Estimation of Independent Component Analysis Systems

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Abstract

Although approaches to Independent Component Analysis (ICA) based on characteristic function are usually deemed theoretically elegant, they are known to suffer from severe implementational challenges because of numerical integration steps or selection of tuning parameters. Leveraging results from the continuum Generalized Method of Moments of Carrasco and Florens (2000), I derive an optimally-weighted objective function that can take a tractable form and thus bypass these concerns. The method shares advantages with characteristic-function approaches; it does not require existence of higher-order moments or parametric restrictions and can achieve asymptotic efficiency. The results are adapted to handle a possible first-step that delivers estimated sensors. Finally, the method delivers a specification test which is valuable in many ICA applications. The method's effectiveness is illustrated through simulations, where the estimator outperforms efficient GMM and fastICA, and an application to the estimation of Structural Vector Autoregressions (SVAR), a popular model in the econometric time series literature.

Keywords: Independent Component Analysis, Structural VAR, characteristic function, continuum GMM.

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1 Introduction

Independent Component Analysis (ICA; Comon (1994); Eriksson and Koivunen (2003*b*)) is a popular method which finds applications in fields as diverse as signal processing, machine learning, or Structural Vector Autoregressions (SVAR).

The standard model posits that an observed vector (of "sensors") at time t, η_t, is generated through $\eta_t = \Theta\varepsilon_t$ where ε_t has independent entries and Θ is an unknown matrix. ε_t, sometimes referred as the "sources", is a vector containing the latent factors that affects the system through the mixing matrix, Θ.

Various methods have been proposed to uncover the unmixing matrix, Θ^{-1}. Early methods typically attempted to maximize a measure of non-normality or use maximum entropy, often making use of third- and fourth-order moments. A popular and fast method based on non-Gaussianity is the fastICA algorithm (Oja and Yuan, 2006). A list of algorithms and applications can be found in in Hyvärinen, Karhunen and Oja (2001).

A broad estimation strategy is to rely on maximum likelihood or related methods. Many papers (Bach and Jordan, 2002; Samarov, Tsybakov et al., 2004; Chen, Bickel et al., 2006; Ilmonen, Paindaveine et al., 2011; Samworth, Yuan et al., 2012; Ablin, Cardoso and Gramfort, 2018) have assumed parametric, smooth, or log-concave densities to devise an estimation strategy for the unmixing matrix. Nevertheless, misspecification bias is a concern as family of distribution is typically unknown and assumptions of smoothness, unimodality, or absence of atoms are not innocuous in applications. In addition, many of these approaches require a choice of tuning parameter or are not straightforward to implement.

I propose a nonparametric approach to estimate the unmixing matrix based on the empirical characteristic function that does not require existence of higher-order moments or distributional restrictions and can achieve asymptotic efficiency. I also explicitely allow η_t to be (consistently) estimated rather than directly observed to account for vanishing noise or an estimation step, as happens for instance in Structural Vector Autoregressions (SVAR) applications where the ICA system is derived through a first-step regression.

A related approach is Eriksson and Koivunen (2003*a*) (see also Chen and Bickel (2005) for the asymptotic properties), who obtained the same identifying equation function in terms of the characteric function. Although this approach shares some of the benefits (such as the absence of parametric restrictions or higher-order moment assumptions), the approach developed in this paper derives straightforward asymptotics, allows for optimal weighting, provide an easy-to-implement objective function that can bypass numerical integration steps, is compatible with noise due a first-step such as autoregressions in SVAR, and delivers a test of the ICA system's validity.

2 Estimation of ICA systems

2.1 Identification of Θ

Identification of systems of linear combinations of unobserved independent variables has been extensively discussed in the literature and extended to general setups (Reiersøl, 1950; Comon, 1994; Bonhomme and Robin, 2009; Ben-Moshe, 2016). In particular, Eriksson and Koivunen (2003*b*) provide general results about identification of ICA systems.

In the square matrix case, the identification result can be easily derived by noting that two observationally equivalent systems (Θ, ε) and $(\Theta^*, \varepsilon^*)$ must satisfy $\Theta\varepsilon =^d \Theta^*\varepsilon^*$ and thus $\varepsilon =^d \Theta^{-1}\Theta^*\varepsilon^*$, where $=^d$ denotes equality in distribution. But if ε is to have independent entries, the Darmois–Skitovich theorem (Darmois, 1953; Skitovitch, 1953) requires trivial linear combinations in the absence of normality. As a result, $\Theta^{-1}\Theta^*$ must be a (possibly scaled) permutation matrix. This well-known result is summarized in the following theorem:

Theorem 2.1 (Identification)**.** *Consider the system $\eta_t = \Theta\varepsilon_t$. Θ is identified up to column scale and permutations from the distribution of η_t if (i) it is invertible, and (ii) the vector ε_t contains independent random variables among which at most one is normal.*

Since identification is obtained only up to scale and column permutations, a normalization and an order are still needed. Absent application-specific knowledge to assign identities to the shocks, the choice can be made out of convenience.

I use a unit norm normalization for each column of Θ and denote the corresponding compact parameter space by $\overline{\Theta}$. The constraint can be explicited through polar coordinates. For instance, in the 2×2 case, $\Theta = \begin{pmatrix} \cos(\gamma_1) & \cos(\gamma_2) \\ \sin(\gamma_1) & \sin(\gamma_2) \end{pmatrix}$, where γ_1, γ_2 lie between 0 and π. The columns can be easily ordered by setting, say, $\gamma_1 < \gamma_2$, with straightforward adaptations to higher-dimensional settings using lexicographic ordering. As a result, the properties can be discussed in terms of γ, with $\Theta = \Theta(\gamma)$.

In what follows, the system is assumed to be identified by imposing conditions (i) and (ii) of Theorem 2.1 and the normalizations described above.

2.2 The class of estimators based on characteristic functions

I will make use of the following notation. I define $\vec{s}$ to be a $1 \times n$ row vector. φ_X denotes the characteristic function of the random vector X, *i.e.* $\varphi_X(\vec{s}) \stackrel{\mathsf{def}}{=} \mathbb{E}[e^{i\vec{s}x}]$. Real and imaginary parts are denoted by $\Re$ and $\Im$, respectively. The j^{th} column of Θ is an $n \times 1$ vector denoted by $\Theta_{\cdot j}$.

2.2.1 The estimator

By independence of sources, the observed variables' distribution is related to the distribution of their unobserved counterparts through

$$\varphi_\eta(\vec{s}) = \prod_{j=1}^{n} \varphi_{\varepsilon_j}(\vec{s}\Theta_{\cdot j}) \tag{1}$$

while each source's characteristic function can be recovered from that of the sensors via

$$\varphi_{\varepsilon_j}(s) = \varphi_\eta(s\Theta^{-1}_{j\cdot}) \tag{2}$$

where $\Theta^{-1}_{j\cdot}$ is the j-th row in Θ^{-1}.

A functional equation for the characteristic function of η in terms of the unknown Θ can be obtained using the last two expressions. First, define $P_j \stackrel{\mathsf{def}}{=} \Theta_{\cdot j}\Theta^{-1}_{j\cdot}$ whose immediate properties are $P_j P_k = \mathbb{1}_{j=k} P_j \ \forall j, k$, $\sum_{j=1}^n P_j = I_n = \sum_{j=1}^n P_j'$, and $\text{rank}(P_j) = 1$.

In addition, the collection of P_j is isomorphic to Θ once a normalization on Θ is imposed. Next, substituting (2) into (1) yields an expression which directly links the characteristic function of η to Θ:

$$\varphi_\eta(\vec{s}) = \prod_{j=1}^{n} \varphi_\eta(\vec{s}P_j) \tag{3}$$

This result was also obtained in Eriksson and Koivunen (2003*a*).

This can be expressed as the condition $q(\vec{s},\gamma) \stackrel{\text{def}}{=} \varphi(\vec{s}) - \prod_{j=1}^{n} \varphi(\vec{s}P_j) = 0$. If one assumes that the characteristic function does not vanish[1] or restricts the analysis to a neighborhood of the origin, then the criterion can be expressed in terms of the cumulant generating function: $\sum_{j=0}^{n} a_j \ln\left(\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}\right) = 0$ with $a_j = (-1)^{\mathbb{1}_{j>0}}$ and $P_0 = I$. Both criteria can be used to form an optimally-weighted estimator of Θ with closely-related expressions; the log form however requires some assumptions to handle zeros.

Let $\hat{\varphi}(\vec{s}) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}\eta_t}$ be the empirical counterpart of φ_η and let $q_T(\vec{s},\gamma) \stackrel{\text{def}}{=} \hat{\varphi}(\vec{s}) - \prod_{j=1}^{n} \hat{\varphi}(\vec{s}P_j)$. Eriksson and Koivunen (2003*a*) consider a criterion based on minimizing integrals of the form

$$\int |q_T(\vec{s},\gamma)|^2 w(\vec{s})\mathrm{d}\vec{s} = \int (\Re q_T(\vec{s},\gamma)^2 + \Im q_T(\vec{s},\gamma)^2) w(\vec{s})\mathrm{d}\vec{s} \tag{4}$$

using some weight function w. They then propose tractable weighting schemes to facilitate integration. The approach is neat but suffers from a couple of shortcomings. First, it does not allow for weighting interactions at different points $\vec{s}$, which precludes efficiency. Second, the weights are chosen sub-optimally in order to avoid numerical integration, which can lead to further efficiency loss.

In analogy with the formation of a quadratic form for estimating equations or Generalized Method of Moments (GMM; Hansen (1982)), consider

$$Q_T(\gamma) = \int\int \begin{pmatrix} \Re q_T(\vec{s},\gamma) & \Im q_T(\vec{s},\gamma) \end{pmatrix} W(\vec{s},\vec{r}) \begin{pmatrix} \Re q_T(\vec{r},\gamma) \\ \Im q_T(\vec{r},\gamma) \end{pmatrix} \pi(\mathrm{d}\vec{r})\pi(\mathrm{d}\vec{s}) \tag{5}$$

where π is a probability measure and W is a (symmetric, positive semi-definite) weighting matrix.

Such a criterion can be induced by a linear operator B by considering the norm $\|Bq_T\|_H$, where H is a Hilbert space of square-integrable functions with scalar product $< f, g >= \int f(\vec{s})\overline{g(\vec{s})}\pi(\vec{s})$. This is similar to Carrasco and Florens (2000, 2002)'s objective function for a continuum of moment conditions, except that q is obtained from a transformation of moments. The following section establishes the asymptotic properties of estimators obtained by minimizing (5) for any weighting matrix. The weighting matrix W_T is allowed to be an estimate of some target matrix W. The results are also made applicable to the case where η_t is not observed but can be obtained from a first step, as in SVAR applications.

[1] Assumptions excluding zeros are common, *e.g.*, literature on nonparametric deconvolution (see Schennach (2004) and references therein).

2.2.2 Asymptotics

As it appears from the consistency proof, it is necessary to exclude sequences toward degenerate matrices if η_t is to be estimated. I make use of the following assumption, which strengthens slightly the invertibility assumption by bounding the matrix an ϵ away from degeneracy.

Definition 2.1 (ϵ-invertibility)**.** The eigenvalues of Θ are bounded away from 0 by ϵ: $|\lambda_i(\Theta)| \geq \epsilon > 0$.

In some applications the η_t are directly observed, so that the assumption is not necessary and can be reduced to the usual full-rank condition on Θ.

In what follows, γ_0 refers to the true value of γ and $q_0(\vec{s}) \stackrel{\text{def}}{=} q(\vec{s}, \gamma_0)$.

The proposed estimator is consistent for the true value γ_0, as summarized by the following Theorem which is proven in the appendix.

Theorem 2.2 (Consistency)**.** *The estimator $\hat{\gamma} \stackrel{\text{def}}{=} \arg\min_\gamma Q_T(\gamma)$ is consistent for γ_0 if (i) there is a consistent estimator, $\hat{\eta}_t$, of η_t that satisfies $\frac{1}{T}\sum_{t=1}^{T}|\hat{\eta}_t - \eta_t| \to^p 0$ and $\Theta = \Theta(\gamma)$ is ϵ-invertible, and (ii) W_T converges uniformly to a positive definite matrix W that satisfies $\sup_{s,t} W < \infty$.*

I now turn to the derivation of the asymptotic distribution of the estimator.

Theorem 2.3 (Asymptotic Normality)**.** *$\hat{\gamma}$ is asymptotically normally distributed. Specifically, if (i) $\hat{\eta}_t$ is obtained from a first-stage where η_t is an independent error term so that $\hat{\eta}_t - \eta_t = w_t(\hat{\beta} - \beta)$ for some consistent estimator $\hat{\beta}$ of β and $\Theta = \Theta(\gamma)$ is ϵ-invertible, (ii) W_T converges uniformly to a positive definite matrix W, and (iii) ε_t has second moments.*

Then, $\sqrt{T}(\hat{\gamma} - \gamma_0) \to^d N(0; BVB')$ where

$$B \stackrel{\text{def}}{=} \left[\int \begin{pmatrix} \Re\frac{\partial q_0}{\partial\gamma}(\vec{r}) & \Im(\frac{\partial q_0}{\partial\gamma}(\vec{r}) \end{pmatrix} W(\vec{r}, \vec{s}) \begin{pmatrix} (\Re\frac{\partial q_0}{\partial\gamma}(\vec{s}))' \\ (\Im\frac{\partial q_0}{\partial\gamma}(\vec{s}))' \end{pmatrix} \pi(d\vec{r})\pi(d\vec{s}) \right]^{-1} \tag{6}$$

and

$$\begin{aligned} V \stackrel{\text{def}}{=} \int\int &\begin{pmatrix} \Re\frac{\partial q_0}{\partial\gamma}(\vec{r}) & \Im\frac{\partial q_0}{\partial\gamma}(\vec{r}) \end{pmatrix} W(\vec{r}, \vec{s}) K(\vec{r}, \vec{s}) \\ &W(\vec{r}, \vec{s})' \begin{pmatrix} (\Re\frac{\partial q_0}{\partial\gamma}(\vec{s}))' \\ (\Im\frac{\partial q_0}{\partial\gamma}(\vec{s}))' \end{pmatrix} \pi(d\vec{r})\pi(d\vec{s}) \end{aligned} \tag{7}$$

where $K(\vec{r}, \vec{s})$ is the covariance function for the real and imaginary parts of q.

As usual, the iid assumption can be weakened to ergodicity and strict stationarity. This is done with a natural adaptation of the proof, noting that results about convergence of characteristic functions have generalizations to ergodic, stationary settings (Feuerverger, 1990). Two corrections may apply to the asymptotic variance. First, the use of a long-run variance might be warranted since uncorrelatedness of sensors does not translate to that of their empirical characteristic functions. Second, estimation of η_t must be accounted for since its disappearance hinges on the vanishing of the term $\frac{\partial q(\gamma_0, \eta_t(\bar{\beta}))}{\partial \beta}(\hat{\beta} - \beta)$ (following notation in the proof), which relied on independence.

Finally, since Θ is usually the parametrization of interest, the Delta method can be applied and asymptotic normality ensues for the corresponding estimator with the same asymptotic variance scaled by $\partial_\gamma \Theta(\gamma)$.

2.2.3 The asymptotic variance

Due to the asymptotic linear representation shown while establishing asymptotic normality, the variance can be approximated via bootstrap. Alternatively, the asymptotic variance can be consistently estimated. Indeed, most terms appearing in its expression have a natural estimator based on the use of $\hat{\gamma}$ in place of γ_0 and the use of the sample counterparts of population quantities. We have

$$\frac{\partial q_0}{\partial \gamma'} = -\varphi(\vec{s}) \sum_{j=1}^{n} \frac{\partial \ln(\mathbb{E}[e^{i\vec{s}P_j\eta_t}])}{\partial \gamma'} = -\varphi(\vec{s}) \sum_{j=1}^{n} \frac{\mathbb{E}[e^{i\vec{s}P_j\eta_t}(\eta_t' \otimes \vec{s})\frac{\partial \operatorname{vec}(P_j)}{\partial \gamma'}]}{\mathbb{E}[e^{i\vec{s}P_j\eta_t}]} \tag{8}$$

As an illustration of the differentiated term, consider the two-dimensional case:

$$\Theta = \begin{pmatrix} \cos(\gamma_1) & \cos(\gamma_2) \\ \sin(\gamma_1) & \sin(\gamma_2) \end{pmatrix} \tag{9}$$

Tedious but straightforward algebra yields

$$\Theta^{-1} = \frac{1}{\sin(\gamma_2 - \gamma_1)} \begin{pmatrix} \sin(\gamma_2) & -\cos(\gamma_2) \\ -\sin(\gamma_1) & \cos(\gamma_1) \end{pmatrix} \tag{10}$$

$$P_1 = \frac{1}{\sin(\gamma_2 - \gamma_1)} \begin{pmatrix} \cos(\gamma_1)\sin(\gamma_2) & -\cos(\gamma_1)\cos(\gamma_2) \\ \sin(\gamma_1)\sin(\gamma_2) & -\sin(\gamma_1)\cos(\gamma_2) \end{pmatrix} \tag{11}$$

$$P_2 = \frac{1}{\sin(\gamma_2 - \gamma_1)} \begin{pmatrix} -\sin(\gamma_1)\cos(\gamma_2) & \cos(\gamma_1)\cos(\gamma_2) \\ -\sin(\gamma_1)\sin(\gamma_2) & \cos(\gamma_1)\sin(\gamma_2) \end{pmatrix} \tag{12}$$

$$\frac{\partial\,\mathrm{vec}(P_1)}{\partial\gamma'} = \frac{1}{\sin^2(\gamma_2-\gamma_1)}\begin{pmatrix} \sin(\gamma_2)\cos(\gamma_2) & -\sin(\gamma_1)\cos(\gamma_1) \\ \sin^2(\gamma_2) & -\sin^2(\gamma_1) \\ -\cos^2(\gamma_2) & \cos^2(\gamma_1) \\ -\sin(\gamma_2)\cos(\gamma_2) & \sin(\gamma_1)\cos(\gamma_1) \end{pmatrix} \tag{13}$$

$$\frac{\partial\,\mathrm{vec}(P_2)}{\partial\gamma'} = \frac{\partial\,\mathrm{vec}(I-P_1)}{\partial\gamma'} = -\frac{\partial\,\mathrm{vec}(P_1)}{\partial\gamma'} \tag{14}$$

Hence, replacing expectations by sample averages and using consistent estimators in place of unknown parameters delivers a consistent estimate of $\frac{\partial q}{\partial\gamma'}$.

It remains to consider the central term in more detail. The (centered) log empirical characteristic function converges to a mean zero process with covariance function $\frac{\varphi(\vec{u}+\vec{v})}{\varphi(\vec{u})\varphi(\vec{v})}-1$ and, since $\overline{\ln(\varphi(\vec{s}))} = \ln(\varphi(-\vec{s}))$, the covariance functions is enough to characterize the complex process.

With $\varphi(\vec{s}) \stackrel{\text{def}}{=} \begin{pmatrix} \varphi_\eta(\vec{s}) \\ \varphi_\eta(P_1\vec{s}) \\ \ldots \\ \varphi_\eta(P_n\vec{s}) \end{pmatrix}$ and $A \stackrel{\text{def}}{=} \begin{pmatrix} a_0 & a_1 & \ldots & a_n \end{pmatrix}$, consider

$$\mathbb{C}ov\left(\begin{pmatrix}\Re q(\gamma_0,\vec{u}) \\ \Im q(\gamma_0,\vec{u})\end{pmatrix};\begin{pmatrix}\Re q(\gamma_0,\vec{v}) \\ \Im q(\gamma_0,,\vec{v})\end{pmatrix}\right) = (I\otimes A)\mathbb{C}ov\left(\begin{pmatrix}\Re\varphi(\vec{u}) \\ \Im\varphi(\vec{u})\end{pmatrix};\begin{pmatrix}\Re\varphi(\vec{v}) \\ \Im\varphi(\vec{v})\end{pmatrix}\right)(I\otimes A')$$

Letting $\vec{u}_j \stackrel{\text{def}}{=} P_j\vec{u}$, properties of the complex-normal distribution yield the relationships

$$\mathbb{C}ov(\Re\varphi(\vec{u}_j);\Re\varphi(\vec{v}_k)) = \frac{1}{2}\Re\left(\frac{\varphi(\vec{u}_j-\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(-\vec{v}_k)} + \frac{\varphi(\vec{u}_j+\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(\vec{v}_k)} - 2\right) \tag{15}$$

$$\mathbb{C}ov(\Re\varphi(\vec{u}_j);\Im\varphi(\vec{v}_k)) = \frac{1}{2}\Im\left(-\frac{\varphi(\vec{u}_j-\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(-\vec{v}_k)} + \frac{\varphi(\vec{u}_j+\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(\vec{v}_k)}\right) \tag{16}$$

$$\mathbb{C}ov(\Im\varphi(\vec{u}_j);\Re\varphi(\vec{v}_k)) = \frac{1}{2}\Im\left(\frac{\varphi(\vec{u}_j-\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(-\vec{v}_k)} + \frac{\varphi(\vec{u}_j+\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(\vec{v}_k)} - 2\right) \tag{17}$$

$$\mathbb{C}ov(\Im\varphi(\vec{u}_j);\Im\varphi(\vec{v}_k)) = \frac{1}{2}\Re\left(\frac{\varphi(\vec{u}_j-\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(-\vec{v}_k)} - \frac{\varphi(\vec{u}_j+\vec{v}_k)}{\varphi(\vec{u}_j)\varphi(\vec{v}_k)}\right) \tag{18}$$

Hence, the use of empirical characteristic functions as an estimate of their population counterparts allows the construction of consistent estimator of the central term of the asymptotic variance.

3 Efficient estimation

3.1 Optimal objective function

This expression is reminiscent of the extension of GMM to a continuum of moment conditions developed by Carrasco and Florens (2000). While $\Re q_T(\vec{s},\gamma)$ are $\Im q_T(\vec{s},\gamma)$ are not *per se* sample moments, they do have a zero asymptotic counterpart and the results of Carrasco and Florens (2000) can be adapted to the present framework.

Specifically, denoting real and imaginary parts of q_T by $g_j, j=1,2$, their objective function $\|B_n(q_T(\gamma))\|$ matches Q_T when B_n is an integral operator $(B_n g)(\vec{s}) = \left(\sum_{l=1}^{2}\int b^{jl}(\vec{s},\vec{r})g_l(\vec{r})\mathrm{d}\vec{r}\right)_{j=1,2}$ that generates a weighting matrix through $W_T^{jk}(\vec{s},\vec{r}) = \sum_{l,l'=1,2}\int b^{jl}(\vec{u},\vec{s})b^{jl}(\vec{u},\vec{r})d\vec{u}$.

As Carrasco and Florens (2000) establish, efficient estimation requires inverting a covariance operator $C : h \to \int K(\vec{r},\vec{s})h(\vec{s})\mathrm{d}\vec{s}$ which is not possible on the whole reference space. They propose a regularized sample version $C_T^{\alpha_T}$ where α_T is a smoothing parameter that disturbs the eigenvalues of C. The choice of α_T has been discussed in subsequent papers, see Carrasco and Kotchoni (2017) and Amengual, Carrasco and Sentana (2020).

Eventually, the optimal estimator minimizes

$$\sum_{m=1}^{T}\frac{\mu_{m;T}}{\mu_{m;T}^2+\alpha_T}\left|< q_T,\phi_{m;T} >\right|^2 \tag{19}$$

where $\mu_{m;T}$ and $\varphi_{m;T}$ are the eigenvalues and eigenfunctions of C_T. Moreover, under the assumptions of Theorem 3.2 and provided $\alpha_T \to 0$ while $\alpha_T^3 T \to \infty$, the expected simplification of the asymptotic variance occurs so that the asymptotic distribution becomes

$$\sqrt{T}(\hat{\gamma}-\gamma_0) \to^d N\left(0;\left|\left|\frac{\partial q}{\partial\gamma}\right|\right|_C^{-2}\right) =^d N\left(0;B\right) \tag{20}$$

where the weighting matrix to compute B is now based on the inverted covariance operator.

Though the more sophisticated objective function implies a more computationally intensive procedure due to integration, the estimator is in practice obtained by minimizing equation (19) and the main computational burden arises from evaluating a matrix and computing its eigen-decomposition. Furthermore, using the efficient form of the estimator carries significant benefits.

It removes the need to specify an arbitrary form of the weighting matrix and furthers efficiency. In particular, Carrasco and Florens (2000) argue that using a continuum of moment conditions allows to close the efficiency gap between GMM and MLE.

3.2 Implementation

Let $\phi_j(\vec{s}) \overset{\text{def}}{=} \prod_{m \neq j} \hat{\varphi}(\vec{s}_m)$ with $\phi_0(\vec{s}) = 1$. The covariance kernel is given by $K(\vec{r}, \vec{s}) = \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j \phi_j(\vec{s}) \phi_k(-\vec{r})(\varphi(\vec{r}_k - \vec{s}_j) - \varphi(\vec{r}_k)\varphi(\vec{s}_j))$, so that the covariance operator reads

$$
\begin{aligned}
(Cg)(\vec{r}) &= \int \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \phi_j(-\vec{s}) \phi_j(\vec{r}) \frac{1}{T} \sum_{\tau=1}^{T} \left(e^{i\vec{r}P_k\eta_\tau} e^{-i\vec{s}P_j\eta_\tau} - \varphi(-\vec{s}_j) e^{i\vec{r}P_k\eta_\tau} \right) g(\vec{s}) \ \pi(\mathrm{d}\vec{s}) \\
&= \frac{1}{T} \sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \phi_k(\vec{r}) \int \sum_{\tau=1}^{T} e^{i\vec{r}_k\eta_\tau} (e^{-i\vec{s}_j\eta_\tau} - \hat{\varphi}(-\vec{s}_j)) \phi_j(\vec{s}) g(\vec{s}) \ \pi(\mathrm{d}\vec{s}) \\
&= \frac{1}{T} \sum_{\tau=1}^{T} c_\tau \sum_{k=0}^{n} a_k \phi_k(\vec{r}) e^{i\vec{r}P_k\eta_\tau} \int \sum_{j=0}^{n} a_j \phi_j(-\vec{s}) \left(e^{-i\vec{s}P_j\eta_\tau} - \hat{\varphi}(-\vec{s}_j) \right) g(\vec{s}) \ \pi(\mathrm{d}\vec{s})
\end{aligned}
$$

which implies that the eigenfunctions g take the form

$$
g(\vec{r}) = \frac{1}{T} \sum_{\tau=1}^{T} c_\tau \sum_{k=0}^{n} a_k \phi_k(\vec{r}) e^{i\vec{r}P_k\eta_\tau} \tag{21}
$$

The eigenvalues and coefficients are given by the eigenvalue-eigenvector pairs of M/T, where the elements of M are given by[2]

$$
M_{\tilde{\tau}\tau} = \int_{\mathbb{R}^n} \overline{\sum_{j=0}^{n} a_j \phi_j(\vec{s})(e^{i\vec{s}_j\eta_\tau} - \hat{\varphi}(\vec{s}_j))} \sum_{k=0}^{n} a_k \phi_k(\vec{s}) e^{i\vec{s}P_k\eta_\tau} \ \pi(\mathrm{d}\vec{s}) \tag{22}
$$

The eigenvector/eigenvalue couples (c, λ) from M/T then allow us to form $\hat{\chi} = E'B$ with $E = c/\sqrt{\lambda T}$ and B is the basis $\{\sum_{k=0}^{n} a_k \phi_k(\vec{r}) e^{i\vec{r}P_k\eta_\tau}\}$.

Then $< f, \hat{\chi} >= E' < f, B >$.

If $\iota' A = 0$, then $\lambda_i = 0$ or $sum(e_i) = 0$. If A is symmetric, it is also true for $A\iota = 0$. This is because $\iota' Ax = \sum \lambda_i (\iota' e_i)(v_i' x) = 0$ where v_i is from the

[2] Alternatively, the matrix whose entries are given by $\int_{\mathbb{R}^n} \overline{\sum_{j=0}^{n} a_j \phi_j(\vec{s}) e^{i\vec{s}_j\eta_\tau}} \sum_{k=0}^{n} a_k \phi_k(\vec{s}) e^{i\vec{s}P_k\eta_\tau} \ \pi(\mathrm{d}\vec{s})$ has the same eigenvectors whenever the associated eigenvalue is nonzero. M is also easily obtained from this matrix by subtracting row averages.

inverted eigenvector matrix. This is true for any x, so one can get $\lambda_i(\iota' e_i) = 0$ for any i by taking x as eigenvector.

Mx=lambda x; if sum(x) is 0 then (M x - M iota iota') x = lambda x too so the eigenvectors are common. This leads to M10 having the same eigenvectors as M00 as long as eigenvalues are nonzero.

The choice of integration measure does not affect asymptotic properties (Carrasco and Kotchoni, 2017) so that ensuring tractability for ease of implementation is the main concern. I propose to integrate with respect to a Gaussian, *i.e.*, $\pi(\mathrm{d}(\vec{s}) = (2\pi)^{-n/2} e^{-\frac{1}{2}\|\vec{s}\|^2}$. Then,

$$\begin{aligned}
&\int\int \overline{\sum_{j=0}^{n} a_j \phi_j(\vec{s}) e^{i\vec{s}_j \eta_\tau}} \sum_{k=0}^{n} a_k \phi_k(\vec{s}) e^{i\vec{s}P_k \eta_\tau} \ \pi(\mathrm{d}\vec{s}) \\
&= \int \frac{\left(\frac{e^{i\vec{s}\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}\eta_t}} - \sum_{j=1}^{n} \frac{e^{i\vec{s}P_j \eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j \eta_t}} \prod_{m\neq j} \hat{\varphi}(\vec{s}_m) \right)}{\left(\frac{e^{i\vec{s}\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}\eta_t}} - \sum_{k=1}^{n} \frac{e^{i\vec{s}P_k \eta_{\tilde{\tau}}}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k \eta_t}} \prod_{m\neq k} \hat{\varphi}(\vec{s}_m) \right)} \ \pi(\mathrm{d}\vec{s}) \\
&= \sum_{j,k\neq 0} \frac{1}{T^{2(n-1)}} \sum_{\{t_m, s_m\}} \exp\left(-0.5 \ \left\| \sum_{m=1}^{n} P_m(\eta_{t_m} - \eta_{s_m}) \right\|^2 \right) \\
&- \sum_{k\neq 0} \frac{1}{T^{n-1}} \sum_{\{s_m\}} \exp\left(-0.5 \ \left\| \eta_\tau - \sum_{m=1}^{n} P_m \eta_{s_m} \right\|^2 \right) \\
&- \sum_{j\neq 0} \frac{1}{T^{n-1}} \sum_{\{t_m\}} \exp\left(-0.5 \ \left\| \sum_{m=1}^{n} P_m \eta_{t_m} - \eta_{\tilde{\tau}} \right\|^2 \right) \\
&+ \exp(-0.5 \ \|\eta_\tau - \eta_{\tilde{\tau}}\|^2)
\end{aligned}$$

where, in the first and third term, $\eta_{t_j} = \eta_\tau$ and, in the first and second term, $\eta_{s_k} = \eta_{\tilde{\tau}}$.

Similarly, one can compute the scalar products that appear in the objective function's expression as

$$\int q_T(\vec{s}) \left(\frac{e^{i\vec{s}\eta_\tau}}{\frac{1}{T}\sum_{t=1}^T e^{i\vec{s}\eta_t}} - \sum_{k=1}^n \frac{e^{i\vec{s}P_k\eta_{\tilde{\tau}}}}{\frac{1}{T}\sum_{t=1}^T e^{i\vec{s}P_k\eta_t}} \prod_{m\neq k} \hat{\varphi}(\vec{s}_m) \right) \pi(\mathrm{d}\vec{s})$$

$$= -\sum_{k=1}^n \frac{1}{T^n} \sum_{t=1}^T \sum_{\{t_m\}} \exp\left(-0.5 \left\| \eta_t - \sum_{m=1}^n P_m \eta_{t_m} \right\|^2 \right)$$

$$+ \sum_{k=1}^n \frac{1}{T^{2n-1}} \sum_{\{t_m, s_m\}} \exp\left(-0.5 \left\| \sum_{m=1}^n P_m(\eta_{t_m} - \eta_{s_m}) \right\|^2 \right)$$

$$+ \frac{1}{T} \sum_{t=1}^T \exp(-0.5 \; \|\eta_t - \eta_\tau\|^2)$$

$$- \frac{1}{T^n} \sum_{\{t_m\}} \exp\left(-0.5 \left\| \sum_{m=1}^n P_m(\eta_{t_m} - \eta_\tau) \right\|^2 \right)$$

where, in the first term, $\eta_{t_k} = \eta_\tau$ and, in the second term, $\eta_{s_k} = \eta_\tau$.

Hence, computing the objective function does not require numerical integration. The averages over all times $\{t_m, s_m\}$ can be high-dimensional in some settings, so that drawing times at random and computing the resulting sample average may be desirable to lower the computational cost.

3.3 Tests

Asymptotic normality provides the basis for usual confidence intervals and tests. Moreover, an advantage of the analogy with GMM is the potential for a specification test, in the spirit of over-identifying restrictions. As detailed in Carrasco and Florens (2000), such a test can be constructed on the basis of

$$\frac{||\sqrt{T} q_T||^2_{C_T^{\alpha_T}} - \sum_{j=1}^T \frac{\mu_{j;T}^2}{\mu_{j;T}^2 + \alpha_T}}{\sqrt{2\sum_{j=1}^T \frac{\mu_{j;T}^4}{(\mu_{j;T}^2+\alpha_T)^2}}} \to^d N(0;1) \tag{23}$$

provided $\alpha_T \sum_{j=1}^T \frac{\mu_{j;T}^4}{(\mu_{j;T}^2+\alpha_T)^2} \to \infty$ and the assumptions for asymptotic normality of the efficient estimator hold.

Such a test is valuable when working with ICA systems as it provides a feedback about the validity of the entire structure.

4 Simulations

I generate samples of η_t through equation $\eta_t = \Theta\epsilon_t$ with various distributions for the epsilons and a sample size of $T = 150$. I compare the performance in recovering the lag polynomial of the efficient estimator of Section 4 to that of efficient GMM based on moment conditions (*i.e.* deriving identifying equations implied by independence under the assumptions that moments up to order 4 exists, see *e.g.*, Guay and Normandin (2018)) and fastICA (Oja and Yuan, 2006). In the forthcoming tables, the corresponding estimators are denoted by log-cf, GMM, and fICA, respectively.

I consider the following distributions for the sources: student with 3 degrees of freedom, uniform on [-1;1], Binomial(20, 0.3), and Gamma(5, 1/7). All distributions are centered as to have mean zero. These distributions account for a variety of cases such as fat tails, skewness, or presence of atoms.

Consider first a student distribution with 3 degrees of freedom. In this case, the student distribution exhibits fat tails and moments higher than 2 do not exist, endangering identification strategies based on higher moments.

Table 1: Student distribution $\nu = 3$

	Bias			Standard deviation			RMSE		
$\Theta_{\cdot 1}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
0.71	-0.05	-0.02	-0.31	0.17	0.22	0.55	0.17	0.22	0.63
0.71	0.01	-0.07	0.01	0.13	0.28	0.17	0.14	0.29	0.17
$\Theta_{\cdot 2}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
-0.50	-0.03	0.03	0.11	0.17	0.28	0.55	0.18	0.28	0.56
0.87	-0.04	-0.05	-0.15	0.10	0.19	0.17	0.11	0.20	0.22

It appears the estimator based on log-empirical characteristic function considerably outperforms both efficient GMM and fastICA estimators when the sources are student distributed. The gains come mostly from a lower standard deviation, though there is some bias reduction especially compared to fastICA.

Now, I turn to uniform and binomial distributions. Both distribution have all their moments but one is continuous and symmetric while the other is discrete and skewed. Both efficient GMM and the characteristic-function based estimator outperform fastICA for the uniform distribution. In the binomial case, the characteristic function based estimator again fares better than both efficient GMM and fastICA, with a considerable reduction in mean square error originating from lower standard deviations.

Table 2: Uniform distribution

	Bias			Standard deviation			RMSE		
$\Theta_{\cdot 1}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
0.71	−0.04	−0.10	−0.28	0.14	0.16	0.57	0.15	0.19	0.63
0.71	0.01	0.06	−0.01	0.12	0.11	0.12	0.12	0.13	0.12
$\Theta_{\cdot 2}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
−0.50	−0.01	−0.06	0.10	0.15	0.20	0.58	0.15	0.21	0.59
0.87	−0.02	−0.07	−0.17	0.08	0.11	0.12	0.08	0.13	0.21

Table 3: Binomial distribution $n = 20, p = 0.3$

	Bias			Standard deviation			RMSE		
$\Theta_{\cdot 1}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
0.71	−0.02	−0.06	−0.30	0.13	0.27	0.51	0.13	0.28	0.60
0.71	0.01	−0.07	−0.01	0.14	0.30	0.31	0.14	0.31	0.31
$\Theta_{\cdot 2}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
−0.50	−0.01	0.00	0.12	0.18	0.32	0.53	0.18	0.32	0.54
0.87	−0.02	−0.11	−0.17	0.08	0.27	0.31	0.08	0.29	0.35

Finally, the last tables show more contrasted results. In the case of a gamma distribution, log-cf and fICA estimators exhibit similar performance in terms of RMSE and tend to be outperformed by efficient GMM. The characteristic function based estimator occasionally displays a greater bias, which reduces its performance with these distributions, at least for some parameters.

Table 4: Gamma distribution $\alpha = 5, \beta = 1/7$

	Bias			Standard deviation			RMSE		
$\Theta_{\cdot 1}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
0.71	-0.20	-0.14	-0.29	0.56	0.34	0.53	0.59	0.36	0.61
0.71	-0.15	-0.02	-0.01	0.35	0.30	0.25	0.39	0.31	0.25
$\Theta_{\cdot 2}$	log-cf	GMM	fICA	log-cf	GMM	fICA	log-cf	GMM	fICA
-0.50	0.02	-0.03	0.11	0.52	0.43	0.54	0.52	0.43	0.55
0.87	-0.23	-0.16	-0.19	0.32	0.36	0.25	0.40	0.42	0.30

5 Application to SVAR

5.1 Structural Vector Autoregression

Structural Vector Autoregressions (SVAR) have attracted a lot of interest in time series econometrics since the pioneering work of Sims (1980). The standard model postulates that some observed state of the economy characterized by a vector of n variables, Y_t, is related to unobserved (stationary) shocks (*e.g.*, monetary or oil shocks) through

$$Y_t = \Theta(L)\varepsilon_t \tag{24}$$

where $\Theta(L)$ is an unknown lag polynomial that represents the impulse response function[3]. $\Theta(L)$ describes the transmission mechanism of shocks to the economy and a subset of its column typically constitutes parameters of interest.

The first step towards estimation of $\Theta(L)$ is usually to perform the vector autoregression $A(L)Y_t = \eta_t$ to recover estimates of the innovation vector, η_t. The fundamentalness assumption states that the span of the shocks and innovations are identical and thus $\eta_t = \Theta\varepsilon_t$ for some invertible matrix Θ. It is well-known in the literature (see for instance Forni, Gambetti and Sala (2019)) that Θ corresponds to the first term in the lag polynomial $\Theta(L)$.

[3]Similarly to ICA systems, shocks as well as their effects on the system are unobserved so that there is a scale indeterminacy: shocks can be arbitrarily re-scaled to get an observationally equivalent system in which the effect of shocks are inversely re-scaled. Hence a normalization is typically imposed, for instance the unit variance normalization (each shock has variance one) or the unit effect normalization ($\Theta_{jj} = 1\ \forall j$) are popular.

Provided one can identify Θ, the whole lag polynomial is recovered as $A(L)^{-1}\Theta$. As a result, the problem is reduced to the system $\eta_t = \Theta\varepsilon_t$. While standard SVAR only assumes that entries in ε_t are uncorrelated, second moments – $\Sigma_\eta = \Theta\Sigma_\varepsilon\Theta'$ – bring too few equations to solve for Θ, even after a normalization. Various solutions have been proposed, among which short-run restrictions (Sims, 1980), long-run restrictions (King et al., 1987; Blanchard and Quah, 1989; Shapiro and Watson, 1988), identification by heteroskedasticity (Rigobon, 2003; Sentana and Fiorentini, 2001; Lewis, 2019), or sign restrictions (Uhlig, 2005). A good recent reference is Kilian and Lütkepohl (2017).

Although these restrictions solve the identification problem, assuming a priori knowledge of numerous ($n(n-1)$) shocks' effects is often an issue, as the transmission mechanism of shocks to the economy is primarily an empirical question. Thus, some authors (Siegfried et al., 2002; Gourieroux, Monfort et al., 2014) have pointed out that $\eta_t = \Theta\varepsilon_t$ can be identified by assuming that ε_t contains non-Gaussian independent variables, bypassing restrictions on Θ and introducing ICA methods to the SVAR literature. Many subsequent studies have followed that road, estimating the model using high-order moments (Guay and Normandin, 2018; Keweloh, 2019), or pseudo-maximum of likelihood Gouriéroux, Monfort and Renne (2017). See also related discussions and methods in Moneta et al. (2013); Herwartz (2015); Lanne, Meitz and Saikkonen (2017).

5.2 Application

I consider a standard SVAR system with monthly data Y_t on real economic, oil price, and stock market growths[4] as in Keweloh (2019). The first-step vector autoregression $A(L)Y_t = \eta_t$ is performed with four lags, as suggested by Akaike's information criterion.

The study is interesting to replicate for two reasons. First, as in many SVAR studies, there might be concerns about the fundamentalness assumption. This could for instance be caused by the presence of additional shocks (*e.g.*, due to measurement error). Though robustness results against non-fundamentalness exist (Sims and Zha, 2006; Sims, 2012; Feve and Jidoud,

[4]Data comes from the following sources:
https://fred.stlouisfed.org/series/INDPRO (industrial production);
https://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=pet&s=r0000¨¨3&f=m (oil);
https://finance.yahoo.com/quote/%5EGSPC?p=%5EGSPC (SP);
https://fred.stlouisfed.org/series/CPIAUCSL (CPI)

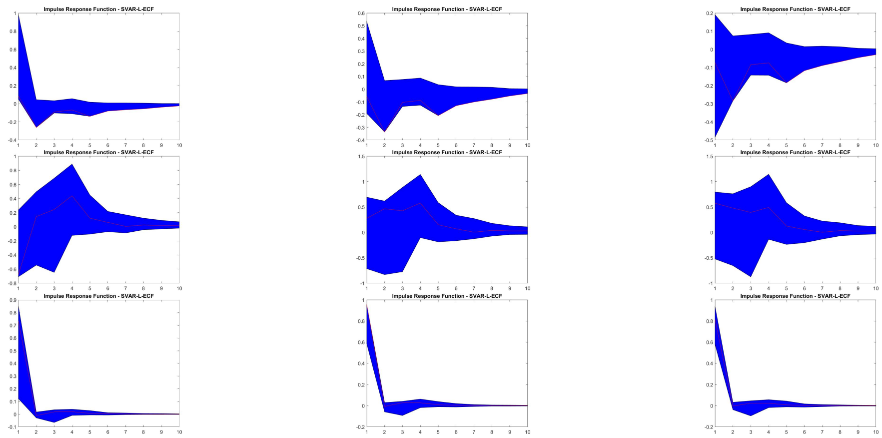

Figure 1: Plots of Impulse Responses Functions. Each column represents the 1-to-10-months impact of a shock on the S&P (first row), oil price (second row), and economic activity (third row). Shaded area depicts 90% bootstrap confidence interval.

2012; Beaudry et al., 2015; Forni, Gambetti and Sala, 2019), it is worthwhile to see if the test detects a problem about the validity of the ICA representation.

Second, shocks might have quite fat tails in practice. For instance, Keweloh (2019) obtains excess kurtosis for all shocks and find that the shock associated to economic activity has a kurtosis above 10. Thus an estimator robust to existence of moment and able to perform accurate estimation in presence of fat tails may be useful.

The object of interest is here the lag polynomial $\Theta(L)$, rather than solely the unmixing matrix. I report the estimated responses to shock in figure 1 and display bootstraped confidence intervals.

Shocks are here subject to the unit norm normalization, so they have the same overall variance over the system. Shocks 2 and 3 have similar variance of about 89, and affect strongly economic activity. The first shock accounts for less of the disturbances to the economic system (variance of 31) and has a lower contemporaneous effect on economic activity; it seems to affect the whole system negatively after a period, but the impact is imprecisely estimated.

The over-identification test' does not reject the null (p-value 0.21), so that there is no evidence against the validity of the ICA representation.

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6 Appendix A: proofs

6.1 Theorem 3.1 (Consistency)

Proof. Consistency follows by verifying assumptions of Theorem 2.1 in Newey and McFadden (1994). The parameter space is compact by construction, identification is established, and the limiting objective function is continuous by dominated convergence. It remains to show uniform convergence in probability.

The empirical characteristic function using a consistent estimator of η_t converges uniformly in probability:

$$
\begin{aligned}
\sup_{\Theta\in\overline{\Theta}}\left|\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\hat{\eta}_t}-\mathbb{E}[e^{i\vec{s}P_j\eta_t}]\right| &= \sup_{\Theta\in\overline{\Theta}}\left|\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\hat{\eta}_t}-\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\eta_t}+\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\eta_t}-\mathbb{E}[e^{i\vec{s}P_j\eta_t}]\right| \\
&\leq \sup_{\Theta\in\overline{\Theta}}\left|\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\hat{\eta}_t}-\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\eta_t}\right|+\sup_{\Theta\in\overline{\Theta}}\left|\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\eta_t}-\mathbb{E}[e^{i\vec{s}P_j\eta_t}]\right| \\
&\leq \sup_{\Theta\in\overline{\Theta}}\left|\frac{1}{T}\sum_{t=1}^{T}e^{i\vec{s}P_j\eta_t}\left(e^{i\vec{s}P_j(\hat{\eta}_t-\eta_t)}-1\right)\right|+o_p(1) \\
&\leq \sup_{\Theta\in\overline{\Theta}}\frac{1}{T}\sum_{t=1}^{T}\left|e^{i\vec{s}P_j(\hat{\eta}_t-\eta_t)}-1\right|+o_p(1) \\
&\leq \sup_{\Theta\in\overline{\Theta}}\frac{1}{T}\sum_{t=1}^{T}|\vec{s}P_j(\hat{\eta}_t-\eta_t)|+o_p(1) \\
&\leq \frac{1}{T}\sum_{t=1}^{T}\|\vec{s}\|\frac{1}{\cos(\frac{\pi}{2}-\epsilon)}|\hat{\eta}_t-\eta_t|+o_p(1) \\
&\to^p 0
\end{aligned}
$$

where the second inequality follows from the uniform law of large numbers, the fourth from $|e^{ix}-1|\leq|x|$, the fifth from ϵ-invertibility, and the convergence is implied by consistency of $\hat{\eta}_t$.

Hence, by Theorem 2.1 in Newey and McFadden (1994), $\hat{\gamma}\to^p\gamma_0$. □

6.2 Theorem 3.2 (Asymptotic Normality)

Proof. Derivation of asymptotic normality follows the approach in Newey and McFadden (1994). By dominated convergence, first-order conditions read

$$\int \begin{pmatrix} \Re\frac{\partial q_T}{\partial \gamma}(\hat\gamma) & \Im(\frac{\partial q_T}{\partial \gamma}(\hat\gamma)) \end{pmatrix} W_T \begin{pmatrix} \Re q_T(\hat\gamma) \\ \Im q_T(\hat\gamma) \end{pmatrix} \mathrm{d}\vec{s} = 0 \tag{25}$$

Applying the mean-value theorem around the true value γ_0 yields

$$\int \begin{pmatrix} \Re\frac{\partial q_T}{\partial \gamma}(\hat\gamma) & \Im(\frac{\partial q_T}{\partial \gamma}(\hat\gamma)) \end{pmatrix} W_T \begin{pmatrix} \Re q_T(\gamma_0) + (\Re\frac{\partial q_T}{\partial \gamma}(\tilde\gamma_R))'(\hat\gamma - \gamma_0) \\ \Im q_T(\gamma_0) + (\Im\frac{\partial q_T}{\partial \gamma}(\tilde\gamma_I))'(\hat\gamma - \gamma_0) \end{pmatrix} \mathrm{d}\vec{s} = 0$$

so that, rearranging

$$\sqrt{T}(\hat\gamma - \gamma_0) = -\left[\int \begin{pmatrix} \Re\frac{\partial q_T}{\partial \gamma}(\hat\gamma) & \Im(\frac{\partial q_T}{\partial \gamma}(\hat\gamma)) \end{pmatrix} W_T \begin{pmatrix} (\Re\frac{\partial q_T}{\partial \gamma}(\tilde\gamma_R))' \\ (\Im\frac{\partial q_T}{\partial \gamma}(\tilde\gamma_I))' \end{pmatrix} \mathrm{d}\vec{s}\right]^{-1}$$
$$\sqrt{T}\int \begin{pmatrix} \Re\frac{\partial q_T}{\partial \gamma}(\hat\gamma) & \Im(\frac{\partial q_T}{\partial \gamma}(\hat\gamma)) \end{pmatrix} W_T \begin{pmatrix} \Re q_T(\gamma_0) \\ \Im q_T(\gamma_0) \end{pmatrix} \mathrm{d}\vec{s}$$

Consider the right-hand side. The first term can be proven to converge uniformly in probability, proceeding as in the uniform convergence step in proving consistency. For the second term, the empirical characteristic function converges to a complex normal stochastic process. This follows from convergence of finite dimensional distributions (by the multivariate central limit theorem and the delta method) and tightness (tightness was proven by Feuerverger, Mureika et al. (1977)[5][6]).

Then convergence of $\sqrt{T}(\frac{1}{T}\sum_{i=1}^{T} e^{i\vec{v}\eta_t} - \mathbb{E}[e^{i\vec{v}\eta_t}])$ to a complex normal stochastic process together with the continuous mapping theorem deliver asymptotic normality. If η_t is known, we directly obtain

$$\sqrt{T}(\hat\gamma - \gamma_0) \to^d N(0; BVB')$$

Interestingly, estimation of η_t does not affect the asymptotic variance. Indeed, estimation of η_t can be accounted for by expanding $q(\gamma_0, \hat\eta_t)$ into $q(\gamma_0, \eta_t(\beta_0)) + \frac{\partial q(\gamma_0, \eta_t(\overline{\beta}))}{\partial \beta}(\hat\beta - \beta)$ by the mean-value theorem, where β is the underlying parameter vector in estimating η_t. The first term corresponds

[5] As pointed out by Csorgo (1981), the result requires slightly stronger conditions than initially thought. Existence of moments larger than 1 suffices.

[6] They also proved almost sure convergence of $\sup_{-K\leq s\leq K}|c_n(s) - c(s)|$, where c denotes the empirical characteristic function and c_n its empirical counterpart, the empirical characteristic function is almost surely bounded away from zero on a neighborhood of the origin for T large enough, and tightness can thus be established for the cumulant generating function as well. As a result, the log-version of the criterion can also be shown to be asymptotically normal

to the case where η_t is observed. If η_t is an error term independent from its regressors w_t, then, by the law of iterated expectations and properties of the P_j, $\frac{\partial q(\gamma_0,\overline{\beta})}{\partial\beta} = \sum_{j=0}^{n} a_j \frac{\mathbb{E}[e^{i\vec{s}P_j\eta_t}(-w_t)P_j'\vec{s}']}{\mathbb{E}[e^{i\vec{s}P_j\eta_t}]} = -\sum_{j=0}^{n} a_j \frac{\mathbb{E}[\mathbb{E}[e^{i\vec{s}P_j\eta_t}|w_t]w_tP_j'\vec{s}']}{\mathbb{E}[e^{i\vec{s}P_j\eta_t}]} = -\mathbb{E}[w_t](\sum_{j=0}^{n} a_jP_j')\vec{s}' = 0$.

□

Appendix B: Additional derivations

I derive the eigenfunctions and eigenvalues in details. Consider the estimated counterpart of the integral operator K:

$$
\begin{aligned}
(\hat{K}g)(\vec{r}) &= \int \begin{pmatrix} \widehat{\mathbb{C}ov}(\Re q_T(\vec{s}), \Re q_T(\vec{r}))g_1(\vec{s}) + \widehat{\mathbb{C}ov}(\Im q_T(\vec{s}), \Re q_T(\vec{r}))g_2(\vec{s}) \\ \widehat{\mathbb{C}ov}(\Re q_T(\vec{s}), \Im q_T(\vec{r}))g_1(\vec{s}) + \widehat{\mathbb{C}ov}(\Im q_T(\vec{s}), \Im q_T(\vec{r}))g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{a_k a_j}{2} \\
&\times \int \begin{pmatrix} \Re\left(\frac{\hat{\varphi}(\vec{s}_j-\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{\hat{\varphi}(\vec{s}_j+\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)} - 2\right) g_1(\vec{s}) + \Im\left(\frac{\hat{\varphi}(\vec{s}_j-\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{\hat{\varphi}(\vec{s}_j+\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_2(\vec{s}) \\ \Im\left(-\frac{\hat{\varphi}(\vec{s}_j-\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{\hat{\varphi}(\vec{s}_j+\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_1(\vec{s}) + \Re\left(\frac{\hat{\varphi}(\vec{s}_j-\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} - \frac{\hat{\varphi}(\vec{s}_j+\vec{r}_k)}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
&= \frac{1}{T} \sum_{\tau=1}^{T} \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{a_k a_j}{2} \\
&\times \int \begin{pmatrix} \Re\left(\frac{e^{i(\vec{s}_j-\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i(\vec{s}_j+\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)} - 2\right) g_1(\vec{s}) + \Im\left(\frac{e^{i(\vec{s}_j-\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i(\vec{s}_j+\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_2(\vec{s}) \\ \Im\left(-\frac{e^{i(\vec{s}_j-\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i(\vec{s}_j+\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_1(\vec{s}) + \Re\left(\frac{e^{i(\vec{s}_j-\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(-\vec{r}_k)} - \frac{e^{i(\vec{s}_j+\vec{r}_k)\eta_\tau}}{\hat{\varphi}(\vec{s}_j)\hat{\varphi}(\vec{r}_k)}\right) g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
=& \frac{1}{T} \sum_{\tau=1}^{T} \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j \\
&\times \int \begin{pmatrix} \Re\left(\left(\frac{e^{-i\vec{r}_k\eta_\tau}}{\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\right)\frac{e^{i\vec{s}_j\eta_\tau}}{2\hat{\varphi}(\vec{s}_j)} - 1\right) g_1(\vec{s}) + \Im\left(\frac{e^{-i\vec{r}_k\eta_\tau}}{\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\right)\frac{e^{i\vec{s}_j\eta_\tau}}{2\hat{\varphi}(\vec{s}_j)} g_2(\vec{s}) \\ \Im\left(-\frac{e^{-i\vec{r}_k\eta_\tau}}{\hat{\varphi}(-\vec{r}_k)} + \frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\right)\frac{e^{i\vec{s}_j\eta_\tau}}{2\hat{\varphi}(\vec{s}_j)} g_1(\vec{s}) + \Re\left(\left(\frac{e^{-i\vec{r}_k\eta_\tau}}{\hat{\varphi}(-\vec{r}_k)} - \frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\right)\frac{e^{i\vec{s}_j\eta_\tau}}{2\hat{\varphi}(\vec{s}_j)}\right) g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
=& \frac{1}{T} \sum_{\tau=1}^{T} \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j \int \begin{pmatrix} \left(\Re\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\Re\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_1(\vec{s}) + \Re\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\Im\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} g_2(\vec{s}) \\ \Im\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\Im i\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} g_1(\vec{s}) + \Im\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}\Re(-i)\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
=& \frac{1}{T} \sum_{\tau=1}^{T} \sum_{k=0}^{n} \sum_{j=0}^{n} a_k a_j \\
&\times \int \begin{pmatrix} \left(\Re\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right)\left(\Re\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_1(\vec{s}) + \left(\Re\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right)\left(\Im\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_2(\vec{s}) \\ \left(\Im\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right)\left(\Re\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_1(\vec{s}) + \left(\Im\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right)\left(\Im\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_2(\vec{s}) \end{pmatrix} \, \mathrm{d}\vec{s} \\
&= \frac{1}{T} \sum_{t=1}^{T} \begin{pmatrix} \Re\sum_{k=0}^{n} a_k \left(\frac{e^{it_k\eta_\tau}}{\hat{\varphi}(t_k)} - 1\right) \\ \Im\sum_{k=0}^{n} a_k \left(\frac{e^{it_k\eta_\tau}}{\hat{\varphi}(t_k)} - 1\right) \end{pmatrix} \\
&\times \left[\int \Re\sum_{j=0}^{n} a_j \left(\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_1(\vec{s}) \, \mathrm{d}\vec{s} + \Im\sum_{j=0}^{n} a_j \left(\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)} - 1\right) g_2(\vec{s}) \, \mathrm{d}\vec{s}\right]
\end{aligned}
$$

noting that $\frac{1}{T}\sum_\tau \frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} = 1$.

This implies that the eigenfunctions g_1 and g_2 take the form

$$g_1(\vec{r}) = \frac{1}{T}\sum_{\tau=1}^{T} c_\tau \Re \sum_{k=0}^{n} a_k \left(\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right) \tag{26}$$

and

$$g_2(\vec{r}) = \frac{1}{T}\sum_{\tau=1}^{T} c_\tau \Im \sum_{k=0}^{n} a_k \left(\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)} - 1\right) \tag{27}$$

Substituting these in the system $(\hat{K}g)(\vec{r}) = \mu g(\vec{r})$, it follows that the coefficients $\{c_\tau^m, \tau = 1,\dots,T\}$ for $m = 1,\dots,T$, form the T eigenvectors of the matrix M with elements

$$M_{\tilde{\tau}\tau} = \Re \int \sum_{j=0}^{n} a_j \left(\frac{e^{i\vec{s}_j\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}} - 1\right) \overline{\sum_{k=0}^{n} a_k \left(\frac{e^{i\vec{s}_k\eta_{\tilde{\tau}}}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}} - 1\right)}\, \mathrm{d}\vec{s}$$

The associated eigenvalues correspond to $T\mu_{m;T}$.

If one desires to integrate on $\mathbb{R}^n$, then we can consider integral with respect to another measure than Lebesgue's. A leading possibility, as in Carrasco and Kotchoni (2017), is to use a density function π as weight.

One then needs

$$M_{\tilde{\tau}\tau} = \Re \int_{\mathbb{R}^n} \sum_{j=0}^{n} a_j \left(\frac{e^{i\vec{s}_j\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}} - 1\right) \overline{\sum_{k=0}^{n} a_k \left(\frac{e^{i\vec{s}_k\eta_{\tilde{\tau}}}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}} - 1\right)}\, \pi(\vec{s})\, \mathrm{d}\vec{s}$$

Note:
- M/T has eigenvalues μ, eigenvectors E.
$\hat{\phi} = \frac{1}{T}\sum_{\tau=1}^{T} c_\tau (\sum_{k=0}^{n} a_k (\frac{e^{i\vec{s}P_k\eta_\tau}}{T^{-1}\sum_\tau^T e^{i\vec{s}P_k\eta_\tau}} - 1))/\|\phi\| = E'B/T/\|\phi\|$ is a a normalized eigenfunction, using E as the eigenvector. B is the base, i.e., the collection of $(\sum_{k=0}^{n} a_k(\frac{e^{i\vec{s}P_k\eta_\tau}}{T^{-1}\sum_\tau^T e^{i\vec{s}P_k\eta_\tau}} - 1))$.
- $\|\phi\| = \sqrt{E'ME/T} = \sqrt{\mu}$
- $< \hat{\phi}, f >= \int f E'B/T/\sqrt{\mu}\Pi(\mathrm{d}\vec{s})$
- $Q = 1/T^2 \sum_\tau \frac{1}{\mu_\tau^2+\alpha} |E'_\tau \int f\overline{B}|^2$.

Now, I derive the eigenfunctions and eigenvalues in details. Consider the integral operator $\hat{K}$ applied to $g : \mathbb{R}^n \to \mathbb{R}^2$

$$(Kg)(\vec{r}) = \int \begin{pmatrix} \mathbb{C}ov(\Re q_T(\vec{s}), \Re q_T(\vec{r}))g_1(\vec{s}) + \mathbb{C}ov(\Im q_T(\vec{s}), \Re q_T(\vec{r}))g_2(\vec{s}) \\ \mathbb{C}ov(\Re q_T(\vec{s}), \Im q_T(\vec{r}))g_1(\vec{s}) + \mathbb{C}ov(\Im q_T(\vec{s}), \Im q_T(\vec{r}))g_1(\vec{s}) \end{pmatrix} \mathrm{d}\vec{s}$$

Doing the algebra on its estimated counterpart yields

$$(\hat{K}g)(\vec{r}) = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix}\Re\sum_{k=0}^{n} a_k\left(\frac{e^{it_k\eta_\tau}}{\hat{\varphi}(t_k)}-1\right)\\ \Im\sum_{k=0}^{n} a_k\left(\frac{e^{it_k\eta_\tau}}{\hat{\varphi}(t_k)}-1\right)\end{pmatrix}$$
$$\times\left[\int \Re\sum_{j=0}^{n} a_j\left(\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)}-1\right)g_1(\vec{s})\,\mathrm{d}\vec{s}+\Im\sum_{j=0}^{n} a_j\left(\frac{e^{i\vec{s}_j\eta_\tau}}{\hat{\varphi}(\vec{s}_j)}-1\right)g_2(\vec{s})\,\mathrm{d}\vec{s}\right]$$

This implies that the eigenfunctions g_1 and g_2 take the form

$$g_1(\vec{r}) = \frac{1}{T}\sum_{\tau=1}^{T} c_\tau \Re\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}-1\right) \tag{28}$$

and

$$g_2(\vec{r}) = \frac{1}{T}\sum_{\tau=1}^{T} c_\tau \Im\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{r}_k\eta_\tau}}{\hat{\varphi}(\vec{r}_k)}-1\right) \tag{29}$$

where the coefficients, $\{c_\tau^m, \tau = 1, \ldots, T\}$ for $m = 1, \ldots, T$, form the T eigenvectors of the matrix M with elements

$$M_{\tilde{\tau}\tau} = \Re\int\sum_{j=0}^{n} a_j\left(\frac{e^{i\vec{s}_j\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}}-1\right)\overline{\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{s}_k\eta_{\tilde{\tau}}}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}}-1\right)}\,\mathrm{d}\vec{s} \tag{30}$$

The associated eigenvalues correspond to $T\mu_{m;T}$.

The scalar products in computing the objective function then reads

$$< q_T, \varphi_{m;T} > = \int\left(\Re\sum_{j=0}^{n} a_j \ln\left(\tfrac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}\right)\quad \Im\sum_{j=0}^{n} a_j \ln\left(\tfrac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}\right)\right)$$
$$\begin{pmatrix}\frac{1}{T}\sum_{\tau=1}^{T} c_\tau^m\Re\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{r}_k\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}}-1\right)\\ \frac{1}{T}\sum_{\tau=1}^{T} c_\tau^m\Im\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{r}_k\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}}-1\right)\end{pmatrix}\,\mathrm{d}\vec{s}$$
$$= \frac{1}{T}\sum_{\tau=1}^{T} c_\tau^m\ \Re\int\sum_{j=0}^{n} a_j \ln\left(\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_j\eta_t}\right)\overline{\sum_{k=0}^{n} a_k\left(\frac{e^{i\vec{r}_k\eta_\tau}}{\frac{1}{T}\sum_{t=1}^{T} e^{i\vec{s}P_k\eta_t}}-1\right)}\,\mathrm{d}\vec{s}$$