

Summary of COMP523 Advanced Algorithm

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Chapter 1

Symmetry Notation

1.1 Asymptotic Notation

Asymptotic notation is a way of describing the limiting behavior of a function when the argument tends towards a particular value or infinity. In computer science, asymptotic notation is frequently used to describe the running time or space usage of an algorithm.

- O -notation: $f(n) = O(g(n))$ if there exist constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.
- Ω -notation: $f(n) = \Omega(g(n))$ if there exist constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.
- Θ -notation: $f(n) = \Theta(g(n))$ if there exist constants c_1, c_2 and n_0 such that $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0$.
- o -notation: $f(n) = o(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.
- ω -notation: $f(n) = \omega(g(n))$ if for any constant $c > 0$, there exists a constant n_0 such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$.

1.2 Comparing Functions

1.2.1 Transitivity

- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ implies $f(n) = \Omega(h(n))$.
- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ implies $f(n) = \Theta(h(n))$.

For example, $n^2 = O(n^3)$ and $n^3 = O(n^4)$ implies $n^2 = O(n^4)$.

1.2.2 Reflexivity

- $f(n) = O(f(n))$.
- $f(n) = \Omega(f(n))$.
- $f(n) = \Theta(f(n))$.

For example, $n^2 = O(n^2)$.

1.2.3 Symmetry

- $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
- $f(n) = \Omega(g(n))$ implies $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ implies $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ implies $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ implies $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ implies $n^3 = \Omega(n^2)$.

1.2.4 Transpose Symmetry

- $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
- $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.
- $f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$.

For example, $n^2 = O(n^3)$ if and only if $n^3 = \Omega(n^2)$.

1.2.5 sum and maximum

$$f_1(n) + f_2(n) + \cdots + f_k(n) = \Theta(\max(f_1(n), f_2(n), \dots, f_k(n)))$$

where k is a constant positive integer.

Let $f_j(n) = j$, $k = n$, then

$$f_1(n) + f_2(n) + \cdots + f_k(n) = n(n+1)/2 = \Theta(n^2)$$

1.2.6 Running time hierarchy

- logarithmic: $O(\log n)$
- linear: $O(n)$
- $n \log n$: $O(n \log n)$
- quadratic: $O(n^2)$
- polynomial: $O(n^k)$
- exponential: $O(c^n)$
- constant: $O(1)$
- superconstant: $\omega(1)$
- sublinear: $o(n)$
- superlinear: $\omega(n)$
- superpolynomial: $\omega(n^k)$
- subexponential: $o(c^n)$

1.3 Expect of algorithms

Correctness: An algorithm is correct if it halts with the correct output for every input instance.

Termination: An algorithm is terminating if it halts for every input instance.

Efficiency: An algorithm is efficient if it halts with the correct output for every input instance and runs in polynomial time.

Chapter 2

Recursion and Divide and Conquer techniques

2.1 Finding Majority in array

The pseudocode of the algorithm is shown in Algorithm 2.1.

Algorithm 1 Finding Majority in array

```
1: procedure MAJORITY( $A$ )
2:    $n \leftarrow$  length of  $A$ 
3:   if  $n = 0$  then
4:     return  $-1$ 
5:   end if
6:   if  $n = 1$  then
7:     return  $A[1]$ 
8:   end if
9:   if  $n \neq 1$  and  $n$  is odd then
10:
11:   end if
12:   Array  $B$  of size  $n/2$ 
13:   set  $j=0$ 
14:   for  $i = 1$  to  $n/2$  do
15:     if  $A[2i - 1] = A[2i]$  then
16:        $B[j] \leftarrow A[2i - 1]$ 
17:        $j \leftarrow j + 1$ 
18:     end if
19:   end for
20:    $m \leftarrow$  MAJORITY( $B$ )
21:    $count \leftarrow 0$ 
22:   for  $i = 1$  to  $n$  do
23:     if  $A[i] = m$  then
24:        $count \leftarrow count + 1$ 
25:     end if
26:   end for
27:   if  $count > n/2$  then
28:     return  $m$ 
29:   else
30:     return  $-1$ 
31:   end if
32: end procedure
```

Correctness:

Lemma: If A has a majority element, then the majority element of A is also the majority element of B .

Base case: $n = 1$, the majority element is $A[1]$.

Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.

Induction step: If A has a majority element, then the majority element of A is also the majority element of B .

Case 1 (A has a majority element m): Then by the lemma, it is also the majority element of B . Then m appears more than $k/2$ times in B . Then m appears more than $(k + 1)/2$ times in A .

Case 2 (A has no majority element): Then B has no majority element. Then A has no majority element.

Proof the lemma:

proof by contradiction. Assume that A has a majority element m and B has a majority element m' , but $m \neq m'$.

Let x be the numbers of occurrence of m in A .

Let y be the numbers of occurrence of m' in B .

Then $2y$ times from pairs that are represented in B by a value different from m' , and $x - 2y$ times, since each occurrence of m in A that is not paired with another occurrence of m in A is paired with an occurrence of m' in B .

In total, this gives $2y + x - 2y = x$ occurrences of m in A , which is a contradiction.

Running time:

Recursive formula for the running time:

$$T(n) \leq T(n/2) + cn$$

where c is a constant.

The solution to the recurrence is $T(n) = O(n)$.

2.2 Searching in logarithmic time

Searching faster with BinarySearch.

It is a particular case of the divide-and-conquer paradigm.

Input: A sorted array A of n elements and a value x .

Output: An index i such that $A[i] = x$ or the special value -1 if x does not appear in A .

Pseudocode is shown in Algorithm 2.2.

Algorithm 2 BinarySearch

```

1: procedure BINARYSEARCH( $x, i, j$ )
2:   if  $i = j$  then
3:     if  $A[i] = x$  then
4:       return  $i$ 
5:     else
6:       return  $-1$ 
7:     end if
8:   else
9:     if  $x = A[\lfloor (i + j)/2 \rfloor]$  then
10:      return  $\lfloor (i + j)/2 \rfloor$ 
11:    else if  $x < A[\lfloor (i + j)/2 \rfloor]$  then
12:      return BINARYSEARCH( $x, i, \lfloor (i + j)/2 \rfloor$ )
13:    else
14:      return BINARYSEARCH( $x, \lfloor (i + j)/2 \rfloor + 1, j$ )
15:    end if
16:  end if
17: end procedure

```

Running time:

The number of comparisons performed by BinarySearch is:

$$T(n) \leq T(n/2) + 4$$

Keep calculate:

$$\begin{aligned}
T(n) &\leq T(n/2) + 4 \\
&\leq T(n/4) + 4 + 4 \\
&\leq T(n/8) + 4 + 4 + 4 \\
&\leq T(n/2^k) + 4k \\
&\leq T(n/2^{\log(n-1)}) + 4\log(n-1) \\
&= T(2) + 4(\log n - 1) \\
&\leq 4\log n - 4 \\
&= 4\log n
\end{aligned}$$

proof $T(n) \leq 4\log n$:

Base case: $n = 1, T(1) = 0 \leq 4\log 1 = 0$.

Induction hypothesis: Assume that the lemma is true for $n = k$, we will prove that the lemma is true for $n = k + 1$.

Induction step: $T(k + 1) \leq 4\log(k + 1)$.

$$\begin{aligned}
T(k + 1) &\leq T(k/2) + 4 \\
&\leq 4\log(k/2) + 4 \\
&= 4\log k - 4 + 4 \\
&= 4\log k \\
&\leq 4\log(k + 1)
\end{aligned}$$

Memory usage:

The memory usage of BinarySearch is:

$$M(n) = O(\log n)$$

Comparing BinarySearch and LinearSearch:

$$\begin{aligned}
T_{\text{BinarySearch}}(n) &= O(\log n) \\
T_{\text{LinearSearch}}(n) &= O(n) \\
T_{\text{BinarySearch}}(n) &= O(\log n) < O(n) = T_{\text{LinearSearch}}(n) \\
M_{\text{BinarySearch}}(n) &= O(\log n) < O(1) = M_{\text{LinearSearch}}(n)
\end{aligned}$$

2.3 Running time of Divide and Conquer algorithms

The Master Theorem:

Suppose that $T(n)$ satisfies the recurrence:

$$T(n) \leq aT(n/b) + cn^d$$

where $a \geq 1, b > 1, c > 0$ and $d \geq 0$ are constants.

Then $T(n)$ has the following asymptotic bounds:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

This theorem is useful for solving recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$ and $f(n)$ is an asymptotically positive function.

Example:

$$T(n) = 8T(n/2) + 100n^2$$

$a = 8$, $b = 2$, $f(n) = 100n^2$, $d = 2$, $\log_b a = \log_2 8 = 3$.

$d = 2 < \log_b a = 3$, so $T(n) = O(n^{\log_b a}) = O(n^3)$.

2.4 Finding pair of points closest to each other

Input: A set P of n points in the plane.

Output: The pair of points in P that are closest to each other.

Pseudocode is shown in Algorithm 2.4. **Running time:**

Algorithm 3 ClosestPair

```

1: procedure CLOSESTPAIR( $P_1, \dots, P_n$ )
2:   Construct  $P_x$  and  $P_y$ .  $P_x$  is sorted by  $x$ -coordinate,  $P_y$  is sorted by  $y$ -coordinate.
3:   return CLOSESTPAIRREC( $P_x, P_y$ )
4: end procedure

```

Algorithm 4 ClosestPairRec

```

1: procedure CLOSESTPAIRREC( $P_x, P_y$ )
2:   if  $|P_x| = |P_y| \leq 3$  then
3:     For each pair of points  $(P_i, P_j)$ , compute  $d(P_i, P_j)$ 
4:     return the pair of points with the smallest distance
5:   end if
6:   Construct  $Q_x, Q_y, R_x$  and  $R_y$ .
7:    $(l_1, l_2) = \text{CLOSESTPAIRREC}(Q_x, Q_y)$ 
8:    $(r_1, r_2) = \text{CLOSESTPAIRREC}(R_x, R_y)$ 
9:    $\delta = \min\{d(l_1, l_2), d(r_1, r_2)\}$ 
10:   $x^* =$  the largest  $x$ -coordinate in  $Q_x$ 
11:   $L = \{(x, y) : x = x^*\}$ 
12:   $S = \{p \in P : p \in L \text{ and } p \text{ is within } \delta \text{ of } L\}$ 
13:  Construct  $S_v$ 
14:  for  $p \in S$  do
15:    Let  $q$  be the point in  $S_v$  closest to  $p$ 
16:    if  $d(p, q) < \delta$  then
17:       $\delta = d(p, q)$ 
18:       $(s_1, s_2) = (p, q)$ 
19:    end if
20:  end for
21:  if  $d(s_1, s_2) < \min\{d(l_1, l_2), d(r_1, r_2)\}$  then
22:    return  $(s_1, s_2)$ 
23:  end if
24:  if  $d(l_1, l_2) < d(r_1, r_2)$  then
25:    return  $(l_1, l_2)$ 
26:  else
27:    return  $(r_1, r_2)$ 
28:  end if
29: end procedure

```

$$T(n) \leq 2T(n/2) + O(n \log n) = O(n \log n)$$

Example:

Chapter 3

Graph Algorithms

3.1 Graph Definitions

Graph: A graph G consists of a set V of vertices and a set E of edges, where each edge is associated with a pair of vertices.

Directed Graph: A directed graph G consists of a set V of vertices and a set E of directed edges, where each directed edge is associated with an ordered pair of vertices.

Undirected Graph: An undirected graph G consists of a set V of vertices and a set E of undirected edges, where each undirected edge is associated with an unordered pair of vertices.

Neighbours of a vertex v : Set of vertices that are connected to v by an edge.

Degree of a vertex v : number of neighbours of v , denoted by $deg(v)$.

Path: A sequence of (non-repeating) nodes with consecutive nodes being connected by an edge.
length = node count - 1 = edge count.

Distance between two nodes: The number of edges in the shortest path between the two nodes.

Graph diameter: The maximum distance between any two nodes in the graph.

Lines, cycles, trees and cliques:

Line: A graph with n vertices and $n - 1$ edges.

Cycle: A graph with n vertices and n edges.

cliques: A graph with n vertices and $n(n - 1)/2$ edges.

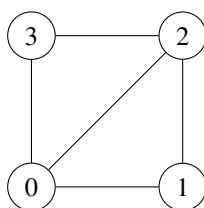
Tree: A graph with n vertices and $n - 1$ edges.

Graph representations:

Adjacency matrix: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge between i and j , and $A_{ij} = 0$ otherwise.

examples of adjacency matrices:

Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Adjacency matrix for directed graphs: A $n \times n$ matrix A where $A_{ij} = 1$ if there is an edge from i to j , and $A_{ij} = 0$ otherwise.

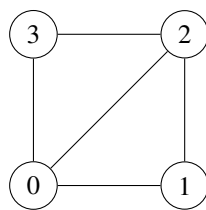
examples of adjacency matrices for directed graphs:
 Given the following graph:



The adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

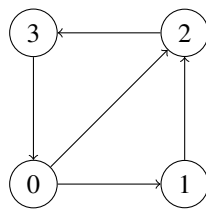
Adjacency list: A list of lists, where the i th list contains the neighbours of vertex i .
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & \\ 0 & 1 & 3 \\ 0 & 2 & \end{bmatrix}$$

Adjacency list for directed graphs: A list of lists, where the i th list contains the neighbours of vertex i .
 Given the following graph:



The adjacency list is:

$$\begin{bmatrix} 1 & 2 \\ 2 & \\ 3 & \\ 0 & \end{bmatrix}$$

Adjacency matrix vs adjacency list:

Adjacency matrix	Adjacency list
$O(1)$ to check if there is an edge between i and j	$O(\min(\deg(i), \deg(j)))$ to check if there is an edge between i and j
$O(n)$ to find the neighbours of i	$O(\deg(i))$ to find the neighbours of i
$O(n^2)$ space	$O(n + m)$ space

3.2 Depth-first search

Depth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of depth-first search:



The depth-first search sequence is:

0, 1, 2, 3, 5, 4

Depth-first search algorithm:

Algorithm 5 Depth-first search algorithm

```
1: procedure DFS( $G, v$ )
2:   for  $e \in V$  do
3:     if  $e$  is unexplored then
4:        $u = \text{head of } e$ 
5:       if  $u$  is unexplored then
6:          $e$  is a tree edge
7:         DFS( $G, u$ )
8:       else
9:          $e$  is a back edge
10:      end if
11:    end if
12:  end for
13: end procedure
```

Running time of depth-first search: $O(n + m)$

3.3 Breadth-first search

Breadth-first search: A graph search algorithm that explores the neighbours of a vertex before exploring the neighbours of its neighbours.

example of breadth-first search:



The breadth-first search sequence starting from vertex 0 is 0, 1, 2, 3, 4, 5.

Breadth-first search algorithm:

Algorithm 6 Breadth-first search algorithm

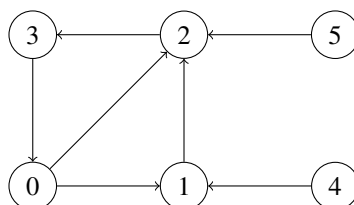
```
1: procedure BFS( $G, s$ )
2:   initial empty list  $L$ 
3:    $L \leftarrow s$ 
4:    $i \leftarrow 0$ 
5:   while  $L[i] \neq \emptyset$  do
6:      $L_{i+1} \leftarrow \text{emptylist}$ 
7:     for  $v \in L[i]$  do
8:       for edges  $(e)$  incident to  $v$  do
9:         if  $e$  is unexplored then
10:            $w \leftarrow$  the other end of  $e$ 
11:           if  $w$  is unexplored then
12:             label  $e$  as a tree edge
13:             add  $w$  to  $L_{i+1}$ 
14:           else
15:             label  $e$  as a cross edge
16:           end if
17:         end if
18:       end for
19:     end for
20:      $i \leftarrow i + 1$ 
21:   end while
22: end procedure
```

Running time of breadth-first search: $O(n + m)$

3.4 Strong Connectivity

Directed graph: A graph where the edges have a direction.

Examples:



DFS and BFS on directed graphs:

Very similar to undirected graphs, except that we only consider edges that go out of a vertex.

Running time is $O(n + m)$

For example graph above the DFS sequence is 0, 1, 2, 3.

The BFS sequence is 0, 1, 2, 3.

3.4.1 Connectivity

Weak connectivity: If we ignore the direction for all edges, there would be a path from any vertex to any other vertex.

Strong Connectivity: For every two nodes u and v , there is a path from u to v and a path from v to u .

3.4.2 Mutual Reachability

Two nodes u and v are mutually reachable if there is a path from u to v and a path from v to u .

Strong connectivity: For every pair of nodes u and v , these two nodes are mutually reachable.

Transitivity: If u is mutually reachable with v and v is mutually reachable with w , then u is mutually reachable with w .

3.4.3 Testing strong connectivity

Algorithm 7 Testing strong connectivity

```
1: procedure TESTSTRONGCONNECTIVITY( $G$ )
2:   define  $G^R$  to be the graph with the same vertices as  $G$  but with all edges reversed
3:   Select a node  $s$  in  $G$ 
4:   BFS( $G, s$ ), BFS( $G^R, s$ )
5:   for each node  $v$  do
6:     if  $v$  is unexplored in either BFS then
7:       return False
8:     end if
9:   end for
10:  return True
11: end procedure
```

3.5 Testing bipartiteness

Bipartite graph: A graph $G = (V, E)$ is bipartite if and only if the vertices can be partitioned into two sets V_1 and V_2 such that every edge has one end in V_1 and the other end in V_2 .

A Graph $G = (V, E)$ is bipartite if and only if it has no odd cycles. (odd cycle: a cycle with odd number of edges)

Testing bipartiteness:

Given a graph $G = (V, E)$, we want to test if G is bipartite.

Given a graph $G = (V, E)$, decide if it is 2-colourable.

Given a graph $G = (V, E)$, decide if it has an odd cycle.

Colouring the nodes It is quite familiar with BFS:

Algorithm 8 Colouring the nodes

```
1: procedure COLOURING( $G, s$ )
2:   initial empty list  $L$ 
3:   initial empty list  $C$ 
4:    $L \leftarrow s$ 
5:    $C[s] \leftarrow red$ 
6:    $i \leftarrow 0$ 
7:   while  $L[i] \neq \emptyset$  do
8:      $L_{i+1} \leftarrow emptylist$ 
9:     for  $v \in L[i]$  do
10:      for edges ( $e$ ) incident to  $v$  do
11:        if  $e$  is unexplored then
12:           $w \leftarrow$  the other end of  $e$ 
13:          if  $w$  is unexplored then
14:            label  $e$  as a tree edge
15:            add  $w$  to  $L_{i+1}$ 
16:            if  $i + 1$  is odd then
17:               $C[w] \leftarrow green$ 
18:            else
19:               $C[w] \leftarrow red$ 
20:            end if
21:          else
22:            label  $e$  as a cross edge
23:            if  $C[v] = C[w]$  then
24:              return False
25:            end if
26:          end if
27:        end if
28:      end for
29:    end for
30:     $i \leftarrow i + 1$ 
31:  end while
32:  for  $e(v, w) \in G$  do
33:    if  $C[v] = C[w]$  then
34:      return False
35:    end if
36:  end for
37:  return True
38: end procedure
```

Running time of colouring the nodes: $O(n + m)$

Correctness of colouring the nodes:

Proof by contradiction.

Suppose that G is not bipartite.

Then G has an odd cycle.

Suppose to the contrary that the algorithm return True.

That means that the algorithm did not detect the odd cycle.

3.6 DAGs and Topological Ordering

DAG: A directed acyclic graph (DAG) is a directed graph with no directed cycles.
examples of DAGs:



Topological ordering: Given a graph $G = (V, E)$, a topological ordering of G is an ordering of the nodes u_1, u_2, \dots, u_n such that for every edge (u_i, u_j) , we have $i < j$.

Intuitively, a topological ordering is an ordering of the nodes such that every edge goes from left to right.

example of topological ordering based on given graph above:

3, 0, 1, 2, 4, 5

Topological ordering implies DAG:

- If G has a topological ordering, then G is a DAG.
- Suppose by contradiction that G has a topological ordering u_1, u_2, \dots, u_n but G also has a cycle C .
- Let u_j be the smallest element of C in the topological ordering.
- Let u_i be its predecessor in C .
- u_i must appear before u_j in the topological ordering.
- This contradicts the fact that u_j is the smallest element of C in the topological ordering.

DAG implies topological ordering:

Proof by induction: Base case: If G has one or two nodes, then G has a topological ordering.

Induction steps: Assume that a DAG up to k nodes has a topological ordering (induction hypothesis). we will prove that a DAG with $k + 1$ nodes has a topological ordering.

- By our lemma, there is at least one source node in G , and let u be the node.
- Put u at the beginning of the topological ordering.
- Consider the graph G' , obtained by G by removing u and its incident edges.
- G' is a DAG with k nodes.
- It has a topological ordering u_1, u_2, \dots, u_k by the induction hypothesis.
- Append this ordering to u to get a topological ordering of G .

Here is the algorithm:

Algorithm 9 Topological Sorting

- 1: **procedure** TOPOLOGICALSORTING(G)
 - 2: set v as source vertex
 - 3: **end procedure**
-