Non-parametric Manifold Regression

Population Shape Regression from Random Design Data

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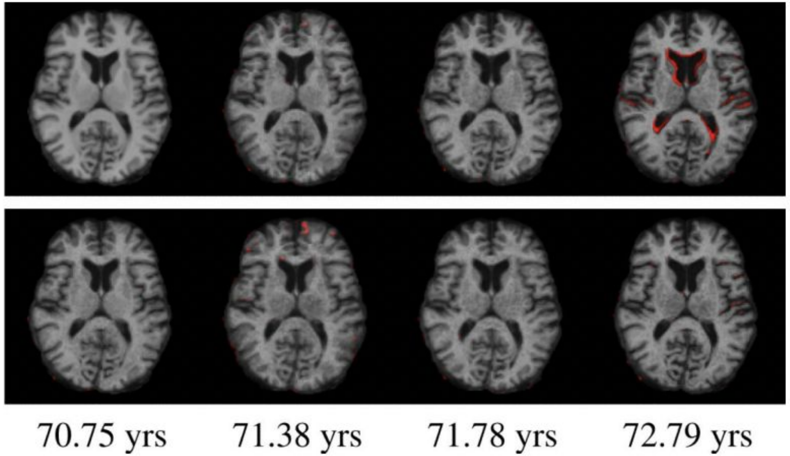
Initial Problem : Longitudinal study



Brain MR Images templates

Statistical Method Overview

- Nadiya -Watson Manifold extension of Frechet Mean

$$\hat{I} = \arg\min_{I \in \mathcal{I}} \frac{\sum_{i=1}^{N} d_{\mathcal{I}}(I, y_i)^2}{N} \quad \Longrightarrow \quad \hat{I}_h(t) = \arg\min_{I \in \mathcal{I}} \frac{\sum_{i=1}^{N} K_h(t - t_i)\, d_{\mathcal{I}}(I, y_i)^2}{\sum_{i=1}^{N} K_h(t - t_i)}$$

Geometric Grounding : LDDMM

Shape Space :

- Manifold : Group action of Diffeomorphisms on the Space of Images
- Local Metric : Defined through kernel k on vector fields of a RKHS V
- Computing geodesic distances :

$$Cost(\phi) = Length(\phi) + \lambda Fidelity(\phi(M_1), M_2)$$

$$d_{\mathcal{I}}(I, J)^2 = \min_{v:\dot{\phi}_t = v_t(\phi_t)} \int_0^1 \|v_t\|_V^2 dt + \frac{\|I(\phi^{-1}) - J\|_{L_2}^2}{\sigma^2}$$

Method

Single subject growth model

$$\min_{v:\dot{\phi}_t=v_t(\phi_t)} \int_0^1 \|v_t\|_V^2 dt + \frac{1}{\sigma^2}\int_0^1 \|I_\alpha(\phi^{-1}) - I_t\|_{L_2}^2 dt$$

$$Av_t = -\frac{1}{\sigma^2}\nabla(I_\alpha \circ \phi^{-1}) \int_t^1 (I_u(\phi_u \circ \phi_t) - I_\alpha(\phi^{-1}))|D(\phi_u \circ \phi_t^{-1})| du$$

Population growth model

$$\min_{v:\dot{\phi}_t=v_t(\phi_t)} \int_0^1 \|v_t\|_V^2 dt + \frac{1}{\sigma^2}\int_0^1 \|I_\alpha(\phi^{-1}) - \hat{I}_t\|_{L_2}^2 dt$$

$$\hat{I}_h(t) = \arg\min_{I \in \mathcal{I}} \frac{\sum_{i=1}^N K_h(t-t_i)\, d_{\mathcal{I}}(I, y_i)^2}{\sum_{i=1}^N K_h(t-t_i)}$$

Implementation

Algorithm 4.1 Intrinsic mean image: LDDMM solution

Input: A collection of N images: $\{I^i\}_{i=1}^N$

Output: A Fréchet mean image $\hat{I}$ and a collection of diffeomorphic transformations $\{\phi^i\}_{i=1}^N$ such that ϕ^i maps I^i onto $\hat{I}$

// Initialize deformations with identity
for $i = 1 : N$ **do**
  $\phi^i \leftarrow \mathrm{Id}_{\mathrm{Diff}_V(\Omega)}$
end for
// Perform gradient descent optimization
repeat
  $\hat{I} = \frac{1}{N}\sum_{i=1}^N I^i_{\phi^i}$ // Update mean image
  for $i = 1 :$ N **do**
    // Compute gradient of functional at each discritized timepoint s
    for $j = 1 :$ T **do**
      $s \leftarrow \frac{j-1}{T}$
      $\nabla_{v_t^i} E_t \leftarrow 2v_t^i - K\left(\frac{2}{\sigma_2}|D\phi^i_{s,1}|\nabla I^i_{\phi^i_{0,s}}(I^i_{\phi^i_{0,s}} - \hat{I}_{\phi^i_{1,s}})\right)$
    end for
    $v^i \leftarrow v^i - \epsilon\nabla_{v^i} E$ // Step along gradient
  end for
until convergence

LDDMM algorithm to solve regression on 2D images

Implementation

Algorithm 4.2 Intrinsic mean image: Greedy solution

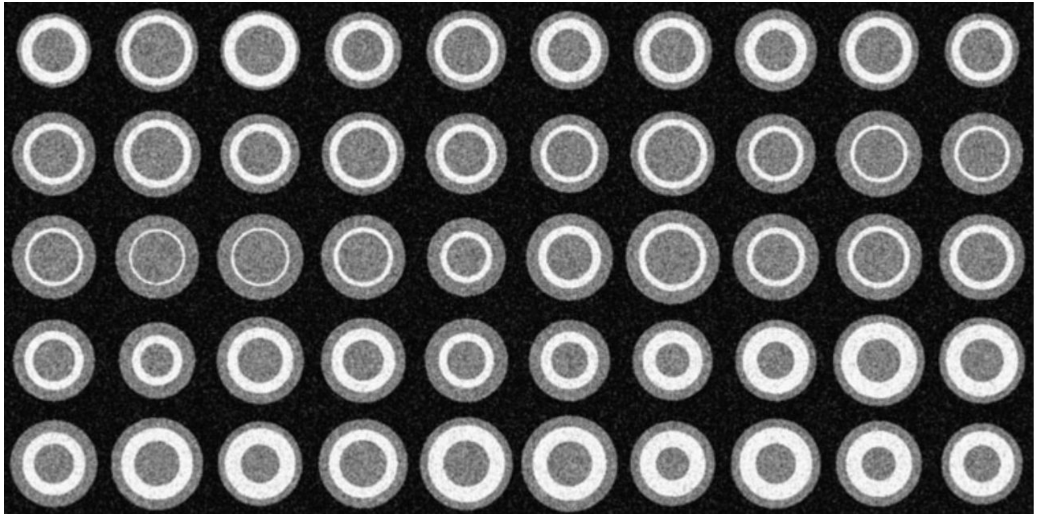
Input: A collection of N images: $\{I^i\}_{i=1}^N$

Output: A Fréchet mean image $\hat{I}$ and a collection of diffeomorphic transformations $\{\phi^i\}_{i=1}^N$ such that ϕ^i maps I^i onto $\hat{I}$

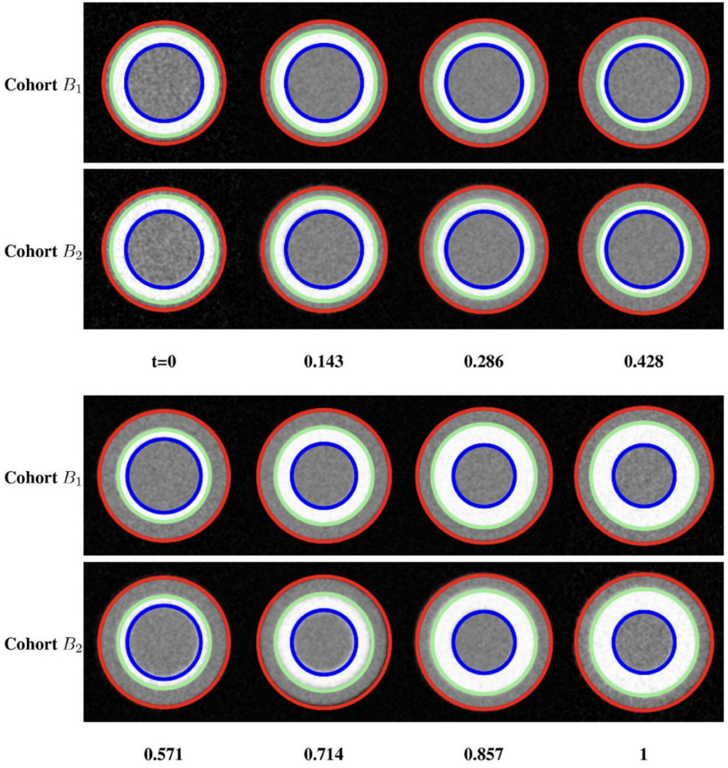
// Initialize deformations with identity
for $i = 1$: number of images **do**
  $\phi^i \leftarrow \mathrm{Id}_{\mathrm{Diff}_V(\Omega)}$
end for
repeat
  $\hat{I} = \frac{1}{M}\sum_{i=1}^{M} I^i_{\phi^i}$ // Update mean image
  for $i = 1$: number of images **do**
    $v_{\mathrm{inc}} \leftarrow K\left(\frac{2}{\sigma^2}\nabla I^i_{\phi^i_{0,1}}(I^i_{\phi^i_{0,1}} - \hat{I})\right)$ // Compute locally optimal velocity
    $\phi^i \leftarrow \mathrm{Exp}_{\phi^i}(\epsilon v_{\mathrm{inc}})$ // Step in direction of incremental velocity
  end for
until convergence

Greedy algorithm to solve regression on 3D images

Results

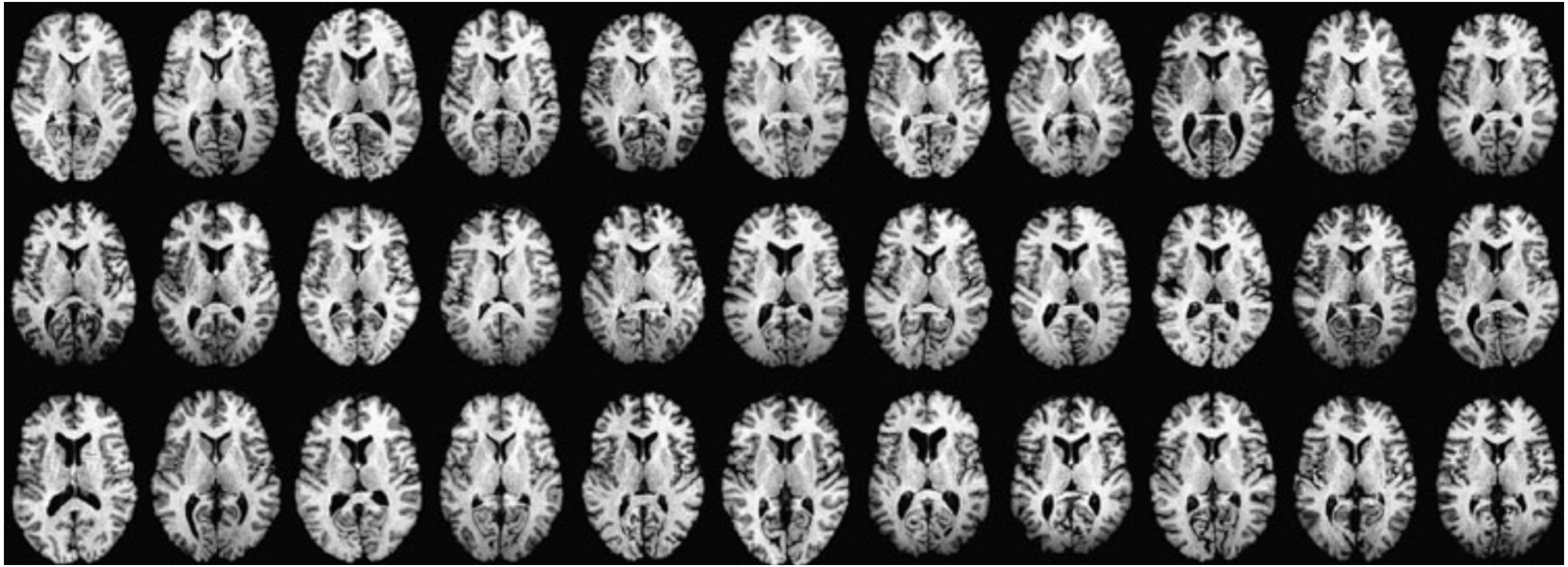


Synthesised images of bull's-eyes as observation data



Regressed images

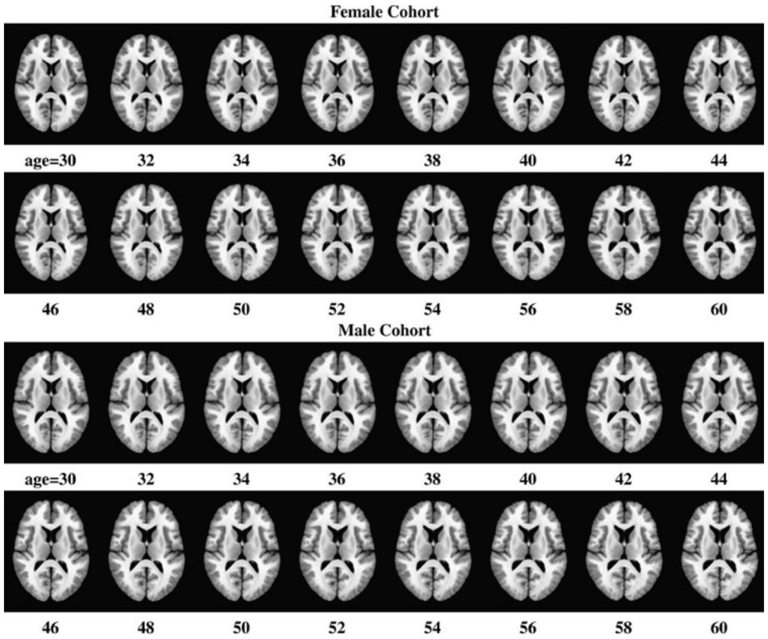
Results



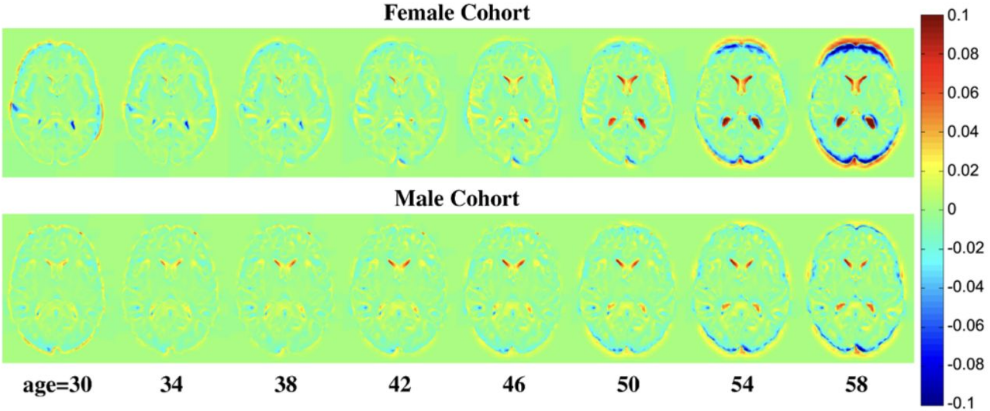
Healthy brain MRIs data set

Results

$$\log \begin{vmatrix} \frac{\partial \dot{g}_t^1}{\partial x^1}(x) & \frac{\partial \dot{g}_t^1}{\partial x^2}(x) & \frac{\partial \dot{g}_t^1}{\partial x^3}(x) \\ \frac{\partial \dot{g}_t^2}{\partial x^1}(x) & \frac{\partial \dot{g}_t^2}{\partial x^2}(x) & \frac{\partial \dot{g}_t^2}{\partial x^3}(x) \\ \frac{\partial \dot{g}_t^3}{\partial x^1}(x) & \frac{\partial \dot{g}_t^3}{\partial x^2}(x) & \frac{\partial \dot{g}_t^3}{\partial x^3}(x) \end{vmatrix}$$



Slices of regressed 3D images



Age-indexed local expansion and contraction

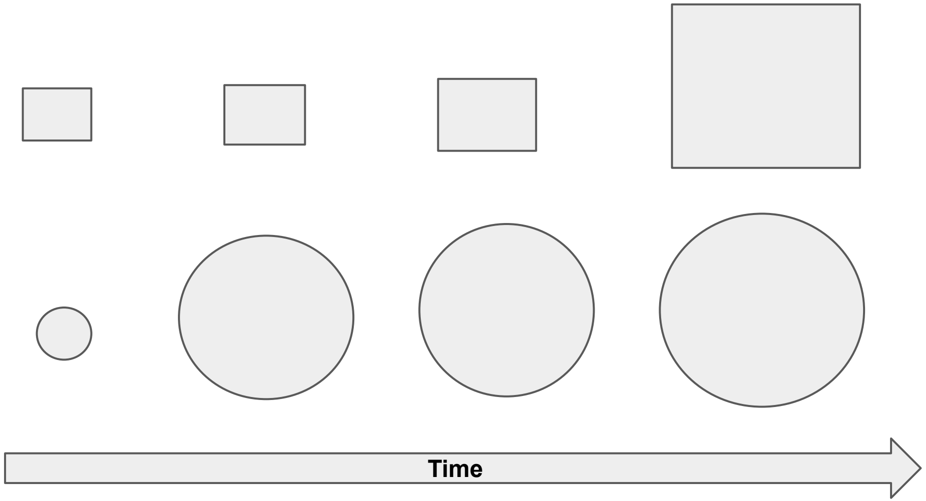
Red : expansion   Blue : contraction

Statistical Limitations

1. No Subject-Specific information:

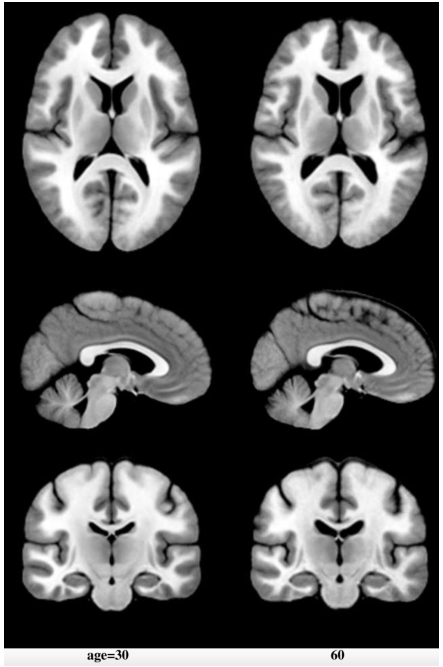
   => Loss of Information

2. Time deformation :

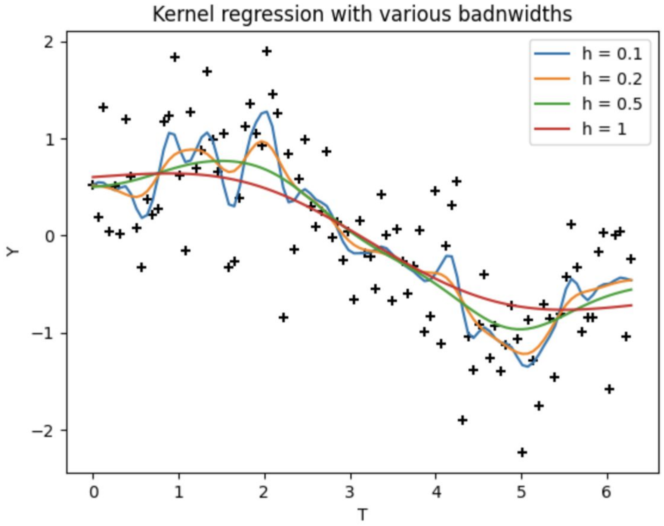


Schematic Growth Scenarios

Computational Limitations



Regressed images of mixed cohorts



Finding best kernel bandwidth can be complicated and/or subjective.

≈ 2 hours of computation time / regressed image

$O(MN \log n)$ M iterations,
N images
n voxels along largest dimension

Sources :

Images where taken from the article and Brad C. Davis’s thesis