Characteristic Functions

Goal: To have smth as useful as pmb generating ftn, but applicable to any r.v. (Not just those that take non-nega integer values

$X: \Omega \to \mathbb{R}$ r.v. $\varphi_X(t) = E e^{itX}$ $i^2=-1$

$\varphi_X(t) = E e^{itX}$, $t \in \mathbb{R}$, where $i^2 = -1$

ex.

Q: $P(X=1) = \frac{1}{2}$, $P(X=-1) = \frac{1}{2}$

A: $\varphi_X(t) = E e^{itX}$

$= \frac{1}{2} e^{it} + \frac{1}{2} e^{-it}$

$= \frac{e^{it} + e^{-it}}{2}$

$= \cos t$

Moral

Review Complex #

Euler's Formula: $e^{it} = \cos t + i \sin t$ (Mod of Complex #: if $z = x + iy$, then $|z| = \sqrt{x^2+y^2}$

$|e^{it}| = |\cos t|^2 + |\sin t|^2 = 1$, $|e^{itx}| = |\cos t|^2 + |\sin t|^2 = 1$

$\varphi_X(t) = E e^{itX}$, so $|\varphi_X(t)| \le 1$ $|\varphi_X(t)| = |E e^{itX}| \le E|e^{itX}| = 1$

if X takes $0, 1, 2, \cdots$ then:

$G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k = E(s^X)$ $G_X(s) = \sum_{k=0}^{\infty} P(X=k) \cdot s^k = E(s^X)$

$\varphi_X(t) = E(e^{itX}) = G_X(e^{it})$ $\varphi_X(t) = E(e^{itX}) = G_X(e^{it})$

ex. If X is cts with density $f_X(x)$, $\varphi_X(t) = E e^{itX} = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$

$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$ ← Fourier transform

$f(x) = \begin{cases} e^{-x} & , x>0 \\ 0 & , \text{o.w.} \end{cases}$

For ex, if X is standard exponential with density: $\varphi(x) = E e^{itX} = \int_0^{\infty} e^{itx} e^{-x}\,dx$

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$= \int_0^\infty e^{(it-1)x}\,dx$$

$e^{it} - 1$

$$\varphi_X(t) = \int_0^\infty e^{itx} e^{-x}\,dx$$

$$= \int_0^\infty e^{(it-1)x}\,dx$$

$$= \frac{1}{it-1} e^{(it-1)x} \Big|_{x=0}^{x=\infty}$$

$$= -\frac{1}{it-1}$$

$$= \frac{1}{1-it}$$

$$= \left(\frac{e^{it}}{e}\right)^x \to 0.$$

$$\left(\frac{e^{it}}{e}\right)^x$$

$e^{it} = \cos t + i \sin t.$

$|e^{it}| = 1$

$|e| = e$

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$$

$$= \int_0^\infty e^{itx} e^{-x}\,dx = \int_0^\infty e^{(it-1)x}\,dx = \frac{1}{1-it}$$

$$\frac{1}{it-1} e^{(it-1)x}\Big|_0^\infty = \frac{-1}{it-1} e^{\cancel{it-1}}$$

Important

If $X_1, \dots, X_n$ are indep and $X = X_1 + \dots X_n$, then:

$$\varphi_X(t) = E e^{itX} = E e^{it(X_1 + \dots X_n)}$$

$$= E(e^{itX_1}) \times \dots E(e^{itX_n})$$

$$= \varphi_{X_1}(t) \times \dots \varphi_{X_n}(t)$$

$$\varphi_X(t) = E e^{itX} = E e^{it(X_1 + \dots X_n)} = E e^{itX_1} \times \dots E e^{itX_n}$$

$$= \varphi_{X_1}(t) \dots \times \varphi_{X_n}(t)$$

Some properties:

$\varphi_X(t) = E e^{itX}$

$\varphi_X(0) = 1$

$$\varphi_X'(t) = (E e^{itX})' = E(e^{itX} iX)$$

$$= E(e^{itX}(iX)^2)$$

$$\varphi_X(0) = E e^{i0X} = E(1) = 1$$

$$\varphi_X'(t) = (E e^{itX})' = E(iX e^{itX})$$

$$\varphi_X'(0) = E(iX) = iE(X)$$

legal if $E(X)$ exists

not true in general. that if $\varphi_X'(0)$ exists, then $E(X)$ exists.

$$\varphi_X''(t) = (E iX e^{itX})' = E[(iX)^2 e^{itX}]$$

$$\varphi_X''(0) = E(-X^2) = -E(X^2)$$ ← legal if $E(X^2)$ exists

Hand Self-test Exercise

If $\varphi_X''(0)$ exists, then $E(X)$, $E(X^2)$ exist.

Important. $\varphi_X(t)$ uniquely determines the cumulative distn ftn.

Theo: (P. Levy, 1887 ~ 1971) $\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\varphi_X(t)\,dt$

$= P(a<X<b) + \frac{1}{2}P(X=a) + \frac{1}{2}P(X=b)$

For any $a<b$,

$$\lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\varphi_X(t)\,dt = P(a<X<b) + \frac{1}{2}P(X=a) + \frac{1}{2}P(X=b)$$

fourier transform $\rightarrow$ $\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$

$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$

$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(y)\,dx$

If X has density, then:

$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\varphi_X(t)\,dt$ $\leftarrow$ inverse fourier transform

$-\infty \sim y$ $\quad y=-\infty$

$-y=+\infty$

$y=0$

$-y=0$ $\quad e^{(it-1)y}$

Pre-ex:

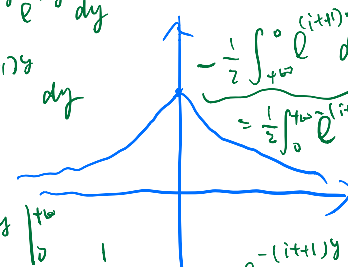
Let Y be "symmetric exponential" r.v. with density $\frac{1}{2}e^{-|Y|}$

$\varphi_Y(t) = \int_{0}^{+\infty} e^{ity}\frac{1}{2}e^{-|y|}\,dy + \frac{1}{2}\int_{-\infty}^{0} e^{ity}e^{y}\,dy$

$\varphi_Y(t) = \int_{-\infty}^{+\infty} e^{ity} \times \frac{1}{2}e^{-|y|}\,dy$

$= \frac{1}{2}\int_{0}^{+\infty} e^{ity}e^{-y}\,dy$

$= \frac{1}{2}\int_{0}^{\infty} e^{ity}e^{-y}\,dy + \frac{1}{2}\int_{-\infty}^{0} e^{ity}e^{y}\,dy$ $= \frac{1}{2}\int_{0}^{+\infty} e^{(it-1)y}\,dy$



$-\frac{1}{2}\int_{+\infty}^{0} e^{(it+1)\cdot(-y)}\,d(-y)$

$= \frac{1}{2}\int_{0}^{+\infty} e^{-(it+1)y}\,dy$

$= \frac{1}{2}\int_{0}^{\infty} e^{ity}e^{-y}\,dy + \frac{1}{2}\int_{0}^{\infty} e^{-ity}e^{-y}\,dy$

$e^{(it-1)y}\,y_0$: $\frac{1}{it-1}e^{(it-1)y}\Big|_0^{+\infty} = \frac{1}{1-it}$ $\qquad$ $-\frac{1}{(it+1)}e^{-(it+1)y}\Big|_0^{\infty} = \frac{1}{it+1}$

$= \frac{1}{2}\cdot\frac{1}{1-it} + \frac{1}{2}\cdot\frac{1}{1+it}$ $\qquad$ $\frac{1}{2}\frac{1}{1-it} + \frac{1}{2}\frac{1}{it+1}$

$= \frac{(1-it)+(1+it)}{2(1-it)(1+it)}$ $\qquad$ $= \frac{1}{2}\frac{it+1+1-it}{1+t^2} = \frac{1}{2}\cdot\frac{2}{1+t^2}$

$= \frac{1}{1+t^2}$

$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ity}\frac{1}{1+t^2}\,dt = \frac{1}{2}e^{-|y|}$

replace y by $-y$

$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-ity}\frac{1}{1+t^2}\,dt = e^{-|y|}$

ex. (Cauchy random variable)

If X has density : $\frac{1}{\pi}\frac{1}{1+x^2}$

$$\int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = e^0 = 1 \quad \Leftarrow \text{density}$$

Note that $E(X)$ does not exist

$Var(X)$ does not exist ($\Leftarrow \int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{x}{1+x^2}\,dx$ diverges)

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = e^{-|t|}$$

Let $X_1,\dots,X_n$ be indep Cauchy. Let $X = \frac{1}{n}(X_1+\cdots+X_n)$

Then X is Cauchy.

Why? (Proof)

$$\begin{aligned}
\varphi_X(t) &= E(e^{itX}) \\
&= E\left(e^{\frac{it(X_1+\cdots+X_n)}{n}}\right) \\
&= E\left(e^{it\frac{X_1}{n}}\times\cdots\, e^{it\frac{X_n}{n}}\right) \\
&= E\left(e^{\frac{itX_1}{n}}\right)\times\cdots E\left(e^{\frac{itX_n}{n}}\right) \\
&= \varphi_{X_1}\left(\tfrac{t}{n}\right)\times\cdots\varphi_{X_n}\left(\tfrac{t}{n}\right) \\
&= e^{-|\frac{t}{n}|}\times\cdots e^{-|\frac{t}{n}|} \\
&= e^{-|t|}
\end{aligned}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ity}\varphi_X(t)\,dt = f_X(x)$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ity}\frac{1}{1+t^2}\,dt = \frac{1}{2}e^{-|y|}$$

$$LHS = \frac{-1}{\pi}\int_{+\infty}^{-\infty} e^{iy(-t)}\frac{1}{1+(-t)^2}\,d(-t)$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{iyt}\frac{1}{1+t^2}\,dt = e^{-|y|}$$

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = e^{-|t|}$$

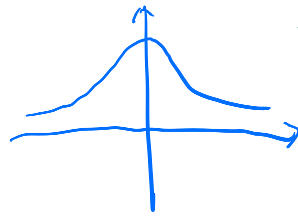
$$\frac{1}{\pi}\frac{1}{1+x^2}$$

$$X = \frac{1}{n}(X_1+\cdots X_n)$$

$$\begin{aligned}
\varphi_X(t) &= Ee^{itX} \\
&= Ee^{it\frac{1}{n}(X_1+\cdots X_n)} \\
&= Ee^{it\frac{1}{n}X_1}\cdots\times Ee^{it\frac{X_n}{n}} \\
&= \varphi_{X_1}\left(\tfrac{t}{n}\right)\times\cdots\varphi_{X_n}\left(\tfrac{t}{n}\right) \\
&= e^{-|\frac{t}{n}|}\times\cdots e^{-|\frac{t}{n}|} \\
&= e^{-|t|}
\end{aligned}$$

Standard Gaussian (Normal) random variable

X has density : $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



need to show : $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} dx = \sqrt{\frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2} dy\right)}$$

$$= \sqrt{\frac{1}{2\pi} \iint_{\text{plane}} e^{-(x^2+y^2)/2} dx\, dy}$$

$x = \tau \cos\varphi \qquad 0 \le \varphi \le 2\pi$

$y = \tau \sin\varphi \qquad 0 \le \tau < \infty$

$dx\,dy = \tau\, d\tau\, d\varphi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\varphi & -\tau\sin\varphi \\ \sin\varphi & \tau\cos\varphi \end{vmatrix}$$

$= \tau\cos^2\varphi + \tau\sin^2\varphi$

$= \tau$

$|J| = |\tau| = \tau$

$$= \sqrt{\frac{1}{2\pi}\left(\int_0^{2\pi} d\varphi\right)\left(\int_0^{\infty} e^{-\tau^2/2}\, \tau\, d\tau\right)}$$

$$= \sqrt{-e^{-\tau^2/2}\Big|_{\tau=0}^{\tau=\infty}}$$

$= \sqrt{1}$

$= 1$

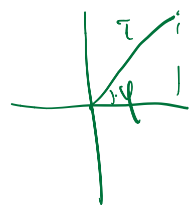
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1$$

$$= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-x^2/2} dx \int_{-\infty}^{+\infty} e^{-y^2/2} dy}$$

$$= \sqrt{\frac{1}{2\pi} \iint_{\text{plane}} e^{-(x^2+y^2)/2} dx\,dy}$$

$x = \tau\cos\varphi, \quad 0 \le \varphi \le 2\pi$

$y = \tau\sin\varphi, \quad 0 \le \tau < \infty$



So $dx\,dy = \tau\, d\tau\, d\varphi$

$$\sqrt{\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\infty} e^{-\tau^2/2}\, \tau\, d\tau}$$

$e^{-x} \to e^{-y}$

$$= \sqrt{\int_0^{\infty} e^{-\tau^2/2}\, d\tfrac{1}{2}\tau^2}$$

$$= \sqrt{-e^{-\tau^2/2}\Big|_{\tau=0}^{\infty}} = \sqrt{1} = 1$$

E(X) and Var(X) for normal RV

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-x^2/2} dx = e^{-x^2/2}\Big|_{x=-\infty}^{x=\infty} = 0$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x\, d(-e^{-x^2/2})$$

$$= \frac{1}{\sqrt{2\pi}} \left(-x e^{-x^2/2}\right)\Big|_{x=-\infty}^{x=\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} dx$$

$= 1$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x\, d(-e^{-x^2/2}) = ?$$

$$= \frac{1}{\sqrt{2\pi}} \left(-x e^{-x^2/2}\right)\Big|_{x=-\infty}^{x=\infty}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1$$

$\sqrt{2\pi}$ —

$$Var(X) = E(X^2) - E^2(X) = 1$$

$E(X) = 0$

$Var(X) = 1$

$\boxed{\text{Important}}$

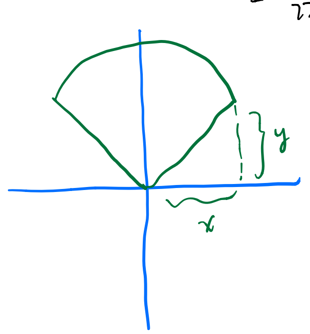
If X, Y are indep standard Gaussian,

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

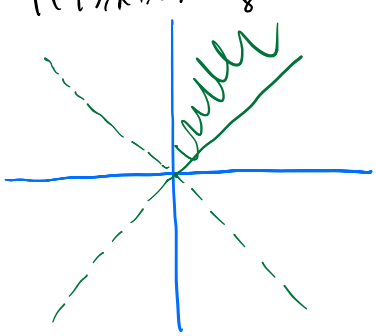
$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$



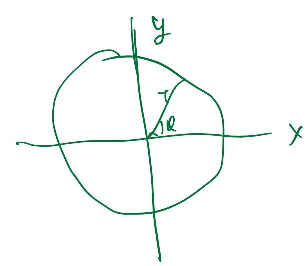
← density depends only on the distance to $(0,0)$ (rotationally invariant)

Ex. If X, Y are indep standard Gaussian, then:

$$P(Y \geq X \geq 0) = \frac{1}{8}$$



$P(Y \geq X \geq 0)$



Important: If X, Y are indep standard Gaussian, $Z = \frac{X^2+Y^2}{2}$

then Z is standard exponential with density:

$$f_Z(z) = \begin{cases} e^{-z} & , z > 0 \\ 0 & , z \le 0 \end{cases}$$

Why? $a > 0$

$$F_Z(a) = P(Z \le a)$$

$$= P\left(\frac{X^2+Y^2}{2} \le a\right)$$

$$= P(X^2+Y^2 \le 2a)$$

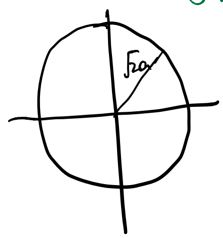
$$= \iint \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, dx\,dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\sqrt{2a}} e^{-\tau^2/2} \, \tau \, d\tau$$

$$= -e^{-\tau^2/2} \Big|_{\tau=0}^{\tau=\sqrt{2a}}$$

$$= 1 - e^{-a}$$

$$f_Z(a) = F_Z'(a) = \begin{cases} e^{-a} & a > 0 \\ 0 & a \le 0 \end{cases}$$



$x = \tau \cos\varphi$

$y = \tau \sin\varphi$

$dx\,dy = \tau \, d\tau \, d\varphi$

$$F_Z(a) = P(Z \le a)$$

$$= P(X^2+Y^2 \le 2a)$$

$$= \iint_{x^2+y^2 \le 2a} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$

$x = \tau \cos\varphi$

$y = \tau \sin\varphi$

$$= \int_0^{\sqrt{2a}} \int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{\tau^2}{2}} \, \tau \, d\tau \, d\varphi$$

$$= \frac{1}{2\pi} \times 2\pi \int_0^{\sqrt{2a}} e^{-\frac{\tau^2}{2}} \, \tau \, d\tau$$

$$= -\int_0^{\sqrt{2a}} e^{\frac{-\tau^2}{2}} \, d\frac{-\tau^2}{2}$$

$$= -\int_0^{-a} e^u \, du$$

$$= \int_{-a}^{0} e^u \, du$$

$$= 1 - e^{-a}$$

$$e^u \Big|_{-a}^{0}$$

$$f_Z(a) = F_Z'(a)$$

$$= \begin{cases} e^{-a} & a > 0 \\ 0 & a \le 0 \end{cases}$$