Tutorial-2: Matrix Algebra

CZ1104 2020-2021

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Problem-1:

Required:

Consider an $m \times n$ matrix A and an $n \times p$ matrix B. Show that if the columns of B are linearly dependent, then so are the columns of AB.

Solution:

If the columns of B are linearly dependent, then there exists a non-zero vector $\boldsymbol{x}$ such that $B\boldsymbol{x} = 0$.
$\Rightarrow A(B\boldsymbol{x}) = 0 \xRightarrow{\text{by associativity}} (AB)\boldsymbol{x} = 0$
Since $\boldsymbol{x}$ is non-zero, columns of AB must be linearly dependent.

Problem-2:

Required:

Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

Solution:

Let $C = AB$, then $CB^{-1} = ABB^{-1} = AI = A$

I.e., A is a product of invertible matrices and, hence, is invertible.

Problem-3:

Required:

Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible. Also, suppose $(A - AX)^{-1} = X^{-1}B$. Solve for X. If you need to invert a matrix, explain why that matrix is invertible.

Solution:

$(A - AX)^{-1} = X^{-1}B \Rightarrow (A - AX) = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$
$\therefore A = AX + B^{-1}X = (A + B^{-1})X$
(B is invertable, because $XX^{-1}B = X(A - AX)^{-1} \Leftrightarrow B = X(A - AX)^{-1}$, where X and $(A - AX)^{-1}$ are invertible.)
The product $(A + B^{-1})X$ is invertible, because A is invertible. Since X is invertible, so is $(A + B^{-1})$
$\therefore (A + B^{-1})^{-1}A = X$

Problem-4:

Required:

(a) Let $A\boldsymbol{x} = 0$ be a homogeneous system with n linear equations and n unknowns. If $A\boldsymbol{x} = 0$ has only the trivial solution, show that for any positive integer k, the system $A^k x = 0$ also has only the trivial solution.

(b) Let $A\boldsymbol{x} = \mathbf{b}$ be any consistent system of linear equations and let $\boldsymbol{x}_1$ be a fixed solution. Show that every solution to the system can be written in the from $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_0$ where $\boldsymbol{x}_0$ is a solution to $A\boldsymbol{x} = 0$.

Solution:

(a) Since $A\boldsymbol{x} = 0$ has only trivial solution, A is invertible. $\therefore$ A^k is also invertible and $A^k\boldsymbol{x} = 0$ has only a trivial solution.

(b) Let $\boldsymbol{x}_1$ be a fixed solution of $A\boldsymbol{x} = \mathbf{b}$ and let $\boldsymbol{x}$ be any other solution. Then $A(\boldsymbol{x} - \boldsymbol{x}_1) = A\boldsymbol{x} - A\boldsymbol{x}_1 = \mathbf{b} - \mathbf{b} = 0$.
$\therefore$ $\boldsymbol{x}_0 = \boldsymbol{x} - \boldsymbol{x}_1$ is a solution to $A\boldsymbol{x} = 0$. $\therefore \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{x}_1$.

Problem-5:

Required:

(a) If the columns of an $n \times n$ matrix A are linearly independent, show that the columns of A^2 span $\mathbb{R}^n$.

(b) Show that if AB is invertible, so is A.

Solution:

(a) **Invertible Matrix Theorem**: If the columns of A are linearly independent, then, since A is square, A is invertible.
$\therefore$ A^2, which is a product of invertible matrices, is also invertible.
$\Rightarrow$ Columns of A^2 span $\mathbb{R}^n$.

(b) Let W be the inverse of AB, then $ABW = I \Rightarrow A(BW) = I$.
Since A is square, A is invertible *[property (9) of Theorem 2.4 IMT]*

Problem-6:

Required:

Using the notion of pivots and free variables only, answer the following questions:

(a) Suppose A is an $n \times n$ matrix with the property that the equation $A\boldsymbol{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why the equation $A\boldsymbol{x} = \mathbf{b}$ has in fact exactly one solution.

(b) Suppose A is an $n \times n$ matrix with the property that the equation $A\boldsymbol{x} = 0$ has only the trivial solution. Explain why the equation $A\boldsymbol{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

Solution:

(a) Since $A\boldsymbol{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$, A has a pivot in each row.
Since A is square, A has a pivot in each column.
$\therefore$ There are no free variables in $A\boldsymbol{x} = \mathbf{b} \Rightarrow$ solution is unique.

(b) Since $A\boldsymbol{x} = 0$ has only the trivial solution, A is invertible.
$\therefore$ $A\boldsymbol{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$, given by $\boldsymbol{x} = A^{-1}\mathbf{b}$.

Problem-7:

Required:

Solve the equation $A\boldsymbol{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

Solution:

Problem-7:

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Solve the equation $A\boldsymbol{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

Solution: $A\boldsymbol{x} = \mathbf{b} \Rightarrow LU\boldsymbol{x} = \mathbf{b}$. If $U\boldsymbol{x} = \mathbf{y}$, $L\mathbf{y} = \mathbf{b}$

Problem-7:

Required:

Solve the equation $Ax = \mathbf{b}$ using the LU factorization given for A:

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Solution: $Ax = \mathbf{b} \Rightarrow LUx = \mathbf{b}$. If $Ux = \mathbf{y}$, $L\mathbf{y} = \mathbf{b}$

Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

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Solve the equation $Ax = \mathbf{b}$ using the LU factorization given for A:

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Solution: $Ax = \mathbf{b} \Rightarrow LUx = \mathbf{b}$. If $Ux = \mathbf{y}$, $L\mathbf{y} = \mathbf{b}$

Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

$$[L\ \mathbf{b}]$$

$$r_2 \leftarrow -\frac{1}{2}r_1 + r_2$$
$$r_3 \leftarrow -\frac{3}{2}r_1 + r_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{bmatrix}$$

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Solution: $A\boldsymbol{x} = \mathbf{b} \Rightarrow LU\boldsymbol{x} = \mathbf{b}$. If $U\boldsymbol{x} = \mathbf{y}$, $L\mathbf{y} = \mathbf{b}$

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$[L\ \mathbf{b}]$

$$r_2 \leftarrow -\frac{1}{2}r_1 + r_2$$
$$r_3 \leftarrow -\frac{3}{2}r_1 + r_3$$

$$r_3 \leftarrow 5r_2 + r_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{bmatrix}$$

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Solve the equation $Ax = \mathbf{b}$ using the LU factorization given for A:

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$\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$

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$r_2 \leftarrow -\frac{1}{2}r_1 + r_2$

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$$\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$$

$[L\ \mathbf{b}]$

$$r_2 \leftarrow -\frac{1}{2}r_1 + r_2 \qquad r_3 \leftarrow -\frac{3}{2}r_1 + r_3$$

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Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

$$\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$$

Step-2: Next, solve $Ux = \mathbf{y}$ for x.

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$$[U \;\; \mathbf{y}]$$

$$\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix}$$

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Step-2: Next, solve $U\boldsymbol{x} = \mathbf{y}$ for $\boldsymbol{x}$.

$$\underset{[U \;\; \mathbf{y}]}{\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix}} \Rightarrow \underset{r_3 \leftarrow -\frac{1}{6} r_3}{\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

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$$\overset{[U \ \mathbf{y}]}{\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix}} \Rightarrow \overset{r_3 \leftarrow -\frac{1}{6} r_3}{\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}} \Rightarrow \overset{\substack{r_1 \leftarrow r_1 - 4r_3 \\ r_2 \leftarrow r_2 + r_3}}{\begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

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Solve the equation $A\boldsymbol{x} = \mathbf{b}$ using the LU factorization given for A:

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Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. $\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$

Step-2: Next, solve $U\boldsymbol{x} = \mathbf{y}$ for $\boldsymbol{x}$.

$$\xrightarrow{r_3 \leftarrow -\frac{1}{6} r_3} \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[r_2 \leftarrow r_2 + r_3]{r_1 \leftarrow r_1 - 4r_3} \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_2 \leftarrow -\frac{1}{2} r_2} \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Solve the equation $A\boldsymbol{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

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Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. $\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$

Step-2: Next, solve $U\boldsymbol{x} = \mathbf{y}$ for $\boldsymbol{x}$.

$$\begin{array}{c} r_1 \leftarrow r_1 - 4r_3 \\ r_2 \leftarrow r_2 + r_3 \end{array} \qquad r_2 \leftarrow -\frac{1}{2}r_2 \qquad r_1 \leftarrow 2r_2 + r_1$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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Solve the equation $A\boldsymbol{x} = \mathbf{b}$ using the LU factorization given for A:

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Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. $\quad \mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$

Step-2: Next, solve $U\boldsymbol{x} = \mathbf{y}$ for $\boldsymbol{x}$. $\quad x = \begin{bmatrix} -5 & 1 & 3 \end{bmatrix}^T$

$$\xrightarrow{}\; \overset{r_2 \leftarrow -\frac{1}{2}r_2}{\Rightarrow \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}} \quad \overset{r_1 \leftarrow 2r_2 + r_1}{\Rightarrow \begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}} \quad \overset{r_1 \leftarrow \frac{1}{2}r_1}{\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

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Step-1: Solving $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. $\mathbf{y} = \begin{bmatrix} 0 & -5 & -18 \end{bmatrix}^T$

Step-2: Next, solve $U\boldsymbol{x} = \mathbf{y}$ for $\boldsymbol{x}$. $x = \begin{bmatrix} -5 & 1 & 3 \end{bmatrix}^T$

$$\Rightarrow \overset{r_2 \leftarrow -\frac{1}{2} r_2}{\begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}} \Rightarrow \overset{r_1 \leftarrow 2r_2 + r_1}{\begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}} \Rightarrow \overset{r_1 \leftarrow \frac{1}{2} r_1}{\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

Problem-8: Spectral Factorization

Required:

Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A.

Solution:

Problem-8: Spectral Factorization

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Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A.

Solution:

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} \\ &= PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1} \end{aligned}$$

$$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \qquad \therefore A^2 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} P^{-1}$$

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Solution:

$$\begin{aligned} A^3 &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)D(P^{-1}P)DP^{-1} \\ &= PDIDIDP^{-1} = PDDDP^{-1} = PD^3P^{-1} \end{aligned}$$

$$D^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^3 & 0 \\ 0 & 0 & (1/3)^3 \end{bmatrix} \qquad \therefore A^3 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/27 \end{bmatrix} P^{-1}$$

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Solution:

$$A^k = \left(PDP^{-1}\right)^k = PD^kP^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (1/3)^k \end{bmatrix} P^{-1}$$