

# Summary of COMP523 Advanced Algorithm

May 10, 2023

# Chapter 1

## Symmetry Notation

### 1.1 Asymptotic Notation

Asymptotic notation is a way of describing the limiting behavior of a function when the argument tends towards a particular value or infinity. In computer science, asymptotic notation is frequently used to describe the running time or space usage of an algorithm.

- $O$ -notation:  $f(n) = O(g(n))$  if there exist constants  $c$  and  $n_0$  such that  $0 \leq f(n) \leq cg(n)$  for all  $n \geq n_0$ .
- $\Omega$ -notation:  $f(n) = \Omega(g(n))$  if there exist constants  $c$  and  $n_0$  such that  $0 \leq cg(n) \leq f(n)$  for all  $n \geq n_0$ .
- $\Theta$ -notation:  $f(n) = \Theta(g(n))$  if there exist constants  $c_1, c_2$  and  $n_0$  such that  $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$  for all  $n \geq n_0$ .
- $o$ -notation:  $f(n) = o(g(n))$  if for any constant  $c > 0$ , there exists a constant  $n_0$  such that  $0 \leq f(n) < cg(n)$  for all  $n \geq n_0$ .
- $\omega$ -notation:  $f(n) = \omega(g(n))$  if for any constant  $c > 0$ , there exists a constant  $n_0$  such that  $0 \leq cg(n) < f(n)$  for all  $n \geq n_0$ .

### 1.2 Comparing Functions

#### 1.2.1 Transitivity

- $f(n) = O(g(n))$  and  $g(n) = O(h(n))$  implies  $f(n) = O(h(n))$ .
- $f(n) = \Omega(g(n))$  and  $g(n) = \Omega(h(n))$  implies  $f(n) = \Omega(h(n))$ .
- $f(n) = \Theta(g(n))$  and  $g(n) = \Theta(h(n))$  implies  $f(n) = \Theta(h(n))$ .

For example,  $n^2 = O(n^3)$  and  $n^3 = O(n^4)$  implies  $n^2 = O(n^4)$ .

#### 1.2.2 Reflexivity

- $f(n) = O(f(n))$ .
- $f(n) = \Omega(f(n))$ .
- $f(n) = \Theta(f(n))$ .

For example,  $n^2 = O(n^2)$ .

### 1.2.3 Symmetry

- $f(n) = O(g(n))$  implies  $g(n) = O(f(n))$ .
- $f(n) = \Omega(g(n))$  implies  $g(n) = \Omega(f(n))$ .
- $f(n) = \Theta(g(n))$  implies  $g(n) = \Theta(f(n))$ .
- $f(n) = o(g(n))$  implies  $g(n) = \omega(f(n))$ .
- $f(n) = \omega(g(n))$  implies  $g(n) = o(f(n))$ .

For example,  $n^2 = O(n^3)$  implies  $n^3 = \Omega(n^2)$ .

### 1.2.4 Transpose Symmetry

- $f(n) = O(g(n))$  if and only if  $g(n) = \Omega(f(n))$ .
- $f(n) = \Theta(g(n))$  if and only if  $g(n) = \Theta(f(n))$ .
- $f(n) = o(g(n))$  if and only if  $g(n) = \omega(f(n))$ .
- $f(n) = \omega(g(n))$  if and only if  $g(n) = o(f(n))$ .

For example,  $n^2 = O(n^3)$  if and only if  $n^3 = \Omega(n^2)$ .

### 1.2.5 sum and maximum

$$f_1(n) + f_2(n) + \cdots + f_k(n) = \Theta(\max(f_1(n), f_2(n), \dots, f_k(n)))$$

where  $k$  is a constant positive integer.

Let  $f_j(n) = j$ ,  $k = n$ , then

$$f_1(n) + f_2(n) + \cdots + f_k(n) = n(n+1)/2 = \Theta(n^2)$$

### 1.2.6 Running time hierarchy

- logarithmic:  $O(\log n)$
- linear:  $O(n)$
- $n \log n$ :  $O(n \log n)$
- quadratic:  $O(n^2)$
- polynomial:  $O(n^k)$
- exponential:  $O(c^n)$
- constant:  $O(1)$
- superconstant:  $\omega(1)$
- sublinear:  $o(n)$
- superlinear:  $\omega(n)$
- superpolynomial:  $\omega(n^k)$
- subexponential:  $o(c^n)$

## 1.3 Expect of algorithms

**Correctness:** An algorithm is correct if it halts with the correct output for every input instance.

**Termination:** An algorithm is terminating if it halts for every input instance.

**Efficiency:** An algorithm is efficient if it halts with the correct output for every input instance and runs in polynomial time.

## Chapter 2

# Recursion and Divide and Conquer techniques

### 2.1 Finding Majority in array

The pseudocode of the algorithm is shown in Algorithm 2.1.

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**Algorithm 1** Finding Majority in array

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```
1: procedure MAJORITY( $A$ )
2:    $n \leftarrow$  length of  $A$ 
3:   if  $n = 0$  then
4:     return  $-1$ 
5:   end if
6:   if  $n = 1$  then
7:     return  $A[1]$ 
8:   end if
9:   if  $n \neq 1$  and  $n$  is odd then
10:
11:   end if
12:   Array  $B$  of size  $n/2$ 
13:   set  $j=0$ 
14:   for  $i = 1$  to  $n/2$  do
15:     if  $A[2i - 1] = A[2i]$  then
16:        $B[j] \leftarrow A[2i - 1]$ 
17:        $j \leftarrow j + 1$ 
18:     end if
19:   end for
20:    $m \leftarrow$  MAJORITY( $B$ )
21:    $count \leftarrow 0$ 
22:   for  $i = 1$  to  $n$  do
23:     if  $A[i] = m$  then
24:        $count \leftarrow count + 1$ 
25:     end if
26:   end for
27:   if  $count > n/2$  then
28:     return  $m$ 
29:   else
30:     return  $-1$ 
31:   end if
32: end procedure
```

---

**Correctness:**

Lemma: If  $A$  has a majority element, then the majority element of  $A$  is also the majority element of  $B$ .

Base case:  $n = 1$ , the majority element is  $A[1]$ .

Induction hypothesis: Assume that the lemma is true for  $n = k$ , we will prove that the lemma is true for  $n = k + 1$ .

Induction step: If  $A$  has a majority element, then the majority element of  $A$  is also the majority element of  $B$ .

Case 1 ( $A$  has a majority element  $m$ ): Then by the lemma, it is also the majority element of  $B$ . Then  $m$  appears more than  $k/2$  times in  $B$ . Then  $m$  appears more than  $(k + 1)/2$  times in  $A$ .

Case 2 ( $A$  has no majority element): Then  $B$  has no majority element. Then  $A$  has no majority element.

**Proof the lemma:**

proof by contradiction. Assume that  $A$  has a majority element  $m$  and  $B$  has a majority element  $m'$ , but  $m \neq m'$ .

Let  $x$  be the numbers of occurrence of  $m$  in  $A$ .

Let  $y$  be the numbers of occurrence of  $m'$  in  $B$ .

Then  $2y$  times from pairs that are represented in  $B$  by a value different from  $m'$ , and  $x - 2y$  times, since each occurrence of  $m$  in  $A$  that is not paired with another occurrence of  $m$  in  $A$  is paired with an occurrence of  $m'$  in  $B$ .

In total, this gives  $2y + x - 2y = x$  occurrences of  $m$  in  $A$ , which is a contradiction.

**Running time:**

Recursive formula for the running time:

$$T(n) \leq T(n/2) + cn$$

where  $c$  is a constant.

The solution to the recurrence is  $T(n) = O(n)$ .

## 2.2 Searching in logarithmic time

Searching faster with BinarySearch.

It is a particular case of the divide-and-conquer paradigm.

**Input:** A sorted array  $A$  of  $n$  elements and a value  $x$ .

**Output:** An index  $i$  such that  $A[i] = x$  or the special value  $-1$  if  $x$  does not appear in  $A$ .

**Pseudocode** is shown in Algorithm 2.2.

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**Algorithm 2** BinarySearch
 

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```

1: procedure BINARYSEARCH( $x, i, j$ )
2:   if  $i = j$  then
3:     if  $A[i] = x$  then
4:       return  $i$ 
5:     else
6:       return  $-1$ 
7:     end if
8:   else
9:     if  $x = A[\lfloor (i + j)/2 \rfloor]$  then
10:      return  $\lfloor (i + j)/2 \rfloor$ 
11:    else if  $x < A[\lfloor (i + j)/2 \rfloor]$  then
12:      return BINARYSEARCH( $x, i, \lfloor (i + j)/2 \rfloor$ )
13:    else
14:      return BINARYSEARCH( $x, \lfloor (i + j)/2 \rfloor + 1, j$ )
15:    end if
16:  end if
17: end procedure

```

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**Running time:**

The number of comparisons performed by BinarySearch is:

$$T(n) \leq T(n/2) + 4$$

Keep calculate:

$$\begin{aligned}
T(n) &\leq T(n/2) + 4 \\
&\leq T(n/4) + 4 + 4 \\
&\leq T(n/8) + 4 + 4 + 4 \\
&\leq T(n/2^k) + 4k \\
&\leq T(n/2^{\log(n-1)}) + 4\log(n-1) \\
&= T(2) + 4(\log n - 1) \\
&\leq 4\log n - 4 \\
&= 4\log n
\end{aligned}$$

proof  $T(n) \leq 4\log n$ :

Base case:  $n = 1, T(1) = 0 \leq 4\log 1 = 0$ .

Induction hypothesis: Assume that the lemma is true for  $n = k$ , we will prove that the lemma is true for  $n = k + 1$ .

Induction step:  $T(k + 1) \leq 4\log(k + 1)$ .

$$\begin{aligned}
T(k + 1) &\leq T(k/2) + 4 \\
&\leq 4\log(k/2) + 4 \\
&= 4\log k - 4 + 4 \\
&= 4\log k \\
&\leq 4\log(k + 1)
\end{aligned}$$

**Memory usage:**

The memory usage of BinarySearch is:

$$M(n) = O(\log n)$$

**Comparing BinarySearch and LinearSearch:**

$$\begin{aligned}
T_{\text{BinarySearch}}(n) &= O(\log n) \\
T_{\text{LinearSearch}}(n) &= O(n) \\
T_{\text{BinarySearch}}(n) &= O(\log n) < O(n) = T_{\text{LinearSearch}}(n) \\
M_{\text{BinarySearch}}(n) &= O(\log n) < O(1) = M_{\text{LinearSearch}}(n)
\end{aligned}$$

## 2.3 Running time of Divide and Conquer algorithms

The Master Theorem:

Suppose that  $T(n)$  satisfies the recurrence:

$$T(n) \leq aT(n/b) + cn^d$$

where  $a \geq 1, b > 1, c > 0$  and  $d \geq 0$  are constants.

Then  $T(n)$  has the following asymptotic bounds:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

This theorem is useful for solving recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

where  $a \geq 1$ ,  $b > 1$  and  $f(n)$  is an asymptotically positive function.

**Example:**

$$T(n) = 8T(n/2) + 100n^2$$

$a = 8$ ,  $b = 2$ ,  $f(n) = 100n^2$ ,  $d = 2$ ,  $\log_b a = \log_2 8 = 3$ .

$d = 2 < \log_b a = 3$ , so  $T(n) = O(n^{\log_b a}) = O(n^3)$ .

## 2.4 Finding pair of points closest to each other

**Input:** A set  $P$  of  $n$  points in the plane.

**Output:** The pair of points in  $P$  that are closest to each other.

**Pseudocode** is shown in Algorithm 2.4. **Running time:**

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### Algorithm 3 ClosestPair

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```

1: procedure CLOSESTPAIR( $P_1, \dots, P_n$ )
2:   Construct  $P_x$  and  $P_y$ .  $P_x$  is sorted by  $x$ -coordinate,  $P_y$  is sorted by  $y$ -coordinate.
3:   return CLOSESTPAIRREC( $P_x, P_y$ )
4: end procedure

```

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### Algorithm 4 ClosestPairRec

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```

1: procedure CLOSESTPAIRREC( $P_x, P_y$ )
2:   if  $|P_x| = |P_y| \leq 3$  then
3:     For each pair of points  $(P_i, P_j)$ , compute  $d(P_i, P_j)$ 
4:     return the pair of points with the smallest distance
5:   end if
6:   Construct  $Q_x, Q_y, R_x$  and  $R_y$ .
7:    $(l_1, l_2) = \text{CLOSESTPAIRREC}(Q_x, Q_y)$ 
8:    $(r_1, r_2) = \text{CLOSESTPAIRREC}(R_x, R_y)$ 
9:    $\delta = \min\{d(l_1, l_2), d(r_1, r_2)\}$ 
10:   $x^* = \text{the largest } x\text{-coordinate in } Q_x$ 
11:   $L = \{(x, y) : x = x^*\}$ 
12:   $S = \{p \in P : p \in L \text{ and } p \text{ is within } \delta \text{ of } L\}$ 
13:  Construct  $S_v$ 
14:  for  $p \in S$  do
15:    Let  $q$  be the point in  $S_v$  closest to  $p$ 
16:    if  $d(p, q) < \delta$  then
17:       $\delta = d(p, q)$ 
18:       $(s_1, s_2) = (p, q)$ 
19:    end if
20:  end for
21:  if  $d(s_1, s_2) < \min\{d(l_1, l_2), d(r_1, r_2)\}$  then
22:    return  $(s_1, s_2)$ 
23:  end if
24:  if  $d(l_1, l_2) < d(r_1, r_2)$  then
25:    return  $(l_1, l_2)$ 
26:  else
27:    return  $(r_1, r_2)$ 
28:  end if
29: end procedure

```

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$$T(n) \leq 2T(n/2) + O(n \log n) = O(n \log n)$$

**Example:**



## Chapter 3

# Graph Algorithms

### 3.1 Graph Definitions

**Graph:** A graph  $G$  consists of a set  $V$  of vertices and a set  $E$  of edges, where each edge is associated with a pair of vertices.

**Directed Graph:** A directed graph  $G$  consists of a set  $V$  of vertices and a set  $E$  of directed edges, where each directed edge is associated with an ordered pair of vertices.

**Undirected Graph:** An undirected graph  $G$  consists of a set  $V$  of vertices and a set  $E$  of undirected edges, where each undirected edge is associated with an unordered pair of vertices.

**Neighbours of a vertex  $v$ :** Set of vertices that are connected to  $v$  by an edge.

**Degree of a vertex  $v$ :** number of neighbours of  $v$ , denoted by  $deg(v)$ .

**Path:** A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.  
length = node count - 1 = edge count.

**Distance between two nodes:** The number of edges in the shortest path between the two nodes.

**Graph diameter:** The maximum distance between any two nodes in the graph.

**Lines, cycles, trees and cliques:**

**Line:** A graph with  $n$  vertices and  $n - 1$  edges.

**Cycle:** A graph with  $n$  vertices and  $n$  edges.

**cliques:** A graph with  $n$  vertices and  $n(n - 1)/2$  edges.

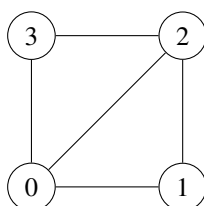
**Tree:** A graph with  $n$  vertices and  $n - 1$  edges.

**Graph representations:**

**Adjacency matrix:** A  $n \times n$  matrix  $A$  where  $A_{ij} = 1$  if there is an edge between  $i$  and  $j$ , and  $A_{ij} = 0$  otherwise.

examples of adjacency matrices:

Given the following graph:

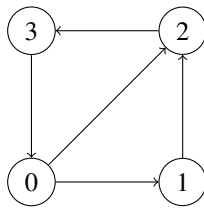


The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

**Adjacency matrix for directed graphs:** A  $n \times n$  matrix  $A$  where  $A_{ij} = 1$  if there is an edge from  $i$  to  $j$ , and  $A_{ij} = 0$  otherwise.

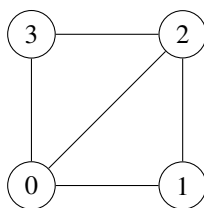
examples of adjacency matrices for directed graphs:  
 Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

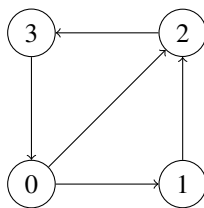
**Adjacency list:** A list of lists, where the  $i$ th list contains the neighbours of vertex  $i$ .  
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \\ 0 & 1 & 3 \\ 0 & 2 & \end{bmatrix}$$

**Adjacency list for directed graphs:** A list of lists, where the  $i$ th list contains the neighbours of vertex  $i$ .  
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 \\ 2 & \\ 3 & \\ 0 & \end{bmatrix}$$

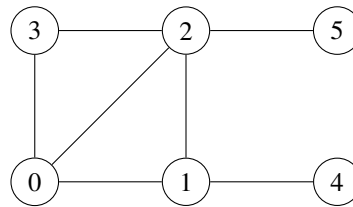
**Adjacency matrix vs adjacency list:**

Adjacency matrix	Adjacency list
$O(1)$ to check if there is an edge between $i$ and $j$	$O(\min(\deg(i), \deg(j)))$ to check if there is an edge between $i$ and $j$
$O(n)$ to find the neighbours of $i$	$O(\deg(i))$ to find the neighbours of $i$
$O(n^2)$ space	$O(n + m)$ space

## 3.2 Depth-first search

**Depth-first search:** A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of depth-first search:



The depth-first search sequence is:

0, 1, 2, 3, 5, 4

**Depth-first search algorithm:**

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**Algorithm 5** Depth-first search algorithm

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```
1: procedure DFS( $G, v$ )
2:   for  $e \in V$  do
3:     if  $e$  is unexplored then
4:        $u = \text{head of } e$ 
5:       if  $u$  is unexplored then
6:          $e$  is a tree edge
7:         DFS( $G, u$ )
8:       else
9:          $e$  is a back edge
10:      end if
11:    end if
12:  end for
13: end procedure
```

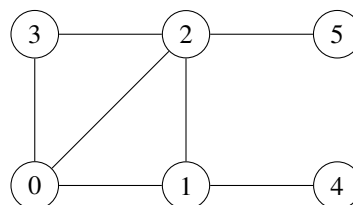
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**Running time of depth-first search:**  $O(n + m)$

## 3.3 Breadth-first search

**Breadth-first search:** A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of breadth-first search:



The breadth-first search sequence starting from vertex 0 is 0, 1, 2, 3, 4, 5.

**Breadth-first search algorithm:**

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**Algorithm 6** Breadth-first search algorithm

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```
1: procedure BFS( $G, s$ )
2:   initial empty list  $L$ 
3:    $L \leftarrow s$ 
4:    $i \leftarrow 0$ 
5:   while  $L[i] \neq \emptyset$  do
6:      $L_{i+1} \leftarrow \text{emptylist}$ 
7:     for  $v \in L[i]$  do
8:       for edges  $(e)$  incident to  $v$  do
9:         if  $e$  is unexplored then
10:            $w \leftarrow$  the other end of  $e$ 
11:           if  $w$  is unexplored then
12:             label  $e$  as a tree edge
13:             add  $w$  to  $L_{i+1}$ 
14:           else
15:             label  $e$  as a cross edge
16:           end if
17:         end if
18:       end for
19:     end for
20:      $i \leftarrow i + 1$ 
21:   end while
22: end procedure
```

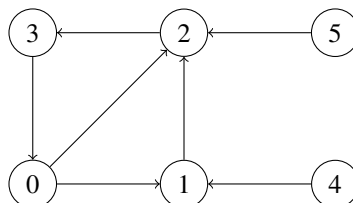
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**Running time of breadth-first search:**  $O(n + m)$

### 3.4 Strong Connectivity

**Directed graph:** A graph where the edges have a direction.

Examples:



**DFS and BFS on directed graphs:**

Very similar to undirected graphs, except that we only consider edges that go out of a vertex.

Running time is  $O(n + m)$

For example graph above the DFS sequence is 0, 1, 2, 3.

The BFS sequence is 0, 1, 2, 3.

#### 3.4.1 Connectivity

**Weak connectivity:** If we ignore the direction for all edges, there would be a path from any vertex to any other vertex.

**Strong Connectivity:** For every two nodes  $u$  and  $v$ , there is a path from  $u$  to  $v$  and a path from  $v$  to  $u$ .

### 3.4.2 Mutual Reachability

Two nodes  $u$  and  $v$  are mutually reachable if there is a path from  $u$  to  $v$  and a path from  $v$  to  $u$ .

**Strong connectivity:** For every pair of nodes  $u$  and  $v$ , these two nodes are mutually reachable.

**Transitivity:** If  $u$  is mutually reachable with  $v$  and  $v$  is mutually reachable with  $w$ , then  $u$  is mutually reachable with  $w$ .

### 3.4.3 Testing strong connectivity

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**Algorithm 7** Testing strong connectivity

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```
1: procedure TESTSTRONGCONNECTIVITY( $G$ )
2:   define  $G^R$  to be the graph with the same vertices as  $G$  but with all edges reversed
3:   Select a node  $s$  in  $G$ 
4:   BFS( $G, s$ ), BFS( $G^R, s$ )
5:   for each node  $v$  do
6:     if  $v$  is unexplored in either BFS then
7:       return False
8:     end if
9:   end for
10:  return True
11: end procedure
```

---

## 3.5 Testing bipartiteness

**Bipartite graph:** A graph  $G = (V, E)$  is bipartite if and only if the vertices can be partitioned into two sets  $V_1$  and  $V_2$  such that every edge has one end in  $V_1$  and the other end in  $V_2$ .

A Graph  $G = (V, E)$  is bipartite if and only if it has no odd cycles. (odd cycle: a cycle with odd number of edges)

**Testing bipartiteness:**

Given a graph  $G = (V, E)$ , we want to test if  $G$  is bipartite.

Given a graph  $G = (V, E)$ , decide if it is 2-colourable.

Given a graph  $G = (V, E)$ , decide if it has an odd cycle.

**Colouring the nodes** It is quite familiar with BFS:

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**Algorithm 8** Colouring the nodes

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```
1: procedure COLOURING( $G, s$ )
2:   initial empty list  $L$ 
3:   initial empty list  $C$ 
4:    $L \leftarrow s$ 
5:    $C[s] \leftarrow red$ 
6:    $i \leftarrow 0$ 
7:   while  $L[i] \neq \emptyset$  do
8:      $L_{i+1} \leftarrow emptylist$ 
9:     for  $v \in L[i]$  do
10:      for edges ( $e$ ) incident to  $v$  do
11:        if  $e$  is unexplored then
12:           $w \leftarrow$  the other end of  $e$ 
13:          if  $w$  is unexplored then
14:            label  $e$  as a tree edge
15:            add  $w$  to  $L_{i+1}$ 
16:            if  $i + 1$  is odd then
17:               $C[w] \leftarrow green$ 
18:            else
19:               $C[w] \leftarrow red$ 
20:            end if
21:          else
22:            label  $e$  as a cross edge
23:            if  $C[v] = C[w]$  then
24:              return False
25:            end if
26:          end if
27:        end if
28:      end for
29:    end for
30:     $i \leftarrow i + 1$ 
31:  end while
32:  for  $e(v, w) \in G$  do
33:    if  $C[v] = C[w]$  then
34:      return False
35:    end if
36:  end for
37:  return True
38: end procedure
```

---

**Running time of colouring the nodes:**  $O(n + m)$

**Correctness of colouring the nodes:**

Proof by contradiction.

Suppose that  $G$  is not bipartite.

Then  $G$  has an odd cycle.

Suppose to the contrary that the algorithm return True.

That means that the algorithm did not detect the odd cycle.

## 3.6 DAGs and Topological Ordering

**DAG:** A directed acyclic graph (DAG) is a directed graph with no directed cycles.  
examples of DAGs:



**Topological ordering:** Given a graph  $G = (V, E)$ , a topological ordering of  $G$  is an ordering of the nodes  $u_1, u_2, \dots, u_n$  such that for every edge  $(u_i, u_j)$ , we have  $i < j$ .

Intuitively, a topological ordering is an ordering of the nodes such that every edge goes from left to right.

example of topological ordering based on given graph above:

3, 0, 1, 2, 4, 5

**Topological ordering implies DAG:**

- If  $G$  has a topological ordering, then  $G$  is a DAG.
- Suppose by contradiction that  $G$  has a topological ordering  $u_1, u_2, \dots, u_n$  but  $G$  also has a cycle  $C$ .
- Let  $u_j$  be the smallest element of  $C$  in the topological ordering.
- Let  $u_i$  be its predecessor in  $C$ .
- $u_i$  must appear before  $u_j$  in the topological ordering.
- This contradicts the fact that  $u_j$  is the smallest element of  $C$  in the topological ordering.

**DAG implies topological ordering:**

Proof by induction: Base case: If  $G$  has one or two nodes, then  $G$  has a topological ordering.

Induction steps: Assume that a DAG up to  $k$  nodes has a topological ordering (induction hypothesis). we will prove that a DAG with  $k + 1$  nodes has a topological ordering.

- By our lemma, there is at least one source node in  $G$ , and let  $u$  be the node.
- Put  $u$  at the beginning of the topological ordering.
- Consider the graph  $G'$ , obtained by  $G$  by removing  $u$  and its incident edges.
- $G'$  is a DAG with  $k$  nodes.
- It has a topological ordering  $u_1, u_2, \dots, u_k$  by the induction hypothesis.
- Append this ordering to  $u$  to get a topological ordering of  $G$ .

Here is the algorithm:

---

**Algorithm 9** Topological Sorting

---

```

1: procedure TOPOLOGICALSORTING( $G$ )
2:   find a source vertex  $u$ 
3:   set  $u$  as the first element of the topological ordering
4:    $G' \leftarrow G$  with  $u$  and its incident edges removed
5:    $L \leftarrow$  TOPOLOGICALSORTING( $G'$ )
6:   append  $L$  to  $u$ 
7: end procedure

```

---

Running time of the algorithm is  $O(n^2)$

**Modified Topological Sorting:**

Running time of the algorithm is  $O(n + m)$

---

**Algorithm 10** Modified Topological Sorting

---

```
1: procedure MODIFIEDTOPOLOGICALSORTING( $G$ )
2:    $L \leftarrow \text{emptylist}$ 
3:    $S \leftarrow$  set of all source vertices
4:   while  $S \neq \emptyset$  do
5:     remove a vertex  $u$  from  $S$ 
6:     append  $u$  to  $L$ 
7:     for each edge  $(u, v)$  do
8:       remove edge  $(u, v)$  from  $G$ 
9:       if  $v$  is a source vertex then
10:        add  $v$  to  $S$ 
11:       end if
12:     end for
13:   end while
14:   if  $G$  has edges then
15:     return  $G$  has a cycle
16:   else
17:     return  $L$ 
18:   end if
19: end procedure
```

---

### 3.7 Finding strongly connected components

**connected components:** A connected component of an undirected graph is subgraph of the graph where any two nodes are connected by a path.

**strongly connected components:** A strongly connected component of a directed graph is a subgraph of the graph where any two nodes are mutually reachable.(mutually reachable: there is a path from  $u$  to  $v$  and a path from  $v$  to  $u$ )

**Finding strongly connected components:**

**Kosaraju's algorithm:**

---

**Algorithm 11** Kosaraju's algorithm

---

```
1: procedure KOSARAJU( $G$ )
2:   Initialise stack  $S$ 
3:   Select a arbitrary node  $s$ 
4:   DFS_tree=DFS( $G, s$ )
5:    $S \leftarrow$  nodes in DFS_tree
6:    $G^R \leftarrow$  nodes in order of  $S$ 
7:   DFS( $G^R, s$ )
8:   return the nodes in the DFS tree
9: end procedure
```

---

**Running time of Kosaraju's algorithm:**  $O(n + m)$

**Correctness of Kosaraju's algorithm:**

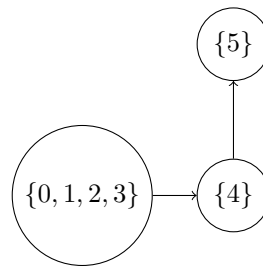
- Define a meta-graph of  $G$ , called  $G^{SCC} = (V^{SCC}, E^{SCC})$ .
- Supposed that  $G$  has strongly connected components (SCCs)  $C_1, C_2, \dots, C_k$ , for some  $k$ .
- $V^{SCC} = \{C_1, C_2, \dots, C_k\}$  contains some of the SCCs of  $G$ .
- There is an edge  $(C_i, C_j)$  in  $E^{SCC}$  if  $G$  contains a directed edge  $(x, y)$  such that  $x \in C_i$  and  $y \in C_j$ , crossing different components.

Examples:





The SCCs are  $\{0, 1, 2, 3\}$  and  $\{4, 5\}$ .  
 The meta-graph is:



# Chapter 4

## Greedy Algorithms

**The greedy approach:**

- The goal is to find a global solution to a problem.
- The solution will be built up in small consecutive steps.
- For each step, we choose the best option available to us at that moment.

### 4.1 Interval Scheduling

**Interval Scheduling:**

A set of requests  $R = \{1, 2, \dots, n\}$ .

- Each request  $i$  has a start time  $s_i$  and a finish time  $f_i$ .
- Alternative view: every request is an interval  $[s_i, f_i]$ .

Two requests  $i$  and  $j$  are compatible if  $[s_i, f_i]$  and  $[s_j, f_j]$  do not overlap.

**Goal:** Find a maximum-size subset of compatible requests.

**Example:**

Interval scheduling.

- Job  $j$  starts at  $s_j$  and finishes at  $f_j$ .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.

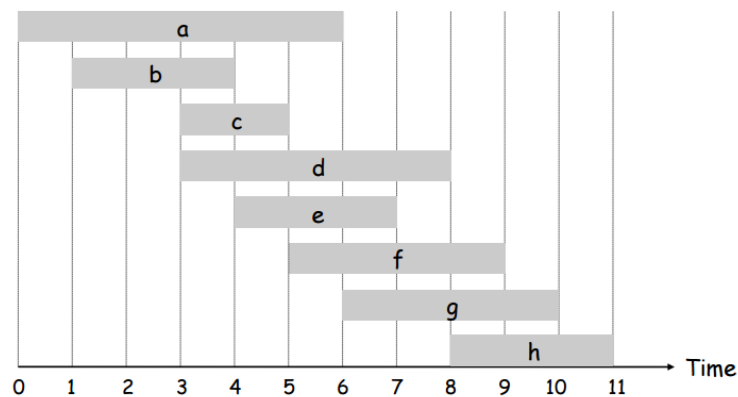


Figure 4.1: Interval Scheduling

### Interval Scheduling Algorithm:

---

**Algorithm 12** Interval Scheduling Algorithm

---

```
1: procedure INTERVALSCHEDULING( $[s_1, f_1], [s_2, f_2], \dots, [s_n, f_n]$ )
2:    $R$  is the set of requests
3:    $A \leftarrow \emptyset$ 
4:   while  $R \neq \emptyset$  do
5:     select a request  $i$  in  $R$  with the smallest finishing time
6:     add  $i$  to  $A$ 
7:     remove all requests from  $R$  that are incompatible with  $i$ 
8:   end while
9:   return  $A$ 
10: end procedure
```

---

**Running time of Interval Scheduling Algorithm:**  $O(n \log n)$

**Correctness of Interval Scheduling Algorithm:** Since the algorithm always selects the request with the smallest finishing time, it is clear that the algorithm will always select a compatible request.

**Arguing optimality:**

## 4.2 Minimum Spanning Trees

Consider a connected graph  $G = (V, E)$ , such that each edge  $e = (v, w)$  of  $E$ , there is an associated cost  $c_e$ .

**Goal:** Find a spanning tree  $T$  of  $E$  so that the graph  $G' = (V, T)$  has minimum cost.

**Example:**



### Greedy approach 1:

- Start with an empty set of edges  $T$ .
- Repeat until  $T$  forms a spanning tree:
  - Select an edge  $e$  of minimum cost.
  - If  $T \cup \{e\}$  does not contain a cycle, then add  $e$  to  $T$ .

### krukals algorithm:

---

**Algorithm 13** Krukals algorithm

---

```
1: procedure KRUKALS( $G$ )
2:    $T \leftarrow \emptyset$ 
3:   while  $T$  is not a spanning tree do
4:     select an edge  $e$  of minimum cost
5:     if  $T \cup \{e\}$  does not contain a cycle then
6:       add  $e$  to  $T$ 
7:     end if
8:   end while
9:   return  $T$ 
10: end procedure
```

---

**Running time of Krukals algorithm:**  $O(m \log n)$

**Greedy approach 2:**

- Start with an empty set of edges  $T$ .
- Start with a node  $s$ .
  - Add an edge  $e = (s, v)$  of minimum cost to  $T$ .
- Repeat until  $T$  forms a spanning tree:

**Prims algorithm:**

---

**Algorithm 14** Prims algorithm

---

```
1: procedure PRIMS( $G$ )
2:    $T \leftarrow \emptyset$ 
3:    $s \leftarrow$  an arbitrary node
4:   while  $T$  is not a spanning tree do
5:     add an edge  $e = (s, v)$  of minimum cost to  $T$ 
6:      $s \leftarrow v$ 
7:   end while
8:   return  $T$ 
9: end procedure
```

---

**Running time of Prims algorithm:**  $O(m \log n)$

**minimum spanning tree of example graph:**

the minimum spanning tree sequence is  $d, a, c, b, e, f, h, g$ .

**Greedy approach 3:**

- Start with the full graph  $G = (V, E)$ .
- Delete an edge from  $G$ 
  - the edge of maximum cost
- Repeat until  $G$  forms a spanning tree:

**Reverse-delete algorithm:**

---

**Algorithm 15** Reverse-delete algorithm

---

```
1: procedure REVERSEDELETE( $G$ )
2:    $T \leftarrow G$ 
3:   while  $T$  is not a spanning tree do
4:     delete an edge  $e$  of maximum cost from  $T$ 
5:   end while
6:   return  $T$ 
7: end procedure
```

---

For when two edges have the same cost, use distinct labels to distinguish them.

**Optimal with Priority Queue:**

Add PQ to Prim's algorithm.

---

**Algorithm 16** Optimal with Priority Queue

---

```
1: procedure OPTIMAL( $G$ )
2:    $T \leftarrow \emptyset$ 
3:    $s \leftarrow$  an arbitrary node
4:    $PQ \leftarrow$  empty priority queue
5:   for each node  $v$  do
6:     add  $v$  to  $PQ$  with key  $\infty$ 
7:   end for
8:   decrease key of  $s$  to 0
9:   while  $PQ$  is not empty do
10:     $v \leftarrow$  node with minimum key in  $PQ$ 
11:    add an edge  $e = (s, v)$  of minimum cost to  $T$ 
12:     $s \leftarrow v$ 
13:    for each edge  $e = (v, w)$  incident to  $v$  do
14:      if  $w$  is in  $PQ$  then
15:        decrease key of  $w$  to  $c_e$ 
16:      end if
17:    end for
18:  end while
19:  return  $T$ 
20: end procedure
```

---

**Running time of Optimal with Priority Queue:**  $O(m \log n)$

## 4.3 Clustering

- a collection of  $n$  objects
- they have different degrees of similarity
- we want to organise them into coherent groups
- there is a notion of distance between objects

**Definition:**

- Given a set  $U$  of  $n$  elements, a  $k$ -clustering of  $U$  is a partition of  $U$  into non-empty subsets  $C_1, C_2, \dots, C_k$ .
- The spacing of a  $k$ -clustering is the minimum distance between any pair of points in different clusters.

**Goal:** Among all possible  $k$ -clusterings, find one with minimum spacing.

**Example:**



**Greedy approach:**

- Pick two objects  $p_i$  and  $p_j$  with minimum distance  $d(p_i, p_j)$ .
- Connect them with an edge  $e = (p_i, p_j)$ .

- Continue like this until we have  $k$  clusters.
- If the edge  $e$  under consideration connects two object  $p_i$  and  $p_j$  already in the same cluster, then discard  $e$ .

**kruskals algorithm:**

---

**Algorithm 17** kruskals algorithm for clustering

---

**Require:** A graph  $G = (V, E)$

**Ensure:** A minimum spanning tree of  $G$  with  $k$  clusters

```

1: procedure KRUSKAL( $G, k$ )
2:    $T \leftarrow \emptyset$ 
3:    $C \leftarrow \{\{v\} \mid v \in V\}$  ▷ Initial clusters
4:   Sort edges in  $E$  in increasing order of weight
5:   for  $\{u, v\} \in E$  do
6:     if  $C$  contains  $k$  clusters then
7:       break
8:     end if
9:     if clusters containing  $u$  and  $v$  are different in  $C$  then
10:       $T \leftarrow T \cup \{\{u, v\}\}$ 
11:      merge clusters containing  $u$  and  $v$  in  $C$ 
12:    end if
13:  end for
14:  return  $T$ 
15: end procedure

```

---

For Given example, the result of divide them into 3 clusters is:

$$\{a, b, c, d\}, \{e, f, h\}, \{g\}$$

# Chapter 5

## Dynamic Programming

**The paradigm of dynamic programming:** Given a problem  $P$ , define a sequence of subproblems, with the following properties:

- The subproblems are ordered from the simplest to the largest
- The largest problem is our original problem  $P$
- The optimal solution of a subproblem can be structured from the optimal solutions of smaller subproblems.

Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution and use it to solve larger subproblems.

### 5.1 Weighted Interval Scheduling

- A set of requests  $R = \{1, 2, \dots, n\}$ .
  - Request  $i$  has a start time  $s_i$  and a finish time  $f_i$ , and a value  $v_i$ .
  - Alternative view: every request is an interval  $[s_i, f_i]$  associated with a value  $v_i$ .
- Two requests  $i$  and  $j$  are compatible if  $[s_i, f_i]$  and  $[s_j, f_j]$  do not overlap.

**build up a solution:**

1. let  $O$  the optimal solution
2.  $O$  contains an optimal solution  $O'$  of the subproblem  $R' = \{1, 2, \dots, i-1\}$
3. in order to find  $O$ , it suffices to look at smaller problems and find  $O(1, 2, \dots, j)$  for some  $j$
4. Let  $O_j$  be a shorthand for  $O(1, 2, \dots, j)$  and let  $OPT(j)$  be its total value.
5. Define  $OPT(0) = 0$
6. Then  $O = O_n$  with value  $OPT(n)$
7.  $OPT(j)$  can be computed from  $OPT(j-1)$
8.  $OPT(j) = \max\{OPT_{p_j} + v_j, OPT(j-1)\}$

---

**Algorithm 18** ComputeOPT

---

```
1: procedure COMPUTEOPT( $j$ )
2:   if  $j = 0$  then
3:     return 0
4:   else
5:     return  $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j-1)\}$ 
6:   end if
7: end procedure
```

---

**Correctness:** ComputeOPT(j) correctly computes  $OPT(j)$  for all  $j = 0, 1, \dots, n$ .

Proof by induction:

**Base case:**  $OPT(0) = 0$  by definition.

**Inductive step:** Assume that it is true for all  $i < j$ . (Induction hypothesis)

return  $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j - 1)\}$

**Running time:**  $\Omega(2^n)$

**Memoization:**

- Compute ComputeOPT(j) for all  $j = 0, 1, \dots, n$ .
- Store it in an accessible place to use again later.
- Keep an array  $M[0, \dots, n]$ .
  - initially  $M[j] = \text{EMPTY}$  for all  $j = 0, 1, \dots, n$ .
  - when ComputeOPT(j) is called,  $M[j] = \text{ComputeOPT}(j)$ .

---

**Algorithm 19** M-ComputeOPT

---

```
procedure M-COMPUTEOPT(j)
  if j = 0 then
    return 0
  else if M[j] is not empty then
    return M[j]
  else
    M[j] ← max{M-COMPUTEOPT(pj) + v(j), M-COMPUTEOPT(j - 1)}
    return M[j]
  end if
end procedure
```

---

**Running time:**  $O(n \log n)$

---

**Algorithm 20** Find-Solution

---

```
procedure FIND-SOLUTION(j)
  if j = 0 then
    return ∅
  else
    if then v(j) + M-COMPUTEOPT(pj) > M-COMPUTEOPT(j - 1)
      return {j} ∪ FIND-SOLUTION(pj)
    else
      return FIND-SOLUTION(j - 1)
    end if
  end if
end procedure
```

---



## Dynamic Programming vs Divide and Conquer:

### Dynamic Programming:

- DP is an optimisation techniques and is only applicable to problems that have optimal substructure.
- DP splits the problem into parts, finds solutions to the parts and joins them.(The parts are not significantly smaller than the original problem and are overlapping.)
- In DP, the subproblems dependency can be represented by a directed acyclic graph.

### Divide and Conquer:

- DC is not normally used for optimisation problems.
- DC splits the problem into parts, finds solutions to the parts and joins them.(The parts are significantly smaller than the original problem and are non-overlapping.)
- In DC, the subproblems dependency can be represented by a tree.

## 5.2 Subset Sum

### Problem Description:

- Given a set of  $n$  items  $1, 2, \dots, n$
- Each item  $i$  has a non-negative weight  $w_i$ .
- Given a bound  $W$ .
- **Goal:** select a subset  $S$  of items such that  $\sum_{i \in S} w_i \leq W$  and  $\sum_{i \in S} w_i$  is maximised.

Dynamic Programming: To find the optimal value of  $OPT(n)$ , we need

- the optimal value of  $OPT(n - 1)$  if item  $n$  is not selected.
- the optimal value of the solution on input  $1, 2, \dots, n-1$  with weight bound  $W - w_n$ .

subproblems:

- Assumptions:
  - $W$  is an integer
  - Every  $w_i$  is an integer
- subproblem for each  $i = 0, 1, \dots, n$  and each integer  $0 \leq w \leq W$ .
- Let  $OPT(i, w)$  be the optimal value of the solution on subset  $1, 2, \dots, i$  with weight bound  $w$ .

---

### Algorithm 21 SubsetSum

---

```
procedure SUBSETSUM( $n, w$ )
  Array  $M[0, \dots, n, 0, \dots, W]$ 
   $M[0, w] = 0$  for each  $w = 0, 1, \dots, W$ 
  for  $i = 1$  to  $n$  do
    for  $w = 0$  to  $W$  do
      if  $w_i > w$  then
         $M[i, w] = M[i - 1, w]$ 
      else
         $M[i, w] = \max\{M[i - 1, w], M[i - 1, w - w_i] + w_i\}$ 
      end if
    end for
  end for
  return  $M[n, W]$ 
end procedure
```

---

**Running time:**  $O(nW)$

## 5.3 knapSack

### Problem Description:

- Given a set of  $n$  items  $1, 2, \dots, n$
- Each item  $i$  has a non-negative weight  $w_i$  and a non-negative value  $v_i$ .
- Given a bound  $W$ .
- **Goal:** select a subset  $S$  of items such that  $\sum_{i \in S} w_i \leq W$  and  $\sum_{i \in S} v_i$  is maximised.

### the fractional knapsack problem:

- Given a set of  $n$  items  $1, 2, \dots, n$
- Each item  $i$  has a non-negative weight  $w_i$  and a non-negative value  $v_i$ .
- Given a bound  $W$ .
- **Goal:** select a fraction  $x_i$  of each item  $i$  such that  $\sum_{i \in S} w_i x_i \leq W$  and  $\sum_{i \in S} v_i x_i$  is maximised.

### The 0/1 knapsack problem: Solution for 0/1 knapsack problem:

---

**Algorithm 22** 0/1 knapsack in dynamic programming

---

```
procedure 0/1 KNAPSACK( $n, W$ )  
  Array  $M[0, \dots, n, 0, \dots, W]$   
   $M[0, w] = 0$  for each  $w = 0, 1, \dots, W$   
  for  $i = 1$  to  $n$  do  
    for  $w = 0$  to  $W$  do  
      if  $w_i > w$  then  
         $M[i, w] = M[i - 1, w]$   
      else  
         $M[i, w] = \max\{M[i - 1, w], M[i - 1, w - w_i] + v_i\}$   
      end if  
    end for  
  end for  
  return  $M[n, W]$   
end procedure
```

---

# Chapter 6

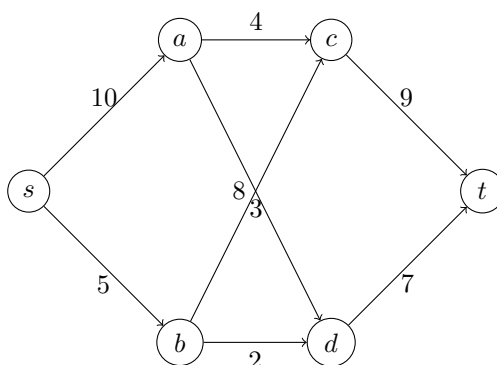
## Network Flow

### 6.1 Network Flow Definitions

**Flow network:** A flow network is a directed graph  $G = (V, E)$  with the following properties:

- Each edge  $(u, v) \in E$  has a non-negative capacity  $c_e$ .
- There is a single source  $s$  in  $V$ .
- There is a single sink  $t$  in  $V$ .
- All other nodes in  $V - \{s, t\}$  are called intermediate nodes.

example:



**Further definitions:**

- The source  $s$  has no incoming edges.
- The sink  $t$  has no outgoing edges.
- There is at least one edge incident to each node.
- All capacities are integers.

**Flow:** An  $(s - t)$  flow is a function  $f : E \rightarrow \mathbb{R}^+$ , mapping each edge  $e$  to a non-negative real number  $f(e)$ . A feasible flow must satisfy the following conditions:

- Capacity: For each edge  $e \in E$ ,  $0 \leq f(e) \leq c_e$ .
- Flow conservation: for each node  $v \in V - \{s, t\}$ , we have

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

The source  $s$  generates flow, and the sink  $t$  absorbs flow.

Value of a flow  $f$ , denoted  $val(f)$ , is the total amount of flow generated by the source  $s$ :

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

Generally, define  $f^{out}(v)$  and  $f^{in}(v)$  for the flow going out of (resp. going into) node  $v$ .

Similarly, define  $f^{out}(S)$  and  $f^{in}(S)$  for sets of nodes  $S$ .

## 6.2 Maximum Flow Problem

**The maximum flow problem:** Given a flow network  $G = (V, E)$ , find a flow of maximum possible value.

**algorithm for maximum flow:**

Idea: push flow forward on edges with leftover capacity, push flow backward on edges that are already carrying flow.

**The residual graph  $G_f$ :**

The residual graph  $G_f$  of  $G$  (also called the flow network) is defined as follows:

- The node set  $V_f$  of  $G_f$  is the same as the node set  $V$ .
- For each edge  $(u, v) \in E$  which  $f(e) < c_e$ , there are  $c_e - f(e)$  "leftover" units of capacity.
  - We will call this number the **residual capacity** of edge  $e$ .
  - We will call the edge  $e$  a forward edge.
- For each edge  $(u, v) \in E$  with  $f(e) > 0$ , there is an edge  $e' = (v, u)$  in  $E_f$  with a capacity of  $f(e)$ . We will call the edge  $e'$  a backward edge.

**Working with residual graphs:**

- Find an  $(s - t)$  path  $P$  in  $G_f$ . This is called an **augmenting path**.
- Define the bottleneck of  $P$ ,
  - Denoted  $\text{bottleneck}(P, f)$
  - to be the minimum residual capacity of any edge in  $P$ .
- Define the augmentation of flow  $f$  into flow  $f'$ 
  - Denoted  $\text{augment}(f, P)$ .

**Augmenting the flow:**

Feasibility of capacity:

---

### Algorithm 23 Augmenting the flow

---

```

1: procedure AUGMENT( $f, P$ )
2:    $b \leftarrow \text{bottleneck}(P, f)$ 
3:   for each edge  $e = (u, v) \in P$  do
4:     if  $e$  is a forward edge then
5:        $f(e) \leftarrow f(e) + b$ 
6:     else
7:        $f(e) \leftarrow f(e) - b$ 
8:     end if
9:   end for
10:  return  $f$ 
11: end procedure

```

---

consider an arbitrary edge  $e = (u, v) \in P$ . Suppose that  $e$  is a forward edge.

$$0 \leq f(e) \leq f'(e) = f(e) + b \leq f(e) + (c_e - f(e)) = c_e$$

Suppose that  $e$  is a backward edge.

$$c_e \geq f(e) \geq f'(e) = f(e) - b \geq f(e) - (f(e) - 0) = 0$$

**The Ford-Fulkerson algorithm:**

---

**Algorithm 24** Max-flow algorithm

---

```

1: procedure MAX-FLOW( $G, s, t$ )
2:    $f(e) \leftarrow 0$  for all edges  $e \in E$ 
3:   while there exists an  $(s - t)$  path  $P$  in  $G_f$  do
4:      $f \leftarrow \text{augment}(f, P)$ 
5:      $f' \leftarrow \text{update}(f)$ 
6:      $G_f \leftarrow \text{update}(G_f, f)$ 
7:   end while
8:   return  $f$ 
9: end procedure

```

---

**Running time of Ford-Fulkerson algorithm:**  $O(mC)$ , where  $C$  is the maximum capacity of any edge in the network.

## 6.3 Min Cut theorem

A cut  $C$  is a partition of the nodes of  $G$  into two sets  $S$  and  $T$  such that  $s \in S$  and  $t \in T$ .

The capacity of a cut  $C = (S, T)$  of a cut  $C$  is the sum of the capacities of the edges "out of"  $S$ : these are edges  $(u, v)$  such that  $u \in S$  and  $v \in T$ .

**The min-cut theorem:** In every flow network, the value of the maximum flow is equal to the capacity of the minimum cut.

**A series of facts:**

Fact 1: Let  $f$  be any  $(s - t)$  flow and let  $(S, T)$  be any cut. Then  $v(f) = f^{out}(S) - f^{in}(S)$ .

1. By definition,  $v(f) = f^{out}(s)$ .
2. By definition  $f^{in}(s) = 0$ .
3. Hence,  $v(f) = f^{out}(s) - f^{in}(s)$ .
4. For every other node  $v \neq s, t$ , we have  $f^{out}(v) = f^{in}(v)$ .
5. Therefore,  $v(f) = \sum_{v \in S} (f^{out}(v) - f^{in}(v))$ .
6. rewrite as  $v(f) = \sum_{v \in S} (f^{out}(v) - f^{in}(v)) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = f^{out}(S) - f^{in}(S)$ .

Fact 2: Let  $f$  be any  $(s - t)$  flow and let  $(S, T)$  be any  $(s - t)$  cut. Then  $v(f) = f^{out}(T) - f^{in}(T)$ .

Fact 3: Let  $f$  be any  $(s - t)$  flow and let  $(S, T)$  be any  $(s - t)$  cut. Then  $v(f) \leq c(S, T)$ .

$$\begin{aligned}
 v(f) &= f^{out}(S) - f^{in}(S) \\
 &\leq f^{out}(S) \\
 &= \sum_{e \text{ out of } S} f(e) \\
 &\leq \sum_{e \text{ out of } S} c(e) \\
 &= c(S, T)
 \end{aligned}$$

Fact 4: Let  $f$  be any  $(s - t)$  flow in  $G$  such that the residual graph  $G_f$  contains no augmenting paths. Then there exists an  $(s - t)$  cut  $(S^*, T^*)$  such that  $v(f) = c(S^*, T^*)$ .

Proving fact 4: In the residual graph  $G_f$ , identify all nodes that are reachable from the source  $s$ . Let  $S^*$  be the set of these nodes, and let  $T^* = V - S^*$ .

1.  $s \in S^*$  and  $t \in T^*$ .
2. No edge of  $G_f$  crosses from  $S^*$  to  $T^*$ .
3. Every edge of  $G_f$  crosses from  $T^*$  to  $S^*$ .
4.  $f^{out}(S^*) = v(f)$ .
5.  $f^{in}(S^*) = 0$ .

$$\begin{aligned}
 v(f) &= f^{out}(S^*) - f^{in}(S^*) \\
 &= \sum_{e \text{ out of } S^*} f(e) - \sum_{e \text{ into } S^*} f(e) \\
 &= \sum_{e \text{ out of } S^*} c(e) - 0 = c(S^*, T^*)
 \end{aligned}$$

Fact 5: If all capacities are integers, then there exists a maximum flow  $f$  for which  $f(e)$  is an integer for every edge  $e$ .

## 6.4 Choosing Better Augmenting Paths

The Edmonds-Karp algorithm:

---

**Algorithm 25** Edmonds-Karp algorithm

---

```

1: procedure EDMONDS-KARP( $G, s, t$ )
2:    $f(e) \leftarrow 0$  for all edges  $e \in E$ 
3:   while there exists an  $(s - t)$  path  $P$  in  $G_f$  do
4:      $P$  is a shortest  $(s - t)$  path
5:      $f \leftarrow \text{augment}(f, P)$ 
6:      $f' \leftarrow \text{update}(f)$ 
7:      $G_f \leftarrow \text{update}(G_f, f)$ 
8:   end while
9:   return  $f$ 
10: end procedure

```

---

**Running time of Edmonds-Karp algorithm:**  $O(nm^2)$ , where  $n$  is the number of nodes and  $m$  is the number of edges in the network.

The shortest path can be found in  $O(m)$  time using BFS.

## 6.5 Modeling with Network Flows

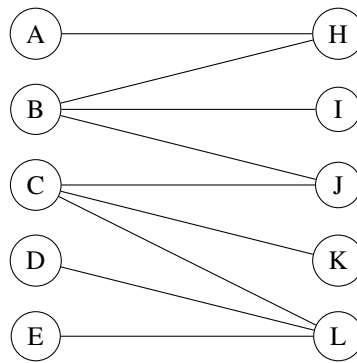
**Bipartite graphs:** A graph  $G = (V, E)$  is bipartite if and only if it can be partitioned into two sets  $A$  and  $B$  such that every edge has one endpoint in  $A$  and one endpoint in  $B$ .

**Maximum bipartite matching:**

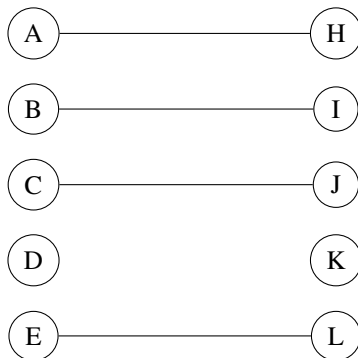
Matching: A subset  $M$  of edges  $E$  such that each node  $v \in V$  appears in at most one edge  $e \in M$ .

Maximum matching: A matching with maximum cardinality.

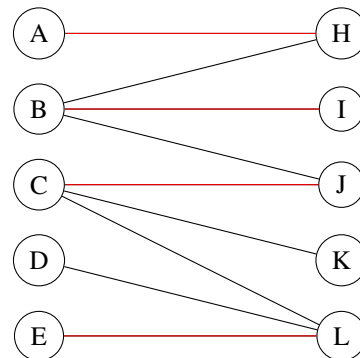
examples of bipartite graphs:



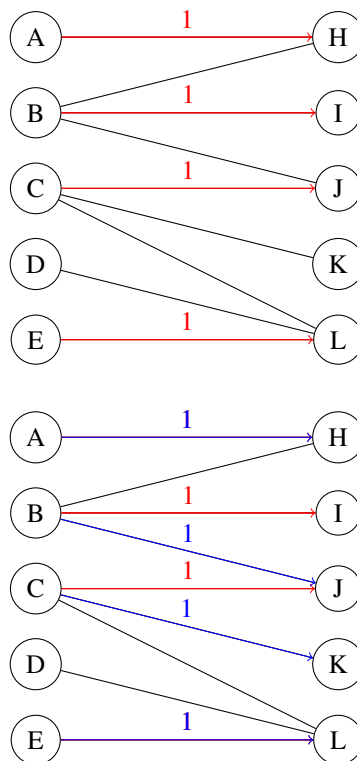
example of maximum bipartite matching:



example of maximal bipartite matching:



From matchings to flows:



### Maximum Flow and Maximum Matching

The size of the maximum matching is equal to the value of the maximum flow.

The edges of  $M$  are the edges that carry flow from  $A$  to  $B$  in the residual network.

Running time:  $O(mn)$

### Baseball Elimination

- Given a set  $S$  of teams
- For each team  $x$  in  $S$ , the current number of wins  $w_x$
- For teams  $x$  and  $y$  in  $S$ , they still have to play  $g_{xy}$  games against each other
- Given a designated team  $z$
- Can  $z$  still win the tournament?

### From Baseball Elimination to flows

- For each pair of teams  $x$  and  $y$ , create a vertex  $v_{xy}$
- For each team  $x$ , create a vertex  $v_x$
- For each pair of teams  $x$  and  $y$ , create an edge  $(s, v_{xy})$  with capacity  $g_{xy}$
- For each team  $x$ , create an edge  $(v_x, t)$  with capacity  $w_z + g_{xz} - w_x$
- For each pair of teams  $x$  and  $y$ , create an edge  $(v_{xy}, v_x)$  with infinite capacity
- For each pair of teams  $x$  and  $y$ , create an edge  $(v_{xy}, v_y)$  with infinite capacity

### Open pit mining

- Given a set  $S$  of blocks
- For each block  $x$  in  $S$ , the value  $v_x$  of the ore in the block
- For each block  $x$  in  $S$ , the cost  $c_x$  of mining the block
- For each block  $x$  in  $S$ , the set  $N_x$  of blocks that are neighbors of  $x$
- Given a designated block  $z$
- What is the maximum value of ore that can be mined?

### From open pit mining to flows

- For each block  $x$  in  $S$ , create a vertex  $v_x$
- For each block  $x$  in  $S$ , create an edge  $(s, v_x)$  with capacity  $v_x$
- For each block  $x$  in  $S$ , create an edge  $(v_x, t)$  with capacity  $c_x$
- For each block  $x$  in  $S$ , create an edge  $(v_x, v_y)$  with infinite capacity for each block  $y$  in  $N_x$



# Chapter 7

## NP-Completeness

### 7.1 NP-Completeness

#### Polynomial time reduction

- Given a problem  $A$  to solve
- Reduce solving  $A$  to solving  $B$
- Assume there is an algorithm  $ALG^B$  that solves  $B$  at cost  $O(1)$
- Construct an algorithm  $ALG^A$  that solves  $A$ , which uses  $ALG^B$  as a subroutine
- If  $ALG^A$  runs in polynomial time, then this is a polynomial time reduction

#### How to work with reductions

Positive: Assume that I want to solve problem  $A$  and I know how to solve problem  $B$ .

I can try come up with a polynomial time reduction  $A \leq^p B$ , which will give me a polynomial time algorithm for  $A$ .

Contrapositive: Assume that there is a problem  $A$  for which it is unlikely that there is a polynomial time algorithm that solves  $A$ .

If I come up with a polynomial time reduction  $A \leq^p B$ , it is also unlikely that there is a polynomial time algorithm that solves  $B$ .

$B$  is "at least as hard to solve as"  $A$ , because if I could solve  $B$ , I could also solve  $A$ .

#### Types of reductions

- Turing reduction:  $A \leq_T B$ 
  - A reduction which solves  $A$  using (potentially many) calls to an oracle for  $B$
  - As known as Cook reduction
- Many-one reduction:  $A \leq_m B$ 
  - A reduction which converts instances of  $A$  to instances of  $B$
  - Also known as Karp reduction

#### Problem classification

Problems in  $P$ :

Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, weighted interval scheduling, etc.

Problems in  $NP$ :

subset sum, knapSack, weighted interval scheduling, Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, etc.

#### NP-hardness

A problem  $B$  is NP-hard if for every problem  $A$  in  $NP$ ,  $A \leq^p B$ .

If every problem in  $NP$  is polynomial time reducible to  $B$ , then this captures the fact that  $B$  is at least as hard as any problem in  $NP$ .

#### 3 SAT

- A CNF formula with  $m$  clauses and  $k$  literals.

$$\varphi = (x_1 \vee \neg x_5 \vee x_3) \wedge (x_2 \vee x_6 \vee \neg x_5) \wedge \cdots \wedge (x_3 \vee x_8 \vee x_{12})$$

- each clause has exactly three literals
- Truth assignment: A value in  $\{0, 1\}$  for each variable  $x_i$
- Satisfying assignment: A truth assignment which makes the formula evaluate to 1
- Computational problem 3 SAT: Decide if the input formula  $\varphi$  has a satisfying assignment.

### 3 SAT is NP-complete

- 3 SAT is in  $NP$ 
  - Given a truth assignment, we can check in polynomial time if it is satisfying
- 3 SAT is NP-hard
  - Given a CNF formula  $\varphi$ , we can construct a polynomial time reduction to 3 SAT

**Proving NP-completeness** Suppose that you are given a problem  $A$  and you want to prove that  $A$  is NP-complete.

First, prove that  $A$  is in  $NP$ .

Usually by observing that a solution is efficiently verifiable.

Then prove that  $A$  is NP-hard.

construct a polynomial time reduction from a known NP-complete problem  $P$ .

## 7.2 NP-completeness of the vertex cover problem

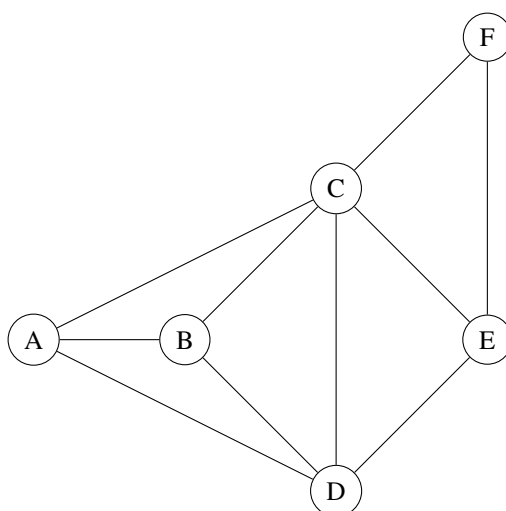
**Vertex cover** Definition: A vertex cover  $C$  of a graph  $G = (V, E)$  is a subset of vertices  $C \subseteq V$  such that for every edge  $e$  in  $E$ , at least one of the endpoints of  $e$  is in  $C$ .

Definition: A minimum vertex cover is a vertex cover of smallest possible size.

Input: A graph  $G = (V, E)$

Output: A minimum vertex cover

**Example**



the vertex cover  $\{A, C, E\}$  is not a minimum vertex cover  
the minimum vertex cover is  $\{A, E\}$ .

**Vertex cover is NP-hard:** construct a polynomial time reduction from 3 SAT to vertex cover.

Let  $\varphi$  be a 3 CNF formula with  $m$  clauses and  $d$  variables.

Construct in polynomial time an instance  $\langle G, k \rangle$  of vertex cover, with  $k = d + 2m$ .

if  $\varphi$  is satisfiable, then  $G$  has a vertex cover of size at most  $k$

Let  $(y_1, y_2, \dots, y_d)$  in  $\{0, 1\}^d$  be a satisfying assignment for  $\varphi$

For the nodes on the top: If  $y_i = 1$ . Include node  $x_i$  in the vertex cover  $C$ , otherwise include node  $\neg x_i$  in  $C$ .

For the nodes on the bottom: in each triangle, choose a node  $x_i$  that has been picked on the top and do not include it in the vertex cover. Include the other two nodes.

if  $\varphi$  is not satisfiable, then  $G$  has no vertex cover of size at most  $k$

Let  $C$  be a vertex cover of size  $k = d + 2m$  in  $G$ .

Since it is a vertex cover, it must include at least two out of three nodes in each "clause gadget" at the bottom. this means that at most  $d$  nodes can be picked on the top.

To satisfy the edges at the top, in each "variable gadget", at least one node must be picked.

## 7.3 Further reductions in NP

### Form optimization to decision

Given an optimization problem  $P$ , introduce a threshold  $k$ .

The decision version  $P_d$  becomes: Given an instance of  $P$  and the threshold  $k$  as input, is there a solution to  $P$  with value at most  $k$ ?

If  $P$  solved in polynomial time, then  $P_d$  is also solved in polynomial time.

If  $P_d$  solved in polynomial time, then  $P$  is also solved in polynomial time.

### NP-complete problems

Independent Set in graph  $G$ : A set of nodes in the graph, such that there is no edge between any two nodes in the set.

Maximum Independent Set: Given a graph  $G$ , find an independent set of maximum size.

Maximum Independent Set(decision version): Given a graph  $G$  and a threshold  $k$ , is there an independent set of size at least  $k$ ?

Set Packing: Given a set  $U$  and a collection of subsets  $S_1, S_2, \dots, S_m$  of  $U$  and a number  $k$ , does there exist a collection of at least  $k$  of these subsets that no two of them intersect?

Set Cover: Given a set  $U$  and a collection of subsets  $S_1, S_2, \dots, S_m$  of  $U$  and a number  $k$ , does there exist a collection of at most  $k$  of these sets whose union is  $U$ ?

3-Dimensional Matching: Given three disjoint sets  $X, Y, Z$  each of size  $n$ , and a collection of triples  $T \subseteq X \times Y \times Z$ , does there exist a set of  $n$  triples in  $T$ , so that each element of  $X \cup Y \cup Z$  appears in exactly one of these triples?

K-Colouring of a graph  $G$ : A function  $f : V \rightarrow \{1, 2, \dots, k\}$  so that for every edge  $(u, v)$  in  $E$ ,  $f(u) \neq f(v)$ .

3-Colouring: Given a graph  $G$ , can we colour the nodes of  $G$  using 3 colours so that no two adjacent nodes have the same colour?

Hamiltonian cycle in a directed graph  $G$ : A cycle in a directed graph that visits every node exactly once.

Hamiltonian path in a directed graph  $G$ : A path in a directed graph that contains every node exactly once.

Hamiltonian Cycle: Given a directed graph  $G$ , does  $G$  contain a Hamiltonian cycle?

Hamiltonian Path: Given a directed graph  $G$ , does  $G$  contain a Hamiltonian path?

Traveling Salesman: Given a complete graph  $G$  with edge weights, and a threshold  $k$ , is there a tour of  $G$  with total weight at most  $k$ ?

A taxonomy of NP-complete problems

Packing problems: Independent Set, Set Packing

Covering problems: Vertex Cover, Set Cover

Partitioning problems: 3-Dimensional Matching, Graph Colouring

Sequence problems: Hamiltonian Cycle, Hamiltonian Path, Traveling Salesman

Numerical problems: Subset Sum, Knapsack

Constraint satisfaction problems: 3 SAT

## Chapter 8

# Liner Programming

**Problem:** A company makes two products,  $X$  and  $Y$ . Each product requires time on two machines,  $A$  and  $B$ .

- Each unit of  $X$  requires 50 minutes on machine  $A$  and 30 minutes on machine  $B$ .
- Each unit of  $Y$  requires 24 minutes on machine  $A$  and 33 minutes on machine  $B$ .
- At the start of the week, there are 30 units of  $X$  and 90 units of  $Y$  in stock.
- The available processing time on machine  $A$  is 40 hours, and on machine  $B$  is 35 hours.
- The demand for  $X$  in this week is 75 units, and for  $Y$  is 95 units.

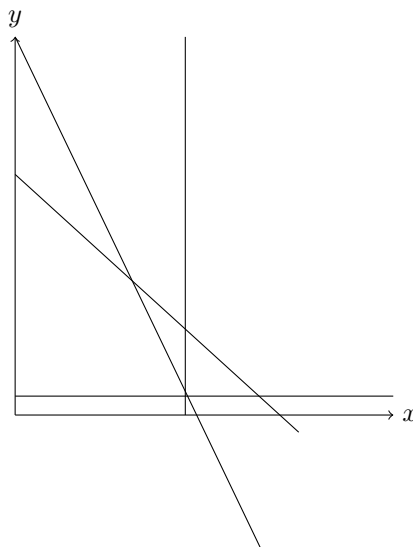
**Goal:** Maximise the combine sum of units of  $X$  and  $Y$  in stock at the end of the week.

$$\text{maximise } (x + 30 - 75) + (y + 90 - 95) = x + y - 50$$

Base in the Description, we can get the following equations:

$$\begin{aligned} &\text{maximize } x + y - 50 \\ &\text{subject to } 50x + 24y \leq 2400 \\ &\quad 30x + 33y \leq 2100 \\ &\quad x \leq 45 \\ &\quad y \leq 5 \end{aligned}$$

**Solving the linear program**



## 8.1 Linear Programming

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^n c_j x_j \\ &\text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \dots, m \\ &&& x_j \geq 0 \text{ for } j = 1, 2, \dots, n \end{aligned}$$

**matrix form**

$$\begin{aligned} &\text{maximize} && \mathbf{c}^T \mathbf{x} \\ &\text{subject to} && \mathbf{Ax} \leq \mathbf{b} \\ &&& \mathbf{x} \geq \mathbf{0} \end{aligned}$$

### Solving the linear program

- To find the optimal solution, it suffices to examine the corners of the feasible region.
- These are the intersection points of the lines defined by the constraints.
- This is the Simplex method.
- Other algorithm for solving LPs: Ellipsoid method, Interior point method.

### Integer Linear Programming

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^n c_j x_j \\ &\text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \dots, m \\ &&& x_j \geq 0 \text{ for } j = 1, 2, \dots, n \\ &&& x_j \in \mathbb{Z} \text{ for } j = 1, 2, \dots, n \end{aligned}$$

### Solving the integer linear program

The corners are not necessarily integer solutions. it does not suffice to look at the corners.  
WE can exhaustively search all possible integer solutions.  
Generally, this is NP-hard.

1. Linear Programming can be solved in polynomial time.
2. Integer Linear Programming Generally cannot be solved in polynomial time.(unless P=NP)

## 8.2 Duality

- Suppose that we have a linear program, which we call the primal.
- We will construct another linear program, which we call the dual.
- The variables of the primal become the constraints of the dual, and vice versa.
- Maximisation becomes minimisation.
- The two linear programs will have a very important connection.

### Primal

$$\begin{aligned} & \text{maximize} && \sum_{j=1}^n c_j x_j \\ & \text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \dots, m \\ & && x_j \geq 0 \text{ for } j = 1, 2, \dots, n \end{aligned}$$

### Dual

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^m b_i y_i \\ & \text{subject to} && \sum_{i=1}^m a_{ij} y_i \geq c_j \text{ for } j = 1, 2, \dots, n \\ & && y_i \geq 0 \text{ for } i = 1, 2, \dots, m \end{aligned}$$

### Weak Duality

Let  $\mathbf{x}$  be any feasible solution to the primal, and let  $\mathbf{y}$  be any feasible solution to the dual. Then

$$value(\mathbf{x}) \leq value(\mathbf{y})$$

### Strong Duality

Let  $\mathbf{x}$  be any feasible solution to the primal, and let  $\mathbf{y}$  be any feasible solution to the dual. if

$$value(\mathbf{x}) = value(\mathbf{y})$$

then  $\mathbf{x}$  and  $\mathbf{y}$  are optimal solutions to the primal and dual respectively.

### LP-relaxation

An LP-relaxation of an integer linear program is a linear program which is identical to the integer linear program, except all the integrality constraints have been removed, or replaced with non-negativity constraints.

Example of Minimum Cut as ILP and LP-relaxation

$$\begin{aligned} & \text{minimize} && \sum_{(u,v) \in E} c_{uv} d_{uv} \\ & \text{subject to} && d_{uv} - z_u + z_v \geq 0 \text{ for each } (u,v) \in E, u \neq s, v \neq t \\ & && d_{su} + z_u \geq 0 \text{ for each } (s,u) \in E \\ & && d_{ut} - z_u \geq 0 \text{ for each } (u,t) \in E \\ & && d_{uv} \geq 0 \text{ for each } (u,v) \in E \\ & && z_u \in \{0, 1\} \text{ for each } u \in V - \{s, t\} \end{aligned}$$

# Chapter 9

## Approximation Algorithms

### 9.1 Approximation Algorithms

#### Why Approximation Algorithms?

- For some problems, there is no known polynomial time algorithm.
- We can design an approximation algorithm
  - It runs in polynomial time
  - Compute a solution that is guaranteed to be close to the optimal solution

#### Methods of approximation algorithms

- Greedy algorithms
- Linear programming and rounding
- Pricing method
- Dynamic programming on round inputs

#### Application: Load Balancing

- We have a set of  $m$  identical machines  $M_1, M_2, \dots, M_m$ .
- We have a set of  $n$  jobs, with job  $j$  having processing time  $t_j$ .
- We want to assign each job to some machine.
- Let  $A(i)$  be the set of jobs assigned to machine  $i$
- The load of machine  $i$  is  $T(i) = \sum_{j \in A(i)} t_j$
- The goal is to minimize the makespan, i.e.,  $T = \max_i T(i)$

The load balancing problem is NP-hard.

#### Greedy Algorithm

- Pick any job
- Assign it to the machine with the smallest load so far
- Remove it from the pile of jobs

#### Arguing about the optimal

- we want to prove that  $T$  is not far from  $T^*$
- We will show that  $T$  is not far from something which is smaller than  $T^*$

- Then it is certainly not far from  $T^*$
- Fundamental idea: Bounding the optimal from below (for minimisation problems) and from above (for maximisation problems).

Consider the total processing time of all jobs.(the sum of all  $t_j$ ).

one of the  $m$  machines must be allocated at least  $\frac{1}{m}$  of the total work.

we have that  $T^* \geq \frac{1}{m} \sum_{j=1}^n t_j$

Every job must be scheduled on some machine.

the makespan is certainly at least the largest processing time  $t_j$  of any job.

we have that  $T^* \geq \max_j t_j$

### The Proof

Consider the final job  $j$  assigned to machine  $M_i$  by the greedy algorithm. Consider the time when this assignment is made.

This was the smallest load among all machines.

Every other machine had load at least  $T_i - t_j$  at this time.

Summing up over all machines, we get

$$\sum_k T_k \geq m(T_i - t_j) \Rightarrow T_i - t_j \leq \frac{1}{m} \sum_k T_k$$

So we got the first lower bound.

$$T^* \geq T_i - t_j \text{ (first lower bound)}$$

Consider a job  $j$  assigned to machine  $M_i$  by the greedy algorithm. Consider the time when this assignment took place.

The load of machine  $j$  was  $T_i - t_j$ .

This was before we added the job.

After we add the job, the load is  $T_i - t_j + t_j$ .

Obviously,  $t_j \leq \max_k t_k \leq T^*$ .

Since  $T^*$  is the optimal makespan, it must be at least as large as the load of the machine with the largest load.

So we got the second lower bound.

$$t_j \leq T^* \text{ (second lower bound)}$$

Combining the two lower bounds, we get

$$T_i \leq 2T^* \text{ (since } j \text{ was the final job)}$$

### Approximation Ratio

Consider a minimization problem  $P$  and an objective  $obj$ .

Here: Load Balancing on identical machines, and makespan.

- Consider an approximation algorithm  $A$ .
- Consider an input  $x$  to  $P$ .
- Let  $obj(A(x))$  be the value of the objective function on the output of  $A$ .

The approximation ratio of  $A$  is defined as

$$\max_{x \in \text{all inputs}} \frac{obj(A(x))}{obj(x)}$$

- In order to prove an upper bound on the approximation ratio, we have to somehow argue about all inputs.
- In order to prove a lower bound on the approximation ratio, we have to argue about a specific input.



For maximum problems, we define the approximation ratio as

$$\max_x \frac{obj(x)}{obj(A(x))}$$

For minimization problems, we define the approximation ratio as

$$\max_x \frac{obj(A(x))}{obj(x)}$$

Convention, to have approximation ratio always  $\geq 1$ .

A better greedy algorithm for load balancing

- Sort the jobs in decreasing order of processing times.
- Pick a job and according to this order.
- Assign it to the machine with the smallest load so far.
- Remove it from the pile of jobs.

---

**Algorithm 26** Greedy Algorithm for Load Balancing

---

```

1: Sort the jobs in decreasing order of processing times.
2:  $T \leftarrow 0$ 
3: for  $j = 1$  to  $n$  do
4:   Assign job  $j$  to the machine with the smallest load so far.
5:    $T \leftarrow \max(T, T(i))$ 
6: end for
7: return  $T$ 

```

---

Consider a job  $j$  that was assigned to machine  $M_i$  by the greedy algorithm. Consider the time when this assignment took place.

The load of machine  $j$  was  $T_i - t_j$ .

This was before we added the job.

After we add the final job, the load is  $T_i - t_j + t_j$ .

we established that  $t_j \leq t_{m+1} \leq \frac{1}{2}T^*$ .

Now we have the third lower bound.

$$t_j \leq \frac{1}{2}T^* \text{ (third lower bound)}$$

Combining the three lower bounds, we get

$$\begin{aligned}
 T_i &\leq \frac{3}{2}T^* \\
 \Rightarrow T &\leq \frac{3}{2}T^*
 \end{aligned}$$

The approximation ratio of the greedy algorithm actually is  $\frac{4}{3}$  with a better analysis.

For the loading balancing problem on identical machines, there is a Polynomial time approximation scheme (PTAS).

An algorithm which, given an input and a parameter  $\epsilon$ , runs in polynomial time and produces an outcome which is  $1 + \epsilon$  far from the optimal.

**Inapproximability**

A PTAS (or an FPTAS, more on that later) is the best approximation we can hope for, for an NP-hard problem.

Often it is not possible to get that close.

Inapproximability  $\alpha$  of a problem  $P$ :

There is no polynomial time algorithm that achieves an approximation ratio better than  $\alpha$ .

## 9.2 The traveling salesman problem

Problem definition:

$n$  cities,  $c_{ij}$  is the cost of traveling from city  $i$  to city  $j$ .

$$c_{ij} = c_{ji} \geq 0, c_{ii} = 0$$

Feasible solution: A tour that visits every city exactly once and returns to the starting city.

Optimal solution: A feasible tour with minimum total cost.

### Graph interpretation

Complete graph on  $n$  vertices.

$c_{ij}$  is the weight of the edge  $(i, j)$ .

Tour: Hamiltonian cycle in the graph.

Goal: A Hamiltonian cycle with minimum total weight.

Metric TSP problem: The cost satisfy the triangle inequality.

$$c_{ik} \leq c_{ij} + c_{jk}$$

### The Nearest Neighbor Algorithm

- Start with a greedy step
- Start with an edge  $(i, j)$  of minimum weight.
- Create a cycle  $(i, j, i)$ .

Generally, Assume that we have already visited a set of cities  $S$ . We find a city  $i \in S$  and a city  $j \notin S$  such that the cost of the edge  $(i, j)$  is minimum possible. We include  $j$  in  $S$  and insert it between  $i$  and the city that follows  $i$  in the cycle. Consider the tour  $(u_1, u_2, \dots, u_n)$  and remove the final that brings us back to  $u_1$ . We will have a spanning tree. So the cost of the tour is at least the cost of the spanning tree.

**Lemma:**  $\text{cost}(\text{optimal tour}) \geq \text{cost}(\text{minimum spanning tree})$ .

We can use Prim Algorithm to find the minimum spanning tree.

Let  $F = \{(u_2, v_2), (u_3, v_3), \dots, (u_n, v_n)\}$  be the set of edges in the minimum spanning tree.

$$\text{Hence } OPT \geq \sum_l c_{u_l, v_l}$$

The cost of the algorithm is at most  $2 \sum_l c_{u_l, v_l}$ .

The cost of the tour between first two nodes is  $2c_{u_1, v_1}$ .

When inserting the city  $j$ , the cost increases by  $c_{i,j} + c_{j,k} - c_{i,k} \leq c_{i,j} + c_{i,k} - c_{i,k} = 2c_{i,j}$ .

### A different perspective

- Start by finding a minimum spanning tree.
- Duplicate each edge of the tree.
- Do a traversal following the edges of the tree.
- Turn this into a tour by shortcutting.

### Eulerian Graph

There is a permutation of the edges  $(i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i_0)$ . If this exists, we can make a traversal of the edges, visiting each edge exactly once.

Property: A graph is Eulerian if and only if it is connected and every vertex has even degree.

Some Facts:

**Fact 1:** the sum of its degrees is even.

**Fact 2:** The total number of even-degree nodes is an even number.

**Fact 3:** The total number of odd-degree nodes is an even number.

**Fact 4:** The number of odd-degree nodes is even.

Constructing an Eulerian graph:

Start from the minimum spanning tree  $T$ .

Consider the set of odd-degree nodes  $O$ .

Find a minimum weight perfect matching of  $O$ . (This can be done in polynomial time.)

Add the edges of the matching to  $T$ .  
The resulting graph is Eulerian.  
Equivalently: The cost of the perfect matching is at most  $OPT/2$ .

#### Bundling the cost of the perfect matching

Claim: There is partial tour containing the cities in  $O$  that has cost at most  $OPT$ .

Consider the optimal tour.(on the whole graph)

Consider a part  $(i, u_1, u_2, \dots, u_l, j)$  of the tour where the cities  $u_1, u_2, \dots, u_l$  are in  $O$ .

"Shortcut" from  $i$  to  $j$  directly, without an increase in cost.

The cheapest of those two has cost at most  $OPT/2$ .

#### The approximation landscape of TSP

The  $\frac{3}{2}$ -approximation algorithm is due to Christofides (1976).

Inapproximability: 1.0045.

## 9.3 Primal-Dual Method

Design a polynomial time approximation algorithm for the weighted vertex cover problem.

#### The Pricing Method

Consider a vertex cover  $S$ .

Every vertex  $i$  has a cost, which is equal to  $w_i$ .

If  $i \in S$ , then the edges that "use it" have to pay for the cost.

Each edge  $e$  in the graph pays a price  $p_e$ .

Given a vertex cover  $S$ , we never ask the edges to pay more than the cost of  $i$ .

$$\sum_{e \in E} p_e \leq w_i$$

**Lemma:** Let  $S$  be any vertex cover and let  $p_e$  be any non-negative fair prices. Then, it holds that:

$$\sum_{e \in E} p_e \leq w(S)$$

**Proof of Lemma:**

1. By fairness, we have that:  $\sum_{e=(i,j)} p_e \leq w_i$

2. Adding up over all nodes, we have

$$\sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S)$$

3. Since  $S$  is a vertex cover, each edge contributes at least one term  $p_e$  to expression.

4. From this, we get that

$$\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e$$

Terminology: We will say that a node  $i$  is "tight" if  $\sum_{e=(i,j)} p_e = w_i$ .

---

#### Algorithm 27 Vertex-Cover-Approximation( $G, w$ )

---

- 1:  $p_e \leftarrow 0$  for all  $e \in E$
  - 2: **while** there is an edge  $e = (i, j)$  such that neither  $i$  nor  $j$  is tight **do**
  - 3:      $p_e \leftarrow \frac{w_i + w_j}{2}$
  - 4: **end while**
  - 5:  $S \leftarrow \{i : i \text{ is tight}\}$
  - 6: **return**  $S$
-

**second lemma:** The set  $S$  and the prices  $p$  returned by the Vertex-Cover-Approximation algorithm satisfy:

$$w(S) \leq 2 \sum_{e \in E} p_e$$

**Proof of second lemma:**

1. Consider a node  $i$  in  $S$ .
  - This means that  $i$  is tight.
  - This means that  $\sum_{e=(i,j)} p_e = w_i$ .
2. Summing up over all nodes in  $S$ , we get that

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e$$

3. An edge  $e = (i, j)$  can be included in the sum at most twice.
4. From this, we get that

$$w(S) = \sum_{i \in S} \sum_{e=(i,j)} p_e \leq 2 \sum_{e \in E} p_e$$

**Correctness:**

1. First,  $S$  is a vertex cover.
2. Suppose that it is not.
3. Then there is an edge  $e = (i, j)$  that is not covered by  $S$ .
4. This means that neither  $i$  nor  $j$  is tight.
5. The algorithm did not terminate, because of "while there is an edge  $e = (i, j)$  such that neither  $i$  nor  $j$  is tight".

**Approximation Ratio:**

- Let  $S^*$  be the minimum weight vertex cover.
- Recall that  $S$  is the vertex cover returned by the algorithm.
- we have:

$$\sum_{e \in E} p_e \leq w(S^*) \text{ and } w(S) \leq 2 \sum_{e \in E} p_e$$

- Combining the two, we get that  $w(S) \leq 2w(S^*)$ .

**Vertex Cover as an ILP:**

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} w_i x_i \\ & \text{subject to} && x_i + x_j \geq 1 \quad \forall (i, j) \in E \\ & && x_i \in \{0, 1\} \quad \forall i \in V \\ & && x_i \geq 0 \quad \forall i \in V \end{aligned}$$

**Vertex Cover LP-relaxation:**

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} w_i x_i \\ & \text{subject to} && x_i + x_j \geq 1 \quad \forall (i, j) \in E \\ & && 0 \leq x_i \quad \forall i \in V \end{aligned}$$

### The Dual

$$\begin{array}{ll} \text{maximize} & \sum_{(i,j) \in E} y_{ij} \\ \text{subject to} & \sum_{(i,j) \in E} y_{ij} \leq w_i \quad \forall i \in V \\ & y_{ij} \geq 0 \quad \forall (i,j) \in E \end{array} \quad \begin{array}{ll} \text{maximize} & \sum_{e \in E} p_e \\ \text{subject to} & \sum_{e=(i,j)} p_e \leq w_i \quad \forall i \in V \\ & p_e \geq 0 \quad \forall e \in E \end{array}$$

### Primal-dual method:

- We start with an infeasible integral solution  $x$  to the primal and a feasible fractional solution  $y$  to the dual.
- We increase the value of some  $y_j$  until some constraint becomes tight.
  - We obtain a better feasible fractional solution  $y$  to the dual.
  - We increase the corresponding variable  $x_i$  of the primal to obtain a still feasible integral solution  $x$  to the primal, which however violates fewer constraints.
- We end up with a feasible integral solution  $x$  to the primal and a feasible fractional solution  $y$  to the dual.
- We compare the two solutions.

## 9.4 Rounding Linear Programs

In last section, we designed an algorithm for the vertex cover problem, which used the following LP-relaxation:

$$\begin{array}{ll} \text{minimize} & \sum_{i \in V} w_i x_i \\ \text{subject to} & x_i + x_j \geq 1 \quad \forall (i,j) \in E \\ & 0 \leq x_i \quad \forall i \in V \\ & x_i \in \{0,1\} \quad \forall i \in V \end{array}$$

### Solving the LP-relaxation:

- We can solve the LP-relaxation in polynomial time, to find an optimal solution.
- The optimal solution is a "fractional" vertex cover, where variables can take values between 0 and 1.
- Is the value of the "fractional" vertex cover, smaller or larger than the value of the minimum weight vertex cover?

We do not know the optimal, so using a lower bound for the optimal.

Now we have a bound:

$$\text{optimal fractional VC weight} \leq \text{optimal VC weight} \leq \text{algorithm VC weight}$$

### Rounding the LP-relaxation:

We round the fractional solution to an integral solution.

- We pick a variable  $x_i$  and round it to 0 or 1.
- If we set everything to 0, it is not a vertex cover.
- If we set everything to 1, we "pay" too much.
- We set variable  $x_i$  to 1 if  $x_i \geq 1/2$  and to 0 otherwise.

**Feasibility:** Claim: The solution that we obtain is a vertex cover. This is, the round solution is a feasible solution to the ILP. Easy Proof: Assume that it is not.

- Then there is an edge  $e = (i,j)$  such that both  $x_i$  and  $x_j$  are 0.

- This means that in the LP-relaxation, we have that  $x_i < 1/2$  and  $x_j < 1/2$ .
- But then the constraint  $x_i + x_j \geq 1$  would be violated, and this would not be a feasible solution to the LP-relaxation.

### Approximation Ratio:

- Claim: The vertex cover obtained from rounding has weight at most twice that of the minimum weight vertex cover.
- Intuition is easy: We included all the vertices for which the LP-relaxation "paid" at least  $1/2$ , and we rounded them to 1, so we "paid" at most twice the optimal.
- One line proof: Let  $x^*$  be the optimal solution to the LP-relaxation, then

$$\sum_i w_i x_i^* \geq \sum_{i \in S} w_i x_i^* \geq \frac{1}{2} \sum_{i \in S} w_i = \frac{1}{2} w(S)$$

The LP-relaxation and round algorithm for vertex cover has approximation ratio of 2. We already know that 2 was possible, from the Pricing method algorithm. In this case, the ILP-relaxation and round algorithm seems conceptually simpler. In other cases, rounding the solution will not be straightforward.

### Limitations of the techniques

The LP-relaxation and round techniques cannot provide approximation ratios better than the integrality gap of the ILP-LP fomulation.

Caution: The integrality gap is a quantity has to do with the formulation of the ILP-LP, not the problem.

- An integrality gap of  $\alpha$  does not mean that it is not possible to design an algorithm with approximation ratio better than  $\alpha$ .
- actually, it does not even mean that LP-relaxation and round cannot give you an algorithm with approximation ratio better than  $\alpha$ .
- It means that with this formulation of the ILP-LP,  $\alpha$  is the best approximation ratio that you can get.

### Inapproximability of VC

- We saw two different algorithms, both with an approximation ratio of 2.
- Could we hope to design a better algorithm?
- The answer to this question depends on our definition of "hope".
- **Known fact 1:** It is impossible to design an algorithm with approximation ratio better than 1.363, unless  $P = NP$ .
- **Known fact 2:** It is impossible to design an algorithm with approximation ratio better than 2, unless Unique Games is an NP hard problem.
- Both facts are quite involved to prove.

### Easier Inapproximability

Definition: A problem  $P$  is strongly NP-hard, when it has a polynomial time reduction from a strongly NP-hard problem. For a strongly NP-hard problem

- There is no Fully Polynomial Time Approximation Scheme (FPTAS).
- There is no pseudopolynomial time algorithm that solves it exactly.

Equivalently, if it remains NP-complete even when all of its number parameters are bounded by a polynomial in the length of the input.

## VC on bipartite graphs

Vertex Cover as an ILP:

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} x_i \\ & \text{subject to} && x_i + x_j \geq 1 \quad \forall (i, j) \in E \\ & && x_i \in \{0, 1\} \quad \forall i \in V \\ & && x \geq 0 \quad \forall i \in V \end{aligned}$$

Vertex Cover as an LP-relaxation:

$$\begin{aligned} & \text{minimize} && \sum_{i \in V} x_i \\ & \text{subject to} && x_i + x_j \geq 1 \quad \forall (i, j) \in E \\ & && x \geq 0 \quad \forall i \in V \end{aligned}$$

The dual of the LP-relaxation:

$$\begin{aligned} & \text{maximize} && \sum_{e \in E} y_e \\ & \text{subject to} && y_e \leq 1 \quad \forall e \in E \\ & && y_e \geq 0 \quad \forall e \in E \end{aligned}$$

The ILP corresponding to the dual:

$$\begin{aligned} & \text{maximize} && \sum_{e \in E} y_e \\ & \text{subject to} && \sum_{e=\{i,j\} \in E} y_e \leq 1 \quad \forall i \in V \\ & && y_e \in \{0, 1\} \quad \forall e \in E \end{aligned}$$

## König's Theorem

In a bipartite graph, the size of the maximum matching is equal to the size of the minimum vertex cover.

- König's proof is constructive: It starts from a maximum matching, and constructs a minimum vertex cover, proving that it is minimum.
- Alternative proof based on total unimodularity.

So we have:

- size of the maximum matching = size of the minimum vertex cover
- size of the minimum vertex cover = size of minimum fractional VC.
- size of the maximum fractional matching = size of the minimum fractional VC.
- size of maximum matching = size of minimum VC.

Solve for VC on bipartite graphs:

- Pick a vertex  $v$  in the graph.
- Remove it to get graph  $G - \{v\}$ .
- Property: If  $v$  is in the minimum vertex cover,  $G - \{v\}$  has a minimum vertex cover of size  $k^* - 1$ .
- Check if the graph  $G - \{v\}$  has a vertex cover of size at most  $k^* - 1$ .

Yes: Include  $v$  in the vertex cover.

No: Do not include  $v$  in the vertex cover.

Then move on to the next vertex.

**Summing up**

Vertex Cover is strongly NP-hard in general graphs.

- In fact, hard to approximate better than 1.363.
- There exist 2-approximate polynomial time algorithms for the problem.

On bipartite graphs, the problem is solvable in polynomial time.