

Summary of COMP523 Advanced Algorithm

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Chapter 1

Symmetry Notation

1.1 Asymptotic Notation

Asymptotic notation is a way of describing the limiting behavior of a function when the argument tends towards a particular value or infinity. In computer science, asymptotic notation is frequently used to describe the running time or space usage of an algorithm.

- O -notation: $f(n) = O(g(n))$ if there exist constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
- Ω -notation: $f(n) = \Omega(g(n))$ if there exist constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
- Θ -notation: $f(n) = \Theta(g(n))$ if there exist constants c_1, c_2 and n_0 such that $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0$.
- o -notation: $f(n) = o(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.
- ω -notation: $f(n) = \omega(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$.

1.2 Comparing Functions

1.2.1 Transitivity

- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ implies $f(n) = \Omega(h(n))$.
- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ implies $f(n) = \Theta(h(n))$.

For example, $n^2 = O(n^3)$ and $n^3 = O(n^4)$ implies $n^2 = O(n^4)$.

1.2.2 Reflexivity

- $f(n) = O(f(n))$.
- $f(n) = \Omega(f(n))$.
- $f(n) = \Theta(f(n))$.

For example, $n^2 = O(n^2)$.

1.2.3 Symmetry

- $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
- $f(n) = \Omega(g(n))$ implies $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ implies $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ implies $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ implies $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ implies $n^3 = \Omega(n^2)$.

1.2.4 Transpose Symmetry

- $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ if and only if $n^3 = \Omega(n^2)$.

1.2.5 sum and maximum

$$f_1(n) + f_2(n) + \cdots + f_k(n) = \Theta(\max(f_1(n), f_2(n), \dots, f_k(n)))$$

where k is a constant positive integer.

Let $f_j(n) = j$, $k = n$, then

$$f_1(n) + f_2(n) + \cdots + f_k(n) = n(n+1)/2 = \Theta(n^2)$$

1.2.6 Running time hierarchy

- logarithmic: $O(\log n)$
- linear: $O(n)$
- $n \log n$: $O(n \log n)$
- quadratic: $O(n^2)$
- polynomial: $O(n^k)$
- exponential: $O(c^n)$
- constant: $O(1)$
- superconstant: $\omega(1)$
- sublinear: $o(n)$
- superlinear: $\omega(n)$
- superpolynomial: $\omega(n^k)$
- subexponential: $o(c^n)$

1.3 Expect of algorithms

Correctness: An algorithm is correct if it halts with the correct output for every input instance.

Termination: An algorithm is terminating if it halts for every input instance.

Efficiency: An algorithm is efficient if it halts with the correct output for every input instance and runs in polynomial time.

Chapter 2

Recursion and Divide and Conquer techniques

2.1 Finding Majority in array

The pseudocode of the algorithm is shown in Algorithm 2.1.

Algorithm 1 Finding Majority in array

```
1: procedure MAJORITY( $A$ )
2:    $n \leftarrow$  length of  $A$ 
3:   if  $n = 0$  then
4:     return  $-1$ 
5:   end if
6:   if  $n = 1$  then
7:     return  $A[1]$ 
8:   end if
9:   if  $n \neq 1$  and  $n$  is odd then
10:
11:   end if
12:   Array  $B$  of size  $n/2$ 
13:   set  $j=0$ 
14:   for  $i = 1$  to  $n/2$  do
15:     if  $A[2i - 1] = A[2i]$  then
16:        $B[j] \leftarrow A[2i - 1]$ 
17:        $j \leftarrow j + 1$ 
18:     end if
19:   end for
20:    $m \leftarrow$  MAJORITY( $B$ )
21:    $count \leftarrow 0$ 
22:   for  $i = 1$  to  $n$  do
23:     if  $A[i] = m$  then
24:        $count \leftarrow count + 1$ 
25:     end if
26:   end for
27:   if  $count > n/2$  then
28:     return  $m$ 
29:   else
30:     return  $-1$ 
31:   end if
32: end procedure
```

Correctness:

Lemma: If A has a majority element, then the majority element of A is also the majority element of B .

Base case: $n = 1$, the majority element is $A[1]$.

Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.

Induction step: If A has a majority element, then the majority element of A is also the majority element of B .

Case 1 (A has a majority element m): Then by the lemma, it is also the majority element of B . Then m appears more than $k/2$ times in B . Then m appears more than $(k + 1)/2$ times in A .

Case 2 (A has no majority element): Then B has no majority element. Then A has no majority element.

Proof the lemma:

proof by contradiction. Assume that A has a majority element m and B has a majority element m' , but $m \neq m'$.

Let x be the numbers of occurrence of m in A .

Let y be the numbers of occurrence of m' in B .

Then $2y$ times from pairs that are represented in B by a value different from m' , and $x - 2y$ times, since each occurrence of m in A that is not paired with another occurrence of m in A is paired with an occurrence of m' in B .

In total, this gives $2y + x - 2y = x$ occurrences of m in A , which is a contradiction.

Running time:

Recursive formula for the running time:

$$T(n) \leq T(n/2) + cn$$

where c is a constant.

The solution to the recurrence is $T(n) = O(n)$.

2.2 Searching in logarithmic time

Searching faster with BinarySearch.

It is a particular case of the divide-and-conquer paradigm.

Input: A sorted array A of n elements and a value x .

Output: An index i such that $A[i] = x$ or the special value -1 if x does not appear in A .

Pseudocode is shown in Algorithm 2.2.

Algorithm 2 BinarySearch

```

1: procedure BINARYSEARCH( $x, i, j$ )
2:   if  $i = j$  then
3:     if  $A[i] = x$  then
4:       return  $i$ 
5:     else
6:       return  $-1$ 
7:     end if
8:   else
9:     if  $x = A[\lfloor (i + j)/2 \rfloor]$  then
10:      return  $\lfloor (i + j)/2 \rfloor$ 
11:    else if  $x < A[\lfloor (i + j)/2 \rfloor]$  then
12:      return BINARYSEARCH( $x, i, \lfloor (i + j)/2 \rfloor$ )
13:    else
14:      return BINARYSEARCH( $x, \lfloor (i + j)/2 \rfloor + 1, j$ )
15:    end if
16:  end if
17: end procedure

```

Running time:

The number of comparisons performed by BinarySearch is:

$$T(n) \leq T(n/2) + 4$$

Keep calculate:

$$\begin{aligned}
T(n) &\leq T(n/2) + 4 \\
&\leq T(n/4) + 4 + 4 \\
&\leq T(n/8) + 4 + 4 + 4 \\
&\leq T(n/2^k) + 4k \\
&\leq T(n/2^{\log(n-1)}) + 4\log(n-1) \\
&= T(2) + 4(\log n - 1) \\
&\leq 4\log n - 4 \\
&= 4\log n
\end{aligned}$$

proof $T(n) \leq 4\log n$:

Base case: $n = 1, T(1) = 0 \leq 4\log 1 = 0$.

Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.

Induction step: $T(k + 1) \leq 4\log(k + 1)$.

$$\begin{aligned}
T(k + 1) &\leq T(k/2) + 4 \\
&\leq 4\log(k/2) + 4 \\
&= 4\log k - 4 + 4 \\
&= 4\log k \\
&\leq 4\log(k + 1)
\end{aligned}$$

Memory usage:

The memory usage of BinarySearch is:

$$M(n) = O(\log n)$$

Comparing BinarySearch and LinearSearch:

$$\begin{aligned}
T_{\text{BinarySearch}}(n) &= O(\log n) \\
T_{\text{LinearSearch}}(n) &= O(n) \\
T_{\text{BinarySearch}}(n) &= O(\log n) < O(n) = T_{\text{LinearSearch}}(n) \\
M_{\text{BinarySearch}}(n) &= O(\log n) < O(1) = M_{\text{LinearSearch}}(n)
\end{aligned}$$

2.3 Running time of Divide and Conquer algorithms

The Master Theorem:

Suppose that $T(n)$ satisfies the recurrence:

$$T(n) \leq aT(n/b) + cn^d$$

where $a \geq 1, b > 1, c > 0$ and $d \geq 0$ are constants.

Then $T(n)$ has the following asymptotic bounds:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

This theorem is useful for solving recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$ and $f(n)$ is an asymptotically positive function.

Example:

$$T(n) = 8T(n/2) + 100n^2$$

$a = 8$, $b = 2$, $f(n) = 100n^2$, $d = 2$, $\log_b a = \log_2 8 = 3$.

$d = 2 < \log_b a = 3$, so $T(n) = O(n^{\log_b a}) = O(n^3)$.

2.4 Finding pair of points closest to each other

Input: A set P of n points in the plane.

Output: The pair of points in P that are closest to each other.

Pseudocode is shown in Algorithm 2.4. **Running time:**

Algorithm 3 ClosestPair

```

1: procedure CLOSESTPAIR( $P_1, \dots, P_n$ )
2:   Construct  $P_x$  and  $P_y$ .  $P_x$  is sorted by  $x$ -coordinate,  $P_y$  is sorted by  $y$ -coordinate.
3:   return CLOSESTPAIRREC( $P_x, P_y$ )
4: end procedure

```

Algorithm 4 ClosestPairRec

```

1: procedure CLOSESTPAIRREC( $P_x, P_y$ )
2:   if  $|P_x| = |P_y| \leq 3$  then
3:     For each pair of points  $(P_i, P_j)$ , compute  $d(P_i, P_j)$ 
4:     return the pair of points with the smallest distance
5:   end if
6:   Construct  $Q_x, Q_y, R_x$  and  $R_y$ .
7:    $(l_1, l_2) = \text{CLOSESTPAIRREC}(Q_x, Q_y)$ 
8:    $(r_1, r_2) = \text{CLOSESTPAIRREC}(R_x, R_y)$ 
9:    $\delta = \min\{d(l_1, l_2), d(r_1, r_2)\}$ 
10:   $x^* = \text{the largest } x\text{-coordinate in } Q_x$ 
11:   $L = \{(x, y) : x = x^*\}$ 
12:   $S = \{p \in P : p \in L \text{ and } p \text{ is within } \delta \text{ of } L\}$ 
13:  Construct  $S_v$ 
14:  for  $p \in S$  do
15:    Let  $q$  be the point in  $S_v$  closest to  $p$ 
16:    if  $d(p, q) < \delta$  then
17:       $\delta = d(p, q)$ 
18:       $(s_1, s_2) = (p, q)$ 
19:    end if
20:  end for
21:  if  $d(s_1, s_2) < \min\{d(l_1, l_2), d(r_1, r_2)\}$  then
22:    return  $(s_1, s_2)$ 
23:  end if
24:  if  $d(l_1, l_2) < d(r_1, r_2)$  then
25:    return  $(l_1, l_2)$ 
26:  else
27:    return  $(r_1, r_2)$ 
28:  end if
29: end procedure

```

$$T(n) \leq 2T(n/2) + O(n \log n) = O(n \log n)$$

Example:

Chapter 3

Graph Algorithms

3.1 Graph Definitions

Graph: A graph G consists of a set V of vertices and a set E of edges, where each edge is associated with a pair of vertices.

Directed Graph: A directed graph G consists of a set V of vertices and a set E of directed edges, where each directed edge is associated with an ordered pair of vertices.

Undirected Graph: An undirected graph G consists of a set V of vertices and a set E of undirected edges, where each undirected edge is associated with an unordered pair of vertices.

Neighbours of a vertex v : Set of vertices that are connected to v by an edge.

Degree of a vertex v : number of neighbours of v , denoted by $\deg(v)$.

Path: A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.
length = node count - 1 = edge count.

Distance between two nodes: The number of edges in the shortest path between the two nodes.

Graph diameter: The maximum distance between any two nodes in the graph.

Lines, cycles, trees and cliques:

Line: A graph with n vertices and $n - 1$ edges.

Cycle: A graph with n vertices and n edges.

cliques: A graph with n vertices and $n(n - 1)/2$ edges.

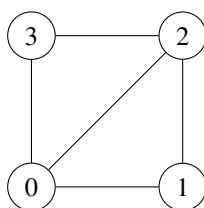
Tree: A graph with n vertices and $n - 1$ edges.

Graph representations:

Adjacency matrix: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge between i and j , and $A_{ij} = 0$ otherwise.

examples of adjacency matrices:

Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Adjacency matrix for directed graphs: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge from i to j , and $A_{ij} = 0$ otherwise.

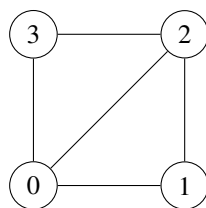
examples of adjacency matrices for directed graphs:
 Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

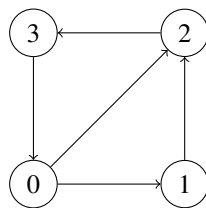
Adjacency list: A list of lists, where the i th list contains the neighbours of vertex i .
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \\ 0 & 1 & 3 \\ 0 & 2 & \end{bmatrix}$$

Adjacency list for directed graphs: A list of lists, where the i th list contains the neighbours of vertex i .
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 \\ 2 & \\ 3 & \\ 0 & \end{bmatrix}$$

Adjacency matrix vs adjacency list:

Adjacency matrix	Adjacency list
$O(1)$ to check if there is an edge between i and j	$O(\min(\deg(i), \deg(j)))$ to check if there is an edge between i and j
$O(n)$ to find the neighbours of i	$O(\deg(i))$ to find the neighbours of i
$O(n^2)$ space	$O(n + m)$ space

3.2 Depth-first search

Depth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of depth-first search:



The depth-first search sequence is:

0, 1, 2, 3, 5, 4

Depth-first search algorithm:

Algorithm 5 Depth-first search algorithm

```
1: procedure DFS( $G, v$ )
2:   for  $e \in V$  do
3:     if  $e$  is unexplored then
4:        $u = \text{head of } e$ 
5:       if  $u$  is unexplored then
6:          $e$  is a tree edge
7:         DFS( $G, u$ )
8:       else
9:          $e$  is a back edge
10:      end if
11:    end if
12:  end for
13: end procedure
```

Running time of depth-first search: $O(n + m)$

3.3 Breadth-first search

Breadth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of breadth-first search:



The breadth-first search sequence starting from vertex 0 is 0, 1, 2, 3, 4, 5.

Breadth-first search algorithm:

Algorithm 6 Breadth-first search algorithm

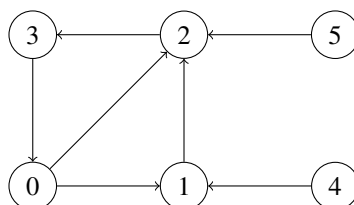
```
1: procedure BFS( $G, s$ )
2:   initial empty list  $L$ 
3:    $L \leftarrow s$ 
4:    $i \leftarrow 0$ 
5:   while  $L[i] \neq \emptyset$  do
6:      $L_{i+1} \leftarrow \text{emptylist}$ 
7:     for  $v \in L[i]$  do
8:       for edges  $(e)$  incident to  $v$  do
9:         if  $e$  is unexplored then
10:            $w \leftarrow$  the other end of  $e$ 
11:           if  $w$  is unexplored then
12:             label  $e$  as a tree edge
13:             add  $w$  to  $L_{i+1}$ 
14:           else
15:             label  $e$  as a cross edge
16:           end if
17:         end if
18:       end for
19:     end for
20:      $i \leftarrow i + 1$ 
21:   end while
22: end procedure
```

Running time of breadth-first search: $O(n + m)$

3.4 Strong Connectivity

Directed graph: A graph where the edges have a direction.

Examples:



DFS and BFS on directed graphs:

Very similar to undirected graphs, except that we only consider edges that go out of a vertex.

Running time is $O(n + m)$

For example graph above the DFS sequence is 0, 1, 2, 3.

The BFS sequence is 0, 1, 2, 3.

3.4.1 Connectivity

Weak connectivity: If we ignore the direction for all edges, there would be a path from any vertex to any other vertex.

Strong Connectivity: For every two nodes u and v , there is a path from u to v and a path from v to u .

3.4.2 Mutual Reachability

Two nodes u and v are mutually reachable if there is a path from u to v and a path from v to u .

Strong connectivity: For every pair of nodes u and v , these two nodes are mutually reachable.

Transitivity: If u is mutually reachable with v and v is mutually reachable with w , then u is mutually reachable with w .

3.4.3 Testing strong connectivity

Algorithm 7 Testing strong connectivity

```
1: procedure TESTSTRONGCONNECTIVITY( $G$ )
2:   define  $G^R$  to be the graph with the same vertices as  $G$  but with all edges reversed
3:   Select a node  $s$  in  $G$ 
4:   BFS( $G, s$ ), BFS( $G^R, s$ )
5:   for each node  $v$  do
6:     if  $v$  is unexplored in either BFS then
7:       return False
8:     end if
9:   end for
10:  return True
11: end procedure
```

3.5 Testing bipartiteness

Bipartite graph: A graph $G = (V, E)$ is bipartite if and only if the vertices can be partitioned into two sets V_1 and V_2 such that every edge has one end in V_1 and the other end in V_2 .

A Graph $G = (V, E)$ is bipartite if and only if it has no odd cycles. (odd cycle: a cycle with odd number of edges)

Testing bipartiteness:

Given a graph $G = (V, E)$, we want to test if G is bipartite.

Given a graph $G = (V, E)$, decide if it is 2-colourable.

Given a graph $G = (V, E)$, decide if it has an odd cycle.

Colouring the nodes It is quite familiar with BFS:

Algorithm 8 Colouring the nodes

```
1: procedure COLOURING( $G, s$ )
2:   initial empty list  $L$ 
3:   initial empty list  $C$ 
4:    $L \leftarrow s$ 
5:    $C[s] \leftarrow red$ 
6:    $i \leftarrow 0$ 
7:   while  $L[i] \neq \emptyset$  do
8:      $L_{i+1} \leftarrow emptylist$ 
9:     for  $v \in L[i]$  do
10:      for edges ( $e$ ) incident to  $v$  do
11:        if  $e$  is unexplored then
12:           $w \leftarrow$  the other end of  $e$ 
13:          if  $w$  is unexplored then
14:            label  $e$  as a tree edge
15:            add  $w$  to  $L_{i+1}$ 
16:            if  $i + 1$  is odd then
17:               $C[w] \leftarrow green$ 
18:            else
19:               $C[w] \leftarrow red$ 
20:            end if
21:          else
22:            label  $e$  as a cross edge
23:            if  $C[v] = C[w]$  then
24:              return False
25:            end if
26:          end if
27:        end if
28:      end for
29:    end for
30:     $i \leftarrow i + 1$ 
31:  end while
32:  for  $e(v, w) \in G$  do
33:    if  $C[v] = C[w]$  then
34:      return False
35:    end if
36:  end for
37:  return True
38: end procedure
```

Running time of colouring the nodes: $O(n + m)$

Correctness of colouring the nodes:

Proof by contradiction.

Suppose that G is not bipartite.

Then G has an odd cycle.

Suppose to the contrary that the algorithm return True.

That means that the algorithm did not detect the odd cycle.

3.6 DAGs and Topological Ordering

DAG: A directed acyclic graph (DAG) is a directed graph with no directed cycles.
examples of DAGs:



Topological ordering: Given a graph $G = (V, E)$, a topological ordering of G is an ordering of the nodes u_1, u_2, \dots, u_n such that for every edge (u_i, u_j) , we have $i < j$.

Intuitively, a topological ordering is an ordering of the nodes such that every edge goes from left to right.

example of topological ordering based on given graph above:

3, 0, 1, 2, 4, 5

Topological ordering implies DAG:

- If G has a topological ordering, then G is a DAG.
- Suppose by contradiction that G has a topological ordering u_1, u_2, \dots, u_n but G also has a cycle C .
- Let u_j be the smallest element of C in the topological ordering.
- Let u_i be its predecessor in C .
- u_i must appear before u_j in the topological ordering.
- This contradicts the fact that u_j is the smallest element of C in the topological ordering.

DAG implies topological ordering:

Proof by induction: Base case: If G has one or two nodes, then G has a topological ordering.

Induction steps: Assume that a DAG up to k nodes has a topological ordering (induction hypothesis). we will prove that a DAG with $k + 1$ nodes has a topological ordering.

- By our lemma, there is at least one source node in G , and let u be the node.
- Put u at the beginning of the topological ordering.
- Consider the graph G' , obtained by G by removing u and its incident edges.
- G' is a DAG with k nodes.
- It has a topological ordering u_1, u_2, \dots, u_k by the induction hypothesis.
- Append this ordering to u to get a topological ordering of G .

Here is the algorithm:

Algorithm 9 Topological Sorting

```

1: procedure TOPOLOGICALSORTING( $G$ )
2:   find a source vertex  $u$ 
3:   set  $u$  as the first element of the topological ordering
4:    $G' \leftarrow G$  with  $u$  and its incident edges removed
5:    $L \leftarrow$  TOPOLOGICALSORTING( $G'$ )
6:   append  $L$  to  $u$ 
7: end procedure

```

Running time of the algorithm is $O(n^2)$

Modified Topological Sorting:

Running time of the algorithm is $O(n + m)$

Algorithm 10 Modified Topological Sorting

```
1: procedure MODIFIEDTOPOLOGICALSORTING( $G$ )
2:    $L \leftarrow \text{emptylist}$ 
3:    $S \leftarrow$  set of all source vertices
4:   while  $S \neq \emptyset$  do
5:     remove a vertex  $u$  from  $S$ 
6:     append  $u$  to  $L$ 
7:     for each edge  $(u, v)$  do
8:       remove edge  $(u, v)$  from  $G$ 
9:       if  $v$  is a source vertex then
10:        add  $v$  to  $S$ 
11:       end if
12:     end for
13:   end while
14:   if  $G$  has edges then
15:     return  $G$  has a cycle
16:   else
17:     return  $L$ 
18:   end if
19: end procedure
```

3.7 Finding strongly connected components

connected components: A connected component of an undirected graph is subgraph of the graph where any two nodes are connected by a path.

strongly connected components: A strongly connected component of a directed graph is a subgraph of the graph where any two nodes are mutually reachable.(mutually reachable: there is a path from u to v and a path from v to u)

Finding strongly connected components:

Kosaraju's algorithm:

Algorithm 11 Kosaraju's algorithm

```
1: procedure KOSARAJU( $G$ )
2:   Initialise stack  $S$ 
3:   Select a arbitrary node  $s$ 
4:   DFS_tree=DFS( $G, s$ )
5:    $S \leftarrow$  nodes in DFS_tree
6:    $G^R \leftarrow$  nodes in order of  $S$ 
7:   DFS( $G^R, s$ )
8:   return the nodes in the DFS tree
9: end procedure
```

Running time of Kosaraju's algorithm: $O(n + m)$

Correctness of Kosaraju's algorithm:

- Define a meta-graph of G , called $G^{SCC} = (V^{SCC}, E^{SCC})$.
- Supposed that G has strongly connected components (SCCs) C_1, C_2, \dots, C_k , for some k .
- $V^{SCC} = \{C_1, C_2, \dots, C_k\}$ contains some of the SCCs of G .
- There is an edge (C_i, C_j) in E^{SCC} if G contains a directed edge (x, y) such that $x \in C_i$ and $y \in C_j$, crossing different components.

Examples:



The SCCs are $\{0, 1, 2, 3\}$ and $\{4, 5\}$.
 The meta-graph is:



Chapter 4

Greedy Algorithms

The greedy approach:

- The goal is to find a global solution to a problem.
- The solution will be built up in small consecutive steps.
- For each step, we choose the best option available to us at that moment.

4.1 Interval Scheduling

Interval Scheduling:

A set of requests $R = \{1, 2, \dots, n\}$.

- Each request i has a start time s_i and a finish time f_i .
- Alternative view: every request is an interval $[s_i, f_i]$.

Two requests i and j are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.

Goal: Find a maximum-size subset of compatible requests.

Example:

Interval scheduling.

- Job j starts at s_j and finishes at f_j .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.

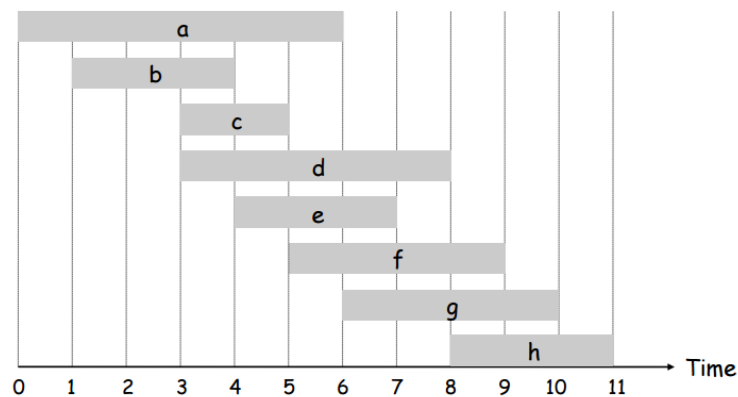


Figure 4.1: Interval Scheduling

Interval Scheduling Algorithm:

Algorithm 12 Interval Scheduling Algorithm

```
1: procedure INTERVALSCHEDULING( $[s_1, f_1], [s_2, f_2], \dots, [s_n, f_n]$ )
2:    $R$  is the set of requests
3:    $A \leftarrow \emptyset$ 
4:   while  $R \neq \emptyset$  do
5:     select a request  $i$  in  $R$  with the smallest finishing time
6:     add  $i$  to  $A$ 
7:     remove all requests from  $R$  that are incompatible with  $i$ 
8:   end while
9:   return  $A$ 
10: end procedure
```

Running time of Interval Scheduling Algorithm: $O(n \log n)$

Correctness of Interval Scheduling Algorithm: Since the algorithm always selects the request with the smallest finishing time, it is clear that the algorithm will always select a compatible request.

Arguing optimality:

4.2 Minimum Spanning Trees

Consider a connected graph $G = (V, E)$, such that each edge $e = (v, w)$ of E , there is an associated cost c_e .

Goal: Find a spanning tree T of E so that the graph $G' = (V, T)$ has minimum cost.

Example:



Greedy approach 1:

- Start with an empty set of edges T .
- Repeat until T forms a spanning tree:
 - Select an edge e of minimum cost.
 - If $T \cup \{e\}$ does not contain a cycle, then add e to T .

krukals algorithm:

Algorithm 13 Krukals algorithm

```
1: procedure KRUKALS( $G$ )
2:    $T \leftarrow \emptyset$ 
3:   while  $T$  is not a spanning tree do
4:     select an edge  $e$  of minimum cost
5:     if  $T \cup \{e\}$  does not contain a cycle then
6:       add  $e$  to  $T$ 
7:     end if
8:   end while
9:   return  $T$ 
10: end procedure
```

Running time of Krukals algorithm: $O(m \log n)$

Greedy approach 2:

- Start with an empty set of edges T .
- Start with a node s .
 - Add an edge $e = (s, v)$ of minimum cost to T .
- Repeat until T forms a spanning tree:

Prims algorithm:

Algorithm 14 Prims algorithm

```
1: procedure PRIMS( $G$ )
2:    $T \leftarrow \emptyset$ 
3:    $s \leftarrow$  an arbitrary node
4:   while  $T$  is not a spanning tree do
5:     add an edge  $e = (s, v)$  of minimum cost to  $T$ 
6:      $s \leftarrow v$ 
7:   end while
8:   return  $T$ 
9: end procedure
```

Running time of Prims algorithm: $O(m \log n)$

minimum spanning tree of example graph:

the minimum spanning tree sequence is d, a, c, b, e, f, h, g .

Greedy approach 3:

- Start with the full graph $G = (V, E)$.
- Delete an edge from G
 - the edge of maximum cost
- Repeat until G forms a spanning tree:

Reverse-delete algorithm:

Algorithm 15 Reverse-delete algorithm

```
1: procedure REVERSEDELETE( $G$ )
2:    $T \leftarrow G$ 
3:   while  $T$  is not a spanning tree do
4:     delete an edge  $e$  of maximum cost from  $T$ 
5:   end while
6:   return  $T$ 
7: end procedure
```

For when two edges have the same cost, use distinct labels to distinguish them.

Optimal with Priority Queue:

Add PQ to Prim's algorithm.

Algorithm 16 Optimal with Priority Queue

```
1: procedure OPTIMAL( $G$ )
2:    $T \leftarrow \emptyset$ 
3:    $s \leftarrow$  an arbitrary node
4:    $PQ \leftarrow$  empty priority queue
5:   for each node  $v$  do
6:     add  $v$  to  $PQ$  with key  $\infty$ 
7:   end for
8:   decrease key of  $s$  to 0
9:   while  $PQ$  is not empty do
10:     $v \leftarrow$  node with minimum key in  $PQ$ 
11:    add an edge  $e = (s, v)$  of minimum cost to  $T$ 
12:     $s \leftarrow v$ 
13:    for each edge  $e = (v, w)$  incident to  $v$  do
14:      if  $w$  is in  $PQ$  then
15:        decrease key of  $w$  to  $c_e$ 
16:      end if
17:    end for
18:  end while
19:  return  $T$ 
20: end procedure
```

Running time of Optimal with Priority Queue: $O(m \log n)$

4.3 Clustering

- a collection of n objects
- they have different degrees of similarity
- we want to organise them into coherent groups
- there is a notion of distance between objects

Definition:

- Given a set U of n elements, a k -clustering of U is a partition of U into non-empty subsets C_1, C_2, \dots, C_k .
- The spacing of a k -clustering is the minimum distance between any pair of points in different clusters.

Goal: Among all possible k -clusterings, find one with minimum spacing.

Example:



Greedy approach:

- Pick two objects p_i and p_j with minimum distance $d(p_i, p_j)$.
- Connect them with an edge $e = (p_i, p_j)$.

- Continue like this until we have k clusters.
- If the edge e under consideration connects two object p_i and p_j already in the same cluster, then discard e .

kruskals algorithm:

Algorithm 17 kruskals algorithm for clustering

Require: A graph $G = (V, E)$

Ensure: A minimum spanning tree of G with k clusters

```

1: procedure KRUSKAL( $G, k$ )
2:    $T \leftarrow \emptyset$ 
3:    $C \leftarrow \{\{v\} \mid v \in V\}$  ▷ Initial clusters
4:   Sort edges in  $E$  in increasing order of weight
5:   for  $\{u, v\} \in E$  do
6:     if  $C$  contains  $k$  clusters then
7:       break
8:     end if
9:     if clusters containing  $u$  and  $v$  are different in  $C$  then
10:       $T \leftarrow T \cup \{\{u, v\}\}$ 
11:      merge clusters containing  $u$  and  $v$  in  $C$ 
12:    end if
13:  end for
14:  return  $T$ 
15: end procedure

```

For Given example, the result of divide them into 3 clusters is:

$$\{a, b, c, d\}, \{e, f, h\}, \{g\}$$

Chapter 5

Dynamic Programming

The paradigm of dynamic programming: Given a problem P , define a sequence of subproblems, with the following properties:

- The subproblems are ordered from the simplest to the largest
- The largest problem is our original problem P
- The optimal solution of a subproblem can be structured from the optimal solutions of smaller subproblems.

Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution and use it to solve larger subproblems.

5.1 Weighted Interval Scheduling

- A set of requests $R = \{1, 2, \dots, n\}$.
 - Request i has a start time s_i and a finish time f_i , and a value v_i .
 - Alternative view: every request is an interval $[s_i, f_i]$ associated with a value v_i .
- Two requests i and j are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.

build up a solution:

1. let O the optimal solution
2. O contains an optimal solution O' of the subproblem $R' = \{1, 2, \dots, i-1\}$
3. in order to find O , it suffices to look at smaller problems and find $O(1, 2, \dots, j)$ for some j
4. Let O_j be a shorthand for $O(1, 2, \dots, j)$ and let $OPT(j)$ be its total value.
5. Define $OPT(0) = 0$
6. Then $O = O_n$ with value $OPT(n)$
7. $OPT(j)$ can be computed from $OPT(j-1)$
8. $OPT(j) = \max\{OPT_{p_j} + v_j, OPT(j-1)\}$

Algorithm 18 ComputeOPT

```
1: procedure COMPUTEOPT( $j$ )
2:   if  $j = 0$  then
3:     return 0
4:   else
5:     return  $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j-1)\}$ 
6:   end if
7: end procedure
```

Correctness: ComputeOPT(j) correctly computes $OPT(j)$ for all $j = 0, 1, \dots, n$.

Proof by induction:

Base case: $OPT(0) = 0$ by definition.

Inductive step: Assume that it is true for all $i < j$. (Induction hypothesis)

return $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j - 1)\}$

Running time: $\Omega(2^n)$

Memoization:

- Compute ComputeOPT(j) for all $j = 0, 1, \dots, n$.
- Store it in an accessible place to use again later.
- Keep an array $M[0, \dots, n]$.
 - initially $M[j] = \text{EMPTY}$ for all $j = 0, 1, \dots, n$.
 - when ComputeOPT(j) is called, $M[j] = \text{ComputeOPT}(j)$.

Algorithm 19 M-ComputeOPT

```
procedure M-COMPUTEOPT(j)
  if j = 0 then
    return 0
  else if M[j] is not empty then
    return M[j]
  else
    M[j] ← max{M-COMPUTEOPT(pj) + v(j), M-COMPUTEOPT(j - 1)}
    return M[j]
  end if
end procedure
```

Running time: $O(n \log n)$

Algorithm 20 Find-Solution

```
procedure FIND-SOLUTION(j)
  if j = 0 then
    return ∅
  else
    if then v(j) + M-COMPUTEOPT(pj) > M-COMPUTEOPT(j - 1)
      return {j} ∪ FIND-SOLUTION(pj)
    else
      return FIND-SOLUTION(j - 1)
    end if
  end if
end procedure
```

Dynamic Programming vs Divide and Conquer:

Dynamic Programming:

- DP is an optimisation techniques and is only applicable to problems that have optimal substructure.
- DP splits the problem into parts, finds solutions to the parts and joins them.(The parts are not significantly smaller than the original problem and are overlapping.)
- In DP, the subproblems dependency can be represented by a directed acyclic graph.

Divide and Conquer:

- DC is not normally used for optimisation problems.
- DC splits the problem into parts, finds solutions to the parts and joins them.(The parts are significantly smaller than the original problem and are non-overlapping.)
- In DC, the subproblems dependency can be represented by a tree.

5.2 Subset Sum

Problem Description:

- Given a set of n items $1, 2, \dots, n$
- Each item i has a non-negative weight w_i .
- Given a bound W .
- **Goal:** select a subset S of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} w_i$ is maximised.

Dynamic Programming: To find the optimal value of $OPT(n)$, we need

- the optimal value of $OPT(n - 1)$ if item n is not selected.
- the optimal value of the solution on input $1, 2, \dots, n-1$ with weight bound $W - w_n$.

subproblems:

- Assumptions:
 - W is an integer
 - Every w_i is an integer
- subproblem for each $i = 0, 1, \dots, n$ and each integer $0 \leq w \leq W$.
- Let $OPT(i, w)$ be the optimal value of the solution on subset $1, 2, \dots, i$ with weight bound w .

Algorithm 21 SubsetSum

```
procedure SUBSETSUM( $n, w$ )
  Array  $M[0, \dots, n, 0, \dots, W]$ 
   $M[0, w] = 0$  for each  $w = 0, 1, \dots, W$ 
  for  $i = 1$  to  $n$  do
    for  $w = 0$  to  $W$  do
      if  $w_i > w$  then
         $M[i, w] = M[i - 1, w]$ 
      else
         $M[i, w] = \max\{M[i - 1, w], M[i - 1, w - w_i] + w_i\}$ 
      end if
    end for
  end for
  return  $M[n, W]$ 
end procedure
```

Running time: $O(nW)$

5.3 knapSack

Problem Description:

- Given a set of n items $1, 2, \dots, n$
- Each item i has a non-negative weight w_i and a non-negative value v_i .
- Given a bound W .
- **Goal:** select a subset S of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximised.

the fractional knapsack problem:

- Given a set of n items $1, 2, \dots, n$
- Each item i has a non-negative weight w_i and a non-negative value v_i .
- Given a bound W .
- **Goal:** select a fraction x_i of each item i such that $\sum_{i \in S} w_i x_i \leq W$ and $\sum_{i \in S} v_i x_i$ is maximised.

The 0/1 knapsack problem: Solution for 0/1 knapsack problem:

Algorithm 22 0/1 knapsack in dynamic programming

```
procedure 0/1 KNAPSACK( $n, W$ )  
  Array  $M[0, \dots, n, 0, \dots, W]$   
   $M[0, w] = 0$  for each  $w = 0, 1, \dots, W$   
  for  $i = 1$  to  $n$  do  
    for  $w = 0$  to  $W$  do  
      if  $w_i > w$  then  
         $M[i, w] = M[i - 1, w]$   
      else  
         $M[i, w] = \max\{M[i - 1, w], M[i - 1, w - w_i] + v_i\}$   
      end if  
    end for  
  end for  
  return  $M[n, W]$   
end procedure
```

Chapter 6

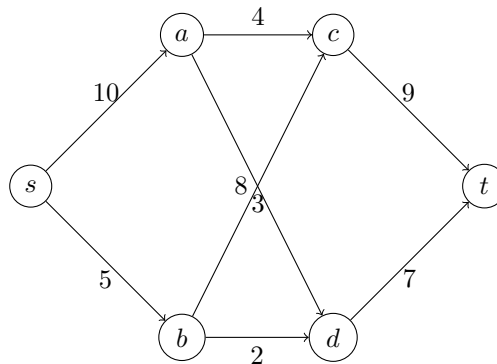
Network Flow

6.1 Network Flow Definitions

Flow network: A flow network is a directed graph $G = (V, E)$ with the following properties:

- Each edge $(u, v) \in E$ has a non-negative capacity c_e .
- There is a single source s in V .
- There is a single sink t in V .
- All other nodes in $V - \{s, t\}$ are called intermediate nodes.

example:



Further definitions:

- The source s has no incoming edges.
- The sink t has no outgoing edges.
- There is at least one edge incident to each node.
- All capacities are integers.

Flow: An $(s - t)$ flow is a function $f : E \rightarrow \mathbb{R}^+$, mapping each edge e to a non-negative real number $f(e)$. A feasible flow must satisfy the following conditions:

- Capacity: For each edge $e \in E$, $0 \leq f(e) \leq c_e$.
- Flow conservation: for each node $v \in V - \{s, t\}$, we have

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

The source s generates flow, and the sink t absorbs flow.

Value of a flow f , denoted $val(f)$, is the total amount of flow generated by the source s :

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

Generally, define $f^{out}(v)$ and $f^{in}(v)$ for the flow going out of (resp. going into) node v .

Similarly, define $f^{out}(S)$ and $f^{in}(S)$ for sets of nodes S .

6.2 Maximum Flow Problem

The maximum flow problem: Given a flow network $G = (V, E)$, find a flow of maximum possible value.

algorithm for maximum flow:

Idea: push flow forward on edges with leftover capacity, push flow backward on edges that are already carrying flow.

The residual graph G_f :

The residual graph G_f of G (also called the flow network) is defined as follows:

- The node set V_f of G_f is the same as the node set V .
- For each edge $(u, v) \in E$ which $f(e) < c_e$, there are $c_e - f(e)$ "leftover" units of capacity.
 - We will call this number the **residual capacity** of edge e .
 - We will call the edge e a forward edge.
- For each edge $(u, v) \in E$ with $f(e) > 0$, there is an edge $e' = (v, u)$ in E_f with a capacity of $f(e)$. We will call the edge e' a backward edge.

Working with residual graphs:

- Find an $(s - t)$ path P in G_f . This is called an **augmenting path**.
- Define the bottleneck of P ,
 - Denoted $bottleneck(P, f)$
 - to be the minimum residual capacity of any edge in P .
- Define the augmentation of flow f into flow f'
 - Denoted $augment(f, P)$.

Augmenting the flow:

Feasibility of capacity:

Algorithm 23 Augmenting the flow

```

1: procedure AUGMENT( $f, P$ )
2:    $b \leftarrow bottleneck(P, f)$ 
3:   for each edge  $e = (u, v) \in P$  do
4:     if  $e$  is a forward edge then
5:        $f(e) \leftarrow f(e) + b$ 
6:     else
7:        $f(e) \leftarrow f(e) - b$ 
8:     end if
9:   end for
10:  return  $f$ 
11: end procedure

```

consider an arbitrary edge $e = (u, v) \in P$. Suppose that e is a forward edge.

$$0 \leq f(e) \leq f'(e) = f(e) + b \leq f(e) + (c_e - f(e)) = c_e$$

Suppose that e is a backward edge.

$$c_e \geq f(e) \geq f'(e) = f(e) - b \geq f(e) - (f(e) - 0) = 0$$

The Ford-Fulkerson algorithm:

Algorithm 24 Max-flow algorithm

```
1: procedure MAX-FLOW( $G, s, t$ )
2:    $f(e) \leftarrow 0$  for all edges  $e \in E$ 
3:   while there exists an  $(s - t)$  path  $P$  in  $G_f$  do
4:      $f \leftarrow \text{augment}(f, P)$ 
5:      $f' \leftarrow \text{update}(f)$ 
6:      $G_f \leftarrow \text{update}(G_f, f)$ 
7:   end while
8:   return  $f$ 
9: end procedure
```

Running time of Ford-Fulkerson algorithm: $O(mC)$, where C is the maximum capacity of any edge in the network.

6.3 Min Cut theorem

A cut C is a partition of the nodes of G into two sets S and T such that $s \in S$ and $t \in T$.

The capacity of a cut $C = (S, T)$ of a cut C is the sum of the capacities of the edges "out of" S : these are edges (u, v) such that $u \in S$ and $v \in T$.

The min-cut theorem: In every flow network, the value of the maximum flow is equal to the capacity of the minimum cut.