Week 4: Advanced Session on Optimization

- How to transform a poorly behaved non-linear formulation
  - Into a well-behaved linear formulation
- Reminder: this is an optional session for the curious
  - You won't be tested on this material
- Agenda
  - Remember IDEA's optimization problem from Session 3
  - Look at the algebraic formulation
  - Graph the "min" statement that calculates revenues and see that is not linear
  - Use the graph to show how to modify the formulation to make it linear
  - Update the algebraic formulation to make it linear
  - Implement the linear optimization problem in Excel

Recall IDEA chooses an order quantity Q for Supplier P

- Demand forecast…for example, if the market were strong
  - Uniformly distributed from 6,000 to 14,000 units
- Revenues and costs for supplier P
  - Sales price = 150€
  - Unit cost = 100€
  - Fixed cost to contract with P = 100,000€
- If IDEA were to order Q and demand turned out to be D…
  - Earn 150€ * min{D , Q} in sales revenue
  - Pay 100€ * Q in unit costs
  - Pay 100,000€ fixed cost
- Profit $\pi = 150 * \min\{D , Q\} - 100 * Q - 100{,}000$
- IDEA decides what Q between 4,000 and 10,000 to order

Spreadsheet example with 10 samples, seed 1234

	A	B	C	D	E	F	G
1	IDEA Linear Optimization.xlsx						
2							
3	Quantity (Q) =	10,000.00	units				
4	Fixed Cost =	100,000	euros				
5	Price =	150	euros per unit				
6	Unit Cost =	100	euros per unit				
7							
8		Demand					
9	Number	Sample (D)	Revenue	Fixed Cost	Variable Cost	Profit	
10	1	6,993.19	1,048,979.16	100,000.00	1,000,000.00	(51,020.84)	
11	2	6,052.00	907,800.53	100,000.00	1,000,000.00	(192,199.47)	
12	3	9,115.57	1,367,336.04	100,000.00	1,000,000.00	267,336	
13	4	8,138.25	1,220,737.33	100,000.00	1,000,000.00	120,737	
14	5	11,629.08	1,500,000.00	100,000.00	1,000,000.00	400,000	
15	6	7,884.09	1,182,613.61	100,000.00	1,000,000.00	82,613.61	
16	7	9,729.12	1,459,367.66	100,000.00	1,000,000.00	359,367.66	
17	8	11,983.58	1,500,000.00	100,000.00	1,000,000.00	400,000.00	
18	9	6,990.26	1,048,539.69	100,000.00	1,000,000.00	(51,460.31)	
19	10	9,248.15	1,387,221.90	100,000.00	1,000,000.00	287,221.90	
20							
21					Average Profit =	162,259.59	

=$B$5*MIN($B$3,B10)

=$B$4

=$B$3*$B$6

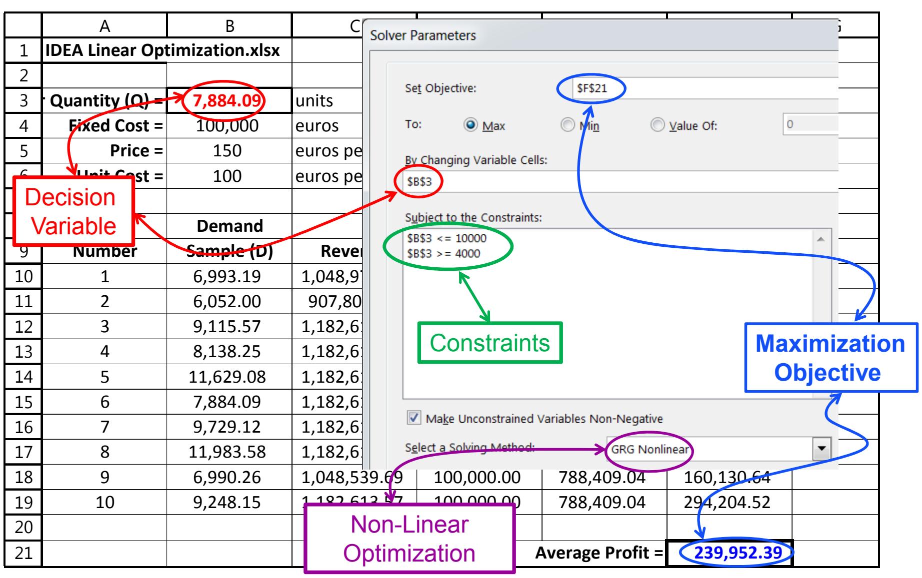
=C10-D10-E10

=AVERAGE(F10:F19)

Decision Variable

Maximization Objective

Optimized version of spreadsheet example

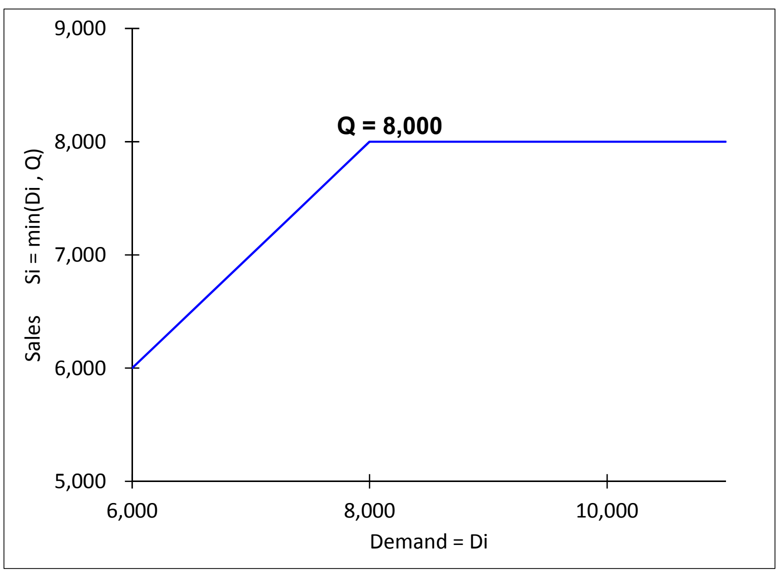


An algebraic formulation of the optimization problem

- We want to find the optimal order quantity: decision variable Q
- So we simulate.   Suppose we generate 10 samples of demand
  - Call the i^{th} sample D_i … where i=1,2,…,10
- If we order Q and the demand sample was D_i
  - Then profit was π_i = 150 * min{D_i , Q} – 100 * Q – 100,000
- We then average the profits over all 10 samples
  - Average $\pi_{average}$ = 1/10 [π_2 + π_2 + … + π_{10}]
- We maximize

$$\max \pi_{average}$$
subject to
$$\pi_i = 150 * \min\{D_i , Q\} - 100 * Q - 100{,}000 \qquad \text{(for i=1,…,10)}$$
$$\pi_{average} = 1/10\, [\pi_2 + \pi_2 + \ldots + \pi_{10}]$$
$$4{,}000 \le Q \le 10{,}000$$

For sales S_i = min{D_i,Q} there's a kink when D_i equals Q



We can avoid this bad type of non-linearity

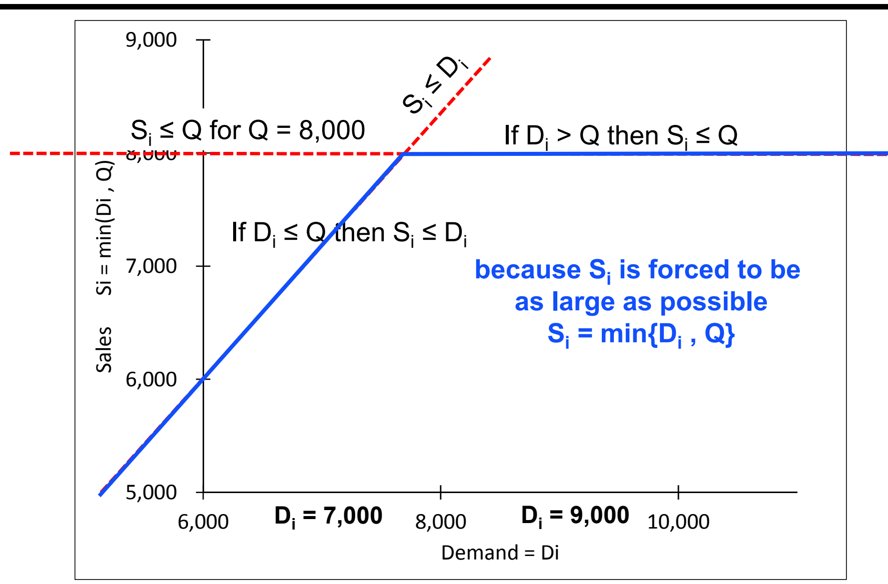
- Original formulation max $\pi_{average}$
  subject to
  $\pi_{average}$ = 1/10 [$\pi_2 + \pi_2 + \ldots + \pi_{10}$]
  π_i = 150 * min{D_i , Q} – 100 * Q - 100,000 (for i=1,…,10)
  4,000 ≤ Q ≤ 10,000

- We'll add 10 decision variables, one for each sample
  - Let S_i be the number of units sold in sample i
  - Now we let have π_i = 150 * S_i – 100 * Q – 100,000 (for i=1,…,10)

- We'll eliminate the 'min's and use two constraints to define each S_i
  - $S_i \le D_i$ (for i=1,…,10)
  - $S_i \le Q$ (for i=1,…,10)

- The constraints work because the optimization maximizes each S_i

The two constraints are evaluated for a given D_i and Q



The full linear formulation

- **Original formulation** max $\pi_{average}$

  subject to

  $\pi_{average}$ = 1/10 [$\pi_2 + \pi_2 + \ldots + \pi_{10}$]

  π_i = 150 * min{D_i , Q} – 100 * Q - 100,000 (for i=1,…,10)

  Q ≥ 0

- **Linear formulation** max $\pi_{average}$

  subject to

  $\pi_{average}$ = 1/10 [$\pi_2 + \pi_2 + \ldots + \pi_{10}$]

  π_i = 150 * min{D_i , Q} – 100 * Q - 100,000 (for i=1,…,10)

  $S_i \leq D_i$ (for i=1,…,10)

  $S_i \leq Q$ (for i=1,…,10)

  Q ≥ 0

Spreadsheet implementation of linear formulation

	A	B	C	D	E	F	G
1	IDEA Linear Optimization.xlsx						
2							
3	Quantity (Q) =	10,000.00	units				
4	Fixed Cost =	100,000	euros				
5	Price =	150	euros per unit				
6	Unit Cost =	100	euros per unit				
7							
8		Demand	Unit				
9	Number	Sample (D)	Sales	Reveue	Fixed Cost	Variable Cost	Profit
10	1	6,993.19	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
11	2	6,052.00	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
12	3	9,115.57	0.00	0.00	100,000.00	1,000,000.0	
13	4	8,138.25	0.00	0.00	100,000.00	1,000,000.0	
14	5	11,629.08	0.00	0.00	100,000.00	1,000,000.0	
15	6	7,884.09	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
16	7	9,729.12	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
17	8	11,983.58	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
18	9	6,990.26	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
19	10	9,248.15	0.00	0.00	100,000.00	1,000,000.00	(1,100,000.00)
20							
21						Average =	(1,100,000.00)

=$B$5*C10

=$B$4

=$B$3*$B$6

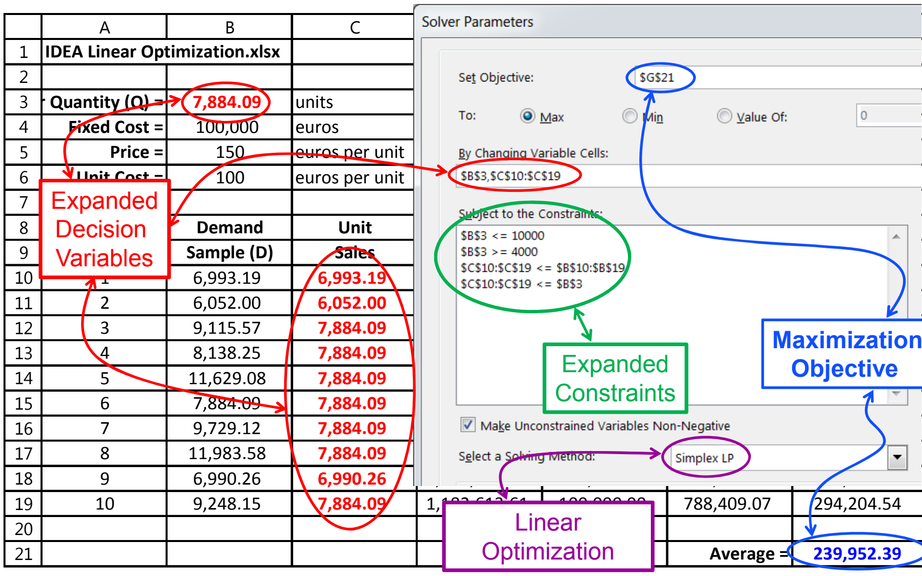
=D10-E10-F10

=AVERAGE(G10:G19)

Decision Variables

Maximization Objective

Optimized version of linear spreadsheet example



Wrap-up for Week 4 Advanced Session 2

- The min(D_i , Q) in our newsvendor formulation was not linear
  - It had a “kink” at D_i = Q
- We added extra decision variables and constraints to work around it
  - Decision variables S_i were the unit sales for each sample i = 1,…,10
  - Constraints $S_i \leq D_i$ so sales would never be more than demand
  - Constraints $S_i \leq Q$ so sales would never be more than the order quantity
- When solver maximizes average profits, each S_i is maximized
  - Even though it would be feasible for an $S_i < D_i$ and $S_i < Q$
  - Maximization forces the S_i to grow until it hits one or the other constraint
- So the problem behaves as if S_i = min(Di , Q)