# Derivatives of Functions

## Learning outcomes

By end of the topic, students will be able to:

- Calculate rate of change and slope of the tangent line.
- Interpret the meaning of the derivative.
- Determine the derivative of elementary functions, using the power rule, product rule, quotient rule and chain rule.

**Introduction**

- Welcome back, dear students!
- This chapter we will explore the first half of calculus known as *differential calculus*. We will study basic differentiation of various functions.
- RECALL from last chapter: **LIMITS.** You need to know how to evaluate limits!

Differentiation is all about finding **rates of change** of one quantity compared to another. They show how fast something is changing at any point.

## *Average Rate of Change*

Mathematicians are often interested in comparing two quantities with different units of measurements. A "*rate*" is a ratio obtained by comparing two quantities with different units of measurements. For example, if a household used 795-kilowatt hours (kwh) of electricity during a 30-day period, then the rate of energy consumption is 26.5-kilowatt hours per day. 26.5 kwh refers only to the average and gives no information about the energy consumption at any specific instant during the 30-day period. We refer to such rates as the "*average rate*".

Average rate refer to the *rate of change* of one quantity relative to another quantity over some fixed interval of given values.

Recall that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two distinct points on a non vertical line ($x_1 \neq x_2$), the slope of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

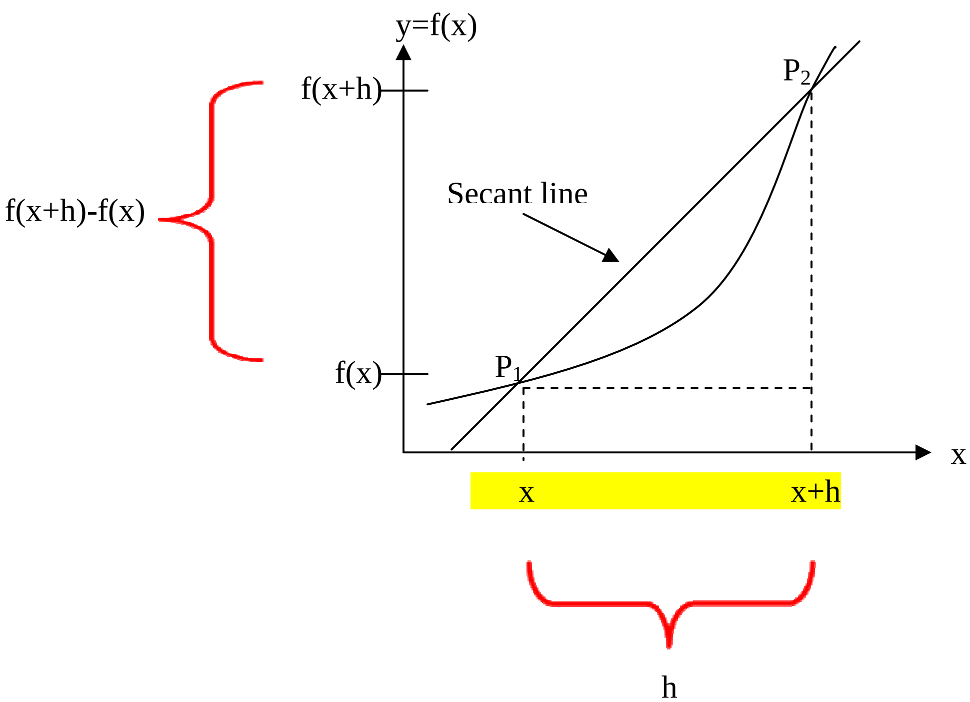
Given two points in a graph, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we define the average rate of change over the interval $x = x_1$ to $x = x_2$ as

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

For $y = f(x)$, the **average rate of change from $x$ to $x + h$** is

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

The above expression is also called a **difference quotient**. It can be interpreted as the **slope of a secant (*the line joining the two points)***. See the picture on the next slide for illustration.




Average rate of change = slope of the secant line through $P_1$ and $P_2$ = $\dfrac{f(x+h) - f(x)}{h}$

**Example 1**
A student writing a term paper notices that she completed her 10th page at 1 AM and her 22nd page at 4 AM. At what rate has she been writing?

**Solution**
The underlying function here is P = P(t), giving the number of pages written t hours after, say, midnight. We know two values of the function, at two specific times: P(1) = 10 pages and P(4) = 22 pages.
We can find the change in value over the three hour period by calculating that DP = 22 – 10 = 12 pages.
Finally, we can find the rate at which the student has been writing by calculating DP/Dt = (22 – 10)/(4 – 1) = 12/3 = 4 pages per hour. We might expect that this rate would decrease as the morning wears on, due a decreased production DP over the next Dt = three hours.

**Example 2**

Suppose $y = x^2 - 2x - 8$. Find the average rate of change of y with respect to x between $x_1 = 1$ and $x_2 = 3$.

**Solution:**

$$f(x_2) = f(3) = (3)^2 - 2(3) - 8 = 9 - 6 - 8 = -5$$

$$f(x_1) = f(1) = (1)^2 - 2(1) - 8 = 1 - 2 - 8 = -9$$

Average rate of change = $\dfrac{(-5)-(-9)}{3-1} = \dfrac{4}{2} = 2$

**Example 3**

If the population of Accra is 1,750,000 in 1980 and 2,750,000 in 1990, over the 10-year period, what is the average rate of change of the population?

**Solution**

Average rate of change = $\dfrac{27500000 - 1750000}{10}$

$= \dfrac{1000000}{10}$

$=$ 100000 per year.

Therefore the population is increasing by 10,000 per year

**Problem 1**

In 1987, the Jermic Manufacturing Company had total sales of $40,000. Three years later, in 1990, total sales were $90,000. What is the average rate of change of the total sales?

Problem 2

Find the average rate of change of each of the following functions over the interval indicated.

a) $f(x) = 2x^2 - x + 3$, from $x = 5$ to $x = 8$.

b) $f(x) = 2x - x^2$, from $x = 2$ to $x = 4$.

c) $f(x) = 3$, from $x = 5$ to $x = 10$.

d) $f(x) = 2 - x^3$, from $x = 1$ to $x = 3$.

In general to determine the average rate of change of one quantity (say *y*) per unit change of another quantity (say *x*), find the difference in the values of *y* at the end and beginning of the interval and divide the result by the length of the interval.

$$average \quad rate = \frac{\text{end value} - \text{beginning value}}{\text{length of interval}}$$

**Average speed**

Let S be the distance traveled by a moving object from some fixed point after t units of time. Assume $S = f(t)$. Then the average speed (also known as average velocity) during a period of time t, ($t = t_1$ to $t = t_2$) is defined as

$$\text{Average speed} = \frac{\text{Distance travel}}{\text{Time elapsed}} = \frac{S(t_2) - S(t_1)}{t_2 - t_1}$$

**Example**

A ball is thrown up into the air from the ground, its distance from the starting point after t seconds is given by the formula $S = 60t - 3t^2$. Find the average speed of the ball from $t = 3$ to $t = 8$.

Solution:

Here $t_1 = 3$ and $t_2 = 8$.

$S(t_2) = S(8) = 60(8) - 3(8)^2 = 480 - 192 = 288$

$S(t_1) = S(3) = 60(3) - 3(3)^2 = 180 - 27 = 153$

Average speed = $\frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{288 - 153}{8 - 3} = \frac{135}{5} = 27 \text{ feet per second}$

**Problem 3**

A ball is thrown up into the air from the ground, its distance from the starting point after t seconds is given by the formula $D = 64t - 16t^2$. Find the average speed of the ball from $t = 2$ to $t = 4$.

## Instantaneous Rate of change

When dealing with average rate, the rate is always over some interval of given values. However, there are situations where we are only interested in the rate at an instance and not over a range of values. *The rate at an instant is called the instantaneous rate*. For example, if a driver looks at his or her speedometer and observes that the needle points at 65 mph, it implies the driver is driving 65 miles per hour *at that instant*.

There's not enough information to compute the slope of a line given just one point on the line. We will use two points as dictated by the slope formula, the first point is given, the second is arbitrary. We then take the limit as the second point moves towards the first. This will give us a well-defined procedure for computing the slope as well as defining what the tangent line is.

Consider the function $y = f(x)$ only near the point $P = (x, f(x))$.
The difference quotient

$$\frac{f(x+h) - f(x)}{h}, h \neq 0$$

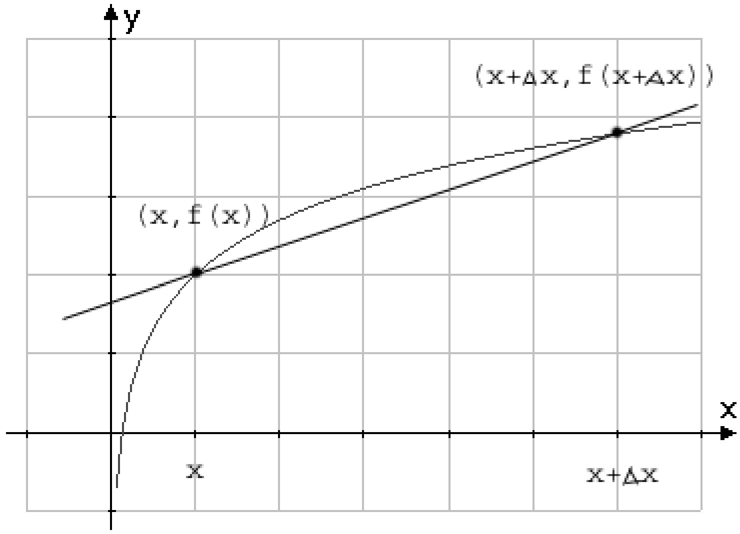
gives the average rate of change of $f$ over the interval $[x, x+h]$. If we make $h$ smaller and smaller, in the limit we obtain the **instantaneous rate of change** of the function at the point $P$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Given $y = f(x)$, the **instantaneous rate of change at $x$** is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ provided that the limit exists. It can be interpreted as the **slope of the tangent** at the point $(x, f(x))$.

Slope of tangent = instantaneous rate of change = $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

We will illustrate this with the diagram that follows. Notice that some books use $\Delta x$ for h to represent the interval.

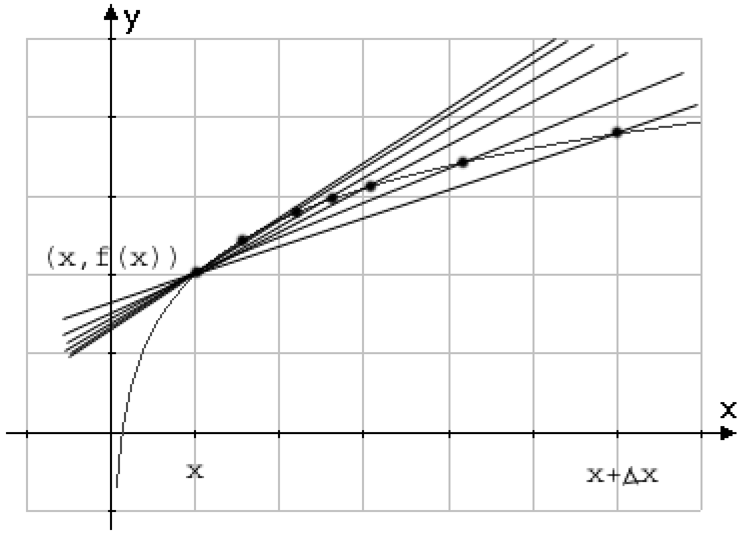



$$slope = m = \frac{y_2 - y_1}{x_2 - x_1}$$

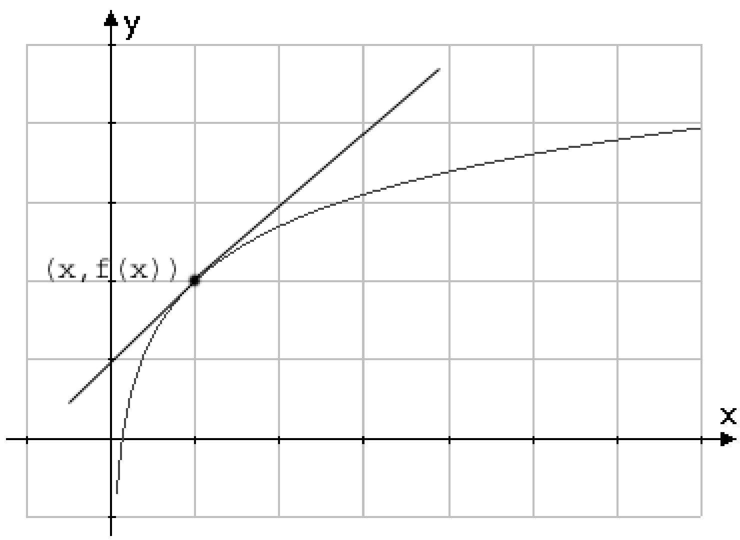
$$m = \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$$

$$m = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Notice that the last formula depends on x and delta x. We are thinking of x as fixed and so the slope is a function of delta x. Now consider the following picture. The idea is that the second point we used to compute the slope was arbitrary. So different people could choose different points and hence get different values for the slope. In order to eliminate this arbitrariness we will get the limiting behavior of the slope as delta x gets smaller and smaller. Hopefully the following picture will help you to see what's going on.




What we are trying to see here is that as delta x goes to zero, the points on the graph, starting with the right most, move towards (x,f(x)). The lines through these points move also but less and less as delta x gets smaller and smaller. There is a limiting line. This line will be called the **tangent line** and its slope is, by definition the limit of the slope formula given above. The graph of the function and its tangent line are as follows.




The slope of the tangent line is then given by the following formula.

$$slope \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

We can compute the right hand side for any function regardless of the geometric interpretation. So this is strictly a mathematical construct. As such we will call it the **derivative** of f(x) with respect to x. It is itself a function of x. We will denote this function by f ' (x).

## The Derivative

For $y = f(x)$, we define the derivative of $f$ at $x$, denoted $f'(x)$, to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{if the limit exists.}$$

If $f'(a)$ exists, we call $f$ **differentiable at *a*.**
If $f'(x)$ exist for each $x$ in the open interval $(a, b)$, then $f$ is said to be **differentiable over (*a*, *b*).**

## Finding the Derivative

To find $f'(x)$, we use a four-step process:

Step 1. Find $f(x + h)$

Step 2. Find $f(x + h) - f(x)$

Step 3. Find $\dfrac{f(x+h) - f(x)}{h}$

Step 4. Find $\displaystyle \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

## Example 1

Find the derivative of $f(x) = x^2 - 3x$.

**Solution**

Step 1. $f(x + h) = (x + h)^2 - 3(x + h) = x^2 + 2xh + h^2 - 3x - 3h$

Step 2. Find $f(x + h) - f(x) = 2xh + h^2 - 3h$

Step 3. Find $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2 - 3h}{h} = 2x + h - 3$

Step 4. Find $\displaystyle \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h - 3 = 2x - 3$

## Example 2

Find the slope of the tangent to the graph of $f(x) = x^2 - 3x$ at $x = 0$, $x = 2$, and $x = 3$.

**Solution:**

In example1 we found the derivative of this function at $x$ to be

$$f'(x) = 2x - 3$$

Hence

$f'(0) = -3$ $\quad$ $f'(2) = 1$, and $\quad$ $f'(3) = 3$

**Example 3**
Compute the derivative of

$$f(x) = x^2$$

**Solution**

$$\begin{aligned}
f'(x) &\equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \left(\Delta x\right)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + \left(\Delta x\right)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(2x + \Delta x\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x + \left(0\right) \\
&= 2x
\end{aligned}$$

We have the following results. $f(x) = x^2 \qquad f'(x) = 2x$

Another way of tackling the same:

$$\begin{aligned} f'(x) &= \lim_{s \to x} \frac{f(s) - f(x)}{s - x} \\ &= \lim_{s \to x} \frac{s^2 - x^2}{s - x} \\ &= \lim_{s \to x} \frac{(s - x)(s + x)}{s - x} \\ &= \lim_{s \to x} s + x \\ &= 2x \end{aligned}$$

You will observe that the formula for finding instantaneous rate is the same as the formula for finding derivative of a function. Therefore, to find the instantaneous rate of change of a function, you just have to find the derivative of the function at some instant.

**Example**

Using the definition of derivative, find the derivative of the following function.

f(x) = 3

Solution:

$$\begin{aligned} f(x) &= 3 \\ f(x + h) &= 3 \\ f(x + h) - f(x) &= 3 - 3 \\ &= 0 \end{aligned}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f'(x) = 0$$

**Problem 6**

Using the definition of derivative, find the derivative of the following functions.

a) f(x) = 5 b) f(x) = 7 c) f(x) = 21

d) f(x) = -12 e) f(x) = 3 f) f(x) = 0

**Example**

Suppose y = $2x^2 - 5x - 4$. Find $\frac{dy}{dx}$ and evaluate this derivative at x = 2.
Solution:

$$f(x + h) = 2(x + h)^2 - 5(x + h) - 4$$
$$= 2x^2 + 4xh + 2h^2 - 5x - 5h - 4$$

$$f(x + h) - f(x) = (2x^2 + 4xh + 2h^2 - 5x - 5h - 4) - (2x^2 - 5x - 4)$$
$$= 4xh + 2h^2 - 5h$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{4xh + 2h^2 - 5h}{h} = \lim_{h\to 0}\frac{h(4x + 2h - 5)}{h}$$
$$= \lim_{h\to 0}(4x + 2h - 5)$$
$$\frac{dy}{dx} = 4x - 5$$

If x = 2, then $f'(2) = 4(2) - 5 = 8 - 5 = 3$.

**Problem 7**

Using the definition of derivative, find the derivative of

(a) (a) $f(x) = 2x - 5$

(b) (b) $f(x) = x^2$

(c) (c) $f(x) = x^2 - x$

(d) (d) $f(x) = x^2 - x - 6$

(e) (e) f(x) = 5$x$

*(f)* *(f)* f(x) = 5 - $x$

(g) (g) Compute $f'(2)$ for each function.

**Example**

Find $y'$ if y = $\frac{2}{x-1}$.
Solution:

$$f(x + h) = \frac{2}{(x+h)-1}$$

$$= \frac{2}{x+h-1}$$

$$f(x + h) - f(x) = \frac{2}{x+h-1} - \frac{2}{x-1}$$

$$= \frac{2(x-1)-2(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{2x-2-2x-2h+2}{(x+h-1)(x-1)}$$

$$= \frac{-2h}{(x+h-1)(x-1)}$$

$$y' = \lim_{h\to 0} \frac{\frac{-2h}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h\to 0} \left[\frac{-2}{(x+h-1)(x-1)}\right]$$

$$= \left[\frac{-2}{(x-1)(x-1)}\right] = \frac{-2}{(x-1)^2}$$

**Problem 8**

Find $y'$

a) y = $\frac{3}{x+1}$ b) y = $\frac{2}{1-x}$ c) y = $\frac{-4}{x-5}$

**Differentiation**

Differentiation provides us with the techniques and procedures for solving problems involving the calculations of instantaneous rates.

- **Symbols used to denote differentiation**

These are the most common symbols used to denote differentiation.

$y'$ --(*read* y prime),
$f'(x)$ -- (*read* f prime of x)
$D_x y$ --(*read* derivative of y with respect to x).
$\frac{dy}{dx}$ --(*read* derivative of y with respect to x).

- **Rules of differentiation**

Using the definition of derivative $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to derive the derivative of a specific function is time-consuming and at times the procedure is very laborious. There are simple formulas that can reduce the time needed to derive derivatives.

**Constant Rule: Derivative of a constant**

If f(x) = k for all x where k is any constant, then $f'(x) = 0$.

***The derivative of a constant is always equal to zero.***

**Example**

Find the derivative of f(x) = 2.

Solution:
Since f(x) = 2 is a constant, $f'(x) = 0$.

**Problem 9**

Find the derivative of the following functions.

a) f(x) = 12. b) f(x) = - 2 c) f(x) = 0

d) f(x) = -21 e) f(x) = 10 f) f(x) = 91

g) f(x) = -321 h) f(x) = 100 i) f(x) = -19

**Power Rule: Derivative of $kx^n$ where n is a positive integer**

If $f(x) = x^n$, where n is any positive integer, then $f'(x) = nx^{n-1}$

Similarly,

If $f(x) = kx^n$, then $f'(x) = knx^{n-1}$, where n is a positive integer.

***The derivative of $x^n$ is equal to the exponent multiplied by the base raised to the exponent minus one.***

**Example**

Find the derivative of $y = x^{12}$.

Solution: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

We have, n = 12, and n-1 = 11 so that $y' = 12x^{11}$.

**Problem 10**

Find the derivative of each of the following functions.

a) $f(x) = x^{20}$ b) $f(x) = x^{23}$ c) $f(x) = x^3$,

d) $f(x) = x^7$ e) $f(x) = x$ f) $f(x) = x^2$

g) $f(x) = x^{21}$ h) $f(x) = x^{13}$ i) $f(x) = x^{19}$,

j) $f(x) = x^{201}$ k) $f(x) = x^{29}$ k) $f(x) = x^{62}$

**Example**

Find the derivative of $y = 3x^5$.

Solution: $y' = 3(5x^4) = 15x^4$.

**Problem 11**

Find the derivative of each of the following functions.

a) $f(x) = 6x^2$ b) $f(x) = 3x^3$ c) $f(x) = -7x^3$,

d) $f(x) = 12x^7$ e) $f(x) = -2x$ f) $f(x) = 2x^5$

g) $f(x) = -3x^{11}$ h) $f(x) = -5x^6$ i) $f(x) = 2x^{19}$,

j) $f(x) = -31x^2$ k) $f(x) = 2x^9$ l) $f(x) = -1x^{22}$

**Derivative of a sum or difference**

Suppose $f'(x)$ and $g'(x)$ exist, then

if $y = f(x) + g(x)$, $y' = f'(x) + g'(x)$.

Similarly,

if $y = f(x) - g(x)$, $y' = f'(x) - g'(x)$.

***The derivative of the sum or difference of any finite number of functions is equal to the sum or difference of the derivatives of the individual functions.***

**Example**

Find the derivative of $y = 3x^3 – 4x^2 – 2x + 11$

Solution:

$$y = 3x^3 – 4x^2 – 2x + 11$$

$$\frac{dy}{dx} = 3(3x^2) – 4(2x^1) – 2(1) + 0$$

$$= 9x^2 – 8x – 2$$

**Problem 12**

Find the derivative of each of the following functions.

a) $f(x) = 6x^2 - 3x + 18$

b) $f(x) = 3x^3 + 3x^2 + 5x - 21$

c) $f(x) = -7x^3 + 2x^5$

d) $f(x) = 12x^6 -2x + 3$

e) $f(x) = 2x^5 - 4x^3 – 3x^2 + 5x – 21$

f) $f(x) = 9x^2 – 8x – 2$

**Derivatives of any polynomial**

Given any polynomial or any expression that can be reduced to a polynomial, we can use the rules above to find its derivative.

**Example**

Find $\frac{dy}{dx}$ if y = $x^2(4x - 3)$.

Solution:

Multiply the two factors to get: $y = 4x^3 - 3x^2$

$$\frac{dy}{dx} = 4(3x^2) - 3(2x)$$

$$\frac{dy}{dx} = 12x^2 - 6x$$

**Problem 13**

Find $\frac{dy}{dx}$.

a) $y = (x- 2)(3x^2 - 2)$.

b) $y = x^2(1 - x)$

c) $y = x(x - 2)$.

d) $y = x^4(2x - x^3)$

e) $y = (x- 1)(x - 2)$.

f) $y = x^8(3 - x)$

**Example**

Find $y'$ if y = $\dfrac{x^5 - 4x^4 + 7x^3}{x^2}$.

Solution:

Rewrite y: $y = \dfrac{x^5}{x^2} - \dfrac{4x^4}{x^2} + \dfrac{7x^3}{x^2}$

$= x^3 - 4x^2 + 7x$

$y' = 3x^2 - 8x + 7.$

**Problem 14**

Find $y'$.

a) $y = \dfrac{x^5 - 5x^3 + 2x}{x}$  b) $y = \dfrac{3x^9 - 5x^6 + 2x^3}{x^3}$

**Derivative of a Product**

Suppose $f'(x)$ and $g'(x)$ exist, then

if y = $f(x) \bullet g(x)$ then $y' = f(x) \bullet g'(x) + g(x) \bullet f'(x)$

***The derivative of a product of two functions is equal to the first function times the derivative of the second function <u>plus</u> the second function times the derivative of the first function.***

$$\frac{dy}{dx} = \left(\begin{matrix} \textit{first} \\ \textit{function} \end{matrix}\right)\left(\begin{matrix} \textit{Derivative of} \\ \textit{second function} \end{matrix}\right) + \left(\begin{matrix} \textit{Second} \\ \textit{function} \end{matrix}\right)\left(\begin{matrix} \textit{Derivative of} \\ \textit{first function} \end{matrix}\right)$$

**Example**

Find $y'$ given y = (7 – 3x)(3 – 5x)

Solution:

There are two ways we could attempt this problem. First multiply the product then differentiate, or use the product rule to differentiate. Using the product rule is much easier in this case.

| | |
|---|---|
| ***First function*** = 7 – 3x | ***Derivative of first function*** = -3 |
| ***Second function*** = 3 – 5x | ***Derivative of second function*** = -5 |

$y'$ = (7 – 3x)(-5) + (3 – 5x)(-3)

$y'$ = -35 + 15x –9 + 15x

$y'$ = - 44 + 30x

**Problem 15**

Find $y'$ using the product rule.

a) $y = (x-1)(3x^2 - 5)$.   b) $y = x^2(1 - 2x)$

c) $y = 6x(x - 7)$.   d) $y = x^4(2 - x^3)$

e) $y = (x + 1)(x - 2)$.   f) $y = (5 - 2x)(1 - 3x)$.

g) $y = 6x(x - 7)$.   h) $y = x^4(2 - x^3)$

i) $y = (x + 1)(x - 2)$.   j) $y = (5 - 2x)(1 - 3x)$.

**Derivative of a Quotient**

Suppose $f'(x)$ and $g'(x)$ exist, then

if $y = \frac{f(x)}{g(x)}$, and $g(x) \neq 0$

then $$y' = \frac{g(x) \bullet f'(x) - f(x) \bullet g'(x)}{[g(x)]^2}$$

***The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.***

$$\frac{dy}{dx} = \frac{(\text{denominator})(\text{derivative } of \text{ numerator}) - (\text{numerator})(\text{derivative } of \text{ denominator})}{(\text{denominator})^2}$$

**Example**

Find $y'$ given y = $\frac{3x+5}{3-5x}$ .
Solution:

| | |
|---|---|
| ***Numerator*** = 3x + 5 | ***Derivative of Numerator*** = 3 |
| ***Denominator*** = 3 – 5x | ***Derivative of Denominator*** = -5 |

$$y' = \frac{(3-5x)(3)-(3x+5)(-5)}{(3-5x)^2}$$

$$y' = \frac{9-15x+15x+25}{(3-5x)^2} = \frac{34}{(3-5x)^2}$$

**Problem 16**

Find $y'$ .

a) y = $\frac{2x-5}{4-7x}$

b) y = $\frac{3-x}{4-x}$

c) y = $\frac{2x}{3-2x}$

d) y = $\frac{-x}{1-3x}$

e) y = $\frac{x-5}{7x+3}$

f) y = $\frac{2x+5}{4x-1}$

g) y = $\frac{x}{x+3}$

h) y = $\frac{2x+5}{x^2}$

**The chain rule**

The chain rule allow us to find the derivative of functions of the form y = $k[f(x)]^n$ (where k is a fixed real number, n is a positive number and f is differentiable function of x) without expanding them.

An example of functions that we can apply the chain rule are y = $(x + 3)^3$, y = $(4x - 1)^{200}$, etc.

We will consider two methods for finding the derivative of functions of the form $f(x) = k[g(x)]^n$

**The first method for finding derivative of the form** $f(x) = k[g(x)]^n$

If g(x) is any differentiable function of x, and if $f(x) = k[g(x)]^n$

then

$$f'(x) = k.n.[g(x)]^{n-1}.g'(x)$$

*The derivative of a function of the form* $f(x) = k[g(x)]^n$ *is equal to the* **constant** *(k)* **times** *the* ***exponent*** *(n)* **times** ***the base function*** *[f(x)]* ***raised to the exponent minus one*** *(n-1)* **times** *the* ***derivative of the base expression*** *g'(x)*

$$f'(x) = \text{(constant)} \bullet \text{(exponent)} \bullet \text{(base function)}^{\text{exponent} - 1} \bullet \text{(derivative of base function)}$$

if k =1 then we have,

$$f'(x) = \text{(exponent)} \bullet \text{(base function)}^{\text{exponent} - 1} \bullet \text{(derivative of base function)}$$

**Example**

Find f'(x) given f(x) = $4(2 - 5x^5)^8$.

Solution:

| | |
|---|---|
| ***1. constant*** = 4 | ***2. exponent*** = 8 |
| ***3. Base function*** = $2 - 5x^5$ | ***4. Derivative of base function*** = $-25x^4$ |

f'(x) = **(constant)** $\bullet$ **(exponent)** $\bullet$ **(base function)**$^{\textbf{exponent} - 1}$ $\bullet$ **(derivative of base function)**

$= (4)(8)(2 - 5x^5)^{8-1}(-25x^4)$

$= -800x^4(2 - 5x^5)^7$

**Example**

Given y = $2\sqrt[3]{3x^2+19}$, find $y'$.

Solution:

Rewrite y as a power function: y = $2\sqrt[3]{3x^2+19}$

$= 2\,(3x^2+19)^{1/3}$

| | |
|---|---|
| ***1. constant*** = 2 | ***2. exponent*** = $\frac{1}{3}$ |
| ***3. Base function*** = $3x^2 + 19$ | ***4. Derivative of base function*** = 6x |

$\frac{dy}{dx}$ **= (constant)** $\bullet$ **(exponent)** $\bullet$ **(base function)**$^{\textbf{exponent} - 1}$ $\bullet$ **(derivative of base function)**

$$y' = 2\left(\frac{1}{3}\right)(3x^2+19)^{\frac{1}{3}-\frac{3}{3}}(6x) = 4x(3x^2+19)^{-\frac{2}{3}} = \frac{4x}{\sqrt[3]{(3x^2+19)^2}}$$

**Example**

Find the derivative of $y = \sqrt{5+x^2}$

Solution:

Rewriting the equation we have,

$y = (5+x^2)^{\frac{1}{2}}$

| | |
|---|---|
| ***1. constant*** = 1 | ***2. exponent*** = $\frac{1}{2}$ |
| ***3. Base function*** = 5 + $x^2$ | ***4. Derivative of base function*** = 2x |

Now find the derivative of y.

$\frac{dy}{dx}$ **= (constant) $\bullet$ (exponent) $\bullet$ (base function)$^{\text{exponent} - 1}$ $\bullet$ (derivative of base function)**

$$y' = (1)(\frac{1}{2})(5+x^2)^{\frac{1}{2}-\frac{2}{2}}(2x) = x(5+x^2)^{-\frac{1}{2}}$$

$$y' = \frac{x}{\sqrt{(5+x^2)}}$$

In the second method, given the function $y = k[g(x)]^n$ we let $u$ = g(x) so that $y = k[u]^n$

Note that u is a function of x, with derivative $\frac{du}{dx}$, and y is a function of u, with derivative $\frac{dy}{du}$.

Therefore , the derivative of y with respect to x, $\frac{dy}{dx}$ will be

$$\frac{dy}{dx} = \frac{du}{dx} * \frac{dy}{du}$$

**Example**

Find the derivative of y = $(x^3 + 2)^4$.

Solution: We let u = $x^3$ + 2 so that y = $u^4$.

Then $\frac{du}{dx}$ = $3x^2$ and $\frac{dy}{du}$ = $4u^3$, since u = $x^3$ + 2, $\frac{dy}{du}$ = $4(x^3 + 2)^3$

So that $\frac{dy}{dx} = \frac{du}{dx} * \frac{dy}{du}$ = $(3x^2)[4(x^3 + 2)^3] = 12x^2(x^3 + 2)^3$

## Problem 17

Find $y'$.

a) $y = (x + 2)^4$

b) $y = (2x^4 + 2)^7$

c) $y = (3 – 2x)^3$

d) $y = (2x + 7)^5$

## Problem 18

Find $y'$.

a) $y = -5(3x - 2)^4$

b) $y = 2(x^4 – 2x)^7$

c) $y = 2(3 + 2x^3)^3$

d) $y = -4(5 - 7x)^5$

e) $y = -3(7 – 2x^3)^5$.

f) $y = -4(1 - x)^3$

## Problem 19

Find $y'$.

a) $y = 7\sqrt[3]{x^3 + 10}$

b) $y = -5\sqrt[3]{x^2 + 1}$

c) $y = \sqrt[3]{3x \ + 7}$

d) $y = -5\sqrt[3]{4x^5 + 7}$

**Problem 20**

Find the derivative of the following functions.

a) $y = \sqrt{1+4x^3}$

b) $y = \sqrt{1-x^3}$

c) $y = \sqrt{1+x^2}$

d) $y = \sqrt{1-2x^2}$

- **Finding derivatives of power functions with negative exponents**

When working with negative integer power, all the rules we have considered so far is valid for a negative integer.

If $y = kx^{-n}$ where $x \neq 0$, then $y' = -knx^{-n-1}$, or $y' = -\dfrac{kn}{x^{n+1}}$

**Example**

Given $y = 5x^{-2}$ find $y'$
Solution:

$$y' = 5(-2)x^{-2-1}$$
$$= -10x^{-3}$$
$$= -\frac{10}{x^3}$$

**Problem 21**

Find $y'$.

a) y = $\frac{2}{3}x^{-3}$ b) a) y = $\frac{2}{5}x^{-5}$

c) y = $\frac{7}{4}x^{-2}$ d) y = $\frac{2}{15}x^{-6}$

**Example**

Given y = $\frac{3}{(x^3+1)^2}$, find $y'$.

Solution: Rewrite y in the form

**y = (numerator)(denominator)$^{\textbf{change the sign of the denominator exponent}}$**

y = $3(x^3 + 1)^{-2}$

$y'$ = $3(-2)\,(x^3 + 1)^{-2-1}(3x^2)$

$y'$ = $-18x^2(x^3 + 1)^{-3}$

$y' = \frac{-18x^2}{(x^3+1)^3}$

**Problem 22**

Find $y'$.

a) y = $\frac{4}{(1-x^2)^2}$ b) y = $\frac{-3}{(x+1)^2}$

c) $y = \frac{3}{(x^3+1)^2}$ d) $y = \frac{-3}{(x+1)^2}$

**Example**

Find the derivative of $y = \frac{8}{\sqrt{4+3x}}$

Solution:

$$y = \frac{8}{(4+3x)^{1/2}}$$

Rewrite y in the form

**y = (numerator)(denominator)$^{\text{change the sign of the denominator exponent}}$**

$$y = 8(4+3x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = k \bullet n \bullet [f(x)]^{n-1} \bullet f'(x)$$

| | |
|---|---|
| ***1. constant*** = 8 | ***2. exponent*** = - $\frac{1}{2}$ |
| ***3. Base function*** = 4 + 3x | ***4. Derivative of base function*** = 3 |

$\frac{dy}{dx}$ = (constant) $\bullet$ (exponent) $\bullet$ (base function)$^{n-1}$ $\bullet$ (derivative of base function)

Now find the derivative of y.

$$y' = (8)(-\frac{1}{2})(4+3x)^{-\frac{1}{2}-\frac{2}{2}}(3) = -12(4+3x)^{-\frac{3}{2}}$$

$$y' = \frac{-12}{\sqrt{(4+3x)^3}}$$

**Problem 23**

Find the derivative of the following functions.

a) $y = \frac{-3}{\sqrt{1-2x}}$ b) $y = \frac{-9}{\sqrt{3-x}}$

**Example**

Find the derivative of the function $f(t) = \frac{1}{(t^2 - 2t - 5)^4}$

Rewrite y in the form

$f$(t) = (numerator)(denominator)$^{\text{change sign of denominator exponent}}$

$f(t) = (t^2 - 2t - 5)^{-4}$

| | |
|---|---|
| *constant* = 1 | *exponent* = - 4 |
| *Base function* = $t^2 - 2t - 5$ | *Derivative of base function* = 2t – 2 |

$\frac{dy}{dx}$ = (constant) • (exponent) • (base function)$^{\text{exponent} - 1}$ • (derivative of base function)

Now find the derivative of f(t).

$f'(t) = (1)(-4)(t^2 - 2t - 5)^{-4-1}(2t - 2)$

$f'(t) = -8(t - 1)(t^2 - 2t - 5)^{-5}$

$f'(t) = \frac{-8(t-1)}{(t^2 - 2t - 5)^5}$

**Problem 24**

Find the derivative of the function

a) $f(x) = \dfrac{3}{(x^2 - 5x - 6)^3}$ b) $f(x) = \dfrac{3}{(x^2 - 6)^3}$

c) $f(x) = \dfrac{-2}{(x^2 - x - 6)^3}$ d) $f(x) = \dfrac{1}{(5x - 6)^4}$

- **Combination of rules**

Sometimes you will need to apply the chain rule within a product or a quotient rule.

**Example**

Find the derivative of the function $f(x) = \dfrac{x}{\sqrt{7 - 3x}}$

Solution:

$$f(x) = \frac{x}{(7 - 3x)^{\frac{1}{2}}}$$

Rewrite y in the form *f*(x) = (numerator)(denominator)$^{\text{change the sign of the denominator exponent}}$

$$f(x) = x(7 - 3x)^{-\frac{1}{2}}$$

Use the product rule to find the derivative of f(x). The formula is,

$$\frac{dy}{dx} = \left(\begin{matrix} first \\ function \end{matrix}\right)\left(\begin{matrix} Derivative\ of \\ second\ function \end{matrix}\right) + \left(\begin{matrix} Second \\ function \end{matrix}\right)\left(\begin{matrix} Derivative\ of \\ first\ function \end{matrix}\right)$$

**1. First function** = x

**2. Derivative of first function** = 1

**3. Second function** = $(7 - 3x)^{-\frac{1}{2}}$

**4. Derivative of second function:** Note that we have to use the chain rule in this case.

| | |
|---|---|
| *constant* = 1 | *exponent* = - $\frac{1}{2}$ |
| *Base function* = 7 - 3x | *Derivative of base function* = -3 |

$\frac{dy}{dx}$ = (constant) $\bullet$ (exponent) $\bullet$ (base function)$^{\text{exponent} - 1}$ $\bullet$ (derivative of base function)

**Derivative of second function** = $(1)(-\frac{1}{2})(7-3x)^{-\frac{1}{2}-\frac{2}{2}}(-3) = \frac{3}{2}(7-3x)^{-\frac{3}{2}}$

Therefore $f'(x) = x[\frac{3}{2}(7-3x)^{-\frac{3}{2}}] + [(7-3x)^{-\frac{1}{2}}](1)$

$$f'(x) = \frac{3x}{2\sqrt{(7-3x)^3}} + \frac{1}{\sqrt{7-3x}}$$

Simplify this expression to get,

$$f'(x) = \frac{3x}{2\sqrt{(7-3x)^3}} + \frac{1}{\sqrt{7-3x}} \bullet \frac{2(7-3x)}{2\sqrt{(7-3x)^2}} = \frac{3x+14-6x}{2\sqrt{(7-3x)^3}} = \frac{-3x+14}{2\sqrt{(7-3x)^3}}.$$

**Problem 25**

Find the derivative of the following functions

a) $f(x) = \frac{-3x}{\sqrt{1-5x}}$

b) $f(x) = \frac{x}{\sqrt{1-x}}$

c) $f(x) = \frac{-x}{\sqrt{3-5x}}$

d) $f(x) = \frac{2x(1-x)}{\sqrt{3+x}}$