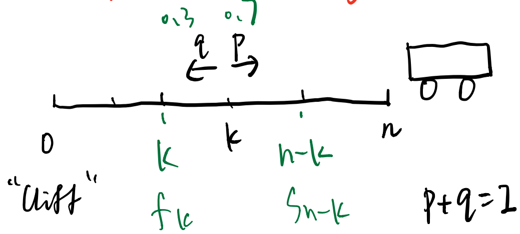
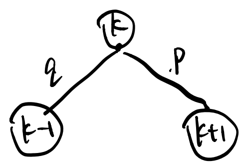
Last time

Random walk with absorbing barriers



S_k = probability of reaching n if we start with k

$S_0 = 0, \; S_n = 1$



$$S_k = pS_{k+1} + qS_{k-1}, \quad k = 1, 2, \cdots n-1$$

Symmetric Case $(p = q = \frac{1}{2})$ $\quad f_k + S_k = 1, \; S_k = \frac{k}{n}, \; f_k = \frac{n-k}{n}.$

$S_k = \frac{k}{n}$,

let f_k be the prob to reach 0

By symmetry, $f_k = S_{n-k} = \frac{n-k}{n}$

$S_k + f_k = 1 \Rightarrow$ the prob of infinite walk $= 0$

If $n \geq 4$ there're uncountably many infinite walks,

If $n = \infty$, then: $S_k = 0, \; f_k = 1$ (Gamble time $\uparrow$, P(lose money) $= 1$

Symmetric Case: $p=q=\frac{1}{2}$

ex. Suppose there the gambler comes with $k=\$99$ and decides to leave when (and if) he has $\$100 = n$.

Then he succeeds with $p=0.99$ $\quad S_{99}=\frac{99}{100}=0.99$

what happens if the gambler repeats it many times.

$$\begin{tikzcd}
& S_k \arrow[dl, "q"'] \arrow[dr, "p"] & \\
S_{k-1} & & S_{k+1}
\end{tikzcd}$$

Non-symmetric (case $p\neq q$)

$S_k = pS_{k+1}+qS_{k-1}$

$S_0=0,\ S_n=1$

Soln: $pS_k+qS_{k+1}=pS_{k+1}+qS_{k-1}$

$\therefore\ p(S_{k+1}-S_k)=q(S_k-S_{k-1})$

$\therefore\ S_{k+1}-S_k=\frac{q}{p}(S_k-S_{k-1})$

$S_2-S_1=\frac{q}{p}(S_1-S_0)$

$S_3-S_2=\frac{q}{p}(S_2-S_1)$

$S_4-S_3=\frac{q}{p}(S_3-S_2)$

$\vdots$

$S_k-S_1=\left(\frac{q}{p}\right)^{k-1}(S_1-S_0)$

$S_k=(S_k-S_{k-1})+(S_{k-1}-S_{k-2})+\cdots(S_1-S_0)+S_0$

$=\left(\frac{q}{p}\right)^{k-1}x+\left(\frac{q}{p}\right)^{k-2}x+\cdots 1\times x$

$=x\left(1+\frac{q}{p}+\cdots\left(\frac{q}{p}\right)^{k-1}\right)$

$S=\dfrac{\left[1-\left(\frac{q}{p}\right)^k\right]}{1-\frac{q}{p}}$

$S_k=qS_{k-1}+pS_{k+1}$

$p+q\quad pS_k+qS_k=qS_{k-1}+pS_{k+1}$

$q(S_k-S_{k-1})=p(S_{k+1}-S_k)$

$S_{k+1}-S_k=\frac{q}{p}(S_k-S_{k-1})$

$S_k=(S_k-S_{k-1})+\cdots\underset{}{\overset{x}{(S_1-S_0)}}+\underset{0}{S_0}$

$=x+\frac{q}{p}x+\cdots\left(\frac{q}{p}\right)^{k-1}x$

$=x\left[1+\frac{q}{p}+\cdots\left(\frac{q}{p}\right)^{k-1}\right]$

$=x\left(\dfrac{1-\left(\frac{q}{p}\right)^k}{1-\frac{q}{p}}\right)$

$S_n=x\left(\dfrac{1-\left(\frac{q}{p}\right)^n}{1-\frac{q}{p}}\right)=1$

$x=\dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^n}$

$$= x\left(\frac{\left(\frac{q}{p}\right)^k - 1}{\frac{q}{p} - 1}\right)$$

$$S_k = \frac{1-\cancel{\frac{q}{p}}}{1-\left(\frac{q}{p}\right)^n} \cdot \frac{1-\left(\frac{q}{p}\right)^k}{1-\cancel{\frac{q}{p}}}$$

$$S_n = x\left(\frac{\left(\frac{q}{p}\right)^n - 1}{\frac{q}{p} - 1}\right) = 1 \Rightarrow x = \frac{\frac{q}{p} - 1}{\left(\frac{q}{p}\right)^n - 1}$$

$$= \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^n}$$

$$S_k = \left(\frac{\frac{q}{p}-1}{\left(\frac{q}{p}\right)^n - 1}\right)\left(\frac{\left(\frac{q}{p}\right)^k - 1}{\frac{q}{p} - 1}\right) = \left(\frac{\left(\frac{q}{p}\right)^k - 1}{\left(\frac{q}{p}\right)^n - 1}\right)$$

$S_k + f_k$

$$= \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(p/q)^{n-k} - 1}{(p/q)^n - 1}$$

$$= \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(q/p)^k - (q/p)^n}{1-(q/p)^n}$$

$$= \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(q/p)^n - \cancel{(q/p)^k}}{(q/p)^n - 1}$$

$= 1$ ✓

$$\boxed{\begin{array}{l}\text{By Symmetry,} \\ f_k = S_{n-k} \text{ (with } p \text{ and } q \text{ swapped)}\end{array}}$$

$$f_k = \frac{(p/q)^{n-k} - 1}{(p/q)^n - 1}$$

$f_k = S_{n-k}$ (with p,q swapped)

$$S_{n-k} = \frac{\left(\frac{p}{q}\right)^{n-k} - 1}{\left(\frac{p}{q}\right)^n - 1} = f_k$$

$$\therefore S_k + f_k = \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(p/q)^{n-k} - 1}{(p/q)^n - 1}$$

$f_k + S_k$

$$= \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(p/q)^{n-k} - 1}{(p/q)^n - 1} = \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{\overset{(q/p)^k}{\cancel{(p/q)^k}} - (q/p)^n}{1-(q/p)^n}$$

$$= \frac{\left(\cancel{\frac{p}{q}}\right)^{n-k} - \left(\frac{p}{q}\right)^n}{1-\left(\frac{p}{q}\right)^n} + \frac{1-\left(\cancel{\frac{p}{q}}\right)^{n-k}}{1-\left(\frac{p}{q}\right)^n} = \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(p/q)^k - (q/p)^n}{1-(q/p)^n}$$

$f_k = S_{n-k}$

$= 1$

$$= \frac{(q/p)^k - 1}{(q/p)^n - 1} + \frac{(q/p)^n - (q/p)^k}{(q/p)^n - 1} = \frac{(q/p)^n - 1}{(q/p)^n - 1} = 1$$

Where if $p = \frac{2}{3}$, $q = \frac{1}{3}$,

$$\xleftarrow{1/3} \quad \xrightarrow{2/3}$$

$k = 1$, $n = \infty$.

$$S_1 = \frac{(q/p)^1 - 1}{(q/p)^\infty - 1} = \frac{(\frac{1}{2}) - 1}{(\frac{1}{2})^\infty - 1} = \frac{1}{2}$$ ← with prob $\frac{1}{2}$ the walk off to infinity

Next Week Homework:

Suppose a gambler comes with $\$k$ and decides to leave with $\$10$.

The gambler has the options of playing $\$1$ at each turn or 1 cent (1c) at each turn. Which is to choose?

Symmetric Case:

$\$1$: $S_k = \frac{k}{n}$

$1c$: $S_{100k} = \frac{100k}{100n} = \frac{k}{n}$

No difference.

Homework: $p > q$, 1c option is better.

$p < q$, $\$1$ option is better.

Random Variables:

$X: \Omega \to R$

$\uparrow$

probability space

For any $a \in R$, the set $\{w \in \Omega : X(w) \leq a\}$ is an event:

X is a random variable.

Self-test exercises:

(1) Show that if X is a r.v., then:

① For all $a \in R$, $\{w \in \Omega : X(w) < a\}$ is an event.

② For all $a, b \in R$

$\{w \in \Omega : b < X(w) < a\}$ is an event.

More generally, for any Borel set $A \subset R$, $\{w \in \Omega : X(w) \subset A\}$ is an event.

(2) Show that if $X, Y : \Omega \to R$ are r.v., then $X+Y$, $X-Y$, XY are also r.v.

$(X+Y)(w)$

$= X(w) + Y(w)$

(3) If $X_n : \Omega \to R$ are r.v. for $n = 1, 2, \ldots$

and $X(w) = \lim_{n \to \infty} X_n(w)$

then $X : \Omega \to R$ is a r.v.

Important example:

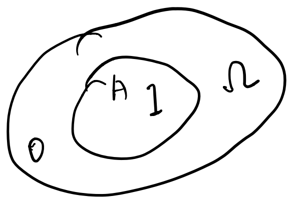
$\Omega \in$ probability space

$A \subset \Omega$ is an event

$I_A : \Omega \to R$

$$I_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{otherwise.} \end{cases}$$

indicator of A



Cumulative Distribution ftns:

$X : \Omega \to \mathbb{R}$ is a r.v.

$F_X : \mathbb{R} \to [0,1]$.

$F_X(a) = P(w \in \Omega : X(w) \le a)$

ex. toss a coin 3 times

$\Omega = \{HHH, HHT, HTH, HTT,$
$THH, THT, TTH, TTT\}$

X : is the # of heads.

$X(HHH) = 3.$