Tutorial-3: Determinants

CZ1104 2020-2021

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Problem-1: Determinant scaling

Required:

If a 3×3 matrix A has $|A| = -1$, find $\left|\frac{1}{2}A\right|$, $|-A|$, $\left|A^2\right|$, $\left|A^{-1}\right|$.

Solution:

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Solution:

Recall two properties: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $|AB| = |A|\,|B|$

(a) $\left|\frac{1}{2}A\right| = \frac{1}{2^3}\,|A| = -\frac{1}{8}$, since each of 3 rows was multiplied by $\frac{1}{2}$

(b) $|-A| = (-1)^3\,|A| = 1$, same as above

(c) $A^2 = AA \Rightarrow \left|A^2\right| = |A|\,|A| = 1$

(d) $AA^{-1} = I \Rightarrow 1 = |I| = \left|AA^{-1}\right| = |A|\,\left|A^{-1}\right| \Rightarrow \left|A^{-1}\right| = \frac{1}{|A|} = -1$

Problem-2: Determinant of a triangular matrix

Required:

Reduce $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ to U to find $\det A$ as the product of pivots.

Solution:

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Solution: Bad solution

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow[r_2 \leftarrow r_2 - r_1]{r_3 \leftarrow r_2 - r_3} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$|A| =$ product of pivots $= 1$

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Reduce $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ to U to find $\det A$ as the product of pivots.

Solution: Good solution

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow[r_2 \leftarrow r_2 - r_1]{r_3 \leftarrow r_3 - r_2} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{-1} \end{bmatrix}$$

$|A| =$ product of pivots $= -1$

Problem-3: Skew symmetric matrix determinant

Required:

Using variables a, b, c, construct a 3×3 skew-symmetric matrix $(A = -A^T)$. Show that the determinant of such a matrix is equal to 0.

Solution:

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Solution: Explicit

All diagonal elements are equal to their negative value $\Rightarrow$ diag$(A) = \vec{0}$.

$$A = \begin{bmatrix} 0 & b & -c \\ -b & 0 & -a \\ c & a & 0 \end{bmatrix} \Rightarrow |A| \begin{aligned}[t] &= 0 \cdot 0 \cdot 0 + a \cdot (-b) \cdot (-c) + (-a) \cdot b \cdot c \\ &\quad - (c \cdot 0 \cdot (-c) + (-b) \cdot b \cdot 0 + 0 \cdot (-a) \cdot a) \\ &= 0 \end{aligned}$$

Problem-3: Skew symmetric matrix determinant

Required:

Using variables a, b, c, construct a 3×3 skew-symmetric matrix $(A = -A^T)$. Show that the determinant of such a matrix is equal to 0.

Solution: Implicit

Since A is skew-symmetric $|A| = \left|-A^T\right| = (-1)^3 \left|A^T\right| = -\left|A^T\right|$
But $|A| = \left|A^T\right|$ always. $\therefore$ must hold $|A| = -\left|A^T\right| = -|A|$.
$\therefore\ |A| = 0$.

Problem-4: Determinants and Geometry (part 1)

Required:

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

Solution:

Problem-4: Determinants and Geometry (part 1)

Required:

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

Solution:

The parallelepiped is determined by the columns of $A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

$\therefore$ The folume of the parallelepiped is $\text{abs}\,(|A|) = \text{abs}\,(-18) = 18$

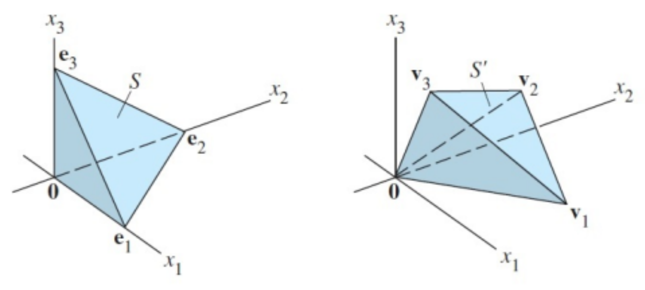
Problem-5: Determinants and Geometry (part 2)

Required:

Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $0, e_1, e_2, e_3$ and let S' be the tetrahedron with vertices at vectors $0, v_1, v_2, v_3$.

(a) Find the standard matrix for the linear transformation that maps S to S'.

(b) Find a formula for the volume of the tetrahedron S'.
Recall, volume of a tetrahedron $= (1/3) \times$ (area of base) $\times$ height.



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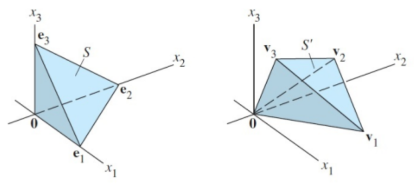

Solution:

The standard matrix for this transformation is
$A = [T(e_1)\ \ T(e_2)\ \ T(e_3)] = [v_1\ \ v_2\ \ v_3]$

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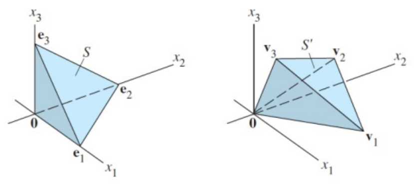

Solution:

Volume of $S = (1/3) \times$ (area of base) $\times$ height
Area of base $= \frac{1}{2} \times 1 \times 1$
$\therefore \text{ Vol}(S) = \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}$
$\therefore \text{ Vol}(S') = |A| \cdot \text{Vol}(S) = \frac{1}{6}\,|A|$