State Estimation

Full Bayes to KF Proof

Author:

Part 1

x_t - state at t

u_t - actuation at t

z_t - measurement at t

Definition of Bayes Filter

1. $p(x_t \mid z_{1:t}, u_{1:t}) = bel(x_t)$

apply bayes rule $p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)}$

2. $\eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \cdot p(x_t \mid z_{1:t-1}, u_{1:t})$

$A = x_t, z_{1:t-1}, u_{1:t-1}$

$B = z_t$

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$P(B) \cdot P(A \mid B) = P(B \mid A) \cdot P(A)$$

$$P(z_t) = \frac{num(z_t)}{N} = \frac{1}{N}$$

$$\frac{1}{n} \cdot P(A \mid B) = P(B \mid A) \cdot P(A)$$

$$P(A \mid B) = n P(B \mid A) \cdot P(A)$$

$\therefore$

$$n \, p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \cdot p(x_t \mid z_{1:t-1}, u_{1:t})$$

3. Markov Assumption that z_t is only dependent of x_t since assumption states past measurements & control are already factored into x_t

$\therefore$

$$n \, p(z_t \mid x_t) \cdot p(x_t \mid z_{1:t-1}, u_{1:t})$$

4. If current state probability is dependent of previous state probability

$$p(\underbrace{x_t \mid z_{1:t-1}, u_{1:t}}^{X_t})$$

by law of total probability

$$P(X_t) = P\left(\frac{X_t}{X_{t-1}}\right) \cdot P(X_{t-1}) \ldots + P\left(\frac{X_t}{X_{t-K}}\right) \cdot P(X_{t-K})$$

$\therefore$

$$p(x_t \mid z_{1:t-1}, u_{1:t}) =$$

$$p(x_t \mid z_{1:t-1}, u_{1:t}) \cdot$$
$$p(x_{t-1} \mid z_{1:t-2}, u_{1:t-1}) \ldots$$

$$=$$

$$\int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \cdot p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \cdot dx_{t-1}$$

$$\int p(x_t \mid x_{t-1}, u_t) \cdot p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \cdot dx_{t-1}$$

$$n\, p(z_t \mid x_t) \cdot \int p(x_t \mid x_{t-1}, u_t) \cdot \underbrace{p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \cdot dx_{t-1}}_{l_{t-1}}$$

$$= bel(x_t)$$

5. simplify

$$n\, p(z_t \mid x_t) \cdot \int p(x_t \mid x_{t-1}, u_t) \cdot bel(x_{t-1})$$

Bayes filter

The bayes filter has prediction and correction

$$\eta\, p(z_t \mid x_t) \cdot \int p(x_t \mid x_{t-1}, u_t) \cdot bel(x_{t-1})$$

prediction

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\text{motion model}} \cdot bel(x_{t-1}) \cdot dx_{t-1}$$

correction

$$bel(x_t) = \eta\, \underbrace{p(z_t \mid x_t)}_{\text{measurement model}} \cdot \overline{bel}(x_t)$$

Part 2 - Kalman Filter in control

Motion model (Linear)

$$Ax + bu = \dot{x}$$

A: Jacobian of dynamics around some linear fixed point

B: Jacobian with respect to actuations

x: state

u: actuation

1. Motion model in Bayes with model defined above

$$p(x_t | x_{t-1}, u_t) = \frac{1}{\sqrt{2\pi R_t}} \cdot e^{\left(\frac{(x-u)^T \cdot R^{-1} \cdot (x-u)}{-2}\right)}$$

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi * \sigma^2}} * e^{-\frac{1}{2} * (\frac{x-\mu}{\sigma})^2}$$

x = input value, μ = dist. mean, σ = dist. standard deviation

$$f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^d * det(\boldsymbol{\Sigma})}} * e^{-\frac{1}{2} * ((\mathbf{x}-\boldsymbol{\mu})^T \cdot inv(\boldsymbol{\Sigma}) \cdot (\mathbf{x}-\boldsymbol{\mu}))}$$

d = dim, **x** = input vector, **μ** = dist. means, **Σ** = covariance matrix

where $R_t = \Sigma$ the covariance of the control input

$$= \frac{1}{\sqrt{2\pi R_t}} \cdot \exp\left(-\frac{1}{2}(x_t - Ax_{t-1} + Bu_t)^T R_t^{-1}(x_t - Ax_{t-1} + Bu_t)\right)$$

$$= \det(2\pi R_t)^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2} \dots\dots\right)$$

Meausurements

C: maps estimates to measurements

Q: covarience of measurements

$$n\, p(z_t \mid x_t) = \det(2\pi Q)^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2}(z_t - Cx_t)^T Q^{-1} (\ldots\ldots)\right)$$

The Kalman Filter Algorithm

$KF(mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$

$$\left.\begin{array}{l} 1: \overline{mu}_t = A_t\, mu_{t-1} + B_t\, u_t \\ 2: \overline{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{array}\right\} \text{propagation}$$

$$\left.\begin{array}{l} 3: K = \overline{\Sigma}_t C_t^T (C_t \overline{\Sigma}_t C_t^T + Q_t)^{-1} \\ 4: mu_t = \overline{mu}_t + K(z_t - C_t \overline{mu}_t) \end{array}\right\} \text{update}$$

$$5\ \ \Sigma_t = (I - K C_t)\overline{\Sigma}_t \text{ — correction}$$

return (mu_t, Σ_t)

$$(I - K(t)) \cdot F_x P F_x^T + C_x$$

$$(I - C(F_x P F_x^T + C_x)$$

Lin conv

$$\Sigma_t =$$

$$(I - (A_t \Sigma_{t-1} A_t^T + R_t) C_t^T (C_t (A_t \Sigma_{t-1} A_t^T + R_t) C_t^T + Q^{-1}) \cdot$$

$$A_t \Sigma_{t-1} A_t^T + R_t$$