Last Time:

Poisson Random Variable

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np = \lambda$$

$$p(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots \quad \lambda > 0$$

$$E(X) = \lambda$$

The # of success in many independent trials, where every trial is successful with small prob, but the expected # of success $\lambda > 0$ is "reasonable".

Typical Example ☆ (quiz)

The # of traffic accidents on W Ave in a random day is Poisson with expectation 3.

$$E(X) = \lambda = 3$$

Q: Probability that there are exactly 3 accidents?

$$p(X=3) = \frac{\lambda^3}{3!} e^{-\lambda} = \frac{3^3}{3!} e^{-3} = \frac{27}{6} e^{-3}$$

Let X be the # of accidents on a random day.

$$p(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad \lambda = E(X) = 3$$

$0! = 1$

$$p(X=3) = \frac{3^3}{3!} e^{-3} \approx 0.22$$

Q: Prob that there is at least one accident?

$$p(X>0) = 1 - p(X=0) = 1 - e^{-\lambda} = 1 - e^{-3} \approx 0.95$$

Q: Conditional Prob that at least 2 given that at least one:

$$p(X \geq 2 \mid X \geq 1) = \frac{p(X \geq 2)}{p(X \geq 1)} = \frac{1 - p(X=0) - p(X=1)}{1 - p(X=0)} = \frac{1 - 4e^{-3}}{1 - e^{-3}} \approx 0.84$$

People VS Collins (1968)

A purse was snatched in LA suburbs

Circumstential case: Malcolm & Janet Collins arrested.

Prosecution: Prob that a random couple match the description is 1/12,000,000

Continued of a second degree robbery. $\lambda = E(X) = \sum_{i=1}^{n} E(X_i) = 5 \times \frac{1}{12}$

On appeal: There are 5,000,000 "eligible" couples. let X be the # of couples matching the descriptions.

It's proper to assume $X \sim P(\lambda)$ since each event happens with small prob, but $E(X)$ is reasonable

X is Poisson with $\lambda = E(X) = \frac{5}{12}$

$$P(X\geq 2 \mid X \geq 1) = \frac{P(X\geq 2)}{P(X\geq 1)} = \frac{1-P(X=0)-P(X=1)}{P(X\geq 1)} = \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{1-e^{-\lambda}} \approx 0.19$$

$\frac{(\lambda_1+\lambda_2)^k}{k!} e^{-(\lambda_1+\lambda_2)}$

Conviction overturned.

$$P(Z=X+Y=k) = \sum_{n=0}^{k} P(X=n, Y=k-n)$$

$$= \sum_{n=0}^{k} P(X=n)P(Y=k-n)$$

$$= \sum_{n=0}^{k} \frac{\lambda_1^n}{n!}e^{-\lambda_1} \frac{\lambda_2^{k-n}}{(k-n)!}e^{-\lambda_2}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{n=0}^{k} \frac{\lambda_1^n \lambda_2^{k-n}}{n!(k-n)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{n=0}^{k} \frac{k!\,\lambda_1^n \lambda_2^{k-n}}{n!(k-n)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1+\lambda_2)^k$$

$\boxed{\text{Important:}}$

If $\underset{\lambda_1}{X}$ and $\underset{\lambda_2}{Y}$ are indep Poisson, then $\underset{\lambda_1+\lambda_2}{Z=X+Y}$ is Poisson.

$$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k=0,1,\dots$$

$$P(Y=k) = \frac{\mu^k}{k!}e^{-\mu}, \quad k=0,1,\dots$$

$$P(Z=n) = P(X+Y=n) = \sum_{k=0}^{n} P(X=k \text{ and } Y=n-k)$$

$$\overset{\text{independence}}{=} \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

$$= \sum_{k=0}^{n} \frac{\lambda^k}{k!}e^{-\lambda} \frac{\mu^{n-k}}{(n-k)!}e^{-\mu}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k}}$$

$$= \frac{(\lambda+\mu)^n}{n!} e^{-(\lambda+\mu)}$$

$\binom{n}{k}$

the Poisson distn with para $(\lambda+\mu)$

Poisson Process

The # of traffic accidents on W Ave in a random day ($= 24$ hs) is Poisson with expectation 3.

Let Y be the # of traffic accidents during 8 h period.

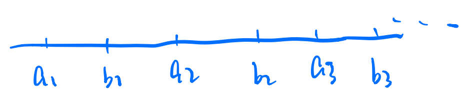
It stands to reason to assume that Y is Poisson with expectation 1.

Poisson Process: Is a family of rvs $X_{[a,b]} : \Omega \to \mathbb{R}$ where $0 \le a < b$ satisfying the axioms:

λ is the same for all $X_{[a,b]}$

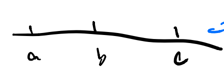
① Each $X_{[a,b]}$ is Poisson with $\underbrace{E X_{[a,b]} = \lambda(b-a)}$ for some $\lambda > 0$ called intensity.

② If intervals $[a_1, b_1], [a_2, b_2] \cdots [a_n, b_n]$, do not overlap, then: $X_{[a_1,b_1]}, X_{[a_2,b_2]}, \cdots X_{[a_n,b_n]}$ are indep.



③ If $a < b < c$, then:

$$\underbrace{X_{[a,b]} + X_{[b,c]} = X_{[a,c]}}$$



It helps that the sum of independent Poisson is Poisson.

$X_{[a,b]}$: the # of accidents in time interval $t \in [a,b]$.

Generally Speaking:

A stochastic (random) process is a family of r.v.s $X_i : \Omega \to \mathbb{R}, i \in I$.

$E[X_{[0,t]}] = \lambda t$

$t = 24$.

$\lambda t = 24 \lambda = 3$

$\boxed{\lambda = \frac{1}{8}}$

Typical Ex (Quiz)

$E X_{[0,24]} = \lambda \times 24 = 3, \lambda = \frac{1}{8}$

$E X_{[0,t]} = \lambda t, \quad P(X_{[0,t]} = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

I begin observing accidents on W Ave.

$P(X_{[0,3]} = 0) = \frac{(\frac{1}{8} \times 3)^0}{0!} e^{-\frac{3}{8}} = e^{-1}$

Q: What's the prob that I have to wait longer than 3 hs for the first accident.

$E X_{[0,t]} = \lambda t$

$P(X_{[0,3]} = 0) = e^{-3\lambda}$

$P(X_{[0,t]} = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

Solu: Let "0" be the time I started observations. Let $X_{[0,t]}$ be the # of accidents in time interval $[0,t]$.

$E X_{[0,t]} = \lambda t, \quad P(X_{[0,t]} = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, \ldots$

I want longer than 8 hours for the first accident $\Leftrightarrow X_{[0,8]} = 0$

$X_{[0,8]} = 0$,

$\left(P(X_{[0,8]} = 0) = e^{-\lambda 8} = e^{-1} \simeq 0.37 \right]$

$P(X_{[0,8]} = 0) = \cancel{8}\, e^{-\lambda 8} = e^{-1}$

$E X_{[0,24]} = \lambda \times 24 = 3 \Rightarrow \lambda = \frac{1}{8}$

$\cancel{P}\ E X_{[0,24]} = 24 \lambda = 3, \lambda = \frac{1}{8}$

$E X_{[0,24]} = \lambda \times 24 = 3 \Rightarrow \lambda = \frac{1}{8}$

Q: Prob that I want longer than 24 h for 3 accidents to happen?

$P[X_{[0,24]} < 3] = P[X_{[0,24]} = 0] + P[X_{[0,24]} = 1] + P[X_{[0,24]} = 2]$

$P(X_{[0,24]} = 0) + P(X_{[0,24]} = 1) = e^{-\lambda 24} + (\lambda 24) e^{-\lambda 24} + \frac{(\lambda 24)^2}{2} e^{-\lambda 24}$

$+ P(X_{[0,24]} = 2) \quad = e^{-3} + 3 e^{-3} + \frac{9}{2} e^{-3} = \frac{17}{2} e^{-3} \simeq 0.42$

$= e^{-3} + 3 e^{-3} + \frac{3^2}{2} e^{-3}$

Q: Expected waiting time till the first accident?

second ?

Let Y be the waiting time till the first accident

For $t > 0$, $F_Y(t) = P(Y \le t) = 1 - P(Y > t)$

$= 1 - P(X_{[0,t]} = 0)$

$= 1 - e^{-\lambda t}$

For $t \le 0$, $F_Y(t) = 0$.

Density $f_Y(t) = F_Y'(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0 & t \le 0 \end{cases}$

$E(Y) = \int_{-\infty}^{+\infty} t f_Y(t)\,dt = \int_0^\infty t \lambda e^{-\lambda t}\,dt$

$= \int_0^\infty t \, d(-e^{-\lambda t})$

$= -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\,dt$

$= \frac{1}{\lambda} \quad (\lambda = \frac{1}{8})$

$= 8$

How to compute fast?

$E(Y^2)$

$= \int_0^{+\infty} e^{-\lambda t} \cdot t^2\,dt$

(Computation trick)

$F_Y(t) = P(Y \le t)$

$= 1 - P(Y > t)$

$= 1 - P(X_{[0,t]} = 0)$

$= 1 - e^{-\lambda t}$

$f_Y(t) = F_Y'(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{o.w.} \end{cases}$

$E(Y) = \int_0^\infty f_Y(t)\, t\,dt$

$= \int_0^\infty \lambda e^{-\lambda t} \cdot t\,dt$

$= \lambda \int_0^\infty e^{-\lambda t} \cdot t\,dt = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$

$\int_0^\infty e^{-\lambda t}\,dt$

$= \frac{1}{-\lambda} e^{-\lambda t}\Big|_0^\infty = \frac{1}{\lambda}$

$\int_0^\infty e^{-\lambda t} \cdot t\,dt$

$= \int_0^\infty e^{-\lambda t} (t) = -\frac{1}{\lambda^2}$

$\int_0^\infty e^{-\lambda t} \cdot t = \frac{1}{\lambda^2}$

Memoryless Property of the waiting time:

one to one

density $\Leftrightarrow$ distn

So if Y has the density of a certain distn, then Y follows this distn.

We should that the waiting time Y has the density $\begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \le 0 \end{cases}$.

For a, b, $P(Y \ge a+b \mid Y \ge a) = \dfrac{P(Y \ge a+b)}{P(Y \ge a)} = \dfrac{e^{-\lambda(a+b)}}{e^{-\lambda(a)}} = e^{-\lambda b} = P(Y \ge b)$

$P(Y \ge x) = \int_x^{+\infty} \lambda e^{-\lambda t}\,dt$

$= e^{-\lambda x}$

waiting time Y has the density: $f_Y(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \le 0 \end{cases}$

$\lambda e^{-\lambda t} \ldots - e^{-\lambda t}\Big|_x^\infty$

$= e^{-\lambda x}$

$P(Y \ge x) = \int_x^\infty \lambda e^{-\lambda t}\,dt = e^{-\lambda x}$

$P(Y \ge a+b \mid Y \ge a)$

$= \dfrac{P(Y \ge a+b)}{P(Y \ge a)} = P(Y \ge b)$

$= \dfrac{1 - P(Y < a+b)}{1 - P(Y < a)}$

Sketch a proof that Poisson Process exists.

$= \dfrac{e^{-\lambda(a+b)}}{e^{-\lambda(a)}} = e^{-\lambda b}$

let $Y_1, Y_2, \cdots Y_k, \cdots$ be indep r.v.s with density $\begin{cases} \lambda e^{-\lambda t} & , t > 0 \\ 0 & , t \leq 0 \end{cases}$

$Y_1 =$ time till the first accident

$Y_2 =$ time between the first and second accidents

$\vdots$

let $Z_k = Y_1 + \cdots Y_k$

Z_k : the waiting time till the k-th accident.

We define:

$X_{[a,b]} = \{ \# \text{ of } k\text{s for which } a < Z_k < b \}$

Get Poisson Process of intensity λ.