CHAPTER 1: LIMITS OF FUNCTIONS

Introduction
In this chapter, the concept of the limit of a function at a point is formally introduced. Rules for computing limits are also given, and some situations are described where the limit does not exist.

Upon completion of this chapter, the student will be able to:

- Compute and understand the meaning and definition of limit of *a function*.
- Compute and understand the meaning of one sided limit of *a function.*
- Evaluate infinite limits and limits at infinity.
- Use limits to determine vertical and horizontal asymptotes of a graph.

1.1 Meaning of Limit
The *limit* is a mathematical tool for *predicting* (or guessing) the value of a function as its independent variable approaches some fixed number. Limits describe what happens to a function f(x) as its variable x approaches a particular number c. The word limit comes from the Latin word *limen*, meaning "threshold." In an attempt to find limit of a function at a particular point, we investigate what the function is "beginning to do" as we get closer and closer to the given point. What we will *predict* that the function will finally do is the *value of the function* at that point.

In finding the limit of a function at a point, we usually use the letter *c* to represent the point in question, and what we actually do is to investigate the actions of the function at points near to *c*.

We use the symbol f(c) to denote the value of the function at *c*. Even if we know or do not know the value of the function at *c*, we do not care, because that is not the issue. The issue is investigating the actions of the functions at points near *c*, the point in question. For this reason, when finding limits, we completely ignore the value of the function at *c*.

1.1.1 Definition

Defn: If f(x) gets closer and closer to a number L as x gets closer and closer to c from either side, then L is the limit of f(x) as x approaches c. This behavior is expressed by writing

$$\lim_{x \to c} f(x) = L$$

In symbols we have, $f(x) \to L$ as $x \to c$, or we simply write $\lim_{x \to c} f(x) = L$

NB: The limit of a function *f* as *x* approaches *c* does not depend on the value of the function at the point *c*.

The $\lim_{x \to c} f(x)$ *exists* if all of the following statements are true;

1. $\lim_{x \to c^-} f(x)$ is definite and has a value.
2. $\lim_{x \to c^+} f(x)$ is definite and has a value.
3. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

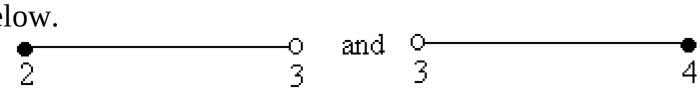
1.2 Ways of finding the limit of a function at a point

Limits of functions can be evaluated by using many different methods including table of values, graphs, simple substitution, factorization, simplification, and rationalization. In this section, we will discuss the above mentioned techniques.

1.2.1 Limits by table of value

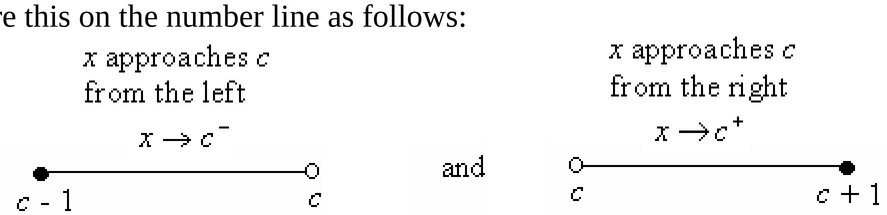
To find the limit of a function at a point by way of table of values, start your investigation from *one point below* the point in question and *one point above* the point in question. For example, to find $\lim_{x \to 3}$ f(x), the point in question is 3, therefore start the investigation from 2 and choose values of *x* from the interval [2, 3), and then start another investigation from 4 by choosing values of *x* from the interval (3, 4].

[2, 3) means 2 is *included* in the interval but 3 is *excluded* from the interval and (3, 4] means 3 is *excluded* from the interval but 4 is *included* in the interval. A picture of this interval is given below.



Suppose we want to find the *limit* of a function as *x* approaches *c* (a point) by means of a table of values. We must first make a table of ordered pairs by choosing values of *x* from the interval [*c*-1, *c*) and from (*c*, *c* +1].

When we consider values of *x* in the interval [c-1, c), we use the phrase "*x* approaches *c* from the left" and written $x \to c^-$. Similarly, When we consider values of *x* in the interval (*c*, *c* +1], we use the phrase "*x* approaches *c* from the right" and written $x \to c^+$. We can picture this on the number line as follows:



The phrase "*x* approaches *c* from the left" is called a one-sided limit. Similarly the phrase "*x* approaches *c* from the right" is also called a one-sided limit. Together, the left and right limits can be combined into a two-sided limit, which is written as $\lim_{x \to c}$ f(x).

In general, $\lim_{x \to c}$ f(x) exits if and only if

a) both $\lim_{x \to c^-}$ f(x) and $\lim_{x \to c^+}$ f(x) exist

b) $\lim_{x \to c^-}$ f(x) = $\lim_{x \to c^+}$ f(x)

c) $\lim_{x \to c}$ f(x) = $\lim_{x \to c^-}$ f(x) = $\lim_{x \to c^+}$ f(x).

Right Hand and Left Hand Limits

The notation

$$\lim_{x \to c^-} \text{f(x)}$$

represents the limit of f(x) as x approaches c from the left (left - hand limit) or from below.

The notation

$$\lim_{x \to c^+} \text{f(x)}$$

represents the limit of f(x) as x approaches c from the right (right - hand limit) or from above.

If the value of the function f(x) approaches the same number L as x approaches c from either direction, then the limit equals L.

Meaning of existence of a limit

We say that

$\lim_{x \to c}$ f(x) exists and

$\lim_{x \to c}$ f(x) = **L** if and only if $\lim_{x \to c^+}$ f(x) = $\lim_{x \to c^-}$ f(x) = **L**

If the limiting values of f(x) are different when x approaches c from each direction, then the function does not approach a limit as x approaches c.
Geometrically, the limit statement

$$\lim_{x \to c} f(x) = \mathbf{L}$$

Means that the height of the graph of y = f(x) approaches L as x approaches c.
One can be tempted to simply substitute the value x = c into f(x) and determine f(c). This actually is a valid way of determining the limit for many but not all functions.

In finding the limit of a function at a point, we may find out that one of the following is a possible conclusion:

a) The limit exists (meaning it has a value), and its value is the same as the value of the function at that point.
b) The limit exists (meaning it has a value), but its value is not the same as the value of the function at that point.
c) The limit does not exist.

When we predict the value of a function as x approaches some fixed point, c. ($x \to c$), the function may have four possible behaviors.
(1) The function f(x) may tend to a *finite number* , say L.
In symbols we have, $f(x) \to L$ as $x \to c$,

or $\lim_{x \to c} f(x) = L$

(2) The function f(x) may be *increasing* without bound (i.e. the function tends to positive infinity, ∞). The function is said to have an *infinite limit*.
In symbols we have, $f(x) \to \infty$ as $x \to c$,

or $\lim_{x \to c} f(x) = \infty$

(3) The function f(x) may be *decreasing* without bound (i.e. the function tends to negative infinity, $-\infty$). The function is said to have an *infinite limit*.
In symbols we have, $f(x) \to -\infty$ as $x \to c$,

or $\lim_{x \to c} f(x) = -\infty$

(4) The function f(x) may neither tend to L, ∞, nor $-\infty$.

If the function f(x) does not approach a real number L as x $\to$ c, then the limit does not exist. Then we write it as $\lim_{x \to c}$ f(x) = DNE (Does not exist).
The function is said to have a *vertical asymptote* at x = c

if either $\lim_{x \to c}$ f(x) = ∞ (i.e. f(x) $\to \infty$ as x $\to$ c)

or $\lim_{x \to c}$ f(x) = $-\infty$ (i.e. f(x) $\to -\infty$ as x $\to$ c)

Example 1

1. Determine whether $\lim_{x \to 5}$ f(x) exists if

$$f(x) = \frac{x^2 - 25}{x - 5}$$

since the denominator equals 0 when x = 5, we can conclude that the function is undefined at this point and it would be tempting to conclude that no limit exists when x = 5. however, this function does approach a limit as x approaches (gets closer to 5), even though the function is not defined at x = 5.
We now write some of the values given to x and the corresponding values acquired by f(x) in a table as x approaches 5 from either side.

Approaching x = 5 from left		Approaching x = 5 from right	
x	**f(x)**	**x**	**f(x)**
4	9	7	12
4.235	9.235	6.31	11.31
4.976	9.976	5.2175	10.2175
4.99998	9.99998	5.0039	10.0039

Since

$\lim_{x \to 5^-}$ f(x) = 10 and $\lim_{x \to 5^+}$ f(x) = 10

then

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = 10$$

Even though the function is undefined when x = 5, the function approaches a value of 10 as the value of x comes closer to 5

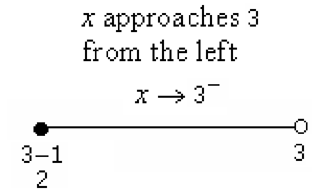
Example 2

a) Illustrate how $\lim_{x \to 3} f(x)$ can be represented on a number line.
b) Give five values of *x* that can be useful in investigating the value of the function at the point in question.

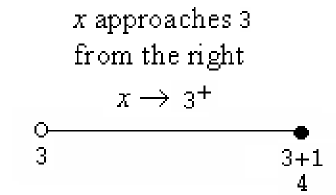
Solution:

$\lim_{x \to 3} f(x)$

a)



and



$c + 1$

b) Example of values of x as x approaches 3 from the left.

2
2.5
2.9
2.99
2.999

Example of values of x as x approaches 3 from the right.

4
3.5
3.1
3.01
3.001

Note that the values of x are choosen in such a way that values closer to the point in question (3) are much more than values closer to the other two endpoints.

Example 3

Given the function

$$f(x) = \begin{cases} 2x & x \le 4 \\ 2x+3 & x > 4 \end{cases}$$

To find $\lim_{x \to 4} f(x)$, we write the table when x approaches 4 from either direction

Approaching x = 4 from left		Approaching x = 4 from right	
x	**f(x) = 2x**	**x**	**f(x) = 2x + 3**
3	6	5	13
3.5	7	4.5	12
3.8	7.6	4.3	11.6
3.9	7.8	4.1	11.2
3.99	7.98	4.01	11.02

Therefore

$\lim_{x \to 4^-} f(x) = 8$ and $\lim_{x \to 4^+} f(x) = 11$

since

$\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$

The function does not approach a limiting value as x approaches 4 and $\lim_{x \to 4} f(x)$ does not exist.

Problem 1

Illustrate how each of the following limits below can be represented on a number line and give five values of *x* that can be useful in investigating the value of the function at the point in question.

a) $\lim\limits_{x\to 6} f(x)$ b) $\lim\limits_{x\to -1} f(x)$

b) $\lim\limits_{x\to -2} f(x)$ c) $\lim\limits_{x\to 0} f(x)$

Example 2

Find $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$ by means of a table. Note that *x* cannot be 1.

Solution:

x approache 1 from the left

$x \to 1^-$

and

x approaches 1 from the right

$x \to 1^+$

x	$\dfrac{x^2-1}{x-1}$
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	$\dfrac{x^2-1}{x-1}$
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

The tables suggest that the values of f(x) get closer and closer to 2 as the values of x get closer and closer to 1 from both sides.

$\dfrac{x^2-1}{x-1} \to 2$ as $x \to 1^-$ and $\dfrac{x^2-1}{x-1} \to 2$ as $x \to 1^+$

we have $\lim\limits_{x\to 1^-}\dfrac{x^2-1}{x-1} = 2$ and $\lim\limits_{x\to 1^+}\dfrac{x^2-1}{x-1} = 2.$

Therefore $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1} = 2.$

Alternatively, you can setup a table like the one below and determine the limits from both sides.

x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2-1}{x-1}$	1	1.5	1.9	1.99	1.999		2.001	2.01	2.1	2.5	3

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = 2$$

Problem 1

a) Find $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$ by means of a table of values.

$\lim_{x\to 2^-}\frac{x^2+x-6}{x-2} =$ $\qquad$ $\lim_{x\to 2^+}\frac{x^2+x-6}{x-2} =$ $\qquad$ $\lim_{x\to 2}\frac{x^2+x-6}{x-2} =$

b) Find $\lim_{x\to 3}\frac{x^2-9}{x-3}$ by means of a table of values.

$\lim_{x\to 3^-}\frac{x^2-9}{x-3} =$ $\qquad$ $\lim_{x\to 3^+}\frac{x^2-9}{x-3} =$ $\qquad$ $\lim_{x\to 3}\frac{x^2-9}{x-3}$

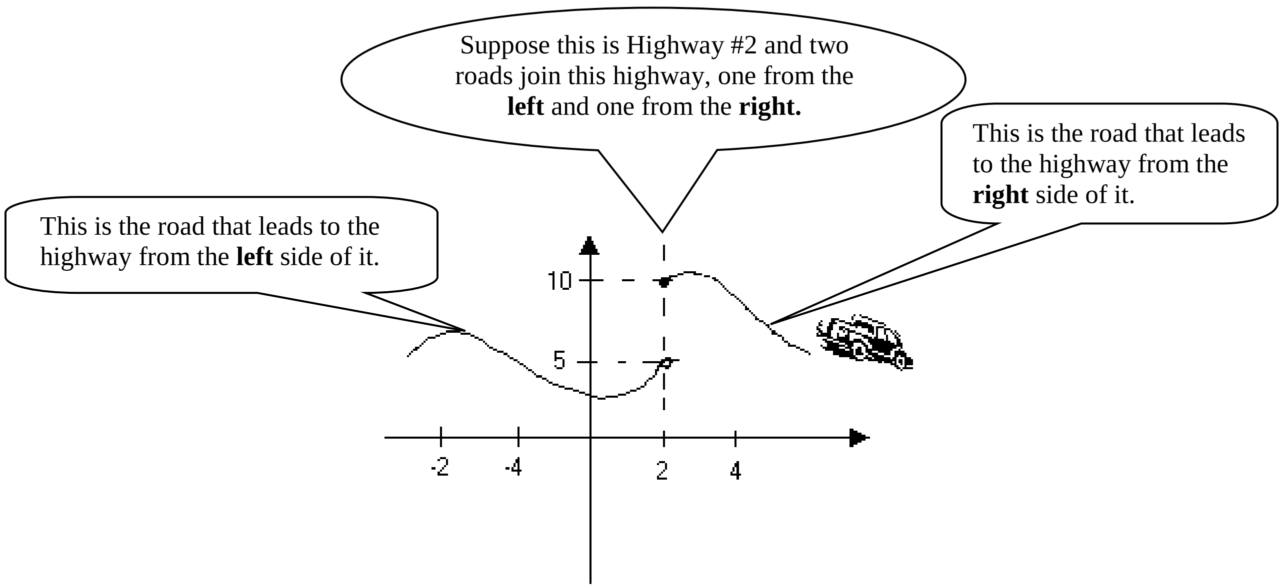
c) Find $\lim_{x\to 1}\frac{1}{x-1}$ by means of a table.

$\lim_{x\to 1^-}\frac{1}{x-1} =$ $\qquad$ $\lim_{x\to 1^+}\frac{1}{x-1} =$ $\qquad$ $\lim_{x\to 1}\frac{1}{x-1} =$

d) If f(x) = $\begin{cases} 2x, & x \le 1 \\ 3-x, & x > 1 \end{cases}$, find $\lim_{x\to 1} f(x)$ by means of a table.

$\lim_{x\to 1^-} f(x)$ = $\lim_{x\to 1^+} f(x)$ = $\lim_{x\to 1} f(x)$ =

1.2.2 Limit by graphing



The graph above can help us understand the notion of a limit. We write x $\to$ 2^- to mean we are driving towards the highway marked #2 (i.e. *x* approaches 2 from the left) from the left road.. We write x $\to$ 2^+ to mean we are driving towards the highway marked #2 (i.e. *x* approaches 2 from the right) from the right road.

Our objective is to find the *point of entry where the two roads enter the highway #2*. We can *predict* that as we get very close and close to the highway *from the left road*, our point of entry, i.e. the height of the highway will be *entry number* 5. Similarly, we can *predict* that as we get very close and close to the highway from the right road, our point of entry, i.e. the height of the highway will be *entry number* 10. Notice that we are predicting our point of entry and we do not necessarily have to enter before we know the entry number.

Using symbols, we can write,

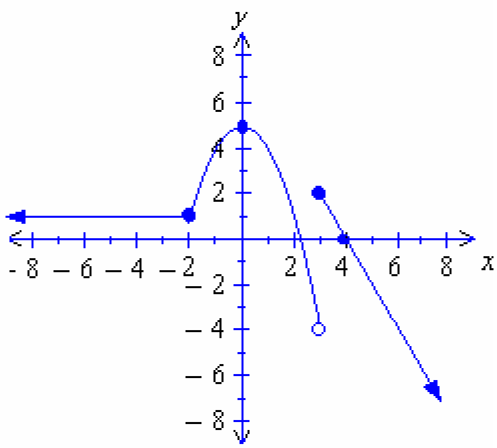
f(x)$\to$5 as x $\to$ 2^- or $\lim_{x\to 2^-}$ f(x) = 5 and

$f(x) \to 10$ as $x \to 2^+$ or $\lim_{x \to 2^+} f(x) = 10$

The $\lim_{x \to 2} f(x)$ does not exit because both the left and right roads do not have the same point of entry. Using our illustration, if the points of entry are different for the left and right roads, then the limit does not exist. For the limit to exits, the point of entry must be the same for both roads.

Example 2

Use the graph below to answer the following questions.



a) $\lim_{x \to -2^-} f(x) = 1$

b) $\lim_{x \to -2^+} f(x) = 1$

c) $\lim_{x \to -2} f(x) = 1$

d) $\lim_{x \to 0^-} f(x) = 5$

e) $\lim_{x \to 0^+} f(x) = 5$

f) $\lim_{x \to 0} f(x) = 5$

g) $\lim_{x \to 3^-} f(x) = -4$

h) $\lim_{x \to 3^+} f(x) = 2$

i) $\lim_{x \to 3} f(x) =$ **DNE**

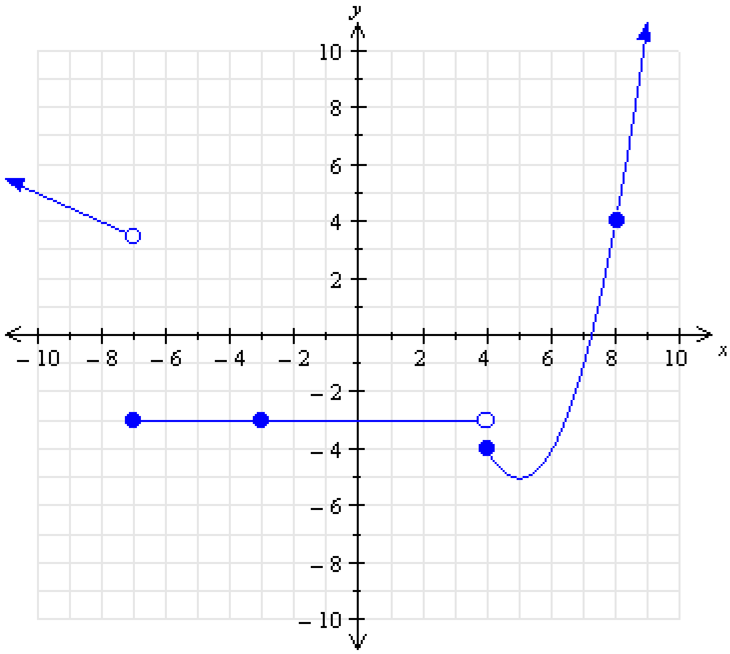
j) $f(-1) = 4$

k) $f(4) = 0$

l) $f(3) = 2$

Problem 2

Use the graph below to answer the following questions.



a) $\lim\limits_{x \to -3^-} f(x) =$

b) $\lim\limits_{x \to -3^+} f(x) =$

c) $\lim\limits_{x \to -3} f(x) =$

d) $\lim\limits_{x \to -5^-} f(x) =$

e) $\lim\limits_{x \to -5^+} f(x) =$

f) $\lim\limits_{x \to 5} f(x) =$

g) $\lim\limits_{x \to 4^-} f(x) =$

h) $\lim\limits_{x \to 4^+} f(x) =$

i) $\lim\limits_{x \to 4} f(x) =$

j) $f(-5) =$

k) $f(4) =$

l) $f(8) =$

m) $f(-3) =$

n) $f(1) =$

We have already discussed finding limits by table of values and graphs. We will now consider finding limits by direct substitution, factorization, simplification, and rationalization.

1.2.3 Limits by substitution

To find the limit of a function, first try direct substitution. Direct substitution is always valid for polynomials and rational functions with nonzero denominators. Do not use direct substitution if it makes a denominator zero.

If p is a polynomial function, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = \dfrac{f(x)}{g(x)}$ and $g(c) \neq 0$, then

$$\lim_{x \to c} r(x) = \frac{f(c)}{g(c)}$$

Example 4

Find the limit: a) $\lim\limits_{x \to 1} 3x^4 - 2x^2 + x - 1$ b) $\lim\limits_{x \to 2} \dfrac{x^2 - x + 2}{x + 2}$

Solution: use direct substitution

a) $\lim_{x \to 1} 3x^4 - 2x^2 + x - 1 = 3(1)^4 - 2(1)^2 + (1) - 1 = 1$

b) $\lim_{x \to 2} \frac{x^2 - x + 2}{x + 2} = \frac{2^2 - 2 + 2}{2 + 2} = \frac{4}{4} = 1$

Problem 4

Find the limit:

a) $\lim_{x \to 1} 2x^3 - x^2 + x - 1$

b) $\lim_{x \to -2} \frac{x^2 + x - 2}{x - 2}$

c) $\lim_{x \to -1} x^4 - 4x^3 + x - 1$

d) $\lim_{x \to -3} \frac{x^2 - x - 6}{x - 3}$

Limit of a constant

$\lim_{x \to c} k = k$

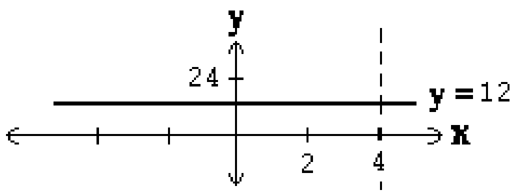
The limit of a constant is a constant.

Example 3

Find the limit: $\lim_{x \to 4} 12.$

Solution:

$\lim_{x \to 4} 12 = 12.$



Problem 3

Find the limit:

a) $\lim_{x \to -3} 10.$

b) $\lim_{x \to 4} 210$

c) $\lim_{x \to -3} -16$

d) $\lim_{x \to 10} 10.$

e) $\lim_{x \to 4} 0$

f) $\lim_{x \to 7} -3$

1.2.4 Limits by factorization

If the limit of f(x) as x approaches c cannot be evaluated through direct substitution, because this will cause denominator to be zero, then try to factor and cancel out the common factors, and then use direct substitution again.

I.e. Any time that direct substitution results in 0/0, then you may factorize to reduce the limit. Examples are as follows.

Example 5

Find the limit: a) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$ b) $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{x+3}$

Solution: Since this is a rational function and direct substitution gives 0/0, try to factor both numerator and denominator and cancel out the common factors.

a) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1}\dfrac{(x+1)(x-1)}{(x-1)} = \lim\limits_{x\to 1}(x+1) = 1+1 = 2$

b) $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{x+3} = \lim\limits_{x\to -3}\dfrac{(x+3)(x-2)}{(x+3)} = \lim\limits_{x\to -3}(x-2) = -3-2 = -5$

Problem 5
Find the limit:

a) $\lim\limits_{x\to -1}\dfrac{x^2-1}{x+1}$

b) $\lim\limits_{x\to -2}\dfrac{x^2+3x-10}{x-2}$

b) $\lim\limits_{x\to 3}\dfrac{x^2-x-6}{x-3}$

d) $\lim\limits_{x\to -5}\dfrac{x^2+3x-10}{x+5}$

c) $\lim\limits_{x\to 5}\dfrac{x-5}{x^2-25}$

d) $\lim\limits_{x\to 6}\dfrac{x-6}{x^2-36}$

e) $\lim\limits_{x\to -4}\dfrac{x^2-16}{x+4}$

f) $\lim\limits_{x\to -2}\dfrac{x^2-x-2}{x-2}$

b) $\lim\limits_{x\to -3}\dfrac{x^2+7x+12}{x+3}$

d) $\lim\limits_{x\to -5}\dfrac{x^2+8x+15}{x+5}$

1.2.5 Limit by rationalization

If the numerator approaches 0 and the denominator also approaches 0 and the numerator or the denominator involves a square root expression, then rationalize the numerator or denominator by multiplying both the numerator and the denominator by the conjugate of the numerator/denominator.

I.e. whenever substitution results in 0/0, and the limit involves a root, you may be able to eliminate terms by rationalizing the expression.

Example 6

(a) Find the limit: $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

Solution:

f(x) = $\frac{\sqrt{x} - 2}{x - 4}$

Multiply numerator and denominator by conjugate of the numerator, $\sqrt{x} + 2$.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \bullet \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{\sqrt{x^2} - 2\sqrt{x} + 2\sqrt{x} - 4}{(x-4)(\sqrt{x}+2)} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$$

Thus,

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

(b) Find the limit: $\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x}$

Solution:

Multiply numerator and denominator by conjugate of the numerator

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x} &= \lim_{x\to 0}\frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{\left(x^2+4\right)-4}{x\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{x^2}{x\left(\sqrt{x^2+4}+2\right)}
\end{aligned}
$$

The nice thing about this is the troublesome factors of x in the numerator and denominator now cancel. This now gives us

$$
\begin{aligned}
\lim_{x\to 0}\frac{x^2}{x\left(\sqrt{x^2+4}+2\right)} &= \lim_{x\to 0}\frac{x}{\left(\sqrt{x^2+4}+2\right)}\\
&= \frac{0}{\sqrt{0^2+4}+2}\\
&= \left(\frac{0}{4}\right)\\
&= 0
\end{aligned}
$$

Problem 6

Find the limit:

a) $\lim_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$

b) $\lim_{x\to 25}\dfrac{\sqrt{x}-5}{x-25}$

c) $\lim_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$

d) $\lim_{x\to 16}\dfrac{\sqrt{x}-4}{x-16}$

Simplifying

Whenever you are dealing with fractions, you may be able to simplify the fraction and reduce a 0/0 result into a term that can be solved.

1.3 Theorem: (Properties of limits)

Let **n** be a positive integer, **k** be a constant, and **f** and **g** be two functions that have limits at **c**. Then:

1. $\lim_{x \to c} k = k$ (The limit of a constant is the constant)

2. $\lim_{x \to c} x = c$

3. $\lim_{x \to c} k\, f(x) = k \cdot \lim_{x \to c} f(x)$ (The limit of a constant times a function is equal to the constant times the limit of the function)

4. $\lim_{x \to c} [\, f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$ (The limit of the sum of the functions is equal to the sum of the limits)

5. $\lim_{x \to c} [\, f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$ (The limit of the difference of the functions is equal to the difference of the limits)

6. $\lim_{x \to c} [\, f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (The limit of the product of the functions is equal to the product of the limits)

7. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$, **provided that** $\lim_{x \to c} g(x) \neq 0$ (The limit of the quotient of the functions is equal to the quotient of the limits, provided that the denominator is not zero)

1.4 Concept of infinity

1.4.1 Indefinite Limit

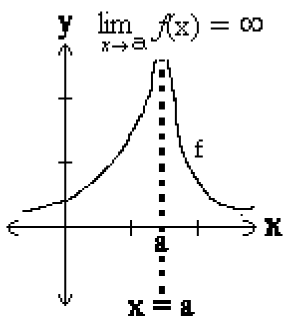
Not every function has a definite limit as *x approaches c*. In fact some functions increases or decreases beyond bounds.

We use the infinity symbols $+\infty$ and $-\infty$ to represent situations where the limit increases or decreases beyond bounds. The symbol ∞ does not represent a real number and cannot be used in any calculations.

We use the symbol $+\infty$ or $-\infty$ when f(x) does not approach a definite limit as $x \to a$, or when we choose the values of x so near to a, making f(x) as large as possible.

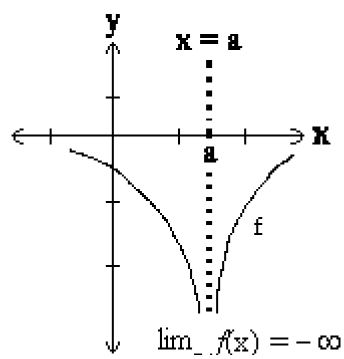
In general, if f(x) grows larger and larger positively without bound, as x approaches a, we have

$$\lim_{x \to a} f(x) = \infty.$$

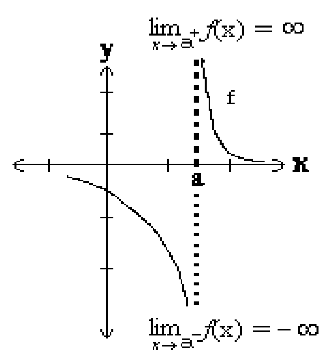


If f(x) grows larger and larger negatively without bound, as x approaches a, we have

$$\lim_{x \to a} f(x) = -\infty.$$



However, when the answer is either $+\infty$ or $-\infty$, we say the ***limit does not exist*** since $+\infty$ or $-\infty$ *are not real numbers*.



Vertical Asymptote: The vertical line x = a is called a <u>vertical asymptote</u> of the graph of f if

$\lim_{x \to a} f(x) = \infty$ **or** $\lim_{x \to a} f(x) = -\infty$ **or**

$\lim_{x \to a^-} f(x) = \infty$ **and/or** $\lim_{x \to a^+} f(x) = \infty$ **or**

$\lim_{x \to a^-} f(x) = -\infty$ **and/or** $\lim_{x \to a^+} f(x) = -\infty$.

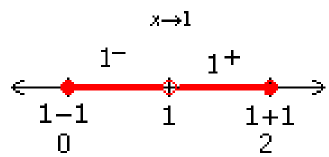
Example 1

Find the limit: $\lim_{x \to 1} \frac{1}{x-1}$

Solution:

Note that the when x =1, the expression $\frac{1}{x-1}$ is undefined. Direct substitution is not possible in this case. In addition, factorization, which is our next option, is not possible. We can only use a table of values to evaluate this limit.

$\lim_{x \to 1} \frac{1}{x-1}$



values of x from the left

x	y
0	-1
0.5	-2
0.9	-100
0.99	-1000
0.999	-10000
	-∞

$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$

values of x from the right

x	y
2	1
1.5	2
1.1	100
1.01	1000
1.001	10000
	∞

$\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$

Thus, $\lim_{x \to 1} \frac{1}{x-1}$ ***does not exist.***

Problem 1

Find the limit:

a) $\lim_{x \to 3^+} \frac{1}{x-3}$

b) $\lim_{x \to -1^-} \frac{1}{x+1}$

c) $\lim_{x \to -2^+} \frac{1}{x+2}$

d) $\lim_{x \to 7^-} \frac{1}{x-7}$

e) $\lim_{x \to 3^-} \frac{1}{x-3}$

f) $\lim_{x \to -1^+} \frac{1}{x+1}$

g) $\lim_{x\to -2^-} \frac{1}{x+2}$

h) $\lim_{x\to 7^+} \frac{1}{x-7}$

i) $\lim_{x\to 3} \frac{1}{x-3}$

j) $\lim_{x\to -1} \frac{1}{x+1}$

k) $\lim_{x\to -2} \frac{1}{x+2}$

l) $\lim_{x\to 3} \frac{1}{x-7}$

m) $\lim_{x\to 3} \frac{x-2}{x-3}$

n) $\lim_{x\to -1} \frac{1-x}{x+1}$

o) $\lim_{x\to 2} \frac{1-x}{x+2}$

p) $\lim_{x\to 3} \frac{1}{x-4}$

1.4.2 Limit to infinity

We know that the symbol $+\infty$ or $-\infty$ does not represent any number. The symbol $x\to\infty$ means that x gets larger and larger positively without bounds. The symbol $x\to-\infty$ means that x gets larger and larger negatively without bounds.

When a limit approaches positive or negative infinity, you can not use direct substitution. The fundamental process that you can use to arrive at a limit is the fact that

$$\lim_{x\to\infty}\frac{1}{x}=0$$

To explain this further, if you have one loaf of bread and you divide it equally among every person on the planet, how much does each person get? The logical answer is nothing. Is it actually nothing? No, but it might as well be.

The solution to most functions is to divide all terms by the largest power in the function.

Finding the infinite limit of a polynomial

To find the infinite limit of a polynomial, identify the term with the highest power and find the indefinite limit of that term. The table below provides an example of the various types of indefinite limits of a polynomial.

Type of Exponent *n*	**General situation**	**Particular situation**
Odd integer	$\lim_{x\to\infty} x^n = +\infty$	$\lim_{x\to\infty} x^3 = +\infty$
Even integer	$\lim_{x\to\infty} x^n = +\infty$	$\lim_{x\to\infty} x^4 = +\infty$
Odd integer	$\lim_{x\to-\infty} x^n$	$\lim_{x\to-\infty} x^3 = -\infty$
Even integer	$\lim_{x\to-\infty} x^n$	$\lim_{x\to-\infty} x^4 = +\infty$

If the term is negative, then you must change the sign of your final answer.

Type of Exponent *n*	**General situation**	**Particular situation**
Odd integer	$\lim_{x\to\infty} -x^n = -\infty$	$\lim_{x\to\infty} -x^3 = -\infty$
Even integer	$\lim_{x\to\infty} -x^n = -\infty$	$\lim_{x\to\infty} -x^4 = -\infty$
Odd integer	$\lim_{x\to-\infty} -x^n$	$\lim_{x\to-\infty} -x^3 = \infty$
Even integer	$\lim_{x\to-\infty} -x^n$	$\lim_{x\to-\infty} -x^4 = -\infty$

Example 2

Find the limit: (a) $\lim_{x\to\infty} x - 3x^3$ (b) $\lim_{x\to-\infty} x + 2x^2$ (c) $\lim_{x\to-\infty} 4x - 7x^5$

Solution: (a) The term with the *highest power* is $-3x^3$.

$$\lim_{x\to\infty} x - 3x^3 = \lim_{x\to\infty} (-3x^3) = -3\lim_{x\to\infty} x^3 = -3(+\infty) = -\infty$$

(b) The term with the highest power is $2x^2$

$$\lim_{x\to-\infty} x + 2x^2 = \lim_{x\to-\infty} 2x^2 = 2\lim_{x\to-\infty} x^2 = 2(+\infty) = +\infty$$

(c) The term with the highest power is $-7x^5$

$$\lim_{x\to-\infty} 4x - 7x^5 = \lim_{x\to-\infty} (-7x^5) = -7\lim_{x\to-\infty} x^5 = -7(-\infty) = +\infty$$

Problem 2

Find the limit:

(a) $\lim_{x\to\infty} (3x+1)$

(b) $\lim_{x\to-\infty} (x^3 - 4)$

(c) $\lim_{x\to\infty} (x - x^3)$

(d) $\lim_{x\to-\infty} (-2x^2 - 4x + 3)$

(e) $\lim_{x\to\infty} (x - x^2)$

f) $\lim_{x\to-\infty} (x + 4x^3)$

(g) $\lim_{x\to\infty} (x - 9x^2)$

(h) $\lim_{x\to-\infty} (-2x^7 - 3)$

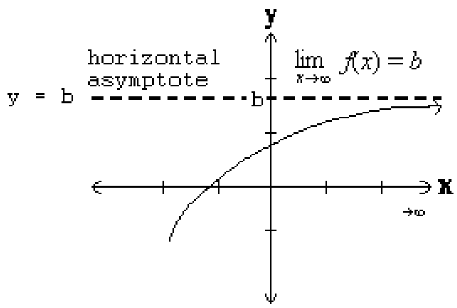
(i) $\lim_{x\to-\infty} (-x^5 - 10x)$

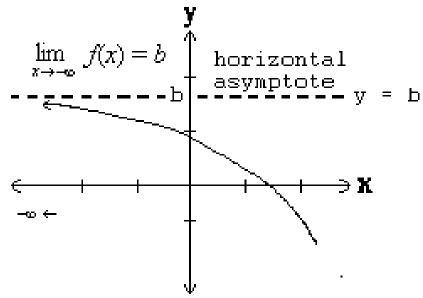
(j) $\lim_{x\to-\infty} (x - 7x^2)$

Sometimes, as x gets larger and larger, $f(x)$ tend to approach a definite limit, say b. We write

$$\lim_{x\to\infty} f(x) = b$$

If either $\lim_{x\to\infty} f(x) = b$ (b is a number) or $\lim_{x\to-\infty} f(x) = b$,
then we call the line y = b a **horizontal asymptote**.





Consider the function f(x) = $\frac{1}{x^2}$:

f(1) = 1
f(10) = 0.01
f(100) = 0.0001
f(1000) = 0.000001
……………………………..
f(1,000,000,000) = 0.000000000000000001

clearly, as x increases, $\frac{1}{x^2}$ is approaching zero, so we write

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

It follows from the above example that,

in general, if k is a constant and n is any positive integer, then

$$\lim_{x\to\infty} \frac{k}{x^n} = 0$$

If k = 2 and n = 1, $\frac{k}{x^n} = \frac{2}{x}$, **then** $\lim_{x\to\infty} \frac{2}{x} = 0.$

If k = 1 and n = 20, $\frac{k}{x^n} = \frac{1}{x^{20}}$ **then** $\lim_{x\to\infty} \frac{1}{x^{20}} = 0$**.**

Example 3

Find the limit: $\lim_{x\to\infty} (3 + \frac{1}{x})$

Solution:

$\lim_{x\to\infty} (3 + \frac{1}{x}) = \lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{1}{x} = 3 + 0 = 3$

Problem 3

Find the limit:

a) $\lim_{x\to\infty} 1 + \frac{1}{x^3}$

b) $\lim_{x\to\infty} (3 - \frac{1}{x})$

c) $\lim_{x\to\infty} \frac{5}{x^7}$

d) $\lim_{x\to-\infty} \frac{1}{x}$

e) $\lim_{x\to-\infty} \frac{-1}{x^2}$

f) $\lim_{x\to\infty} 2 + \frac{1}{x}$

g) $\lim_{x\to-\infty} \frac{17}{x^{200}}$

h) $\lim_{x\to-\infty} \frac{1}{x^2}$

i) $\lim_{x\to-\infty} \frac{1}{x^{10}}$

j) $\lim_{x\to\infty} 1 + \frac{1}{\sqrt{x}}$

k) $\lim_{x\to-\infty} \frac{1}{x-1}$

l) $\lim_{x\to-\infty} \frac{1}{2-x^2}$

If the limit of the numerator and the limit of the denominator does not have a definite value as x approaches $\pm\infty$, ***divide the numerator and denominator by the highest power of x in the function and then take the limit of each term***.

Example 4

$$\lim_{x\to\infty} \frac{3x^2-4x-5}{2x^2-3x-6}$$

Solution:

Since the highest term of the denominator is x^2, divide each term by x^2.

$$\lim_{x\to\infty} \frac{3x^2-4x-5}{2x^2-3x-6} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}-\frac{4x}{x^2}-\frac{5}{x^2}}{\frac{2x^2}{x^2}-\frac{3x}{x^2}-\frac{6}{x^2}} = \lim_{x\to\infty} \frac{3-\frac{4}{x}-\frac{5}{x^2}}{2-\frac{3}{x}-\frac{6}{x^2}} = \frac{3-0-0}{2-0-0} = \frac{3}{2}$$

Problem 4

Find the limit:

a) $\lim_{x\to\infty} \frac{2-5x}{3+2x}$

b) $\lim_{x\to\infty} \frac{2x^2+5x-3}{5x^2-3x+1}$

c) $\lim_{x\to\infty} \frac{x}{x^2+2}$

d) $\lim_{x\to\infty} \frac{5x-3}{7x^2+4x+1}$

e) $\lim_{x\to-\infty} \frac{1-x}{2x-3}$

f) $\lim_{x\to\infty} \frac{x^2+10x+24}{-2x^2-x+9}$

g) $\lim_{x\to-\infty} \frac{1-x}{2x-3}$

h) $\lim_{x\to\infty} \frac{2}{x^2-x-12}$

Exercise 1. Find $\lim_{x\to\infty} \sqrt[3]{\frac{x^2+3}{27x^2-1}}$.

Exercise 2. Find $\lim_{x\to-\infty} \frac{x-2}{\sqrt{x^2+1}}$.

Exercise 3. Find $\lim_{x\to\infty} \frac{x^2+1}{2x-3}$.

Exercise 4. Find $\lim_{x\to\infty} 2x+1-\sqrt{4x^2+5}$.