Summary of COMP523 Advanced Algorithm

May 8, 2023

Chapter 1

Symmetry Notation

1.1 Asymptotic Notation

Asymptotic notation is a way of describing the limiting behavior of a function when the argument tends towards a particular value or infinity. In computer science, asymptotic notation is frequently used to describe the running time or space usage of an algorithm.

- O-notation: $f(n) = O(g(n))$ if there exist constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
- Ω-notation: $f(n) = \Omega(g(n))$ if there exist constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
- Θ-notation: $f(n) = \Theta(g(n))$ if there exist constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.
- o-notation: $f(n) = o(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.
- ω-notation: $f(n) = \omega(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$.

1.2 Comparing Functions

1.2.1 Transitivity

- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ implies $f(n) = \Omega(h(n))$.
- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ implies $f(n) = \Theta(h(n))$.

For example, $n^2 = O(n^3)$ and $n^3 = O(n^4)$ implies $n^2 = O(n^4)$.

1.2.2 Reflexivity

- $f(n) = O(f(n))$.
- $f(n) = \Omega(f(n))$.
- $f(n) = \Theta(f(n))$.

For example, $n^2 = O(n^2)$.

1.2.3 Symmetry

- $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
- $f(n) = \Omega(g(n))$ implies $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ implies $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ implies $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ implies $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ implies $n^3 = \Omega(n^2)$.

1.2.4 Transpose Symmetry

- $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ if and only if $n^3 = \Omega(n^2)$.

1.2.5 sum and maximum

$$f_1(n) + f_2(n) + \cdots + f_k(n) = \Theta(\max(f_1(n), f_2(n), \ldots, f_k(n)))$$

where k is a constant positive integer.
Let $f_j(n) = j$, $k = n$, then

$$f_1(n) + f_2(n) + \cdots + f_k(n) = n(n+1)/2 = \Theta(n^2)$$

1.2.6 Running time hierarchy

- logarithmic: $O(\log n)$
- linear: $O(n)$
- $n \log n$: $O(n \log n)$
- quadratic: $O(n^2)$
- polynomial: $O(n^k)$
- exponential: $O(c^n)$
- constant: $O(1)$
- superconstant: $\omega(1)$
- sublinear: $o(n)$
- superlinear: $\omega(n)$
- superpolynomial: $\omega(n^k)$
- subexponential: $o(c^n)$

1.3 Expect of algorithms

Correctness: An algorithm is correct if it halts with the correct output for every input instance.
Termination: An algorithm is terminating if it halts for every input instance.
Efficiency: An algorithm is efficient if it halts with the correct output for every input instance and runs in polynomial time.

Chapter 2

Recursion and Divide and Conquer techniques

2.1 Finding Majority in array

The pesudocode of the algorithm is shown in Algorithm 2.1.

Algorithm 1 Finding Majority in array

```
1:  procedure MAJORITY(A)
2:      n ← length of A
3:      if n = 0 then
4:          return −1
5:      end if
6:      if n = 1 then
7:          return A[1]
8:      end if
9:      if n1 and n is odd then
10:
11:     end if
12:     Array B of size n/2
13:     set j=0
14:     for i = 1 to n/2 do
15:         if A[2i − 1] = A[2i] then
16:             B[j] ← A[2i − 1]
17:             j ← j + 1
18:         end if
19:     end for
20:     m ← MAJORITY(B)
21:     count ← 0
22:     for i = 1 to n do
23:         if A[i] = m then
24:             count ← count + 1
25:         end if
26:     end for
27:     if count > n/2 then
28:         return m
29:     else
30:         return −1
31:     end if
32: end procedure
```

Correctness:
Lemma: If A has a majority element, then the majority element of A is also the majority element of B.
Base case: $n = 1$, the majority element is $A[1]$.
Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.
Induction step: If A has a majority element, then the majority element of A is also the majority element of B.
Case 1 (A has a majority element m): Then by the lemma, it is also the majority element of B. Then m appears more than $k/2$ times in B. Then m appears more than $(k + 1)/2$ times in A.
Case 2 (A has no majority element): Then B has no majority element. Then A has no majority element.
Proof the lemma:
proof by contradiction. Assume that A has a majority element m and B has a majority element m', but $m \neq m'$.
Let x be the numbers of occurrence of m in A.
Let y be the numbers of occurrence of m' in B.
Then $2y$ times from pairs that are represented in B by a value different from m', and $x - 2y$ times, since each occurrence of m in A that is not paired with another occurrence of m in A is paired with an occurrence of m' in B.
In total, this gives $2y + x - 2y = x$ occurrences of m in A, which is a contradiction.
Running time:
Recursive formula for the running time:

$$T(n) \leq T(n/2) + cn$$

where c is a constant.
The solution to the recurrence is $T(n) = O(n)$.

2.2 Searching in logarithmic time

Searching faster with BinarySearch.
It is a particular case of the divide-and-conquer paradigm.
Input: A sorted array A of n elements and a value x.
Output: An index i such that $A[i] = x$ or the special value -1 if x does not appear in A.
Pseudocode is shown in Algorithm 2.2.

Algorithm 2 BinarySearch

```
1:  procedure BINARYSEARCH(x, i, j)
2:      if i = j then
3:          if A[i] = x then
4:              return i
5:          else
6:              return −1
7:          end if
8:      else
9:          if x = A[⌊(i + j)/2⌋] then
10:             return ⌊(i + j)/2⌋
11:         else if x < A[⌊(i + j)/2⌋] then
12:             return BINARYSEARCH(x, i, ⌊(i + j)/2⌋)
13:         else
14:             return BINARYSEARCH(x, ⌊(i + j)/2⌋ + 1, j)
15:         end if
16:     end if
17: end procedure
```

Running time:
The number of comparisons performed by BinarySearch is:

$$T(n) \leq T(n/2) + 4$$

Keep calculate:

$$\begin{aligned}
T(n) &\leq T(n/2) + 4 \\
&\leq T(n/4) + 4 + 4 \\
&\leq T(n/8) + 4 + 4 + 4 \\
&\leq T(n/2^k) + 4k \\
&\leq T(n/2^{\log(n-1)}) + 4\log(n-1) \\
&= T(2) + 4(\log n - 1) \\
&\leq 4\log n - 4 \\
&= 4\log n
\end{aligned}$$

proof $T(n) \leq 4\log n$:
Base case: $n = 1, T(1) = 0 \leq 4\log 1 = 0$.
Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.
Induction step: $T(k+1) \leq 4\log(k+1)$.

$$\begin{aligned}
T(k+1) &\leq T(k/2) + 4 \\
&\leq 4\log(k/2) + 4 \\
&= 4\log k - 4 + 4 \\
&= 4\log k \\
&\leq 4\log(k+1)
\end{aligned}$$

Memory usage:
The memory usage of BinarySearch is:

$$M(n) = O(\log n)$$

Comparing BinarySearch and LinearSearch:

$$\begin{aligned}
T_{\text{BinarySearch}}(n) &= O(\log n) \\
T_{\text{LinearSearch}}(n) &= O(n) \\
T_{\text{BinarySearch}}(n) = O(\log n) &< O(n) = T_{\text{LinearSearch}}(n) \\
M_{\text{BinarySearch}}(n) = O(\log n) &< O(1) = M_{\text{LinearSearch}}(n)
\end{aligned}$$

2.3 Running time of Divide and Conquer algorithms

The Master Theorem:
Suppose that $T(n)$ satisfies the recurrence:

$$T(n) \leq aT(n/b) + cn^d$$

where $a \geq 1$, $b > 1$, $c > 0$ and $d \geq 0$ are constants.
Then $T(n)$ has the following asymptotic bounds:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

This theorem is useful for solving recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$ and $f(n)$ is an asymptotically positive function.
Example:

$$T(n) = 8T(n/2) + 100n^2$$

$a = 8$, $b = 2$, $f(n) = 100n^2$, $d = 2$, $\log_b a = \log_2 8 = 3$.
$d = 2 < \log_b a = 3$, so $T(n) = O(n^{\log_b a}) = O(n^3)$.

2.4 Finding piar of points closest to each other

Input: A set P of n points in the plane.
Output: The pair of points in P that are closest to each other.
Pseudocode is shown in Algorithm 2.4. **Running time**:

Algorithm 3 ClosestPair

```
1: procedure ClosestPair(P_1, ..., P_n)
2:     Construct P_x and P_y. P_x is sorted by x-coordinate, P_y is sorted by y-coordinate.
3:     return ClosestPairRec(P_x, P_y)
4: end procedure
```

Algorithm 4 ClosestPairRec

```
1: procedure ClosestPairRec(P_x, P_y)
2:     if |P_x| = |P_y| ≤ 3 then
3:         For each pair of points (P_i, P_j), compute d(P_i, P_j)
4:         return the pair of points with the smallest distance
5:     end if
6:     Construct Q_x, Q_y, R_x and R_y.
7:     (l_1, l_2) = ClosestPairRec(Q_x, Q_y)
8:     (r_1, r_2) = ClosestPairRec(R_x, R_y)
9:     δ = min{d(l_1, l_2), d(r_1, r_2)}
10:    x* = the largest x-coordinate in Q_x
11:    L = (x, y) : x = x*
12:    S = {p ∈ P : p ∈ L and p is within δ of L}
13:    Construct S_v
14:    for p ∈ S do
15:        Let q be the point in S_v closest to p
16:        if d(p, q) < δ then
17:            δ = d(p, q)
18:            (s_1, s_2) = (p, q)
19:        end if
20:    end for
21:    if d(s_1, s_2) < min{d(l_1, l_2), d(r_1, r_2)} then
22:        return (s_1, s_2)
23:    end if
24:    if d(l_1, l_2) < d(r_1, r_2) then
25:        return (l_1, l_2)
26:    else
27:        return (r_1, r_2)
28:    end if
29: end procedure
```

$T(n) \leq 2T(n/2) + O(n \log n) = O(n \log n)$
Example:

Chapter 3

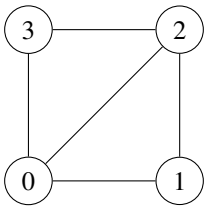
Graph Algorithms

3.1 Graph Definitions

Graph: A graph G consists of a set V of vertices and a set E of edges, where each edge is associated with a pair of vertices.
Directed Graph: A directed graph G consists of a set V of vertices and a set E of directed edges, where each directed edge is associated with an ordered pair of vertices.
Undirected Graph: An undirected graph G consists of a set V of vertices and a set E of undirected edges, where each undirected edge is associated with an unordered pair of vertices.
Neighbours of a vertex v: Set of vertices that are connected to v by an edge.
Degree of a vertex v: number of neighbours of v, denoted by $deg(v)$.
Path: A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.
length = node count - 1 = edge count.
Distance between two nodes: The number of edges in the shortest path between the two nodes.
Graph diameter: The maximum distance between any two nodes in the graph.

Lines,cycles,trees and cliques:
Line: A graph with n vertices and $n-1$ edges.
Cycle: A graph with n vertices and n edges.
cliques: A graph with n vertices and $n(n-1)/2$ edges.
Tree: A graph with n vertices and $n-1$ edges.

Graph representations:
Adjacency matrix: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge between i and j, and $A_{ij} = 0$ otherwise.
examples of adjacency matrices:
Given the following graph:

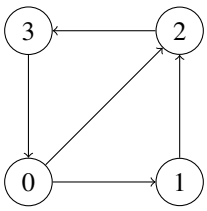


The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Adjacency matrix for directed graphs: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge from i to j, and $A_{ij} = 0$ otherwise.

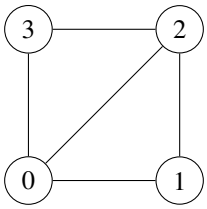
examples of adjacency matrices for directed graphs:
Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

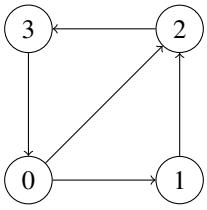
Adjacency list: A list of lists, where the ith list contains the neighbours of vertex i.
Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \\ 0 & 1 & 3 \\ 0 & 2 & \end{bmatrix}$$

Adjacency list for directed graphs: A list of lists, where the ith list contains the neighbours of vertex i.
Given the following graph:



The adjacency list is:

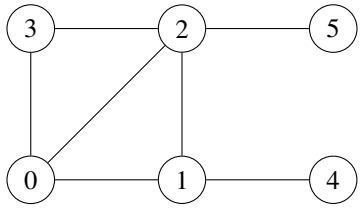
$$\begin{bmatrix} 1 & 2 \\ 2 & \\ 3 & \\ 0 & \end{bmatrix}$$

Adjacency matrix vs adjacency list:

Adjacency matrix	Adjacency list
$O(1)$ to check if there is an edge between i and j	$O(min(deg(i), deg(j)))$ to check if there is an edge between i and j
$O(n)$ to find the neighbours of i	$O(deg(j))$ to find the neighbours of i
$O(n^2)$ space	$O(n + m)$ space

3.2 Depth-first search

Depth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.
example of depth-first search:



The depth-first search sequence is:

$$0, 1, 2, 3, 5, 4$$

Depth-first search algorithm:

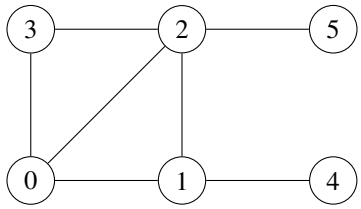
Algorithm 5 Depth-first search algorithm

```
1:  procedure DFS(G, v)
2:      for e ∈ V do
3:          if e is unexplored then
4:              u = head of e
5:              if u is unexplored then
6:                  e is a tree edge
7:                  DFS(G, u)
8:              else
9:                  e is a back edge
10:             end if
11:         end if
12:     end for
13: end procedure
```

Running time of depth-first search: $O(n + m)$

3.3 Breadth-first search

Breadth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.
exaqmple of breadth-first search:



The breadth-first search sequence starting from vertex 0 is 0, 1, 2, 3, 4, 5.
Breadth-first search algorithm:

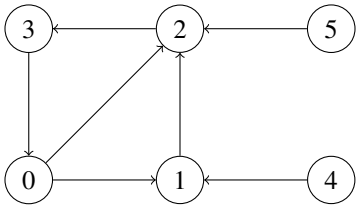
Algorithm 6 Breadth-first search algorithm

```
1:  procedure BFS(G, s)
2:      initial empty list L
3:      L ← s
4:      i ← 0
5:      while L[i] ≠ ∅ do
6:          L_{i+1} ← emptylist
7:          for v ∈ L[i] do
8:              for edges (e) incident to v do
9:                  if e is unexplored then
10:                     w ← the other end of e
11:                     if w is unexplored then
12:                         label e as a tree edge
13:                         add w to L_{i+1}
14:                     else
15:                         label e as a cross edge
16:                     end if
17:                 end if
18:             end for
19:         end for
20:         i ← i + 1
21:     end while
22: end procedure
```

Running time of breadth-first search: $O(n + m)$

3.4 Strong Connectivity

Directed graph: A graph where the edges have a direction.
Examples:



DFS and BFS on directed graphs:
Very similar to undirected graphs, except that we only consider edges that go out of a vertex.
Running time is $O(n + m)$
For example graph above the DFS sequence is 0, 1, 2, 3.
The BFS sequence is 0, 1, 2, 3.

3.4.1 Connectivity

Weak connectivity:If we ignore the direction for all edges, there would be a pah from any vertex to any other vertex.
Strong Connectivity:For every two nodes u and v, there is a path from u to v and a path from v to u.

3.4.2 Mutual Reachability

Two nodes u and v are mutually reachable if there is a path from u to v and a path from v to u.
Strong connectivity:For every pair of nodes u and v, these two nodes are mutually reachable.
Transitivity:If u is mutually reachable with v and v is mutually reachable with w, then u is mutually reachable with w.

3.4.3 Testing strong connectivity

Algorithm 7 Testing strong connectivity

```
1:  procedure TESTSTRONGCONNECTIVITY(G)
2:      define G^R to be the graph with the same vertices as G but with all edges reversed
3:      Select a node s in G
4:      BFS(G, s),BFS(G^R, s)
5:      for each node v do
6:          if v is unexplored in either BFS then
7:              return False
8:          end if
9:      end for
10:     return True
11: end procedure
```

3.5 Testing bipartiteness

Bipartite graph: A graph $G = (V, E)$ is bipartite if any only if the vertices can be partitioned into two sets V_1 and V_2 such that every edge has one end in V_1 and the other end in V_2.
A Graph $G = (V, E)$ is bipartite if and only if it has no odd cycles.(odd cycle: a cycle with odd number of edges)
Testing bipartiteness:
Given a graph $G = (V, E)$, we want to test if G is bipartite.
Given a graph $G = (V, E)$, decide if it is 2-colourable.
Given a graph $G = (V, E)$, decide if it has an odd cycle.
Colouring the nodes It is quite familiar with BFS:

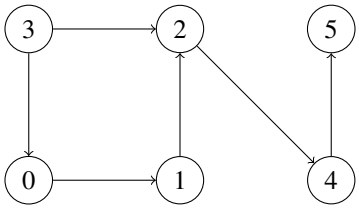
Algorithm 8 Colouring the nodes

```
1: procedure COLOURING(G, s)
2:     initial empty list L
3:     initial empty list C
4:     L ← s
5:     C[s] ← red
6:     i ← 0
7:     while L[i] ≠ ∅ do
8:         L_{i+1} ← emptylist
9:         for v ∈ L[i] do
10:            for edges (e) incident to v do
11:                if e is unexplored then
12:                    w ← the other end of e
13:                    if w is unexplored then
14:                        label e as a tree edge
15:                        add w to L_{i+1}
16:                        if i + 1 is odd then
17:                            C[w] ← green
18:                        else
19:                            C[w] ← red
20:                        end if
21:                    else
22:                        label e as a cross edge
23:                        if C[v] = C[w] then
24:                            return False
25:                        end if
26:                    end if
27:                end if
28:            end for
29:        end for
30:        i ← i + 1
31:    end while
32:    for e(v, w) ∈ G do
33:        if C[v] = C[w] then
34:            return False
35:        end if
36:    end for
37:    return True
38: end procedure
```

Running time of colouring the nodes: $O(n+m)$
Correctness of colouring the nodes:
Proof by contradiction.
Suppose that G is not bipartite.
Then G has an odd cycle.
Suppose to the contrary that the algorithm return True.
That means that the algorithm did not detect the odd cycle.

3.6 DAGs and Topological Ordering

DAG: A directed acyclic graph (DAG) is a directed graph with no directed cycles.
examples of DAGs:



Topological ordering:Given a graph$G = (V, E)$, a topological ordering of G is an ordering of the nodes $u_1, u_2, \ldots, u_n$ such that for every edge (u_i, u_j), we have $i < j$.
Intutively, a topological ordering is an ordering of the nodes such that every edge goes from left to right.
example of topological ordering based on given graph above:

$$3, 0, 1, 2, 4, 5$$

Topological ordering implies DAG:

- If G has a topological ordering, then G is a DAG.
- Suppose by contradiction that G has a topological ordering $u_1, u_2, \ldots, u_n$ but G also has a cycle C.
- Let u_j be the smallest element of C in the topological ordering.
- Let u_i be its predecessor in C.
- u_i must appear before u_j in the topological ordering.
- This contradicts the fact that u_j is the smallest element of C in the topological ordering.

DAG implies topological ordering:
Proof by induction: Base case: If G has one or two nodes, then G has a topological ordering.
Induction steps: Assume that a DAG up to k nodes has a topological ordering(induction hypothesis). we will prove that a DAG with $k + 1$ nodes has a topological ordering.

- By our lemma, there is at least one source node in G, and let u be the node.
- Put u at the beginning of the topological ordering.
- Consider the graph G', obtained by G by removing u and its incident edges.
- G' is a DAG with k nodes.
- It has a topological ordering $u_1, u_2, \ldots, u_k$ by the induction hypothesis.
- Append this ordering to u to get a topological ordering of G.

Here is the algorithm:

Algorithm 9 Topological Sorting

```
1: procedure TOPOLOGICALSORTING(G)
2:     find a source vertex u
3:     set u as the first element of the topological ordering
4:     G' ← G with u and its incident edges removed
5:     L ← TOPOLOGICALSORTING(G')
6:     append L to u
7: end procedure
```

Running time of the algorithm is $O(n^2)$
Modified Topological Sorting:
Running time of the algorithm is $O(n + m)$

Algorithm 10 Modified Topological Sorting

```
1: procedure MODIFIEDTOPOLOGICALSORTING(G)
2:     L ← emptylist
3:     S ← set of all source vertices
4:     while S ≠ ∅ do
5:         remove a vertex u from S
6:         append u to L
7:         for each edge (u, v) do
8:             remove edge (u, v) from G
9:             if v is a source vertex then
10:                add v to S
11:            end if
12:        end for
13:    end while
14:    if G has edges then
15:        return G has a cycle
16:    else
17:        return L
18:    end if
19: end procedure
```

3.7 Finding strongly connected components

connected components: A connected component of an undirected graph is subgraph of the graph where any two nodes are connected by a path.
strongly connected components: A strongly connected component of a directed graph is a subgraph of the graph where any two nodes are mutually reachable.(mutually reachable: there is a path from u to v and a path from v to u)
Finding strongly connected components:
Kosaraju's algorithm:

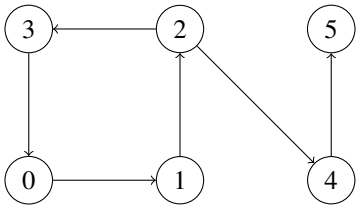
Algorithm 11 Kosaraju's algorithm

```
1: procedure KOSARAJU(G)
2:     Initialise stack S
3:     Select a arbitrary node s
4:     DFS_tree=DFS(G, s)
5:     S ← nodes in DFS_tree
6:     G^R ← nodes in order of S
7:     DFS(G^R, s)
8:     return the nodes in the DFS tree
9: end procedure
```

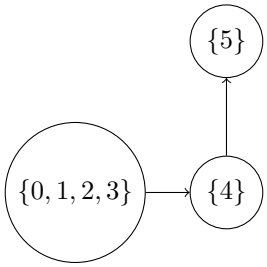
Running time of Kosaraju's algorithm: $O(n+m)$
Correctness of Kosaraju's algorithm:

- Define a meta-graph of G, called $G^{SCC} = (V^{SCC}, E^{SCC})$.
- Supposed that G has strongly connected components (SCCs) $C_1, C_2, \ldots, C_k$, for some k.
- $V^{SCC} = \{C_1, C_2, \ldots, C_k\}$ contains some of the SCCs of G.
- There is an edge (C_i, C_j) in E^{SCC} if G contains a directed edge (x, y) such that $x \in C_i$ and $y \in C_j$, crossing different components.

Examples:



The SCCs are $\{0, 1, 2, 3\}$ and $\{4, 5\}$.
The meta-graph is:



Chapter 4

Greedy Algorithms

The greedy approach:

- The goal is to find a global solution to a problem.
- The solution will be built up in small consecutive steps.
- For each step, we choose the best option available to us at that moment.

4.1 Interval Scheduling

Interval Scheduling:
A set of requests$R = \{1, 2, \ldots, n\}$.

- Each request i has a start time s_i and a finish time f_i.
- Alternative view: every request is an interval $[s_i, f_i]$.

Two requests i and j are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.
Goal: Find a maximum-size subset of compatible requests.
Example:

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.

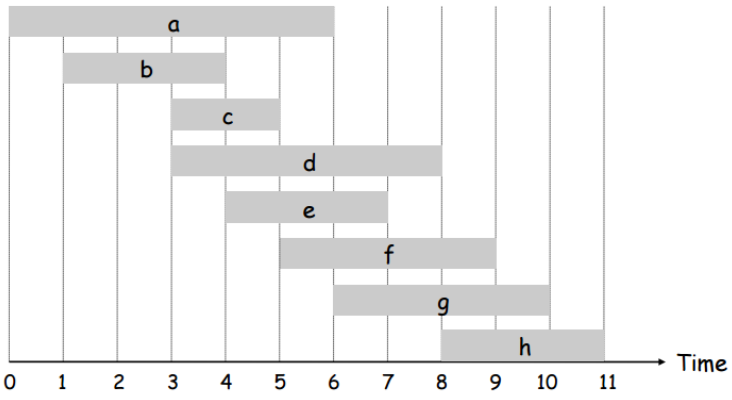


Figure 4.1: Interval Scheduling

Interval Scheduling Algorithm:

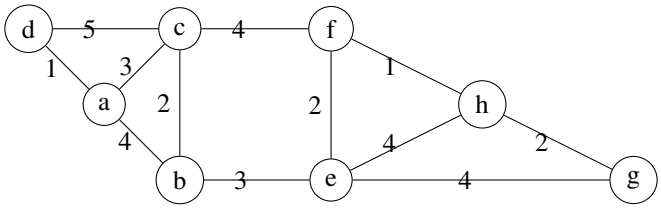
Algorithm 12 Interval Scheduling Algorithm

```
1: procedure IntervalScheduling([s_1, f_1], [s_2, f_2], ..., [s_n, f_n])
2:     R is the set of requests
3:     A ← ∅
4:     while R ≠ ∅ do
5:         select a request i in R with the smallest finishing time
6:         add i to A
7:         remove all requests from R that are incompatible with i
8:     end while
9:     return A
10: end procedure
```

Running time of Interval Scheduling Algorithm: $O(n \log n)$
Correctness of Interval Scheduling Algorithm: Since the algorithm always selects the request with the smallest finishing time, it is clear that the algorithm will always select a compatible request.
Arguing optimality:

4.2 Minimum Spanning Trees

Consider a connected graph $G = (V, E)$, such that each edge $e = (v, w)$ of E, there is an associated cost c_e.
Goal: Find a spanning tree T of E so that the graph $G' = (V, T)$ has minimum cost.
Example:



Greedy approach 1:

- Start with an empty set of edges T.
- Repeat until T forms a spanning tree:
  - Select an edge e of minimum cost.
  - If $T \cup \{e\}$ does not contain a cycle, then add e to T.

krukals algorithm:

Algorithm 13 Krukals algorithm

```
1: procedure Krukals(G)
2:     T ← ∅
3:     while T is not a spanning tree do
4:         select an edge e of minimum cost
5:         if T ∪ {e} does not contain a cycle then
6:             add e to T
7:         end if
8:     end while
9:     return T
10: end procedure
```

Running time of Krukals algorithm: $O(m \log n)$
Greedy approach 2:

- Start with an empty set of edges T.
- Start with a node s.
  - Add an edge $e = (s, v)$ of minimum cost to T.
- Repeat until T forms a spanning tree:

Prims algorithm:

Algorithm 14 Prims algorithm

```
1: procedure PRIMS(G)
2:     T ← ∅
3:     s ← an arbitrary node
4:     while T is not a spanning tree do
5:         add an edge e = (s, v) of minimum cost to T
6:         s ← v
7:     end while
8:     return T
9: end procedure
```

Running time of Prims algorithm: $O(m \log n)$
minimum spanning tree of example graph:
the minimum spanning tree sequence is d, a, c, b, e, f, h, g.

Greedy approach 3:

- Start with the full graph $G = (V, E)$.
- Delete an edge from G
  - the edge of maximum cost
- Repeat until G forms a spanning tree:

Reverse-delete algorithm:

Algorithm 15 Reverse-delete algorithm

```
1: procedure REVERSEDELETE(G)
2:     T ← G
3:     while T is not a spanning tree do
4:         delete an edge e of maximum cost from T
5:     end while
6:     return T
7: end procedure
```

For when two edges have the same cost, use distinct labels to distinguish them.

Optimal with Priorty Queue:
Add PQ to Prim's algorithm.

Algorithm 16 Optimal with Priorty Queue

```
1: procedure OPTIMAL(G)
2:     T ← ∅
3:     s ← an arbitrary node
4:     PQ ← empty priority queue
5:     for each node v do
6:         add v to PQ with key ∞
7:     end for
8:     decrease key of s to 0
9:     while PQ is not empty do
10:        v ← node with minimum key in PQ
11:        add an edge e = (s, v) of minimum cost to T
12:        s ← v
13:        for each edge e = (v, w) incident to v do
14:            if w is in PQ then
15:                decrease key of w to c_e
16:            end if
17:        end for
18:    end while
19:    return T
20: end procedure
```

Running time of Optimal with Priorty Queue: $O(m \log n)$

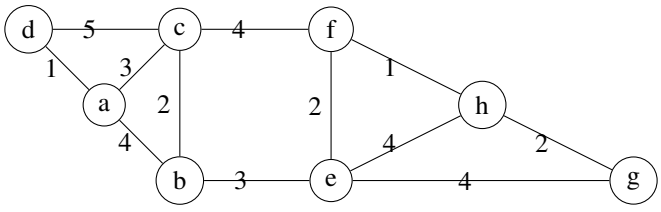
4.3 Clustering

- a collection of n objects
- they have different degrees of similarity
- we want to organise them into coherent groups
- there is a notion of distance between objects

Definition:

- Given a set U of n elements, a $k-$clustering of U is a partition of U into non-empty subsets $C_1, C_2, \ldots, C_k$.
- The spacing of a $k-$clustering is the minimum distance between any pair of points in different clusters.

Goal:Among all possible $k-$clusterings, find one with minimum spacing.
Example:



Greedy approach:

- Pick two objects p_i and p_j with minimum distance $d(p_i, p_j)$.
- Connect them with an edge $e = (p_i, p_j)$.

- Continue like this until we have k clusters.
- If the edge e under consideration connects two object p_i and p_j already in the same cluster, then discard e.

kruskals algorithm:

Algorithm 17 kruskals algorithm for clustering

Require: A graph $G = (V, E)$
Ensure: A minimum spanning tree of G with k clusters

```
1:  procedure KRUSKAL(G, k)
2:      T ← ∅
3:      C ← {{v} | v ∈ V}                                   ▷ Initial clusters
4:      Sort edges in E in increasing order of weight
5:      for {u, v} ∈ E do
6:          if C contains k clusters then
7:              break
8:          end if
9:          if clusters containing u and v are different in C then
10:             T ← T ∪ {{u, v}}
11:             merge clusters containing u and v in C
12:         end if
13:     end for
14:     return T
15: end procedure
```

For Given example, the result of divide them into 3 clusters is:

$$\{a, b, c, d\}, \{e, f, h\}, \{g\}$$

Chapter 5

Dynamic Programming

The paradigm of dynamic programming: Given a problem P, define a sequence of subproblems, with the following properties:

- The subproblems are ordered from the simplest to the largest
- The largest problem is our original problem P
- The optimal solution of a subproblem can be structured from the optimal solutions of smaller subproblems.

Solve the subproblems from the smallest to the largest. When you solve a subproblem, store the solution and use it to solve larger subproblems.

5.1 Weighted Interval Scheduling

- A set of requests $R = \{1, 2, \ldots, n\}$.
  - Request i has a start time s_i and a finish time f_i, and a value v_i.
  - Alternative view: every request is an interval $[s_i, f_i]$ associated with a value v_i.
- Two requests i and j are compatible if $[s_i, f_i]$ and $[s_j, f_j]$ do not overlap.

build up a solution:

1. let O the optimal solution
2. O contains an optimal solution O' of the subproblem $R' = \{1, 2, \ldots, i-1\}$
3. in order to find O, it suffices to look at smaller problems and find $O(1, 2, \ldots, j)$ for some j
4. Let O_j be a shorthand for $O(1, 2, \ldots, j)$ and let $OPT(j)$ be its total value.
5. Define $OPT(0) = 0$
6. Then $O = O_n$ with value $OPT(n)$
7. $OPT(j)$ can be computed from $OPT(j-1)$
8. $OPT(j) = \max\{OPT_{p_j} + v_j, OPT(j-1)\}$

Algorithm 18 ComputeOPT

```
1: procedure COMPUTEOPT(j)
2:     if j = 0 then
3:         return 0
4:     else
5:         return max{COMPUTEOPT(p_j) + v(j), COMPUTEOPT(j − 1)}
6:     end if
7: end procedure
```

Correctness: ComputeOPT(j) correctly computes $OPT(j)$ for all $j = 0, 1, \ldots, n$.
Proof by induction:
Base case: $OPT(0) = 0$ by definition.
Inductive step: Assume that it is true for all $i < j$.(Induction hypothesis)
return $\max\{\text{COMPUTEOPT}(p_j) + v(j), \text{COMPUTEOPT}(j-1)\}$
Running time: $\Omega(2^n)$
Memoization:

- Compute ComputeOPT(j) for all $j = 0, 1, \ldots, n$.
- Store it in an accessible place to use again later.
- Keep an array $M[0, \ldots, n]$.
  - initially $M[j] = \text{EMPTY}$ for all $j = 0, 1, \ldots, n$.
  - when ComputeOPT(j) is called, $M[j] =$ ComputeOPT(j).

Algorithm 19 M-ComputeOPT

procedure M-COMPUTEOPT(j)
  if $j = 0$ **then**
    return 0
  else if $M[j]$ is not empty **then**
    return $M[j]$
  else
    $M[j] \leftarrow \max\{\text{M-COMPUTEOPT}(p_j) + v(j), \text{M-COMPUTEOPT}(j-1)\}$
    return $M[j]$
  end if
end procedure

Running time: $O(n \log n)$

Algorithm 20 Find-Solution

procedure FIND-SOLUTION(j)
  if $j = 0$ **then**
    return $\emptyset$
  else
    if then$v(j) + \text{M-COMPUTEOPT}(p_j) > \text{M-COMPUTEOPT}(j-1)$
      return $\{j\} \cup \text{FIND-SOLUTION}(p_j)$
    else
      return $\text{FIND-SOLUTION}(j-1)$
    end if
  end if
end procedure

Dynamic Programming vs Divide and Conquer:

Dynamic Programming:

- DP is an optimisation techniques and is only applicable to problems that have optimal substructure.
- DP splits the problem into parts, finds solutions to the parts and joins them.(The parts are not significantly smaller than the original problem and are overlapping.)
- In DP, the subproblems dependency can be represented by a directed acyclic graph.

Divide and Conquer:

- DC is not normally used for optimisation problems.
- DC splits the problem into parts, finds solutions to the parts and joins them.(The parts are significantly smaller than the original problem and are non-overlapping.)
- In DC, the subproblems dependency can be represented by a tree.

5.2 Subset Sum

Problem Description:

- Given a set of n items $1, 2, \ldots, n$
- Each item i has a non-negative weight w_i.
- Given a bound W.
- **Goal**: select a subset S of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} w_i$ is maximised.

Dynamic Programming: To find the optimal value of $OPT(n)$, we need

- the optimal value of $OPT(n-1)$ if item n is not selected.
- the optimal value of the solution on input 1,2,...,n-1 with weight bound $W - w_n$.

subproblems:

- Assumptions:
  - W is an integer
  - Every w_i is an integer
- subproblem for each $i = 0, 1, \ldots, n$ and each integer $0 \leq w \leq W$.
- Let $OPT(i, w)$ be the optimal value of the solution on subset $1, 2, \ldots, i$ with weight bound w.

Algorithm 21 SubsetSum

```
procedure SUBSETSUM(n, w)
    Array M[0, ..., n, 0, ..., W]
    M[0, w] = 0 for each w = 0, 1, ..., W
    for i = 1 to n do
        for w = 0 to W do
            if w_i > w then
                M[i, w] = M[i − 1, w]
            else
                M[i, w] = max{M[i − 1, w], M[i − 1, w − w_i] + w_i}
            end if
        end for
    end for
    return M[n, W]
end procedure
```

Running time: $O(nW)$

5.3 knapSack

Problem Description:

- Given a set of n items $1, 2, \ldots, n$
- Each item i has a non-negative weight w_i and a non-negative value v_i.
- Given a bound W.
- **Goal**: select a subset S of items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximised.

the fractional knapsack problem:

- Given a set of n items $1, 2, \ldots, n$
- Each item i has a non-negative weight w_i and a non-negative value v_i.
- Given a bound W.
- **Goal**: select a fraction x_i of each item i such that $\sum_{i \in S} w_i x_i \leq W$ and $\sum_{i \in S} v_i x_i$ is maximised.

The 0/1 knapsack problem: Solution for 0/1 knapsack problem:

Algorithm 22 0/1 knapsack in dynamic programming

```
procedure 0/1 KNAPSACK(n, W)
    Array M[0, ..., n, 0, ..., W]
    M[0, w] = 0 for each w = 0, 1, ..., W
    for i = 1 to n do
        for w = 0 to W do
            if w_i > w then
                M[i, w] = M[i − 1, w]
            else
                M[i, w] = max{M[i − 1, w], M[i − 1, w − w_i] + v_i}
            end if
        end for
    end for
    return M[n, W]
end procedure
```

Chapter 6

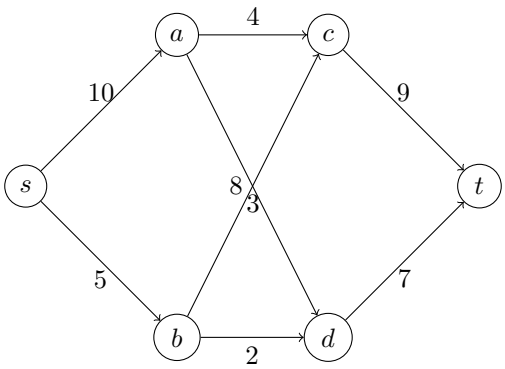
Network Flow

6.1 Network Flow Definitions

Flow network: A flow network is a directed graph $G = (V, E)$ with the following properties:

- Each edge $(u, v) \in E$ has a non-negative capacity c_e.
- There is a single source s in V.
- There is a single sink t in V.
- All other nodes in $V - \{s, t\}$ are called intermediate nodes.

example:



Further definitions:

- The source s has no incoming edges.
- The sink t has no outgoing edges.
- There is at least one edge incident to each node.
- All capacities are integers.

Flow: An $(s - t)$ flow is a function $f : E \to \mathbb{R}^+$, mapping each edge e to a non-negative real number $f(e)$.
A feasible flow must satisfy the following conditions:

- Capacity: For each edge $e \in E$, $0 \leq f(e) \leq c_e$.
- Flow conservation: for each node $v \in V - \{s, t\}$, we have

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

The source s generates flow, and the sink t absorbs flow.
Value of a flow f, denoted $val(f)$, is the total amount of flow generated by the source s:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

Generally, define $f^{out}(v)$ and $f^{in}(v)$ for the flow going out of(resp. going into) node v.
Similarly, define $f^{out}(S)$ and $f^{in}(S)$ for sets of nodes S.

6.2 Maximum Flow Problem

The maximum flow problem: Given a flow network $G = (V, E)$, find a flow of maximum possible value.
algorithm for maximum flow:
Idea: push flow forward on edges with leftover capacity, push flow backward on edges that are already carrying flow.
The residual graph G_f:
The residual graph G_f of G(also called the flow network) is defined as follows:

- The node set V_f of G_f is the same as the node set V.
- For each edge $(u, v) \in E$ which $f(e) < c_e$, there are $c_e - f(e)$ "leftover" units of capacity.
  - We will call this number the **residual capacity** of edge e.
  - We will call the edge e a forward edge.
- For each edge $(u, v) \in E$ with $f(e) > 0$, there is an edge $e' = (v, u)$ in E_f with a capacity of $f(e)$.We will call the edge e' a backward edge.

Working with residual graphs:

- Find an $(s - t)$ path P in G_f. This is called an **augmenting path**.
- Define the bottleneck of P,
  - Denoted bottleneck(P, f)
  - to be the minimum residual capacity of any edge in P.
- Define the augmentation of flow f into flow f'
  - Denoted augment(f, P).

Augmenting the flow:
Feasibility of capacity:

Algorithm 23 Augmenting the flow

```
1:  procedure AUGMENT(f, P)
2:      b ← bottleneck(P, f)
3:      for each edge e = (u, v) ∈ P do
4:          if e is a forward edge then
5:              f(e) ← f(e) + b
6:          else
7:              f(e) ← f(e) − b
8:          end if
9:      end for
10:     return f
11: end procedure
```

consider an arbitrary edge $e = (u, v) \in P$. Suppose that e is a forward edge.

$$0 \le f(e) \le f'(e) = f(e) + b \le f(e) + (c_e - f(e)) = c_e$$

Suppose that e is a backward edge.

$$c_e \geq f(e) \geq f'(e) = f(e) - b \geq f(e) - (f(e) - 0) = 0$$

The Ford-Fulkerson algorithm:

Algorithm 24 Max-flow algorithm

```
1: procedure MAX-FLOW(G, s, t)
2:     f(e) ← 0 for all edges e ∈ E
3:     while there exists an (s − t) path P in G_f do
4:         f ← augment(f, P)
5:         f' ← update(f)
6:         G_f ← update(G_f, f)
7:     end while
8:     return f
9: end procedure
```

Running time of Ford-Fulkerson algorithm: $O(mC)$, where C is the maximum capacity of any edge in the network.

6.3 Min Cut theorem

A cut C is a partition of the nodes of G into two sets S and T such that $s \in S$ and $t \in T$.
The capacity of a cut $C = (S, T)$ of a cut C is the sum of the capacities of the edges "out of" S: these are edges (u, v) such that $u \in S$ and $v \in T$.

The min-cut theorem: In every flow network, the value of the maximum flow is equal to the capacity of the minimum cut.
A series of facts:
Fact 1: Let f be any $(s - t)$ flow and let (S, T) be any cut. Then $v(f) = f^{out}(S) - f^{in}(S)$.

1. By definition, $v(f) = f^{out}(s)$.
2. By definition $f^{in}(s) = 0$.
3. Hence, $v(f) = f^{out}(s) - f^{in}(s)$.
4. For every other node $v \neq s, t$, we have $f^{out}(v) = f^{in}(v)$.
5. Therefore, $v(f) = \sum_{v \in S}(f^{out}(v) - f^{in}(v))$.
6. rewrite as $v(f) = \sum_{v \in S}(f^{out}(v) - f^{in}(v)) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = f^{out}(S) - f^{in}(S)$.

Fact 2: Let f be any $(s - t)$ flow and let (S, T) be any $(s - t)$ cut. Then $v(f) = f^{out}(T) - f^{in}(T)$.
Fact 3: Let f be any $(s - t)$ flow and let (S, T) be any $(s - t)$ cut. Then $v(f) \leq c(S, T)$.

$$\begin{aligned} v(f) &= f^{out}(S) - f^{in}(S) \\ &\leq f^{out}(S) \\ &= \sum_{e \text{ out of } S} f(e) \\ &\leq \sum_{e \text{ out of } S} c(e) \\ &= c(S, T) \end{aligned}$$

Fact 4: Let f be any $(s-t)$ flow in G such that the residual graph G_f contains no augmenting paths. Then there exists an $(s-t)$ cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
Proving fact 4: In the residual graph G_f, identify all nodes that are reachable from the source s. Let S^* be the set of these nodes, and let $T^* = V - S^*$.

1. $s \in S^*$ and $t \in T^*$.
2. No edge of G_f crosses from S^* to T^*.
3. Every edge of G_f crosses from T^* to S^*.
4. $f^{out}(S^*) = v(f)$.
5. $f^{in}(S^*) = 0$.

$$\begin{aligned} v(f) &= f^{out}(S^*) - f^{in}(S^*) \\ &= \sum_{e \text{ out of } S^*} f(e) - \sum_{e \text{ into } S^*} f(e) \\ &= \sum_{e \text{ out of } S^*} c(e) - 0 \qquad\qquad = c(S^*, T^*) \end{aligned}$$

Fact 5: If all capacities are integers, then there exists a maximum flow f for which $f(e)$ is an integer for every edge e.

6.4 Choosing Better Augmenting Paths

The Edmonds-Karp algorithm:

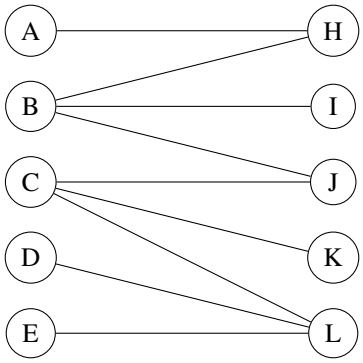
Algorithm 25 Edmonds-Karp algorithm

```
1:  procedure EDMONDS-KARP(G, s, t)
2:      f(e) ← 0 for all edges e ∈ E
3:      while there exists an (s − t) path P in G_f do
4:          P is a shortest (s − t) path
5:          f ← augment(f, P)
6:          f' ← update(f)
7:          G_f ← update(G_f, f)
8:      end while
9:      return f
10: end procedure
```

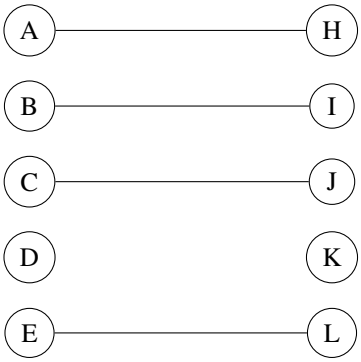
Running time of Edmonds-Karp algorithm: $O(nm^2)$, where n is the number of nodes and m is the number of edges in the network.
The shortest path can be found in $O(m)$ time using BFS.

6.5 Modeling with Network Flows

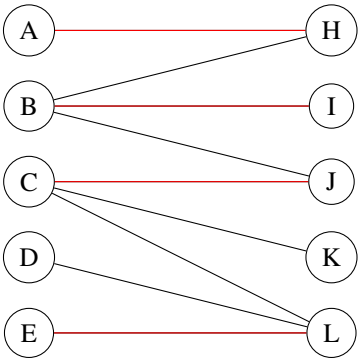
Bipartite graphs: A graph $G = (V, E)$ is bipartite if any only if it can be partitioned into two sets A and B such that every edge has one endpoint in A and one endpoint in B.
Maximum bipartite matching:
Matching: A subset M of edges E such that each node $v \in V$ appears in at most one edge $e \in M$.
Maximum matching: A matching with maximum cardinality.
examples of bipartite graphs:



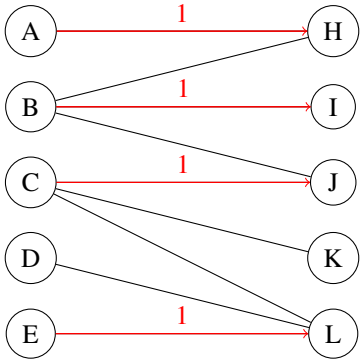
example of maximum bipartite matching:

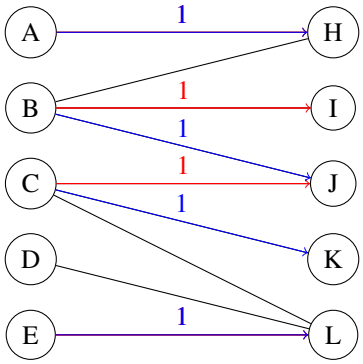


exapmle of maximal bipartite matching:



From matchings to flows:





Maximum Flow and Maximum Matching
The size of the maximum matching is equal to the value of the maximum flow.

The edges of M are the edges that carry flow form A to B in the residual network.
Running time: $O(mn)$

Baseball Elimination

- Given a set S of teams
- For each team x in S, the current number of wins w_x
- For teams x and y in S, they still have to play g_{xy} games against each other
- Given a designated team z
- Can z still win the turnament?

From Baseball Elimination to flows

- For each pair of teams x and y, create a vertex v_{xy}
- For each team x, create a vertex v_x
- For each pair of teams x and y, create an edge (s, v_{xy}) with capacity g_{xy}
- For each team x, create an edge (v_x, t) with capacity $w_z + g_{xz} - w_x$
- For each pair of teams x and y, create an edge (v_{xy}, v_x) with infinite capacity
- For each pair of teams x and y, create an edge (v_{xy}, v_y) with infinite capacity

Open pit mining

- Given a set S of blocks
- For each block x in S, the value v_x of the ore in the block
- For each block x in S, the cost c_x of mining the block
- For each block x in S, the set N_x of blocks that are neighbors of x
- Given a designated block z
- What is the maximum value of ore that can be mined?

From open pit mining to flows

- For each block x in S, create a vertex v_x
- For each block x in S, create an edge (s, v_x) with capacity v_x
- For each block x in S, create an edge (v_x, t) with capacity c_x
- For each block x in S, create an edge (v_x, v_y) with infinite capacity for each block y in N_x

Chapter 7

NP-Completeness

7.1 NP-Completeness

Polynomial time reduction

- Given a problem A to solve
- Reduce solving A to solving B
- Assume there is an algorithm ALG^B that solves B at cost $O(1)$
- Construct an algorithm ALG^A that solves A, which uses ALG^B as a subroutine
- If ALG^A runs in polynomial time, then this is a polynomial time reduction

How to work with reductions
Positive: Assume that I want to solve problem A and I know how to solve problem B.

I can try come up with a polynomial time reduction $A \leq^p B$, which will give me a polynomial time algorithm for A.
Contrapositive: Assume that there is a problem A for which it is unlikely that there is a polynomial time algorithm that solves A.

If I come up with a polynomial time reduction $A \leq^p B$, it is also unlikely that there is a polynomial time algorithm that solves B.

B is "at least as hard to solve as" A, because if I could solve B, I could also solve A.
Types of reductions

- Turing reduction: $A \leq_T B$
  - A reduction which solves A using (potentially many) calls to an oracle for B
  - As known as Cook reduction
- Many-one reduction: $A \leq_m B$
  - A reduction which converts instances of A to instances of B
  - Also known as Karp reduction

Problem classification
Problems in P:

Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, weighted interval scheduling, etc.
Problems in NP:

subset sum, knapSack, weighted interval scheduling, Searching, sorting, minimum spanning tree, graph traversal, maximum flow, minimum cut, etc.
NP-hardness
A problem B is NP-hard if for every problem A in NP, $A \leq^p B$.

If every problem in NP is polynomial time reducible to B, then this captures the fact that B is at least as hard as any problem in NP.