Last time:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Assign: Read Sec 1.7

$$P = \frac{\binom{13}{3}\cdot\binom{39}{2}}{\binom{52}{5}}$$

e.g. Poker hand 5 cards out of 52.

(a) Probability that a random poker hand contains exactly 3 spades.

A: $\frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} \simeq 0.08$

\# 3 spades → $\binom{13}{3}$

$\binom{39}{2}$ ← # 2 non-spades

$\binom{52}{5}$ ← # all poker hands

$$P = \frac{\binom{13}{3} \times \binom{39}{2}}{\binom{52}{5}}$$

(b) Two pair

$\{a, a, b, b, c\}$

a, b, c are distinct values.

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \times 11 \times 4}{\binom{52}{5}}$$

\# ways to choose $\{a, b\}$ → $\binom{13}{2}$

\# ways to choose two cards of value a and b → $\binom{4}{2}\binom{4}{2}$

\# to choose C → 11

\# ways to choose the card of value c → 4

$$P = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2} \times 11 \times 4}{\binom{52}{5}}$$

4 ← # colors of c.

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \times 11 \times 4}{\binom{52}{5}}$$

$$\frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

(c) Choose: $\{a, a, a, b, b\}$

$$\frac{13 \times 12 \times \binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \simeq 0.0044$$

choose the value of A → 13

choose the value of b → 12

choose cards of value a → $\binom{4}{3}$

choose cards of value b → $\binom{4}{2}$

$$P = \frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

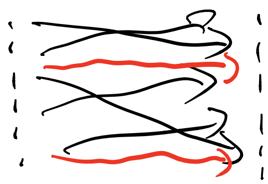
$$\frac{13 \times 12 \times \binom{4}{3}\binom{4}{2}}{\binom{52}{}}$$

The Problems of letters

Someone wrote n personalized letters to n different people. Addressed n envelops and put letters into the envelop at random. (Each envelop containing one letter) and mailed.

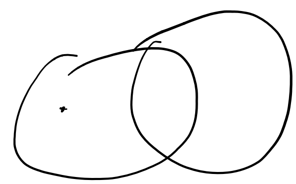
Q: Prob that at least someone got the intended letter?

Ω: probability space



n letters n people

$\overline{\quad\quad}$
$n!$



$|\Omega| = n!$

1st letters has n possibilities

2nd ——— $(n-1)$ ———

$\vdots$

n th ——— 1 ———

$P(\text{at least one correct}) = P(\bigcup_{i=1}^{n} A_i)$

$= \sum_{i=1}^{n} P(A_i) - \sum_{1\le i_1 \le i_2 \le n} P(A_{i_1} \cap A_{i_2})$

$+ \cdots (-1)^{k-1} \sum_{1\le i_1 \le i_2 \cdots i_k} P(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k})$

$P(\bigcup_{i=1}^{n} A_i)$

$= \sum_{i=1}^{n} P(A_i) - \sum_{1\le i\le j\le n} P(A_i \cap A_j)$

first term: $n \times \frac{(n-1)!}{n!} = 1$

Second: $\frac{n!}{2!(n-2)!} \times \frac{(n-2)!}{n!} = \frac{1}{2!}$

$A_i \subset \Omega$: the ith letter goes to the right (ith) person.

$\therefore P = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots$

We want:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i_1 \le i_2 \le n} P(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^{k-1} \sum_{1\le i_1 \le i_2 \cdots i_k \le n} P(A_{i_1} \cap \cdots A_{i_k})$$

$$P(A_i) = \frac{(n-1)!}{n!}$$

$$P(A_{i_1} \cap A_{i_2}) = \frac{(n-2)!}{n!}$$

$\vdots$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots A_{i_k}) = \frac{(n-k)!}{n!}$$

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$x = -1$, then:

$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$\Downarrow$

$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = 1 - e^{-1} \simeq 0.63$

$$P(A_{i_1} \cap A_{i_2} \cdots A_{in}) = \frac{0!}{n!} = \frac{1}{n!} \quad \text{(only one way to mail all envelopes correctly)}$$

$$\therefore P\Big(\bigcup_{i=1}^{n} A_i\Big) = n \times \frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \cdots (-1)^{k-1}\binom{n}{k}\frac{(n-k)!}{n!} \cdots$$

$$+(-1)^{n-1}\, 1 \, \frac{1}{n!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad = 1 - \frac{1}{2!} + \cdots + (-1)^{k-1}\frac{1}{k!} + \cdots (-1)^{n-1}\frac{1}{n!} \approx 1 - \frac{1}{e} \approx 0.63$$

for large n.

$$A: \ 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}$$

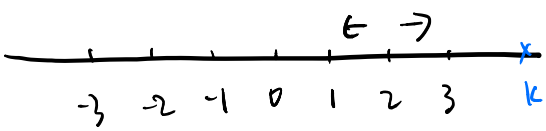
Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^k}{k!} + \cdots$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

Random walks ("Discrete Brown Motion")

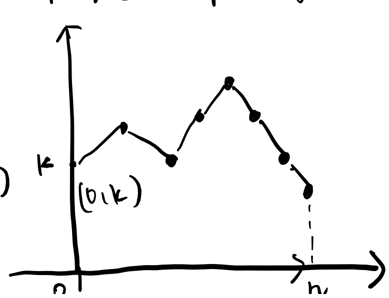


Moving either to the left or to the right at random (one unit at unit time)

Gambler playing compulsively, betting \$1 at a time.

trajectory:

Start at $(0,k)$



Q: How many trajectories start at $(0,k)$ end at (n,t)

||

How many walks of n steps start at k end at t?

up : # steps up

down : # steps down

$\text{up} + \text{down} = n$

$\text{up} - \text{down} = k - t$

$$\left.\begin{aligned} \text{up} + \text{down} &= n \\ \text{up} - \text{down} &= t - k \end{aligned}\right\} \rightarrow \begin{aligned} \text{up} &= \frac{n+t-k}{2} \\ \text{down} &= \frac{n-t+k}{2} \end{aligned}$$

$\text{up} = \frac{n-k+t}{2}$, $\text{down} = \frac{n+k-t}{2}$

A. $\binom{n}{\frac{n+t-k}{2}}$

$\binom{n}{\frac{n-k+t}{2}}$

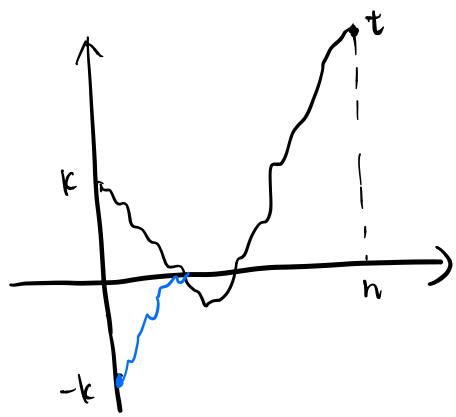
If $\frac{n+t-k}{2}$, $\frac{n+t-k}{2}$ are not integers, $\rightarrow$ no trajectory (e.g. $(0,1) \rightarrow (1,1)$)

X trajectory

Reflection Principle

Suppose that k and t are positive int.

trajectories that start at $(0,k)$ and end at (n,t) and visit the x-axis
(cross or touch)



= # trajectories going from $(0,-k)$ to (n,t)

Why? Pick any trajectory from $(-k,0)$ to (n,t) reflect in the x-axis its interval till the first intersection.

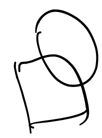
$A_1 \subset A_2 \subset \cdots$

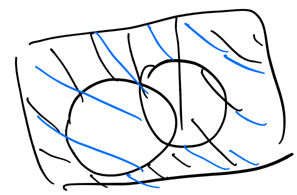
$B_1 = A_1,\ B_2 = A_2 - A_1 \cdots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) = P(A_1) + P(A_2 - A_1) + P(A_3 - A_2) \cdots$$

$$= P(A_1) + P(A_2) - P(A_1) + P(A_3) - P(A_2) + \cdots$$

$$= \lim_{n\to\infty} P(A_n)$$





$A_1 \supset A_2 \supset A_3 \cdots$

$A_1^c \subset A_2^c \subset A_3^c \cdots \to \lim_{n\to\infty} P(A_i^c) = P\left(\bigcup_{i=1}^{\infty} A_i^c\right)$

$\lim_{n\to\infty} P(A_i) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$

$$\underbrace{1 - \lim_{n\to\infty} P(A_i^c)} = 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right)$$

$$\lim_{n\to\infty} P(A_i) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$$