# Summary of COMP523 Advanced Algorithm

May 9, 2023

# Chapter 1

# Symmetry Notation

## 1.1 Asymptotic Notation

Asymptotic notation is a way of describing the limiting behavior of a function when the argument tends towards a particular value or infinity. In computer science, asymptotic notation is frequently used to describe the running time or space usage of an algorithm.

- $O$-notation: $f(n) = O(g(n))$ if there exist constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
- $\Omega$-notation: $f(n) = \Omega(g(n))$ if there exist constants $c$ and $n_0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
- $\Theta$-notation: $f(n) = \Theta(g(n))$ if there exist constants $c_1$, $c_2$ and $n_0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.
- $o$-notation: $f(n) = o(g(n))$ if for any constant $c > 0$, there exists a constant $n_0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.
- $\omega$-notation: $f(n) = \omega(g(n))$ if for any constant $c > 0$, there exists a constant $n_0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$.

## 1.2 Comparing Functions

### 1.2.1 Transitivity

- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ implies $f(n) = \Omega(h(n))$.
- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ implies $f(n) = \Theta(h(n))$.

For example, $n^2 = O(n^3)$ and $n^3 = O(n^4)$ implies $n^2 = O(n^4)$.

### 1.2.2 Reflexivity

- $f(n) = O(f(n))$.
- $f(n) = \Omega(f(n))$.
- $f(n) = \Theta(f(n))$.

For example, $n^2 = O(n^2)$.

### 1.2.3 Symmetry

- $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
- $f(n) = \Omega(g(n))$ implies $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ implies $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ implies $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ implies $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ implies $n^3 = \Omega(n^2)$.

### 1.2.4 Transpose Symmetry

- $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ if and only if $n^3 = \Omega(n^2)$.

### 1.2.5 sum and maximum

$$f_1(n) + f_2(n) + \cdots + f_k(n) = \Theta(\max(f_1(n), f_2(n), \ldots, f_k(n)))$$

where $k$ is a constant positive integer.
Let $f_j(n) = j$, $k = n$, then

$$f_1(n) + f_2(n) + \cdots + f_k(n) = n(n+1)/2 = \Theta(n^2)$$

### 1.2.6 Running time hierarchy

- logarithmic: $O(\log n)$
- linear: $O(n)$
- $n \log n$: $O(n \log n)$
- quadratic: $O(n^2)$
- polynomial: $O(n^k)$
- exponential: $O(c^n)$
- constant: $O(1)$
- superconstant: $\omega(1)$
- sublinear: $o(n)$
- superlinear: $\omega(n)$
- superpolynomial: $\omega(n^k)$
- subexponential: $o(c^n)$

## 1.3 Expect of algorithms

**Correctness**: An algorithm is correct if it halts with the correct output for every input instance.
**Termination**: An algorithm is terminating if it halts for every input instance.
**Efficiency**: An algorithm is efficient if it halts with the correct output for every input instance and runs in polynomial time.

# Chapter 2

# Recursion and Divide and Conquer techniques

## 2.1 Finding Majority in array

The pesudocode of the algorithm is shown in Algorithm 2.1.

**Algorithm 1** Finding Majority in array

**procedure** MAJORITY$(A)$
    $n \leftarrow$ length of $A$
    **if** $n = 0$ **then**
        **return** $-1$
    **end if**
    **if** $n = 1$ **then**
        **return** $A[1]$
    **end if**
    **if** $n1$ and $n$ is odd **then**
    **end if**
    Array $B$ of size $n/2$
    set $j$=0
    **for** $i = 1$ to $n/2$ **do**
        **if** $A[2i-1] = A[2i]$ **then**
            $B[j] \leftarrow A[2i-1]$
            $j \leftarrow j+1$
        **end if**
    **end for**
    $m \leftarrow$ MAJORITY$(B)$
    $count \leftarrow 0$
    **for** $i = 1$ to $n$ **do**
        **if** $A[i] = m$ **then**
            $count \leftarrow count + 1$
        **end if**
    **end for**
    **if** $count > n/2$ **then**
        **return** $m$
    **else**
        **return** $-1$
    **end if**
**end procedure**

**Correctness**:
Lemma: If $A$ has a majority element, then the majority element of $A$ is also the majority element of $B$.
Base case: $n = 1$, the majority element is $A[1]$.
Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.
Induction step: If $A$ has a majority element, then the majority element of $A$ is also the majority element of $B$.
Case 1 ($A$ has a majority element $m$): Then by the lemma, it is also the majority element of $B$. Then $m$ appears more than $k/2$ times in $B$. Then $m$ appears more than $(k + 1)/2$ times in $A$.
Case 2 ($A$ has no majority element): Then $B$ has no majority element. Then $A$ has no majority element.
**Proof the lemma**:
proof by contradiction. Assume that $A$ has a majority element $m$ and $B$ has a majority element $m'$, but $m \neq m'$.
Let $x$ be the numbers of occurrence of $m$ in $A$.
Let $y$ be the numbers of occurrence of $m'$ in $B$.
Then $2y$ times from pairs that are represented in $B$ by a value different from $m'$, and $x - 2y$ times, since each occurrence of $m$ in $A$ that is not paired with another occurrence of $m$ in $A$ is paired with an occurrence of $m'$ in $B$.
In total, this gives $2y + x - 2y = x$ occurrences of $m$ in $A$, which is a contradiction.
**Running time**:
Recursive formula for the running time:

$$T(n) \leq T(n/2) + cn$$

where $c$ is a constant.
The solution to the recurrence is $T(n) = O(n)$.

## 2.2 Searching in logarithmic time

Searching faster with BinarySearch.
It is a particular case of the divide-and-conquer paradigm.
**Input**: A sorted array $A$ of $n$ elements and a value $x$.
**Output**: An index $i$ such that $A[i] = x$ or the special value $-1$ if $x$ does not appear in $A$.
**Pseudocode** is shown in Algorithm 2.2.

**Algorithm 2** BinarySearch

**procedure** BINARYSEARCH$(x, i, j)$
    **if** $i = j$ **then**
        **if** $A[i] = x$ **then**
            **return** $i$
        **else**
            **return** $-1$
        **end if**
    **else**
        **if** $x = A[\lfloor (i + j)/2 \rfloor]$ **then**
            **return** $\lfloor (i + j)/2 \rfloor$
        **else if** $x < A[\lfloor (i + j)/2 \rfloor]$ **then**
            **return** BINARYSEARCH$(x, i, \lfloor (i + j)/2 \rfloor)$
        **else**
            **return** BINARYSEARCH$(x, \lfloor (i + j)/2 \rfloor + 1, j)$
        **end if**
    **end if**
**end procedure**

**Running time**:
The number of comparisons performed by BinarySearch is:

$$T(n) \leq T(n/2) + 4$$

Keep calculate:

$$\begin{aligned}
T(n) &\leq T(n/2) + 4 \\
&\leq T(n/4) + 4 + 4 \\
&\leq T(n/8) + 4 + 4 + 4 \\
&\leq T(n/2^k) + 4k \\
&\leq T(n/2^{\log(n-1)}) + 4\log(n-1) \\
&= T(2) + 4(\log n - 1) \\
&\leq 4\log n - 4 \\
&= 4\log n
\end{aligned}$$

proof $T(n) \leq 4\log n$:
Base case: $n = 1$, $T(1) = 0 \leq 4\log 1 = 0$.
Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k+1$.
Induction step: $T(k+1) \leq 4\log(k+1)$.

$$\begin{aligned}
T(k+1) &\leq T(k/2) + 4 \\
&\leq 4\log(k/2) + 4 \\
&= 4\log k - 4 + 4 \\
&= 4\log k \\
&\leq 4\log(k+1)
\end{aligned}$$

**Memory usage**:
The memory usage of BinarySearch is:

$$M(n) = O(\log n)$$

**Comparing BinarySearch and LinearSearch**:

$$T_{\text{BinarySearch}}(n) = O(\log n)$$
$$T_{\text{LinearSearch}}(n) = O(n)$$
$$T_{\text{BinarySearch}}(n) = O(\log n) < O(n) = T_{\text{LinearSearch}}(n)$$
$$M_{\text{BinarySearch}}(n) = O(\log n) < O(1) = M_{\text{LinearSearch}}(n)$$

## 2.3 Running time of Divide and Conquer algorithms

The Master Theorem:
Suppose that $T(n)$ satisfies the recurrence:

$$T(n) \leq aT(n/b) + cn^d$$

where $a \geq 1$, $b > 1$, $c > 0$ and $d \geq 0$ are constants.
Then $T(n)$ has the following asymptotic bounds:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

This theorem is useful for solving recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$ and $f(n)$ is an asymptotically positive function.
**Example**:

$$T(n) = 8T(n/2) + 100n^2$$

$a = 8$, $b = 2$, $f(n) = 100n^2$, $d = 2$, $\log_b a = \log_2 8 = 3$.
$d = 2 < \log_b a = 3$, so $T(n) = O(n^{\log_b a}) = O(n^3)$.

## 2.4 Finding piar of points closest to each other

**Input**: A set $P$ of $n$ points in the plane.
**Output**: The pair of points in $P$ that are closest to each other.
**Pseudocode** is shown in Algorithm 2.4. **Running time**:

**Algorithm 3** ClosestPair

**procedure** ClosestPair($P_1, \ldots, P_n$)
    Construct $P_x$ and $P_y$. $P_x$ is sorted by $x$-coordinate, $P_y$ is sorted by $y$-coordinate.
    **return** ClosestPairRec($P_x, P_y$)
**end procedure**

**Algorithm 4** ClosestPairRec

**procedure** ClosestPairRec($P_x, P_y$)
    **if** $|P_x| = |P_y| \leq 3$ **then**
        For each pair of points $(P_i, P_j)$, compute $d(P_i, P_j)$
        **return** the pair of points with the smallest distance
    **end if**
    Construct $Q_x$, $Q_y$, $R_x$ and $R_y$.
    $(l_1, l_2)$ = ClosestPairRec($Q_x, Q_y$)
    $(r_1, r_2)$ = ClosestPairRec($R_x, R_y$)
    $\delta = \min\{d(l_1, l_2), d(r_1, r_2)\}$
    $x^*$ = the largest $x$-coordinate in $Q_x$
    $L = (x, y) : x = x^*$
    $S = \{p \in P : p \in L$ and $p$ is within $\delta$ of $L\}$
    Construct $S_v$
    **for** $p \in S$ **do**
        Let $q$ be the point in $S_v$ closest to $p$
        **if** $d(p, q) < \delta$ **then**
            $\delta = d(p, q)$
            $(s_1, s_2)$ = $(p, q)$
        **end if**
    **end for**
    **if** $d(s_1, s_2) < \min\{d(l_1, l_2), d(r_1, r_2)\}$ **then**
        **return** $(s_1, s_2)$
    **end if**
    **if** $d(l_1, l_2) < d(r_1, r_2)$ **then**
        **return** $(l_1, l_2)$
    **else**
        **return** $(r_1, r_2)$
    **end if**
**end procedure**

$T(n) \leq 2T(n/2) + O(n \log n) = O(n \log n)$
**Example**:

# Chapter 3

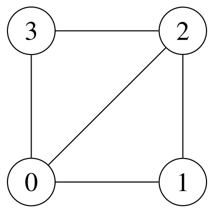
# Graph Algorithms

## 3.1 Graph Definitions

**Graph**: A graph $G$ consists of a set $V$ of vertices and a set $E$ of edges, where each edge is associated with a pair of vertices.
**Directed Graph**: A directed graph $G$ consists of a set $V$ of vertices and a set $E$ of directed edges, where each directed edge is associated with an ordered pair of vertices.
**Undirected Graph**: An undirected graph $G$ consists of a set $V$ of vertices and a set $E$ of undirected edges, where each undirected edge is associated with an unordered pair of vertices.
**Neighbours of a vertex** $v$: Set of vertices that are connected to $v$ by an edge.
**Degree of a vertex** $v$: number of neighbours of $v$, denoted by $deg(v)$.
**Path**: A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.
length = node count - 1 = edge count.
**Distance between two nodes**: The number of edges in the shortest path between the two nodes.
**Graph diameter**: The maximum distance between any two nodes in the graph.

**Lines,cycles,trees and cliques**:
**Line**: A graph with $n$ vertices and $n-1$ edges.
**Cycle**: A graph with $n$ vertices and $n$ edges.
**cliques**: A graph with $n$ vertices and $n(n-1)/2$ edges.
**Tree**: A graph with $n$ vertices and $n-1$ edges.

**Graph representations**:
**Adjacency matrix**: A $n \times n$ matrix $A$ where $A_{ij} = 1$ if there is an edge between $i$ and $j$, and $A_{ij} = 0$ otherwise.
examples of adjacency matrices:
Given the following graph:

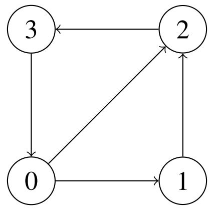



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

**Adjacency matrix for directed graphs**: A $n \times n$ matrix $A$ where $A_{ij} = 1$ if there is an edge from $i$ to $j$, and $A_{ij} = 0$ otherwise.

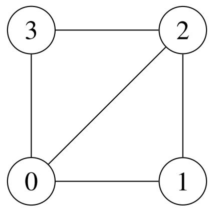
examples of adjacency matrices for directed graphs:
Given the following graph:




The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

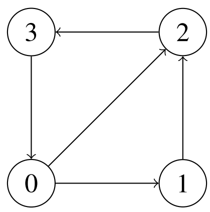
**Adjacency list**: A list of lists, where the $i$th list contains the neighbours of vertex $i$.
Given the following graph:




The adjacency list is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \\ 0 & 1 & 3 \\ 0 & 2 & \end{bmatrix}$$

**Adjacency list for directed graphs**: A list of lists, where the $i$th list contains the neighbours of vertex $i$.
Given the following graph:




The adjacency list is:

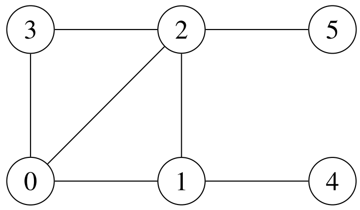
$$\begin{bmatrix} 1 & 2 \\ 2 & \\ 3 & \\ 0 & \end{bmatrix}$$

**Adjacency matrix vs adjacency list**:

| Adjacency matrix | Adjacency list |
|---|---|
| $O(1)$ to check if there is an edge between $i$ and $j$ | $O(min(deg(i), deg(j)))$ to check if there is an edge between $i$ and $j$ |
| $O(n)$ to find the neighbours of $i$ | $O(deg(j))$ to find the neighbours of $i$ |
| $O(n^2)$ space | $O(n + m)$ space |

## 3.2 Depth-first search

**Depth-first search**: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.
example of depth-first search:




The depth-first search sequence is:

$$0, 1, 2, 3, 5, 4$$

**Depth-first search algorithm**:

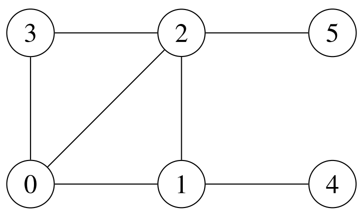
**Algorithm 5** Depth-first search algorithm

**procedure** DFS$(G, v)$
  **for** $e \in V$ **do**
    **if** $e$ is unexplored **then**
      $u$ = head of $e$
      **if** $u$ is unexplored **then**
        $e$ is a tree edge
        DFS$(G, u)$
      **else**
        $e$ is a back edge
      **end if**
    **end if**
  **end for**
**end procedure**

**Running time of depth-first search**: $O(n + m)$

## 3.3 Breadth-first search

**Breadth-first search**: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.
exaqmple of breadth-first search:




The breadth-first search sequence starting from vertex 0 is 0, 1, 2, 3, 4, 5.
**Breadth-first search algorithm**:

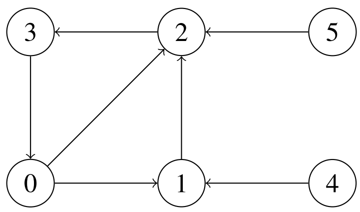
**Algorithm 6** Breadth-first search algorithm

**procedure** BFS($G, s$)
    initial empty list $L$
    $L \leftarrow s$
    $i \leftarrow 0$
    **while** $L[i] \neq \emptyset$ **do**
        $L_{i+1} \leftarrow emptylist$
        **for** $v \in L[i]$ **do**
            **for** edges $(e)$ incident to $v$ **do**
                **if** $e$ is unexplored **then**
                    $w \leftarrow$ the other end of $e$
                    **if** $w$ is unexplored **then**
                        label $e$ as a tree edge
                        add $w$ to $L_{i+1}$
                    **else**
                        label $e$ as a cross edge
                    **end if**
                **end if**
            **end for**
        **end for**
        $i \leftarrow i + 1$
    **end while**
**end procedure**

**Running time of breadth-first search**: $O(n + m)$

## 3.4 Strong Connectivity

**Directed graph**: A graph where the edges have a direction.
Examples:




**DFS and BFS on directed graphs**:
Very similar to undirected graphs, except that we only consider edges that go out of a vertex.
Running time is $O(n + m)$
For example graph above the DFS sequence is 0, 1, 2, 3.
The BFS sequence is 0, 1, 2, 3.

### 3.4.1 Connectivity

**Weak connectivity**:If we ignore the direction for all edges, there would be a pah from any vertex to any other vertex.
**Strong Connectivity**:For every two nodes $u$ and $v$, there is a path from $u$ to $v$ and a path from $v$ to $u$.

### 3.4.2 Mutual Reachability

Two nodes $u$ and $v$ are mutually reachable if there is a path from $u$ to $v$ and a path from $v$ to $u$.
**Strong connectivity**:For every pair of nodes $u$ and $v$, these two nodes are mutually reachable.
**Transitivity**:If $u$ is mutually reachable with $v$ and $v$ is mutually reachable with $w$, then $u$ is mutually reachable with $w$.

### 3.4.3 Testing strong connectivity

**Algorithm 7** Testing strong connectivity

**procedure** TESTSTRONGCONNECTIVITY($G$)
    define $G^R$ to be the graph with the same vertices as $G$ but with all edges reversed
    Select a node $s$ in $G$
    BFS($G, s$),BFS($G^R, s$)
    **for** each node $v$ **do**
        **if** $v$ is unexplored in either BFS **then**
            **return** False
        **end if**
    **end for**
    **return** True
**end procedure**

## 3.5 Testing bipartiteness

**Bipartite graph**: A graph $G = (V, E)$ is bipartite if any only if the vertices can be partitioned into two sets $V_1$ and $V_2$ such that every edge has one end in $V_1$ and the other end in $V_2$.
A Graph $G = (V, E)$ is bipartite if and only if it has no odd cycles.(odd cycle: a cycle with odd number of edges)
**Testing bipartiteness**:
Given a graph $G = (V, E)$, we want to test if $G$ is bipartite.
Given a graph $G = (V, E)$, decide if it is 2-colourable.
Given a graph $G = (V, E)$, decide if it has an odd cycle.
**Colouring the nodes** It is quite familiar with BFS:

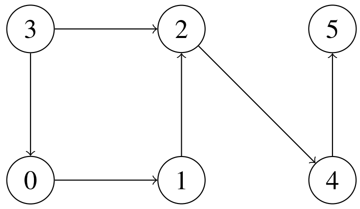
**Algorithm 8** Colouring the nodes

```
procedure COLOURING(G, s)
    initial empty list L
    initial empty list C
    L ← s
    C[s] ← red
    i ← 0
    while L[i] ≠ ∅ do
        L_{i+1} ← emptylist
        for v ∈ L[i] do
            for edges (e) incident to v do
                if e is unexplored then
                    w ← the other end of e
                    if w is unexplored then
                        label e as a tree edge
                        add w to L_{i+1}
                        if i + 1 is odd then
                            C[w] ← green
                        else
                            C[w] ← red
                        end if
                    else
                        label e as a cross edge
                        if C[v] = C[w] then
                            return False
                        end if
                    end if
                end if
            end for
        end for
        i ← i + 1
    end while
    for e(v, w) ∈ G do
        if C[v] = C[w] then
            return False
        end if
    end for
    return True
end procedure
```

**Running time of colouring the nodes**: $O(n + m)$
**Correctness of colouring the nodes**:
Proof by contradiction.
Suppose that $G$ is not bipartite.
Then $G$ has an odd cycle.
Suppose to the contrary that the algorithm return True.
That means that the algorithm did not detect the odd cycle.

## 3.6 DAGs and Topological Ordering

**DAG**: A directed acyclic graph (DAG) is a directed graph with no directed cycles.
examples of DAGs:




**Topological ordering**:Given a graph$G = (V, E)$, a topological ordering of $G$ is an ordering of the nodes $u_1, u_2, \ldots, u_n$ such that for every edge $(u_i, u_j)$, we have $i < j$.
Intutively, a topological ordering is an ordering of the nodes such that every edge goes from left to right.
example of topological ordering based on given graph above:

$$3, 0, 1, 2, 4, 5$$

**Topological ordering implies DAG**:

- If $G$ has a topological ordering, then $G$ is a DAG.
- Suppose by contradiction that $G$ has a topological ordering $u_1, u_2, \ldots, u_n$ but $G$ also has a cycle $C$.
- Let $u_j$ be the smallest element of $C$ in the topological ordering.
- Let $u_i$ be its predecessor in $C$.
- $u_i$ must appear before $u_j$ in the topological ordering.
- This contradicts the fact that $u_j$ is the smallest element of $C$ in the topological ordering.

**DAG implies topological ordering**:
Proof by induction: Base case: If $G$ has one or two nodes, then $G$ has a topological ordering.
Induction steps: Assume that a DAG up to $k$ nodes has a topological ordering(induction hypothesis). we will prove that a DAG with $k + 1$ nodes has a topological ordering.

- By our lemma, there is at least one source node in $G$, and let $u$ be the node.
- Put $u$ at the beginning of the topological ordering.
- Consider the graph $G'$, obtained by $G$ by removing $u$ and its incident edges.
- $G'$ is a DAG with $k$ nodes.
- It has a topological ordering $u_1, u_2, \ldots, u_k$ by the induction hypothesis.
- Append this ordering to $u$ to get a topological ordering of $G$.

Here is the algorithm:

**Algorithm 9** Topological Sorting

**procedure** TOPOLOGICALSORTING($G$)
  find a source vertex $u$
  set $u$ as the first element of the topological ordering
  $G' \leftarrow G$ with $u$ and its incident edges removed
  $L \leftarrow$ TOPOLOGICALSORTING($G'$)
  append $L$ to $u$
**end procedure**

Running time of the algorithm is $O(n^2)$
**Modified Topological Sorting**:
Running time of the algorithm is $O(n + m)$

**Algorithm 10** Modified Topological Sorting

**procedure** MODIFIEDTOPOLOGICALSORTING($G$)
    $L \leftarrow emptylist$
    $S \leftarrow$ set of all source vertices
    **while** $S \neq \emptyset$ **do**
        remove a vertex $u$ from $S$
        append $u$ to $L$
        **for** each edge $(u, v)$ **do**
            remove edge $(u, v)$ from $G$
            **if** $v$ is a source vertex **then**
                add $v$ to $S$
            **end if**
        **end for**
    **end while**
    **if** $G$ has edges **then**
        **return** $G$ has a cycle
    **else**
        **return** $L$
    **end if**
**end procedure**

## 3.7 Finding strongly connected components

**connected components**: A connected component of an undirected graph is subgraph of the graph where any two nodes are connected by a path.
**strongly connected components**: A strongly connected component of a directed graph is a subgraph of the graph where any two nodes are mutually reachable.(mutually reachable: there is a path from $u$ to $v$ and a path from $v$ to $u$)
**Finding strongly connected components**:
**Kosaraju's algorithm**:

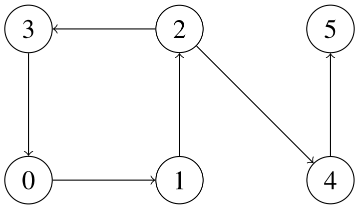
**Algorithm 11** Kosaraju's algorithm

**procedure** KOSARAJU($G$)
    Initialise stack $S$
    Select a arbitrary node $s$
    DFS_tree=DFS($G, s$)
    $S \leftarrow$ nodes in DFS_tree
    $G^R \leftarrow$ nodes in order of $S$
    DFS($G^R, s$)
    **return** the nodes in the DFS tree
**end procedure**

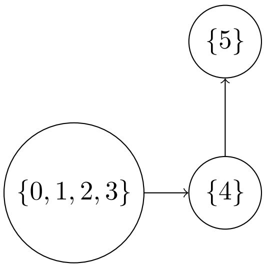
**Running time of Kosaraju's algorithm**: $O(n + m)$
**Correctness of Kosaraju's algorithm**:

- Define a meta-graph of $G$, called $G^{SCC} = (V^{SCC}, E^{SCC})$.
- Supposed that $G$ has strongly connected components (SCCs) $C_1, C_2, \ldots, C_k$, for some $k$.
- $V^{SCC} = \{C_1, C_2, \ldots, C_k\}$ contains some of the SCCs of $G$.
- There is an edge $(C_i, C_j)$ in $E^{SCC}$ if $G$ contains a directed edge $(x, y)$ such that $x \in C_i$ and $y \in C_j$, crossing different components.

Examples:




The SCCs are $\{0, 1, 2, 3\}$ and $\{4, 5\}$.
The meta-graph is:



# Chapter 4

# Greedy Algorithms

**The greedy approach**:

- The goal is to find a global solution to a problem.
- The solution will be built up in small consecutive steps.
- For each step, we choose the best option available to us at that moment.

## 4.1 Interval Scheduling

**Interval Scheduling**:
A set of requests$R = \{1, 2, \ldots, n\}$.

- Each request $i$ has a start time $s_i$ and a finish time $f_i$.
- Alternative view: every request is an interval $[s_i, f_i]$.

Two requests $i$ and $j$ are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.
**Goal**: Find a maximum-size subset of compatible requests.
**Example**:

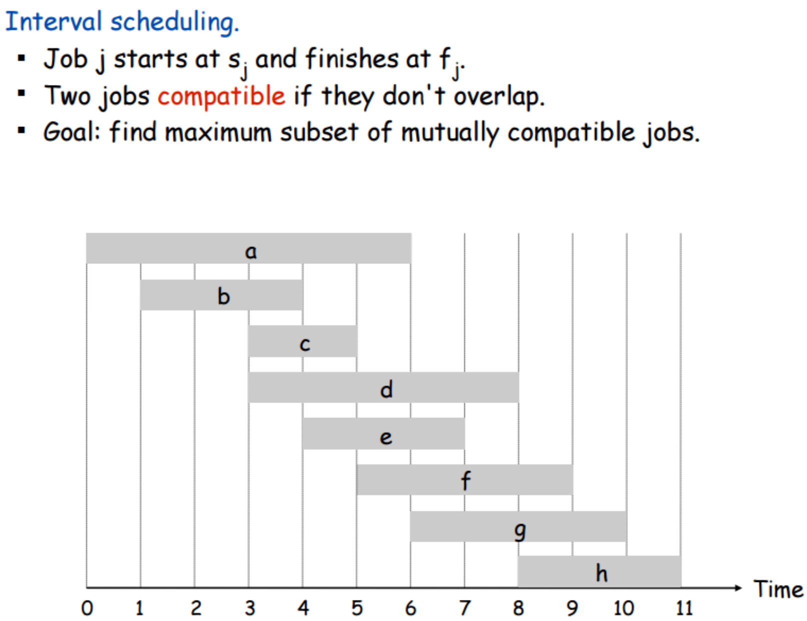



Figure 4.1: Interval Scheduling

**Interval Scheduling Algorithm**:

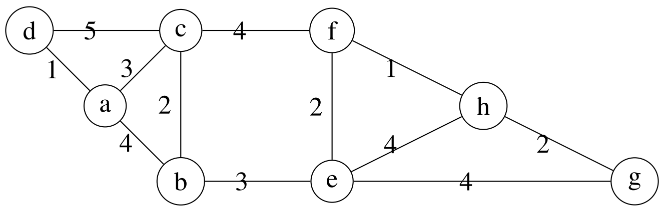
**Algorithm 12** Interval Scheduling Algorithm

```
procedure INTERVALSCHEDULING([s_1, f_1], [s_2, f_2], ..., [s_n, f_n])
    R is the set of requests
    A ← ∅
    while R ≠ ∅ do
        select a request i in R with the smallest finishing time
        add i to A
        remove all requests from R that are incompatible with i
    end while
    return A
end procedure
```

**Running time of Interval Scheduling Algorithm**: $O(n \log n)$
**Correctness of Interval Scheduling Algorithm**: Since the algorithm always selects the request with the smallest finishing time, it is clear that the algorithm will always select a compatible request.
**Arguing optimality**:

## 4.2 Minimum Spanning Trees

Consider a connected graph $G = (V, E)$, such that each edge $e = (v, w)$ of $E$, there is an associated cost $c_e$.
**Goal**: Find a spanning tree $T$ of $E$ so that the graph $G' = (V, T)$ has minimum cost.
**Example**:




**Greedy approach 1**:

- Start with an empty set of edges $T$.
- Repeat until $T$ forms a spanning tree:
  - Select an edge $e$ of minimum cost.
  - If $T \cup \{e\}$ does not contain a cycle, then add $e$ to $T$.

**krukals algorithm**:

**Algorithm 13** Krukals algorithm

```
procedure KRUKALS(G)
    T ← ∅
    while T is not a spanning tree do
        select an edge e of minimum cost
        if T ∪ {e} does not contain a cycle then
            add e to T
        end if
    end while
    return T
end procedure
```

**Running time of Krukals algorithm**: $O(m \log n)$
**Greedy approach 2**:

- Start with an empty set of edges $T$.
- Start with a node $s$.
  - Add an edge $e = (s, v)$ of minimum cost to $T$.
- Repeat until $T$ forms a spanning tree:

**Prims algorithm**:

**Algorithm 14** Prims algorithm

```
procedure PRIMS(G)
    T ← ∅
    s ← an arbitrary node
    while T is not a spanning tree do
        add an edge e = (s, v) of minimum cost to T
        s ← v
    end while
    return T
end procedure
```

**Running time of Prims algorithm**: $O(m \log n)$
**minimum spanning tree of example graph**:
the minimum spanning tree sequence is $d, a, c, b, e, f, h, g$.

**Greedy approach 3**:

- Start with the full graph $G = (V, E)$.
- Delete an edge from $G$
  - the edge of maximum cost
- Repeat until $G$ forms a spanning tree:

**Reverse-delete algorithm**:

**Algorithm 15** Reverse-delete algorithm

```
procedure REVERSEDELETE(G)
    T ← G
    while T is not a spanning tree do
        delete an edge e of maximum cost from T
    end while
    return T
end procedure
```

For when two edges have the same cost, use distinct labels to distinguish them.

**Optimal with Priorty Queue**:
Add PQ to Prim's algorithm.

**Algorithm 16** Optimal with Priorty Queue

```
procedure OPTIMAL(G)
    T ← ∅
    s ← an arbitrary node
    PQ ← empty priority queue
    for each node v do
        add v to PQ with key ∞
    end for
    decrease key of s to 0
    while PQ is not empty do
        v ← node with minimum key in PQ
        add an edge e = (s, v) of minimum cost to T
        s ← v
        for each edge e = (v, w) incident to v do
            if w is in PQ then
                decrease key of w to c_e
            end if
        end for
    end while
    return T
end procedure
```

**Running time of Optimal with Priorty Queue**: $O(m \log n)$

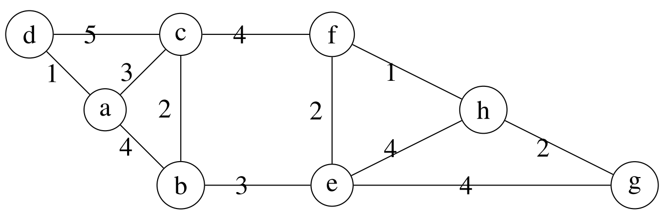
## 4.3 Clustering

- a collection of $n$ objects
- they have different degrees of similarity
- we want to organise them into coherent groups
- there is a notion of distance between objects

**Definition**:

- Given a set $U$ of $n$ elements, a $k-$clustering of $U$ is a partition of $U$ into non-empty subsets $C_1, C_2, \ldots, C_k$.
- The spacing of a $k-$clustering is the minimum distance between any pair of points in different clusters.

**Goal**:Among all possible $k-$clusterings, find one with minimum spacing.
**Example**:




**Greedy approach**:

- Pick two objects $p_i$ and $p_j$ with minimum distance $d(p_i, p_j)$.
- Connect them with an edge $e = (p_i, p_j)$.

- Continue like this until we have $k$ clusters.
- If the edge $e$ under consideration connects two object $p_i$ and $p_j$ already in the same cluster, then discard $e$.

**kruskals algorithm**:

**Algorithm 17** kruskals algorithm for clustering

**Require:** A graph $G = (V, E)$
**Ensure:** A minimum spanning tree of $G$ with $k$ clusters

```
procedure KRUSKAL(G, k)
    T ← ∅
    C ← {{v} | v ∈ V}                                  ▷ Initial clusters
    Sort edges in E in increasing order of weight
    for {u, v} ∈ E do
        if C contains k clusters then
            break
        end if
        if clusters containing u and v are different in C then
            T ← T ∪ {{u, v}}
            merge clusters containing u and v in C
        end if
    end for
    return T
end procedure
```

For Given example, the result of divide them into 3 clusters is:

$$\{a, b, c, d\}, \{e, f, h\}, \{g\}$$

# Chapter 5

# Dynamic Programming

**The paradigm of dynamic programming**: Given a problem $P$, define a sequence of subproblems, with the following properties:

- The subproblems are ordered from the simplest to the largest
- The largest problem is our original problem $P$
- The optimal solution of a subproblem can be structured from the optimal solutions of smaller subproblems.

Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution and use it to solve larger subproblems.

## 5.1 Weighted Interval Scheduling

- A set of requests $R = \{1, 2, \ldots, n\}$.
  - Request $i$ has a start time $s_i$ and a finish time $f_i$, and a value $v_i$.
  - Alternative view: every request is an interval $[s_i, f_i]$ associated with a value $v_i$.
- Two requests $i$ and $j$ are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.

**build up a solution**:

1. let $O$ the optimal solution
2. $O$ contains an optimal solution $O'$ of the subproblem $R' = \{1, 2, \ldots, i-1\}$
3. in order to find $O$, it suffices to look at smaller problems and find $O(1, 2, \ldots, j)$ for some $j$
4. Let $O_j$ be a shorthand for $O(1, 2, \ldots, j)$ and let $OPT(j)$ be its total value.
5. Define $OPT(0) = 0$
6. Then $O = O_n$ with value $OPT(n)$
7. $OPT(j)$ can be computed from $OPT(j-1)$
8. $OPT(j) = \max\{OPT_{p_j} + v_j, OPT(j-1)\}$

**Algorithm 18** ComputeOPT

```
procedure COMPUTEOPT(j)
    if j = 0 then
        return 0
    else
        return max{COMPUTEOPT(p_j) + v(j), COMPUTEOPT(j − 1)}
    end if
end procedure
```

**Correctness**: ComputeOPT(j) correctly computes $OPT(j)$ for all $j = 0, 1, \ldots, n$.
Proof by induction:
**Base case**: $OPT(0) = 0$ by definition.
**Inductive step**: Assume that it is true for all $i < j$.(Induction hypothesis)
return $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j-1)\}$
**Running time**: $\Omega(2^n)$
**Memoization**:

- Compute ComputeOPT(j) for all $j = 0, 1, \ldots, n$.
- Store it in an accessible place to use again later.
- Keep an array $M[0, \ldots, n]$.
  - initially $M[j] =$ EMPTY for all $j = 0, 1, \ldots, n$.
  - when ComputeOPT(j) is called, $M[j] =$ ComputeOPT(j).

**Algorithm 19** M-ComputeOPT

```
procedure M-COMPUTEOPT(j)
    if j = 0 then
        return 0
    else if M[j] is not empty then
        return M[j]
    else
        M[j] ← max{M-COMPUTEOPT(p_j) + v(j), M-COMPUTEOPT(j − 1)}
        return M[j]
    end if
end procedure
```

**Running time**: $O(n \log n)$

**Algorithm 20** Find-Solution

```
procedure FIND-SOLUTION(j)
    if j = 0 then
        return ∅
    else
        if  thenv(j) + M-COMPUTEOPT(p_j) > M-COMPUTEOPT(j − 1)
            return {j} ∪ FIND-SOLUTION(p_j)
        else
            return FIND-SOLUTION(j − 1)
        end if
    end if
end procedure
```

**Dynamic Programming vs Divide and Conquer**:

**Dynamic Programming**:

- DP is an optimisation techniques and is only applicable to problems that have optimal substructure.
- DP splits the problem into parts, finds solutions to the parts and joins them.(The parts are not significantly smaller than the original problem and are overlapping.)
- In DP, the subproblems dependency can be represented by a directed acyclic graph.

**Divide and Conquer**:

- DC is not normally used for optimisation problems.
- DC splits the problem into parts, finds solutions to the parts and joins them.(The parts are significantly smaller than the original problem and are non-overlapping.)
- In DC, the subproblems dependency can be represented by a tree.

## 5.2 Subset Sum

**Problem Description**:

- Given a set of $n$ items $1, 2, \ldots, n$
- Each item $i$ has a non-negative weight $w_i$.
- Given a bound $W$.
- **Goal**: select a subset $S$ of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} w_i$ is maximised.

Dynamic Programming: To find the optimal value of $OPT(n)$, we need

- the optimal value of $OPT(n-1)$ if item $n$ is not selected.
- the optimal value of the solution on input 1,2,...,n-1 with weight bound $W - w_n$.

subproblems:

- Assumptions:
  - $W$ is an integer
  - Every $w_i$ is an integer
- subproblem for each $i = 0, 1, \ldots, n$ and each integer $0 \leq w \leq W$.
- Let $OPT(i, w)$ be the optimal value of the solution on subset $1, 2, \ldots, i$ with weight bound $w$.

**Algorithm 21** SubsetSum

```
procedure SUBSETSUM(n, w)
    Array M[0, ..., n, 0, ..., W]
    M[0, w] = 0 for each w = 0, 1, ..., W
    for i = 1 to n do
        for w = 0 to W do
            if w_i > w then
                M[i, w] = M[i − 1, w]
            else
                M[i, w] = max{M[i − 1, w], M[i − 1, w − w_i] + w_i}
            end if
        end for
    end for
    return M[n, W]
end procedure
```

**Running time**: $O(nW)$

## 5.3 knapSack

**Problem Description**:

- Given a set of $n$ items $1, 2, \ldots, n$
- Each item $i$ has a non-negative weight $w_i$ and a non-negative value $v_i$.
- Given a bound $W$.
- **Goal**: select a subset $S$ of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximised.

**the fractional knapsack problem**:

- Given a set of $n$ items $1, 2, \ldots, n$
- Each item $i$ has a non-negative weight $w_i$ and a non-negative value $v_i$.
- Given a bound $W$.
- **Goal**: select a fraction $x_i$ of each item $i$ such that $\sum_{i \in S} w_i x_i \leq W$ and $\sum_{i \in S} v_i x_i$ is maximised.

**The 0/1 knapsack problem**: Solution for 0/1 knapsack problem:

**Algorithm 22** 0/1 knapsack in dynamic programming

```
procedure 0/1 KNAPSACK(n, W)
    Array M[0, ..., n, 0, ..., W]
    M[0, w] = 0 for each w = 0, 1, ..., W
    for i = 1 to n do
        for w = 0 to W do
            if w_i > w then
                M[i, w] = M[i − 1, w]
            else
                M[i, w] = max{M[i − 1, w], M[i − 1, w − w_i] + v_i}
            end if
        end for
    end for
    return M[n, W]
end procedure
```

# Chapter 6

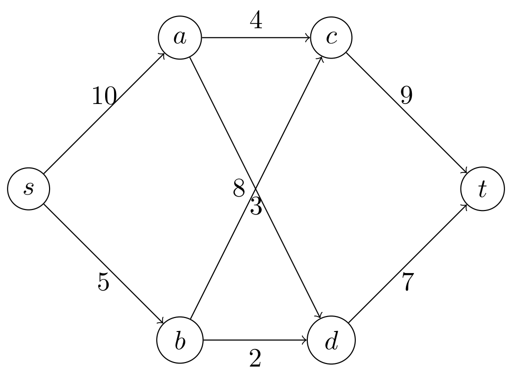
# Network Flow

## 6.1 Network Flow Definitions

**Flow network**: A flow network is a directed graph $G = (V, E)$ with the following properties:

- Each edge $(u, v) \in E$ has a non-negative capacity $c_e$.
- There is a single source $s$ in $V$.
- There is a single sink $t$ in $V$.
- All other nodes in $V - \{s, t\}$ are called intermediate nodes.

example:




**Further definitions**:

- The source $s$ has no incoming edges.
- The sink $t$ has no outgoing edges.
- There is at least one edge incident to each node.
- All capacities are integers.

**Flow**: An $(s - t)$ flow is a function $f : E \to \mathbb{R}^+$, mapping each edge $e$ to a non-negative real number $f(e)$. A feasible flow must satisfy the following conditions:

- Capacity: For each edge $e \in E$, $0 \leq f(e) \leq c_e$.
- Flow conservation: for each node $v \in V - \{s, t\}$, we have

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

The source $s$ generates flow, and the sink $t$ absorbs flow.
Value of a flow $f$, denoted $val(f)$, is the total amount of flow generated by the source $s$:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

Generally, define $f^{out}(v)$ and $f^{in}(v)$ for the flow going out of(resp. going into) node $v$.
Similarly, define $f^{out}(S)$ and $f^{in}(S)$ for sets of nodes $S$.

## 6.2 Maximum Flow Problem

**The maximum flow problem**: Given a flow network $G = (V, E)$, find a flow of maximum possible value.
**algorithm for maximum flow**:
Idea: push flow forward on edges with leftover capacity, push flow backward on edges that are already carrying flow.
**The residual graph $G_f$**:
The residual graph $G_f$ of $G$(also called the flow network) is defined as follows:

- The node set $V_f$ of $G_f$ is the same as the node set $V$.
- For each edge $(u, v) \in E$ which $f(e) < c_e$, there are $c_e - f(e)$ "leftover" units of capacity.
  - We will call this number the **residual capacity** of edge $e$.
  - We will call the edge $e$ a forward edge.
- For each edge $(u, v) \in E$ with $f(e) > 0$, there is an edge $e' = (v, u)$ in $E_f$ with a capacity of $f(e)$.We will call the edge $e'$ a backward edge.

**Working with residual graphs**:

- Find an $(s - t)$ path $P$ in $G_f$. This is called an **augmenting path**.
- Define the bottleneck of $P$,
  - Denoted bottleneck$(P, f)$
  - to be the minimum residual capacity of any edge in $P$.
- Define the augmentation of flow $f$ into flow $f'$
  - Denoted augment$(f, P)$.

**Augmenting the flow**:
Feasibility of capacity:

**Algorithm 23** Augmenting the flow

**procedure** AUGMENT$(f, P)$
  $b \leftarrow$ bottleneck$(P, f)$
  **for** each edge $e = (u, v) \in P$ **do**
    **if** $e$ is a forward edge **then**
      $f(e) \leftarrow f(e) + b$
    **else**
      $f(e) \leftarrow f(e) - b$
    **end if**
  **end for**
  **return** $f$
**end procedure**

consider an arbitrary edge $e = (u, v) \in P$. Suppose that $e$ is a forward edge.

$$0 \leq f(e) \leq f'(e) = f(e) + b \leq f(e) + (c_e - f(e)) = c_e$$

Suppose that $e$ is a backward edge.

$$c_e \geq f(e) \geq f'(e) = f(e) - b \geq f(e) - (f(e) - 0) = 0$$

**The Ford-Fulkerson algorithm**:

**Algorithm 24** Max-flow algorithm

**procedure** MAX-FLOW$(G, s, t)$
    $f(e) \leftarrow 0$ for all edges $e \in E$
    **while** there exists an $(s - t)$ path $P$ in $G_f$ **do**
        $f \leftarrow$ augment$(f, P)$
        $f' \leftarrow$ update$(f)$
        $G_f \leftarrow$ update$(G_f, f)$
    **end while**
    **return** $f$
**end procedure**

**Running time of Ford-Fulkerson algorithm**: $O(mC)$, where $C$ is the maximum capacity of any edge in the network.

## 6.3 Min Cut theorem

A cut $C$ is a partition of the nodes of $G$ into two sets $S$ and $T$ such that $s \in S$ and $t \in T$.
The capacity of a cut $C = (S, T)$ of a cut $C$ is the sum of the capacities of the edges ”out of” $S$: these are edges $(u, v)$ such that $u \in S$ and $v \in T$.

**The min-cut theorem**: In every flow network, the value of the maximum flow is equal to the capacity of the minimum cut.
**A series of facts**:
Fact 1: Let $f$ be any $(s - t)$ flow and let $(S, T)$ be any cut. Then $v(f) = f^{out}(S) - f^{in}(S)$.

1. By definition, $v(f) = f^{out}(s)$.
2. By definition $f^{in}(s) = 0$.
3. Hence, $v(f) = f^{out}(s) - f^{in}(s)$.
4. For every other node $v \neq s, t$, we have $f^{out}(v) = f^{in}(v)$.
5. Therefore, $v(f) = \sum_{v \in S}(f^{out}(v) - f^{in}(v))$.
6. rewrite as $v(f) = \sum_{v \in S}(f^{out}(v) - f^{in}(v)) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = f^{out}(S) - f^{in}(S)$.

Fact 2: Let $f$ be any $(s - t)$ flow and let $(S, T)$ be any $(s - t)$ cut. Then $v(f) = f^{out}(T) - f^{in}(T)$.
Fact 3: Let $f$ be any $(s - t)$ flow and let $(S, T)$ be any $(s - t)$ cut. Then $v(f) \leq c(S, T)$.

$$\begin{aligned} v(f) &= f^{out}(S) - f^{in}(S) \\ &\leq f^{out}(S) \\ &= \sum_{e \text{ out of } S} f(e) \\ &\leq \sum_{e \text{ out of } S} c(e) \\ &= c(S, T) \end{aligned}$$

Fact 4: Let $f$ be any $(s-t)$ flow in $G$ such that the residual graph $G_f$ contains no augmenting paths. Then there exists an $(s-t)$ cut $(S^*, T^*)$ such that $v(f) = c(S^*, T^*)$.
Proving fact 4: In the residual graph $G_f$, identify all nodes that are reachable from the source $s$. Let $S^*$ be the set of these nodes, and let $T^* = V - S^*$.

1. $s \in S^*$ and $t \in T^*$.
2. No edge of $G_f$ crosses from $S^*$ to $T^*$.
3. Every edge of $G_f$ crosses from $T^*$ to $S^*$.
4. $f^{out}(S^*) = v(f)$.
5. $f^{in}(S^*) = 0$.

$$\begin{aligned} v(f) &= f^{out}(S^*) - f^{in}(S^*) \\ &= \sum_{e \text{ out of } S^*} f(e) - \sum_{e \text{ into } S^*} f(e) \\ &= \sum_{e \text{ out of } S^*} c(e) - 0 \qquad\qquad = c(S^*, T^*) \end{aligned}$$

Fact 5: If all capacities are integers, then there exists a maximum flow $f$ for which $f(e)$ is an integer for every edge $e$.

## 6.4 Choosing Better Augmenting Paths

**The Edmonds-Karp algorithm**:

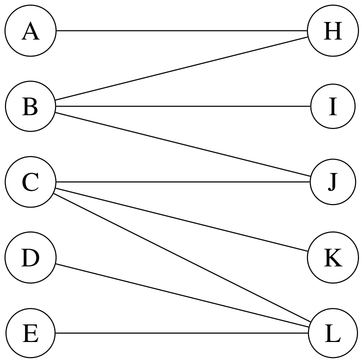
**Algorithm 25** Edmonds-Karp algorithm

**procedure** EDMONDS-KARP$(G, s, t)$
    $f(e) \leftarrow 0$ for all edges $e \in E$
    **while** there exists an $(s-t)$ path $P$ in $G_f$ **do**
        $P$ is a shortest $(s-t)$ path
        $f \leftarrow$ augment$(f, P)$
        $f' \leftarrow$ update$(f)$
        $G_f \leftarrow$ update$(G_f, f)$
    **end while**
    **return** $f$
**end procedure**

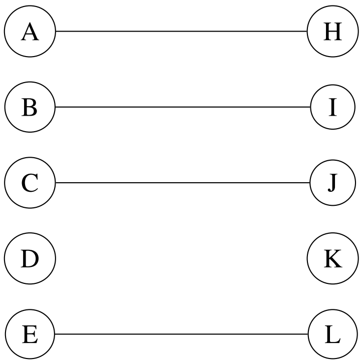
**Running time of Edmonds-Karp algorithm**: $O(nm^2)$, where $n$ is the number of nodes and $m$ is the number of edges in the network.
The shortest path can be found in $O(m)$ time using BFS.

## 6.5 Modeling with Network Flows

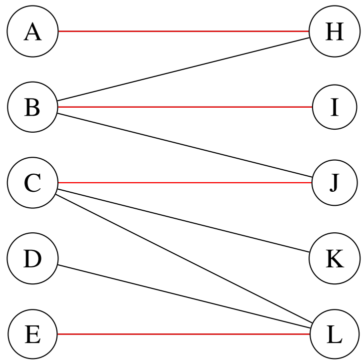
**Bipartite graphs**: A graph $G = (V, E)$ is bipartite if any only if it can be partitioned into two sets $A$ and $B$ such that every edge has one endpoint in $A$ and one endpoint in $B$.
**Maximum bipartite matching**:
Matching: A subset $M$ of edges $E$ such that each node $v \in V$ appears in at most one edge $e \in M$.
Maximum matching: A matching with maximum cardinality.
examples of bipartite graphs:



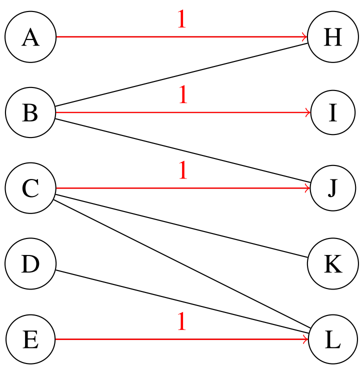

example of maximum bipartite matching:

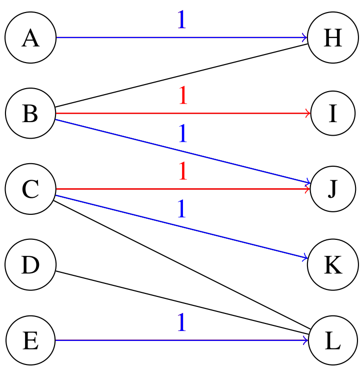



exapmle of maximal bipartite matching:




From matchings to flows:




**Maximum Flow and Maximum Matching**
The size of the maximum matching is equal to the value of the maximum flow.

The edges of $M$ are the edges that carry flow form $A$ to $B$ in the residual network.
Running time: $O(mn)$

**Baseball Elimination**

- Given a set $S$ of teams
- For each team $x$ in $S$, the current number of wins $w_x$
- For teams $x$ and $y$ in $S$, they still have to play $g_{xy}$ games against each other
- Given a designated team $z$
- Can $z$ still win the turnament?

**From Baseball Elimination to flows**

- For each pair of teams $x$ and $y$, create a vertex $v_{xy}$
- For each team $x$, create a vertex $v_x$
- For each pair of teams $x$ and $y$, create an edge $(s, v_{xy})$ with capacity $g_{xy}$
- For each team $x$, create an edge $(v_x, t)$ with capacity $w_z + g_{xz} - w_x$
- For each pair of teams $x$ and $y$, create an edge $(v_{xy}, v_x)$ with infinite capacity
- For each pair of teams $x$ and $y$, create an edge $(v_{xy}, v_y)$ with infinite capacity

**Open pit mining**

- Given a set $S$ of blocks
- For each block $x$ in $S$, the value $v_x$ of the ore in the block
- For each block $x$ in $S$, the cost $c_x$ of mining the block
- For each block $x$ in $S$, the set $N_x$ of blocks that are neighbors of $x$
- Given a designated block $z$
- What is the maximum value of ore that can be mined?

**From open pit mining to flows**

- For each block $x$ in $S$, create a vertex $v_x$
- For each block $x$ in $S$, create an edge $(s, v_x)$ with capacity $v_x$
- For each block $x$ in $S$, create an edge $(v_x, t)$ with capacity $c_x$
- For each block $x$ in $S$, create an edge $(v_x, v_y)$ with infinite capacity for each block $y$ in $N_x$

# Chapter 7

# NP-Completeness

## 7.1 NP-Completeness

**Polynomial time reduction**

- Given a problem $A$ to solve
- Reduce solving $A$ to solving $B$
- Assume there is an algorithm $ALG^B$ that solves $B$ at cost $O(1)$
- Construct an algorithm $ALG^A$ that solves $A$, which uses $ALG^B$ as a subroutine
- If $ALG^A$ runs in polynomial time, then this is a polynomial time reduction

**How to work with reductions**
Positive: Assume that I want to solve problem $A$ and I know how to solve problem $B$.
I can try come up with a polynomial time reduction $A \leq^p B$, which will give me a polynomial time algorithm for $A$.
Contrapositive: Assume that there is a problem $A$ for which it is unlikely that there is a polynomial time algorithm that solves $A$.
If I come up with a polynomial time reduction $A \leq^p B$, it is also unlikely that there is a polynomial time algorithm that solves $B$.
$B$ is "at least as hard to solve as" $A$, because if I could solve $B$, I could also solve $A$.
**Types of reductions**

- Turing reduction: $A \leq_T B$
  - A reduction which solves $A$ using (potentially many) calls to an oracle for $B$
  - As known as Cook reduction
- Many-one reduction: $A \leq_m B$
  - A reduction which converts instances of $A$ to instances of $B$
  - Also known as Karp reduction

**Problem classification**
Problems in $P$:
Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, weighted interval scheduling, etc.
Problems in $NP$:
subset sum, knapSack, weighted interval scheduling, Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, etc.
**NP-hardness**
A problem $B$ is NP-hard if for every problem $A$ in $NP$, $A \leq^p B$.
If every problem in $NP$ is polynomial time reducible to $B$, then this captures the fact that $B$ is at least as hard as any problem in $NP$.
**3 SAT**

- A CNF formula with $m$ clauses and $k$ literals.

$$\varphi = (x_1 \vee \neg x_5 \vee x_3) \wedge (x_2 \vee x_6 \vee \neg x_5) \wedge \cdots \wedge (x_3 \vee x_8 \vee x_{12})$$

- each clause has exactly three literals
- Truth assignment: A value in $\{0, 1\}$ for each variable $x_i$
- Satisfying assignment: A truth assignment which makes the formula evaluate to 1
- Computational problem 3 SAT: Decide if the input formula $\varphi$ has a satisfying assignment.

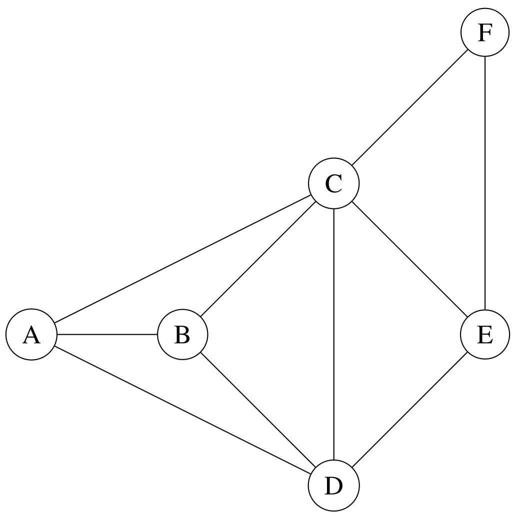
**3 SAT is NP-complete**

- 3 SAT is in $NP$
  - Given a truth assignment, we can check in polynomial time if it is satisfying
- 3 SAT is NP-hard
  - Given a CNF formula $\varphi$, we can construct a polynomial time reduction to 3 SAT

**Proving NP-completeness** Suppose that you are given a problem $A$ and you want to prove that $A$ is NP-complete.
First, prove that $A$ is in $NP$.
Usually by observing that a solution is efficiently verifiable.
Then prove that $A$ is NP-hard.
construct a polynomial time reduction from a known NP-complete problem $P$.

## 7.2 NP-completeness of the vertex cover problem

**Vertex cover** Definition: A vertex cover $C$ of a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that for every edge $e$ in $E$, at least one of the endpoints of $e$ is in $C$.
Definition: A minimum vertex cover is a vertex cover of smallest possible size.
Input: A graph $G = (V, E)$
Output: A minimum vertex cover
**Example**




the vertex cover $\{A, C, E\}$ is not a minimum vertex cover
the minimum vertex cover is $\{A, E\}$.

**Vertex cover is NP-hard**: construct a polynomial time reduction from 3 SAT to vertex cover.

Let $\varphi$ be a 3 CNF formula with $m$ clauses and $d$ variables.

Construct in polynomial time an instance $< G, k >$ of vertex cover, with $k = d + 2m$.

if $\varphi$ is satisfiable, then $G$ has a vertex cover of size at most $k$

Let $(y_1, y_2, \dots, y_d)$ in $\{0, 1\}^d$ be a satisfying assignment for $\varphi$

For the nodes on the top: If $y_i = 1$. Include node $x_i$ in the vertex cover $C$, otherwise include node $\neg x_i$ in $C$.

For the nodes on the bottom: in each triangle, choose a node $x_i$ that has been picked on the top and do not include it in the vertex cover. Include the other two nodes.

if $\varphi$ is not satisfiable, then $G$ has no vertex cover of size at most $k$

Let $C$ be a vertex cover of size $k = d + 2m$ in $G$.

Since it is a vertex cover, it must include at least two out of three nodes in each "clause gadget" at the bottom.

this means that at most $d$ nodes can be picked on the top.

To satisfy the edges at the top, in each "variable gadget", at least one node must be picked.

## 7.3 Further reductions in NP

**Form optimization to decision**

Given an optimization problem $P$, introcude a threshold $k$.

The decision version $P_d$ becomes: Given an instance of $P$ and the threshold $k$ as input, is there a solution to $P$ with value at most $k$?

If $P$ solved in polynomial time, then $P_d$ is also solved in polynomial time.

If $P_d$ solved in polynomial time, then $P$ is also solved in polynomial time.

**NP-complete problems**

Independent Set in graph $G$: A set of nodes in the graph, such that there is no edge between any two nodes in the set.

Maximum Independent Set: Given a graph $G$, find an independent set of maximum size.

Maximum Independent Set(desision version): Given a graph $G$ and a threshold $k$, is there an independent set of size at least $k$?

Set Packing: Given a set $U$ and a collection of subsets $S_1, S_2, \dots, S_m$ of $U$ and a number $k$, does there exist a collection of at least $k$ of these subsets that no two of them intersect?

Set Cover: Given a set $U$ and a collection of subsets $S_1, S_2, \dots, S_m$ of $U$ and a number $k$, does there exist a collection of at most $k$ of these sets whose union is $U$?

3-Dimensional Matching: Given three disjoint sets $X, Y, Z$ each of size $n$, and a collection of triples $T \subseteq X \times Y \times Z$, does there exist a set of $n$ triples in $T$, so that each element of $X \cup Y \cup Z$ appears in exactly one of these triples?

K-Colouring of a graph $G$: A function $f : V \to \{1, 2, \dots, k\}$ so that for every edge $(u, v)$ in $E$, $f(u) \neq f(v)$.

3-Colouring: Given a graph $G$, can we colour the nodes of $G$ using 3 colours so that no two adjacent nodes have the same colour?

Hamiltionian cycle in a directed graph $G$: A cycle in a directed graph that visits every node exactly once.

Hamiltionian path in a directed graph $G$: A path in a directed graph that contains every node exactly once.

Hamiltionian Cycle: Given a directed graph $G$, does $G$ contain a Hamiltionian cycle?

Hamiltionian Path: Given a directed graph $G$, does $G$ contain a Hamiltionian path?

Traveling Salesman: Given a complete graph $G$ with edge weights, and a threshold $k$, is there a tour of $G$ with total weight at most $k$?

A texonomy of NP-complete problems