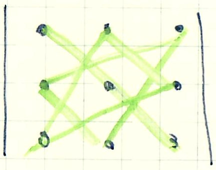
Q2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution:

To find eigenvalues:

$$\begin{vmatrix} 0-\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} =$$



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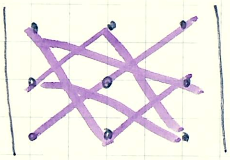

$$= (-\lambda)(2-\lambda)^2 + (-1)\cdot 1\cdot 1 + (-1)\cdot 1 \cdot 1 - \left(-(2-\lambda) + (-\lambda)\cdot 1\cdot 1 + (-1)\cdot 1\cdot (2-\lambda)\right) = (-\lambda)(2-\lambda)^2 - 2 + \lambda + 2(2-\lambda) = (2-\lambda)(\lambda^2 - 2\lambda - 1 + 2) = (2-\lambda)(\lambda^2 - 2\lambda + 1) = (2-\lambda)(\lambda-1)^2$$

Eigenvalues: 1, 2.

To find eigenvectors:

$\lambda = 1$

$$\begin{pmatrix} 0-1 & -1 & -1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0$$

2 free variables $\Rightarrow$ dim(solutions) = 2 : span$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$\lambda = 2$

$$\begin{pmatrix} 0-2 & -1 & -1 \\ 1 & 2-2 & 1 \\ 1 & 1 & 2-2 \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{cases} x + z = 0 \\ x + y = 0 \end{cases} \iff \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

Answer: eigenvalue $\lambda = 1$ 2 eigenvectors span$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

eigenvalue $\lambda = 2$ & eigenvectors span$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$

Q1 b) Is $\lambda = 4$ an eigenvalue of $\begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{pmatrix}$? If so, find 1 corresponding eigenvector.

Solution:

To check eigenvalue:

$$\begin{vmatrix} 3\boxed{-4} & 0 & -1 \\ 2 & 3\boxed{-4} & 1 \\ -3 & 4 & 5\boxed{-4} \end{vmatrix} = \begin{vmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{vmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 + R_1} \begin{vmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ -4 & 4 & 0 \end{vmatrix} =$$

$$\xrightarrow{R_3 \leftarrow R_3 + 4R_2} \begin{vmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\lambda = 4$ is eigenvalue

To find eigenvector

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \underset{3)\, R_3 \leftarrow R_3 + \text{new}R_2}{\overset{1)\, R_2 \leftarrow R_2 + R_1 \quad 2)\, R_3 \leftarrow R_3 + R_1}{\Longleftrightarrow}} \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} -x - z = 0 \\ x - y = 0 \end{cases} \Longleftrightarrow \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Answer: $\lambda = 4$ is eigenvalue & for example $\begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$ is resp. eigenvector.

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Q1(a) Is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$? If so, find the eigenvalue.

Solution:

To check eigenvector:

$$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 6\cdot(-2) + 7\cdot 1 \\ 3\cdot 1 + 3\cdot(-2) + 7\cdot 1 \\ 5\cdot 1 + 6\cdot(-2) + 5\cdot 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = \boxed{(-2)}\cdot\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Answer: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector & resp. eigenvalue -2

Q11: Compute A^8, where $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$

Step 1. Find eigenvalues.

$$\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 2\cdot(-3) = \lambda^2 - 3\lambda - 4 + 6 =$$
$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$$

Eigenvalues: 1, 2.

Step 2. Find eigenvectors

$\lambda = 1$ $\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x - y = 0 \Leftrightarrow \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\lambda = 2$ $\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 2x - 3y = 0 \Leftrightarrow \text{span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$

Set $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$ is linear independent $\Rightarrow$ there is basis of $\mathbb{R}^2$ of eigenvectors of A.

Step 3. $A = PDP^{-1}$: $P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$P^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}\cdot\frac{1}{\det P} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}\cdot(-1) = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$A^8 = PD^8P^{-1} = \begin{pmatrix} 766 & -765 \\ 510 & -509 \end{pmatrix}$$

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