Tutorial-1: Systems of Linear Equations

CZ1104 2020-2021

Presented by: Svetlana (Lana) Obraztsova

Problem-1: Existence of non-trivial solutions

Required

Find the values of k for which the following system has non-trivial solutions

$$\begin{array}{rcrcrcl} x & + & 5y & + & 3z & = & 0 \\ 5x & + & y & - & kz & = & 0 \\ x & + & 2y & + & kz & = & 0 \end{array}$$

Solution:

Problem-1: Existence of non-trivial solutions

Required

Find the values of k for which the following system has non-trivial solutions

Augmented Matrix

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 5 & 1 & -k & 0 \\ 1 & 2 & k & 0 \end{bmatrix} \Leftarrow \begin{array}{rcrcrcr} x & + & 5y & + & 3z & = & 0 \\ 5x & + & y & - & kz & = & 0 \\ x & + & 2y & + & kz & = & 0 \end{array}$$

Solution: via Gaussian elimination

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Solution: via Gaussian elimination

$$\begin{array}{c} r_2 \leftarrow r_2 - 5r_1 \\ r_3 \leftarrow r_3 - r_1 \\ \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -k-15 & 0 \\ 0 & -3 & k-3 & 0 \end{bmatrix} \end{array} \Rightarrow \begin{array}{c} r_3 \leftarrow 8r_3 - r_2 \\ \\ \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -k-15 & 0 \\ 0 & 0 & 9k-9 & 0 \end{bmatrix} \end{array}$$

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Non-trivial solution $\Rightarrow$ infinitely many solution for homogeneous system

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$$\begin{array}{c} r_2 \leftarrow r_2 - 5r_1 \\ r_3 \leftarrow r_3 - r_1 \end{array} \qquad\qquad r_3 \leftarrow 8r_3 - r_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -k-15 & 0 \\ 0 & -3 & k-3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -k-15 & 0 \\ 0 & 0 & 9k-9 & 0 \end{bmatrix}$$

Non-trivial solution $\Rightarrow$ infinitely many solution for homogeneous system

$\therefore$ The last row should be all zeros $\Rightarrow 9k - 9 = 0 \Rightarrow k = 1$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

Problem-2: Gaussian elimination

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Use Gaussian elimination and back substitution to solve the following system of linear equations

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow$$

In matrix form

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Augmented matrix

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form — Augmented matrix

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

$$r_1 \leftarrow \frac{1}{2} r_1$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

$$r_1 \leftarrow \frac{1}{2} r_1 \qquad\qquad r_2 \leftarrow r_2 - r_1, \quad r_3 \leftarrow r_3 - r_1$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 5/2 & 3 & 3/2 \end{bmatrix}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow$$

In matrix form

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Augmented matrix

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

$$r_1 \leftarrow \frac{1}{2} r_1$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} \Rightarrow$$

$$r_2 \leftarrow r_2 - r_1$$
$$r_3 \leftarrow r_3 - r_1$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 5/2 & 3 & 3/2 \end{bmatrix} \Rightarrow$$

$$r_3 \leftarrow r_3 - 5r_2$$

$$\underbrace{\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix}}_{\text{row-echelon form}}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow$$

In matrix form

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Augmented matrix

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Reduce augmented matrix to row-echelon form

$$r_1 \leftarrow \frac{1}{2} r_1 \qquad\qquad r_2 \leftarrow r_2 - r_1,\; r_3 \leftarrow r_3 - r_1 \qquad\qquad r_3 \leftarrow r_3 - 5r_2$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 5/2 & 3 & 3/2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix}}_{\text{row-echelon form}}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-1: Gaussian elimination done. Now apply back substitution

$$r_1 \leftarrow \tfrac{1}{2} r_1 \qquad r_2 \leftarrow r_2 - r_1, \; r_3 \leftarrow r_3 - r_1 \qquad r_3 \leftarrow r_3 - 5r_2$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 5/2 & 3 & 3/2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix}}_{\text{row-echelon form}}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-2: Apply back substitution

Row-echelon form

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

System

$$\begin{aligned} x + \frac{3}{2}y + 2z &= 1/2 \\ \frac{1}{2}y + z &= 1/2 \\ -2z &= -1 \end{aligned}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x+3y+4z &= 1 \\ x+2y+3z &= 1 \\ x+4y+5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-2: Apply back substitution

Row-echelon form

System

Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \qquad \begin{aligned} x+\tfrac{3}{2}y+2z &= 1/2 \\ \tfrac{1}{2}y+z &= 1/2 \\ -2z &= -1 \end{aligned} \qquad \Rightarrow z = 1/2$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x+3y+4z &= 1 \\ x+2y+3z &= 1 \\ x+4y+5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

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Step-2: Apply back substitution

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System

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$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{aligned} x+\tfrac{3}{2}y+2z &= 1/2 \\ \tfrac{1}{2}y+z &= 1/2 \\ -2z &= -1 \end{aligned} \quad \Rightarrow z = 1/2$$

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Solution

Step-2: Apply back substitution

Row-echelon form

System

Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{aligned} x+\tfrac{3}{2}y+2z &= 1/2 \\ \tfrac{1}{2}y+z &= 1/2 \\ -2z &= -1 \end{aligned} \quad \begin{aligned} & \\ &\Rightarrow \tfrac{1}{2}y+\tfrac{1}{2}=\tfrac{1}{2} \\ &\Rightarrow z = 1/2 \end{aligned}$$

Problem-2: Gaussian elimination

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system of linear equations | In matrix form | Augmented matrix

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Solution

Step-2: Apply back substitution

Row-echelon form | System | Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \qquad \begin{aligned} x + \tfrac{3}{2}y + 2z &= 1/2 & & \\ \tfrac{1}{2}y + z &= 1/2 & &\Rightarrow y = 0 \\ -2z &= -1 & &\Rightarrow z = 1/2 \end{aligned}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x+3y+4z &= 1 \\ x+2y+3z &= 1 \\ x+4y+5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-2: Apply back substitution

Row-echelon form

System

Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{aligned} x+\frac{3}{2}y+2z &= 1/2 \\ \frac{1}{2}y+z &= 1/2 \\ -2z &= -1 \end{aligned} \quad \begin{aligned} &\Rightarrow x+0+2*\frac{1}{2}=\frac{1}{2} \\ &\Rightarrow y=0 \\ &\Rightarrow z=1/2 \end{aligned}$$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following system of linear equations

In matrix form

Augmented matrix

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-2: Apply back substitution

Row-echelon form

System

Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{aligned} x + \tfrac{3}{2}y + 2z &= 1/2 \\ \tfrac{1}{2}y + z &= 1/2 \\ -2z &= -1 \end{aligned} \quad \begin{aligned} &\Rightarrow x = -\tfrac{1}{2} \\ &\Rightarrow y = 0 \\ &\Rightarrow z = 1/2 \end{aligned}$$

$\therefore x = -\frac{1}{2}, y = 0, z = \frac{1}{2}$

Problem-2: Gaussian elimination

Required

Use Gaussian elimination and back substitution to solve the following

system of linear equations | In matrix form | Augmented matrix

$$\begin{aligned} 2x+3y+4z &= 1 \\ x+2y+3z &= 1 \\ x+4y+5z &= 2 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

Solution

Step-2: Apply back substitution

Row-echelon form | System | Outcome

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{aligned} x+\tfrac{3}{2}y+2z &= 1/2 \\ \tfrac{1}{2}y+z &= 1/2 \\ -2z &= -1 \end{aligned} \quad \begin{aligned} &\Rightarrow x = -\tfrac{1}{2} \\ &\Rightarrow y = 0 \\ &\Rightarrow z = 1/2 \end{aligned}$$

$\therefore x = -\frac{1}{2}, y = 0, z = \frac{1}{2}$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

Solution:

(a) Run Gaussian elimination and back-substitution

(b) Last row will give equation in z and a

   * For some values of a, the equation will have no solution
   * For some values of a it will hold for any z
   * For some values of a it will give a unique z

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

Solution: (a) Run Gaussian elimination and back-substitution

Augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

Solution: (a) Run Gaussian elimination and back-substitution

Augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$$

$r_2 \leftarrow -3r_1 + r_2$
$r_3 \leftarrow -4r_1 + r_3$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array}$$

Solution: (a) Run Gaussian elimination and back-substitution

$$r_2 \leftarrow -3r_1 + r_2 \qquad\qquad r_3 \leftarrow -1r_2 + r_3$$
$$r_3 \leftarrow -4r_1 + r_3$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & 10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Solution: (a) Run Gaussian elimination and back-substitution

$$r_3 \leftarrow -1r_2 + r_3$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & 10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

Back Substitution

$$\begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \Rightarrow \begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Solution: (b) Last row equation in z and a

$$(a^2-16)z = a-4 \Rightarrow (a+4)(a-4)z = a-4$$

(a) If $a=-4$, this equation is $0z=-8$, which has no solution

(b) If $a=4$, this equation is $0z=0$, and z is a free variable

(c) If $a \neq \pm 4$, the equation's solution is $z=\frac{1}{a+4}$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \Rightarrow \begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Solution: (b) Last row equation in z and a

$$(a^2-16)z = a-4 \Rightarrow (a+4)(a-4)z = a-4$$

(a) If $a=-4$, the system has no solution

(b) If $a=4$, this equation is $0z=0$, and z is a free variable

(c) If $a \neq \pm 4$, the equation's solution is $z = \frac{1}{a+4}$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \Rightarrow \begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Solution: (b) Last row equation in z and a

$$(a^2-16)z = a-4 \Rightarrow (a+4)(a-4)z = a-4$$

(a) If $a=-4$, the system has no solution

(b) If $a=4$, the system has infinitely many solutions

(c) If $a \neq \pm 4$, the equation's solution is $z = \frac{1}{a+4}$

Problem-3: Number of solutions

Required

Determine the value of a for which the following linear system has:
(i) no solutions, (ii) infinite number of solutions, (ii) exactly one solution

$$\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & (a^2-14)z & = & a+2 \end{array} \Rightarrow \begin{array}{rcl} x+2y-3z & = & 4 \\ -7y+14z & = & 10 \\ (a^2-16)z & = & a-4 \end{array}$$

Solution: (b) Last row equation in z and a

$$(a^2-16)z = a-4 \Rightarrow (a+4)(a-4)z = a-4$$

(a) If $a = -4$, the system has no solution

(b) If $a = 4$, the system has infinitely many solutions

(c) If $a \neq \pm 4$, the system has exactly one solution

Problem-4: Span

Required

Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$.

Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Is $b \in W$? How many vectors are in W?

Problem-4: Span

Required

Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$.

Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Is $b \in W$? How many vectors are in W?

Solution:

To determine if $\mathbf{b}$ is in $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, is to check if there exist scalars x_1, x_2, x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$. I.e. check that the equation $Ax = b$ is consistent: form an augmented matrix $[A, b]$ and reduce to row-echelone form

Problem-4: Span

Required

Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$.

Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Is $b \in W$? How many vectors are in W?

Solution: form an augmented matrix $[A, b]$ and reduce to row-echelon

$$\underset{\text{Augmented matrix}}{\begin{array}{cccc} \multicolumn{4}{c}{\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix}} \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}} \Rightarrow \overset{r_3 \leftarrow 2r_1 + r_2}{\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix}} \Rightarrow \overset{r_3 \leftarrow r_3 + 2r_2}{\begin{bmatrix} \underbrace{\begin{matrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{3} & -2 \\ 0 & 0 & \boxed{-1} \end{matrix}}_{A'} & \begin{matrix} 4 \\ 1 \\ 2 \end{matrix} \end{bmatrix}}$$

Each row of A' has a pivot $\Rightarrow$ the system is consistent $\Rightarrow \mathbf{b} \in W$

There are infinitely many vectors in W

Problem-5: Span properties

Required

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppowe that $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Problem-5: Span properties

Required

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppowe that $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Solution

Since $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3, such that:

$$\begin{aligned} \mathbf{u} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ \mathbf{v} &= d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \end{aligned}$$

Hence, $\mathbf{u} + \mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3$

Problem-5: Span properties

Required

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppowe that $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Solution

Since $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3, such that:

$$\begin{aligned} \mathbf{u} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\ \mathbf{v} &= d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\ \text{Hence, } \mathbf{u} + \mathbf{v} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\ &= (c_1 + d_1)\mathbf{w}_1 + (c_2 + d_2)\mathbf{w}_2 + (c_3 + d_3)\mathbf{w}_3 \end{aligned}$$

Problem-5: Span properties

Required

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppowe that $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Solution

Since $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3, such that:

$$\begin{aligned}
\mathbf{u} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\
\mathbf{v} &= d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\
\text{Hence, } \mathbf{u} + \mathbf{v} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\
&= (c_1 + d_1)\mathbf{w}_1 + (c_2 + d_2)\mathbf{w}_2 + (c_3 + d_3)\mathbf{w}_3
\end{aligned}$$

Since $(c_1 + d_1), (c_2 + d_2), (c_3 + d_3) \in \mathbb{R}$ are scalars,

the vector $\mathbf{u} + \mathbf{v}$ is in $Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

Problem-5: Span properties

Required

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppowe that $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Solution

Since $\mathbf{u}, \mathbf{v} \in Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3, such that:

$$\begin{aligned}
\mathbf{u} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \\
\mathbf{v} &= d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\
\text{Hence, } \mathbf{u} + \mathbf{v} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\
&= (c_1 + d_1)\mathbf{w}_1 + (c_2 + d_2)\mathbf{w}_2 + (c_3 + d_3)\mathbf{w}_3
\end{aligned}$$

Since $(c_1 + d_1), (c_2 + d_2), (c_3 + d_3) \in \mathbb{R}$ are scalars,

the vector $\mathbf{u} + \mathbf{v}$ is in $Span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

Problem-6: Span definitions

Required

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

(a) How many rows of A contain a pivot position? Does the equation $A\boldsymbol{x} = \mathbf{b}$ have a solution for each $\mathbf{b} \in \mathbb{R}^4$?

(b) Do the columns of B span $\mathbb{R}^4$? Does the equation $B\boldsymbol{x} = \mathbf{y}$ have a solution for each $\mathbf{y} \in \mathbb{R}^4$?

(c) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

(d) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

Problem-6: Span definitions

Required: (a) How many rows of A contain a pivot position? Does the equation $A\boldsymbol{x} = \mathbf{b}$ have a solution for each $\mathbf{b} \in \mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution: Tranform A to row-echelon form, check pivots

$$A \Rightarrow \overset{\substack{r_2 \leftarrow r_1 + r_2 \\ r_4 \leftarrow r_4 - 2r_2}}{\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix}} \Rightarrow \overset{\substack{r_3 \leftarrow r_3 + 2r_2 \\ r_4 \leftarrow r_4 + 3r_2}}{\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}} \Rightarrow \overset{r_3 \leftrightarrow r_4}{\begin{bmatrix} \boxed{1} & 3 & 0 & 3 \\ 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & \boxed{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Since every row of A does *not* contain a pivot position, $A\boldsymbol{x} = \mathbf{b}$ does not have a solution for each $\mathbf{b} \in \mathbb{R}^4$

Problem-6: Span definitions

Required: (b) Do the columns of B span $\mathbb{R}^4$? Does the equation $B\boldsymbol{x} = \mathbf{y}$ have a solution for each $\mathbf{y} \in \mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution: Tranform B to row-echelon form, check pivots

$$B \Rightarrow \overset{\substack{r_3 \leftarrow r_3 - r_1 \\ r_4 \leftarrow 2r_1 + r_4}}{\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix}} \Rightarrow \overset{\substack{r_3 \leftarrow r_2 + r_3 \\ r_4 \leftarrow r_4 - 2r_3}}{\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}} \Rightarrow \overset{r_3 \leftrightarrow r_4}{\begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & 0 & 0 & \boxed{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Since not every row of B contains a pivot position, $B\boldsymbol{x} = \mathbf{y}$ does **not** have a solution for every $\mathbf{y} \in \mathbb{R}^4$.

Problem-6: Span definitions

Required: (b) Do the columns of B span $\mathbb{R}^4$? Does the equation $B\boldsymbol{x} = \mathbf{y}$ have a solution for each $\mathbf{y} \in \mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution: Tranform B to row-echelon form, check pivots

$$B \Rightarrow \overset{\substack{r_3 \leftarrow r_3 - r_1 \\ r_4 \leftarrow 2r_1 + r_4}}{\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix}} \Rightarrow \overset{\substack{r_3 \leftarrow r_2 + r_3 \\ r_4 \leftarrow r_4 - 2r_3}}{\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}} \Rightarrow \overset{r_3 \leftrightarrow r_4}{\begin{bmatrix} \boxed{1} & 3 & -2 & 2 \\ 0 & \boxed{1} & 1 & -5 \\ 0 & 0 & 0 & \boxed{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Columns of B do **not** span $\mathbb{R}^4$. For columns to span $\mathbb{R}^4$, every $\mathbf{y} \in \mathbb{R}^4$ must satisfy $B\boldsymbol{x} = \mathbf{y}$ for some $\boldsymbol{x}$.

Problem-6: Span definitions

Required: (c) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are either all true or all false:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Problem-6: Span definitions

Required: (c) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are equivalent:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Problem-6: Span definitions

Required: (c) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are equivalent:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Since A does not have a pivot position in every row, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do not span $\mathbb{R}^4$.

Problem-6: Span definitions

Required: (d) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are equivalent:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Problem-6: Span definitions

Required: (d) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are equivalent:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Problem-6: Span definitions

Required: (d) Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

Solution

For a matrix P of size $m \times n$ the following statements are equivalent:

(i) For each $\mathbf{b} \in \mathbb{R}^m$, $P\boldsymbol{x} = \mathbf{b}$ has a solution

(ii) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of P

(iii) The columns of P span $\mathbb{R}^m$

(iv) P has a pivot position in every row

Same as for A, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of B.

Also, the columns of B do not span $\mathbb{R}^3$, because they are in $\mathbb{R}^{\mathbf{4}}$

Problem-7: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Problem-7a: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Solution

Solution is a line. That is, there is 1 free variable.

Problem-7a: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Solution

Solution is a line. That is, there is 1 free variable.

Consider 1 out of 2 possibilities $A = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$.

Problem-7a: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Solution

Solution is a line. That is, there is 1 free variable.
Consider 1 out of 2 possibilities $A = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$.
Vector $\vec{x} = (4, 1)$ satisfies $A\vec{x} = 0$, so, $4c + d = 0 \ \Leftrightarrow d = -4c$.
$d = 4, c = -1$ is one of possible solutions. Problem solution is

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$$

Problem-7a: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Solution

Solution is a line. That is, there is 1 free variable.
Consider 1 out of 2 possibilities $A = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$.
Vector $\vec{x} = (4, 1)$ satisfies $A\vec{x} = 0$, so, $4c + d = 0 \Leftrightarrow d = -4c$.
$d = 4, c = -1$ is one of possible solutions. Problem solution is

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$$

MCQ variant of problem 7a

Required

Which a 2×2 matrix A is such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin?

(a) $A = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 8 \\ -3 & 12 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & -8 & 0 \\ -4 & 7 & 0 \\ 1 & -3 & 4 \end{bmatrix}$

MCQ variant of problem 7a

Required

Which a 2×2 matrix A is such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin?

(a) $A = \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 8 \\ -3 & 12 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & -8 & 0 \\ -4 & 7 & 0 \\ 1 & -3 & 4 \end{bmatrix}$

Problem-7b: Solution set geometry

Required

Construct a 2×2 matrix A such that the solution set of the equation $A\boldsymbol{x} = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b} \in \mathbb{R}^2$ such that the solution set of $A\boldsymbol{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

Solution

Solution is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\boldsymbol{x} = 0$.

From previous solution: In matrix A there is the only one row with pivot element. That is, only 2 statements possible:

(a) No solutions.

(b) ~~1 solution.~~

(c) Infinitely many solutions (line parallel to the solution set of $A\boldsymbol{x} = 0$).

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Set of solutions empty $\Leftrightarrow$ second coordinate is non-zero. For example, $(1, 1)$.

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Problem-8: Solution set size

Required

Suppose A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z} \in \mathbb{R}^3$ such that the equation $A\boldsymbol{x} = \mathbf{z}$ has a unique solution? Why?

Solution

In matrix A there is at least one row without pivot element.

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In matrix A there is at least one row without pivot element.

That is, only 2 statements possible:

(a) No solutions: after reducing to row-echelon form one of zero-row corresponds to non-zero element of right side vector.

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Problem-9: Linear dependence

Required

Find the value of h for which the vectors $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ are linearly *dependent*.

Solution

To study the linear dependence of three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \vec{0}]$.

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \xrightarrow[r_3 \leftarrow r_3 - \frac{1}{2} r_1]{r_2 \leftarrow 2r_1 + r_2} \begin{bmatrix} \boxed{2} & -6 & 8 & 0 \\ 0 & \boxed{-5} & 16 + h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 0$ has a free variable and, hence, non-trivial solution no matter the value of h. So, the vectors are *linearly dependent* for *all* values of h.

Problem-10: Basis Transformations

Required

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$,$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$, and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution

By definition $\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$.

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$T(5\mathbf{e}_1 - 3\mathbf{e}_2) = T(5\mathbf{e}_1) - T(3\mathbf{e}_2) = 5\,T(\mathbf{e}_1) - 3\,T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 =$

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$$= 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$$

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Problem-11: Rotations and reflections

Required

Find the standard matrix of the linear transformation

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first rotates points through ${}^{-3\pi}/_{4}$ (clock-wise) and then reflects points through the horizontal x-axis.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first reflects points through the horizontal x-axis and then reflects points through the line $y = x$. Show that the transformation is merely a rotation about the origin.
What is the angle of rotation?