# Tutorial-4: Vector Spaces

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# Problem-1: Matrix Subspace

## Required:

An $n \times n$ matrix $A$ is said to be symmetric if $A^T = A$. Let $S$ be the set of all $3 \times 3$ symmetric matrices. Show that $S$ is a subspace of $M_{3\times 3}$ , the vector space of all $3 \times 3$ matrices.

## Solution:

Check for the requirements of a subspace:

(i) Zero "vector" is the $3 \times 3$ zero matrix, which is symmetric and, hence, is in $S$.

(ii) Let $A$ and $B$ in $S$, with $A = A^T$ and $B = B^T$.
$(A + B)^T = A^T + B^T = A + B$. $\therefore A + B$ is symmetric and in $S$.

(iii) Let $A$ be in $S$ and $c$ a scalar. $(cA)^T = cA^T = cA$.
$\therefore cA$ is also symmetric and, hence, in $S$.

# Problem-2: Affine Spaces

## Required:

(a) Let $P$ be the plane in $\mathbb{R}^3$ with equation $x + y - 2z = 4$. Find two vectors in $P$ and check that their sum is **not** in $P$.

(b) Let $P_0$ be the plane through $(0,0,0)$ and parallel to $P$. Write the equation for $P_0$. Find two vectors in $P_0$ and check that their sum is **in** $P_0$.

## Solution:

(a) The plane does **not** go through $(0,0,0)$. The sum of $(4,0,0)$ and $(0,4,0)$ is **not** on the plane.

(b) The parallel plane $P_0$ has the quation $x + y - 2z = 0$. Pick two points, say, $(2,0,1)$ and $(0,2,1)$. Their sum $(2,2,2)$ is in $P_0$.

# Problem-3: Subspace intersection

## Required:

Let $H$ and $K$ be subspaces of a vector space $V$. The intersection of $H$ and $K$, written as $H \cap K$, is the set of vectors $v \in V$ that belong to both $H$ and $K$. Show that $H \cap K$ is a subspace of $V$.

## Solution:

Check for the requirements of a subspace:

(i) Both $H$ and $K$ contain the zero vector, because they are subspaces of $V$. Thus, zero vector of $V$ is in $H \cap K$.

(ii) Let $\vec{u}$ and $\vec{v}$ be in $H \cap K$. Then $\vec{u}$ and $\vec{v}$ are in $H$. Since $H$ is a subspace, $\vec{u} + \vec{v}$ is in $H$. Likewise, $\vec{u}$ and $\vec{v}$ are in $K$, and, since $K$ is a subspace, $\vec{u} + \vec{v}$ is in $K$. Thus $\vec{u} + \vec{v}$ is in $H \cap K$.

(iii) Let $\vec{u}$ be in $H \cap K$. Then $\vec{u}$ is in $H$. Since $H$ is a subspace, $c\vec{u}$ is in $H$. Likewise, $\vec{u}$ is in $K$, and $K$ is a subspace, so that $c\vec{u}$ is in $K$. Thus, $c\vec{u}$ is in $H \cap K$.

Therefore, $H \cap K$ is a subspace.

# Problem-4: Vector Space Proper

## Required:

Determine if the following set is a vector space:

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \; : \; \begin{array}{lcl} a - 2b & = & 4c \\ 2a & = & c + 3d \end{array} \right\}$$

## Solution:

The set $W$ is the set of all solutions to the homogeneous equations:

$$\begin{array}{rcl} a - 2b - 4c & = & 0 \\ 2a - c - 3d & = & 0 \end{array}$$

I.e., $W = \mathbf{N}(A)$, the null space of $A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}$.

$\therefore$ $W$ is a subspace of $\mathbb{R}^4$ and is a vector space.

# Problem-5: Matrices and Spaces (part 1)

## Required:

Find the matrix $A$ if the following set is $\mathbf{C}(A)$:

$$\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \text{ real} \right\}$$

## Solution:

An element in this set may be written as:

$$r \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = A \begin{bmatrix} r \\ s \\ t \end{bmatrix},$$

where $r, s, t$ are any real numbers.

$\therefore$ The set is $\mathbf{C}(A)$, the column space of $A$.

# Problem-6: Matrices and Spaces (part 2)

## Required:

For the matrix $D = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, find a nonzero vector in $\mathbf{N}(D)$ and a nonzero vector in $\mathbf{C}(D)$.

## Solution:

Either column of $D$ is a non-zero vector in $\mathbf{C}(D)$. To find a non-zero vector in $\mathbf{N}(D)$, find the general solution of $D\boldsymbol{x} = 0$ in terms of the free variables.

$$\begin{bmatrix} D & \vec{0} \end{bmatrix} = \begin{bmatrix} 2 & -6 & 0 \\ -1 & 3 & 0 \\ -4 & 12 & 0 \\ 3 & -9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 - 3x_2 = 0 \Rightarrow x_1 = 3x_2,$$

where $x_2$ is a free variable. Say, $x_2 = 1$, then $\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is in $\mathbf{N}(D)$.

# Problem-7: Subspace basis (part 1)

## Required:

Find the basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

## Solution:

Let $A = [1, 2, 1]$. We wish to find a basis for $\mathbf{N}(A)$. The general solution of $A\boldsymbol{x} = 0$ in terms of free variables is $x = -2y - z$.

$$\therefore \ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for $\mathbf{N}(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

# Problem-8: Subspace basis (part 2)

## Required:

Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = 0$. Find a basis for $H$.

## Solution:

Since $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = 0$, each of the vectors is a linear combination of the others. Thus, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span $H$.
None of the vectors is a multiple of any of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent.
Therefore, each set forms a basis for $H$.

# Problem-9: Vector Space of Polynomials

## Required:

Consider the polynomials $p_1(t) = 1 + t, p_2(t) = 1 - t$ and $p_3(t) = 2$ (for all $t$). By inspection, write a linear dependence relation among $p_1$, $p_2$ and $p_3$. Then find a basis for $Span\{p_1, p_2, p_3\}$.

## Solution:

By inspection, $p_3 = p_1 + p_2$ or $p_1 + p_2 - p_3 = \vec{0}$.
By the Spanning Set Theorem, $Span\{p_1, p_2, p_3\} = Span\{p_1, p_2\}$. Since neither $p_1$ nor $p_2$ is a multiple of the other, they are linearly independent.
$\therefore \{p_1, p_2\}$ is a basis for $Span\{p_1, p_2, p_3\}$.

# Problem-10: Coordinate systems

## Required:

Use an inverse matrix to find the $\mathcal{B}$-coordinate of the vector $\boldsymbol{x}$, i.e., $[\boldsymbol{x}]_{\mathcal{B}}$, for $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and $\boldsymbol{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

## Solution:

Since $P_{\mathcal{B}}^{-1}$ converts $\boldsymbol{x}$ into its $\mathcal{B}$-coordinate vector:

$$[\boldsymbol{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\boldsymbol{x} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

# Problem-11: Subspace dimension

## Required:

Find the dimension of the subspace $H$ of $\mathbb{R}^2$ spanned by
$\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

## Solution:

$$A = \begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix} \xrightarrow{r_2 \leftarrow \frac{2}{5} r_2 + r_1} \begin{bmatrix} \boxed{2} & -4 & -3 \\ 0 & 0 & \boxed{-3/5} \end{bmatrix}$$

The matrix has 2 pivot columns

$\therefore \dim\left(\mathbf{C}(A)\right)$ [which is the dimension of $H$] is 2.

# Problem-12: Null and Column (Image) Spaces (part 1)

## Required:

Determine the dimensions of $\mathbf{N}(A)$ and $\mathbf{C}(A)$ for $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

## Solution:

The matrix is in row echelon form.

There are 2 pivot columns.

$\therefore \dim \mathbf{C}(A) = 2.$

There are 2 columns without pivots $\Rightarrow A\boldsymbol{x} = 0$ has two free variables.

$\therefore \dim \mathbf{N}(A) = 2.$

# Problem-13: Null and Column (Image) Spaces (part 2)

## Required:

If a $3 \times 8$ matrix $A$ has rank $3$, find $\dim \mathbf{N}(A), \dim \mathbf{C}(A^T)$, and rank of $A^T$.

## Solution:

By the Rank Theorem, $\dim \mathbf{N}(A) = 8 - rank(A) = 8 - 3 = 5$

$\dim \underbrace{\mathbf{C}(A^T)}_{\text{row space}} = rank(A) = 3$

Since $rank(A^T) = \dim \mathbf{C}(A^T) = 3$

# Problem-14: Null and Column (Image) Spaces (part 3)

## Required:

Suppose the solutions of a homogeneous system of $5$ linear equations in $6$ unknowns are all multiples of $1$ nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations?

## Solution:

Consider the system $A\boldsymbol{x} = 0$, where $A$ is $5 \times 6$.
Since there is only $1$ non-zero solution, $\dim \mathbf{N}(A) = 1$.
From the Rank Theorem, $rank(A) = 6 - \dim \mathbf{N}(A) = 5$.
$\therefore \dim \mathbf{C}(A) = rank(A) = 5$.
Since $\mathbf{C}(A)$ is a subspace of $\mathbb{R}^5$, $\mathbf{C}(A) = \mathbb{R}^5$.
$\Rightarrow$ every vector $\mathbf{b} \in \mathbb{R}^5$ is also in $\mathbf{C}(A)$
$\Rightarrow$ $A\boldsymbol{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$.

# Problem-15: More on rank

## Required:

Verify that the rank of $\mathbf{u}\mathbf{v}^T \leq 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

## Solution:

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$$

Each column of $\mathbf{u}\mathbf{v}^T$ is a multiple of $\mathbf{u} \Rightarrow \dim \mathbf{C}(\mathbf{u}\mathbf{v}^T) = 1$,
unless $a = b = c = 0$, in which case $\mathbf{u}\mathbf{v}^T$ is the $3 \times 3$ zero matrix and $\dim \mathbf{C}(\mathbf{u}\mathbf{v}^T) = 0$.
In either case, $rank(\mathbf{u}\mathbf{v}^T) = \dim \mathbf{C}(\mathbf{u}\mathbf{v}^T) \leq 1$.

# Problem-16: More on coordinates

## Required:

Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for $V$ and suppose that $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change-of-coordinate matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Find $[\boldsymbol{x}]_{\mathcal{B}}$ for $\boldsymbol{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

## Solution:

# Problem-16: More on coordinates

## Required:

Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for $V$ and suppose that $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change-of-coordinate matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Find $[\boldsymbol{x}]_{\mathcal{B}}$ for $\boldsymbol{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

## Solution:

(a) $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$

$$\therefore\ [\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, [\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, [\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\therefore P_{\mathcal{B}\leftarrow\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

# Problem-16: More on coordinates

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Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for $V$ and suppose that $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change-of-coordinate matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Find $[\boldsymbol{x}]_{\mathcal{B}}$ for $\boldsymbol{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

## Solution:

(b) $\boldsymbol{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3 \;\Rightarrow\; [\boldsymbol{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$.

$$[\boldsymbol{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}[\boldsymbol{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$