Adaptive Model Predictive Control

June 5, 2021

1 Reference Frames

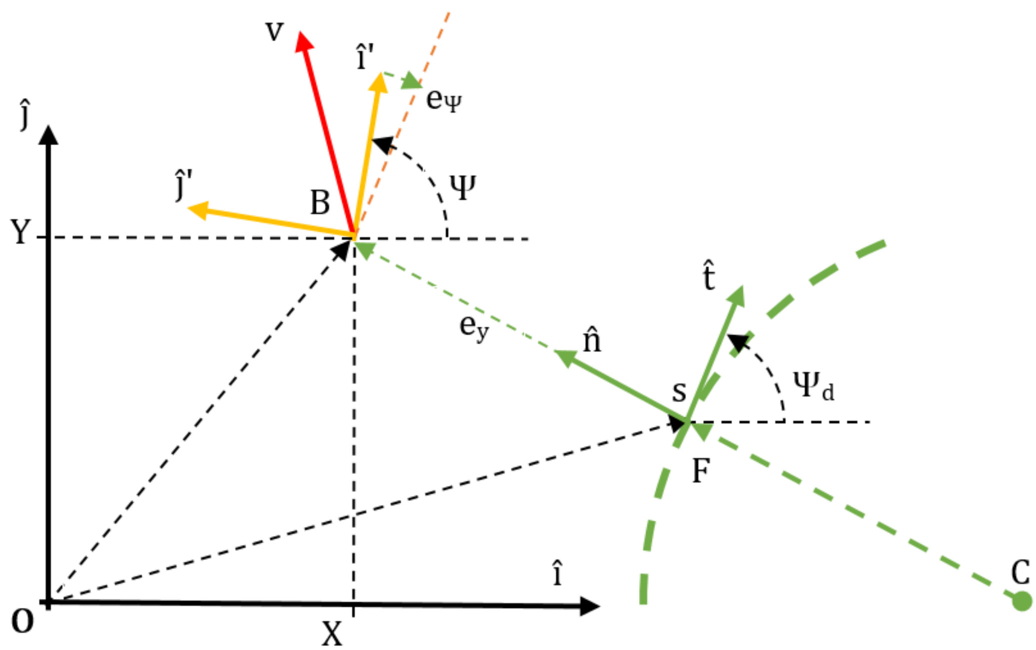

Figure 1: Global, body and track frames

It is considered a moving body frame B in a plane with respect to an inertial global frame O, and a Frenet frame F.

1.1 Inertial-Body transform in cartesian cordiantes

1.1.1 Position transform

The position of a point attached to the frame body B with respect to the global (inertial) frame O is given by the elementary rotation:

$$\vec{r}^O = R_B^O \vec{r}^B \tag{1}$$

where:

$$R_B^O = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \tag{2}$$

1.1.2 Velocity transform

The velocity of the rotating body frame B, expressed with respect to the inertial global frame O, is:

$$\begin{aligned} \dot{\vec{r}}^O &= \frac{d}{dt}(r^O) = \frac{d}{dt}\left(R_B^O \vec{r}^B\right) = \frac{dR_B^O}{dt}\vec{r}^B + R_B^O \frac{d}{dt}(\vec{r}^B) = \dot{R}_B^O \vec{r}^B + R_B^O \dot{\vec{r}}_B^B \\ &= \vec{\omega}^O \times R_B^O \vec{r}^B + R_B^O \vec{v}_B^B = \vec{\omega}^O \times \vec{r}^O + \vec{v}_B^O \end{aligned} \tag{3}$$

where: $\dot{R}_B^O = \omega^O \times R_B^O$
$R_B^O \dot{\vec{r}}_B^B = R_B^O \vec{v}_B^B = \vec{v}_B^O$

then:

$$\vec{v} = \vec{v}_B + \vec{\omega} \times \vec{r} \tag{4}$$

Velocity decomposition

$$\begin{aligned} \vec{v} &= \vec{v}_B + \vec{\omega} \times \vec{r} \\ &= v_{B_x}\hat{i} + v_{B_y}\hat{j} + \omega\hat{k} \times (x\hat{i} - y\hat{j}) \\ &= v_{B_x}\hat{i} + v_{B_y}\hat{j} + \omega x(\hat{k} \times \hat{i}) - \omega y(\hat{k} \times \hat{j}) \\ &= v_{B_x}\hat{i} + v_{B_y}\hat{j} + \omega x\hat{j} + \omega y\hat{i} \end{aligned} \tag{5}$$

then:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_{B_x} + \omega y \\ v_{B_y} + \omega x \end{bmatrix} \tag{6}$$

1.1.3 Acceleration transform

The acceleration of the rotating body frame B, expressed with respect to the inertial global frame O, is:

$$
\begin{aligned}
\ddot{\vec{r}}^O &= \frac{d}{dt}\left(\vec{v}_B^O\right) + \frac{d}{dt}\left(\vec{\omega}^O \times \vec{r}^O\right) \\
&= \frac{d}{dt}\left(R_O^B \vec{v}_B^B\right) + \frac{d}{dt}\left(\vec{\omega}^O \times R_O^B \vec{r}^B\right) \\
&= \dot{R}_O^B \vec{v}_B^B + R_O^B \dot{\vec{v}}_B^B + \dot{\vec{\omega}}^O \times R_O^B \vec{r}^B + \vec{\omega}^O \times \dot{R}_O^B \vec{r}^B + \vec{\omega}^O \times R_O^B \vec{v}_B^B \\
&= \vec{\omega}^O \times \vec{v}_B^O + \vec{a}_B^O + \dot{\vec{\omega}}^O \times \vec{r}^O + \vec{\omega}^O \times (\vec{\omega}^O \times \vec{r}^O) + \vec{\omega}^O \times \vec{v}_B^O \\
&= \vec{a}_B^O + \vec{\omega}^O \times (\vec{\omega}^O \times \vec{r}^O) + 2\vec{\omega}^O \times \vec{v}_B^O + \dot{\vec{\omega}}^O \times \vec{r}^O
\end{aligned}
\tag{7}
$$

where: $\dot{R}_B^O = \omega^O \times R_B^O$
$\vec{v}_B^O = R_O^B \vec{v}_B^B$
$R_O^B \dot{\vec{v}}_B^B = R_O^B \vec{a}_B^B = \vec{a}_B^O$

then:

$$
\vec{a} = \vec{a}_B + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_B + \dot{\vec{\omega}} \times \vec{r} \tag{8}
$$

Acceleration decomposition Negletting angular accellerations and the Coriolis components, the equation can be rewritten as:

$$
\begin{aligned}
\vec{a} &= \vec{a}_B + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
&= a_{B_x}\hat{i} + a_{B_y}\hat{j} + \omega\hat{k} \times (\omega\hat{k} \times (x\hat{i} - y\hat{j})) \\
&= a_{B_x}\hat{i} + a_{B_y}\hat{j} + \omega^2 x(\hat{k} \times (\hat{k} \times \hat{i})) - \omega^2 y(\hat{k} \times (\hat{k} \times \hat{j})) \\
&= a_{B_x}\hat{i} + a_{B_y}\hat{j} - \omega^2 x\hat{i} + \omega^2 y\hat{j} \\
&= a_{B_x}\hat{i} + a_{B_y}\hat{j} - \omega v_{B_y}\hat{i} + \omega v_{B_x}\hat{j}
\end{aligned}
\tag{9}
$$

where: $v_{B_x} = \omega y$
$v_{B_y} = \omega x$

then:

$$
\begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a_{B_x} - \omega v_{B_y} \\ a_{B_y} + \omega v_{B_u} \end{bmatrix} \tag{10}
$$

1.2 Body-Frenet transform in curvilinear coordiantes

The body frame B can be expressed with respect to a Frenet frame F on a defined track, in curvilinear cooridnates.

1.2.1 Position transform

The position of the body frame B with respect to the Frenet frame F, expressed in the curvilinear coordinates are:

$$\begin{aligned} s &= \rho\theta \\ \vec{e}_y &= \vec{OB} - \vec{OF} = \vec{FB} \end{aligned} \tag{11}$$

where the deviation from the body heading and the track curvature is:

$$e_\psi = \psi - \psi_d \tag{12}$$

The rotation matrix transform from the Frenet frame F to the global frame O:

$$R_O^F(\psi_d) = \begin{bmatrix} \cos\psi_d & -\sin\psi_d \\ \sin\psi_d & \cos\psi_d \end{bmatrix} \tag{13}$$

1.2.2 Velocity transform

Deriving the position transorm equations:

$$\begin{aligned} \dot{\vec{s}} &= \frac{d}{dt}(\rho\theta)\hat{t} = (\frac{d\rho}{dt}\theta + \rho\frac{d\theta}{dt})\hat{t} = (\rho\frac{d\theta}{dt})\hat{t} = \rho\omega_d\hat{t} \\ \dot{\vec{e}}_y &= \frac{d}{dt}(\vec{OB} - \vec{OF}) = \frac{d}{dt}(\vec{FB}) \end{aligned} \tag{14}$$

from which can be obtained the track angular velocity:

$$\omega_d = \frac{\dot{s}}{\rho} = \dot{s}c_c \tag{15}$$

where: $\rho :$ radius of curvature
$c_c = 1/\rho :$ curvature

Referring to the picture 1:

$$\vec{OB} = \vec{OF} + \vec{FB} \tag{16}$$

deriving with respect the time:

$$
\begin{aligned}
&\frac{d}{dt}(\vec{OB})^F = \frac{d}{dt}(\vec{OF})^F + \frac{d}{dt}(\vec{FB})^F + \vec{\omega}_d^O \times (\vec{FB}) \\
\Rightarrow &R_F^O(\psi_d)R_O^B(\psi)\frac{d}{dt}(\vec{OB})^B = \frac{d}{dt}(\vec{OF})^F + \frac{d}{dt}(\vec{FB})^F + \vec{\omega}_d^O \times (\vec{FB}) \\
\Rightarrow &R_F^O(\psi_d)R_O^B(\psi)\vec{v}_{OB}^B = \dot{s}\hat{t} + \dot{e}_y\hat{n} - \dot{s}e_y c_c\hat{t}
\end{aligned}
\tag{17}
$$

where: $\frac{d}{dt}(\vec{OB})^F = R_F^O(\psi_d)\frac{d}{dt}(\vec{OB})^O = R_F^O(\psi_d)R_O^B(\psi)\frac{d}{dt}(\vec{OB})^B$
$\frac{d}{dt}(\vec{OB})^B = \vec{v}_{OB}^B$
$\frac{d}{dt}(\vec{OF})^F = \dot{s}\hat{t}$
$\frac{d}{dt}(\vec{FB})^F = \dot{e}_y\hat{n}$
$\vec{\omega}_d^O \times (\vec{FB}) = \dot{s}c_c\hat{k} \times e_y\hat{n} = -\dot{s}e_y c_c\hat{t}$

Multiplying the matrices and exploiting the trigonometric sum of angles:

$$
\begin{aligned}
R_F^O(\psi_d)R_O^B(\psi) &= \begin{bmatrix} \cos\psi_d & \sin\psi_d \\ -\sin\psi_d & \cos\psi_d \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \\
&= \begin{bmatrix} \cos(\psi - \psi_d) & -\sin(\psi - \psi_d) \\ \sin(\psi - \psi_d) & \cos(\psi - \psi_d) \end{bmatrix} \\
&= \begin{bmatrix} \cos e_\psi & -\sin e_\psi \\ \sin e_\psi & \cos e_\psi \end{bmatrix} = R_F^B(e_\psi)
\end{aligned}
\tag{18}
$$

Replacing the matrix transform above in the previus equation:

$$
R_F^B(e_\psi)\vec{v}_{OB}^B = \dot{s}\hat{t} + \dot{e}_y\hat{n} - \dot{s}e_y c_c\hat{t} = R_F^B(e_\psi)\vec{v}_{OB}^B = \dot{s}(1 - e_y c_c)\hat{t} + \dot{e}_y\hat{n}
\tag{19}
$$

and decomposing along the Frenet frame axes:

$$
\begin{bmatrix} \dot{s} \\ \dot{e}_y \end{bmatrix} = R_F^B(e_\psi) \begin{bmatrix} v_x \\ v_y \end{bmatrix} \begin{bmatrix} \frac{1}{1 - c_c e_y} \\ 0 \end{bmatrix}
\tag{20}
$$

while the deviation angle velocity:

$$
\vec{\dot{e}}_\psi = \frac{d}{dt}(\psi - \psi_d)\hat{k} = \vec{\omega} - \vec{\omega}_d = \vec{\omega} - \vec{\dot{s}}c_c
\tag{21}
$$

1.3 Inertial-Frenet-Body transfrom

1.3.1 Rotation matrix

Rotation matrix between the inertial/global frame O, body frame B and Frenet frame F:

$$\begin{aligned} R_B^O(\psi) &= \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \\ R_F^O(\psi_d) &= \begin{bmatrix} \cos\psi_d & -\sin\psi_d \\ \sin\psi_d & \cos\psi_d \end{bmatrix} \\ R_B^F(e_\psi) &= \begin{bmatrix} \cos e_\psi & -\sin e_\psi \\ \sin e_\psi & \cos e_\psi \end{bmatrix} \end{aligned} \tag{22}$$

1.3.2 Coordinate system transform

$$\begin{aligned} s &= \vec{r}_{OF}^{F} \\ \vec{e}_y &= \vec{r}_{FB}^{F} \\ e_\psi &= \psi - \psi_d \end{aligned} \tag{23}$$

The curvilinear abscissa s can be expressed in polar coordinates

$$s = \rho\theta \tag{24}$$

where: $\rho:$ curve radius
$c_c = 1/\rho:$ curvature
$\theta:$ angle travelled

which can be derived with respect to the time, considering a constant curvature:

$$\vec{\dot{s}} = \frac{d}{dt}(\rho\theta)\hat{t} = \rho\frac{d}{dt}(\theta)\hat{t} = \rho\omega_d\hat{t} \tag{25}$$

then the angular velocity:

$$\omega_d = \frac{\dot{s}}{\rho} = \dot{s}c_c \tag{26}$$

1.3.3 Position

$$\vec{r}_{OB}^{O} = \vec{r}_{OF}^{O} + \vec{r}_{FB}^{O} = \vec{r}_{OF}^{O} + R_F^O(\psi_d)\vec{r}_{FB}^{F} \tag{27}$$

1.3.4 Velocity

$$
\begin{aligned}
\vec{v}_B^O &= R_B^O(\psi)\vec{v}_B^B \\
\vec{v}_B^F &= R_B^F(e_\psi)\vec{v}_B^B
\end{aligned}
\tag{28}
$$

Deriving the position vector $\vec{r}_{OB}^O$ with respect the time:

$$
\begin{aligned}
&\frac{d}{dt}(\vec{r}_{OB}^O) = \frac{d}{dt}(\vec{r}_{OF}^O) + \frac{d}{dt}(R_F^O(\psi_d)\vec{r}_{FB}^F) \\
\Rightarrow &\vec{v}_B^O = \frac{d}{dt}(\vec{r}_{OF}^O) + \dot{R}_F^O(\psi_d)\vec{r}_{FB}^F + R_F^O(\psi_d)\dot{\vec{r}}_{FB}^F \\
\Rightarrow &\vec{v}_B^O = \dot{\vec{r}}_{OF}^O + \vec{\omega}_F^O \times R_F^O(\psi_d)\vec{r}_{FB}^F + R_F^O(\psi_d)\dot{\vec{r}}_{FB}^F \\
\Rightarrow &R_O^F(\psi_d)\vec{v}_B^O = R_O^F(\psi_d)\dot{\vec{r}}_{OF}^O + \vec{\omega}_F^O \times \vec{r}_{FB}^F + \dot{\vec{r}}_{FB}^F \\
\Rightarrow &R_O^F(\psi_d)R_B^O(\psi)\vec{v}_B^B = \dot{\vec{r}}_{OF}^F + \vec{\omega}_F^O \times \vec{r}_{FB}^F + \dot{\vec{r}}_{FB}^F \\
\Rightarrow &R_F^B F(e_\psi)\vec{v}_B^B = \dot{\vec{r}}_{OF}^F + \vec{\omega}_F^O \times \vec{r}_{FB}^F + \dot{\vec{r}}_{FB}^F
\end{aligned}
\tag{29}
$$

where: $R_F^B F(e_\psi) = R_O^F(\psi_d)R_B^O(\psi)$

Passing from the cartesian coordinates to the curvilinear coordinates:

$$
\begin{aligned}
&R_F^B(e_\psi)\vec{v}_B^B = \vec{\dot{s}} + \vec{\omega}_d \times \vec{e}_y + \dot{\vec{e}}_y \\
\Rightarrow &R_F^B(e_\psi)\vec{v}_B^B = \dot{s}\hat{t} + \dot{s}c_c\hat{k} \times e_y\hat{n} + \dot{e}_y\hat{n} \\
\Rightarrow &R_F^B(e_\psi)\vec{v}_B^B = \dot{s}\hat{t} + \dot{s}e_yc_c(-\hat{t}) + \dot{e}_y\hat{n} \\
\Rightarrow &R_F^B(e_\psi)\vec{v}_B^B = \dot{s}(1 - e_yc_c)\hat{t} + \dot{e}_y\hat{n}
\end{aligned}
\tag{30}
$$

which can be decompoed in the Frenet frame axes:

$$
\begin{bmatrix} \dot{s} \\ \dot{e}_y \end{bmatrix} = R_F^B(e_\psi) \begin{bmatrix} v_x \\ v_y \end{bmatrix} \begin{bmatrix} \frac{1}{1-c_ce_y} \\ 0 \end{bmatrix}
\tag{31}
$$

while the deviation angle velocity:

$$
\dot{\vec{e}}_\psi = \frac{d}{dt}(\psi - \psi_d)\hat{k} = \vec{\omega} - \vec{\omega}_d = \vec{\omega} - \dot{s}c_c\hat{k}
\tag{32}
$$

2 Bicycle dynamic model

Ref: [1] (pg. 27), [3]

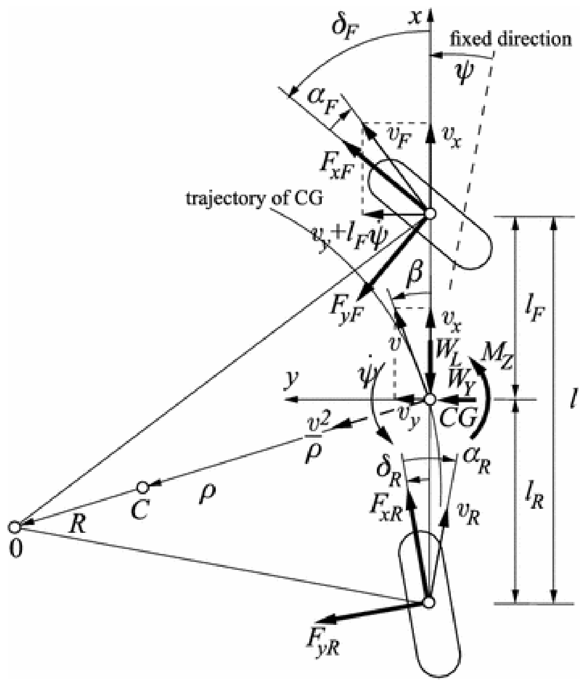

Figure 2: Bicycle dynamics

From equations 10, changing the notation as follow:

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a_{B_x} - \omega v_{B_x} \\ a_{B_y} + \omega v_{B_y} \end{bmatrix} \longrightarrow \begin{bmatrix} a_x^O \\ a_y^O \end{bmatrix} = \begin{bmatrix} \dot{v}_x - \omega_z v_y \\ \dot{v}_y + \omega_z v_x \end{bmatrix} \tag{33}$$

it is possible to write the dynamic equation for a bicycle:

$$\begin{aligned} ma_x^O &= m(\dot{v}_x - \omega_z v_y) = F_{xf} + F_{xr} \\ ma_y^O &= m(\dot{v}_y + \omega_z v_x) = F_{yf} + F_{yr} \\ I_z \dot{\omega}_z &= l_f F_{yf} - l_r F_{yr} \end{aligned} \tag{34}$$

where: m : mass
l_f, l_r : front and rear wheel distance from the center of mass
F_{xf}, F_{xr} : Forces on x body axis of front (f) and rear (r) wheel
F_{yf}, F_{yr} : Forces on y body axis of front (f) and rear (r) wheel
in which the forces can be decomposed in the tire-fixed frame:

$$\begin{cases} F_{x,i} = F_{l,i}cos(\delta_i) - F_{s,i}sin(\delta_i) \\ F_{y,i} = F_{l,i}sin(\delta_i) + F_{s,i}cos(\delta_i) \end{cases} \tag{35}$$

where: $F_{l,i} = \{F_{lf}, F_{lr}\}$: Force on longitudinal axis of front (f) and rear (r) wheel
$F_{s,i} = \{F_{sf}, F_{sr}\}$: Force on perpendicular axis of front (f) and rear (r) wheel
$\delta_i = \{delta_f, delta_r\}$: Front (f) and rear (r) wheel steering angles

Replacing the equations 35 into 34, it is obtained the complete formulation:

$$\begin{aligned} \dot{v}_x &= \frac{1}{m}(F_{lf}\cos(\delta_f) - F_{sf}\sin(\delta_f) + F_{lr}\cos(\delta_r) - F_{sr}\sin(\delta_r)) + \omega_z v_y \\ \dot{v}_y &= \frac{1}{m}(F_{lf}\sin(\delta_f) + F_{sf}\cos(\delta_f) + F_{lr}\sin(\delta_r) + F_{sr}\cos(\delta_r)) - \omega_z v_x \\ \dot{\omega}_z &= \frac{1}{I_z}(l_f(F_{lf}\sin(\delta_f) + F_{sf}\cos(\delta_f)) - l_r(F_{lr}\sin(\delta_r) + F_{sr}\cos(\delta_r))) \end{aligned} \tag{36}$$

Sideslip angle Ref: [1] (pg. 27)

The sideslip angle is defined as:

$$\alpha_* = \delta_* - \theta_* \tag{37}$$

where: $\theta_* = \arctan\left(\frac{v_{x*}}{v_{y*}}\right)$
v_{x*}, v_{y*} : longitudinal and lateral wheel velocities in fixed body-frame
While v_{x*} coincides with the CoM longitudinal velocity, the v_{y*} is the sum of the CoM lateral velocity and the tangential velocity of the wheel respect to the CoM due to the angular velocity, then:

$$\begin{aligned} v_{xf} &= v_{xr} = v_x \\ v_{yf} &= v_y + l_f\omega_z \\ v_{yr} &= v_y - l_r\omega_z \end{aligned} \tag{38}$$

where: v_{xf}, v_{yf} : front wheel velocities in fixed body-frame
v_{xr}, v_{yr} : rear wheel velocities in fixed body-frame
Replacing the equation above in the first one, it is obtained the sideslip angle of the front

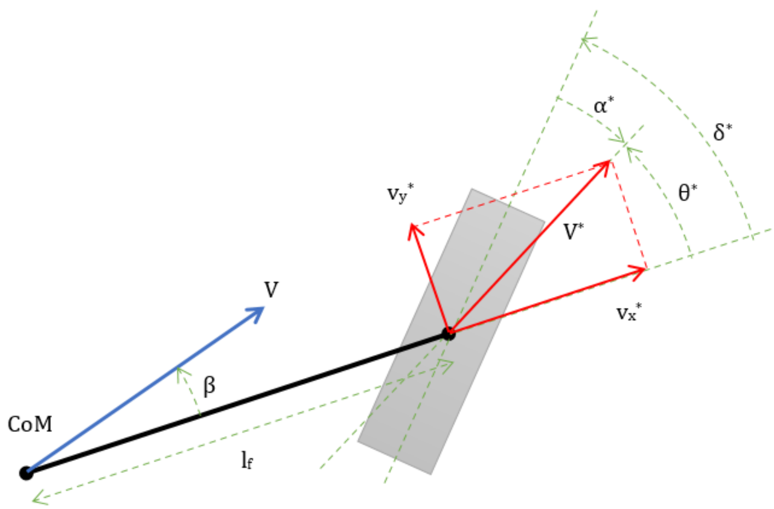

Figure 3: Sideslip angle

and the rear wheel:

$$\begin{aligned}\alpha_f &= \delta_f - \arctan\left(\frac{v_y + l_f\omega_z}{v_x}\right)\\ \alpha_r &= \delta_r - \arctan\left(\frac{v_y - l_r\omega_z}{v_x}\right)\end{aligned} \tag{39}$$

2.1 Tire's linear dynamic

Ref: [3]
For small sideslip angles, the wheel lateral force can be well approximated proportional to the sideslip angle itself:

$$\begin{aligned}F_{sf} &= C_f(F_z, \mu)\alpha_f\\ F_{sr} &= C_r(F_z, \mu)\alpha_r\end{aligned} \tag{40}$$

where C_f, C_r is the "Cornering stiffness" of each wheel, depending on the normal force acting on the wheel and the friction coefficient with the surface.
Here, an estimate of this coefficient has been done evaluating the dynamic evolution of the nonlinear plant model, with different values of the cornering stiffness, until a realistic behaviour has been achieved.
To notice is that small values of the cornering stiffness, make difficult the steering, while too high values generate excessively high lateral forces also for small sideslip angles, which

are incompatible with the reality.

3 Bicycle dynamics in a curvilinear reference for MPC prediction

From the previous sections, a dynamic model of the bicycle in a curvilinear reference can be implemented under the following assumptions:

- Small sideslip angles ($\alpha \leq 10° \rightarrow \sin(\Delta) \simeq \Delta, \cos(\Delta) \simeq 1$)
- Front steering wheel command ($\delta_r = 0$)
- Rear traction ($F_{lr} = T, F_{lf} = 0$)
- The forces are multiplied by a factor of 2 in order to take into account the physics of 4 wheels

the dynamics in 36 can be semplified as:

$$\begin{aligned} \dot{v}_x &= \frac{1}{m}(-F_{sf}\sin(\delta_f) + T) + \omega_z v_y \\ \dot{v}_y &= \frac{1}{m}(F_{sf}\cos(\delta_f) + F_{sr}) - \omega_z v_x \\ \dot{\omega}_z &= \frac{1}{I_z}(l_f F_{sf}\cos(\delta_f) - l_r F_{sr}) \end{aligned} \tag{41}$$

while the sideslip angle, from 39, under the assumption of small angles, is:

$$\begin{aligned} \alpha_f &= \delta_f - \arctan\left(\frac{v_y + l_f\omega_z}{v_x}\right) \simeq \delta_f - \frac{v_y + l_f\omega_z}{v_x} \\ \alpha_r &= -\arctan\left(\frac{v_y - l_r\omega_z}{v_x}\right) \simeq -\frac{v_y - l_r\omega_z}{v_x} \end{aligned} \tag{42}$$

The tire's forces are then:

$$\begin{aligned} F_{sf} &= 2C_f\alpha_f \simeq 2C_f\left(\delta_f - \frac{v_y + l_f\omega_z}{v_x}\right) \\ F_{sr} &= 2C_r\alpha_r \simeq 2C_r\left(-\frac{v_y - l_r\omega_z}{v_x}\right) \end{aligned} \tag{43}$$

While the link between the dynamics from the cartesian coordinate into the curvilinear cooridnate from 20 is:

$$\begin{aligned}
\dot{s}(t) &= \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} \\
\dot{e}_y(t) &= v_x \sin e_\psi + v_y \cos e_\psi \\
\dot{e}_\psi(t) &= \omega_z - \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} c_c(s)
\end{aligned} \tag{44}$$

Putting toghether 41 43 44 it is obtained the complete dynamics:

$$\begin{aligned}
\dot{v}_x &= a + \frac{2}{m}\left(-C_f\left(\delta_f - \frac{v_y + l_f \omega_z}{v_x}\right)\sin(\delta_f)\right) + v_y \omega_z \\
\dot{v}_y &= \frac{2}{m}\left(C_f\left(\delta_f - \frac{v_y + l_f \omega_z}{v_x}\right)\cos(\delta_f) + C_r\left(-\frac{v_y - l_r \omega_z}{v_x}\right)\right) - v_x \omega_z \\
\dot{\omega}_z &= \frac{2}{I_z}\left(l_f C_f\left(\delta_f - \frac{v_y + l_f \omega_z}{v_x}\right)\cos(\delta_f) - l_r C_r\left(-\frac{v_y - l_r \omega_z}{v_x}\right)\right) \\
\dot{s}(t) &= \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} \\
\dot{e}_y(t) &= v_x \sin e_\psi + v_y \cos e_\psi \\
\dot{e}_\psi(t) &= \omega_z - \frac{v_x \cos e_\psi - v_y \sin e_\psi}{1 - c_c(s) e_y} c_c(s)
\end{aligned} \tag{45}$$

where: $\xi = \begin{bmatrix} v_x & v_y & \omega_z & s & e_y & e_\psi \end{bmatrix}^T$
$u = \begin{bmatrix} \delta_f & a \end{bmatrix}^T$

4 MPC

4.1 Model linearization

The nonlinear system has to be linearized around the operative point $p = [\xi_0, u_0]$. The system can be approximated through the Taylor expansion, stopped to the first derivative:

$$\frac{d}{dt}(\Delta\xi(t)) \approx \frac{\partial f}{\partial \xi}\Delta\xi(t) + \frac{\partial f}{\partial u}\Delta u(t) \Rightarrow \Delta\dot{\xi}(t) \approx A(p)\Delta\xi(t) + B(p)\Delta u(t) \tag{46}$$

Changing the notation for semplicity, the new state/input definition is defined as the state error with respect the operative point:

$$\dot{\xi}(t) = A(p)\xi(t) + B(p)u(t) \tag{47}$$

where: $\xi = \begin{bmatrix}\Delta v_{x0} & \Delta v_{y0} & \omega_{z0} & \Delta s_0 & \Delta e_{y0} & \Delta e_{\psi 0}\end{bmatrix}^T$
$u = \begin{bmatrix}\Delta\delta_{f0} & \Delta a_0\end{bmatrix}^T$
$A(p) = \frac{\partial f}{\partial \xi}$
$B(p) = \frac{\partial f}{\partial u}$

Linearized QP matrices

$$A = J_f(x) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix}
\frac{\partial \dot{v}_x}{\partial v_x} & \frac{\partial \dot{v}_x}{\partial v_y} & \frac{\partial \dot{v}_x}{\partial \omega_z} & \frac{\partial \dot{v}_x}{\partial s} & \frac{\partial \dot{v}_x}{\partial e_y} & \frac{\partial \dot{v}_x}{\partial e_\psi} \\
\frac{\partial \dot{v}_y}{\partial v_x} & \frac{\partial \dot{v}_y}{\partial v_y} & \frac{\partial \dot{v}_y}{\partial \omega_z} & \frac{\partial \dot{v}_y}{\partial s} & \frac{\partial \dot{v}_y}{\partial e_y} & \frac{\partial \dot{v}_y}{\partial e_\psi} \\
\frac{\partial \dot{\omega}_z}{\partial v_x} & \frac{\partial \dot{\omega}_z}{\partial v_y} & \frac{\partial \dot{\omega}_z}{\partial \omega_z} & \frac{\partial \dot{\omega}_z}{\partial s} & \frac{\partial \dot{\omega}_z}{\partial e_y} & \frac{\partial \dot{\omega}_z}{\partial e_\psi} \\
\frac{\partial \dot{s}}{\partial v_x} & \frac{\partial \dot{s}}{\partial v_y} & \frac{\partial \dot{s}}{\partial \omega_z} & \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial e_y} & \frac{\partial \dot{s}}{\partial e_\psi} \\
\frac{\partial \dot{e}_y}{\partial v_x} & \frac{\partial \dot{e}_y}{\partial v_y} & \frac{\partial \dot{e}_y}{\partial \omega_z} & \frac{\partial \dot{e}_y}{\partial s} & \frac{\partial \dot{e}_y}{\partial e_y} & \frac{\partial \dot{e}_y}{\partial e_\psi} \\
\frac{\partial \dot{e}_\psi}{\partial v_x} & \frac{\partial \dot{e}_\psi}{\partial v_y} & \frac{\partial \dot{e}_\psi}{\partial \omega_z} & \frac{\partial \dot{e}_\psi}{\partial s} & \frac{\partial \dot{e}_\psi}{\partial e_y} & \frac{\partial \dot{e}_\psi}{\partial e_\psi}
\end{bmatrix} \tag{48}$$

$$\frac{\partial \dot{v}_x}{\partial v_x} = \frac{2C_f \sin(\delta_{f0})(v_{y0} + l_f\omega_{z0})}{m v_{x0}^2}$$

$$\frac{\partial \dot{v}_x}{\partial v_y} = \frac{2C_f \sin(\delta_{f0})}{m v_{x0}} + \omega_{z0}$$

$$\frac{\partial \dot{v}_x}{\partial \omega_z} = \frac{2C_f l_f \sin(\delta_{f0})}{m v_{x0}} + v_{y0}$$

$$
\begin{aligned}
\frac{\partial \dot{v}_y}{\partial v_x} &= -\frac{2}{m v_{x0}^2}\left[C_f(v_{y0}+l_f\omega_{z0})\cos(\delta_{f0}) + C_r(v_{y0}-l_r\omega_{z0})\right] - \omega_{z0} \\
\frac{\partial \dot{v}_y}{\partial v_y} &= -\frac{2}{m v_{x0}}\left[C_f\cos(\delta_{f0}) + C_r\right] \\
\frac{\partial \dot{v}_y}{\partial \omega_z} &= -\frac{2}{m v_{x0}}\left[C_f l_f\cos(\delta_{f0}) - C_r l_r\right] - v_{x0} \\
\frac{\partial \dot{\omega}_z}{\partial v_x} &= \frac{2}{I_z v_{x0}^2}\left[C_f l_f(v_{y0}+l_f\omega_{z0})\cos(\delta_{f0}) - C_r l_r(v_{y0}-l_r\omega_{z0})\right] \\
\frac{\partial \dot{\omega}_z}{\partial v_y} &= \frac{2}{I_z v_{x0}}\left[-C_f l_f\cos(\delta_{f0}) + C_r l_r\right] \\
\frac{\partial \dot{\omega}_z}{\partial \omega_z} &= \frac{2}{I_z v_{x0}}\left[-C_f l_f^2\cos(\delta_{f0}) - C_r l_r^2\right] \\
\frac{\partial \dot{s}}{\partial s} &= 0 \\
\frac{\partial \dot{s}}{\partial e_y} &= \frac{v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0})}{(1-c_c e_{y0})^2} c_c \\
\frac{\partial \dot{s}}{\partial e_\psi} &= \frac{-v_{x0}\sin(e_{\psi 0}) - v_{y0}\cos(e_{\psi 0})}{(1-c_c e_{y0})^2} c_c \\
\frac{\partial \dot{e}_y}{\partial s} &= 0 \\
\frac{\partial \dot{e}_y}{\partial e_y} &= 0 \\
\frac{\partial \dot{e}_y}{\partial e_\psi} &= v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0}) \\
\frac{\partial \dot{e}_\psi}{\partial s} &= 0 \\
\frac{\partial \dot{e}_\psi}{\partial e_y} &= \frac{-v_{x0}\cos(e_{\psi 0}) - v_{y0}\sin(e_{\psi 0})}{(1-c_c e_{y0})^2} c_c^2 \\
\frac{\partial \dot{e}_\psi}{\partial e_\psi} &= \frac{v_{x0}\sin(e_{\psi 0}) + v_{y0}\cos(e_{\psi 0})}{(1-c_c e_{y0})} c_c
\end{aligned}
$$

$$\frac{\partial \dot{v}_x}{\partial s} = \frac{\partial \dot{v}_x}{\partial e_y} = \frac{\partial \dot{v}_x}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{v}_y}{\partial s} = \frac{\partial \dot{v}_y}{\partial e_y} = \frac{\partial \dot{v}_y}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{\omega}_z}{\partial s} = \frac{\partial \dot{\omega}_z}{\partial e_y} = \frac{\partial \dot{\omega}_z}{\partial e_\psi} = 0$$
$$\frac{\partial \dot{s}}{\partial v_x} = \frac{\cos(e_{\psi 0})}{1 - c_c e_{y0}}$$
$$\frac{\partial \dot{s}}{\partial v_y} = -\frac{\sin(e_{\psi 0})}{1 - c_c e_{y0}}$$
$$\frac{\partial \dot{s}}{\partial \omega_z} = 0$$
$$\frac{\partial \dot{e_y}}{\partial v_x} = \sin(e_{\psi 0})$$
$$\frac{\partial \dot{e_y}}{\partial v_y} = \cos(e_{\psi 0})$$
$$\frac{\partial \dot{e_y}}{\partial \omega_z} = 0$$
$$\frac{\partial \dot{e_\psi}}{\partial v_x} = -\frac{\cos(e_{\psi 0})}{1 - c_c e_{y0}} c_c$$
$$\frac{\partial \dot{\psi}}{\partial v_y} = \frac{\sin(e_{\psi 0})}{1 - c_c e_{y0}} c_c$$
$$\frac{\partial \dot{e_\psi}}{\partial \omega_z} = 1$$

$$B = J_f(u) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{v}_x}{\partial \delta_f} & \frac{\partial \dot{v}_x}{\partial a} \\ \frac{\partial \dot{v}_y}{\partial \delta_f} & \frac{\partial \dot{v}_y}{\partial a} \\ \frac{\partial \dot{\omega}_z}{\partial \delta_f} & \frac{\partial \dot{\omega}_z}{\partial a} \\ \frac{\partial \dot{s}}{\partial \delta_f} & \frac{\partial \dot{s}}{\partial a} \\ \frac{\partial \dot{e_y}}{\partial \delta_f} & \frac{\partial \dot{e_y}}{\partial a} \\ \frac{\partial \dot{e_\psi}}{\partial \delta_f} & \frac{\partial \dot{\psi}}{\partial a} \end{bmatrix} \tag{49}$$

$$\frac{\partial \dot{v}_x}{\partial \delta_f} = -\frac{2C_f}{m}\left[\sin(\delta_{f0}) + \left(\delta_{f0} - \frac{v_{y0} + l_f}{v_{x0}}\right)\cos(\delta_{f0})\right]$$
$$\frac{\partial \dot{v}_x}{\partial a} = 1$$
$$\frac{\partial \dot{v}_y}{\partial \delta_f} = \frac{2C_f}{m}\left[\left(-\delta_{f0} + \frac{v_{y0} + l_f w_{z0}}{v_{x0}}\right)\sin(\delta_{f0}) + \cos(\delta_{f0})\right]$$
$$\frac{\partial \dot{v}_y}{\partial a} = 0$$
$$\frac{\partial \dot{\omega}_z}{\partial \delta_f} = \frac{2C_f l_f}{m}\left[\left(-\delta_{f0} + \frac{v_{y0} + l_f w_{z0}}{v_{x0}}\right)\sin(\delta_{f0}) + \cos(\delta_{f0})\right]$$
$$\frac{\partial \dot{\omega}_z}{\partial a} = 0$$
$$\frac{\partial \dot{s}}{\partial \delta_f} = \frac{\partial \dot{s}}{\partial a} = 0$$
$$\frac{\partial \dot{e_y}}{\partial \delta_f} = \frac{\partial \dot{e_y}}{\partial a} = 0$$
$$\frac{\partial \dot{e_\psi}}{\partial \delta_f} = \frac{\partial \dot{e_\psi}}{\partial a} = 0$$

4.2 Discretization

The system model is discretized according the Euler approximation:

$$A_d = I + AT_s B_d = A * T_s \tag{50}$$

where: $Ts:$ MPC sampling time

4.3 Prediction

At k-th sampled, the predicted state is:

$$\xi_{i+1|k} = A_d(p)\xi_{i|k} + B_d(p)u_{i|k}, \quad \xi_{0|k} = \xi_k \tag{51}$$

State prediction:

$$\bar{\xi}_k = \bar{A}_d(p)\xi_k + \bar{B}_d(p)U_k \tag{52}$$

where: $\bar{\xi}_k = \begin{bmatrix} \xi_{1|k} \\ \vdots \\ \xi_{N_p|k} \end{bmatrix} \in \mathbb{R}^{n_x N_p}; \quad U_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N_c-1|k} \end{bmatrix} \in \mathbb{R}^{n_u N_c};$

$$\bar{A}_d(p) = \begin{bmatrix} A_d(p) \\ A_d^2(p) \\ \vdots \\ A_d^{N_p}(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x};$$

$$\bar{B}_d(p) = \begin{bmatrix} B_d(p) & 0^{n_x,n_u} & \cdots & 0^{n_x,n_u} \\ A_d(p)B_d(p) & B_d(p) & \cdots & 0^{n_x,n_u} \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_c}(p)B_d(p) & A_d^{N_c-1}(p)B_d(p) & \cdots & B_d(p) \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N_p-1}(p)B_d(p) & A_d^{N_p-2}(p)B_d(p) & \cdots & A_d^{N_p-N_c}(p)B_d(p) \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_u N_c}$$

4.4 Optimization

4.4.1 Objective function

$$\begin{aligned} J(\xi_k) &= \sum_{i=1}^{N_p} \left\| \xi_{i|k} - ref_{i|k} \right\|_Q^2 + \sum_{i=0}^{N_c-1} \left\| u_{i|k} \right\|_R^2 \\ &= \left\| \bar{\xi}_k - ref_k \right\|_{\bar{Q}}^2 + \left\| U_k \right\|_{\bar{R}}^2 \\ &= U_k^T \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d \right) U_k + 2 \left(\bar{A}_d \xi_k - ref_k \right)^T \bar{Q} \bar{B}_d U_k \\ &+ \left(\bar{A}_d \xi_k - ref_k \right)^T \bar{Q} \left(\bar{A}_d \xi_k - ref_k \right) \end{aligned} \tag{53}$$

where: $ref_k = \begin{bmatrix} ref_{1|k} & \cdots & ref_{N_p|k} \end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

$$\bar{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q \end{bmatrix} \in \mathbb{R}^{n_x N_p, n_x N_p}$$

$$\bar{R} = \begin{bmatrix} \bar{\bar{R}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_u N_p, n_u N_p}; \; \bar{\bar{R}} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix} \in \mathbb{R}^{n_u N_c, n_u N_c}$$

4.4.2 Objective function

State constraints

$$\xi_{min} \leq \xi_{i|k} \leq \xi_{max} \Rightarrow \begin{cases} -\bar{B}_d(p)U_k \leq -\bar{\xi}_{min} + \bar{A}_d(p)\xi_k \\ \bar{B}_d(p)U_k \leq \bar{\xi}_{max} - \bar{A}_d(p)\xi_k \end{cases} \tag{54}$$

where: $\bar{\xi}_{min} = \begin{bmatrix}\xi_{min} & \cdots & \xi_{min}\end{bmatrix}^T$; $\bar{\xi}_{max} = \begin{bmatrix}\xi_{max} & \cdots & \xi_{max}\end{bmatrix}^T \in \mathbb{R}^{n_x N_p}$

Input constraints

$$u_{min} \leq u_{i|k} \leq u_{max} \Rightarrow \begin{cases} -U_k \leq U_{min} \\ U_k \leq U_{max} \end{cases} \tag{55}$$

where: $U_{min} = \begin{bmatrix}u_{min} & \cdots & u_{min}\end{bmatrix}^T$; $U_{min} = \begin{bmatrix}u_{max} & \cdots & u_{max}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$

Input rate constraints

$$\Delta u_{min} \leq \Delta u_{i|k} \leq \Delta u_{max} \Rightarrow \begin{cases} -U_k \leq -\Delta U_{min} - U_{k-1} \\ U_k \leq \Delta U_{max} + U_{k-1} \end{cases} \tag{56}$$

where: $\Delta U_{min} = \begin{bmatrix}\Delta u_{min} & \cdots & \Delta u_{min}\end{bmatrix}^T$; $\Delta U_{min} = \begin{bmatrix}\Delta u_{max} & \cdots & \Delta u_{max}\end{bmatrix}^T \in \mathbb{R}^{n_u N_p}$
$\Delta u_{i|k} = u_{i|k} - u_{i|k-1} \rightarrow \Delta U_k = U_k - U_{k-1}$
$\Delta U_k = \begin{bmatrix}\Delta u_{0|k} & \cdots & \Delta u_{N_c-1|k}\end{bmatrix}^T$
$U_{k-1} = \begin{bmatrix}\Delta u_{0|k-1} & \cdots & \Delta u_{N_c-1|k-1}\end{bmatrix}^T$

4.5 QP formulation

$$\begin{aligned} \min_{U_k} \quad & U_k^T H(p) U_k + 2f^T(p) U_k + g(p) \\ \text{s.t.} \quad & A_{ineq}(p) U_k \leq b_{ineq}(p) \end{aligned} \tag{57}$$

where: $H(p) = \left(\bar{R} + \bar{B}_d^T \bar{Q} \bar{B}_d\right)$
$f(p) = \left(\bar{A}_d \xi_k - ref_k\right)^T \bar{Q} \bar{B}_d$
$g(p) = \left(\bar{A}_d \xi_k - ref_k\right)^T \bar{Q} \left(\bar{A}_d \xi - ref_k\right)$
$$A_{ineq}(p) = \begin{bmatrix} -\bar{B}_d \\ \bar{B}_d \\ -I^{n_u N_p} \\ I^{n_u N_p} \\ -I^{n_u N_p} \\ I^{n_u N_p} \end{bmatrix}, \quad b_{ineq}(p) = \begin{bmatrix} -\bar{\xi}_{min} + \bar{A}_d \xi_k \\ \bar{\xi}_{max} - \bar{A}_d \xi_k \\ -U_{min} \\ U_{max} \\ -\Delta U_{min} - U_{k-1} \\ \Delta U_{max} + U_{k-1} \end{bmatrix}$$

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