$X_{[a,b]}: \Omega \to \mathbb{R}$

① $X_{[a,b]}$ is Poisson with $E X_{[a,b]} = \lambda(b-a)$

$$P(X_{[a,b]} = k) = \frac{[\lambda(b-a)]^k}{k!} e^{-\lambda(b-a)}, \quad k = 0,1,2,\dots$$

② If $[a_1,b_1], [a_2,b_2], \dots [a_n,b_n]$ are pairwise & non-overlapping,

then $X_{[a_1,b_1]}, X_{[a_2,b_2]}, \dots X_{[a_n,b_n]}$ are indep.

③ If $0 \le a \le c < b$

$X_{[a,c]} + X_{[c,b]} = X_{[a,b]}$

$E X_{[0,24]} = 24\lambda = 3 \Rightarrow \lambda = \frac{1}{8}$

ex. Accidents on W str (3 accid are every day).

Accidents on Main (4 acci are every day)

Q: Start observation at $t=0$, prob that we first see the acci on Main St and then on W?

(Assume Independent Poisson Processes)

Solu: let W be the time till the first accident on W.

M ——— on Main.

$P(M < W) = ?$ $\quad \frac{4}{7}$

Densities of M and W.

Density of M:

let $X_{[0,t]}$ be the # of accidents on main between time of 0 and t.

$$P(X_{[0,t]} = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0,1,\dots$$

$E X_{[0,24]} = 4 = \lambda \times 24 \Rightarrow \lambda = \frac{1}{6}, \Rightarrow P(X_{[0,t]} = k) = \frac{(t/6)^k}{k!} e^{-t/6}$

$t>0$,

$F_M(t) = P(M \le t) = 1 - P(M>t)$

$= 1 - P(X_{[0,t]} = 0)$

$= 1 - e^{-t/6}$

$f_M(t) = F_M'(t) = \begin{cases} \frac{1}{6} e^{-t/6} & , t>0 \\ 0 & , o.w. \end{cases}$

$F_M(t) = P(M \le t)$

$= 1 - P(M>t)$

$= 1 - P(X_{[0,t]} = 0)$

$= 1 - e^{-t/6}$ $\boxed{t>0}$

$f_M(t) = \begin{cases} \frac{1}{6} e^{-t/6} & t>0 \\ 0 & o.w. \end{cases}$

Density of W:

Let $Y_{[0,t]}$ be the # of accidents on W between 0 and t.

$P(Y_{[0,t]} = k) = \frac{(\mu t)^k}{k!} e^{-\mu t}$, $k = 0,1,2,\ldots$

$E\, Y_{[0,24]} = 24\mu = 3$, $\Rightarrow \mu = \frac{1}{8}$

$P(Y_{[0,t]} = k) = \frac{(t/8)^k}{k!} e^{-t/8}$, $k = 0,1,2,\ldots$

$t>0$,

$F_W(t) = P(W \le t)$

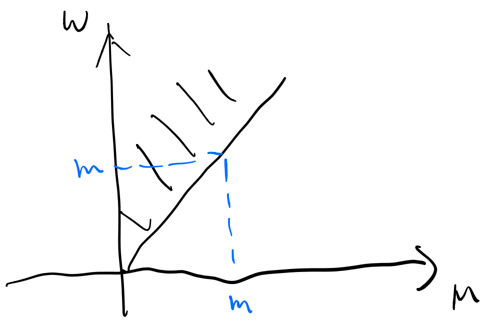
$= 1 - P(W>t)$

$= 1 - P(Y_{[0,t]} = 0)$

$= 1 - e^{-\mu t}$

$= 1 - e^{-t/8}$

$f_W(t) = F_W'(t) = \begin{cases} \frac{1}{8} e^{-t/8} & t>0 \\ 0 & o.w \end{cases}$



Joint density (independence):

$$f_{M,W}(x,y) = f_M(m)\, f_W(w)$$

joint density ftn

$$= \begin{cases} \frac{1}{6} e^{-m/6} \times \frac{1}{8} e^{-w/8} & , m, w > 0 \\ 0. & \text{o.w.} \end{cases}$$

$$P(W > M) = \iint_{w > m} \frac{1}{6} e^{-m/6} \times \frac{1}{8} e^{-w/8}\, dm\, dw$$

Calculus

$$= \int_0^{+\infty} \frac{1}{6} e^{-m/6} \left(\int_m^{+\infty} \frac{1}{8} e^{-w/8}\, dw \right) dm$$

$$= \int_0^{+\infty} \frac{1}{6} e^{-m/6} \left(-e^{-w/8} \Big|_{w=m}^{+\infty} \right)$$

$$= \int_0^{+\infty} \frac{1}{6} e^{-m/6} \times e^{-m/8}\, dm$$

$$= \frac{1}{6} \int_0^{+\infty} e^{\frac{-7m}{24}}\, dm$$

$$= \frac{4}{7}$$

Q: Expected time till the first accident? (On main or W)

$T = \min(M, W)$

Density of T? $(t>0)$

$$F_T(t) = P\{T \le t\}$$
$$= 1 - P\{T > t\}$$
$$= 1 - P\{M > t \text{ and } W > t\}$$
independence
$$\overset{\downarrow}{=} 1 - P\{M > t\}\, P(W > t)$$
$$= 1 - P(X_{[0,t]} = 0)\, P(Y_{[0,t]} = 0)$$
$$= 1 - e^{-\lambda t} e^{-\mu t}$$
$$= 1 - e^{-(\lambda+\mu)t}$$

$$F_T(t) = P(T \le t)$$
$$= 1 - P(T > t)$$
$$= 1 - P(M > t, W > t)$$
$$= 1 - P(M > t)\, P(W > t)$$
$$= 1 - P(X_{[0,t]} = 0)\, P(Y_{[0,t]} = 0)$$
$$= 1 - e^{-\lambda t} e^{-\mu t}$$
$$= 1 - e^{-(\lambda+\mu)t}$$
$$f_T(t) = e^{-(\lambda+\mu)t}(\lambda+\mu) \quad t>0$$

$$f_T(t) = F_T'(t) = \begin{cases} (\lambda+\mu)\, e^{-(\lambda+\mu)t}, & t>0 \\ 0, & t \le 0 \end{cases}$$

$$= \begin{cases} \frac{7}{24} e^{-\frac{7}{24}t} & t>0 \\ 0 & \text{o.w.} \end{cases}$$

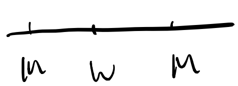
$$E(T) = \int_0^{+\infty} t\, \frac{7}{24} e^{-\frac{7}{24}t}\, dt = \frac{24}{7}$$

Observation: The min of independent exponential r.v. is also exponential.

Q: the prob of the following event:

Accident on Main, $\underbrace{}$ Accidents on W

Second Accident on Main? $P(M_1 < W < M_2)$



M_1: ← waiting time till the first accident on Main

W: ← - - - - - - - - - - - - - - - - - - - W

M_2: ← - - - - - - - - - - second - - - - - Main

"Z" ← time between the first and second accident on Main

$P(M_1 < W < M_1 + Z)$ M W $\frac{4}{7} \times \frac{3}{7} = \frac{12}{49}$

For poisson process, M_1, W, Z are indep. (Memoryless Property)

Z has the same density as M_1.

$f_{M_1}(m_1) = \frac{1}{6} e^{-m_1/6}, \quad m_1 > 0$

$f_W(w) = \frac{1}{8} e^{-w/8}, \quad w > 0$

$f_Z(z) = \frac{1}{6} e^{-z/6}, \quad z > 0$

$$\iiint_{M_1 < W < M_1 + Z} f_{M_1}(m_1) f_W(w) f_Z(z) \, dm_1 \, dw \, dz$$

$$= \int_0^{+\infty} \frac{1}{6} e^{-m_1/6} \int_0^{+\infty} \frac{1}{6} e^{-z/6} \int_{m_1}^{m_1+z} \frac{1}{8} e^{-w/8} \, dw \, dz \, dm_1$$

$$= \underbrace{\frac{4}{7}}_{\text{prob that first acci happens on Main}} \times \underbrace{\frac{3}{7}}_{\text{prob that after that the first happens on W.}} = \frac{12}{49}$$