Independent Random Variables

$X_i : \Omega \to \mathbb{R},\ i \in I$
$A_i = \{w \in \Omega : X_i(w) \le a_i\}$

$X_i : \Omega \to \mathbb{R}$, $i \in I$ are r.v.s.

We say they're indep if for any real $a_i : i \in I$, the events $A_i = \{w \in \Omega : X_i(w) \le a_i\}$ are indep.

intuition: X_i take values "independently".

We don't prove (hard self-test exercise).

But we'll use: For any Borel sets $B_i \subset \mathbb{R}$, the events $\{w \in \Omega : X_i(w) \in B_i\}$ are indep.

We don't prove, but we'll use (self-test exercise).

If each X_i takes countably many (finitely many) values, then X_i are indep.

$\Leftrightarrow$ For any $a_i \in \mathbb{R}$, the events $A_i = \{w \in \Omega : X_i(w) = a_i\}$ are indep.

$\not\Leftarrow$ Not true if X_i take uncountably many values.

ex. I roll 2 fair dice $\Omega = \{(i,j) : \begin{matrix} 1 \le i \le 6 \\ 1 \le j \le 6 \end{matrix}\}$.

X_1 = the # shown by first die

X_2 = the # shown by the second die

$X_1(i,j) = i$, $X_2(i,j) = j \leftarrow X_1, X_2$ are indep.

Generally, we can view indep r.vs as each acting on its own coordinate:

Product of prob spaces:

let $\Omega_i : i \in I$ be a family of prob spaces.

$\Omega = \prod_{i \in I} \Omega_i$

Outcomes: $w = (w_i : w_i \in \Omega_i$ for all $i \in I)$

Events: "Cylindrical Events"

Pick events: $A_{i_1} \subset \Omega_{i_1}, A_{i_2} \subset \Omega_{i_2}, \cdots A_{i_n} \subset \Omega_{i_n}$,

Consider: $A_{i_1} \subset \Omega_{i_1}, A_{i_2} \subset \Omega_{i_2}, \cdots A_{i_n} \subset \Omega_{i_n}$

$$A = \{ w = (w_i) : \begin{array}{l} w_{i_1} \in A_{i_1} \\ w_{i_2} \in A_{i_2} \\ \vdots \\ w_{i_n} \in A_{i_n} \end{array} \}$$

$$A = \{ w = (w_i) : \begin{array}{l} w_{i_1} \subset A_{i_1} \\ \vdots \\ w_{i_n} \subset A_{i_n} \end{array} \}$$

$$P(A) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_n})$$

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Add all events obtained from cylindrical events by repeated operations of countable unions, intersections complement, etc.

Hard self-test exercise: It works, you get Ω as a prob space.

A way to view indep r.vs: $X_i : \Omega \to \mathbb{R}$

$X_i(w) = X_i(w_i)$ ← depends only on the ith coordinate.

Important:

If $X_1, \cdots X_n : \Omega \to \mathbb{R}$ are indep, then $E(X_1 X_2 \cdots X_n) = E(X_1) \cdots E(X_n)$.

Meaning that if $E(X_1), E(X_2), \cdots E(X_n)$ exist, then: $E(X_1 \cdots X_n)$ exists and.

Why?

Check for X_1, X_2.

If X_1, X_2 are simple, then X_1 takes values $a_1 \cdots a_n$; X_2 takes values $b_1 \cdots b_n$.

let $A_i = \{ w \in \Omega : X_1(w) = a_i \}$ $\quad A_i = \{ w \in \Omega : X_1(w) = a_i \}$

$B_j = \{ w \in \Omega : X_2(w) = b_j \}$ $\quad B_j = \{ w \in \Omega : X_2(w) = b_j \}$

$X_1 = \sum_{i=1}^n a_i 1_{A_i}$, $X_2 = \sum_{j=1}^m b_j 1_{B_j}$

All pairs A_i, B_j are indep. then:

$$E(X_1 X_2) = E\Big(\sum_{i=1}^n a_i 1_{A_i} \sum_{j=1}^m b_j 1_{B_j}\Big)$$

$$= E\Big(\sum_{\substack{i=1\ldots n\\ j=1\ldots m}} a_i b_j 1_{A_i} 1_{B_j}\Big)$$

$$= \sum_{\substack{i=1\ldots n\\ j=1\ldots n}} a_i b_j E(1_{A_i \cap B_j})$$

$$= \sum_{\substack{i=1\ldots n\\ j=1\ldots n}} a_i b_j P(A_i \cap B_j)$$

$\downarrow$ A_i, B_j indep

$$= \sum_{\substack{i=1\ldots n\\ j=1\ldots n}} a_i b_j P(A_i) P(B_j)$$

$$= \Big(\sum_{i=1}^n a_i P(A_i)\Big)\Big(\sum_{j=1}^m b_j P(B_j)\Big)$$

$$= E(X) E(Y)$$

$X_1 = \sum_{i=1}^n a_i 1_{A_i}$ $\qquad 1_{A_i \cap B_j} = \begin{cases} 1, & \omega \in A_i \cap B_j \\ 0 & \ldots \end{cases}$

$X_2 = \sum_{j=1}^m b_j 1_{B_j}$ $\qquad E(\cdot) = P(\omega \ldots)$

$$E(X_1 X_2) = E\Big(\sum_{i=1}^n a_i 1_{A_i} \sum_{j=1}^m b_j 1_{B_j}\Big)$$

$$= \sum_{i=1}^n \sum_{j=1}^m a_i b_j E(1_{A_i} 1_{B_j})$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i b_j E(1_{A_i \cap B_j})$$

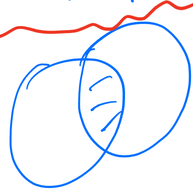
$$= \sum_{i=1}^n \sum_{j=1}^n a_i b_j P(A_i \cap B_j)$$

$$= \sum_{i=1}^n \sum_{j=1}^m a_i b_j P(A_i) P(B_j)$$

$$= \underbrace{\sum_{i=1}^n a_i P(A_i)}_{E(X_1)} \underbrace{\sum_{j=1}^n b_j P(B_j)}_{E(X_2)}$$

In general, take limit (self-test exercise)

Briefly, if $P(A \cap B) = P(A) P(B)$, then events A, B are indep.



A_i - - - - - - iff: $P(\bigcap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$

In particular, events A and B are indep iff: $P(A \cap B) = P(A) \times P(B)$

Equivalently, vars X and Y are indep iff $P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$

Random Vectors

$X_1, \cdots X_n : \Omega \to R$ r.v

$X = (X_1, \cdots X_n)$ as a random vector.

$F_X(a_1 \cdots a_n) = P(X_1 \le a_1, X_2 \le a_2, \cdots X_n \le a_n)$

$= P((X_1 \le a_1) \cap (X_2 \le a_2) \cdots \cap (X_n \le a_n))$

$= F_{X_1}(a_1) \cdots$

Cumulative distribution ftn:

$$F_X : R^n \to [0,1]$$

$$F_X(a_1, \cdots a_n) = P(X_1 \le a_1, X_2 \le a_2, \cdots X_n \le a_n)$$

$$= P((X_1 \le a_1) \cap (X_2 \le a_2) \cap \cdots (X_n \le a_n))$$

$X_1, X_2, \cdots X_n$ are indep $\Leftrightarrow F_X(a_1 \cdots a_n) = F_{X_1}(a_1) \cdots F_{X_n}(a_n)$

A random vector $X = (X_1, X_2, \cdots X_n)$ is cts if there is a ftn: $f(X_1, X_2, \cdots X_n)$, $R_n \to R^+$ (non-negative) called joint density, integrable, and for any Borel $B \subset R^n$, $P(X \in B) = \int_B f(x_1, \cdots x_n)\, dx_1 \cdots dx_n$.

Self-test exercise: $X_1, \cdots X_n$ are indep $\Leftrightarrow f_X(x_1, \cdots x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

doable

$\Rightarrow$: Since $X_1 \cdots X_n$ are indep, we know $F_X(a_1 \cdots a_n) = F_{X_1}(a_1) \cdots F_{X_n}(a_n)$

(partial derivative) $\dfrac{\partial^n LHS}{\partial a_1 \partial a_2 \cdots \partial a_n} = f_X(a_1, a_2, \cdots a_n)$; $\dfrac{\partial^n RHS}{\partial a_1 \partial a_2 \cdots \partial a_n} = f_{X_1}(a_1) \cdots f_{X_n}(a_n)$

Since $LHS = RHS$, $\dfrac{\partial^n LHS}{\partial a_1 \cdots \partial a_n} = \dfrac{\partial^n RHS}{\partial a_1 \cdots \partial a_n}$, $\Rightarrow f_X(a_1, a_2, \cdots a_n) = f_{X_1}(a_1) \cdots f_{X_n}(a_n)$

Important:

If X is a random n-vector, and $g : R^n \to R$ is Borel measurable.

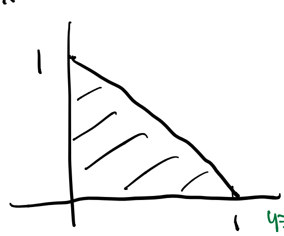
Then $g(X)$ is a random variable if X is cts with joint density $f_X(x_1, \cdots x_n)$, then $E\, g(X) = \int_{R^n} g(x) f_X(x)\, dx$,

$f_X(x_1 \cdots x_n)$

$E\, g(x)$

$= \int g(x) f_X(x)\, dx$

ex.



(X, Y) is uniform in the triangle.

$$f(x,y) = \begin{cases} c & \text{if } (x,y) \in \triangle \\ 0 & \text{o.w} \end{cases}$$

① What is c? $y=-x+1$

We must have: $1=\int_\Delta c\,dx\,dy \Rightarrow c=2$

② What is $E(XY)$

$$\int_\Delta xy\times 2\,dx\,dy = 2\int_0^1 x\left(\int_0^{1-x} y\,dy\right)dx$$

$$=2\int_0^1 x\,\frac{(1-x)^2}{2}\,dx$$

$$=\int_0^1 x(1-x)^2\,dx$$

$$=\frac{1}{12}$$

Moral: Multivariate Calculus.

③ $E(X^2+Y^2)=\int_\Delta (x^2+y^2)\times 2\,dx\,dy$

④ $E(\sin(X+Y))=\int_\Delta \sin(x+y)\,2\,dx\,dy$

$$1=\int_\Delta c\,dx\,dy = c\cdot\frac{1}{2},\quad \underline{c=2}$$

$$\int_0^1 c\int_0^{-x+1} \cancel{c}\,dy\,dx = 1.$$

$$\int_0^1 c(-x+1)\,dx = 1$$

$$\int_0^1 -cx+c\,dx=1$$

原: $-\frac{c}{2}x^2+cx\Big|_0^1 = \frac{-c}{2}+c=\frac{c}{2}=1$

$\boxed{c=2}$

$$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,f(x,y)\,dy\,dx$$

$2y$ 原: $y^2\Big|_0^{-x+1}$

$$=\int_0^1 x\int_0^{-x+1} y\cdot 2\,dy\,dx$$

$$=\int_0^1 x(-x+1)^2\,dx$$

$x-2x^2+x^3$ 原:

$$=\int_0^1 x(1-x)^2\,dx$$

$\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4\Big|_0^1$

$$=\int_0^1 x(1-2x+x^2)\,dx$$

$=\frac{1}{2}-\frac{2}{3}+\frac{1}{4}$

$$=\int_0^1 x-2x^2+x^3\,dx$$

$=\frac{6-8+3}{12}=\frac{1}{12}$

$$=\frac{1}{12}$$

$$E(X)=\int_{-\infty}^{+\infty} x\underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}_{f_X(x)}\,dx$$

$$=\underline{E(X)}$$

Poisson Random Variable

Emergence: Suppose that we perform $\underset{\sim}{n}$ indep experiments, each success with prob p.

many but discrete

Let X be the # of successes.

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

What happens if $n \to \infty$, $p \to 0$ and $np \to \lambda > 0$.

$$P(X=k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

p to the k

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \times \cdots \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \left(1-\frac{\lambda}{n}\right)^{-k}$$

k is fixed, $n \to \infty$, then:

$$LHS = 1 \times \cdots 1 \times \frac{1}{k!} \lambda^k e^{-\lambda} \times 1 = \frac{\lambda^k}{k!} e^{-\lambda}$$

$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ $k = 0, 1, 2, \cdots$

We say that X is Poisson with para $\lambda > 0$ if $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, \cdots$

Intuition: X is the # of success in great many independent trials, where the prob of success in each trial is very small. But the expected # of success $\lambda > 0$ is reasonable.

Check:

$$\sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$x=0$.

$$E(X) = \sum_{k=0}^{\infty} k P(X=k) = \sum_{k=1}^{\infty} k P(X=k) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda} \cdot \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \boxed{\lambda}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a)$$

So $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!}$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

k: real value.

fix k. Compute LHS over n.

① take sup

② $\lim_{k \to \infty}$ + pattern for sequence