Tutorial-6

CZ1104 2020-2021

Presented by: Svetlana (Lana) Obraztsova

Problem-1

Required:

Find orthogonal basis of A from the 3 vectors below:

$$\boldsymbol{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

Solution:

1. First vector $\mathbf{v}_1 = \boldsymbol{x}_1$. $Span\{\mathbf{v}_1\} = Span\{\boldsymbol{x}_1\}$
2. Second vector $\mathbf{v}_2$ should be: $\mathbf{v}_2 \perp \mathbf{v}_1$ and $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$.

$$\mathbf{v}_2 = \boldsymbol{x}_2 - Proj_{\mathbf{v}_1}\boldsymbol{x}_2 = \boldsymbol{x}_2 - \frac{\mathbf{v}_1 \bullet \boldsymbol{x}_2}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{0\cdot1+1\cdot1+1\cdot1+1\cdot1}{1^2+1^2+1^2+1^2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$$

Problem-1

Required:

Find orthogonal basis of A from the 3 vectors below:

$$\boldsymbol{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

Solution:

3. Third vector v_3 should be: $\mathbf{v}_3 \perp \mathbf{v}_1$, $\mathbf{v}_3 \perp \mathbf{v}_2$ and $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$.

$$\mathbf{v}_3 = \boldsymbol{x}_3 - Proj_{\mathbf{v}_1}\boldsymbol{x}_3 - Proj_{\mathbf{v}_2}\boldsymbol{x}_3 = \boldsymbol{x}_3 - \frac{\mathbf{v}_1 \bullet \boldsymbol{x}_3}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{v}_2 \bullet \boldsymbol{x}_3}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 =$$

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{0\cdot 1 + 0\cdot 1 + 1\cdot 1 + 1\cdot 1}{1^2+1^2+1^2+1^2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{0\cdot(-\frac{3}{4}) + 0\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{4}}{\frac{3}{4}^2 + \frac{1}{4}^2 + \frac{1}{4}^2 + \frac{1}{4}^2}\begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix} = \begin{bmatrix}-\frac{36}{84}\\-\frac{44}{84}\\\frac{40}{84}\\\frac{40}{84}\end{bmatrix}$$

$Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{x}_3\} = Span\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$.

Problem-1

Required:

Find orthogonal basis of A from the 3 vectors below:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

1. First vector $\mathbf{v}_1 = \boldsymbol{x}_1$. $Span\{\mathbf{v}_1\} = Span\{\boldsymbol{x}_1\}$
2. Second vector $\mathbf{v}_2$ should be: $\mathbf{v}_2 \perp \mathbf{v}_1$ and $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$.

$$\mathbf{v}_2 = 4(\boldsymbol{x}_2 - Proj_{\mathbf{v}_1}\boldsymbol{x}_2) = 4(\boldsymbol{x}_2 - \frac{\mathbf{v}_1 \bullet \boldsymbol{x}_2}{\|\mathbf{v}_1\|^2}\mathbf{v}_1) = 4(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0\cdot1+1\cdot1+1\cdot1+1\cdot1}{1^2+1^2+1^2+1^2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem-1

Required:

Find orthogonal basis of A from the 3 vectors below:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

3. Third vector v_3 should be: $\mathbf{v}_3 \perp \mathbf{v}_1$, $\mathbf{v}_3 \perp \mathbf{v}_2$ and $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$.

$$\mathbf{v}_3 = \boldsymbol{x}_3 - Proj_{\mathbf{v}_1} \boldsymbol{x}_3 - Proj_{\mathbf{v}_2} \boldsymbol{x}_3 = \boldsymbol{x}_3 - \frac{\mathbf{v}_1 \bullet \boldsymbol{x}_3}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\mathbf{v}_2 \bullet \boldsymbol{x}_3}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}{1^2 + 1^2 + 1^2 + 1^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0 \cdot (-3) + 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}{3^2 + 1^2 + 1^2 + 1^2} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Problem-2

Required:

Form matrix A with 3 column vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ in Problem-1. Find its QR factorization.

Solution:

1. Gram–Schmidt process for $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Almost done in Problem-1. To finalize:

   We have: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$

   To simplify calculations: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$

   Normalize vectors and put them into matrix Q.

   $$Q = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Problem-2

Required:

Form matrix A with 3 column vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ in Problem-1. Find its QR factorization.

Solution:

2. By theorem 12 (see lecture slides) $Q^T Q = I$. Thus,
$Q^T A = Q^T(QR) = (Q^T Q)R = IR = R$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Problem-3

Required:

Find a least-squares solution of $A\boldsymbol{x} = \mathbf{b}$. $A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$

Solution:

Columns ($\mathbf{c}_1$ and $\mathbf{c}_2$) of A are orthogonal ($\mathbf{c}_1 \cdot \mathbf{c}_2 = 1 \cdot (-6) + 1 \cdot (-2) + 1 \cdot 1 + 7 \cdot 7.$) Thus,

$$\widehat{\mathbf{b}} = Proj_{Col\ A}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{c}_1}{\mathbf{c}_1\cdot\mathbf{c}_1}\mathbf{c}_1 + \frac{\mathbf{b}\cdot\mathbf{c}_2}{\mathbf{c}_2\cdot\mathbf{c}_2}\mathbf{c}_2 = \frac{8}{4}\mathbf{c}_1 + \frac{45}{90}\mathbf{c}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ \frac{1}{2} \\ \frac{7}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ \frac{5}{2} \\ \frac{11}{2} \end{bmatrix}$$

Now, we need to solve $A\widehat{\boldsymbol{x}} = \widehat{\mathbf{b}}$. It is trivial, because of the way of finding $\widehat{\mathbf{b}}$:

$$A \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ \frac{5}{2} \\ \frac{11}{2} \end{bmatrix}.$$

That is $\widehat{\boldsymbol{x}} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$.

Problem-4

Required:

Let $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Compute $A\mathbf{u}$ and $A\mathbf{v}$, and compare them with $\mathbf{b}$. Is it possible that at least one of $\mathbf{u}$ or $\mathbf{v}$ could be a least-squares solution of $A\boldsymbol{x} = \mathbf{b}$?

Solution:

$A\mathbf{u} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$ and $A\mathbf{v} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$.

So, $\mathbf{b} - A\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$ and $\mathbf{b} - A\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$.

Therefore, $\|\mathbf{b} - A\mathbf{u}\| = \|\mathbf{b} - A\mathbf{v}\| = \sqrt{24}$.

Orthogonal projection is unique closest point in $Col\ A$ to $\mathbf{b} \Rightarrow$ neither $\mathbf{u}$ nor $\mathbf{v}$ can be a least squares solutions of $A\boldsymbol{x} = \mathbf{b}$.

Problem-7

Required:

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fit the given data points:$(1, 0), (2, 1), (4, 2), (5, 3)$.

Solution:

Construct matrix of the system: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ (y is a number and $\mathbf{y}$ is a vector.)

A is constructed such that $A(\beta_0, \beta_1)^T = \mathbf{y}$ if we put all equations together.

Problem-7

Required:

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fit the given data points:$(1,0), (2,1), (4,2), (5,3)$.

Solution:

To solve problem: Step 1. $A^T A(\beta_0, \beta_1)^T = A^T \mathbf{y}$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}$$

$$A^T \mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}.$$

$$A^T A(\beta_0, \beta_1)^T = A^T \mathbf{y} \Leftrightarrow \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} (\beta_0, \beta_1)^T = \begin{bmatrix} 6 \\ 25 \end{bmatrix}.$$

Problem-7

Required:

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fit the given data points:$(1,0),(2,1),(4,2),(5,3)$.

Solution:

Normal equation: $A^T A(\beta_0, \beta_1)^T = A^T \mathbf{y} \Leftrightarrow \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} (\beta_0, \beta_1)^T = \begin{bmatrix} 6 \\ 25 \end{bmatrix}$.

$det(A) \neq 0 \quad \Rightarrow \quad \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 25 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 46 & -12 \\ -12 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 25 \end{bmatrix} ==$
$\frac{1}{40} \begin{bmatrix} -24 \\ 28 \end{bmatrix}$

Best line: $y = -0.6 + 0.7x$.