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# Improving control over unobservables with network data\*

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## Abstract

Unobserved variables often threaten the causal interpretation of empirical estimates. An opportunity to alleviate this concern lies in network datasets, which provide a rich source of information about individual characteristics insofar as they influence network formation. This paper develops the idea of controlling for unobserved confounders by leveraging network structures that exhibit homophily, a frequently observed tendency to form edges with similar nodes. This is formally accomplished under two main frameworks. First, I introduce a concept of *asymptotic homophily*, according to which individuals' selectivity is at scale with the size of the potential connection pool. This contributes to the network formation literature with a model that can accommodate common features of empirical networks such as homophily, sparsity, and clustering, and allows me to show that an estimator that considers neighbors as a comparison group is consistent for the Conditional Average Treatment Effect (CATE). I then consider a setting without *asymptotic homophily* and show how selecting connected individuals whose observed characteristics made such a connection less likely delivers an estimator with similar properties. Overall, the method allows for nonparametric treatment effect inference for both CATE and Average Treatment Effect (ATE) under a version of unconfoundedness that conditions on unobservables, which is often more credible than selection on observables alone. In an application, I recover an estimate of the effect of parental involvement on students' test scores that is greater than that of OLS, arguably due to the estimator's ability to account for unobserved ability and motivation.

**Keywords:** Selection on unobservables, Networks, Homophily, Treatment effect.

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# 1 Introduction

Estimating the effect of a treatment is a frequent goal in economics and social sciences. A common challenge is the presence of unobserved confounders that threaten the validity of the unconfoundedness assumption, which is typically necessary to perform inference with standard methods. Tools that strengthen control over variables that affect outcomes but are hard to measure, such as ability, culture, work ethic, tastes, *etc.*, are thus particularly valuable.

Recently, networks and datasets with a spatial structure have become increasingly available to researchers, providing new avenues for research. Homophily or assortative matching is a ubiquitous feature of empirical networks: nodes tend to associate with similar nodes (Lazarsfeld, Merton et al., 1954; Clark and Ayers, 1992; McPherson, Smith-Lovin and Cook, 2001; Currarini, Jackson and Pin, 2009; Moody, 2001; Boucher and Mourifié, 2017; Dzemski, 2019).

Homophily generates opportunities to create unobservable-adjusted comparison groups. For instance, if we are interested in the effect of parental involvement on student test scores, we may be concerned about unobserved confounding from differences in student ability. However, if students of similar ability are likely to be friends with each other (Clark and Ayers, 1992; Burgess et al., 2011; Boutwell, Meldrum and Petkovsek, 2017), the omitted variable bias can be reduced by comparing connected students.

The paper develops the idea that homophilic networks can be exploited to derive a consistent estimator of treatment effects in the presence of unobserved confounders. This is formally done under two main frameworks: either the network is asymptotically homophilic – homophily is the core mechanism of network formation or provides a good model fit to it – or the network at least features homophily in unobservables.

In the former case, I let the link formation process vary with the size of the network: people become pickier to limit the number of connections or improve their average quality as the network expands. This is consistent with the common view that the average degree should not increase proportionally to the size of the network and that most networks are sparse. As people are able to form increasingly better matches with a larger pool of potential neighbors, they become more selective because of decreasing benefits per additional match, preference for quality of matches, or limited resources to devote to additional connections.

This accomplishes two things. First, the approach provides an asymptotic approximation that accommodates sparsity and that does not render the mechanism of network formation negligible in the limit; selectivity is at scale with the size of the connection pool. As such, it contributes to the network formation literature with a model that can accommodate common features of empirical networks such as homophily, sparsity, clustering, etc. Second, it establishes consistency and asymptotic normality for the

estimators that use (possibly high-order) connections or connections in common as comparison groups.

In an alternative framework, I only assume some preference for homophilic matching in the unobserved variables. This allows for an arbitrary functional form in the observed covariates involved in link formation and, possibly, a link formation function independent of the size of the network. In this scenario, comparison groups can be derived by comparing people who are different on observables but connect nevertheless. This hinges on the following intuition: if there is no observed rationale for two people being friends, the reason for their friendship likely lies in the unobserved world. If two people are connected despite their observables indicating such a link was unlikely, they are more likely to be close in terms of unobservables. By suitably manipulating a discrepancy in observables and letting it grow with sample size, one can recover consistent estimators.

I provide results that allow for the estimation of the Conditional Average Treatment Effect (CATE), which provides a way to describe the heterogeneity of the treatment effect for sampled individuals. The conditional average effect may be the end goal of the analysis (when a specific unit is targeted for treatment or policy) or may be a prelude to aggregation to the Average Treatment Effect (ATE).

I define a general form of CATE estimator as a function of a group of counterfactual observations to be determined, then propose different choices to deal with different empirical issues. In all cases, estimators isolate increasingly better counterfactuals as to recover the CATE asymptotically. I show that the proposed estimators of the (C)ATE are asymptotically normal, enabling statistical inference.

Although results pertain to nonparametric estimators, the intuition is valid in parametric specifications and similar results are achievable under similar or weaker conditions. Propensity score analysis – and then possibly doubly robust estimators – could also be developed.

Finally, I demonstrate the feasibility and effectiveness of the method through both simulations and an empirical application. In the application, I obtain an estimate of the effect of parental involvement on students’ test scores that suggests a greater impact than OLS does, arguably due to the estimator’s ability to account for unobserved ability and motivation.

**Related literature** The paper is at the intersection of the literature on networks ([Jackson, 2010](#); [Graham, 2015](#); [De Paula, 2017](#); [Newman, 2018](#)), in particular those featuring homophilic network formation ([Boucher, 2015](#); [Graham, 2016, 2017](#); [Demirer, 2019](#); [Mele, 2022](#)), and estimation of treatment effects ([Imbens, 2004](#); [Imbens and Wooldridge, 2009](#); [Imbens and Rubin, 2015](#)), both of which considerably grew in size over the last decades.

A closely related paper is [Auerbach \(2019, 2022\)](#) who considers a partially linear outcome regression where the nonlinear term depends on an unobserved variable. Us-

ing information from a network whose formation hinges on the unobserved variable, he was able to recover consistent estimates of regression coefficients under general assumptions. See also Goldsmith-Pinkham and Imbens (2013); Johnsson and Moon (2021) who use related frameworks and analyze peer effects and Demirer (2019) who provides partial identification results in linear models under homophilic behavior and proposes a nomenclature for homophily.

The present paper considers a more general outcome equation in a potential outcome framework, at the expense of generality in the network formation process. Specifically, I consider a nonparametric treatment effect setup, but I impose more structure on network formation, especially homophily in the unobservables. The nonparametric approach allows us to deal with the common concerns of treatment effect heterogeneity and nonlinearities. In addition, the method circumvents the need to define and estimate equivalent classes and allows the use of higher-order neighbors or friends in common through triangular inequality relationships. This allows for intuitive estimators that are easy to implement.

## 2 Improving control over unobservables using network data

### 2.1 Notation and assumptions

The sample is a cross-section of  $n$  individuals. The treatment status of individual  $i$ ,  $T_i \in \{0, 1\}$ , and the corresponding outcome,  $Y_i = Y_i(T_i)$  with the potential outcome notation (Neyman, 1923; Rubin, 1974), are observed. As the notation for the outcome suggests, the Stable Unit Treatment Value Assumption (SUTVA) is maintained throughout<sup>1</sup>.

The covariates,  $X = (X^o, X^u) \in \mathcal{X}^o \times \mathcal{X}^u \stackrel{\text{def}}{=} \mathcal{X} \subset \mathbb{R}^d$ , are divided into observed variables,  $X^o$ , and unobserved variables,  $X^u$ . There is a norm  $\|\cdot\|$  on  $\mathbb{R}^d$  (with some abuse of notation, this will be used to represent the norm on  $\mathcal{X}^o$  or  $\mathcal{X}^u$ ), typically Euclidean. I focus on continuously distributed covariates  $X$ , though discrete variables can be accommodated – typically under weaker conditions since concerns such as asymptotic bias disappear. The joint density of the covariates is denoted by  $f$ .  $B_r(x)$  denotes a ball of radius  $r$  centered at  $x$ .

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<sup>1</sup>The possible existence of peer effects or interactions among individuals requires careful modelling, starting with a modification of what a potential outcome is. In particular, the problem of interference has recently received attention and a growing literature addresses (versions of) the issue and identifies direct and indirect effects under various relaxations of SUTVA (Sofrygin and van der Laan, 2016; Aronow and Samii, 2017; Arpino, Benedictis and Mattei, 2017; Liu et al., 2019; Forastiere, Airolti and Mealli, 2020; Jackson, Lin and Yu, 2020; Sánchez-Becerra, 2021; Huber and Steinmayr, 2021). Results on peer effect identification with a potentially endogenous network discussed in linear models by Goldsmith-Pinkham and Imbens (2013); Johnsson and Moon (2021). Although homophilic restrictions may still carry identifying power in such settings, adapting estimators and establishing their properties is to be done on a case by case basis and is beyond the scope of this paper.

Draws of  $(Y_i, T_i, X_i)$  are i.i.d. and realizations of a random variable are denoted by the corresponding lower-case letter.  $C$  represents a generic constant satisfying  $0 < C < \infty$ .

A network is given through a (binary) weighting/link matrix  $W$ , of size  $(n \times n)$ . The neighborhood  $\mathcal{N}(i)$  refers to the links, friends, or connections of the node or individual  $i$ , *i.e.*  $\mathcal{N}(i) \stackrel{\text{def}}{=} \{j \in \{1, \dots, n\} | W_{ij} = 1\}$ ,  $\mathcal{N}_t(i)$  denotes neighbors with a specific treatment status  $t$ , *i.e.*  $\mathcal{N}_t(i) \stackrel{\text{def}}{=} \{j \in \{1, \dots, n\} | W_{ij} = 1, T_j = t\}$ . These definitions extend to higher-order neighbors, say of order  $m$ , which are denoted by  $\mathcal{N}_t^m(i)$ . Connections in common are given by  $\mathcal{N}_t(i; j) \stackrel{\text{def}}{=} \mathcal{N}_t(i) \cap \mathcal{N}_t(j)$ . A  $c$  superscript is used to denote the complement of a set.

The goal is to conduct inference about treatment effects. In particular, I develop inference methods for the Conditional Average Treatment Effect (CATE),  $\text{CATE}(x_i) \stackrel{\text{def}}{=} \mathbb{E}[Y_i(1) - Y_i(0) | X_i = x_i]$ , and then for the Average Treatment Effect (ATE),  $\text{ATE} \stackrel{\text{def}}{=} \mathbb{E}[Y_i(1) - Y_i(0)]$ . Although I focus on average treatment effects, the method can be exploited to obtain, *e.g.*, quantiles of treatment effects. The usual statement about omitting ‘almost surely’ qualifiers, in particular pertaining to conditional expectations, applies.

The following core assumptions are maintained throughout the paper:

**Assumption 2.1** (Core).

- a) Unconfoundedness:  $(Y_i(1), Y_i(0)) \perp\!\!\!\perp T_i | X_i$
- b) Overlap:  $0 < C < \mathbb{P}[T_i = 1 | X_i] < 1 - C < 1$

These two assumptions are ubiquitous in the treatment effect literature, though this version of unconfoundedness conditions on  $X$  instead of  $X^o$ . It is thus only assumed that treatment is independent of potential outcomes when conditioned on individual characteristics, including unobserved ones. Since covariates that may influence selection into treatment such as ability, work ethic, or personal preferences are typically unobserved, this is often a valuable relaxation: selection on some unobservables is allowed.

## 2.2 Network formation

Let  $i, j$  be two individuals and  $i \neq j$ . I focus on link-formation models of the type

$$W_{ij} = 1 \iff \eta_{ij} \leq w_n(h(X_i^o; X_j^o) + \|X_i^u - X_j^u\|) \quad (1)$$

where  $w_n : \mathbb{R}^+ \rightarrow [0; 1]$  is a decreasing function that satisfies  $\lim_{x \rightarrow \infty} w_n(x) = 0$ . Typically,  $w_n$  would decrease with  $n$  to accommodate some sparsity in the network, *e.g.*,  $w_n(x) = \max\{1 - s_n x, 0\}$  or  $e^{-s_n \frac{1}{2} x^2}$ . The function  $h$  is arbitrary but known and  $\eta_{ij} = \eta_{ji}$  are independent uniform<sup>2</sup> shocks, drawn independently of  $(X_i, X_j, T_i, T_j, Y_i, Y_j)$ .

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<sup>2</sup>Since one can apply an inverse cumulative distribution function on both sides to generate any distribution, the uniform assumption is made without loss of generality.

Such dyadic network formation processes are common in the literature, *e.g.*, [Graham \(2017\)](#); [Auerbach \(2019\)](#); [Johnsson and Moon \(2021\)](#). Compared to more general specifications (for instance, [Auerbach \(2019, 2022\)](#) uses  $\eta_{ij} \leq \mathbb{1}_{w(X_i, X_j)}$  and only makes a weak continuity assumption on  $w$ ), model (1) adds some separability and homophily in the unobservables. Although the homophilic structure is an additional assumption, it may often be plausible given the empirical prevalence of homophily. The other main feature of this network formation is the explicit dependence of  $w$  on network size, which is often natural as the mean degree of a node is rarely expected to scale with the size of the network ([Jackson, 2010](#); [Newman, 2018](#)). I will provide an explicit model of such dependence, which is useful to describe sorting behavior that hinges on the number of potential connections and to limit the asymptotic density of the network.

The function  $h$  may feature homophily as well<sup>3</sup> as in Subsection 2.3<sup>4</sup> in which case  $h(X_i^o; X_j^o) = \|X_i^o - X_j^o\|$ . In some applications, it may be of interest to allow for another functional form and thus to leave  $h$  flexible, as in Subsection 2.4. For instance, some work relationships may warrant complementarity in background education, in which case there is non-(possibly anti-) homophilic selection in a covariate.

The model can be given the usual interpretation of ‘link creation under a mutual positive utility of forming a link’ ( $w - \eta$  then reflecting utility), see, *e.g.*, [Jackson \(2010\)](#); [Goldsmith-Pinkham and Imbens \(2013\)](#); [Graham \(2017\)](#) where people derive more utility from interacting with similar individuals, or rationalizes the idea that people with similar characteristics are more likely to meet and thus to form a connection. Nevertheless, since the network is primarily seen as information to draw from, this rationale may not be necessary. For instance, if individuals end up developing similar characteristics after randomly forming connections, a researcher that observes the network after covariates have evolved could use the present methods. In other words, (1) need not be the structural equation for network formation but should approximate the relationship between links and covariates at the time of observation.

I first explore the case of *asymptotic homophily*, *i.e.*, homophilic behavior is the core mechanism of network formation and the selectivity of the individuals is tied to the size of their potential matching pool.

The *asymptotic homophily* case provides an asymptotic theory when homophilic behavior is pronounced relative to network size and formalizes the intuition that connections among individuals can be used to form comparison groups. It also shows the identifying power of homophilic restrictions under simpler conditions than the results of the later sections and provides a network formation process compatible with many empirically-observed features.

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<sup>3</sup>In this case in particular, it may make sense to consider variables whose variance has been normalized to put them on the same scale. Nevertheless, the results hold if the norms weight each dimension differently as to reflect stronger selection in some covariates.

<sup>4</sup>More precisely, this Subsection combines observables and unobservables under a single norm, *i.e.*,  $\|X_i - X_j\|$  which can, but need not, correspond to the sum of two norms.

An alternative framework is later provided in Subsection 2.4, possibly letting the link formation be independent of sample size. This is also compatible with some degree of sparsity in the network even with a link formation function independent of  $n$  as it is possible to include variables such as a physical distance that decreases the likelihood of a connection between the newly added observations and previous ones, in a way to be tailored to the unfolding geography of the application.

## 2.3 asymptotic homophily

### 2.3.1 The asymptotic homophily framework

Suppose people associate based on homophily so that the probability of a connection is decreasing in  $\|X_i - X_j\|$  and, as the network expands, they become pickier as to limit the number of connections or improve their quality. As people are able to form increasingly better matches with a larger pool of potential neighbors, they may become more selective because of decreasing benefits per additional match, preference for quality of matches, or limited resources to devote to additional connections.

To reflect this behavior, the sequence of functions  $w_n$  must satisfy two conditions. First, the sequence must be decreasing in order to decrease the probability of forming connections as  $n$  rises. Homophily further suggests that, in addition, people penalize dissimilar individuals increasingly more harshly so that the average match quality (in terms of homophilic preferences) increases.

Functions of the form  $w_n(x) \geq g(s_n x)$  are consistent with such behavior – homophily becomes more prevalent as  $n$  rises – irrespective of the exact form of  $w_n$  (or  $g$ ). Although a weak equality suffices for some results, it will be convenient to replace it by an equality for some derivations and to impose some regularity conditions for asymptotic results. I thus adopt the following definitions:

**Definition 2.1** (asymptotic homophily). a) Network formation is asymptotically homophilic if  $W_{ij} = \mathbb{1}_{\eta_{ij} \leq w_n(\|X_i - X_j\|)}$ ,  $w_n(x) \geq g(s_n x)$ , where  $g : [0; \infty[ \rightarrow [0; 1]$  is decreasing and  $\lim_{n \rightarrow \infty} s_n = \infty$ .  
b) Network formation is regularly asymptotically homophilic if  $W_{ij} = \mathbb{1}_{w_n(\|X_i - X_j\|) \geq \eta_{ij}}$ ,  $w_n(x) = g(s_n x)$ , where  $g : [0; \infty[ \rightarrow [0; 1]$  is a decreasing function such that  $0 < \int_{\mathbb{R}^d} g(\|y\|) dy < \infty$ , and  $\lim_{n \rightarrow \infty} n s_n^{-d} = \infty$ .

Part a) formalizes the idea of *asymptotic homophily*; part b) provides regularity conditions for asymptotic results. The definition is new and the resulting formation mechanism is consistent with empirical regularities of social networks (Jackson, 2010):



sparsity<sup>5</sup>, transitivity/clustering<sup>6</sup>, degree heterogeneity<sup>7</sup>, and homophily.

Increasing homophily, if not taken literally, formalizes the notion that homophilic behavior is pronounced relative to sample size in the spirit of a drifting sequence; the degree to which individuals are selective is in a sense preserved as we proceed to an asymptotic approximation. These asymptotics thus provide a good approximation to the finite sample properties of the data in settings where (i) sample averages are taken over enough observations for law-of-large-numbers approximations to be meaningful and (ii) the degree of sorting that individuals applied when selecting matches was sizable compared to the pool of potential connections. In particular, asymptotic homophily asymptotics approximate the case where people are substantially more likely to match with people who are “close” to them than people who are far away: the ratio  $\mathbb{P}[W_{ij} = 1 | \|X_i - X_j\| < C] / \mathbb{P}[W_{ij} = 1 | \|X_i - X_j\| > C]$  diverges as  $n$  gets large.

### 2.3.2 Comparison groups

Under a asymptotically homophilic network formation process, it is possible to derive estimators whose bias asymptotically disappears, even if some confounders are unobserved. Given a comparison group  $\mathcal{C}_i$  for individual  $i$ , I define a CATE estimator as

$$\widehat{\text{CATE}}(x_i; \mathcal{C}_i) \stackrel{\text{def}}{=} \frac{1}{|\mathcal{C}_{i1}|} \sum_{j \in \mathcal{C}_{i1}} Y_j(T_j) - \frac{1}{|\mathcal{C}_{i0}|} \sum_{j \in \mathcal{C}_{i0}} Y_j(T_j) \quad (2)$$

where  $\mathcal{C}_{it} \stackrel{\text{def}}{=} \mathcal{C}_i \cap \{j | T_j = t\}$ . Note that the comparison group will vary with sample size, although the dependence is left implicit in the notation.

A first idea is to rely on friends to construct a comparison group, *i.e.*, set  $\mathcal{C}_i = \mathcal{N}(i)$ . This creates a CATE estimator based on the difference between treated friends and non-treated friends. Thanks to triangular inequality relationships and the nature of homophily, however, one can use further information from the friendship network.

For instance, one can consider friends of friends or higher-order friendships:

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<sup>5</sup>It is natural to let the average degree of a node be constant or grow only slowly for most applications. For instance, the average number of friends is typically viewed as constant or slowly increasing as the network expands, which requires the probability of forming a link to decrease with the size of the network. Letting the degree increase, albeit slowly, allows one to take advantage of asymptotic approximations.

<sup>6</sup>Intuitively, clustering occurs because group of similar individuals tend to form connections. Moreover, as shown in the Appendix, the clustering coefficient does not vanish asymptotically by contrast to Poisson random graphs (Erdős, Rényi et al., 1960) or configuration models (Bender, Canfield and McKay, 1990).

<sup>7</sup>Because of the influence of covariates, the expected degree varies across individuals. The underlying density affects the degree distribution as discussed in the Appendix.



$$\begin{aligned}\widehat{\text{CATE}}(x_i; \cup_{m=1}^M \mathcal{N}^m(i)) &= \frac{1}{|\cup_{m=1}^M \mathcal{N}_1^m(i)|} \sum_{j \in \cup_{m=1}^M \mathcal{N}_1^m(i)} Y_j(T_j) \\ &\quad - \frac{1}{|\cup_{m=1}^M \mathcal{N}_0^m(i)|} \sum_{j \in \cup_{m=1}^M \mathcal{N}_0^m(i)} Y_j(T_j)\end{aligned}\tag{3}$$

for some highest order of friendship  $M$ . For  $M = 1$ , this is the simple estimator that compares treated friends and non-treated friends.

Although this estimator reduces the variance by averaging over more observations, it may be more biased. An alternative estimator relies on friends in common:

$$\begin{aligned}\widehat{\text{CATE}}(x_i; \{j \mid |\mathcal{N}(i; j)| > \tau\}) &= \frac{1}{|\{j \mid |\mathcal{N}_1(i; j)| \geq \tau\}|} \sum_{j \in \{j \mid |\mathcal{N}_1(i; j)| \geq \tau\}} Y_j(T_j) \\ &\quad - \frac{1}{|\{j \mid |\mathcal{N}_0(i; j)| \geq \tau\}|} \sum_{j \in \{j \mid |\mathcal{N}_0(i; j)| \geq \tau\}} Y_j(T_j)\end{aligned}\tag{4}$$

for some minimum number of common connections  $\tau$ .

Although both estimators allow for consistent estimation of treatment effects, the composition of the underlying comparison groups can differ significantly. Therefore, it can be a useful robustness check to compare their treatment effect estimates, especially if one is concerned about peer effects or related issues that would likely affect these estimators in different ways.

As an illustration, consider Figure 1 where the comparison group for individual  $i$  with unobserved characteristics  $x_i \in \mathbb{R}^2$  is depicted in red and the remaining observations in black. The leftmost picture is the infeasible<sup>8</sup> estimator where the researcher is to select all observations below a certain distance. Next on the right, friends are used as the comparison group, providing a noisy version of (i): selected observations tend to fall close to  $x_i$ , but some close observations are unselected while observations farther away may be selected nevertheless.

In the third picture, one can see the effect of picking friends of friends. This allows us to make use of more observations, which will reduce the variance of the estimator, but there may be a tendency to grab more observations outside the sphere too. Finally, the last picture selects individuals who have at least 2 friends in common with  $i$ . This typically reduces the bias compared to the previous estimator but selects fewer observations.

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<sup>8</sup>Except in the extreme network formation process in which individuals select friends deterministically conditional on covariates:  $w(x) = \mathbf{1}_{x < C}$ . Then, the first two pictures become identical.

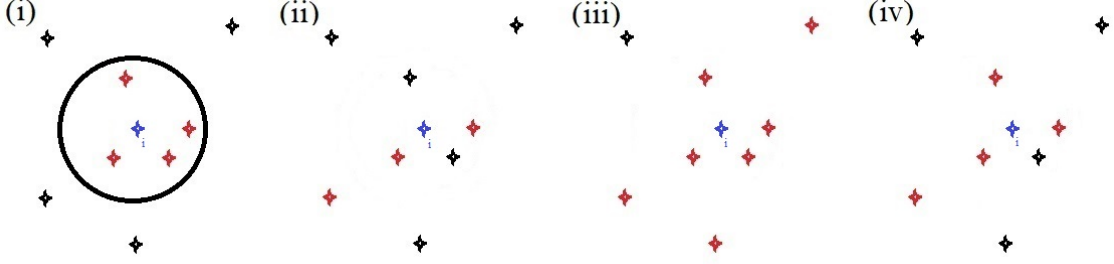


Figure 1: Comparison among possible comparison groups for individual  $i$ . Observations are represented by stars (blue =  $i$ ; red = selected as counterfactual; black = not selected as counterfactual). From left to right, (i) (Infeasible) individuals within a given distance of  $x_i$ , (ii) individuals who are friends with  $i$ , (iii) individuals who are friend with  $i$  or friend of a friend of  $i$ , (iv) individuals who have 2 friends in common with  $i$ .

### 2.3.3 CATE inference

Under *asymptotic homophily*, standard asymptotics are directly achievable with estimators such as (3) or (4). In what follows, I leave the vector of covariates  $x$  undecomposed as manipulations on observables are not needed to secure consistency and asymptotic normality. The main possible benefit of observing some covariate is the opportunity for bias reduction or controlling for variables that are not part of network formation, both of which are discussed in the Appendix. Observables will be used explicitly to manage networks that are not asymptotically homophilic; a detailed discussion is the object of Subsection 2.4.

Asymptotic homophilic behavior ensures that the bias of the proposed CATE estimators disappears. As for the variance, its collapse hinges on averaging over increasingly many observations. The asymptotic behavior of the number of counterfactuals depends on the speed of convergence of  $w_n$ , of which the following lemma provides a formal analysis.

**Lemma 2.1.** *The number of connections for  $i$  exceeds any real number with probability approaching one if  $\lim_{n \rightarrow \infty} n \int_{\mathbb{R}^d} w_n(\|x_j - x_i\|) f(x_j) dx_j = \infty$ .*

*Under regular asymptotic homophily, this holds, and moreover*

a) *If  $w_n \leq 1 - C$  for all sufficiently large  $n$ , the probability of forming a connection of order up to  $M$  is at least  $O\left(s_n^{-d} (ns_n^{-d})^{M-1}\right)$ . The number of connections for  $i$  of order up to  $M$  exceeds any real number with probability approaching one.*

b) *The probability of  $i$  and  $j$  having at least  $\tau$  friends in common is at least  $O(s_n^{-d})$ . The number of individuals with whom  $i$  has at least  $\tau$  friends in common exceeds any real number with probability approaching one.*

The lemma, proven in the appendix, provides conditions under which the number of connections grows to infinity. It also specifies the rates at which the probabilities of

forming a connection up to the  $M$ -th or having  $\tau$  friends in common decrease under *asymptotic homophily*, and when all corresponding counts go to infinity. With overlap, the lemma ensures the number of treated and untreated connections both grow to infinity.

The condition in the lemma states that  $w_n$  must not increase too fast to ensure that connections are still being formed. Basically, individuals must not become pickier at too high a rate for their neighborhood size to keep increasing. This is natural in many networks (*e.g.*, friendship network) as the mean degree is often viewed as constant or slowly increasing.

The formal criterion analyses the integral  $\int_{\mathbb{R}^d} w_n(\|x\|)f(x + x_i) dx$ , suggesting that link functions that do not depend on network size or grow uniformly slowly enough such as  $w_n(x) = s_n^{-1}g(x)$  induce unbounded friend counts.

*Regular asymptotic homophily* is not only consistent with a growing count but also further implies a matching quality improvement that is absent when  $w_n$  is constant or grows uniformly. Specifically, a sequence such as  $w_n(x) = s_n^{-1}g(x)$  would stabilize the ‘posterior’ distribution  $f_{X_j|j \in \mathcal{N}(i)}$ ; it does not imply that friends are closer on average in larger networks.

As a result, *regular asymptotic homophily* will be key in securing consistency properties. An important part in establishing these is the analysis of the bias

$$\begin{aligned} B_i &\stackrel{\text{def}}{=} \mathbb{E}[Y_j(1)|j \in \mathcal{C}_i, T_j = 1] - \mathbb{E}[Y_j(1)|X_j = x_i] \\ &\quad - (\mathbb{E}[Y_j(0)|j \in \mathcal{C}_i, T_j = 0] - \mathbb{E}[Y_j(0)|X_j = x_i]) \end{aligned}$$

which will be shown to disappear under various conditions depending on the comparison group. Specifically, the following conditions on  $w_n$  will ensure that the bias vanishes.

**Assumption 2.2** (Hölder continuity of CATE and convergence of link function). a)  $\text{CATE}(x)$  is Hölder continuous with exponent  $\alpha$  on a neighborhood of  $x_i$ , *i.e.* for any  $x, y$  in the neighborhood  $\|\text{CATE}(y) - \text{CATE}(x)\| \leq C\|y - x\|^\alpha$  for some  $\alpha > 0$ . b) For some  $\varepsilon_n \downarrow 0$ , either  $\mathcal{C}_i = \cup_{m=1}^M \mathcal{N}^m(i)$  and  $\sum_{m=1}^M (w_n(\frac{\varepsilon_n}{m}))^m = o(s_n^{-d})$ , or  $\mathcal{C}_i = \{j | |\mathcal{N}(i; j)| \geq \tau\}$  and  $(w_n(\frac{\varepsilon_n}{2}))^\tau = o(s_n^{-d})$ .

Hölder continuity is a standard assumption that imposes a mild degree of smoothness in the CATE. Part b) of the assumption restricts the way  $w_n$  converges to 1; it requires a sufficiently fast convergence away from the origin.

The second part of the next theorem obtains a clean rate of  $O(s_n^{-2})$ , valid even for the simple first-order friend comparison group. The basic requirement is smoothness of the underlying functions of  $x$ , in particular existence of derivatives. Specifically,

**Assumption 2.3** (Existence of Derivatives). The conditional expectation  $\mathbb{E}[Y_i(t)|X_i = x]$  and the propensity score  $\mathbb{P}[T_i = 1|X_i = x]$  are differentiable.

Now, the consistency theorem reads

**Theorem 2.1** (Consistency). *a) Suppose  $\lim_{n \rightarrow \infty} n \int_{\mathbb{R}^d} w_n(\|x\|) f(x + x_i) dx = \infty$ , and  $\mathbb{E}[Y_j(t)^2] < \infty$  for  $t = 0, 1$ .*

*Then,  $\widehat{\text{CATE}}(x_i; \mathcal{C}_i)$  is consistent for  $\text{CATE}(x_i)$  under Assumption 2.2. Moreover, the bias satisfies  $\mathbb{B}_i = O(\varepsilon_n^\alpha + s_n^d R)$  with  $R = \sum_{m=1}^M w_n(\frac{\varepsilon_n}{m})^m$  and  $R = w_n(\varepsilon_n/2)^\tau$ , respectively, for  $\varepsilon_n \downarrow 0$  as in Assumption 2.2.*

*b) If network formation is regularly asymptotically homophilic, Existence of derivatives holds, and the covariate density has bounded second-order derivatives, then the estimator that simply uses friends, i.e.  $\widehat{\text{CATE}}(x_i; \mathcal{N}(i))$ , is consistent with bias  $\mathbb{B}_i = O(s_n^{-2})$ .*

The theorem is proven in the appendix. The main difficulty in part a) is to derive an expression for the bias that can subsequently be bounded via homophilic assumptions and triangular inequalities. In the second part, the existence of derivatives allows the use of Taylor expansions so that the derivation shares similarities with nonparametric kernel analysis, though the noisy matching through  $w_n$  renders the problem non-standard.

An intuition for the nature and order of the bias can be gathered from the (infeasible) kernel estimator averaging observations within a shrinking sphere centered at  $x_i$ . The present estimators constitute noisy versions of the kernel estimator with  $s_n^{-1}$  playing a role analogous to that of a bandwidth; their biases vanish at a similar rate - possibly slightly more slowly depending on the counterfactual selection strategy and the underlying smoothness.

To sum up, consistency is secured provided homophilic behavior is preserved as sample size grows and  $w_n$  does not drop too fast with sample size. If  $w_n$  does not decrease to zero (so that there is still an asymptotic bias due to different covariates) or does so too quickly (so that the friend count shrinks), the estimator is no longer consistent. Because mean degrees are often viewed as constant or slowly increasing with network size, these situation are seldom empirically relevant. When the network is dense – the probability of forming a link does not drop to 0 – inference results can be obtained using the method of Subsection 2.4.

Now, I finalize the current analysis with an asymptotic normality result: CATE estimators are asymptotically normal at  $x_i$  under the assumptions for consistency. Formally,

**Theorem 2.2** (Asymptotic Normality). *Suppose the Consistency assumptions hold. Then, for a bias  $\mathbb{B}_i$  at location  $x_i$ , the (conditional) asymptotic distribution reads*

$$\sqrt{|\mathcal{C}_i|}(\widehat{\text{CATE}}(x_i; \mathcal{C}_i) - \text{CATE}(x_i) - \mathbb{B}_i) \rightarrow^d \mathcal{N}(0; V)$$

where  $V = \frac{\mathbb{V}[Y_j(1)|X_j=x_i]}{\mathbb{P}[T_j=1|X_j=x_i]^2} + \frac{\mathbb{V}[Y_j(0)|X_j=x_i]}{\mathbb{P}[T_j=0|X_j=x_i]^2}$ .

Moreover,  $\mathbb{B}_i$  is asymptotically negligible if  $\text{CATE}(x)$  is Hölder continuous with exponent  $\alpha$  on a neighborhood of  $x_i$  and one of the following holds:

(i)  $\mathcal{C}_i = \cup_{m=1}^M \mathcal{N}^m(i)$  with  $\sum_{m=1}^M w_n(\frac{n^{-\gamma}n}{m})^m = o(\lambda_n^{\frac{2M}{n}})$  for  $\gamma > \frac{1}{2\alpha}$  or  
(ii)  $\mathcal{C}_i = \{j \mid |\mathcal{N}(i; j)| \geq \tau\}$  and  $s_n^d w_n(n^{-\gamma})^\tau = o(\lambda_n^{-1/2})$  and  $\gamma > \frac{1}{2\alpha}$  or  
(iii) Network formation is regularly asymptotically homophilic, Existence of Derivatives holds, the covariate density has bounded second-order derivatives, and  $\frac{\sqrt{\lambda_n}}{s_n^2} \rightarrow 0$  for  $\lambda_n \stackrel{\text{def}}{=} n s_n^{-d}$ .

The proof of the theorem is given in the appendix and is mostly a consequence of the consistency proof, with an analysis of the bias convergence rate.

The size of the comparison group grows at the rate  $\lambda_n$ . If  $\lambda_n$  grows too fast, the estimator is still consistent but inference requires handling the bias, which I discuss in the Appendix. In social networks such as friendship networks, however, it is often reasonable to let the average degree grow only slowly (Newman, 2018) so that a slow growth of  $\lambda_n$  such as  $\ln(n)$  can be empirically justified. Nevertheless, Appendix B discusses a few strategies to deal with a possible asymptotic bias.

The variance can be estimated with the same kind of truncation methods, allowing for statistical inference.

Finally, the consistent estimation of treatment effects at a given  $x_i$  suggests that it is possible to perform inference about the average treatment treatment effects. This is the next main result: the average treatment effect can also be estimated under unobservable-robust unconfoundedness.

### 2.3.4 ATE inference

Given the last two theorems, one obtains an estimator  $\widehat{\text{ATE}}$  by averaging over a CATE estimator at all  $x_i$ . The resulting ATE estimator is then consistent and asymptotically normal under the theorems' conditions and regularity conditions.

Specifically, consider  $\frac{1}{n} \sum_{i=1}^n \widehat{\text{CATE}}(x_i, \mathcal{C}_i)$  and collect the terms involving  $Y_i(T_i)$  for each  $i$  to obtain  $\frac{1}{n} \sum_{i=1}^n \left( \mathbb{1}_1(T_i) \sum_{j \in \mathcal{C}_i} \frac{1}{|\mathcal{C}_{j1}|} - \mathbb{1}_0(T_i) \sum_{j \in \mathcal{C}_i} \frac{1}{|\mathcal{C}_{j0}|} \right) Y_i(T_i)$ <sup>9</sup>. The weights are well-defined as long as  $\mathcal{C}_{jt}$  is non-empty, which occurs with probability approaching one. By including  $j$  into its comparison group and increasing the count  $|\mathcal{C}_{j(1-T_j)}|$  by one unit, the weights are also well-defined in finite samples and the estimator will have appropriate asymptotic behavior.

Formally, this leads to the estimator

$$\widehat{\text{ATE}} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^n (\hat{\omega}_{i1} - \hat{\omega}_{i0}) Y_i(T_i) \quad (5)$$

where  $\hat{\omega}_{it} \stackrel{\text{def}}{=} \mathbb{1}_t(T_i) \sum_{j \in \mathcal{C}_i} \frac{1}{1 + |\mathcal{C}_{jt}|}$ . Its asymptotic distribution is described by the next theorem:

---

<sup>9</sup>This last expression holds as long as  $i \in \mathcal{C}_i$  implies  $j \in \mathcal{C}_i$ , which is true for the comparison groups previously discussed.

**Theorem 2.3.** Suppose  $\mathbb{E}[Y_j(t)^2] < \infty$ , condition (iii) of Theorem 2.2 holds,  $\frac{\sqrt{n}}{s_n^2} \rightarrow 0$ , for  $t \in \{0, 1\}$ , and the density of covariate satisfies  $f \geq C > 0$ . Then,

$$\sqrt{n}(\widehat{\text{ATE}} - \text{ATE}) \rightarrow^d N\left(0; \mathbb{E}\left[\frac{\mathbb{V}[Y_i(1)|X_i]}{p(X_i)} + \frac{\mathbb{V}[Y_i(0)|X_i]}{1-p(X_i)}\right] + \mathbb{V}[\text{CATE}(X_i)]\right)$$

where  $\omega_i \stackrel{\text{def}}{=} \omega_{i1} - \omega_{i0}$ ,  $\omega_{it} \stackrel{\text{def}}{=} \frac{\mathbb{1}_t(T_i)}{\mathbb{P}[T_i=t|X_i]}$  for  $t \in \{0, 1\}$ .

The theorem is proven in the appendix. The overall proof strategy is similar to that of theorem 1 in [Hirano, Imbens and Ridder \(2003\)](#). The result is not straightforward, however, because the weights are here constructed differently using the network structure so that some expressions must be adapted and the convergence properties of the weights must be derived.

It can be seen that the asymptotic variance reaches the semiparametric efficiency bound for ATE estimation as derived in [Hirano, Imbens and Ridder \(2003\)](#). This theorem allows one to conduct inference about the average treatment effect under (unobservable-robust) unconfoundedness in an asymptotically efficient way.

## 2.4 Homophily in unobservables

This section considers a link formation model that only assumes homophilic behavior in unobservables and is suitable for dense networks. The method provides insight about how to create sub-groups that are increasingly close in terms of unobservables and can be adapted to deal with some empirical concerns such as matching on treatment status. Although dense networks are less frequent, this approach is thus valuable as it covers additional network structure and applications of interest.

Now, links are formed as  $W_{ij} = \mathbb{1}_{\eta_{ij} \leq w(h(X_i^o; X_j^o) + \|X_i^u - X_j^u\|)}$ . The function  $w$  does not explicitly depend on  $n$  anymore, though some dependence could be accounted for. The function  $h$  need not be homophilic nor separable in the observed  $X^o$  but is assumed known. Most results would apply with minor modifications if a lower bound with the relevant properties can be obtained. The dimension of  $\mathcal{X}^u$  is denoted by  $d_u$ .

If the observables are relevant to the outcome equation, they can be controlled for with standard methods. For instance, one might restrict the sample to the  $j$ 's whose distance  $\|X_j^o - x_i^o\|$  is low enough or use bias-reduction techniques such as kernel smoothing. To get the main point, I will abstract from observables in the outcome equation and illustrate how to control for the unobserved components. One can consider the outcomes  $Y$  in what follows as a pre-controlled version, for instance with  $Y = \mathbb{1}_{\|X^o - x_i^o\| < \varepsilon} \tilde{Y}$  or  $Y = K((X^o - x_i^o)/b)/(\sum_j K((X_j^o - x_i^o)/b)) \tilde{Y}$  where  $\tilde{Y}$  is from the original sample. In the former case of simple truncation, it is straightforward to adapt the results.

I consider  $\frac{1}{|\mathcal{C}_{i1}(\kappa)|} \sum_{j \in \mathcal{C}_{i1}(\kappa)} Y_j(T_j) - \frac{1}{|\mathcal{C}_{i0}(\kappa)|} \sum_{j \in \mathcal{C}_{i0}(\kappa)} Y_j(T_j)$ , where the comparison group now depends on a truncation parameter  $\kappa$ . It will play a key role by placing a lower bound on  $h$ , inducing a closer distribution of unobservables among friends.

The main idea is that if there is no observed rationale for two people being friends, the reason for their friendship likely lies in the unobserved world. Then in the present model<sup>10</sup>, people that are friends despite a high value of  $h$  are less likely to differ strongly on unobservables. To see this, consider the case where people reject friendship with anyone whose quality of match is too poor (*i.e.*,  $w(x) = 1$  for  $x$  large enough). Then, two friends with an  $h$  close to the boundary must have close unobservables since a high discrepancy in unobservables would have brought  $h + \|X_i^u - X_j^u\|$  above the threshold.

Specifically, I consider  $\mathcal{C}_i(\kappa) \stackrel{\text{def}}{=} \{j \in \mathcal{N}(i) | h(x_i^o, x_j^o) > \kappa\}$ . The estimator then truncates the sums to select individuals whose observed characteristics make them unlikely to be friends.  $\kappa$  is viewed as a sequence converging to  $\infty$  at a rate to be determined.

Using this strategy, a counterpart to Theorem 2.2 can be derived with  $\kappa$ -truncation replacing *asymptotic homophily*.

**Theorem 2.4.** *Suppose  $\text{CATE}(x)$  is Hölder continuous with exponent  $\alpha$  on a neighborhood of  $x_i$ ,  $1 - w$  has bounded support, and there exists a sequence  $b_n$  such that  $\kappa$ -truncation satisfies  $w(\kappa + b_n)b_n^{d_u+1} = O(s_n^{-d})$  and a sequence  $\varepsilon_n \downarrow 0$  satisfying  $\kappa + \varepsilon_n > \sup\{\text{supp}\{w\}\}$  eventually and  $\sqrt{\lambda_n}\varepsilon_n^\alpha$ .*

*Then, the estimator  $\widehat{\text{CATE}}(x_i; \mathcal{C}_i(\kappa))$  satisfies*

$$\sqrt{|\mathcal{C}_i|}(\widehat{\text{CATE}}(x_i; \mathcal{C}_i(\kappa)) - \text{CATE}(x_i)) \rightarrow^d \mathcal{N}(0; V)$$

The condition  $w(\kappa + b_n)b_n^{d_u+1} = O(s_n^{-d})$  restricts the speed at which  $\kappa$  can increase so that the number of observations used in estimating the CATE keeps growing. The first two terms pertain to the behavior of the  $w$  function; the second pertains to the space in which unobservables live.

The term  $w(\kappa + b_n)$  comes from the increasing cost of truncating as potential connections are accepted at decreasing rates. In the presence of a discontinuity at the end of the support, *i.e.*,  $w(x) = a\mathbb{1}_{x \leq D}$  for  $a \in [0; 1[$  and  $D \in \mathbb{R}^+$ , this term disappears. The second term,  $b_n^{d_u+1}$ , is the result of forcing unobservables in a  $b_n$ -ball using values of  $h$  lying between  $\kappa$  and  $\kappa + b_n$ .

A natural estimator of the end of support, if unknown, is the highest value of  $h$  among all  $i, j$  satisfying  $W_{ij} = 1$ . One can extend the theorem to unbounded support of  $1 - w$ , provided  $1 - w$  decreases sufficiently quickly (in practice,  $w = e^{-x^2}$  provides a sufficiently fast decay). In this case, a term  $\rho_h(\kappa + b_n/2)$ , where  $\rho_h$  is (a lower bound on) the tail decay of  $h$  conditional on  $X_j^u \in B_r(x_i^u)$  for some  $r$ , arises. The reason is the tail decay of the density while one seeks increasingly larger values of  $h$ .

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<sup>10</sup>One could imagine variations on the network formation process, where this intuition would be rationalized slightly differently, leading to another estimator. One instance is briefly discussed in appendix B.



The conditions on  $\varepsilon_n$  ensure that the bias disappears sufficiently fast to allow standard inference. A more primitive statement is  $w(\kappa + \varepsilon_n) = o(s_n^{-d})$ , which mirrors conditions such as those in Assumption 1, part b), but one can focus on  $\kappa + \varepsilon_n$  eventually crossing the end of the support of  $1 - w$  if it is finite.

Again, one can consider variations on the above scheme, for instance by including friends of friends or truncating based on the number of friends in common.

A possible concern about the previous estimator is that  $h$  may feature homophilic behavior as well, in which case there is a conflict between requiring closeness in observables (to improve counterfactual quality) and difference in observables (to induce closeness in unobservables) at once.

The issue can be addressed in various ways. Often, there are variables excluded from the outcome equations that induce the necessary variation in  $h$ . A common example is physical distance, which typically affects the likelihood of friendship but not the outcome.

Alternatively, it is possible to work with ‘triangles’ of friends to control the overall distance (in terms of both observables and unobservables) directly. Compared to the previous  $\kappa$ -truncation, this strategy allows selecting observations whose values of  $h$  are low.

Adapt the comparison group  $\mathcal{C}_i(\kappa)$  to be defined as  $\{j \in \mathcal{N}(i) | \exists k \in \mathcal{N}(i) \cap \mathcal{N}(j), h_{ik} > \kappa, h_{jk} > \kappa\}$ . Members of this group are close in observables by selection and more likely to be close in unobservable because they have a common friend with  $i$  with relatively high observed differences. Formally,

**Theorem 2.5.** *The estimator  $\widehat{\text{CATE}}(x_i; \mathcal{C}_i(\kappa))$  satisfies*

$$\sqrt{|\mathcal{C}_i|}(\widehat{\text{CATE}}(x_i; \mathcal{C}_i(\kappa)) - \text{CATE}(x_i)) \rightarrow^d \mathcal{N}(0; V)$$

*if  $\text{CATE}(x)$  is Hölder continuous with exponent  $\alpha$  on a neighborhood of  $x_i$ ,  $h$  is homophilic, and the sequence  $\kappa$  satisfies  $w(\kappa + b_n)b_n^{d_u+1}\rho_h(\kappa + b_n/2)^2 = O(\frac{\lambda_n}{n^2})$  and  $w(\kappa + \varepsilon)w(\kappa) = o(\frac{\lambda_n}{n^2})$  where  $\rho_h$  is (a lower bound on) the tail decay of  $h$  conditional on  $X_j^u \in B_r(x_i^u)$  for some  $r$ .*

Note that relying on triangles of friends also provides further opportunities in terms of bias management, in case  $\lambda_n$  grows too fast for the bias to remain asymptotically negligible. Indeed, it is possible to smooth along both observed and unobserved dimensions by using the (noisy) measure of distance in the covariates suggested by triangular inequalities.

### 3 Simulations

I assess the performance of the estimators through simulations. I consider various outcome equations and network formation processes and measure the resulting root

mean square error (RMSE). For comparison, the RMSE of standard estimators – using available observed variables – is also provided. Specifically, ordinary least squares and kernel regression on observables will be used as benchmarks.

Variables are generated as follows: a random vector  $V$ , whose components are uniform, triangular, and sum of 3 uniforms, is used to construct

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 & 0 \\ -0.1 & 1 & 0.4 \\ 0 & -0.6 & 0.7 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

so that there is a non-negligible correlation structure among the components of  $X$ . In particular  $\text{corr}(X_1, X_2) = 0.28$ ,  $\text{corr}(X_1, X_3) = -0.26$ , and  $\text{corr}(X_2, X_3) = -0.32$ . The first two variables are observed, but the last one is not, *i.e.*,  $X^o = (X_1, X_2)$  and  $X^u = X_3$ .

The propensity score follows a logistic distribution with argument  $X\beta$ , where  $\beta = (0.5 \ 0.5 \ \beta_3)'$ , and treatment status is then drawn conditional on  $X$ . This ensures the propensity score varies widely, often ranging between 0.05 and 0.95. The parameter  $\beta_3$  controls the selection on unobservables. Three values will be considered: 0, 0.5, 1. In the first case, the probability of being treated does not change with  $X_3$ ; the second case puts the unobserved variable on a similar footing as the observed ones while the last value reflects a setting in which the most relevant covariate was unavailable. Thus, the performance of traditional methods that cannot account for the unobserved component is expected to deteriorate as  $\beta_3$  increases.

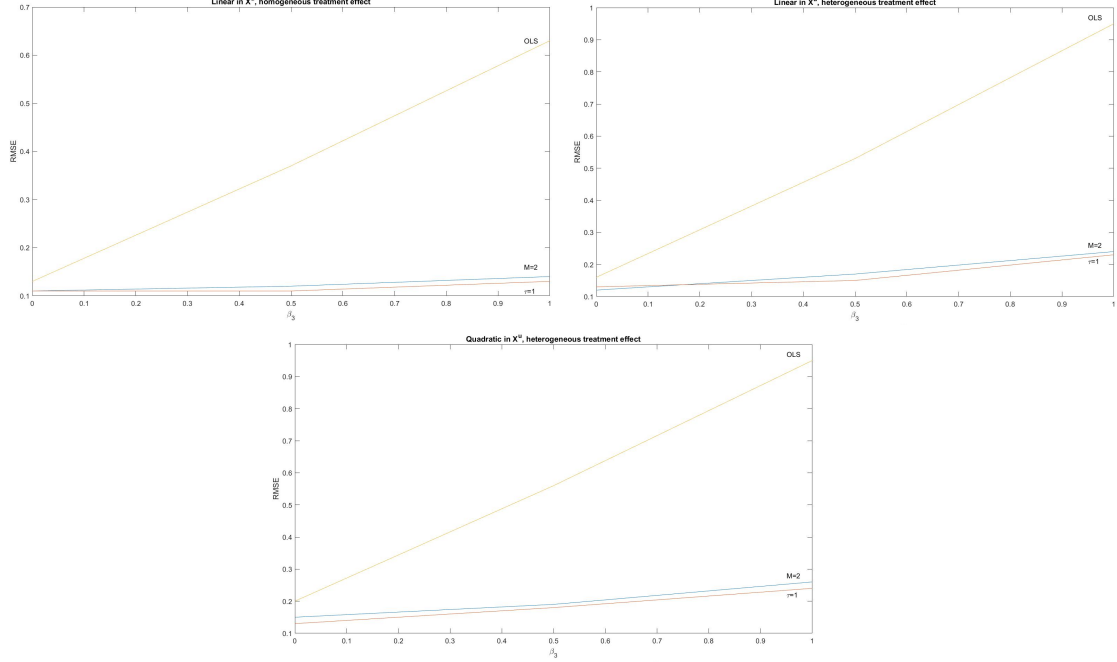
I consider three types of outcome equations:

- Homogeneous treatments effects with linear impact of unobservables
- Heterogeneous treatments effects with linear impact of unobservables
- Heterogeneous treatments effects with quadratic impact of unobservables

The default network formation processes is  $w(x) = e^{-\frac{1}{2}x^2}$ . I also consider other versions such as  $w(x) = \mathbb{1}_{x < 1}$  or  $w(x) = \max\{1 - x, 0\}$  in additional simulations. The sample size is  $n = 500$  and  $s_n$  is calibrated so that the average number of friends is roughly five as in the application. As  $n$  rises,  $s_n$  then evolves at rate  $n/\ln(n)$ .

The first function is simple and unlikely to represent an exact network formation process. It is however a useful benchmark: it constitutes an ‘ideal’ (noiseless) kernel smoothing scenario in which the network provides a deterministic truncation conditional on covariates. Furthermore, it can also be viewed as an approximation to a smooth function with a steep increase around a critical threshold. This could occur for instance when people are not allowed to, have limited possibility to, or do not wish to form ties outside of their community or some defining characteristics, or when the costs of forming friendships start rising sharply outside some comfort zone.

The full results for the ATE are displayed in appendix C, detailing the performance of the estimators as  $M$  or  $\tau$  vary. For easier review, I report the performance of the most relevant specifications  $M = 3$ ,  $\tau = 1$ , and OLS graphically below for 3 types of outcome equations.



OLS generally performs well when there is no selection on unobservables but its performance quickly deteriorates as  $\beta_3$  increases. By contrast, the estimators developed in the paper do not suffer so much and tend to dominate OLS across specifications in terms of ATE estimation. In the presence of unobserved confounding, the properties of OLS become quickly unappealing, especially when the unobserved variable has additional non-linear impacts on the outcome equation as in scenario  $B$ .

In the CATE case, one has to select an individual whose treatment effect is of interest. I focus on an individual with average observed characteristics<sup>11</sup> ( $X^o = 0$  and  $X^u$  drawn conditioning on  $X^o = 0$ ) as to target  $E[Y_i(1) - Y_i(0)|X_i^o = 0]$ .

The results are displayed in Appendix C. Standard methods perform particularly well with outcome equation  $A$  and  $\beta_3 = 0$ . In this case, there is no selection on unobservables and the treatment effect varies only according to observed characteristics. If one of these elements is modified, however, the performance deteriorates. Moreover,

<sup>11</sup> Alternative target individuals could be (i) an individual with average characteristics, including unobservables, or (ii) randomly chosen individuals. Although they lead to similar results, the latter is less meaningful here since the friend count drops frequently to 0 for individuals farther from the norm. I provide the results from randomly sampling the individual of reference for a larger dataset of 1,000 in the appendix.

the kernel estimator generally struggles because of the unobserved variable – even when it affects only the outcome and not selection into treatment.

By contrast, the estimators that leverage network information are able to perform similarly across specifications, regardless of the strength of selection in unobservables. Here, the ability to average over a larger number of observations is particularly valuable because the number of friends is small and one targets the CATE. This explains why larger values of  $M$  and lower values of  $\tau$  perform better: when the number of peers tends to be small, the estimator benefits from lowering the variance. This also explains the better performance of the higher-order friend version of the estimator. Overall, using friends of order up to 3 tends to deliver the best results and to dominate standard methods that use only observables except in the first scenario.

The type of network link function appears to have little impact on the performance. To the extent that the estimators can be interpreted as using (noisy) kernel estimates, this mirrors the result about the (lack of) influence of the choice of kernel.

## 4 Application

I provide an application of the method to the estimation of the effect of parental involvement on students’ test scores. I use the dataset from the project “Attitudes and Relationships among Primary and High School Students”, see [Portella and Kirschbaum \(2022\)](#). The dataset contains information about 4409 Brazilian high-school students, their beliefs, and friendship ties among them.

The outcome of interest is the average grade, ranging from 0 to 10, and the treatment is the level of parent support. The average is taken over math, Portuguese, English, history, geography, and art grades<sup>12</sup>. For comparability across school years, I normalize the grade by subtracting the mean over students in a given year. The dataset contains a (self-reported) score for parent support which ranges from around -3 to 1; the treatment is dichotomized by truncating around the mean of 0 and the goal is to estimate the ATE.

Although some covariates are available – age, gender, race, religion, whether parents are employed, poverty, and importance of study for the child, they are likely at best proxies for the underlying causes. Thus, omitted variable bias remains a concern, especially due to the usual unobserved ability.

The sign of the omitted variable bias is far from straightforward in this instance, even assuming that the importance of study score controls adequately for motivation. Intuitively, children with lower ability may require more attention but are also more likely through heredity to have less able parents, who may be less inclined or able to help.

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<sup>12</sup>Although additional subjects are available, including them would require dropping a substantial number of observations because of missing data.

Formally, a possible model for grade,  $y_i$ , would be  $y_i = F(a_i; m_i; s_i)$  where  $F$  is an unknown function,  $a_i$  is ability,  $m_i$  is the level of (intrinsic) motivation<sup>13</sup>, and  $s_i$  is parent support. Denoting parents’ ability by  $A_i$ , one could model  $a_i = A_i + \text{noise}_i$  and  $s_i = \delta_1 a_i + \delta_2 A_i + \text{noise}_i$  with independent noises. This would imply  $s_i = (\delta_1 + \delta_2) a_i + \text{noise}_i$ , where the signs of the deltas are likely to be negative and positive, respectively, leading to an indeterminate sign for the correlation. Hence, it is not obvious whether parents pay more attention to children with higher or lower ability. Moreover, the function  $F$  is unlikely to be linear since, *e.g.*, ability may increase the return to motivation and parent support and there may be nonlinear returns to motivation and/or ability.

As a result, not only is the OLS estimate likely to be unreliable, but it is also hard to figure out the direction of the bias. This motivates the use of alternative methods that can account for the presence of unobservables.

There is evidence that people, teenagers in particular, sort on intelligence. Some studies (Clark and Ayers, 1992; Burgess et al., 2011; Boutwell, Meldrum and Petkovsek, 2017) have documented homophilic matching on various measures of intelligence among teenagers. According to Boutwell, Meldrum and Petkovsek (2017), “preadolescent friendship dyads are robustly correlated on measures of general intelligence”.

It is thus plausible that the high school students in the sample have accounted for ability when forming friendship ties, suggesting an avenue for correcting the ability bias through the estimator developed in this paper.

There is some evidence for homophilic matching on some of the observed covariates: age, gender, the importance of study, and poverty. This can be seen in the average distance in the covariate, which is substantially reduced when the average is computed within the peer group<sup>14</sup>.

Table 1: Average distance in observables

	Age	Gender	Study score	Poverty score
Average distance (overall)	2.92	0.50	0.96	0.82
Average distance (friends)	0.64	0.26	0.84	0.76

Combined with the small number of peers, which averages four, and the goal of estimating the ATE, this suggests that the *asymptotic homophily* asymptotics developed in section 2.3 provides the relevant framework. With a limited friend count<sup>15</sup>,

<sup>13</sup>Parental involvement may affect grades directly and indirectly through increased effort. The target is the total effect that includes the mediated effect, but conditions on intrinsic motivation and ability.

<sup>14</sup>By contrast, the average distance decreases by less than 2% for other variables such as race or religious affiliation, suggesting these variables were not relevant for network formation in this sample.

<sup>15</sup>The number of people with no reported friend is relatively high, but most of these students are also missing covariates. This suggests zero friend counts are more indicative of a missing data problem

using the higher-order friendship version of the estimator makes intuitive sense and is supported by simulation results.

For each individual, I use the friends up to the third order as the comparison group, whose average size is fifteen, and compute  $\widehat{\text{ATE}}$ .

According to OLS, the ATE is 0.06 with no statistical significance at conventional levels (se 0.05). The estimator developed in the paper, however, suggests a higher effect of 0.21 (se 0.05). Because of the ability to control for unobserved ability – including potential nonlinear interactions, the latter estimate may be more reasonable. Going from a low to a high level of parent support increases the mean grade by almost a quarter of a point on average.

Table 2: Estimates and standard errors

	Estimated ATE	standard error
$\widehat{\text{ATE}}$	0.21	0.05
OLS	0.06	0.05

**Remark 1:** running a regression without the parent’s employment status and the poverty score increases the treatment effect estimate from OLS to 0.13. Going back to the model  $s_i = \delta_1 a_i + \delta_2 \text{parents' ability}_i + \text{noise}_i$ , this may indicate that OLS may be *more* biased upon controlling for parent’s employment and poverty because these variables may act as proxies for parent ability and thus increase the conditional correlation between ability and parent support.

**Remark 2:** if one believes that the importance of study score accurately reflects motivation level, results in average distance from Table 1 may suggest that the estimate would benefit from a stricter control on this variable. A simple way<sup>16</sup> to address this concern is to estimate the effect on high and low score (dichotomizing at the mean, which is 0) students and then aggregate. This results in a slightly lower average treatment effect estimate of  $0.2175 \cdot 1453/2786 + 0.0740 \cdot 1333/2786 = 0.15$ .

**Remark 3:** Switching to  $M = 4$  or considering people with at least two friends in common ( $\tau = 2$ ) as the comparison group delivers similar results: the effect is estimated to be 0.21 or 0.17. Because the estimators rely on comparison groups with different relationships to the individual, this can alleviate concerns about peer effects driving the effect.

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than of a general asocial behavior. Consequently, I restrict the sample to the sample used for OLS, which also facilitates comparability. The effective sample size is then 2777.

<sup>16</sup>If one wants to incorporate all controls, a possibility is to run a regression of  $y_i$  on the treatment and relevant controls for each  $i$  using observations in  $\mathcal{C}_i$ , then to form an optimal weighted average of the treatment effect estimates as described in Appendix B. Using all controls, this delivers treatment effect estimate of 0.15 (standard error 0.07)

## References

- Aronow, Peter M, and Cyrus Samii.** 2017. “Estimating average causal effects under general interference, with application to a social network experiment.” *The Annals of Applied Statistics*, 11(4): 1912–1947.
- Arpino, Bruno, Luca De Benedictis, and Alessandra Mattei.** 2017. “Implementing propensity score matching with network data: the effect of the General Agreement on Tariffs and Trade on bilateral trade.” *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 66(3): 537–554.
- Auerbach, Eric.** 2019. “Identification and estimation of a partially linear regression model using network data.” *arXiv preprint arXiv:1903.09679*.
- Auerbach, Eric.** 2022. “Identification and estimation of a partially linear regression model using network data.” *Econometrica*, 90(1): 347–365.
- Bender, Edward A, E Rodney Canfield, and Brendan D McKay.** 1990. “The asymptotic number of labeled connected graphs with a given number of vertices and edges.” *Random Structures & Algorithms*, 1(2): 127–169.
- Boucher, Vincent.** 2015. “Structural homophily.” *International Economic Review*, 56(1): 235–264.
- Boucher, Vincent, and Ismael Mourifié.** 2017. “My friend far, far away: a random field approach to exponential random graph models.” *The econometrics journal*, 20(3): S14–S46.
- Boutwell, Brian B, Ryan C Meldrum, and Melissa A Petkovsek.** 2017. “General intelligence in friendship selection: A study of preadolescent best friend dyads.” *Intelligence*, 64: 30–35.
- Burgess, Simon, Eleanor Sanderson, Marcela Umaña-Aponte, et al.** 2011. *School ties: An analysis of homophily in an adolescent friendship network*. Centre for Market and Public Organisation.
- Chao, Min-Te, and WE Strawderman.** 1972. “Negative moments of positive random variables.” *Journal of the American Statistical Association*, 67(338): 429–431.
- Clark, ML, and Marla Ayers.** 1992. “Friendship similarity during early adolescence: Gender and racial patterns.” *The journal of psychology*, 126(4): 393–405.
- Currarini, Sergio, Matthew O Jackson, and Paolo Pin.** 2009. “An economic model of friendship: Homophily, minorities, and segregation.” *Econometrica*, 77(4): 1003–1045.



- Demirer, Mert.** 2019. “Partial Identification of Linear Models Using Homophily in Network Data.” *Working paper*.
- De Paula, Aureo.** 2017. “Econometrics of network models.” Vol. 1, 268–323, Cambridge University Press Cambridge.
- Dzanski, Andreas.** 2019. “An empirical model of dyadic link formation in a network with unobserved heterogeneity.” *Review of Economics and Statistics*, 101(5): 763–776.
- Erdős, Paul, Alfréd Rényi, et al.** 1960. “On the evolution of random graphs.” *Publ. math. inst. hung. acad. sci.*, 5(1): 17–60.
- Forastiere, Laura, Edoardo M Airoidi, and Fabrizia Mealli.** 2020. “Identification and estimation of treatment and interference effects in observational studies on networks.” *Journal of the American Statistical Association*, 1–18.
- Goldsmith-Pinkham, Paul, and Guido W Imbens.** 2013. “Social networks and the identification of peer effects.” *Journal of Business & Economic Statistics*, 31(3): 253–264.
- Graham, Bryan S.** 2015. “Methods of identification in social networks.” *Annu. Rev. Econ.*, 7(1): 465–485.
- Graham, Bryan S.** 2016. “Homophily and transitivity in dynamic network formation.” National Bureau of Economic Research.
- Graham, Bryan S.** 2017. “An econometric model of network formation with degree heterogeneity.” *Econometrica*, 85(4): 1033–1063.
- Hirano, Keisuke, Guido W Imbens, and Geert Ridder.** 2003. “Efficient estimation of average treatment effects using the estimated propensity score.” *Econometrica*, 71(4): 1161–1189.
- Huber, Martin, and Andreas Steinmayr.** 2021. “A framework for separating individual-level treatment effects from spillover effects.” *Journal of Business & Economic Statistics*, 39(2): 422–436.
- Imbens, Guido W.** 2004. “Nonparametric estimation of average treatment effects under exogeneity: A review.” *Review of Economics and statistics*, 86(1): 4–29.
- Imbens, Guido W, and Donald B Rubin.** 2015. *Causal inference in statistics, social, and biomedical sciences*. Cambridge University Press.
- Imbens, Guido W, and Jeffrey M Wooldridge.** 2009. “Recent developments in the econometrics of program evaluation.” *Journal of economic literature*, 47(1): 5–86.

- Jackson, Matthew O.** 2010. *Social and economic networks*. Princeton university press.
- Jackson, Matthew O, Zhongjian Lin, and Ning Neil Yu.** 2020. “Adjusting for peer-influence in propensity scoring when estimating treatment effects.” *Available at SSRN 3522256*.
- Johnsson, Ida, and Hyungsik Roger Moon.** 2021. “Estimation of peer effects in endogenous social networks: Control function approach.” *Review of Economics and Statistics*, 103(2): 328–345.
- Lazarsfeld, Paul F, Robert K Merton, et al.** 1954. “Friendship as a social process: A substantive and methodological analysis.” *Freedom and control in modern society*, 18(1): 18–66.
- Liu, Lan, Michael G Hudgens, Bradley Saul, John D Clemens, Mohammad Ali, and Michael E Emch.** 2019. “Doubly robust estimation in observational studies with partial interference.” *Stat*, 8(1): e214.
- McPherson, Miller, Lynn Smith-Lovin, and James M Cook.** 2001. “Birds of a feather: Homophily in social networks.” *Annual review of sociology*, 27(1): 415–444.
- Mele, Angelo.** 2022. “A structural model of homophily and clustering in social networks.” *Journal of Business & Economic Statistics*, 40(3): 1377–1389.
- Moody, James.** 2001. “Race, school integration, and friendship segregation in America.” *American journal of Sociology*, 107(3): 679–716.
- Newman, Mark.** 2018. *Networks*. Oxford university press.
- Neyman, Jerzy.** 1923. “On the application of probability theory to agricultural experiments. Essay on principles.” *Ann. Agricultural Sciences*, 1–51.
- Portella, Alysson, and Charles Kirschbaum.** 2022. “Replication Data for: Racial Social Norms among Brazilian Students: Academic Performance, Social Status and Racial Identification.”
- Rubin, Donald B.** 1974. “Estimating causal effects of treatments in randomized and nonrandomized studies.” *Journal of educational Psychology*, 66(5): 688.
- Sánchez-Becerra, Alejandro.** 2021. “Spillovers, Homophily, and Selection into Treatment: The Network Propensity Score.” *Job market paper*.
- Sofrygin, Oleg, and Mark J van der Laan.** 2016. “Semi-parametric estimation and inference for the mean outcome of the single time-point intervention in a causally connected population.” *Journal of causal inference*, 5(1).

## Appendix A: Proofs

### 4.1 Lemma 2.1

*Proof.* Consider a sample of  $(n + 1)$  observations, including  $i$ . By independence, the degree of individual  $i$  satisfies  $d_i \stackrel{\text{def}}{=} |\mathcal{N}(i)| = \sum_{j \neq i, j=1}^{n+1} \mathbb{1}_{\eta_{ij} \leq w_n(\|x_i - x_j\|)} \sim \mathcal{B}(n, p_n)$  with  $p_n \stackrel{\text{def}}{=} \mathbb{E}[w_n(\|x_i - X_j\|)] = \int_{\mathbb{R}^d} w_n(\|x_j - x_i\|) f(x_j) dx_j$  by the law of iterated expectation and distributional properties of  $\eta$ . If  $np_n \rightarrow C > 0$ , the friend count asymptotically follows a Poisson distribution with parameter  $C$ , while slower sequences  $p_n$  induce an unbounded friend count: by Chebyshev inequality we have for any  $N$  and large sufficiently large  $n$

$$\begin{aligned} \mathbb{P}[\mathcal{N}(i) \leq N] &\leq \mathbb{P}[|\mathcal{N}(i) - np_n| \geq np_n - N] \\ &\leq \frac{np_n(1 - p_n)}{(np_n - N)^2} \\ &= \frac{1 - p_n}{np_n(1 - \frac{N}{np_n})^2} \end{aligned}$$

and thus  $\mathbb{P}[\mathcal{N}(i) \leq N] \rightarrow 0$  as long as  $np_n = n \int_{\mathbb{R}^d} w_n(\|x_j - x_i\|) f(x_j) dx_j \rightarrow \infty$ .

This is the case when the network formation is regularly asymptotically homophilic. Indeed, consider

$$\begin{aligned} \int_{\mathbb{R}^d} w_n(\|x - x_i\|) f(x) dx &= \int_{\mathbb{R}^d} g(s_n \|x - x_i\|) f(x) dx \\ &\geq \underline{f} \int_{B_{\underline{r}}(x_i)} g(s_n \|x - x_i\|) dx \\ &\geq \frac{\underline{f}}{s_n^d} \int_{B_{s_n \underline{r}}(0)} g(\|y\|) dy \\ &\geq \frac{\underline{f}}{s_n^d} \int_{B_{\underline{r}}(0)} g(\|y\|) dy \end{aligned}$$

Since  $\int_{\mathbb{R}^d} g(\|y\|) dy > 0$  and  $g$  is decreasing,  $\int_{B_{\underline{r}}(0)} g(\|y\|) dy > 0$  (otherwise,  $\int_{B_{s_n \underline{r}}(0)} g(\|y\|) dy = 0$  implies  $g \equiv 0$ , contradicting the former assertion). Thus, using  $ns_n^{-d} \rightarrow \infty$ , *regular asymptotic homophily* suffices to establish  $\lim_{n \rightarrow \infty} n \int_{\mathbb{R}^d} w_n(\|x_j - x_i\|) f(x_j) dx_j = \infty$ .

For part a), write  $w_n(\|x_i - x_j\|)$  as  $w_{ij}$  and let  $\varepsilon > 0$  be small enough for  $g(\varepsilon) > 0$  (such an  $\varepsilon$  exists: otherwise  $g = 0$  except possibly at the origin, contradicting  $\int_{\mathbb{R}^d} g(\|y\|) dy > 0$ ).

Then, for  $n$  large,  $j$  is a friend of  $m$ -th order with probability

$$\begin{aligned}
& \sum_{j_0, \dots, j_m} \mathbb{P}[W_{j_k j_{k+1}} = 1, W_{j_k j_l} = 0 \mid l \neq k \pm 1] \\
&= (n-1) \cdots (n-m+1) \mathbb{E}[\mathbb{E}[\mathbb{1}\{w_{j_k j_{k+1}} \geq \eta_{j_k j_{k+1}}, w_{j_k j_l} \geq \eta_{j_k j_l}, \forall k, l > k+1 \mid \{X_{j_k}\}\}]] \\
&= \frac{(n-1)!}{(n-m)!} \mathbb{E} \left[ \prod_{k=0}^{m-1} w_{j_k j_{k+1}} \prod_{l \neq k \pm 1} (1 - w_{j_k j_l}) \right] \\
&\geq C \frac{(n-1)!}{(n-m)!} \int_{(\mathbb{R}^d)^m} \prod_{k=0}^{m-1} g(s_n \|x_{j_k} - x_{j_{k+1}}\|) \prod_{k=1}^m f(x_{j_k}) \prod_{l > k+1} w_{j_k j_l} d \left( \prod_{k=1}^m x_{j_k} \right) \\
&\geq C \frac{(n-1)!}{(n-m)!} \int_{\left( B_{\frac{\min\{\varepsilon, r\}}{2s_n}}(x_i) \right)^m} g(\varepsilon)^m \prod_{k=1}^m f(x_{j_k}) d \left( \prod_{k=1}^m x_{j_k} \right) \\
&\geq C \frac{(n-1)!}{(n-m)!} \int_{\left( B_{\frac{\min\{\varepsilon, r\}}{2s_n}}(x_i) \right)^m} 1 d \left( \prod_{k=1}^m x_{j_k} \right) \\
&= C \frac{(n-1)!}{(n-m)!} s_n^{-md}
\end{aligned}$$

Noting that  $C$  depends on  $m$  only through a factor  $c^m$ ,  $c \stackrel{\text{def}}{=} \underline{f} w_{\frac{\pi^{d/2}}{\Gamma(d/2+1)}} g(\varepsilon)$ , a lower bound for connections up to order  $M$  is obtained from

$$\begin{aligned}
\sum_{m=1}^M c^m s_n^{-dm} \frac{(n-1)!}{(n-m)!} &\geq s_n^{-d} \sum_{m=1}^M (c s_n^{-d} (n-M+1))^{m-1} \\
&= s_n^{-d} \frac{(s_n^{-d} c (n-M+1))^M - 1}{s_n^{-d} c (n-M+1) - 1} \\
&= O \left( s_n^{-d} (n s_n^{-d})^{M-1} \right)
\end{aligned}$$

The second part of the statement follows directly from  $|\cup_{m=1}^M \mathcal{N}^m(i)| \geq \mathcal{N}(i) > M$  for any  $M$  with probability approaching one.

Finally, for part b), the probability of interest is bounded from below as

$$\begin{aligned}
\mathbb{P}[|\mathcal{N}_t(i; j)| \geq \tau] &\geq \mathbb{P}[|\mathcal{N}_t(i; j)| \geq \tau \mid X_j \in B_{s_n^{-1}}(x_i)] \mathbb{P}[X_j \in B_{s_n^{-1}}(x_i)] \\
&\geq \mathbb{P}[|\mathcal{N}_t(i; j)| \geq \tau \mid X_j \in B_{s_n^{-1}}(x_i)] C s_n^{-d}
\end{aligned}$$

The number of friends in common, conditional on  $X_j$  being in the ball  $B_{s_n^{-1}}(x_i)$ , follows a binomial distribution with parameters  $(n-1, \mathbb{P}[W_{il} = W_{jl} = 1 \mid X_j \in B_{s_n^{-1}}(x_i)])$ . The probability is bounded from below by  $C s_n^{-d}$ .

Indeed, noting that  $\|x_j - x_k\| \leq s_n^{-1} + \|x_i - x_k\|$  when  $x_j \in B_{s_n^{-1}}(x_i)$ ,

$$\begin{aligned}
& \mathbb{P}[w_{ik}w_{jk}|X_j \in B_{s_n^{-1}}(x_i)] \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w_{ik}w_{jk} \frac{f(x_j)}{\int_{B_{s_n^{-1}}(x_i)} f} \frac{\mathbb{1}_{B_{s_n^{-1}}(x_i)} f}{\int_{B_{s_n^{-1}}(x_i)} f} f(x_k) dx_j dx_k \\
&\geq C s_n^d \int_{\mathbb{R}^d} \int_{B_{s_n^{-1}}(x_i)} g(s_n \|X_k - x_i\|) g(1 + s_n \|X_k - x_i\|) f(x_j) f(x_k) dx_j dx_k \\
&\geq C s_n^{-d} \int_{\mathbb{R}^d} g(\|y\|) g(1 + \|y\|) f\left(\frac{y}{s_n} + x_i\right) dy \\
&\geq C s_n^{-d} O(1)
\end{aligned}$$

where the change of variable  $s_n(x_k - x_i) \rightarrow y$  was used.

Since the Bernoulli trials from the binomial are conducted  $n - 1$  times with a probability at least of order  $s_n^{-d}$ , the probability of having at least  $\tau$  friends in common approaches one as long as  $n s_n^{-d} \rightarrow \infty$  as in a). Thus,  $\mathbb{P}[|\mathcal{N}_t(i; j)| \geq \tau] \geq C \mathbb{P}[X_j \in B_{s_n^{-1}}(x_i)] = C s_n^{-d}$ .

□

## 4.2 Theorem 2.1

*Proof. Part a)* Lemma 2.1 ensures asymptotics apply. By the overlap assumption, the number of treated friends and untreated friends both grow to infinity.

Consider the treated group. Letting  $\tilde{Y}_{j;n}$  denote i.i.d. draws from  $Y(1)|\mathcal{C}_{i1}$ , we have  $\sup_n \mathbb{E}[(\tilde{Y}_{j;n})^2] < \infty$  since  $\mathbb{E}[Y_j(1)^2|\mathcal{C}_{i1}] \rightarrow \mathbb{E}[Y_j(1)^2|X_j = x_i, T_j = 1] = \mathbb{E}[Y_j(1)^2|X_j = x_i]$  by continuity (this is shown in details for  $Y_j(1)$  below) and unconfoundedness. Then,  $\frac{1}{c_{i1}} \sum_{j \in \mathcal{C}_{i1}} (Y_j(T_j) - \mathbb{E}[Y_j(1)|\mathcal{C}_{i1}]) + \mathbb{E}[Y_j(1)|\mathcal{C}_{i1}] - \mathbb{E}[Y_j(1)|X_j = x_i] \rightarrow^p 0$  by the law of large numbers for triangular arrays and continuity, which is formally established by bounding  $\mathbb{E}[Y_j(1)|\mathcal{C}_{i1}] - \mathbb{E}[Y_j(1)|X_j = x_i]$  as follows.

The expectation can be divided into an integral over a  $\varepsilon$ -ball centered at  $x_i$  and an integral over its complement. Within the ball,

$$\begin{aligned}
& \left| \int_{B_\varepsilon(x_i)} (\mathbb{E}[Y_j(1)|X = x] - \mathbb{E}[Y_j(1)|X = x_i]) f_{X|\mathcal{C}_i, T} dx \right| \\
&\leq \sup_{B_\varepsilon} |\mathbb{E}[Y_j(1)|X = x] - \mathbb{E}[Y_j(1)|X = x_i]| \\
&= O(\varepsilon^\alpha)
\end{aligned}$$

by Hölder continuity.

The integral outside the ball,  $\int_{B_\varepsilon^c(x_i)} (\mathbb{E}[Y_j(1)|X = x] - \mathbb{E}[Y_j(1)|X = x_i]) f_{X|\mathcal{C}_i, T} dx$ , can be bounded using information about the composition of  $\mathcal{C}_i$ . I proceed with the two groups separately.

For the first choice of counterfactuals, *i.e.*,  $\mathcal{C}_i = \cup_{m=1}^M \mathcal{N}^m(i)$ , note that  $\varepsilon < \|x_j - x_i\| = \|\sum_{k=0}^{m-1} x_{j_{k+1}} - x_{j_k}\| \leq \sum_{k=0}^{m-1} \|x_{j_{k+1}} - x_{j_k}\|$  and thus outside the ball

$$\begin{aligned}
f_{\mathcal{C}_i|X_j, T_j} &= \sum_{m=1}^M \mathbb{P}[W_{ij}^m = 1, W_{ij}^{m'} = 0 \mid m' < m | X, T] \\
&= \sum_{m=1}^M \mathbb{E}[\mathbb{P}[W_{ij}^m = 1, W_{ij}^{m'} = 0 \mid m' < m | \{X_{j_k}\}, T] | X_j, T_j] \\
&\leq \sum_{m=1}^M \int \prod_{k=0}^{m-1} w_{j_{k+1}j_k} \prod_{l \neq k} (1 - w_{j_k j_l}) f_{X_{-j} | X_j} \\
&\leq \sum_{m=1}^M w_n \left( \frac{\varepsilon}{m} \right)^m
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_{B_\varepsilon^c(x_i)} (\mathbb{E}[Y_j(1) | X = x] - \mathbb{E}[Y_j(1) | X = x_i]) f_{X|\mathcal{C}_i, T} dx \right| \\
&\leq \int_{B_\varepsilon^c(x_i)} |\mathbb{E}[Y_j(1) | X = x] - \mathbb{E}[Y_j(1) | X = x_i]| \frac{\mathbb{P}[\mathcal{C}_i | X = x] f_{X, T}(x, 1)}{\mathbb{P}[\mathcal{C}_i, T = 1]} dx \\
&\leq \frac{\sum_{m=1}^M \left( w_n \left( \frac{\varepsilon}{m} \right) \right)^m (\mathbb{E}[|\mathbb{E}[Y_j(1) | X]|] + \mathbb{E}[Y_j(1) | X = x_i])}{\left( \sum_{m=1}^M \sum_{j_0, \dots, j_m} \mathbb{P}[W_{j_k j_{k+1}} = 1, W_{j_k j_l} = 0 \mid l \neq k+1] \right)} \\
&\leq C \frac{n}{\lambda_n M} \sum_{m=1}^M \left( w_n \left( \frac{\varepsilon}{m} \right) \right)^m
\end{aligned}$$

Hence, the integrals are  $O(\varepsilon_n^\alpha)$  and  $O(\frac{n}{\lambda_n M} \sum_{m=1}^M (1 - w_n(\frac{\varepsilon_n}{m}))^m)$ , respectively, and the bias disappears provided  $\sum_{m=1}^M (1 - w_n(\frac{\varepsilon_n}{m}))^m = o(\lambda_n \frac{M}{n})$ . Applying the same reasoning to the non-treated gives the result for the first choice of comparison group.

For the second comparison group, the posterior (leaving the conditioning on  $T = t$  implicit) reads

$$\begin{aligned}
f_{X_j|\mathcal{C}_i} &= \frac{\mathbb{P}[\mathcal{C}_i|X_j]f(x)}{\int \mathbb{P}[\mathcal{C}_i|X_j]f(x) \, dx} \\
&= \frac{\mathbb{E}[\mathbb{P}[W_{i1} = W_{j1} = \dots = W_{im} = W_{jm} = 1|X_1, \dots, X_m, X_j]]f(x)}{\int \mathbb{E}[\mathbb{P}[W_{i1} = W_{j1} = \dots = W_{im} = W_{jm} = 1|X_1, \dots, X_m, X_j]]f(x) \, dx} \\
&= \frac{\prod_{\tau=1}^m \int w_n(\|x_i - x_k\|)w_n(\|x_j - x_k\|)f(x_k)dx_k f(x)}{\prod_{\tau=1}^m \int w_n(\|x_i - x_k\|)w_n(\|x_j - x_k\|)f(x_k)dx_k f(x) \, dx} \\
&\leq C s_n^d \sum_{m=\tau}^{n-2} \left( w_n \left( \frac{\varepsilon_n}{2} \right) \right)^m \\
&= C s_n^d \left( w_n \left( \frac{\varepsilon_n}{2} \right) \right)^\tau O(1)
\end{aligned}$$

so that we want  $(w_n(\frac{\varepsilon_n}{2}))^\tau = o(s_n^{-d})$

**Part b)** one can compute

$$\begin{aligned}
\mathbb{E}[Y_j(t)|\mathcal{C}_i, T_j = t] &= \mathbb{E}[\mathbb{E}[Y_j(t)|X_j]|\mathcal{C}_i, T_j = t] \\
&= \int_{\mathbb{R}^d} \mathbb{E}[Y_j(t)|x_j] f_{x_j|\mathcal{C}_i, x_i, t} dx_j \\
&= \int_{\mathbb{R}^d} \mathbb{E}[Y_j(t)|x_j] f_{x_j|\mathcal{C}_i, x_i, t} dx_j \\
&= \int_{\mathbb{R}^d} \mathbb{E}[Y_j(t)|x_j] \frac{w_{ij} p_t(x_j) f_{x_j}}{\int_{\mathbb{R}^d} w_{ik} p_t(x_k) f_{x_k} dx_k} dx_j \\
&= \int_{\mathbb{R}^d} \mathbb{E}[Y_j(t)|x_i + y/s_n] \frac{g(\|y\|) p_t(x_i + y/s_n) f_{x_i+y/s_n}}{\int_{\mathbb{R}^d} g(\|z\|) p_t(x_i - z/s_n) f_{x_i-z/s_n} dz} dy \\
&= \mathbb{E}[Y_j(t)|x_i] \int_{\mathbb{R}^d} \frac{g(\|y\|)}{\int_{\mathbb{R}^d} g(\|z\|) dz} dy + O(s_n^{-1}) \int_{\mathbb{R}^d} g(\|y\|) y dy + O(s_n^{-2}) \\
&= \mathbb{E}[Y_j(t)|x_i] + O(s_n^{-2})
\end{aligned}$$

using the changes of variable  $y = s_n(x_j - x_i)$  and  $z = s_n(x_k - x_i)$  and a first-order expansion in densities, expectation, and propensity scores (noting that  $\frac{A+a_n}{B+b_n} = \frac{A}{B} \left(1 + \frac{a_n}{A}\right) \left(1 - \frac{b_n}{B+b_n}\right) = \frac{A}{B} + O(a_n) + O(b_n) + O(a_n b_n)$ ) and that second-order derivatives are bounded.

□



### 4.3 Theorem 2.2

*Proof.* By Lindeberg-Feller's central limit theorem, where Lindeberg's condition follows from the dominated convergence theorem,

$$\begin{aligned} & \sqrt{|\mathcal{C}_i|} \left( \frac{\frac{1}{|\mathcal{C}_{i1}|} \sum_{j \in \mathcal{C}_{i1}} (Y_j(T_j) - \mathbb{E}[Y_j(1)|\mathcal{C}_i, T_j = 1])}{\frac{1}{|\mathcal{C}_{i0}|} \sum_{j \in \mathcal{C}_{i0}} (Y_j(T_j) - \mathbb{E}[Y_j(0)|\mathcal{C}_i, T_j = 0])} \right) \\ & \rightarrow^d \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} \frac{\mathbb{V}[Y_j(1)|X_j=x_i]}{\mathbb{P}[T_j=1|X_j=x_i]^2} & 0 \\ 0 & \frac{\mathbb{V}[Y_j(0)|X_j=x_i]}{\mathbb{P}[T_j=0|X_j=x_i]^2} \end{pmatrix} \right) \end{aligned}$$

It is now required that  $\sqrt{n}(\mathbb{E}[Y_j(t)|\mathcal{C}_i, T_j = t] - \mathbb{E}[Y_j(t)|X_j = x_i]) \rightarrow 0$  sufficiently fast. From the consistency proof, it is seen that the bias disappears faster than root  $n$  when  $\varepsilon_n = n^{-\gamma}$  under the conditions of the theorem.  $\square$

### 4.4 Theorem 2.3

The overall proof strategy is similar to that of theorem 1 in [Hirano, Imbens and Ridder \(2003\)](#). Because the weights are here constructed differently using the network structure, however, some expressions must be adapted and the convergence properties of the weights must be derived.

The proof establishes that the estimator is  $\sqrt{n}$ -equivalent to the efficient score that uses the true propensity score and conditional expectation. As in [Hirano, Imbens and Ridder \(2003\)](#), I present the proof for  $Y(0) \equiv 0$  to shorten the notation.

Let  $\hat{\theta}_i \stackrel{\text{def}}{=} \sum_{j \in \mathcal{C}_i} \frac{1}{1+|\mathcal{C}_{j1}|}$  and  $\theta_{ni} \stackrel{\text{def}}{=} \mathbb{E}[\hat{\theta}_i | X_i = x_i]$ . Also let  $\theta(x) \stackrel{\text{def}}{=} \frac{1}{p(x)}$  where  $p(x) \stackrel{\text{def}}{=} \mathbb{P}[T_i = 1 | X_i = x]$ . The result to be established is then

$$\left\| \sqrt{n}(\hat{\text{ATE}} - \text{ATE}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n (\theta_i T_i Y_i - \text{ATE} - \mathbb{E}[Y_i(1)|X_i])(\theta_i T_i - 1) \right\| = o_p(1) \quad (6)$$

This implies asymptotic linearity and leads to the asymptotic variance presented in the theorem (for the case  $\mathbb{V}[Y_i(0)|X_i = x] \equiv 0$ ). The left-hand-side is proven to be  $o_p(1)$  be

adding a subtracting a few terms and bounding the differences separately. Specifically,

$$\begin{aligned}
& \sqrt{n}(\hat{\text{ATE}} - \text{ATE}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n (\theta_i T_i Y_i - \text{ATE} - \mathbb{E}[Y_i(1)|X_i])(\theta_i T_i - 1)) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{\theta}_i T_i Y_i - \theta_i T_i Y_i + \frac{\theta_i}{\hat{\theta}_i} T_i Y_i (\theta_i - \hat{\theta}_i)) \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( -\frac{\theta_i}{\hat{\theta}_i} T_i Y_i (\theta_i - \hat{\theta}_i) + \int \frac{\mu_1(x)}{\hat{\theta}(x)} (\theta(x) - \hat{\theta}(x)) F(dx) \right) \\
&- \sqrt{n} \int \frac{\mu_1(x)}{\hat{\theta}(x)} (\theta(x) - \hat{\theta}(x)) F(dx) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{\delta}(X_i) \frac{T_i - p_n(X_i)}{\sqrt{p_n(X_i)(1 - p_n(X_i))}} \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n (\tilde{\delta}_n - \delta_n)(T_i - p_n(X_i)) \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n (\delta_n(T_i - p(X_i)) - \delta_0(T_i - p(X_i)))
\end{aligned} \tag{7}$$

where, with  $F$  denoting the cumulative distribution function of  $X$ ,

$$\tilde{\delta}_n \stackrel{\text{def}}{=} - \int \mathbb{E}[Y|Z = z] \hat{p}(z) \sum_{j=1}^n \left[ \frac{\mathbb{1}_{\mathcal{C}_z}(X_j) \mathbb{1}_{\mathcal{C}_j}(X_i)}{(\tilde{p}_n(z) + |\mathcal{C}_j|^{-1})^2} \right] dF(z) \tag{8}$$

$$\delta_n \stackrel{\text{def}}{=} - \int \mathbb{E}[Y|Z = z] p(z) \mathbb{E} \left[ \frac{|\mathcal{C}_z|}{|\mathcal{C}_j|^2 (p_n(z) + |\mathcal{C}_j|^{-1})^2} \middle| j \in \mathcal{C}_x \cap \mathcal{C}_i \right] dF(z) \tag{9}$$

$$\delta_0 \stackrel{\text{def}}{=} - \frac{\mathbb{E}[Y_i|X_i]}{p(X_i)} \tag{10}$$

A few preliminary results are useful to establish the bounds. Let  $X$  follow a Binomial distribution with parameters  $(n, p)$ . It follows from definition that

$$\begin{aligned}
\mathbb{E} \left[ \frac{1}{1+X} \right] &= \sum_{x=0}^n \frac{1}{1+x} \frac{n}{x! (n-x)!} p^x (1-p)^{n-x} \\
&= \frac{1}{(n+1)p} \sum_{k=1}^n \binom{n+1}{k} p^k (1-p)^{n+1-k} \\
&= \frac{1 - (1-p)^{n+1}}{(n+1)p}
\end{aligned} \tag{11}$$

This is a particular case of the general result in [Chao and Strawderman \(1972\)](#), who obtain a closed-form expression for  $\mathbb{E} \left[ \frac{1}{C+X} \right]$ . Note that from Jensen inequality, we also have the lower bound  $\mathbb{E} \left[ \frac{1}{1+X} \right] \geq \frac{1}{1+np}$ .

Next, because  $X$  is positive,

$$\begin{aligned}
\mathbb{E} \left[ \frac{1}{(1+X)^2} \right] &= \mathbb{E} \left[ \frac{1}{(1+X)(2+X)} \right] + \mathbb{E} \left[ \frac{1}{(1+X)^2(2+X)} \right] \\
&\leq 2 \mathbb{E} \left[ \frac{1}{(1+X)(2+X)} \right] \\
&= 2 \frac{1}{(n+1)(n+2)p^2} \sum_{x=0}^n \frac{(n+2)!}{(x+2)!(n-x)!} p^{x+2} (1-p)^{n-x} \\
&= 2 \frac{1}{(n+1)(n+2)p^2} (1 - (1-p)^{n+2} - (1-p)^{n+1} p(n+2)) \\
&\leq \frac{2}{(n+1)(n+2)p^2}
\end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned}
\mathbb{V} \left[ \frac{1}{1+X} \right] &\leq \frac{2}{(n+1)(n+2)p^2} - \frac{1}{(1+np)^2} \\
&= \frac{1}{(np)^2} \left( \frac{2n^2}{(n+1)(n+2)} - \left( \frac{np}{1+np} \right)^2 \right) \\
&\leq \frac{1}{(np)^2} (2-1) \\
&= \frac{1}{(np)^2}
\end{aligned} \tag{13}$$

As a result for a success probability that depends on an underlying variable  $z$ ,

$$\begin{aligned}
\mathbb{P} \left[ \sup_z \left| \frac{1}{1+X} - \frac{1 - (1-p(z))^{n+1}}{(n+1)p(z)} \right| \geq \varepsilon \right] &= \frac{\mathbb{E} \left[ \sup_z \left| \frac{1}{1+X} - \frac{1 - (1-p(z))^{n+1}}{(n+1)p(z)} \right|^2 \right]}{\varepsilon^2} \\
&= \sup_z \frac{\mathbb{V} \left[ \frac{1}{1+X} \right]}{\varepsilon^2} \\
&\leq \sup_z \frac{1}{(np(z)\varepsilon)^2}
\end{aligned} \tag{14}$$

using Chebyshev, dominated convergence, and previous results. The reciprocal of the binomial converges uniformly if the probability of success remains bounded away from 0.

Now, notice that

$$\begin{aligned}
\mathbb{V} \left[ \mathbb{1}_{\mathcal{C}_i}(j) \frac{1}{1 + |\mathcal{C}_{jt}|} \right] &= \mathbb{E} \left[ \mathbb{1}_{\mathcal{C}_i}(j) \frac{1}{(1 + |\mathcal{C}_{jt}|)^2} \right] - \left( \mathbb{E} \left[ \mathbb{1}_{\mathcal{C}_i}(j) \frac{1}{1 + |\mathcal{C}_{jt}|} \right] \right)^2 \\
&= \mathbb{E} \left[ \frac{1}{(1 + |\mathcal{C}_{jt}|)^2} \middle| j \in \mathcal{C}_i \right] - \left( \mathbb{E} \left[ \frac{1}{1 + |\mathcal{C}_{jt}|} \middle| j \in \mathcal{C}_i \right] \right)^2 \mathbb{P}[j \in \mathcal{C}_i] \\
&= \mathbb{V} \left[ \mathbb{1}_{\mathcal{C}_i}(j) \frac{1}{1 + |\mathcal{C}_{jt}|} \middle| j \in \mathcal{C}_i \right] \mathbb{P}[j \in \mathcal{C}_i]
\end{aligned} \tag{15}$$

Note that probabilities of belonging to the comparison group, appropriately scaled, are  $O(1)$ . They are bounded and bounded away from 0 using that  $f \geq C > 0$ . Therefore,

$$\lim_{n \rightarrow \infty} \sup_x |\hat{\theta}(x) - \theta(x)| = 0 \tag{16}$$

Moreover, expanding  $\theta_n(x)$  as in the proof of Theorem 2.1 part b), one obtains

$$|\theta_n(x) - \theta(x)| \leq C s_n^{-2} \tag{17}$$

These two results, together with overlap, bounded densities, and conditions on the rates, allow to use the bounds in Hirano, Imbens and Ridder (2003) to imply that (7) is  $O_p(\sqrt{n} s_n^{-2}) = o_p(1)$ .

## 4.5 Theorem 2.4

*Proof.* To ease notation, the argument  $\kappa$  is omitted in all instances of  $\mathcal{C}_{it}(\kappa)$ . I also make use of the following shorthands:  $\Delta_{ij}^u \stackrel{\text{def}}{=} \|x_j^u - x_i^u\|$ ,  $h_{ij} \stackrel{\text{def}}{=} h(X_j^o, X_i^o)$ .

Decompose the centered mean of the group as

$$\frac{1}{|\mathcal{C}_{it}|} \sum_{j \in \mathcal{C}_{it}} Y_j(T_j) - \mathbb{E}[Y(t)|x_i] = \frac{1}{|\mathcal{C}_{it}|} \sum_{j \in \mathcal{C}_{it}} Y_j(T_j) - \mathbb{E}[Y_j(t)|\mathcal{C}_{it}] + \mathbb{E}[Y_j(t)|\mathcal{C}_{it}] - \mathbb{E}[Y(t)|x_i]$$

for  $t \in \{0, 1\}$ .

The first term depends on sample fluctuations and converges (in probability to 0 and in distribution once re-scaled) provided the number of observations in the sum grows to infinity. The second term is a bias term and disappears under regularity conditions and the truncation  $h_{ij} > \kappa$  with  $\kappa \rightarrow \infty$ .

Thus, the main steps prove that (i) the number of observations in the sum grows to infinity and (ii) the bias disappears, for some sequence  $\kappa \rightarrow \infty$ . Consider the

probability of an observation belonging to  $\mathcal{C}_i$  first:

$$\begin{aligned}
\mathbb{P}[\mathcal{C}_i] &\geq C \int_{h_{ij} > \kappa} w_{ij} f(x_j) dx_j \\
&\geq C \int_{\Delta_{ij}^u \leq \frac{b_n}{2}, \kappa < h_{ij} < \kappa + \frac{b_n}{2}} w(\kappa + b_n) f(x_j) dx_j \\
&= C w(\kappa + b_n) \mathbb{P} \left[ \Delta_{ij}^u \leq \frac{b_n}{2}, \kappa < h_{ij} < \kappa + \frac{b_n}{2} \right] \\
&= C w(\kappa + b_n) \mathbb{P} \left[ \Delta_{ij}^u \leq \frac{b_n}{2} \right] \mathbb{P} \left[ \kappa < h_{ij} < \kappa + \frac{b_n}{2} \middle| \Delta_{ij}^u \leq \frac{b_n}{2} \right] \\
&\geq C w(\kappa + b_n) b_n^{d_u+1}
\end{aligned}$$

Following previous proofs in the *asymptotic homophily* case, it suffices to let  $\kappa \rightarrow \infty$  slowly enough as to induce a rate of  $\frac{\lambda_n}{n}$  for the above probability.

Next, let's turn to the bias term. It reads

$$\begin{aligned}
|\mathbb{E}[Y_j(T_j)|\mathcal{C}_{it}] - \mathbb{E}[Y_j(t)|x_i]| &= |\mathbb{E}[\mathbb{E}[Y_j(T_j)|X_j]|\mathcal{C}_{it}] - \mathbb{E}[Y_j(t)|x_i]| \\
&= \left| \int_{\mathbb{R}^d} (\mathbb{E}[Y_j(t)|x_j] - \mathbb{E}[Y_j(t)|x_i]) f_{X_j|\mathcal{C}_{it}}(x_j) dx_j \right| \\
&\leq \left| \int_{B_\varepsilon(x_i)} (\mathbb{E}[Y_j(t)|x_j] - \mathbb{E}[Y_j(t)|x_i]) f_{X_j|\mathcal{C}_{it}}(x_j) dx_j \right| \\
&\quad + \left| \int_{B_\varepsilon^c(x_i)} (\mathbb{E}[Y_j(t)|x_j] - \mathbb{E}[Y_j(t)|x_i]) f_{X_j|\mathcal{C}_{it}}(x_j) dx_j \right| \\
&\leq C \varepsilon^\alpha \\
&\quad + \frac{1}{\mathbb{P}[\mathcal{C}_{it}]} \int_{B_\varepsilon^c(x_i)} (\mathbb{E}[Y_j(t)|x_j] - \mathbb{E}[Y_j(t)|x_i]) \mathbb{P}[\mathcal{C}_{it}|x_j] f_{X_j,T_j}(x_j, t) dx_j \\
&\leq C \varepsilon^\alpha \\
&\quad + \frac{C}{\mathbb{P}[\mathcal{C}_{it}]} \int_{B_\varepsilon^c(x_i), h_{ij} > \kappa} (\mathbb{E}[Y_j(t)|x_j] - \mathbb{E}[Y_j(t)|x_i]) (1 - w_{ij}) f_{X_j,T_j}(x_j, t) dx_j \\
&\leq C \varepsilon^\alpha + \frac{Cn}{\lambda_n} w(\kappa + \varepsilon) (\mathbb{E}[|\mathbb{E}[Y_j(t)|X_j]| | T_j = t] + |\mathbb{E}[Y_j(t)|x_i]|)
\end{aligned}$$

so that, with  $\varepsilon \downarrow 0$ , the bias disappears as  $\kappa$  rises provided  $w(\kappa + \varepsilon) = o(\frac{\lambda_n}{n})$ .  $\square$

## 4.6 Theorem 2.5

*Proof.* The proof proceeds along the same line as that of the previous theorem, going through the same main two steps. The only changes come from modifying the bounds on the probability that an individual belongs to the comparison group and on that probability conditional on  $X_j$ .

Noting that  $\mathbb{P}[\mathcal{C}_{it}|X_j] = n\mathbb{E}[w_{ik}w_{ikjk}\mathbb{1}_{h_{ik}>\kappa, h_{jk}>\kappa}|X_j]$ , the probability that an individual belongs to the comparison group can be bounded as follows:

$$\begin{aligned}\mathbb{P}[\mathcal{C}_{it}] &\geq Cn(w(\kappa + b_n))^2 \mathbb{P}\left[h_{ik} \in \left[\kappa, \kappa + \frac{b_n}{2}\right], h_{jk} \in \left[\kappa, \kappa + \frac{b_n}{2}\right], \Delta_{ik}^u \leq \frac{b_n}{2}, \Delta_{jk}^u \leq \frac{b_n}{2}\right] \\ &\geq Cn\left(w(\kappa + b_n)b_n^{d_u+1}\rho_h\left(\kappa + \frac{b_n}{2}\right)\right)^2\end{aligned}$$

where we used

$$\begin{aligned}\mathbb{P}\left[\Delta_{ik}^u \leq \frac{b_n}{2}, \Delta_{jk}^u \leq \frac{b_n}{2}\right] &= \mathbb{P}\left[\Delta_{ik}^u \leq \frac{b_n}{2}\right] \mathbb{P}\left[\Delta_{jk}^u \leq \frac{b_n}{2} \middle| \Delta_{ik}^u \leq \frac{b_n}{2}\right] \\ &\geq C(b_n/2)^{d_u}(b_n/2)^{d_u}\end{aligned}$$

and a similar reasoning for the term involving  $h$ , under the homophily assumption.

Finally, on  $B_\varepsilon^c(x_i^u)$ , we get an upper bound for the probability that an individual belongs to the comparison group conditional on  $X_j$  from

$$\mathbb{P}[\mathcal{C}_{it}|X_j] = n\mathbb{E}[w_{ik}w_{ikjk}\mathbb{1}_{h_{ik}>\kappa, h_{jk}>\kappa}|X_j] \leq Cnw\left(\kappa + \frac{\varepsilon}{2}\right)w(\kappa)$$

□

## Appendix B: Further results

### 4.7 Clustering

Assume the covariate density is bounded and continuous almost everywhere. The clustering coefficient then satisfies

$$\begin{aligned}
C &\stackrel{\text{def}}{=} \mathbb{P}[W_{jk} = 1 | W_{ij} = W_{ik} = 1] \\
&= \mathbb{E}[\mathbb{P}[W_{jk} = 1 | W_{ij} = W_{ik} = 1, X_j, X_k] | W_{ij} = W_{ik} = 1] \\
&= \mathbb{E}[\mathbb{P}[W_{jk} = 1 | X_j, X_k] | W_{ij} = W_{ik} = 1] \\
&= \mathbb{E}[g(s_n \|x_j - x_k\|) | W_{ij} = W_{ik} = 1] \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s_n \|x_j - x_k\|) f_{X_j, X_k | W_{ij}=1, W_{ik}=1}(x_j, x_k) dx_j dx_k \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s_n \|x_j - x_k\|) \frac{\mathbb{P}[W_{ij} = 1 = W_{ik} | X_j = x_j, X_k = x_k] f(x_k) f(x_j)}{\mathbb{P}[W_{ij} = 1 = W_{ik}]} dx_j dx_k \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s_n \|x_j - x_k\|) \frac{\mathbb{E}[\mathbb{P}[W_{ij} = 1 = W_{ik} | X_i = x_i, X_j = x_j, X_k = x_k] | X_j = x_j, X_k = x_k]}{\mathbb{E}[\mathbb{P}[W_{ij} = 1 = W_{ik} | X_i = x_i, X_j = x_j, X_k = x_k]]} \\
&\quad f(x_k) f(x_j) dx_j dx_k \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(s_n \|x_j - x_k\|) \int_{\mathbb{R}^d} g(s_n \|x_i - x_j\|) g(s_n \|x_i - x_k\|) f(x_i) dx_i}{\int_{\mathbb{R}^{3d}} g(s_n \|x_i - x_k\|) g(s_n \|x_i - x_j\|) f(x_i) f(x_j) f(x_k) dx_i dx_j dx_k} f(x_j) f(x_k) dx_j dx_k \\
&= \frac{s_n^{-3d}}{s_n^{-3d}} \int_{\mathbb{R}^{3d}} \frac{g(\|\hat{x}_j - \hat{x}_k\|) g(\|\hat{x}_i - \hat{x}_j\|) g(\|\hat{x}_i - \hat{x}_k\|) f\left(\frac{\hat{x}_i}{\sqrt{s_n}}\right) f\left(\frac{\hat{x}_j}{\sqrt{s_n}}\right) f\left(\frac{\hat{x}_k}{\sqrt{s_n}}\right)}{\int_{\mathbb{R}^{3d}} g(\|\hat{x}_i - \hat{x}_k\|) g(\|\hat{x}_i - \hat{x}_j\|) f\left(\frac{\hat{x}_i}{\sqrt{s_n}}\right) f\left(\frac{\hat{x}_j}{\sqrt{s_n}}\right) f\left(\frac{\hat{x}_k}{\sqrt{s_n}}\right) d\hat{x}_i d\hat{x}_j d\hat{x}_k} d\hat{x}_i d\hat{x}_j d\hat{x}_k \\
&\rightarrow \frac{\int_{\mathbb{R}^{3d}} g(\|\hat{x}_j - \hat{x}_k\|) g(\|\hat{x}_i - \hat{x}_j\|) g(\|\hat{x}_i - \hat{x}_k\|) d\hat{x}_i d\hat{x}_j d\hat{x}_k}{\int_{\mathbb{R}^{3d}} g(\|\hat{x}_i - \hat{x}_k\|) g(\|\hat{x}_i - \hat{x}_j\|) d\hat{x}_i d\hat{x}_j d\hat{x}_k}
\end{aligned}$$

where a hat represent a change of variable such as  $\hat{x} = s_n x$ .

#### 4.7.1 A note on degree heterogeneity

One feature of empirical networks is heterogeneity in degree, *i.e.*, the number of connections varies widely from one individual to another. The asymptotic homophily framework provides two sources of such heterogeneity: (i) for given characteristics  $x_i$ , the noise  $\eta_{ij}$  generates variation across individuals with similar attributes, and (ii) variations in characteristics induce various probabilities of forming connections; individuals who have more common characteristics have an easier time forming connections so that the expected degree is not constant.

Thus, the model accommodates degree heterogeneity. Assuming the covariate density is bounded and continuous almost everywhere, note that the expected degree of individual  $i$  conditional on his or her or its covariates satisfies



$$\begin{aligned}
\frac{1}{ns_n^{-d}} \mathbb{E}[d_i] &= \frac{n-1}{ns_n^{-d}} \int g(s_n \|x_i - x_j\|) f(x_j) dx_j \\
&= \frac{n-1}{n} \int g(\|y\|) f\left(x_i + \frac{y}{s_n}\right) dy \\
&\rightarrow \int g(\|y\|) dy f(x_i)
\end{aligned} \tag{18}$$

so that the (properly scaled) expected degree is determined through the underlying density.

## 4.8 Bias management

I briefly outline various strategies to deal with such a fast growth of  $\lambda_n$  that the bias affects the asymptotics.

First, note that the analysis focuses on continuous random variables, which are responsible for asymptotic biases in nonparametric estimation. If some of the variables are discrete, they do not contribute to potential bias issues and dimensionality can be reduced accordingly.

If some variables are observed, it is possible to use more conventional estimators on the observed side and to truncate the sums involved in the construction of the corresponding estimators to account for unobservables. The use of parametric or high-order kernels type of estimators for the observed part allows one to relegate bias issues to the unobserved world. Some of these possibilities are explored in Appendix B.

As a result, the bias mostly creates difficulties when unobservables are high-dimensional.

Then, one can try and obtain an approximation or a bound on the bias to perform inference. A way to capture the magnitude of the bias is to specify (bounds on) partial effects, distributions, and link formation.

For instance, suppose  $f$  is normal with mean  $\mu$  and variance  $\Sigma = H^{-1}$ , while  $w_n(\|x\|) = e^{-s_n \|x\|^2/2}$  (it often makes sense to standardize the vector  $X$  for such network formation models). The numerator of the ‘posterior’ distribution of  $X_j$  given a friendship link with  $i$  reads

$(2\pi)^{d/2} |\Sigma|^{-1/2} e^{-1/2(x-\mu)'H(x-\mu)} e^{-1/2s_n \|x-x_i\|^2}$  Re-arranging, we obtain  $(2\pi)^{d/2} |\Sigma|^{-1/2} e^{-1/2(x-\bar{\mu})'\bar{H}(x-\bar{\mu})} e^{-1/2(x_i-\mu)'(H-H\bar{H}^{-1}H)(x_i-\mu)}$  where  $\bar{H} \stackrel{\text{def}}{=} H + s_n I$ ;  $\bar{\mu} \stackrel{\text{def}}{=} x_i - \bar{H}^{-1}H(x_i - \mu)$  Therefore, the posterior distribution is normal with mean  $\bar{\mu}$  and variance  $\bar{H}^{-1}$ .

Remark that obtaining a closed-form solution typically hinges on specifying a link formation  $w$  that combines well with the underlying density  $f$ , which is algebraically equivalent to the use of conjugate priors in Bayesian statistics.

Specifying a bound on the CATE then delivers an approximation to the bias, which allows one to perform inference by extending the normal confidence intervals of Theorem 2.2. In practice, it may be of interest to assess how large the Hölder constant, how fat the tails of unobservable distributions, or how large the derivatives of underlying functions would have to be for the confidence interval to include 0. One can also get a sense of the range of possible bias values by varying inputs and then discussing resulting confidence intervals.

## 4.9 Alternative specifications

The insight that motivated the estimators analyzed in the main text can be formalized in various ways depending on the application. In addition to the choice of nonparametric method one can also apply the main insight to more parametrized models, *e.g.*, logit propensity score or regression, to simplify the analysis, construct doubly-robust estimators, and possibly make use of more observations. I illustrate the idea in the regression case.

### 4.9.1 Regression

Consider the *asymptotic homophily* framework with a linear outcome equation:

$$y_i = \alpha + z_i' \gamma + \tau_i T_i + \varepsilon_i$$

where  $z$  are additional controls, *i.e.*, variables that do not influence network formation<sup>17</sup>.

One can perform a regression for each observation among their comparison group. Regressing  $y$  on  $z, T$  among each  $\mathcal{C}_i$  delivers an estimator  $\hat{\tau}_i$ . The estimation error can be decomposed into a bias and a variance term as  $\hat{\tau}_i - \tau_i = \hat{\tau}_i - \tau_{ni} + \tau_{ni} - \tau_i$ , where  $\tau_{ni} \stackrel{\text{def}}{=} \mathbb{E}[\tau_i | \mathcal{C}_i]$ . By standard arguments, the asymptotic distribution reads

$$\sqrt{n}(\hat{\tau}_i - \tau_i) \rightarrow \mathcal{N}(0, \Sigma) \tag{19}$$

provided second moments exist and  $\sqrt{n}/s_n^2 \rightarrow 0$ .

The estimates will converge to  $(\alpha + z_i' \beta, \text{CATE})$  as the sample size rises.

### 4.9.2 Common interest formation process

Many types of network formation have been investigated in the literature and homophilic behavior has different implications depending on the linking model. In general, homophily allows us to extract some information to refine counterfactuals, albeit with different quality and asymptotic behavior.

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<sup>17</sup>Note that controlling for variables that affect network formation is not needed. In fact, controlling can be counter-productive: homophily brings regressors towards the same value, inducing collinearity with the intercept in the limit.

For another example that still uses dyadic link formation, consider a common interest formation process:  $i$  and  $j$  are friends if  $|X_{ik} - X_{jk}| < D_k$  (or possibly random uniforms  $\eta_{ijk}$  replacing  $D$ ) for some  $k$ , where  $k$  ranges over the covariate indices. In this case, two persons become friends if they have some kind of common interest or have some characteristic in common.

This model rationalizes the idea that, if two persons have apparently nothing in common, it is likely they share some unobserved common interest, ability, or taste. If only one  $X_{ik}$  is unobserved then two individuals differing sufficiently on observables must be great counterfactuals for each other in terms of this  $X_{ik}$ . Such link formation processes have different implications for CATE estimation than the earlier link formation model. With univariate unobservables and  $D = D_n = o(n^{-\beta})$ ,  $\beta < 1$  a simple consistent estimator follows readily, while it is not possible to handle multivariate unobservables and truncation based on observables is helpless.

#### 4.9.3 A note on directed and weighted links

It is generally possible to use the estimators in the case of directed or weighted links. The former would no longer assume  $\eta_{ij} = \eta_{ji}$ , while the latter could use a model such as  $W_{ij} = (w_n(h(X_i^o, X_j^o) + \|X_i^u - X_j^u\|) + \eta_{ij})/2$ . Since the information contained in the network is richer in the cases, one can likely refine the estimators, further information on the distribution of unobservables, or specification tests. An analysis of these cases is beyond the scope of this paper but would constitute a valuable and interesting exercise.