Last time

$X_n \xrightarrow{D} X$ if $\underbrace{F_{X_n}(a)}_{\text{CDF}} \to F_X(a)$ whenever F_X is cts at a.

Levy's continuity theorem: If $\underbrace{\varphi_{X_n}(t)}_{\text{characteristic ftn}} \to \varphi(t)$, and φ is cts at $t=0$, then $\varphi(t) = \varphi_X(t)$ for some X and $X_n \xrightarrow{D} X$

Law of Large # (One of Many)

Let X_n be a sequence of i.i.d. and $E(X_n) = b$.

Let $Y_n = \frac{1}{n}\sum_{k=1}^{n} X_n$

Theorem: $Y_n \xrightarrow{D} b$, "b" constant ftn

Proof: Look at $\varphi_{X_n}(t) = \varphi(t) \to$ the characteristic ftn of X_n

Then: $\varphi_{Y_n}(t) = E e^{itY_n} = E e^{it\left(\frac{X_1 + \cdots X_n}{n}\right)}$

$$= E\left(e^{\frac{it}{n}X_1} \times \cdots e^{\frac{it}{n}X_n}\right)$$

$$\overset{\text{indep}}{=} E(e^{\frac{it}{n}X_1}) \times \cdots E(e^{\frac{it}{n}X_n}) = \varphi^n\left(\frac{t}{n}\right)$$

$(1 + \frac{ibt}{n})^n$

Taylor Approximation:

$\varphi(t) = \varphi(0) + \varphi'(0)\, t + \partial(t)\, t$, where $\partial(t) \to 0$ as $t \to 0$

$\varphi(0) = 1$, $\varphi'(0) = iE(X_n) = ib$

So $\varphi_{Y_n}(t) = \left(1 + \frac{ibt}{n} + \underbrace{\partial\left(\frac{t}{n}\right)\frac{t}{n}}_{\text{very small}}\right)^n$

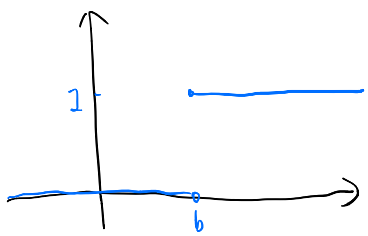
$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$

$\left(1 + \frac{ibt}{n}\right)^n \to e^{ibt}$

$(1 + \frac{ibt}{n})^n \xrightarrow{n\to\infty} \underbrace{e^{ibt}}_{\text{characteristic ftn of random variable equal to } b \text{ everywhere.}}$ since $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$

Levy's Theorem: $Y_n \xrightarrow{D} b$

CDF of b: $F(a) = P(b \le a) = \begin{cases} 1 & \text{if } a \ge b \\ 0 & \text{if } a < b \end{cases}$



$F_n(a) \to F(a)$ for all $a \ne b$.

$\uparrow$

CDF of Y_n

$\forall\, \varepsilon > 0$, $\lim_{n\to\infty} P(Y_n \ge b + \varepsilon) = 0$,

$\lim_{n\to\infty} P(Y_n \le b - \varepsilon) = 0$

So $P(Y_n = b)$ all bets are off.

$X_i = \begin{cases} 1, & p = \frac{1}{2} \\ -1, & p = \frac{1}{2} \end{cases}$

$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_k$

$E\, Y_k = b = 0$,

$P(Y_n = 0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \underbrace{2^{-n} \binom{2m}{m}}_{\approx \frac{1}{\sqrt{n}}} & \text{if } n = 2m \text{ is even} \end{cases}$

How fast the convergence is?

Chebyshev, Bernstein, and others are about that

They require more than just having expectation.

Why do we need $\varphi(t)$ cts at $t=0$ in Levy's continuity theorem?

Here is what can happen:

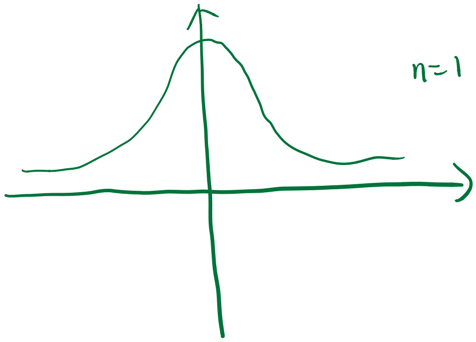
Suppose that X_n is Gaussian with $E(X_n) = 0$ and $Var(X_n) = n$.

$\varphi_{X_n}(t) = e^{-nt^2/2} \xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{if } t \ne 0 \\ 1 & \text{if } t = 0 \end{cases}$

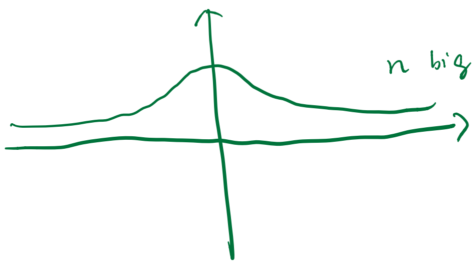
So $\varphi(t) = \begin{cases} 0 & \text{for } t \ne 0 \\ 1 & \text{for } t = 0 \end{cases}$

not cts at $t=0$

$X_n \xrightarrow{D}$ doesn't converge anywhere in distn.



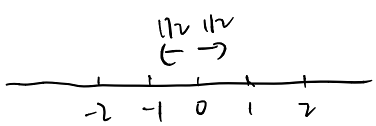
graph for pdf



The condition that $\varphi(t)$ is cts at $t=0$ prevents the distn from "disappearing" in the limit.

Towards Brownian Motion

Intuition: Consider symmetric random walk starting at $x=0$ without barriers.



$X = X_1 + \cdots X_n$, $X_k = \begin{cases} 1 & , p=1/2 \\ -1 & , p=1/2 \end{cases}$

position at the n-th step

Now: Make each step $\frac{1}{\sqrt{n}}$ (instead of 1)

And make each step in $\frac{1}{n}$ second (instead of 1 second)

Position after n steps: $Y_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow{D} N(0,1)$

Central Limit Theorem

Now try to describe the whole movement.

Wiener Process

$X_t : \Omega \to \mathbb{R}$ r.v. $t \geq 0$

① $X_0 = 0$

② If $0 \leq t_1 < t_2 < \cdots < t_n$

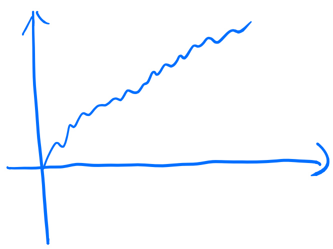
then random variables : $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots X_{t_n} - X_{t_{n-1}}$ are indep. } indep increments

intuition : $X_{t_k} - X_{t_{k-1}}$ displacement from time t_{k-1} to t_k

③ For each $0 \leq t_1 < t_2$, the r.v $X_{t_2} - X_{t_1}$ is Gaussian with

$E(X_{t_2} - X_{t_1}) = 0$ and $Var(X_{t_2} - X_{t_1}) = t_2 - t_1$

④ $P\{ w \in \Omega : \text{the ftn } t \to X_t(w) \text{ is cts} \} = 1$



Theorem (Wiener, Levy)

Wiener process exists.

Theorem : $P\{ w \in \Omega : \text{the ftn } t \to X_t(w) \text{ is not differentiable at any } t \} = 1$

Sanity check : $X_{t_3} - X_{t_1} = \underbrace{(X_{t_3} - X_{t_2}) + (X_{t_2} - X_{t_1})}_{\text{indep}}$

$$Var(X_{t_3} - X_{t_1}) = Var(X_{t_3} - X_{t_2}) + Var(X_{t_2} - X_{t_1})$$

$$t_3 - t_1 = (t_3 - t_2) + (t_2 - t_1)$$