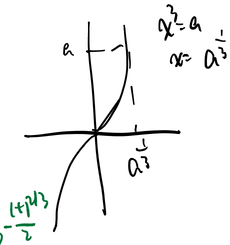


$f_Y(a) = F_Y'(a) = F_X'(a^{1/3})$

$= f_X(a^{1/3}) \cdot \frac{1}{3} \frac{1}{a^{\frac{2}{3}}}$

$F_X(a) = \int_{-\infty}^{a} f_X(t)\,dt \Rightarrow F_X'(a) = f_X(a)$

a function of a.



A: $f_Y(t) = \frac{1}{3\sqrt{2\pi}\,|t|^{2/3}} e^{-\frac{|t|^{2/3}}{2}}$

$f_X(a^{1/3}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^{\frac{2}{3}}}{2}} \cdot \frac{1}{3} \frac{1}{a^{\frac{2}{3}}} = \frac{e^{-\frac{|t|^{2/3}}{2}}}{3\sqrt{2\pi}\,|t|^{2/3}}$

~~$P_Y(t) =$~~

$F_Y(t) = P(Y \le t)$

$= P(X^3 \le t)$

$= P(X \le \sqrt[3]{t})$

$= F_X(t^{\frac{1}{3}})$

$f_Y(t) = F_Y'(t) = f_X(t^{\frac{1}{3}}) \frac{1}{3} \frac{1}{t^{\frac{2}{3}}}$

ex. Suppose X is a r.v. with density $f_X(t) = \begin{cases} \frac{1}{2} & \text{for } -1 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

let $Y = X^2$.

Q: Density of Y

$F_Y(a) = P(Y \le a) = P(X^2 \le a)$

If a is negative, then $P(X^2 \le a) = 0$

If $a \ge 0$, $P(|X| \le \sqrt{a}) = P(-\sqrt{a} \le X \le \sqrt{a})$

$= F_X(\sqrt{a}) - F_X(-\sqrt{a})$

$f_Y(a) = F_Y'(a) = \frac{1}{2\sqrt{a}} f_X(\sqrt{a}) + \frac{1}{2\sqrt{a}} f_X(-\sqrt{a})$

chain rule (a is the variable here)

A: $f_Y(a) = \begin{cases} \frac{1}{2\sqrt{a}} & 0 < a \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$\boxed{0 \le t \le 1}$

$\underline{t > 0}$

pt Y: $F_Y(t) = P(Y \le t)$

$= P(X^2 \le t)$

$-\frac{1}{2}\frac{1}{\sqrt{t}}$

$= P(-\sqrt{t} \le X \le \sqrt{t})$

$= \cancel{F_X(} F_X(\sqrt{t}) - F_X(-\sqrt{t})$

$F_Y'(t) = f_Y(t) = f_X(\sqrt{t}) \cdot \frac{1}{2}\frac{1}{\sqrt{t}} + f_X(-\sqrt{t}) \frac{1}{2\sqrt{t}}$

$= \frac{1}{2\sqrt{t}} (f_X(\sqrt{t}) + f_X(-\sqrt{t}))$

$= \begin{cases} \frac{1}{2\sqrt{t}} \times \frac{1}{2} \times 2 = \frac{1}{2\sqrt{t}} & 0 < t \le 1 \end{cases}$

10. ou.