The Seismic Inverse Problem: From Unknowns to Results

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Seismic Sources - Contents

The nature of problems in the world around us.

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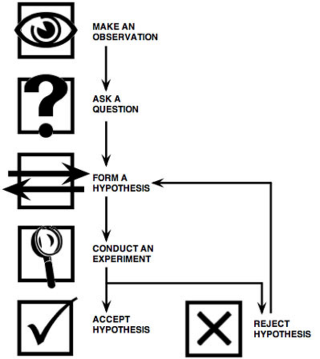
The nature of problems in the world around us.

- **In some way all the problems of Science are inverse Problems.**
- **One goes about looking for the laws of nature in the following way:**

1. Guess what is going on.
2. Compute the consequences of this guess.
3. Compare the consequences of the guess to what is observed in real-life (this may require an experiment).
4. If the consequences of the guess agree with what is observed in real life then the guess is right and we have learnt something important.
5. If the computed consequences and the observations disagree, then something must be changed.

The nature of problems in the world around us.

Figure: The method used in answering problems



The nature of problems in the world around us continued...

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The nature of problems in the world around us continued...

- The forward problem is concerned with the physical laws involved in this process. I.e: the forward problem is concerned with the initial guess.
- The modelling problem is concerned with computing the consequences of the initial guess.
- The experimental problem is concerned with interpreting the results of experiments and making observations.
- The inverse problem comes into play when the calculated results do not match up with what is observed.

So something has gone wrong.

What happens when the calculation and the observation disagree.

- If the computed consequences and the observations disagree, then something must be changed.

So something has gone wrong.

What happens when the calculation and the observation disagree.

- **If the computed consequences and the observations disagree, then something must be changed.**
- **The first question : where did things go wrong?**

1. Maybe we don't understand the problem correctly and are using the incorrect equations.
2. Maybe there is something incorrect with the model we are using to do calculations.

So where are we in Mine Seismology?

- **Worst case scenario:**

1. We do not understand the problem correctly.
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- **Better Scenario :**

1. We know that we understand the problem correctly.
2. We know that our computational models are not correct.
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- **Best Scenario : Our calculations and the observations agree.**

What we know:

- **We know the equations governing seismic waves:**

$$u_n(\vec{x},t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\vec{\xi},\tau) G_{in}(\vec{\xi},t-\tau;\vec{x},0)\,dV$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\vec{\xi},t-\tau;\vec{x},0)\,T_i(\vec{u}(\vec{\xi},\tau),\vec{n})$$
$$- u_i(\vec{\xi},\tau)\,c_{ijkl}n_j G_{kn,l}(\xi,t-\tau;\vec{x},0)\}\}dS(\vec{\xi})$$

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- **Where the Green's function** $G_{in}(\vec{x},t;\vec{\xi},\tau)$ **satisfies:**

$$\rho \frac{\partial^2}{\partial t^2} G_{in} = \delta_{in}\delta(\vec{x}-\vec{\xi})\delta(t-\tau) + \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial}{\partial x_l} G_{kn}\right)$$

Many Problems we can only tackle numerically:

An Example of doing things numerically:

- **When modelling it is usually the case that one needs some help from a computer.**
- **As an example of how this is handled let us look at how the computer estimates partial derrivatives, using an example from elasticity:**

$$\rho \frac{\partial}{\partial t} v_x = \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} + \frac{\partial}{\partial z}\sigma_{xz} + f_x$$

The derivatives are replaced by the numerical estimates:

$$1^{st} \text{ order : } \frac{\partial}{\partial x} f(x) = \frac{f(x+h) - f(x)}{h}$$

$$2^{nd} \text{ order : } \frac{\partial}{\partial x} f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

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- We are rather confident that these are right and have a good understanding of seismic waves.
- However what we are not sure of is our model. If we are modelling a mine we cannot know the velocities, densities and other parameters everywhere.
- The question we ask is then the inverse problem. Given that we know the equations and that we have observations what (if anything) can we say about our model?

The Answer:

- **The answer: Given what we know we should be able to make our model perfect (in theory)!**

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- **The answer: Given what we know we should be able to make our model perfect (in theory)!**
- **Unfortunately reality is slightly less kind than this. However one can still make the model very good.**

Outline

Fermat's principle and traveltime functionals.

Definitions:

- **Functional: A functional is a map from a function space to a field of numbers (here we can assume we're working with real numbers). Think of this like a function, but instead of putting in a number and getting another number out, one puts in a function and gets out a number.**

Fermat's principle and traveltime functionals.

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- **Functional: A functional is a map from a function space to a field of numbers (here we can assume we're working with real numbers). Think of this like a function, but instead of putting in a number and getting another number out, one puts in a function and gets out a number.**
- **Consider the following Example:**

For : $f(x) = x^2$

$x = 1 \qquad f(1) = 1$

$x = 2 \qquad f(2) = 4$

$x = 3 \qquad f(3) = 9$

For : $f[\rho(x)] = \int_0^1 \rho(x)dx$

$\rho(x) = x \qquad f[x] = \frac{1}{2}$

$\rho(x) = x^2 \qquad f[x^2] = \frac{1}{3}$

$\rho(x) = 8 \qquad f[8] = 8$

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- **Slowness: Slowness is the inverse of velocity and will be denoted by $\vec{s}(\vec{x})$.**
- **Traveltime functional: A functional that will spit out the traveltime for a seismic wave given some path and a function describing the slowness of the medium.**

Fermat's Principle:

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- **Fermat's Principle of least time states that the path which is taken is the path which minimises the traveltime. I.e. the traveltime of a wave traversing any two points is given by:**

$$\tau^*(\vec{x}) = \min_{P \in Paths} \int_P \vec{s}(\vec{x})\, dl^P$$

Outline

The beginnings of a solution.

- So let us begin asking how we can approach the inverse problem. The most appropriate place to start is, as always to make the problem as simple as possible.

Elementary Back-Projection.

The beginnings of a solution.

- So let us begin asking how we can approach the inverse problem. The most appropriate place to start is, as always to make the problem as simple as possible.
- Let us begin by using the straight-ray approximation. Imagine you throw a pebble into a large pond. The waves created move out in every direction with equal velocity. The shortest route for a wave to travel between two points is the straight line connecting these two points. This is the idea behind the straight ray approximation.

Intuition

Figure: Intuition behind the straight ray approximation

Elementary Back-Projection II

- **Following Fermat's Principle, we know that the observed traveltime is the minimum possible. If we can get the traveltime in our model to be close to the observed traveltime, we know that our model is close to reality. In this light we minimise:**

$$\chi(\vec{m}) = \sum_{\substack{\text{P-wave,}\\ \text{S-wave}}} \sum_{i=1}^{n_r} \{ | T^i_{\text{obs.}} - t^i_{\text{calc.}} - t_0 | \}$$

Where $t^i_{\text{calc.}} = \dfrac{d_i}{v_i}$, $\quad t_0 = $ event-time

$\vec{m} =$ The model used. $\quad n_r =$ The number of receivers

- **In reality we do not (usually) know the value for t_0, but we can circumvent this difficulty by making use of the trick:**

$$t_0 = T^i_{\text{obs.}} - t^i_{\text{calc.}} + \varepsilon^i = \overline{T_{\text{obs.}}} - \overline{t_{\text{calc.}}} + \bar{\varepsilon}$$

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- **This trick leaves us to minimise:**

$$\chi(m) = \sum_{\substack{\text{P-wave,}\\ \text{S-wave}}} \sum_{i=1}^{n_r} \{ |T^i_{\text{obs.}} - \overline{T_{\text{obs.}}} - (t^i_{\text{calc.}} - \overline{t_{\text{calc.}}})| \}$$

Double difference methods.

For Straight rays

A little bit more detail

- We are not limited to having a model with constant velocity throughout. We can get better results if we split our model up into different cells each having a different velocity.

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- We are not limited to having a model with constant velocity throughout. We can get better results if we split our model up into different cells each having a different velocity.
- Naturally in such a situation, we are sure to have errors. Double difference is one of the methods used to minimise these errors.
- The idea behind the method is rather simple. One looks at pairs of seismic events which are close together. The velocity errors along the portion of the path which is common cancel.

The idea of Double difference:

A little bit more detail

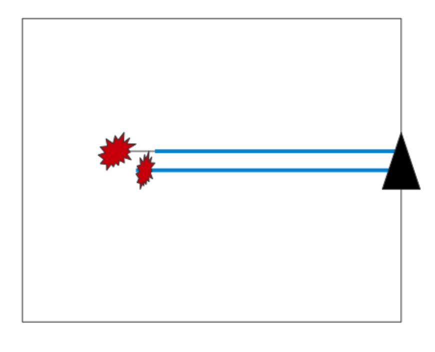

Figure: The errors along the common paths will cancel

Double difference methods continued.

- **Consider a pair of events *i* and *j*, received at station *k*. Letting:**

$$\vec{m}^i = \begin{bmatrix} \vec{x}^i \\ \tau^i \end{bmatrix}$$

Where:

$\vec{x}^i$ = The event location according to the current model

$\tau^{event\ i}$ = origin time of event i according to the current model

$t_{station\ k}(\vec{x}^i)$ = travel time

$T_{station\ k}(\vec{m}^{event\ i})$ = Arrival time of event i at station k

$\vec{m}_0$ = The “true” event location and origin time.

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- **Making use of the first order Taylor expansion, one can write....**

$$\left(T_k(\vec{m}^i) - T_k(\vec{m}^j)\right) - \left(T_k(\vec{m}_0^i) - T_k(\vec{m}_0^j)\right) = \frac{\partial \vec{x}^i}{\partial x^i}\delta x^i + \frac{\partial \vec{x}^i}{\partial y^i}\delta y^i + \frac{\partial \vec{x}^i}{\partial z^i}\delta z^i + \delta\tau^i - \frac{\partial \vec{x}^j}{\partial x^j}\delta x^j - \frac{\partial \vec{x}^j}{\partial y^j}\delta y^j - \frac{\partial \vec{x}^j}{\partial z^j}\delta z^j - \delta\tau^j$$

- **One may form this into a system of equations for all pairs of events which are nearby one another.**

Using the system of equations:

- **This system of equations can then be written in the form: $\mathbf{A}\vec{x} = \vec{b}$.**

Where:
A is a $m \times 4n$ matrix,
A contains the partial derivatives
m is the number of double difference equations
n is the number of events
$\vec{b} =$ Observed $-$ calculated arriaval times.

$$\vec{x} = \begin{bmatrix} \Delta x^1 \\ \Delta y^1 \\ \Delta z^1 \\ \Delta \tau^1 \\ \vdots \\ \Delta x^n \\ \Delta y^n \\ \Delta z^n \\ \Delta \tau^n \end{bmatrix}$$

- **One then uses a conjugate-gradient algorithm to minimise $\mathbf{A}\vec{x} - \vec{b}$. This is done by iteratively changing the model vector $\vec{x}$.**

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- **The procedure is as follows:**

1. Create some data, with a velocity model known to us.
2. Look for the seismic events, use elementary back projection to locate them.
3. Use Double Difference to re-locate the events more accurately than before.

The velocity model:

From reference [3]

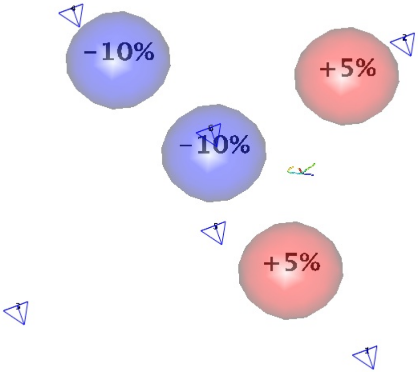

Figure: $V_p = 5000\ m.s^{-1}$, $V_s = 3200\ m.s^{-1}$

Results for Simulation 1:

From reference [3]

- Here the arrival times were noiselessly computed from the true velocity model. The initial locations are estimated on a single-event basis using the homogeneous (incorrect) velocity model. The error is deterministic.

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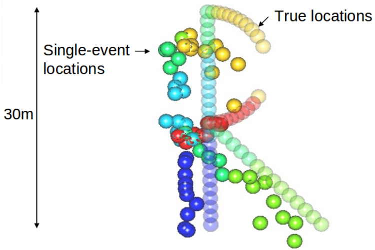

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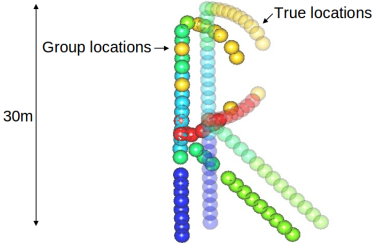

Figure: **After**

Results for Simulation 2:

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- Here random error was added to the arrival times computed from the true velocity model. The initial locations were estimated on an single-event basis using the homogeneous (incorrect) velocity model. There is deterministic as well as random error.

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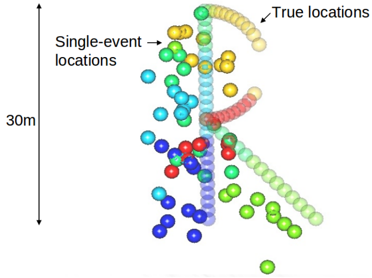

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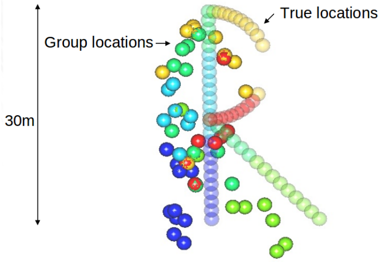

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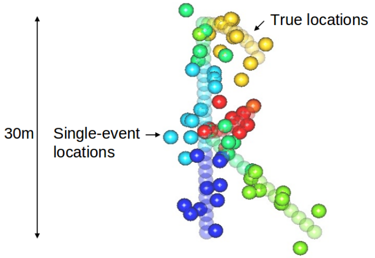

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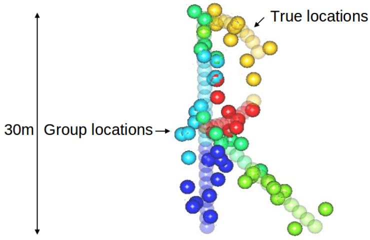

Figure: **After**

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- **In an environment where the assumed velocity model is incorrect the double difference does well in restoring the configurations of seismic events. That is the locations of seismic events relative to one another.**
- **However there are still the following problems:**

1. Many events may not be close enough together to use this method.
2. Even though the method does well in restoring the configuration, it falls short in finding the actual event locations.
3. The method is bad for large variations in velocity.
4. Errors in processing lead to drastic loss of accuracy,

Outline

The problem with straight-ray methods:

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- **We have seen Fermat's least time principle. There is nothing straight about it!**
- **Briefly recall Snell's law.**

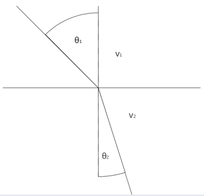

Figure: **Snell's Law where** $v_1 > v_2$

$$v_2 \sin(\theta_1) = v_1 \sin(\theta_2)$$

The sort of thing that can go wrong when using straight ray methods (incorrectly):

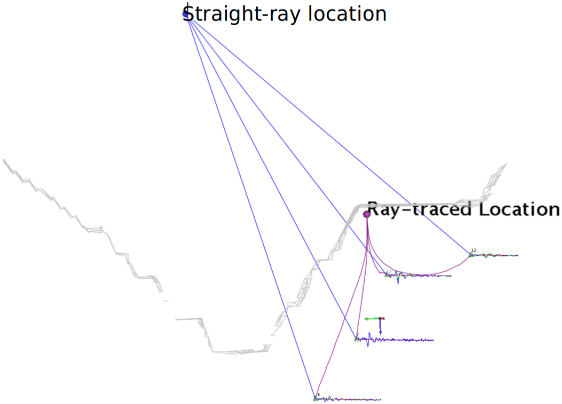

Figure: An open pit mine - an example of what can go wrong

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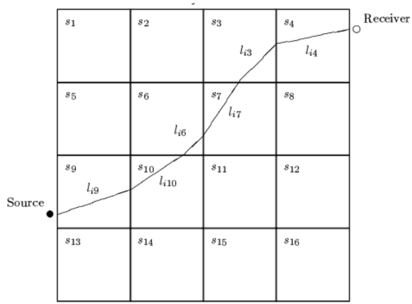
- We have seen Fermat's least time principle. There is nothing straight about it!
- Given Snell's law there is *No Way* that one can say: We are using straight rays in a model where the velocity is not constant.
- It is obvious then that straight-ray methods are just an approximation.

Basic ray-tracing

From reference [2]:

- **Let us again break up our models into plenty of little cells, and let us assume that in each of these little cells has a constant velocity. Now we no longer pretend the rays are straight.**

Figure: **A simple model**



How does one handle a model like this?

Handling of these kinds of models:

- **Consider the traveltime integral:**

$$t^i = \int_{P_i} \vec{s}(\vec{x})\, dl^{P_i}$$

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- Given that the blocks of the model have constant slowness, the length of the i^{th} ray path through the j^{th} cell is given by:

$$l_{ij} = \int_{P_i \cap cell_j} dl^{P_i}$$

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- **In which case the above simplifies to :**

$$t_i = l_{ij} s_j$$

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- If we are lucky (also depending on how accurate we want to be), we may have enough information to invert l_{ij} directly. For real problems however this is not a realistic approach.
- What one is then forced to do is to proceed from the source in the forward direction and see where one ends up. At each point the ray has to be consistent with Fermat's principle of least traveltime, and hence also the ray equation. A brief derivation of the ray equation follows:

$$t = \int_P s(\vec{x}(\lambda))dl(\lambda) = \int_{\lambda(A)}^{\lambda(B)} f(\vec{x}, \dot{\vec{x}})d\lambda$$

Ray Shooting

Where: $\dot{\vec{x}} = \frac{d\vec{x}}{d\lambda}$ and $f(\vec{x}, \dot{\vec{x}}) = s(\vec{x})|\dot{\vec{x}}|$

- **Fermat's principle tells us that $\delta t = 0$ for variations in λ (the derivative of a function is 0 at the minimum). This leads to the ray equation:**

$$\vec{\nabla} s = \frac{d}{dl}\left(s\frac{d}{dl}\vec{x}\right)$$

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$$\vec{\nabla} s = \frac{d}{dl}\left(s\frac{d}{dl}\vec{x}\right)$$

- **Which in 2D is:**

$$\frac{d\theta}{dl} = \frac{1}{s}\left(\frac{\partial s}{\partial y}\cos(\theta) - \frac{\partial s}{\partial x}\sin(\theta)\right)$$

Problems with shooting

Things are far from perfect

- Consider the picture again. Some of the rays barely cut through the cell. There is every chance that this little piece of the cell gives a bad representation of the average slowness in the cell. However this problem can be fixed by using "fat rays" that sample more of the cell.

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- Consider the picture again. Some of the rays barely cut through the cell. There is every chance that this little piece of the cell gives a bad representation of the average slowness in the cell. However this problem can be fixed by using "fat rays" that sample more of the cell.
- You have no idea where you're going to end up from where you start. You have to keep going until you find a path that gets you from where you started to where you want to end up.

Where does the ray end?

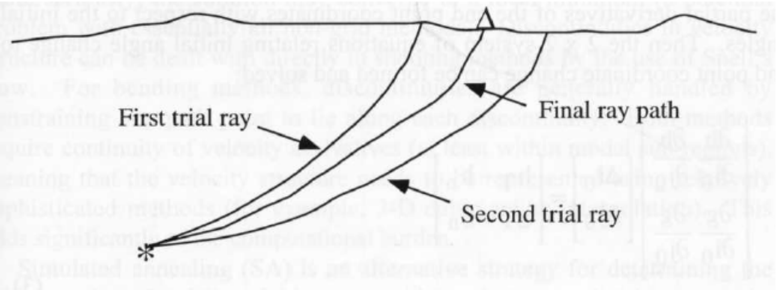

Figure: **From [Julian & Gubbins, 1977]**

More problems with shooting

Things are far from perfect

- The above problem is actually a lot more serious than it sounds. This is because one *does not* in general know where the seismic sources are. So one has to keep shooting rays from various points where the seismic sources may be to the known receivers to minimise the traveltime.

More problems with shooting

Things are far from perfect

- The above problem is actually a lot more serious than it sounds. This is because one *does not* in general know where the seismic sources are. So one has to keep shooting rays from various points where the seismic sources may be to the known receivers to minimise the traveltime.
- These problems cause ray-tracing methods such as the one above to be very computationally inefficient, and severely limits their practicality.

Outline

Bending methods:

Bending

- **Bending is another method that can be used to get around the fact that the ray paths bend. In bending an initial (possibly curved) path between two end points (the source and the receiver) is estimated. The path is then slowly perturbed (bent) until the traveltime is minimised.**

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- Bending is another method that can be used to get around the fact that the ray paths bend. In bending an initial (possibly curved) path between two end points (the source and the receiver) is estimated. The path is then slowly perturbed (bent) until the traveltime is minimised.
- Bending methods are generally more efficient than the ray shooting methods mentioned above.

Bending methods continued:

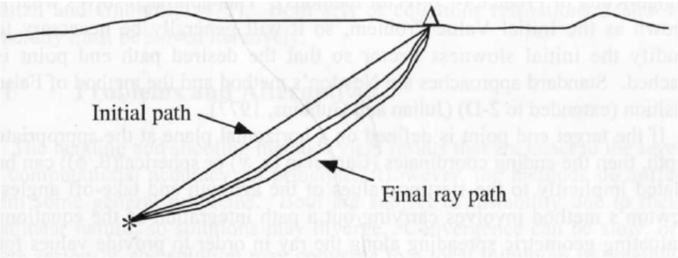

Figure: An example of a bent ray path, [Um & Thurber, 1987]

Shortfalls of bending methods:

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Shortfalls of bending methods:

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- Bending methods require the velocity model to be smooth (the first derivatives must be continuous) or else they become numerically unstable.
- In open pit mines and for caving air/rock or rock/muck-pile interfaces are sharp, so good velocity models are unlikely to be smooth.

Fast Marching methods:

Fast Marching

- **In this method a finite-difference alogrithm is used to track the movement of the wave front.**

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- In this method a finite-difference alogrithm is used to track the movement of the wave front.
- The wave front evolves according to the solution of the Eikonal equation $|\vec{\nabla} T| = \frac{1}{v}$. In the high-frequency approximation (there are no S-P or P-S conversions).

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- The wave front evolves according to the solution of the Eikonal equation $|\vec{\nabla} T| = \frac{1}{v}$. In the high-frequency approximation (there are no S-P or P-S conversions).
- This method gains its speed by intelligently updating nodes. But this requires that all velocities are either positive or negative.

Fast Marching methods:

Fast Marching

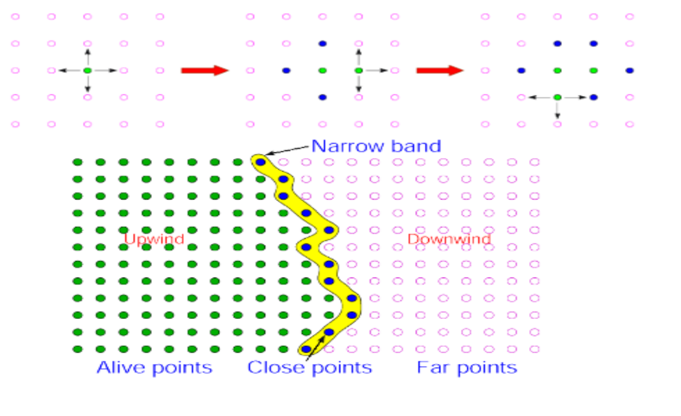

Figure: Movement of a point on the wavefront, [Sethian, 1996]

Shortfalls of fast marching methods:

Drawbacks of Fast Marching

- **This method uses assumes the high-frequency limit, so low frequency effects are not handled properly.**

Shortfalls of fast marching methods:

Drawbacks of Fast Marching

- This method uses assumes the high-frequency limit, so low frequency effects are not handled properly.
- This method is computationally expensive. The non-linearity of the method means matrix methods cannot be used.

Outline

Some Background:

Some background

- **Functional derivatives: A derivative is a mathematical operation. It says if A depends on B, how does object A change given an infinitesimal change in B. Usually A is a (vector) function and B is a (vector) variable. But there is no reason why A can't be a functional of B.**

Consider the functional: $F[\rho] = \int_V A(\rho(\vec{r}))d^3r$

Some Background:

Some background

The functional derivative $\frac{\delta F}{\delta \rho}(\vec{r})$ **is defined so that:**

$$F[\rho+\delta\rho]-F[\rho]=\int_V \left\{A(\rho(\vec{r})+\delta\rho(\vec{r}))-A(\rho(\vec{r}))\right\}d^3r$$

$$F[\rho+\delta\rho]-F[\rho]=\int_V \frac{\delta F}{\delta\rho}(\vec{r})\delta\rho(\vec{r})$$ **, to first order in** $\delta\rho(\vec{r})$

- **Consider the following as an example:**

If $F[\rho]=\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}$

Then : $\frac{\delta F[\rho]}{\delta\rho(\vec{r}')}(\vec{r})=\frac{1}{4\pi\epsilon_0|\vec{r}-\vec{r}'|}$

The Idea of Full Waveform Inversion:

Why only traveltimes.

- **Consider the following seismogram**

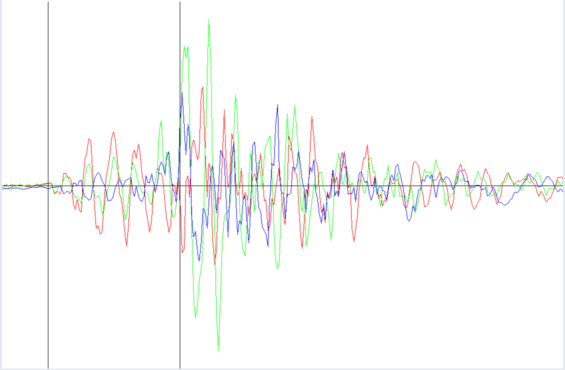

Figure: Your average seismogram

The Idea of Full Waveform Inversion continued...

Why only traveltimes.

- ***Look at all that information!*** **And so far we've only been using the P-wave and S-wave arrival times. Why are we throwing away so much information?**

The Idea of Full Waveform Inversion continued...

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- **Careful consideration of what we've been doing, will reveal that we've been comparing the information that our model spits out to the recorded data, and manipulating our model accordingly.**

The Idea of Full Waveform Inversion continued...

Why only traveltimes.

- ***Look at all that information!*** **And so far we've only been using the P-wave and S-wave arrival times. Why are we throwing away so much information?**
- **Careful consideration of what we've been doing, will reveal that we've been comparing the information that our model spits out to the recorded data, and manipulating our model accordingly.**
- **But *surely* if our model spits out an identical waveform to what is recorded at the receivers, then we've got things spot on.**

Outline

Putting the Idea into action:

- **The first thing to be done is to find a good way to measure how far our model is from the data. Consider the following cost function (technically a functional):**

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- **The first thing to be done is to find a good way to measure how far our model is from the data. Consider the following cost function (technically a functional):**

$$\chi[\vec{m}] = \frac{1}{2}\sum_{r=1}^{N}\int_{0}^{T} \|\vec{s}(\vec{x}_r, t, \vec{m}) - \vec{d}(\vec{x}_r, t)\|^2 dt$$

Where: $\vec{m}$ **is the model vector, containing all the parameters describing our model.**

$\vec{s}(\vec{x}_r, t, \vec{m}) =$ **The Synthetic seismograms i.e.: the output of our model**

$\vec{d}(\vec{x}_r, t) =$ **The Data i.e.: the seismograms we record.**

$[0, T] =$ **The observation time**

Putting the Idea into action, continued:

- **We have a good measure of how well/badly our model fits the data. If $\chi[\vec{m}] = 0$, then we know that our model is perfect. If $\chi[\vec{m}]$ is large, then we know that our model is bad.**

Putting the Idea into action, continued:

- **We have a good measure of how well/badly our model fits the data. If $\chi[\vec{m}] = 0$, then we know that our model is perfect. If $\chi[\vec{m}]$ is large, then we know that our model is bad.**
- **The following questions are now open to us:**

1. How does the "cost" change when we change the model?
2. How can we use this information to improve our model (i.e. minimise the "cost")?

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- **It turns out that these questions are related and developing the answers leads to important results.**

Some Details:

Towards full waveform inversion.

- To understand how the “cost” changes when one changes the model, one simply evaluates the functional derivative $\frac{\delta\chi[\vec{m}]}{\delta\vec{m}}$.

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Towards full waveform inversion.

- **To understand how the "cost" changes when one changes the model, one simply evaluates the functional derivative $\frac{\delta\chi[\vec{m}]}{\delta\vec{m}}$.**
- **Using the chain rule it is seen that:**

$$\delta\chi = \sum_{r=1}^{N} \int_{0}^{T} [s_i(\vec{x}_r, t, \vec{m}) - d_i(\vec{x}_r, t)]\delta s_i(\vec{x}_r, t, \vec{m})$$

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- The next step is to find $\frac{\delta s_i(\vec{x}_r,t,\vec{m})}{\delta\vec{m}}$. To do this one needs to make use of the Born Approximation.

Another step along the way

- **Recall the Seismic wave equation:**

$$\rho \frac{\partial^2}{\partial t^2} u_i^{(0)} - \frac{\partial}{\partial x_j} c_{ijkl} \epsilon_{kl}^{(0)} = 0$$

This can we written as: $\mathcal{L}^{(0)} u_i^{(0)} = 0$

The Born Approximation:

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- **The Born approximation says that a small small perturbation in the medium ($\mathcal{L}^{(0)} \to \mathcal{L}^{(0)} + \delta\mathcal{L}$) will cause a small perturbation in the wavefield ($u_i^{(0)} \to u_i^{(0)} + \delta u_i$). Thus the following equation must be satisfied:**

$$(\mathcal{L}^{(0)} + \delta\mathcal{L})(u_i^{(0)} + \delta u_i) = 0$$

$$\Rightarrow \mathcal{L}^{(0)} \delta u_i = -\delta\mathcal{L} u_i^{(0)} + \mathcal{O}(\delta^2)$$

The Born Approximation continued...

- **And hence** $\delta u_i = -(\mathcal{L}^{(0)})^{-1}\delta\mathcal{L}u_i^{(0)} + \mathcal{O}(\delta^2)$ **.**

The Born Approximation continued...

- **And hence** $\delta u_i = -(\mathcal{L}^{(0)})^{-1}\delta\mathcal{L}u_i^{(0)} + \mathcal{O}(\delta^2)$ **.**
- **But we have from earlier that:**

$$
\begin{aligned}
u_n(\vec{x},t) = &\int_{-\infty}^{\infty} d\tau \iiint_V f_i(\vec{\xi},\tau)G_{in}(\vec{\xi},t-\tau;\vec{x},0)dV \\
&+ \int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\vec{\xi},t-\tau;\vec{x},0)T_i(\vec{u}(\vec{\xi},\tau),\vec{n}) \\
&-u_i(\vec{\xi},\tau)c_{ijkl}n_jG_{kn,l}(\xi,t-\tau;\vec{x},0)\}\}dS(\vec{\xi}) \\
&= (\mathcal{L}^{(0)})^{-1}f_i
\end{aligned}
$$

The Born Approximation continued...

- **And hence** $\delta u_i = -(\mathcal{L}^{(0)})^{-1}\delta\mathcal{L}u_i^{(0)} + \mathcal{O}(\delta^2)$ **.**
- **But we have from earlier that:**

$$u_n(\vec{x},t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\vec{\xi},\tau)G_{in}(\vec{\xi},t-\tau;\vec{x},0)dV$$
$$+\int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\vec{\xi},t-\tau;\vec{x},0)T_i(\vec{u}(\vec{\xi},\tau),\vec{n})$$
$$-u_i(\vec{\xi},\tau)c_{ijkl}n_jG_{kn,l}(\xi,t-\tau;\vec{x},0)\}\}dS(\vec{\xi})$$
$$=(\mathcal{L}^{(0)})^{-1}f_i$$

- **So, we know what** $(\mathcal{L}^{(0)})^{-1}$ **is. Thus we can say:**

More Details:

$$\delta s_i(\vec{x},t) = -\int_0^t \int_V \Big[\delta\rho(\vec{x}')\, G_{ij}(\vec{x},\vec{x}';t-t')\, \frac{\partial^2}{\partial t'^2} s_j(\vec{x}',t')$$

$$+\delta c_{ijkl}(\vec{x}')\, \frac{\partial}{\partial x'_j} G_{ij}(\vec{x},\vec{x}';t-t')\, \frac{\partial}{\partial x'_l} s_m(\vec{x}',t')\Big]\, d^3\vec{x}'\, dt'$$

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$$\delta s_i(\vec{x},t) = -\int_0^t \int_V \Big[\delta\rho(\vec{x}')G_{ij}(\vec{x},\vec{x}';t-t')\frac{\partial^2}{\partial t'^2}s_j(\vec{x}',t')$$

$$+\delta c_{ijkl}(\vec{x}')\frac{\partial}{\partial x_j'}G_{ij}(\vec{x},\vec{x}';t-t')\frac{\partial}{\partial x'_l}s_m(\vec{x}',t')\Big]d^3\vec{x}'dt'$$

- **Substituting this into the expression for $\delta\chi$ one finds:**

$$\delta\chi = \sum_{r=1}^{N}\int_0^T [s_i(\vec{x}_r,t,\vec{m}) - d_i(\vec{x}_r,t)]$$

$$\times \int_0^t \int_V \Big\{\delta\rho(\vec{x}')G_{ij}(\vec{x}_r,\vec{x}';t-t')\frac{\partial^2}{\partial t'^2}s_j(\vec{x}',t')$$

$$+\,\delta c_{ijkl}(\vec{x}')\frac{\partial}{\partial x_j'}G_{ij}(\vec{x}_r,\vec{x}';t-t')\frac{\partial}{\partial x'_l}s_m(\vec{x}',t')\Big\}d^3\vec{x}'dt'$$

- **Doing some Algebra one finds terms of the form:**

$$\Phi_k(\vec{x}', t') = \sum_{r=1}^{N} \int_{t'}^{T} G_{ki}(\vec{x}', \vec{x}_r; t - t') \left[s_i(\vec{x}_r, t) - d_i(\vec{x}_r, t) \right] dt$$

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- **Let us define the waveform adjoint source:**

$$f_i^{\dagger}(\vec{x}, t) = \sum_{r=1}^{N} \left[s_i(\vec{x}_r, T - t) - d_i(\vec{x}_r, T - t) \right] \delta(\vec{x} - \vec{x}_r)$$

The waveform adjoint field:

- **The waveform adjoint field is given by:**

$$s_k^{\dagger}(\vec{x}', t') = \int_0^{t'} \int_V G_{ki}(\vec{x}', \vec{x}; t' - t) f_i^{\dagger}(\vec{x}, t) d^3\vec{x} dt$$

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- **What is amazing about the waveform adjoint field is that it satisfies the following equation:**

$$\rho \frac{\partial^2}{\partial t^2} s_i^{\dagger}(\vec{x}, t) - \frac{\partial}{\partial x_j} c_{ijkl} \frac{\partial}{\partial x_k} s_l^{\dagger}(\vec{x}, t) =$$
$$= \sum_{r=1}^{N} \left[s_i(\vec{x}_r, T - t) - d_i(\vec{x}_r, T - t) \right] \delta(\vec{x} - \vec{x}_r) = f_i^{\dagger}(\vec{x}, t)$$

Observations about the waveform adjoint field

- Note that the above equation is exactly the seismic wave equation, with the waveform adjoint source as the source of the seismic waves.

Interpreting the waveform adjoint field

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Observations about the waveform adjoint field

- Note that the above equation is exactly the seismic wave equation, with the waveform adjoint source as the source of the seismic waves.
- The waveform adjoint source is the time-reversed differences between the synthetics and the data located at the receiving stations.
- We are now in a position where we can explicitly state how the "cost" changes when the model changes.

Outline

How “cost” changes with model perturbations:

- **Carrying out the calculation above one eventually finds:**

$$\delta\chi = \int_V \left[K_\rho(\vec{x})\delta\ln(\rho(\vec{x})) + K_{c_{ijkl}}(\vec{x})\delta\ln(c_{ijkl}(\vec{x}))\right] d^3\vec{x}$$

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- **Where the kernels $K_\rho(\vec{x})$ and $K_{c_{ijkl}}(\vec{x})$ are defined as follows:**

$$K_\rho(\vec{x}) = -\int_0^T \rho(\vec{x}) {s_i}^\dagger(\vec{x}, T-t)\frac{\partial^2}{\partial t^2} s_i(\vec{x}, t)dt$$

$$K_{c_{ijkl}}(\vec{x}) = -\int_0^T \epsilon^\dagger_{jk}(\vec{x}, T-t) c_{jklm}(\vec{x})\epsilon_{lm}(\vec{x}, t)dt \qquad \textbf{[NO } \sum \textbf{]}$$

Where : $\epsilon_{lm} = \frac{1}{2}\left(\frac{\partial}{\partial x_l}s_m + \frac{\partial}{\partial x_m}s_l\right)$ $\quad \epsilon^\dagger_{jk} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}{s_k}^\dagger + \frac{\partial}{\partial x_k}{s_j}^\dagger\right)$

More Kernels:

- **In an isotropic model one finds:**

$$\delta\chi = \int_V \Big[K_{\rho(\kappa\mu)}(\vec{x})\delta\ln(\rho(\vec{x})) + K_{\mu(\kappa\rho)}\delta\ln(\mu(\vec{x}))$$

$$+K_{\kappa(\mu\rho)}(\vec{x})\delta\ln(\kappa(\vec{x}))\Big] d^3\vec{x}$$

Where:

$$K_{\mu(\kappa\rho)}(\vec{x}) = -\int_0^T 2\mu(\vec{x})D_{ij}^{\dagger}(\vec{x}, T-t)D_{ij}(\vec{x}, t)dt$$

$$K_{\kappa(\mu\rho)}(\vec{x}) = -\int_0^T \kappa(\vec{x})[\frac{\partial}{\partial x_j}s_j{}^{\dagger}(\vec{x}, T-t)][\frac{\partial}{\partial x_i}s_i(\vec{x}, t)]dt$$

$$D_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \qquad \textbf{and} \qquad D_{ij}^{\dagger} = \varepsilon_{ij}^{\dagger} - \frac{1}{3}\varepsilon_{kk}^{\dagger}\delta_{ij}$$

More Kernels:

- **Which can also be written in term of shear (β) and compressional (α) wave velocities as:**

$$\delta\chi = \int_V \left[K_{\rho(\alpha\beta)}(\vec{x})\delta\ln(\rho(\vec{x})) + K_{\beta(\alpha\rho)}\delta\ln(\beta(\vec{x})) + K_{\alpha(\beta\rho)}(\vec{x})\delta\ln(\alpha(\vec{x}))\right] d^3\vec{x}$$

- **Where**

$$K_{\rho(\alpha\beta)} = K_\rho + K_{\kappa(\mu\rho)} + K_{\mu(\kappa\rho)}$$

$$K_{\beta(\alpha\rho)} = 2\left(K_{\mu(\kappa\rho)} - \frac{4}{3}\frac{\mu}{\kappa}\right)K_{\kappa(\mu\rho)}$$

$$K_{\alpha(\beta\rho)} = 2(1 + \frac{4}{3}\frac{\mu}{\kappa})K_{\kappa(\mu\rho)}$$

The Kernel for moment source perturbations:

- **There is one more kernel I'd like to show you, and that is the kernel corresponding to moment source perturbations:**

$$\delta\chi = \int_0^T \iint_\Sigma \epsilon_{ij}^{\dagger}(\vec{x}, T - t)\delta m_{ij}(\vec{x}, t) d^2\vec{x} dt$$

- **What is particularly important about this is that it means that full waveform inversion can be used for controlled sources (where we know the source time function and source location) as well as naturally occurring sources (where the source time function and source location are unknown - but there is a large amount of recorded information).**

But what is the point of these Kernels?

Why did we go through all of that?

- **What do we know about χ? Obviously χ has a global minimum when the model is correct. Also any perturbation away from the minimum increases the value of χ. The derivatives should be 0 when at the minimum, and small nearby and large further away.**

But what is the point of these Kernels?

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- **What do we know about χ? Obviously χ has a global minimum when the model is correct. Also any perturbation away from the minimum increases the value of χ. The derivatives should be 0 when at the minimum, and small nearby and large further away.**
- **The Kernels are functional derivatives. These Kernels then, which encode the interaction between the forward moving wavefield and the backward moving adjoint wavefield "paint a picture" of the accuracy of our model. Where they're big the model is bad, where they are small, the model is good.**

A Picture is worth a thousand words:

From reference [1]

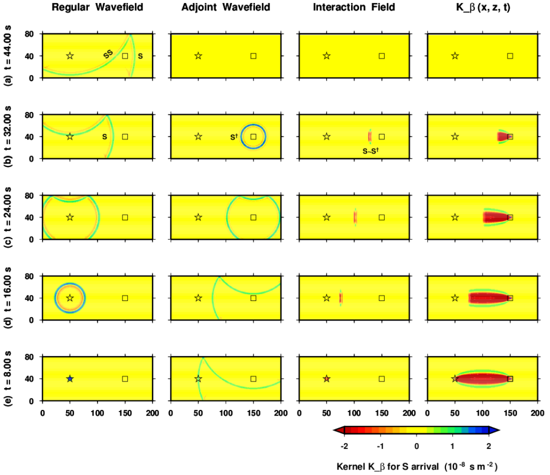

Figure: The Kernel Paints a picture

Kernels indicating model accuracy:

From reference [1]

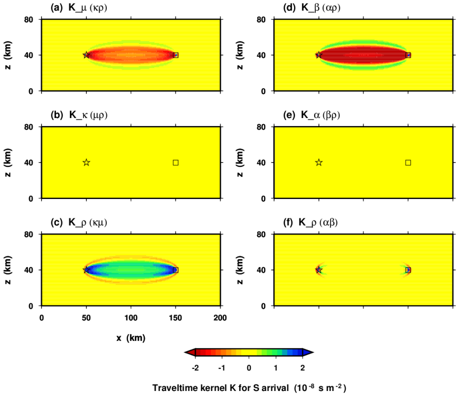

Figure: A kernel picture-gallery

Sensitivity Kernel for a reflected wave:

From reference [1]

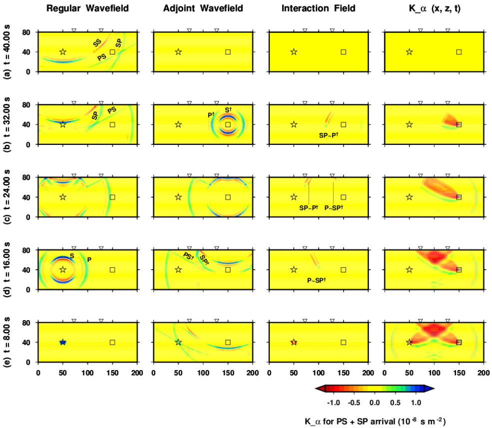

Figure: Compressional wavespeed sensitivity kernel for a reflected wave

A brief overview of the iterative procedure.

The Kernels in action

- **These Kernels also give us a way to iteratively improve our model! The procedure follows:**

1. For $k = 0$, $\vec{p}^{(0)} = -\frac{\partial \chi}{\partial \vec{m}} = -\vec{g}^{(0)}$. If at any stage $\|\vec{p}\| < \varepsilon$, then our model is accurate.
2. Find a scalar $\lambda^{(k)}$ that minimises the function $\tilde{\chi}(\lambda) = \chi(\vec{m}^{(0)} + \lambda \vec{p}^{(k)})$.
3. Update the model as follows: $\vec{m}^{(k+1)} = \vec{m}^{(0)} + \lambda^{(k)} \vec{p}^{(k)}$, calculate $\vec{g}^{(k+1)} = \frac{\partial}{\partial \vec{m}^{(k+1)}} \chi(\vec{m}^{(k+1)})$.
4. Update $\vec{p}$ as follows: $\vec{p}^{(k+1)} = -\vec{g}^{(k+1)} + \beta_{k+1} \vec{p}^{(k)}$. Where $\beta_{k+1} = \frac{\vec{g}^{(k+1)} \cdot (\vec{g}^{(k+1)} - \vec{g}^{(k)})}{\vec{g}^{(k)} \cdot \vec{g}^{(k)}}$.
5. If $\|\vec{p}^{(k+1)}\| < \varepsilon$ then the model $\vec{m}^{(k+1)}$ is the one we want. Else we restart from 2.

Iterative model improvement:

From reference [2]

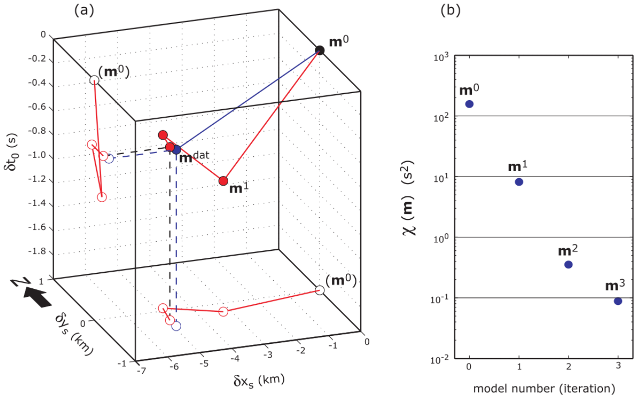

Figure: An Example of basic source inversion.

The point of Full Waveform inversion.

Why I am showing this to you.

- **I hope that I've managed to express the following to you:**

1. No matter how good straight-ray methods may be, they give erroneous results for inhomogeneous media.
2. Bending methods, although overcoming some of the problems associated with straight-ray methods, have have generally proved computationally infeasible.
3. Full Waveform inversion allows one to make the model only as accurate as one would like it to be (one has this control, as one has control over the parameter ε)
4. By doing things properly in the forward direction we should avoid running into nasty surprises.

Closing

Goodbye!

- I also hope that you'll agree with me that this method holds great promise for improving the accuracy of our models.

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Goodbye!

- I also hope that you'll agree with me that this method holds great promise for improving the accuracy of our models.
- This technique has been used to study problems in crustal seismology. So ***why not*** try to modify and adapt this method for applications mine seismology. Currently this is what we are trying to do.

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