Tutorial-5

CZ1104 2020-2021

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Problem-1

Required:

Compute the quantities in exercises using the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$$

4) $\frac{1}{\mathbf{u} \bullet \mathbf{u}} \mathbf{u}$

5) $\frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v}$

7) $\|\mathbf{w}\|$

Solution:

4) $\frac{1}{\mathbf{u} \bullet \mathbf{u}} \mathbf{u} = \frac{1}{(-1)^2 + 2^2} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5} \\ \frac{2}{5} \end{bmatrix}$

5) $\frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}} \mathbf{v} = \frac{(-1) \cdot 4 + 2 \cdot 6}{4^2 + 6^2} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{8}{52} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{8}{13} \\ \frac{12}{13} \end{bmatrix}$

7) $\|\mathbf{w}\| = \sqrt{\mathbf{w} \bullet \mathbf{w}} = \sqrt{3^2 + 1^2 + 5^2} = \sqrt{35}$

Problem-3

Required:

Let $\mathbf{W}$ be a subspace of $\mathbb{R}^n$, and let $\mathbf{W}^\perp$ be the set of all vectors orthogonal to $\mathbf{W}$. Show that $\mathbf{W}^\perp$ is a subspace of $\mathbb{R}^n$.

Solution:

Check for the requirements of a subspace:

(i) $\mathbf{W}^\perp$ contains the zero vector, because it is orthogonal to every vector.

(ii) Let $\mathbf{z}$ be in $\mathbf{W}^\perp$, $\mathbf{u}$ be any vector from $\mathbf{W}$ and c be any scalar.
$c\mathbf{z} \bullet \mathbf{u} = c(\mathbf{z} \bullet \mathbf{u}) = c0 = 0$. Thus, $c\mathbf{z} \in \mathbf{W}^\perp$.

(iii) Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $\mathbf{W}^\perp$. Then for any $\vec{u} \in \mathbf{W}$,
$(\mathbf{z}_1 + \mathbf{z}_2) \bullet \mathbf{u} = (\mathbf{z}_1 \bullet \mathbf{u}) + (\mathbf{z}_2 \bullet \mathbf{u}) = 0 + 0 = 0$. Thus, $\mathbf{z}_1 + \mathbf{z}_2 \in \mathbf{W}^\perp$.

Therefore, $\mathbf{W}^\perp$ is a subspace.

Problem-2

Required:

In exercises all vectors are in $\mathbb{R}^n$. Mark each statement True or False.

a) $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$.

b) For any scalar c, $\mathbf{u} \bullet (c\mathbf{v}) = c(\mathbf{u} \bullet \mathbf{v})$.

c) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then and $\mathbf{v}$ are orthogonal.

d) For square matrix A, vectors in $\mathbf{Col}(A)$ are orthogonal to vectors in $\mathbf{N}(A)$.

e) If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace $\mathbf{W}$ and if $\boldsymbol{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\boldsymbol{x}$ is in $\mathbf{W}^{\perp}$.

Solution:

a) $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$.
By definition.

b) For any scalar c, $\mathbf{u} \bullet (c\mathbf{v}) = c(\mathbf{u} \bullet \mathbf{v})$.
$\mathbf{u} \bullet (c\mathbf{v}) = u_1(cv_1) + \ldots + u_n(cv_n) = c(u_1 v_1) + \ldots + c(u_n v_n) = c(\mathbf{u} \bullet \mathbf{v})$

Problem-2

Required:

In exercises all vectors are in $\mathbb{R}^n$. Mark each statement True or False.

c) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-bv$, then and bv are orthogonal.

Solution:

c) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then and $\mathbf{v}$ are orthogonal.

By definition $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

$$d(\mathbf{u}, \mathbf{v}) = d(\mathbf{u}, -\mathbf{v}) \Leftrightarrow \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\| \Leftrightarrow$$
$$\Leftrightarrow \|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \Leftrightarrow$$
$$\Leftrightarrow (u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots (u_n - v_n)^2 = (u_1 + v_1)^2 + (u_2 + v_2)^2 + \ldots (u_n + v_n)^2 \Leftrightarrow$$
$$\Leftrightarrow 0 = 4u_1 v_1 + 4u_2 v_2 + \ldots + 4u_n v_n \Leftrightarrow 4\mathbf{u} \bullet \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \bullet \mathbf{v} = 0$$

Problem-2

Required:

In exercises all vectors are in $\mathbb{R}^n$. Mark each statement True or False.

d) For square matrix A, vectors in $\mathbf{Col}(A)$ are orthogonal to vectors in $\mathbf{N}(A)$.

e) If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace $\mathbf{W}$ and if $\boldsymbol{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\boldsymbol{x}$ is in $\mathbf{W}^\perp$.

Solution:

d) For square matrix A, vectors in $\mathbf{Col}(A)$ are orthogonal to vectors in $\mathbf{N}(A)$.
Counterexample:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

e) If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace $\mathbf{W}$ and if $\boldsymbol{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\boldsymbol{x}$ is in $\mathbf{W}^\perp$.
Any vector in $Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is $a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p$.

$$\boldsymbol{x} \bullet (a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p) = a_1(\boldsymbol{x} \bullet \mathbf{v}_1) + \ldots + a_p(\boldsymbol{x} \bullet \mathbf{v}_p) = 0 + \ldots + 0 = 0$$

Thus, $\boldsymbol{x}$ is orthogonal to any vector in $\mathbf{W}$ and $\boldsymbol{x} \in \mathbf{W}^\perp$.

Problem-4

Required:

Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.

Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in $Span\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

Solution:

$Proj_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$.

In our case: $Proj_{\mathbf{u}}\mathbf{y} = \frac{7 \cdot 4 + 6 \cdot 2}{4^2 + 2^2} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

The vector component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is $\mathbf{y} - proj_{\mathbf{u}}\mathbf{y}$. That is,
$\begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Check correctness: $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 4 \cdot (-1) + 2 \cdot 2 = 0$

Problem-5

Required:

Determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

a) $\mathbf{v} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

b) $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Solution:

a) Orthogonality: $\mathbf{v} \bullet \mathbf{u} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \bullet \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{3} \cdot (-\frac{1}{2}) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2}$.

$\mathbf{v}$ and $\mathbf{u}$ are orthogonal.
Normal vectors: $\|\mathbf{v}\|^2 = \mathbf{v} \bullet \mathbf{v} = (\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2 = \frac{1}{3}$ No.
$\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u} = (\frac{1}{2})^2 + 0^2 + (\frac{1}{2})^2 = \frac{1}{2}$ No.

Normalize $\mathbf{v}$ and $\mathbf{u}$: $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \sqrt{3} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$; $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \sqrt{2} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$.

Problem-5

Required:

Determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

a) $\mathbf{v} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

b) $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Solution:

b) Orthogonality: $\mathbf{v} \bullet \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 0 \cdot 0 + 1 \cdot (-1) + 0 \cdot 0 = -1.$

$\mathbf{v}$ and $\mathbf{u}$ are not orthogonal.

Problem-6.1

Required:

Mark each statement True or False.

a) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b) If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d) A matrix with orthonormal columns is an orthogonal matrix.

e) If L is a line through 0 and if $\mathbf{y}$ is the orthogonal projection of $\hat{\mathbf{y}}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

Problem-6.1

Required:

Mark each statement True or False.

a) Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b) If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

Solution:

a) For example, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

b) Formulas for coordinates of vector $\mathbf{y}$ in orthogonal basis $\{\mathbf{e}_i\}_{i:1,\ldots,n}$: $\mathbf{y}_i = \frac{\mathbf{y} \bullet \mathbf{e}_i}{\|\mathbf{e}_i\|^2}$ for $i : 1, \ldots, n$.

Problem-6.1

Required:

Mark each statement True or False.

c) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d) A matrix with orthonormal columns is an orthogonal matrix.

e) If L is a line through 0 and if $\mathbf{y}$ is the orthogonal projection of $\hat{\mathbf{y}}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

Solution:

c) $\mathbf{v} \bullet \mathbf{u} = 0 \Leftrightarrow a\mathbf{v} \bullet b\mathbf{u} = 0$ if and only if $ab \neq 0$.

d) Orthogonal matrix should be square.

e) Distance is $\mathbf{y} - \hat{\mathbf{y}}$.

Problem-6.2

Required:

Mark each statement True or False.

a) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b) If a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ has a property that $\mathbf{u}_i \bullet \mathbf{u}_j = 0$ for all $i \neq j$, then S is an orthonormal set.

c) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\boldsymbol{x} \to A\boldsymbol{x}$ preserves lengths.

d) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e) An orthogonal matrix is invertible.

Problem-6.2

Required:

Mark each statement True or False.

a) Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b) If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has a property that $\mathbf{u}_i \bullet \mathbf{u}_j = 0$ for all $i \neq j$, then S is an orthonormal set.

Solution:

a) For example, $\vec{0}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

False: Not every orthogonal set of **nonzero vectors** in $\mathbb{R}^n$ is linearly independent.

b) Orthogonal, but not orthonormal.

Problem-6.2

Required:

Mark each statement True or False.

c) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\boldsymbol{x} \to A\boldsymbol{x}$ preserves lengths.

d) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e) An orthogonal matrix is invertible.

Solution:

d) $Proj_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{c(\mathbf{y} \bullet \mathbf{u})}{c^2\|\mathbf{u}\|^2}c\mathbf{u} = \frac{\mathbf{y} \bullet c\mathbf{u}}{\|c\mathbf{u}\|^2}c\mathbf{u} = Proj_{c\mathbf{u}}\mathbf{y}$

e) A is orthogonal $\Leftrightarrow A^TA = I$.

Problem-7

Required:

Show that the orthogonal projection of a vector $\mathbf{y}$ onto a line L through the origin in $\mathbb{R}^2$ does not depend on the choice of nonzero $\mathbf{u}$ in L used in formula for $\hat{\mathbf{y}}$.

Solution:

See problem 6.2.d).

Problem-8

Required:

The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\mathbf{y}$ to the nearest point in W. Find the distance from $\mathbf{y}$ to $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$, where
$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Solution:

$dist(\mathbf{y}, W) = \|\mathbf{y} - proj_W \mathbf{y}\|$.
Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis of W then $proj_W \mathbf{y} = proj_{\mathbf{u}_1} \mathbf{y} + proj_{\mathbf{u}_2} \mathbf{y} =$
$\frac{(-1)\cdot 5 + (-5)\cdot(-2) + 10\cdot 1}{5^2+2^2+1^2} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{(-1)\cdot 1 + (-5)\cdot 2 + 10\cdot(-1)}{1^2+2^2+1^2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{15}{30} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-21}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$.

$$\mathbf{y} - proj_W \mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$dist(\mathbf{y}, W) = \left\| \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \right\| = \sqrt{0^2 + 3^2 + 6^2} = \sqrt{45}.$$