Fluid Mechanics

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Abstract

This project has a goal to **simulate** the flow of perfect and real **fluids**. This projects will be evaluated on the following skills:

- Modeling and simulating perfect fluids’ behavior leaking from a tank
- Applying Bernoulli’s and Torricelli’s theorems
- Modeling and simulating real fluids’ motion using Lagrangian method (Smoothed Particle Hydrodynamic)
- Studying the effect of modifying fluid properties regarding its behavior

Contents

1. Perfect fluid: Water tower

For this part we were tasked with making a simulation of a dripping perfect incompressible fluid. More precisely we had to make a water tower with a hole from where water would leak.
The height of the water tower, the size of the hole, the volumetric mass and the added pressure on water are tweakable.

1.1 What is the horizontal section S of the tank?

Edges of the tank are $4m$ long therefore its horizontal section S is $16m^2$ wide.

1.2 Determine the initial velocity Vorif_0 of the fluid flow from the orifice

$$P_a + \rho \cdot g \cdot Z_a + \frac{(\rho \cdot V_a)^2}{2} = P_b + \rho \cdot g \cdot Z_b + \frac{(\rho \cdot V_b)^2}{2}$$

with a being the orifice and b being the top of the tank

$$\begin{array}{ll} P_a = P_b = P_{atm} & H = 4m \\ Zb = Za + H & \rho = 1000kg.m^-3 \\ V_b \approx 0m.s^{-1} & g = 9.81m.s^-2 \end{array}$$

$$\frac{\rho \cdot {V_a}^2}{2} = P_b - P_a + \rho \cdot g \cdot Z_b - \rho \cdot g \cdot Z_a + \frac{\rho \cdot {V_b}^2}{2}$$

$$\frac{\rho \cdot {P_a}^2}{2} = \rho \cdot g \cdot H \equiv V_a = \sqrt{2 \cdot g \cdot H}$$

$$Vorif_0 = \sqrt{2 \cdot 9.81 \cdot 4} = \frac{3 \cdot \sqrt{218}}{5} = 8.86m.s^-1$$

1.3 For this velocity Vorif_0, find the maximum distance at which the fluid is projected

Knowing that :

$$v0y = 0m.s^-1 \qquad x0 = 0m$$

We find that :

$$D = \frac{v0x \cdot v0y + v0x \cdot \sqrt{v0y^2 + 2 \cdot g \cdot y0}}{g} + x0$$

$$D = \frac{v0x \cdot \sqrt{2 \cdot g \cdot y0}}{g}$$

$$v0x = \sqrt{2 \cdot g \cdot H}$$

$$D = \frac{\sqrt{g} \cdot \sqrt{H} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{g} \cdot \sqrt{y0}}{g}$$

$$D = \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{y0}$$

$$D = \sqrt{16} \cdot \sqrt{y0}$$

$$D = 4 \cdot \sqrt{y0}$$

1.4 Find the time needed to empty the water tank.

Knowing that :

$$V = -S \cdot dh \qquad Qv = s \cdot v = s \cdot Va$$

$$-S \cdot dh = s \cdot Va \cdot dt$$

We find that :

$$-S \cdot dh = s \cdot \sqrt{2 \cdot g \cdot H} \cdot dt$$

$$\frac{dh}{\sqrt{H}} = \frac{s \cdot \sqrt{2} \cdot \sqrt{g}}{-S} \cdot dt$$

$$\int_H^0 \frac{1}{\sqrt{H}} \cdot dh = \frac{s \cdot \sqrt{2 \cdot g}}{-S} \cdot \int_0^T 1dt$$

$$\left[2 \cdot \sqrt{H}\right]_H^0 = \frac{s\sqrt{2 \cdot g}}{-S} \cdot [t]_0^T$$

Knowing that :

$$S = 16 \qquad H = 4 \qquad s = 4 \cdot 10^{-4}$$

We find that :

$$-2\sqrt{H} = \frac{S\sqrt{2 \cdot g}}{-s} \cdot T$$

$$T = \frac{2 \cdot S \cdot \sqrt{H}}{s \cdot \sqrt{2g}}$$

$$T = \frac{2 \cdot 16 \cdot \sqrt{4}}{4 \cdot 10^{-4} \cdot \sqrt{2 \cdot 9.81}}$$

$$T = 36122s \approx 602min \approx 10h$$

1.5 Find the literal expressions h(t) and Vorif (t)

$$\frac{dh}{\sqrt{H}} = \frac{s \cdot \sqrt{2 \cdot g}}{-S} \cdot t$$

$$\int_{H}^{h(t)} \frac{1}{\sqrt{H}}\, dh \ = \frac{s \cdot \sqrt{2 \cdot g}}{-S} \cdot t$$

$$2 \cdot \left[\sqrt{H}\right]_{H}^{h(t)} = \frac{s \cdot \sqrt{2 \cdot g}}{-S} \cdot t$$

$$2 \cdot (\sqrt{h(t)} - \sqrt{H}) = \frac{s \cdot \sqrt{2 \cdot g}}{-S} \cdot t$$

$$\sqrt{h(t)} = \frac{s \cdot \sqrt{2 \cdot g}}{-2S} \cdot t + \sqrt{H}$$

$$h(t) = (\frac{s \cdot \sqrt{2 \cdot g}}{-2S} \cdot t + \sqrt{H})^2$$

$$v(t) = \sqrt{2 \cdot g \cdot h(t)}$$

1.6 Simulate the change in fluid level in the tank using the equation h(t)

$$OrificeArea = \frac{OrificeDiameter^2}{4} \cdot \pi$$

$$Height = (\frac{OrificeArea \cdot \sqrt{2 \cdot g} \cdot t}{-2 \cdot S} + \sqrt{HeightWaterTower})^2$$

1.9 Draw a curve showing the variation of the volume flow Qv_orif as a function of time

$$Speed = \sqrt{\frac{2 \cdot Pressure}{\rho} + 2 \cdot g \cdot Height}$$

$$x = Speed \cdot t \cdot PositionHole.x$$

$$y = \frac{-g \cdot t^2}{2 \cdot PositionHole.y}$$

$$LandingTime = \frac{\sqrt{2 * g * 3.4}}{g}$$

1.10 Vorif(t) with rho, P, s and H

$$P_a + \rho \cdot g \cdot Z_a + \frac{(\rho \cdot V_a)^2}{2} = P_b + \rho \cdot g \cdot Z_b + \frac{(\rho \cdot V_b)^2}{2}$$

with $P_b - P_a = P$ **and** $V_b \approx 0m.s^{-1}$

$$\rho \cdot g \cdot Z_a + \frac{\rho \cdot {P_a}^2}{2} = P \cdot \rho \cdot g \cdot Z_b$$

$$\frac{\rho \cdot {P_a}^2}{2} = P + \rho \cdot g \cdot Z_b - \rho \cdot g \cdot Z_a$$

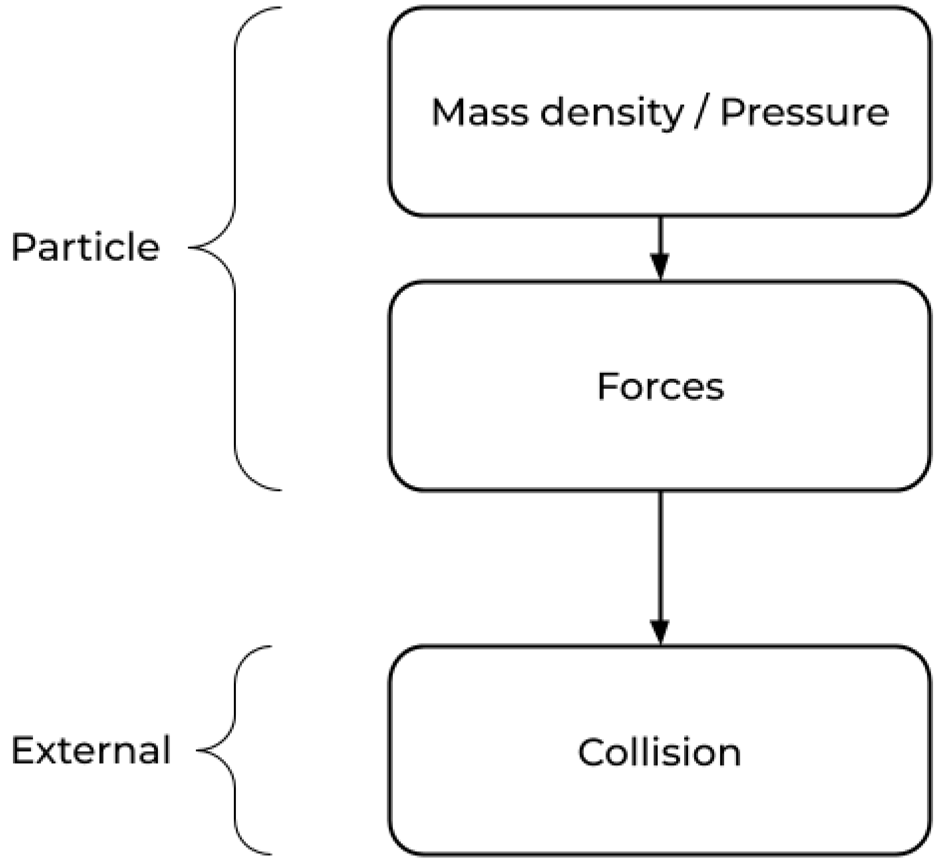
$$\frac{{V_a}^2}{2} = \frac{P}{\rho} + g \cdot Z_b - g \cdot Z_a$$

$$\frac{{V_a}^2}{2} = \frac{P}{g} + g \cdot h(t)$$

$${V_a}^2 = \frac{2 \cdot P}{\rho} + 2 \cdot g \cdot h(t)$$

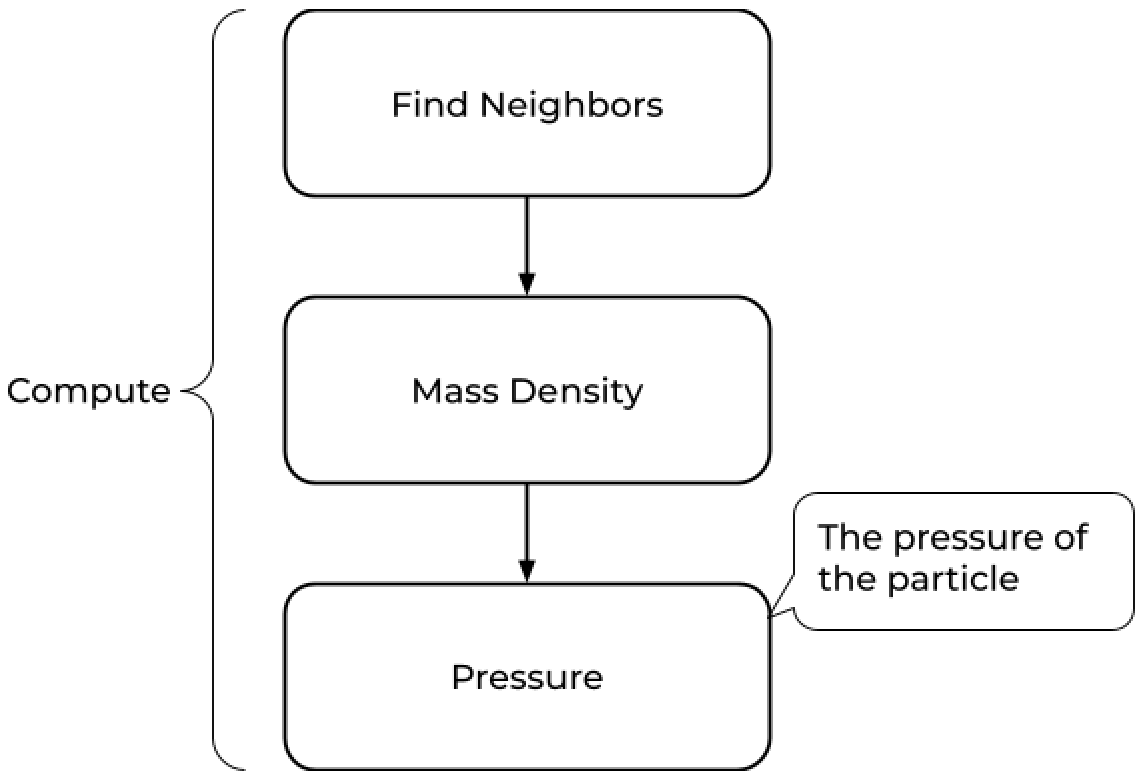
$$V_a = \sqrt{2(\frac{P}{\rho} + g \cdot h(t))}$$

2. Real fluid : SPH



To simulate our fluid as realistically as possible we followed a three steps process itself subdivided in smaller steps.
First of all we define and calculate internal variables then we apply force to it before finally solving our collision when it occurs.
As our simulation is not well optimized, we made it run at a 20th of the real time.

2.1 Internal Variables



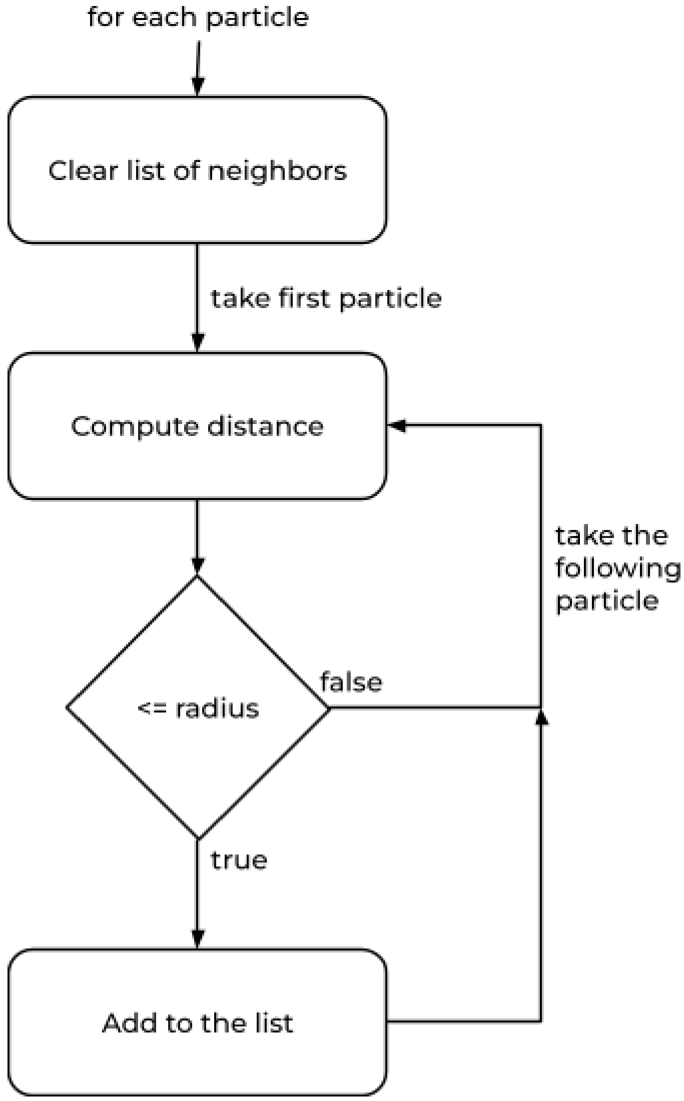
First of all, we need to find all the particles close to our current particle. Each particle possesses a list of all the particles. We deduce which one of those are close enough before adding them to our list of neighbors.

For the mass density: we sum the kernels to each neighbor and then we multiply it by the particle mass.

Finally we calculate the pressure. More precisely we multiply a predefined gas stiffness by the difference of our current particle density and the rest density of the fluid.

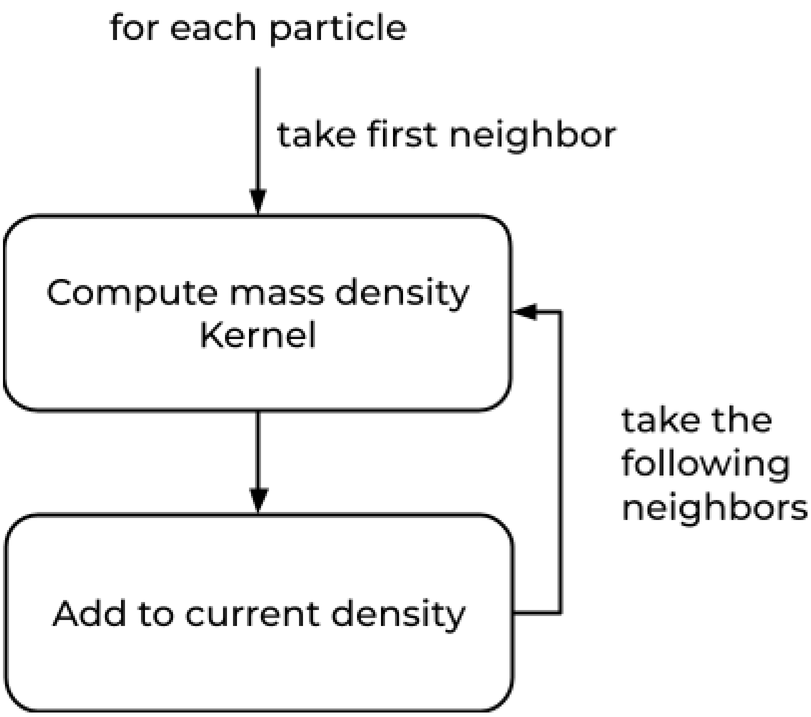
All these steps are done for each of our particles.

2.1.1 Find Neighbors



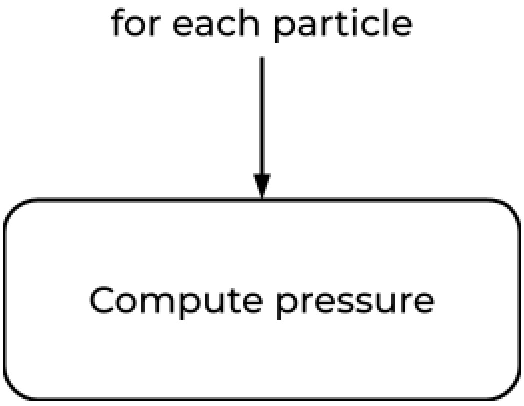
$$Distance = (PotentialNeighborsParticle.position - CurrentParticle.position).magnitude$$

2.1.2 Mass Density



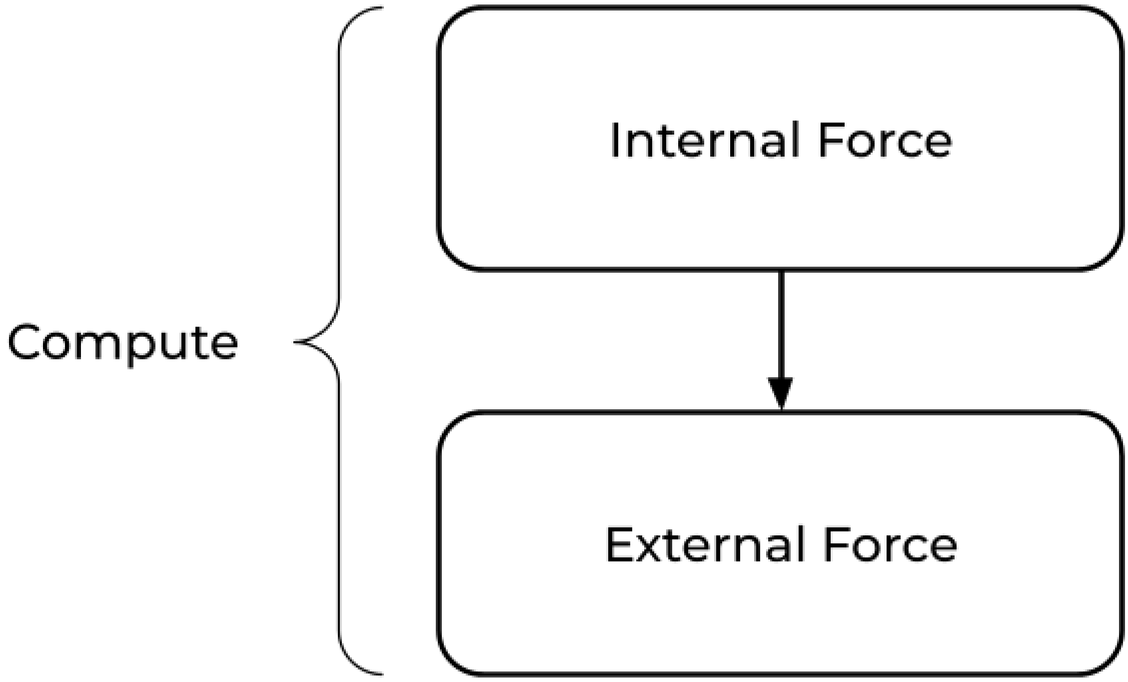
$$MassDensity = \frac{315}{64 \cdot \pi \cdot radius^9 \cdot (radius^2 - distance^2)^3}$$

2.1.3 Pressure



$$Pressure = GasStiffness \cdot (MassDensity - restDensity)$$

2.2 Force



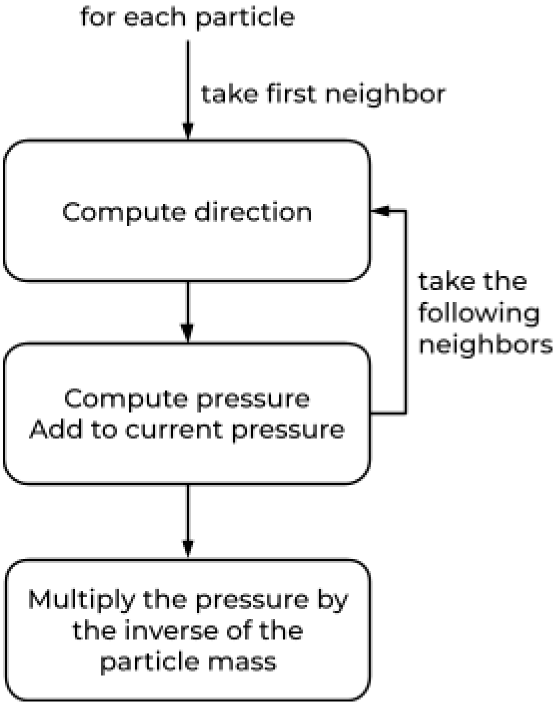
For the internal force, we need the pressure force and the viscosity force.
For the pressure, we calculate the direction between the current particle and its neighbors. Then we deduce pressure force thanks to the pressure previously found for the current particle and its neighbor as well as the mass density and the mass.
For the viscosity, we use the speed of those particles, particle mass and the mass density of our neighbors. All those forces accumulate for each neighbor.

For our external forces, we only have gravity.

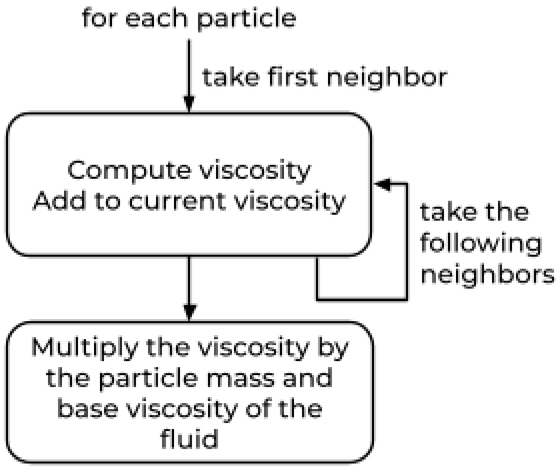
Finally we calculate the acceleration which is equal to the sum of the internal and external forces; in order to find the speed and ultimately the particle position.
To get our speed we multiply the acceleration by delta time.
For the position we multiply the speed by delta time.

2.2.1 Internal Forces

Pressure:



Viscosity:



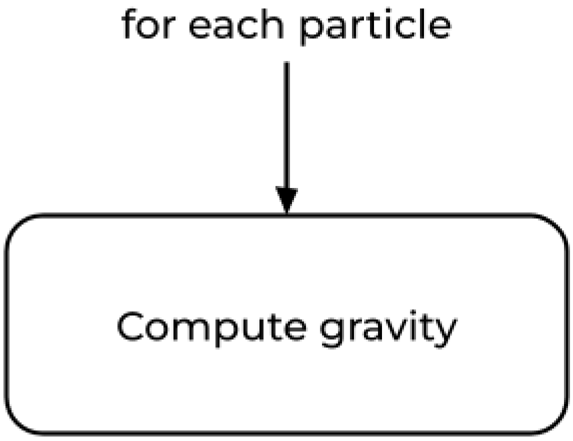
$$PressureForce = \frac{CurrentParticle.pressure + Neighbor.pressure}{2 \cdot Neighbor.MassDensity^2} \cdot GradientKernel$$

$$GradientKernel = \frac{-45}{(\pi \cdot radius^6)} \cdot direction \cdot (radius - distance)^2$$

$$ViscosityForce = \frac{(Neighbor.speed - CurrentParticle.speed)}{MassDensity} \cdot LaplacianKernel$$

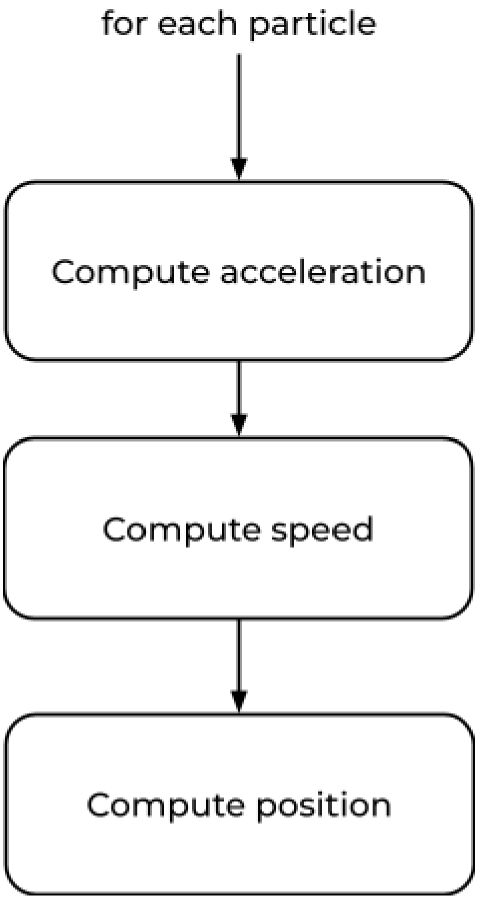
$$LaplacianKernel = \frac{45}{\pi \cdot radius^6} \cdot (radius - distance)$$

2.2.2 External Forces



$$CurrentParticle.Gravity = Gravity$$

2.3 Acceleration, Speed and Position

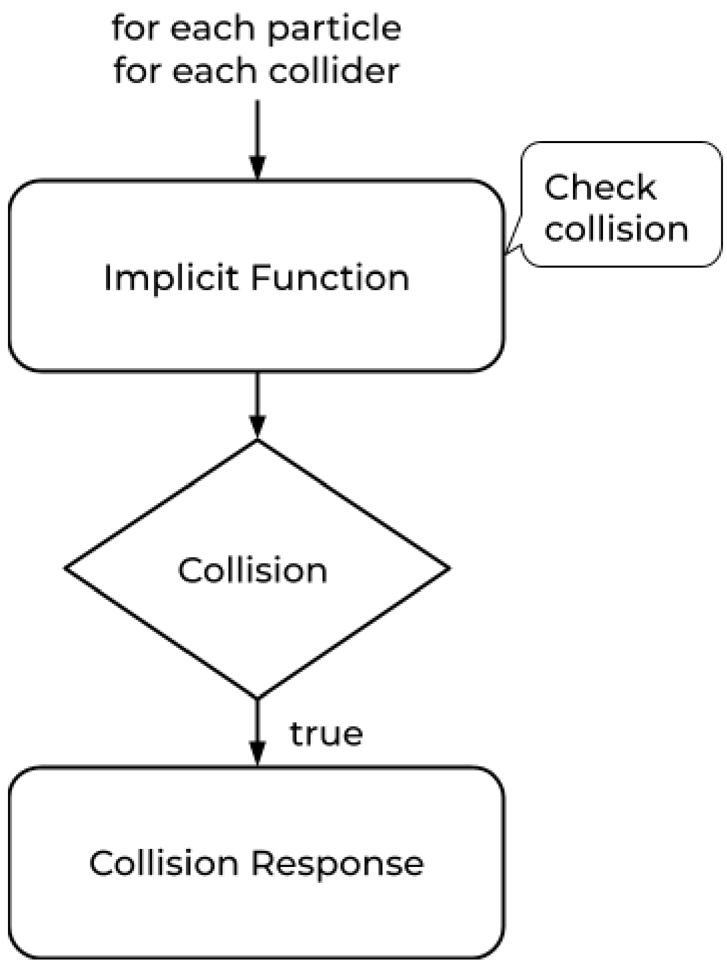


$$Acceleration = \frac{InternalForces}{Particle.MassDensity} + ExternalForces$$

$$Speed = Acceleration \cdot DeltaTime$$

$$Position = Speed \cdot DeltaTime$$

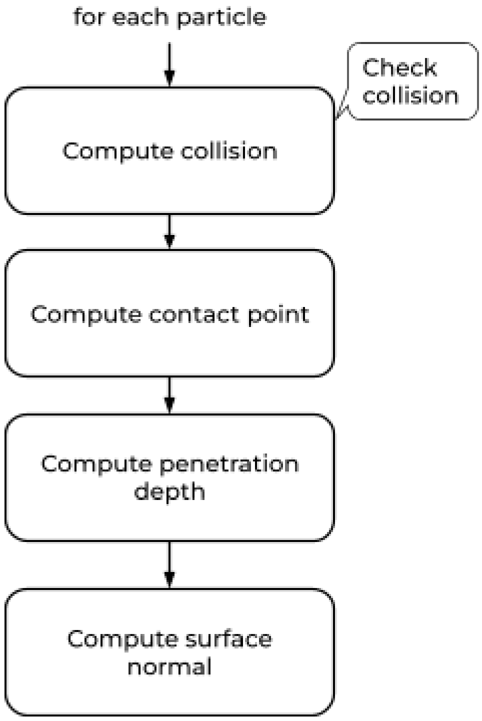
2.4 Collision



We first need to check if we have a collision.
We then compute the contact point, the penetration depth and the surface normal.
Finally, we modify speed and position consequently.

2.4.1 Implicit Function

2.4.1.1 Implicit Function - Sphere Collider



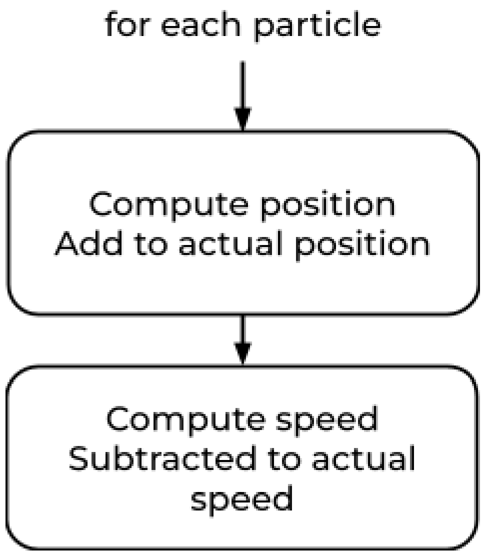
$$Collision = Distance^2 - (Radius + ParticleScale)^2$$

$$ContactPoint = Position + Radius \cdot Direction$$

$$PenetrationDepth = |(Direction.magnitude - Radius - ParticleScale)|$$

$$SurfaceNormal = Sign(Collision) \cdot Direction$$

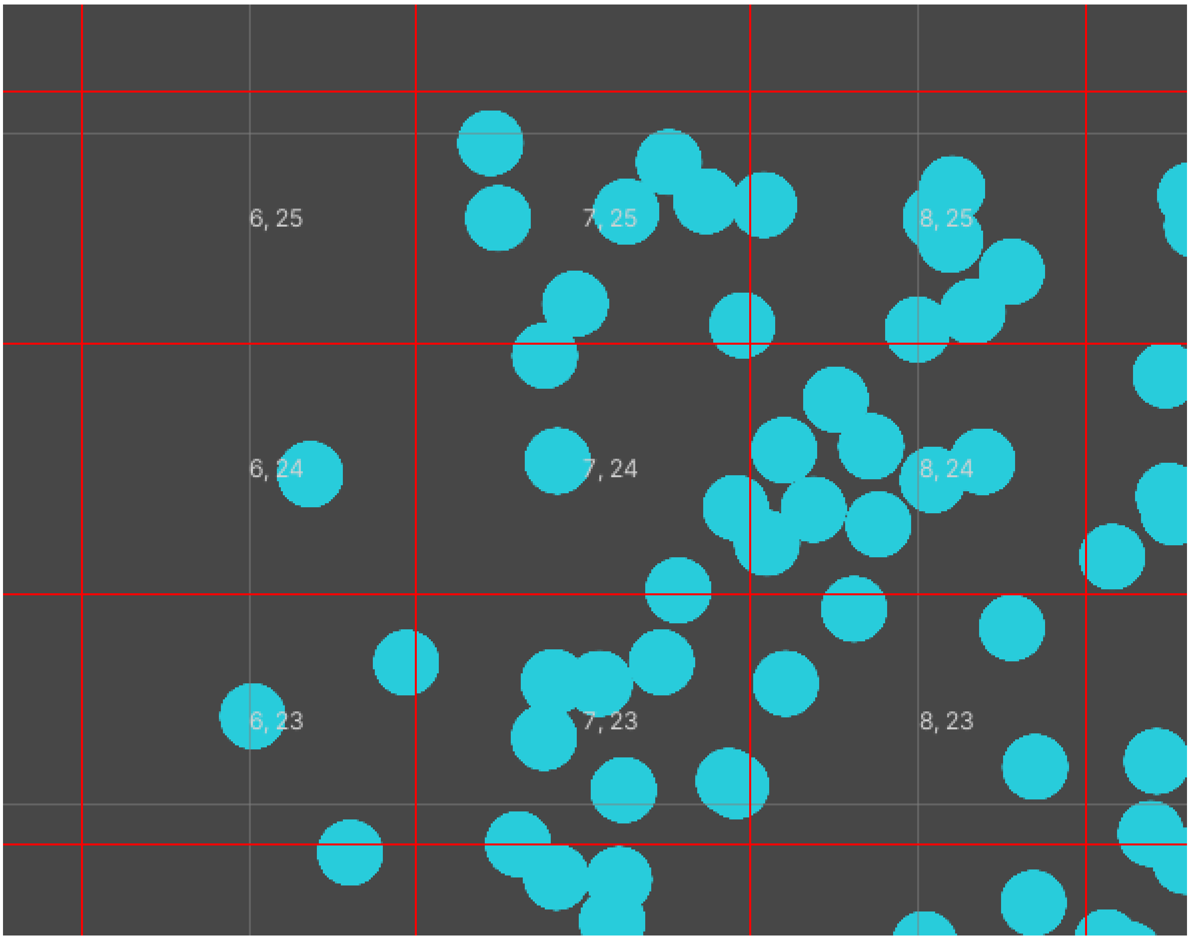
2.4.2 Collision Response



$$Position = PenetrationDepth \cdot SurfaceNormal$$

$$Speed = \frac{1 + Restitution \cdot Depth}{DeltaTime \cdot ||Speed||} \cdot DotProduct(Speed, Normal) \cdot Normal$$

2.5 Potential evolution



Each cell has its list of particles and each has a reference to its cell.
For example:

- **cell 6:25 has no particle**
- **cell 6:24 has 1 particle**

When you want to add other particles to the current particle neighbor, you need to check particles in the same cell and in the surrounding cells.
For example: if you want to check particles in the cell 7:24, you will check all particles in 7:24, 6:25, 7:25, 8:25, 6:24, 8:24, 6:23, 7:23 and 8:23