Chapter 13 Partial Derivative

§13-1. Functions of several variables

Recall

$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y) : x, y \in \mathbb{R}\}$

$\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R} = \{(x,y,z) : x, y, z \in \mathbb{R}\}$.

$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_1, \cdots, x_n \in \mathbb{R}\}$

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^n \to \mathbb{R}$

$f(x,y) = xy$ $\quad$ $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $\quad$ $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$

$f: \mathbb{R} \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad$ $f: \mathbb{R}^n \to \mathbb{R}$

$y = f(x)$ $\quad$ $z = f(x,y)$ $\quad$ $w = f(x,y,z)$ $\quad$ $w = f(x_1, \cdots, x_n)$.

Def.

$D \subseteq \mathbb{R}^n$, a subset of $\mathbb{R}^n$

$w = f(x_1, \cdots, x_n) : D \to \mathbb{R}$, a real-valued function

(i) D is the domain of f

(ii) the set of w is the range of f

(iii) w is the dependent variable (output variable) of f

(iv) $x_1, \cdots, x_n$ are independent variable (input variable) of f

Domains and ranges.

Note: The domain of a function is assumed to be the largest set for which the defining rule generates real number.

(Ex. $f(x) = \frac{1}{x}$ domain is $\mathbb{R} \setminus \{0\}$).

Example 1.

Function	Domain	Range
$z = \sqrt{y - x^2}$	$\{(x,y) \mid y \geq x^2\}$	$[0, \infty)$
$z = \frac{1}{xy}$	$\{(x,y) \mid xy \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	$\{x,y \mid x,y \in \mathbb{R}^2\}$	$[-1, 1]$
$w = \sqrt{x^2+y^2+z^2}$	$\mathbb{R}^3$	$[0, \infty)$
$w = \frac{1}{x^2+y^2+z^2}$	$\mathbb{R}^3 - \{(0,0,0)\}$	$(0, \infty)$ 正
$w = xy \ln z$	$\{(x,y,z) \mid z > 0, x, y \in \mathbb{R}\}$	$\mathbb{R}$

Functions of two variables:

Def. in- $S - \partial S$

$S \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$, $r > 0$ $B(x_0, y_0)$

(1) $D((x_0,y_0), r) = \{(x,y) \in \mathbb{R}^2 : |(x,y)-(x_0,y_0)| < r\}$

(2) (x_0,y_0) is an interior pt. of S if $\exists\, r>0$ s.t. $D((x_0,y_0),r) \subseteq S$

(3) (x_0,y_0) is a boundary pt. of S if $D((x_0,y_0),r) \cap S \neq \phi$ and $D((x_0,y_0),r) \cap (\mathbb{R}^2 \setminus S) \neq \phi$, $\forall\, r > 0$

(4) $\{(x,y) \in \mathbb{R}^2 : (x,y)$ is an interior pt. of $S\} \equiv \operatorname{Int} S$ is the interior of S.

(5) $\partial S \equiv \{(x,y) :$ 〃 〃 a boundary pt. of $S\}$: : boundary of S.

$\mathbb{R}$: both [(6) S is open if $S = \operatorname{Int} S$ interior

(7) S is closed if $\partial S \subseteq S$.

(8) S is bdd if $S \subseteq D((0,0), r)$ for some $r > 0$ 大 disk

(9) S is unbdd if S is not bdd.

Example 2.

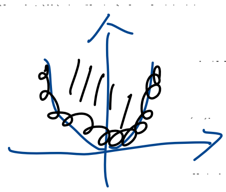
Describe the doman of the function $f(x,y) = \sqrt{y - x^2}$

Sol:

In S $\{(x,y) \mid y > x^2\}$

∂S $\{(x,y) \mid y = x^2\} \subseteq D = \{(x,y) \mid y \geq x^2\}$

closed and unbdd



Graphs, level curve, and contour of functions of two variables:

Def:

$f(x,y) : D^{\subseteq \mathbb{R}^2} \to \mathbb{R}$, $c \in \mathbb{R}$.

$\{(x,y) \in D : f(x,y) = c\}$ is a level $^{\mathbb{R}^2}$ curve of f $\mathbb{R}^3$ 投影到 x-y 平面

$\{(x,y,f(x,y)) \in \mathbb{R}^3 : (x,y) \in D\}$ is the graph of f

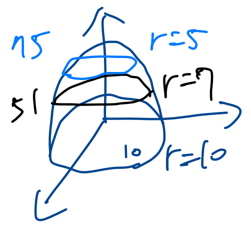
$z = f(x,y)$ is called a surface

$\{(x,y,c) \in \mathbb{R}^3 : f(x,y) = c\}$ is a contour curve of f.
$\mathbb{R}^3$

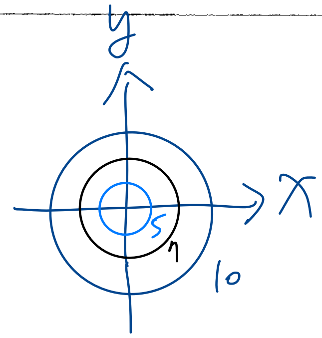
Example 3.

Graph $f(x,y) = 100 - x^2 - y^2$ and plot the level curves $f(x,y) = 0, 51, 75$.

sol.



Contour



Functions of three variables:

Def.

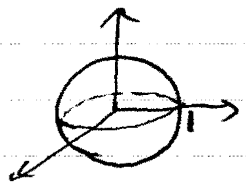
$f(x,y,z) : D \xrightarrow{\subseteq \mathbb{R}^3} \mathbb{R}, \quad c \in \mathbb{R}$

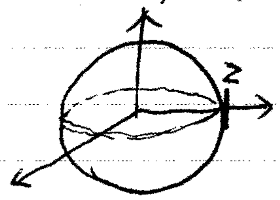
$\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = c\}$ is a level surface of f.

Example 4.

$f(x,y,z) = \sqrt{x^2+y^2+z^2} : \mathbb{R}^3 \to \mathbb{R}$.

$f(x,y,z) = c = 1$. $\qquad c = z = f(x,y,z)$





Def.

$S \subseteq \mathbb{R}^3, \quad (x_0,y_0,z_0) \in \mathbb{R}^3, \quad r > 0$.

(1) $D((x_0,y_0,z_0), r) = \{(x,y,z) : |(x,y,z) - (x_0,y_0,z_0)| < r\}$

(2) (x_0,y_0,z_0) is an interior pt. of S if $\exists r > 0$ s.t. $D((x_0,y_0,z_0), r) \subseteq S$.

(3) (x_0,y_0,z_0) is a boundary pt. of S if $\forall r > 0$, $D((x_0,y_0,z_0), r) \cap S \neq \phi$ and $D((x_0,y_0,z_0), r) \cap (\mathbb{R}^3 \setminus S) \neq \phi$

(4) The interior of S is $\text{Int}\, S = \{$all interior pts of $S\}$

(5) The boundary of S is $\partial S = \{$all boundary pts of $S\}$.

(6) S is open if $S = \text{Int}\, S$

(7) S is closed if $\partial S \subseteq S$.

Example

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$ is closed

$\text{Int}\, S = \{(x,y,z) : x^2+y^2+z^2 < 1\}$

$\partial S = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ $\qquad \partial S \in S$ closed

level surface

§13-2. Limits and continuity in higher dimensions

Def.

$D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$, $(x_0, y_0) \in D \cup \partial D$. $\sqrt{(x-x_0)^2+(y-y_0)^2}$

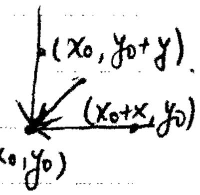
$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall\, 0 < |(x,y)-(x_0,y_0)| < \delta$

$\Rightarrow |f(x,y) - L| < \varepsilon$.

Note

(1) If $f: [a,b] \to \mathbb{R}$, $x_0 \in (a,b)$, $\lim\limits_{x\to x_0} f(x) = L$

$\Rightarrow \lim\limits_{x\to x_0^-} f(x) = L = \lim\limits_{x\to x_0^+} f(x)$. (i.e. if $\lim\limits_{x\to x_0^-} f(x) \neq \lim\limits_{x\to x_0^+} f(x) \Rightarrow$ not exist)



(2) If $f: \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

$$\lim_{x\to 0} f(x_0+x, y_0) = L = \lim_{y\to 0} f(x_0, y_0+y) = \lim_{h\to 0} f(x_0+h, y_0+h)$$

$$= \lim_{x\to 0} f(x_0+x, y_0+x^2) = \lim_{x\to 0} f(x_0+x, y_0+mx)$$

(i.e. The limits of any two directions are diffenent $\Rightarrow$ the limit does not exist.).

Example

Prove $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ if (a) $f(x,y) = x$, (b) $f(x,y) = y$, (c) $f(x,y) \equiv k$

pf:

Theorem 1 (properties of limits)

If $L, M, k \in \mathbb{R}$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$, $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$.

Then (1) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$

(2) $\lim\limits_{(x,y)\to(x_0,y_0)} (k \cdot f(x,y)) = kL$

(3) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$

(4) if $M \neq 0$, $\Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$

(5) $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$, $\forall n \geq 1$

(6) $\lim\limits_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = L^{\frac{1}{n}}$, $\forall n \geq 1$ and $L \geq 0$ if n even

Example 1.

(a) $\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \frac{3}{-1} = -3$

(b) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = 5$

Example 2.

Find $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Sol:

$$x\left(\frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}\right) = x(\sqrt{x}+\sqrt{y}) = 0$$

Example 3.

Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists

Sol:

$$0 \le \left|\frac{4xy^2}{x^2+y^2}\right| \le 4|x|$$

$$\lim_{(x,y)\to(0,0)} 4|x| = 0 \quad \Rightarrow \quad \left|\frac{4xy^2}{x^2+y^2}\right| = 0 \quad \Rightarrow \quad \frac{4xy^2}{x^2+y^2} = 0$$

找到任兩方向 value different , limit doesn't exist.

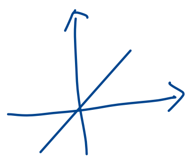
Example 4.

Does $\lim_{(x,y)\to(0,0)} \frac{y}{x}$ exist?

$$\lim_{x\to x_0} f(x) = L$$

$$\Leftrightarrow \lim_{x\to x_0^-} f(x) = L = \lim_{x\to x_0^+} f(x)$$

Sol:



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x}} \frac{y}{x} = \lim_{x\to 0}\frac{x}{x} = 1 \qquad \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=2x}} \frac{y}{x} = \lim_{x\to 0}\frac{2x}{x} = 2$$

插入 Ex.6

Continuity

Def.

(1) $f(x,y)$ is conti. at (x_0,y_0) if (i) $(x_0,y_0) \in$ domain

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

(2) f is conti. if f is conti. at (x_0,y_0), $\forall\, (x_0,y_0) \in$ domain.

Note. (0) $f(x,y)$ is conti. at $(x_0,y_0) \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) = f\left(\lim_{(x,y)\to(x_0,y_0)}(x,y)\right)$.

(1) If f, g are conti. at (x_0,y_0), and $k \in \mathbb{R}$

$\Rightarrow f \pm g,\ kf,\ f \cdot g,\ f/g\ (g(x_0,y_0) \neq 0),\ f^n$ are conti. at (x_0,y_0).

(2) $p(x,y)$, a poly., is conti. on $\mathbb{R}^2$.

(3) p, g are poly. $\Rightarrow p/g$ is conti. on $\mathbb{R}^2 \setminus \{(x,y) : g(x,y)=0\}$.

Example 5.

Show that $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$ is conti. on $\mathbb{R}^2 \setminus \{(0,0)\}$

pf. $\forall (x,y) \in \mathbb{R}^2 - \{(0,0)\} \Rightarrow (x,y) \neq (0,0)$

$2xy$ and x^2+y^2 are conti on $\mathbb{R}^2$

$\Rightarrow f(x,y) = \frac{2xy}{x^2+y^2}$ is conti on $\mathbb{R}^2 - \{(0,0)\}$

show $f(x,y)$ is not cont at $(0,0)$

$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} f(x,y) = \lim_{x\to 0} \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2} = 1 \neq$ along $y=2x \Rightarrow \frac{4}{5}$

limit doesn't exist Hence f is not conti at $(0,0)$

Example 6

Show that $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ does not exist.

pf: along $y=x^2$

$\frac{2x^4}{x^4+x^4} = 1$

(1) $y=mx$

(2) $y=mx^2$

$\boxed{\text{與 } m \text{ 無關}}$

$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=2x^2}} \frac{4x^4}{x^4+4x^4} = \frac{4}{5}$ Hence ... does not exist

Note

In Example 6, $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2mx^3}{x^4+m^2x^2} = \lim_{x\to 0} \frac{2mx}{m^2+x^2} = 0,\ \forall m.$

i.e. If $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x,y) = L,\ \forall m \not\Rightarrow$ the limit exists.

mx^2 ? mx^3 ?

Thm (Continuity of composites)

$f(x,y) : D \to \mathbb{R}$; $g(x) : I \to \mathbb{R}$, $f(D) \subseteq I$

If f is conti. at (x_0, y_0), g is conti. at $f(x_0, y_0)$ $\Rightarrow$ $h(x,y) \equiv g(f(x,y))$ is conti. at (x_0, y_0).

Example

e^{x-y} , $\cos \frac{xy}{x^2+1}$, $\ln(1+x^2y^2)$. are conti. on $\mathbb{R}^2$.

Functions of more than two variable.

$f(x,y,z) = \ln(x+y+z)$, $g(x,y,z) = \frac{y \sin z}{x-1}$

$$\lim_{(x,y,z)\to(1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^0}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$$

Note

The def. of limit and continuity are same with functions of two variables

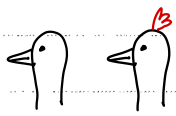
Recall.

(1) $f : [a,b] \to \mathbb{R}$ is conti. $\Rightarrow$ f has an abs. max. value and an abs. min. value

(2) $f : S \to \mathbb{R}$ is $\boxed{\text{conti}}$, $S \subseteq \mathbb{R}$ is $\boxed{\text{closed and bdd}}$ $\Rightarrow$:

(3) $f(x_1, \cdots, x_n) : S \to \mathbb{R}$ is conti. and $S \subseteq \mathbb{R}^n$ is closed and bdd

$\Rightarrow$:



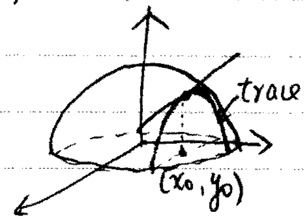
pf of thm:

$$\lim_{(x,y)\to(x_0,y_0)} g(f(x,y)) = g\left(\lim_{(x,y)\to(x_0,y_0)} f(x,y)\right) = g(f(x_0,y_0)) = h(x_0,y_0)$$

#

§13-3. Partial derivatives

Recall



$R \subseteq \mathbb{R}^2$, an open region, $(x_0, y_0) \in R$, $f: R \to \mathbb{R}$
Fix $y = y_0$, then $f(x, y_0)$ is a curve with one variable x, i.e. $f(x, y_0) = g(x)$.

$$\Rightarrow g'(x_0) = \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$$

Ex. $f(x,y) = 10 - x^2 - y^2 + xy \Rightarrow g(x) = f(x, y_0) = 10 - y_0^2 - x^2 + y_0 x \Rightarrow g'(x_0) = -2x_0 + y_0$

$$\Rightarrow \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = -2x_0 + y_0$$

Def.

The partial derivative of $f(x,y)$ w.r.t. x at the point (x_0, y_0) is
$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$ provided the limit exists.

Ex. $\frac{\partial}{\partial x}(2x^2 - 3xy + y^2)\Big|_{(1,2)} = 4x - 3y\Big|_{(1,2)} = 4 - 6 = -2$.

$4x - 3y\Big|_{(1,2)} = -2$

Note.

(1) $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \frac{d}{dx} f(x, y_0)\Big|_{x = x_0}$

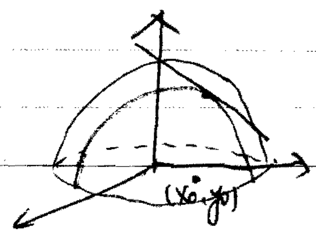
(2) $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$ is the slope of the tangent line of the curve $f(x, y_0)$ at (x_0, y_0) in the plane $y = y_0$.

(3) Notations: $z = f(x,y)$

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = f_x(x_0, y_0) = \frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}, \qquad f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z_x$$

Def.

The partial derivative of $f(x,y)$ w.r.t. y at (x_0, y_0) is
$$\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = \frac{d f(x_0, y)}{dy}\Big|_{y = y_0} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$
provided the limit exists.



Note: (2) (3)

$\frac{df}{dx} = f_x = z_x$

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Calculations:

對 x 偏微，只有 x 是變數，其它都是數字。

〃 y 〃 ，〃 y 〃 ，〃 。

Example

$f(x,y) = x^2 + 3xy + y - 1$. Find $\frac{\partial f}{\partial x}(4,-5)$, $\frac{\partial f}{\partial y}(4,-5)$.

Sol:

$2x + 3y \big|_{(4,-5)} = -7$

$3x + 1 \big|_{(4,-5)} = 13$

Example 2.

$f(x,y) = y \sin xy$. Find $\frac{\partial f}{\partial y}$.

Sol:

$\sin xy + xy \cos xy$

Example 3.

$f(x,y) = \frac{2y}{y + \cos x}$. Find f_x , f_y .

Sol:

$f_x = \frac{-2y(-\sin x)}{(y+\cos x)^2}$ $\qquad$ $f_y = \frac{2(y+\cos x) - 2y}{(y+\cos x)^2}$

Example 4.

The equation $yz - \ln z = x + y$ defines z as a function of x and y.

Find $\partial z / \partial x$.

Sol:

$y z' - \frac{z'}{z} = 1$

$z'(y - \frac{1}{z}) =$

$y z_x - \frac{z_x}{z} = 1$

$z_x (y - \frac{1}{z}) = 1$

$z_x \cdot \frac{yz-1}{z} = 1$ $\qquad$ $z_x = \frac{z}{yz-1}$

Example 5.

The plane $x=1$ intersects $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,5)$.

Sol: $\frac{dz}{dy} = z_y = 2y = 4$ #

$1+y^2$

Example 6.

$f(x,y,z) = x\sin(y+3z)$. Find $\frac{\partial f}{\partial z}$.

Sol:

$x\cos(y+3z)\cdot 3$

Example 7.

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find $\frac{\partial R}{\partial R_2}$ when $R_1 = 30, R_2 = 45, R_3 = 90$

$\quad\quad\quad\; x \quad\;\; y \quad\;\; z$

Sol:

$\frac{dR}{dy}$

$-R^{-2} R_y = -y^{-2}$

$R_y = \frac{R^2}{y^2}$

$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{1}{15}$

$R = 15$

$y = 45$

$R_y = \frac{1}{9}$ #

Note.

If $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist $\not\Rightarrow$ f is conti. at (x_0,y_0).

Example 8.

$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } x=0 \text{ or } y=0 \end{cases}$

(a) Find the limit of f as $(x,y) \to (0,0)$ along $y=x$.

(b) Prove f is not conti. at $(0,0)$.

(c) Show that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

Sol:

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$\lim_{h\to 0} \frac{(h,0)-(0,0)}{h} = 0$

Second-order partial derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = (f_x)_y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

Example 9.

$f(x,y) = x\cos y + y e^x$. Find f_{xx}, f_{xy}, f_{yy}, f_{yx}.

$ye^x \quad -\sin y + e^x \quad -x\cos y \quad -\sin y + e^x$

Sol: $f_x = \cos y + y e^x$

$f_y = -x\sin y + e^x$

Question: $\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) \overset{?}{\cancel{=}} \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)$.

Ans: In general, not! Need some conditions on f.

Theorem.

$R \subseteq \mathbb{R}^2$, open region, $(a,b) \in R$. $f: R \to \mathbb{R}$.

If f_x, f_y, f_{xy}, f_{yx} exist and conti. on $R \Rightarrow f_{xy}(a,b) = f_{yx}(a,b)$.

Example 10. $w = xy + \frac{e^y}{y^2+1}$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial w}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x + \frac{e^y(y^2+1) - e^y \cdot 2y}{(y^2+1)^2} \\ \Rightarrow \frac{\partial^2 w}{\partial y \partial x} = 1 = \frac{\partial^2 w}{\partial x \partial y} \end{array}\right\} \text{are all conti.}$$

High order partial derivative.

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \qquad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

Example 11.

$f(x,y,z) = 1 - 2xy^2z + x^2y$. Find f_{yxyz}

Sol:

$-4xyz + x^2$

$-4z \qquad -4$

$-4yz + 2x$

Differentiability:

Recall.

$f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$.

f is diff. at $x_0 \Leftrightarrow \lim_{h\to0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0)$.

$\Rightarrow \frac{1}{h}\left(f(x_0+h)-f(x_0)-f'(x_0)\cdot h\right)\to 0$ as $h\to0$.

If we let $\varepsilon(h)=\left(f(x_0+h)-f(x_0)-f'(x_0)h\right)/h$

Then " f is diff. at $x_0 \Leftrightarrow \varepsilon(h)\to0$ as $h\to0$ "

We write, $\Delta x=h$ and $\Delta y=f(x_0+\Delta x)-f(x_0)$.

$\Rightarrow \Delta y=f'(x_0)\cdot\Delta x+\varepsilon\Delta x$ in which $\varepsilon\to0$ as $\Delta x\to0$

Taylor formula: $\Delta y=f'(x_0)\cdot\Delta x+\frac{f''(c)}{2}(\Delta x)^2 \Rightarrow$ i.e. $\varepsilon=\frac{1}{2}f''(c)\Delta x$

$\Rightarrow \varepsilon\to0$ as $\Delta x\to0$

Now,

$R\subseteq\mathbb{R}^2$, open region, $(x_0,y_0)\in R$

$f:R\to\mathbb{R}$.

We write $\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$,

If $\Delta z=f_x(x_0,y_0)\cdot\Delta x+f_y(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, and

$\varepsilon_1,\varepsilon_2\to0$ as both $\Delta x,\Delta y\to0$

We say that f is diff. at (x_0,y_0), write $Df(x_0,y_0)=\left(\frac{\partial f}{\partial x}(x_0,y_0),\frac{\partial f}{\partial y}(x_0,y_0)\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left(\nabla f(x_0,y_0)\right)$.

Def.

(1) A function $z=f(x,y)$ is diff. at (x_0,y_0) if $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist and Δz satisfies $\Delta z=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$ in which each of $\varepsilon_1,\varepsilon_2\to0$ as both $\Delta x,\Delta y\to0$.

(2) f is diff. if f is diff. at every point in domain.

(3) In this case, the graph of f is a smooth surface.

Thm.

(1) If f_x and f_y exist and conti. at $(x_0,y_0)\Rightarrow f$ is diff. at (x_0,y_0).

(2) f is diff. at $(x_0,y_0)\Rightarrow f$ is conti. at (x_0,y_0).

i.e. $\lim_{\Delta x,\Delta y\to0}\Delta z=0$.

Cor.

$R\subseteq\mathbb{R}^2$, open region, $f:R\to\mathbb{R}$.

If f_x,f_y exist and conti. on $R\Rightarrow f$ is diff. on R.

and $Df(x,y)=\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)\right)$

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$$f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0, y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \varepsilon \to 0 \text{ as } \Delta x, \Delta y \to 0.$$

Taylor formula:

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \frac{1}{2} f_{xx}(c_1,c_2)(\Delta x)^2 + \frac{1}{2} f_{yy}(c_1,c_2)(\Delta y)^2 + f_{xy}(c_1,c_2)\Delta x \cdot \Delta y.$$

i.e. $\varepsilon_1 = \frac{1}{2} f_{xx}(c_1,c_2)\Delta x + \frac{1}{2} f_{xy}(c_1,c_2)\Delta y$

$\varepsilon_2 = \frac{1}{2} f_{yy}(c_1,c_2)\Delta y + \frac{1}{2} f_{xy}(c_1,c_2)\Delta x$

$\Rightarrow \varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

§13-4. The chain rule

Theorem

$w=f(x,y)$ is diff., $x=x(t)$, $y=y(t)$ are diff.

$\Rightarrow w(t)=f(x(t),y(t))$ is a diff. function of t and

$$\frac{dw}{dt}=f_x(x(t),y(t))\cdot x'(t)+f_y(x(t),y(t))\cdot y'(t)$$

i.e. $$\frac{dw}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot\left(\frac{dx}{dt},\frac{dy}{dt}\right)$$

pf.

$\forall t$, let $P_0=(x(t_0),y(t_0))$

$\because w=f(x,y)$ is diff.

$\Rightarrow \Delta w=f_x(P_0)\cdot\Delta x+f_y(P_0)\cdot\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$, where

$\varepsilon_1,\varepsilon_2\to 0$ as $\Delta x,\Delta y\to 0$ $\qquad \Delta x=x(t)-x(t_0)$, $\Delta y=y(t)-y(t_0)$

$\Rightarrow \dfrac{\Delta w}{\Delta t}=f_x(P_0)\cdot\dfrac{\Delta x}{\Delta t}+f_y(P_0)\cdot\dfrac{\Delta y}{\Delta t}+\varepsilon_1\dfrac{\Delta x}{\Delta t}+\varepsilon_2\dfrac{\Delta y}{\Delta t}$ $\quad(\Delta t=t-t_0)$

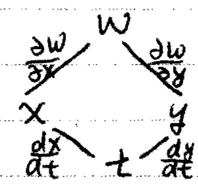
Let $\Delta t\to 0$ (i.e. $t\to t_0$) $\xRightarrow{x,y\text{ diff.}\Rightarrow x,y\text{ conti.}}$ $\Delta x\to 0$ and $\Delta y\to 0$ $\Rightarrow \varepsilon_1,\varepsilon_2\to 0$

$\Rightarrow \left.\dfrac{dw}{dt}\right|_{t=t_0}=f_x(P_0)\cdot x'(t_0)+f_y(P_0)\cdot y'(t_0)+0\cdot x'(t_0)+0\cdot y'(t_0)$

i.e. $w'(t_0)=f_x(x(t_0),y(t_0))\cdot x'(t_0)+f_y(x(t_0),y(t_0))\cdot y'(t_0)$. ※

Note

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$



Example 1. $\quad x'y+xy'$

$w=xy$, $x=\cos t$, $y=\sin t$, Find $\left.\dfrac{dw}{dt}\right|_{t=\frac{\pi}{2}}$

Sol:

$(y,x)\cdot(-\sin t,\cos t)=-y\sin t+x\cos t=-y=-\sin t=-1$

Note: $w=\cos t\cdot\sin t=\frac{1}{2}\sin 2t\Rightarrow \dfrac{dw}{dt}=\cos 2t$

Theorem

$w=f(x,y,z)$ is diff. and x,y,z are diff. functions of t

$\Rightarrow w$ is a diff. function of t and $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial w}{\partial y}\cdot\dfrac{dy}{dt}+\dfrac{\partial w}{\partial z}\cdot\dfrac{dz}{dt}$

$x'y + xy' + z'$

$-\sin t + \cos t + 1$

Example 2.

$-\sin t \qquad \cos t \qquad 1$

Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. Find $w'(0)$.

Sol:

$(y, x, 1)\cdot(-\sin t, \cos t, 1) = -\sin^2 t + \cos^2 t + 1 = \cos 2t + 1 = 2$

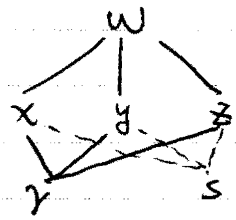
Theorem

$w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$ are diff.

$\Rightarrow \frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ exist and
$$\begin{cases} \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r} \\ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} \end{cases}$$

Note

$$\frac{\partial w}{\partial r} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)\cdot\left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right)$$
$$\frac{\partial w}{\partial s} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)\cdot\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right)$$



$2r$

Example 3

$x' + 2y' + 2z' = \frac{1}{s} + \cancel{2r}\ 4r + \cancel{2z}$

Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

Sol:

(1) $(1, 2, 2z)\cdot\left(\frac{1}{s}, 2r, 2\right) = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$

$-\frac{r}{s^2} + 2\times\frac{1}{s} + 0$

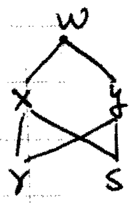
(2) $(1, 2, 2z)\cdot\left(-rs^{-2}, \frac{1}{s}, 0\right) = \frac{-r}{s^2} + \frac{2}{s} = \frac{-r+2s}{s^2}$

$=$

Note

If $w = f(x, y)$, $x = g(r, s)$, $y = h(r, s)$

$\Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$, $\quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$.



Example

Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ if $w = x^2 + y^2$, $x = r - s$, $y = r + s$.

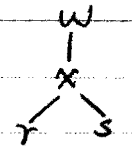
Sol:

$(2x, 2y)\cdot(1, 1) = 2x + 2y = 4r$

$(2x, 2y)\cdot(-1, 1) = -2x + 2y = 4s$

Note.

If $w = f(x)$, $x = g(r,s) \Rightarrow \frac{\partial w}{\partial r} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial r}$, $\frac{\partial w}{\partial s} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial s}$



Implicit differentiation revisited.

(1) $F(x,y)$ is diff.

(2) $F(x,y)=0$ defines $y=h(x)$, a diff. function of x

Ex: $x^2+y=0 \Rightarrow y=-x^2$

$\Rightarrow w = F(x, h(x)) = 0$, $\forall x$

$\Rightarrow \quad 0 = \frac{dw}{dx} = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = F_x + F_y\cdot\frac{dy}{dx} \Rightarrow \underset{\overset{\|}{h'(x)}}{\frac{dy}{dx}} = -\frac{F_x}{F_y}$

Theorem

Suppose $F(x,y)$ is diff. and $F(x,y)=0$ defines y as a diff. function of x. If $F_y \neq 0 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$.

Example

Find $\frac{dy}{dx}$ if $y^2-x^2-\sin xy = 0$

$F_x = -2x - y\cos xy$

$F_y = 2y - x\cos xy$

Sol:

$F_x = -2x - y\cos xy$

$F_y = 2y - x\cos xy$

$y' = \frac{2x+y\cos xy}{2y - x\cos xy}$

$\frac{2x+y\cos xy}{2y-x\cos xy}$

$2y\cdot y' - 2x - \cos xy\,(y + y'x) = 0$

Note.

If $F(x,y,z)$ is diff. and $F(x,y,z)=0$ defines $z=f(x,y)$ is diff.

$\Rightarrow F(x,y,f(x,y)) = 0$, $\forall x, y$.

$\Rightarrow \quad 0 = \frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

Similarly, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}$

Example.

Find $\frac{\partial z}{\partial x}(0,0,0)$ and $\frac{\partial z}{\partial y}(0,0,0)$ if $x^3+z^2+ye^{xz}+z\cos y = 0$

$0 \qquad -1$

Sol:

$F_x = 3x^2 + ye^{xz}\cdot z = 0 \qquad F_x(0,0,0) = 0$

$F_z = 2z + xye^{xz} + \cos y \Rightarrow 1$

$F_y = e^{xz} - z\sin y \Rightarrow 1$

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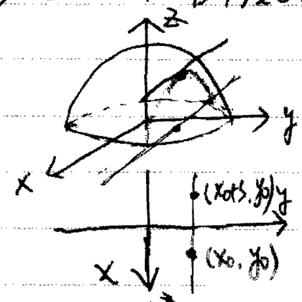
If w is a diff function of x, y, z, u, v

x, y, z, u, v are diff. functions of p, q, r, s, t.

$$\Rightarrow \frac{\partial w}{\partial p} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}\right) \cdot \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p}, \frac{\partial u}{\partial p}, \frac{\partial v}{\partial p}\right).$$

§13-5. Directional derivatives and gradient vectors.

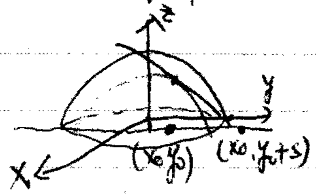


$z = f(x, y)$ (Ex. $z = 10 - x^2 - y^2$).

Fix $y = y_0$

$\Rightarrow \lim_{x \to x_0} \dfrac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = f_x(x_0, y_0) = \lim_{s \to 0} \dfrac{f(x_0+s, y_0) - f(x_0, y_0)}{s}$

i.e. $x = x_0 + s$, $y = y_0$ — is the rate of change of f in the direction $\vec{i} = (1, 0)$

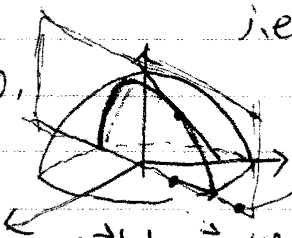


Fix $x = x_0$

$\Rightarrow \lim_{y \to y_0} \dfrac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = f_y(x_0, y_0) = \lim_{s \to 0} \dfrac{f(x_0, y_0+s) - f(x_0, y_0)}{s}$

i.e. $x = x_0$, $y = y_0 + s$, is the rate of change of f in the direction $\vec{j} = (0, 1)$

Now,



$\to x = x_0 + s u_1$, $y = y_0 + s u_2$, $u_1^2 + u_2^2 = 1$

$|\vec{u}| = 1$, $\vec{u} = (u_1, u_2)$ $\Rightarrow \lim_{s \to 0} \dfrac{f(x_0 + s u_1, y_0 + s u_2) - f(x_0, y_0)}{s}$ is the rate of change of f in the direction $\vec{u}$.

Def.

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = (u_1, u_2)$ is the number

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + s u_1, y_0 + s u_2) - f(x_0, y_0)}{s}$$ provided the limit exists.

Note

(1) The direction derivative $\left(\frac{df}{ds}\right)_{\vec{u}, P_0}$ is denoted by $(D_{\vec{u}} f)_{P_0}$

(2) If $\vec{u} = \vec{i} = (1, 0) \Rightarrow (D_{\vec{i}} f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}$

(3) If $\vec{u} = \vec{j} = (0, 1) \Rightarrow (D_{\vec{j}} f)_{P_0} = \left(\frac{\partial f}{\partial y}\right)_{P_0}$

Example 1. $2x+y$, x

$f(x, y) = x^2 + xy$, $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $P_0(1, 2)$. Find $(D_{\vec{u}} f)_{P_0}$

Sol:

$\lim_{s \to 0} \dfrac{f(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}) - f(1,2)}{s} = \lim_{s \to 0} \dfrac{(1+\frac{s}{\sqrt{2}})^2 + (1+\frac{s}{\sqrt{2}})(2+\frac{s}{\sqrt{2}}) - 3}{s}$

$= \lim_{s \to 0} \dfrac{1 + \sqrt{2}s + \frac{s^2}{2} + 2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2} - 3}{s} = \lim_{s \to 0} \left(\sqrt{2} + \frac{3}{\sqrt{2}} + s\right) = \sqrt{2} + \frac{3}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

$f_x = 2x+y = 4$ $\quad (4,1)\cdot\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right) = \frac{5}{\sqrt2}$

$f_y = x = 1$

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Note

$f(x,y)$, $x = x_0 + su_1$, $y = y_0 + su_2$ are diff. functions of s.

By chain rule $\Rightarrow$
$$\left(\frac{df}{ds}\right)_{\vec u, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\left(\frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y}\right)_{P_0}\cdot\left(\frac{dy}{ds}\right)$$
$$= \left(\frac{\partial f}{\partial x}\right)_{P_0}\cdot u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}\cdot u_2$$
$$= \left(\left(\frac{\partial f}{\partial x}\right)_{P_0}, \left(\frac{\partial f}{\partial y}\right)_{P_0}\right)\cdot \vec u$$

Def

The gradient vector of $f(x,y)$ at a point $P_0(x_0,y_0)$ is
$$(\nabla f)_{P_0} = \left(\left(\frac{\partial f}{\partial x}\right)_{P_0}, \left(\frac{\partial f}{\partial y}\right)_{P_0}\right) \in \mathbb{R}^2.$$
i.e. $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Theorem

$R \subseteq \mathbb{R}^2$, open region, $f(x,y): R \to \mathbb{R}$ is diff. on R, $P_0(x_0,y_0) \in R$

$\Rightarrow \left(\frac{df}{ds}\right)_{\vec u, P_0} \left(= (D_{\vec u} f)_{P_0}\right) = (\nabla f)_{P_0}\cdot \vec u$, for any unit vector $\vec u \in \mathbb{R}^2$.

i.e. $D_{\vec u} f = \nabla f \cdot \vec u$

Example 2 Find the derivation of

$f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec v = (3,-4)$ unit vector

Sol.

$\vec u = \left(\frac{3}{5}, \frac{-4}{5}\right)$

$D_{\vec u} f\big|_{(2,0)} = (1,2)\cdot\left(\frac{3}{5}, \frac{-4}{5}\right)$

$f_x = e^y - y\sin xy \quad f_x(2,0) = 1$

$f_y = xe^y - x\sin xy \quad f_y(2,0) = 2$

$\Rightarrow \nabla f(2,0) = (1,2)$

$= \frac{3-8}{5} = -1$

Note

$\because D_{\vec u} f = \nabla f\cdot\vec u = |\nabla f|\cdot|\vec u|\cdot\cos\theta = |\nabla f|\cdot\cos\theta$

$\Rightarrow$ (i) $\theta = 0 \underset{\vec u = \frac{\nabla f}{|\nabla f|}}{\Rightarrow} D_{\vec u} f = |\nabla f|$ max. $\Rightarrow f$ increases most rapidly in direction $\frac{\nabla f}{|\nabla f|}$

(ii) $\theta = \pi \overset{\vec u = -\frac{\nabla f}{|\nabla f|}}{\Rightarrow} D_{\vec u} f = -|\nabla f|$ min. $\Rightarrow f$ decreases most rapidly " " $-\frac{\nabla f}{|\nabla f|}$

(iii) $\theta = \frac{\pi}{2} \underset{\vec u \perp \nabla f}{\Rightarrow} D_{\vec u} f = 0 \Rightarrow f$ zero change in direction $\vec u \perp \nabla f$

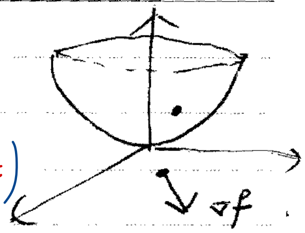
Example 3

Find the direction in which $f(x,y)=\frac{x^2}{2}+\frac{y^2}{2}$

(a) increases most rapidly at $(1,1)$, and $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

(b) decreases " " " " , $\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$

(c) What are the directions of zero change in f at $(1,1)$.

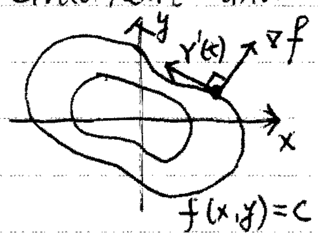


Sol: $f_x(x,y)=x=1$ $\qquad |\nabla f(x,y)|=\sqrt{2}$

$f_y(x,y)=y=1$

$\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right),\ \left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

Gradient and tangents to level curves



$f(x,y)=c$ defined a level curve $r(t)=(x(t),y(t))$

$\Rightarrow f(x(t),y(t))=c,\ \forall t$

$$\Rightarrow 0=\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot(x'(t),y'(t))$$
$$=\nabla f\cdot r'(t)$$

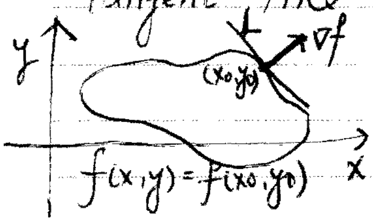
i.e. $\nabla f(x(t_0),y(t_0))$ is normal to the tangent vector $r'(t_0)$

i.e. normal to the curve.

Conclusion:

$\nabla f(x_0,y_0)$ is normal to the level curve $f(x,y)=f(x_0,y_0)$ through (x_0,y_0).

Tangent line



$$(x,y)\in L \Leftrightarrow (x-x_0,y-y_0)\perp \underbrace{\nabla f(x_0,y_0)}_{}\ \ \left(\nabla f(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))\right)$$

$$\Leftrightarrow f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$

tangent line

Example 4

Find an equation for the tangent to the ellipse $\frac{x^2}{4}+y^2=2$ at $(-2,1)$

Sol: $f_x(-2,1)=\frac{1}{2}x=-1$

$f_y(-2,1)=2y=2$

$-(x+2)+2(y-1)=0$

$-x-2+2y-2=0$

$2y-x=4$

$-x+2y=4$

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Algebra rules for gradients

f, g are diff. real-valued functions.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. $\nabla(kf) = k \nabla f$
3. $\nabla(fg) = g(\nabla f) + f \cdot (\nabla g)$
4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Example 5.

$f(x,y) = x - y$, $g(x,y) = 3y$

If $\nabla f = (1,-1)$, $\nabla g = (0,3)$.

$\Rightarrow \nabla(f-g) = (1,-4)$, $\nabla(f \cdot g) = (x-y)(0,3) + 3y(1,-1) = (3y, 3x-6y)$

$\underset{x-4y}{\overset{\|}{}}$ $\qquad$ $3xy - 3y^2 \Rightarrow (3y, 3x-6y)$

Three variables :

$f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$, $\vec{u} = (u_1, u_2, u_3)$ unit vector

$\Rightarrow$ (1) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

(2) $D_{\vec u} f = \nabla f \cdot \vec{u} = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z}$

(3) $D_u f = |\nabla f| \cos\theta$

$\Rightarrow$ (1) f increases most rapidly in the direction $\frac{\nabla f}{|\nabla f|}$ and $D_{\nabla f} f = |\nabla f|$

(2) f decreases 〃 〃 〃 〃 〃 $-\frac{\nabla f}{|\nabla f|}$ and $D_{\nabla f} f = -|\nabla f|$

(3) $D_{\vec u} f = 0$, $\forall \vec{u} \perp \nabla f$.

Example 6 .

$f(x,y,z) = x^3 - xy^2 - z$, $P_0(1,1,0)$,

(a) Find the derivative of f at P_0 in the direction $\vec{U} = (2,-3,6)$

(b) In what direction does f change most rapidly at P_0 and what are the rates of change in these directions ?

Sol :

$f_x(1,1,0) = 3x^2 - y^2 = 2$

$f_y(1,1,0) = -2xy = -2$

$f_z(1,1,0) = -1$

1. $2 \times \frac{2}{7} + (-2) \times \frac{(-3)}{7} + (-1)\frac{6}{7} = \frac{4}{7}$

2. $\left(\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}\right)$ $|\nabla f| = 3$

$\left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ $-|\nabla f| = -3$

The chain rule for path

$\gamma(t) = (x(t), y(t), z(t))$ is a smooth path

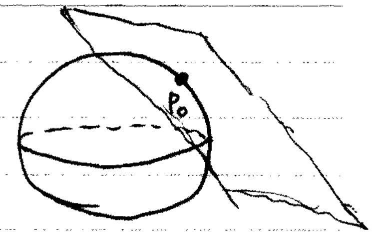
$w = f(\gamma(t))$, a diff. real-valued function

$$\xRightarrow{\text{Chain rule}} \frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \nabla f(\gamma(t))\cdot\gamma'(t) .$$

The $\underbrace{\text{derivative}}_{\text{of } f}$ along a path : $\frac{d}{dt} f(\gamma(t)) = \nabla f(\gamma(t)) \cdot \gamma'(t)$

§13-6. Tangent planes and differentials.



Recall.

$S: f(x,y,z)=c$, a level surface.

$P_0=(x_0,y_0,z_0)\in S$ satisfies $f(x_0,y_0,z_0)=c$.

$\gamma(t)=(x(t),y(t),z(t))$, a curve on $f(x,y,z)=c$ through P_0

i.e. $P_0=\gamma(t_0)$ for some t_0.

$\Rightarrow$ $c=f(\gamma(t))$, $\forall t$ $\Rightarrow$ $0=\nabla f(\gamma(t))\cdot\gamma'(t)$, $\forall t$.

If $t=t_0$ $\Rightarrow$ $\nabla f(P_0)\cdot\gamma'(t_0)=0$.

i.e. $\nabla f(P_0)\perp\gamma'(t_0)$, $\forall$ curve γ through P_0

i.e. $\underbrace{\text{all tangent lines}}_{\text{a plane}}$ at P_0 normal to $\nabla f(P_0)$.

Def.

A level surface $S: f(x,y,z)=c$, where f is diff.

$P_0=(x_0,y_0,z_0)\in S$ (i.e. $f(x_0,y_0,z_0)=c$).

The tangent plane at P_0 of f is the plane through P_0 normal to $\nabla f|_{P_0}$

The normal line of S at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$

Note.

(1) Tangent plane to $f(x,y,z)=c$ at $P_0=(x_0,y_0,z_0)$

$$f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$$

(2) Normal line to $f(x,y,z)=c$ at $P_0=(x_0,y_0,z_0)$

$$x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$$

Example 1.

Find the tangent plane and normal line of $f(x,y,z)=x^2+y^2+z-9=0$ at $P_0=(1,2,4)$.

Sol:

$f_x(1,2,4)=2x=2$

$f_y(1,2,4)=2y=4$

$f_z(1,2,4)=1$

$2x+4y+z=13$

$$\begin{cases} x=1+2t \\ y=2+4t \\ z=4+t \end{cases}$$

Tangent plane of a surface $z = f(x,y)$ at (x_0, y_0).

Let $F(x,y,z) = f(x,y) - z$

The surface $z = f(x,y)$ = the level surface $F(x,y,z) = 0$.

The point $P_0 = (x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$.

$\because$ $F_x = f_x$, $F_y = f_y$, $F_z = -1$

$\Rightarrow \nabla F|_{P_0} = (f_x(x_0,y_0), f_y(x_0,y_0), -1)$.

Tangent plane of $z = f(x,y)$ at (x_0,y_0) = tangent plane of $F(x,y,z)=0$ at (x_0,y_0,z_0)

: $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$

Example 2

$F(x,y,z) = x\cos y - ye^x - z$

Find the plane tangent to $z = x\cos y - ye^x$ at $(0,0,0)$.

Sol: $f_x(0,0,0) = \cos y - ye^x = 1$

$f_y(0,0,0) = -x\sin y - e^x = -1$

$f_z(0,0,0) = -1$

$x - y - z = 0$

Example 3.

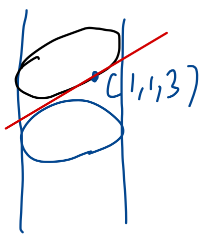
$\vec{n} : (1,0,1)$

The surfaces $f(x,y,z) = x^2+y^2-2=0$ and $g(x,y,z) = x+z-4=0$ meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1,1,3)$.

Sol:

$\vec{n} \Rightarrow \nabla f\,(2x, 2y, 0) \Rightarrow (2,2,0)$

$\begin{cases} x = 1+t \\ y = 1-t \\ z = 3-t \end{cases}$



$\begin{matrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}$ $\quad$ $\begin{matrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}$

$(2,-2,-2) = (1,-1,-1)$

不用教

Estimating change in a specific direction.

Recall

$f(x) : \mathbb{R} \to \mathbb{R}$, diff. $x_0 \in \mathbb{R}$

$df = f'(x_0)\, ds$ ← x_0 move a small distance ds

The value of f changes

estimation

Now, $f(x_1,\cdots,x_n): \mathbb{R}^n \to \mathbb{R}$, $n=2$ or 3, diff.
a point P_0 moves a small distance ds in a direction $\vec{u}$,
To estimate the change of f is the formula:

$$df = \left(\nabla f|_{P_0} \cdot \vec{u}\right) ds.$$

不用該

Example 4.
Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,1,0)$ straight toward $P_1(2,2,-2)$.
Sol:

$$\nabla f = (y\cos x,\ \sin x + 2z,\ 2y) \Rightarrow \nabla f|_{P_0} = (1,0,2).$$

$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{(2,1,-2)}{3}$$

$$\Rightarrow df = \left(\tfrac{2}{3} + 0 - \tfrac{4}{3}\right)\cdot 0.1 = -\frac{0.2}{3} \approx -0.067 \text{ unit}.$$

Linearization
Recall.
$f(x): \mathbb{R} \to \mathbb{R}$ diff, $x_0 \in \mathbb{R}$ $\Rightarrow$ $L(x) = f(x_0) + f'(x_0)(x-x_0)$
$y = L(x)$ is the tangent line of f at x_0.
Now,
$f(x,y): \mathbb{R}^2 \to \mathbb{R}$, $(x_0,y_0) \in \mathbb{R}^2$
$\because f$ is diff. $\Rightarrow f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \varepsilon_1\Delta x + \varepsilon_2\Delta y$
We define $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

Def.
$f(x,y)$ is diff. The linearization of f at (x_0,y_0) is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The approximation $f(x,y) \approx L(x,y)$ is the standard linear approximation of f at (x_0,y_0).

Note
$z = L(x,y)$ is the tangent plane of $z = f(x,y)$ at (x_0,y_0)

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Example 5.

$4-6+2+3$

Find the linearization of $f(x,y)=x^2-xy+\frac{1}{2}y^2+3$ at $(3,2)$.

Sol: $f_x(3,2)=2x-y=4$

$f_y(3,2)=-x+y=-1$

$L(x,y)=8+4(x-3)-(y-2)$

$=4x-y-2$

不用教

Estimate the error $E(x,y)\equiv f(x,y)-L(x,y)$.

Recall

$f(x)=I\to\mathbb{R}$, twice diff., $x_0\in I$

Taylor formula: $f(x)=\underbrace{f(x_0)+f'(x_0)(x-x_0)}_{L(x)}+\frac{f''(c)}{2}(x-x_0)^2$

If $|f''|\le M$ on $I \Rightarrow |f(x)-L(x)|\le\frac{1}{2}M\cdot|x-x_0|^2$

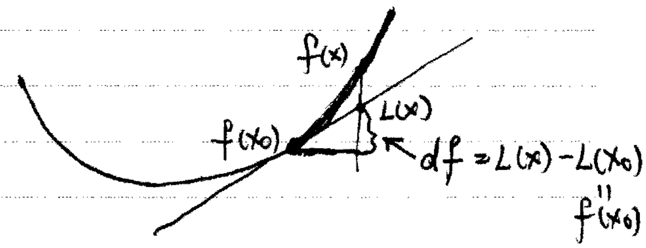
Now,

$D\subseteq\mathbb{R}^2$, open, $(x_0,y_0)\in D$, $f(x,y): D\to\mathbb{R}$, f_{xx}, f_{xy}, f_{yx} exist and conti. on D. If $|f_{xx}|\le M$, $|f_{xy}|\le M$, $|f_{yy}|\le M$ on R for some $M>0$.

$R\subseteq D$, $R = [x_0-r, x_0+r]\times[y_0-h, y_0+h]$

$\Rightarrow |E(x,y)|\le\frac{1}{2}M\cdot(|x-x_0|+|y-y_0|)^2$

Differentials.



Recall

$f(x): I\to\mathbb{R}$, diff. $a\in I$

$\Delta f = f(a+\Delta x)-f(a)$

$df = f'(a)dx = f'(a)\Delta x = L(x)-L(a)=\Delta L$ estimate Δf

Now,

$D\subseteq\mathbb{R}^2$, open, $(x_0,y_0)\in D$, $f(x,y): D\to\mathbb{R}$, diff.

$\Delta x = x-x_0$, $\Delta y = y-y_0$

$L(x_0+\Delta x, y_0+\Delta y)=L(x,y)=\underbrace{f(x_0,y_0)}_{=L(x_0,y_0)}+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y$

$\Rightarrow \Delta L = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$

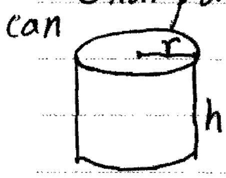
$df = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$.

↑ a estimation of the change of f

Def.

If we move from (x_0, y_0) to (x_0+dx, y_0+dy) nearby, the resulting change $df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$ in the linearization of f is called the total differential of f.

Example 6



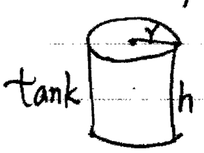
$r = 1\text{ cm}$ $\quad dr = +0.03$

$h = 5\text{ cm}$ $\quad dh = -0.1$

Estimate the resulting absolute change in the volume of the can.

Sol:

Example 7.



$r = 0.5\text{ m}$ $\quad$ How sensitive are the tanks' volumes to small

$h = 2.5\text{ m}$ $\quad$ variations in height and radius.

Sol:

Functions of more than two variables

1. The linearization of $f(x,y,z)$ at $P_0(x_0, y_0, z_0)$ is
$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0).$$
2. If $|f_{xx}|, |f_{xy}|, |f_{xz}|, |f_{yz}|, |f_{yy}|, |f_{zz}| \le M$
$$\Rightarrow |E(x,y,z)| = |L(x,y,z) - f(x,y,z)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0| + |z-z_0|\right)^2$$
3. $df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$. (total differential of f).

Example

$f(x,y,z) = x^2 - xy + 3\sin z$, $\quad R \equiv \{(x,y,z) : |x-2| \le 0.01,\ |y-1| \le 0.02,\ |z| \le 0.01\}$

Find the linearization of f at $(2, 1, 0)$ and an upper bound for the error incurred in replacing f by L on R.

Sol:

§13-7 Extreme values and saddle points

Recall

$f:[a,b]\to\mathbb{R}$, $\quad x_0\in(a,b)$

$f(x_0)$ is a local max. (min.) if $f(x_0)\ge f(x)$ $(\le)$ $\forall x\in(x_0-r,x_0+r)$ for some $r>0$

If $f(x_0)$ is a local extrema and $f'(x_0)$ exists $\Rightarrow f'(x_0)=0$.

x_0 is a critical pt. of f if $f'(x_0)=0$ or does not exist.

Find extrema: Step 1: find all critical pts of f. (i.e. Solve $f'(x)=0$)

Step 2: $f'(x_0)=0$, $f''(x_0)>0 \Rightarrow$ local min.

$\qquad\qquad\qquad\qquad <0 \Rightarrow$ local max.

$\qquad\qquad\qquad\qquad =0 \Rightarrow$ no conclusion

Def.

$D\subseteq\mathbb{R}^2$, a region, $(a,b)\in D$, $f(x,y): D\to\mathbb{R}$.

1. $f(a,b)$ is a local maximum if $f(a,b)\ge f(x,y)$, $\forall (x,y)\in D((a,b),r)\cap D$ for some $r>0$. Domain
2. 〃 〃 〃 〃 minimum if $f(a,b)\le f(x,y)$, 〃

Theorem.

$D\subseteq\mathbb{R}^2$, a region, $(a,b)\in \operatorname{Int}D$, $f(x,y): D\to\mathbb{R}$

If $f(a,b)$ is a local extrema and $f_x(a,b)$, $f_y(a,b)$ exist

$\Rightarrow f_x(a,b)=0$ and $f_y(a,b)=0$ $\quad \nabla f(a,b)=\vec{0}$

pf:

May assume $f(a,b)$ is a local max.

$\Rightarrow \exists r>0$ s.t. $f(a,b)\ge f(x,y)$, $\forall (x,y)\in D((a,b),r)$

$\Rightarrow f(a,b)\ge f(x,b)$, $\forall x\in(a-r,a+r)$

Let $g(x)\equiv f(x,b) \Rightarrow g(a)$ is a local max. $\Rightarrow g'(a)=0$

$\Rightarrow 0=g'(a)=f_x(a,b)$

Similaly, $f(a,b)\ge f(a,y)$, $\forall y\in(b-r,b+r) \Rightarrow f_y(a,b)=0$

Note.

The tangent plane to the surface $z=f(x,y)$ at (a,b) is

$$\underset{0}{\underset{\shortparallel}{f_x(a,b)}}(x-a)+\underset{0}{\underset{\shortparallel}{f_y(a,b)}}(y-b)-(z-f(a,b))=0 \Rightarrow \boxed{z=f(a,b)}$$

a horizontal plane

Def.

$f(x,y): D \to \mathbb{R}$, $(a,b) \in \text{Int}\, D$

(1) (a,b) is a critical point of f if ① $f_x(a,b) = 0 = f_y(a,b)$ or ② at least one of $f_x(a,b)$ and $f_y(a,b)$ does not exist.

(2) (a,b) is a saddle point of f if (a,b) is a critical pt. but $f(a,b)$ is not a local extrema.

i.e. $\forall r > 0$, $\exists (x_1,y_1), (x_2,y_2) \in D((a,b),r) \cap D$ s.t. $f(x_1,y_1) > f(a,b) > f(x_2,y_2)$.

$(a, b, f(a,b))$ is a saddle pt. of the surface $z = f(x,y)$

Example

Find the local extreme value of f.

(1) $f(x,y) = x^2 + y^2 - 4y + 9$ $x^2 + (y-2)^2 + 5$

(2) $f(x,y) = y^2 - x^2$

Sol.

$\nabla f = (2x, 2y-4) = \vec{0} \Rightarrow x = 0, y = 2$ $f(0,2) = 5$ is a local min

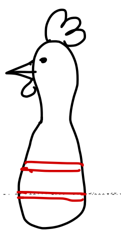
$\nabla f = (-2x, 2y) = \vec{0} =$ $x=0, y=0$ $f(x,0) = -x^2 \le 0$ $f(0,0)$ max

$f(0,y) = y^2 \ge 0$ min

$f_{xx} = -2$

$f_{yy} = 2$ $-4 - 0$

$f_{xy} = 0$



saddle point

Theorem

$f(x,y): D \to \mathbb{R}$, $D((a,b),r) \subseteq D$. $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist and conti. on on $D((a,b),r)$ and $f_x(a,b) = 0 = f_y(a,b)$.

(1) If $f_{xx}(a,b) < 0$ and $(f_{xx} f_{yy} - f_{xy}^2)(a,b) > 0 \Rightarrow f(a,b)$ is a local max.

(2) If $f_{xx}(a,b) > 0$ and 〃 $\Rightarrow$ 〃 〃 〃 〃 min.

(3) If $(f_{xx} f_{yy} - f_{xy}^2)(a,b) < 0 \Rightarrow$ (a,b) is a saddle pt. of f.

(4) If $(f_{xx} f_{yy} - f_{xy}^2)(a,b) = 0 \Rightarrow$ no conclusion

Note

$$f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

discriminant

Hessian of f at (a,b)

Example

Find the local extrema of f.

(1) $f(x,y)=xy-x^2-y^2-2x-2y+4$ max $f(-2,-2)=8$ is local max

(2) $f(x,y)=3y^2-2y^3-3x^2+6xy$

Sol:

$\nabla f(x,y)=(y-2x-2,\ x-2y-2)$

$y-2x-2=0$
$x-2y-2=0$

$y=-2$
$x=-2$

$f_{xx}=-2$

$f_{xy}=1$

$f_{yy}=-2$

$\nabla f(x,y)=(-6x+6y,\ 6y-6y^2+6x)$

$x=y$
$y=0 \lor y=2$

$\begin{cases}(0,0) & \text{saddle point} \\ (2,2) & \end{cases}$

$f_{xx}=-6$

$f_{xy}=6$

$f_{yy}=6-12y$

$-36+72y-36$

$f(2,2)=8$ local max

$f(0,0)=0$ saddle point

Example.

Find the critical points of $f(x,y)=10xye^{-(x^2+y^2)}$, and classify each point.

$f_x=10ye^{-(x^2+y^2)}-20x^2ye^{-(x^2+y^2)}=0$

$f_y=10xe^{-(x^2+y^2)}-20xy^2e^{-(x^2+y^2)}=0$

$\begin{cases} y-2x^2y=0 & y(1-2x^2)=0 \\ x-2xy^2=0 & x(1-2y^2)=0 \end{cases}$

$f_{xx}=-20ye^{-(x^2+y^2)}-40xye^{-(x^2+y^2)}+40x^3ye^{-(x^2+y^2)}$

(1) $y=0$ $x=0$ $(0,0)$

(2) $y\neq 0$ $x=\pm\frac{1}{\sqrt{2}} \Rightarrow y=\pm\frac{1}{\sqrt{2}}$

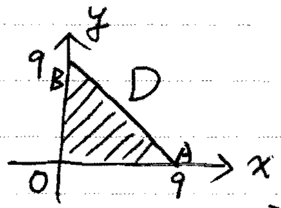
$(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ $(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ $(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$ $(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$

Absolute Maxima and minima on closed bdd regions

Step 1: For Int D, find all critical pt and local extrema.

Step 2: For ∂D, $f(x,y) = g(x), h(y)$ on interval. Find extrema of g or h.

Step 3: Comparison all local extrema



Example

$D = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 9\}$

Find the abs. max. and min. of $f(x,y) = 2+2x+4y-x^2-y^2$ on D.

邊界是 critical

Sol.

Int $f_x = 2-2x$ $x=1$

$f_y = 4-2y$ $y=2$

$f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 0$

$f(1,2) = 7$ max

$y = 9-x$

$k(x) = f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2$

$k'(x) = -4x+16$ $x=4$

$f(4,5) = -11$

會考

bound $g(x) = f(x,0) = 2+2x-x^2$ $0 \leq x \leq 9$

$g'(x) = 2-2x = 0$ $x=1$

$g(1) = f(1,0) = 3$

$g(0) = f(0,0) = 2$

$g(9) = f(9,0) = -61$

$h(y) = f(0,y) = 4y-y^2+2$ $0 \leq y \leq 9$

$h'(y) = 4-2y = 0$ $y=2$

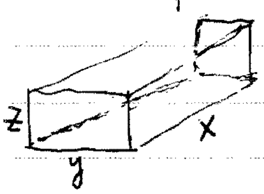
$f(0,2) = 6$

$h(9) = -43$

max $= 7$

min $= -61$

Example



$x+2y+2z = 270$.

Find the largest volum $V = xyz$.

Sol: $f(x,y,z) = xyz$ find max

$z=0$ 不合

$V = 270yz - 2y^2z - 2yz^2$

$V_y = 270z - 4yz - 2z^2 = 2z(135-2y-z) = 0$

$V_z = 270y - 2y^2 - 4yz = 2y(135-y-2z) = 0$

$\begin{cases} 135-2y-z=0 \\ 135-y-2z=0 \end{cases}$

$y=45$

$z=45$

$x=90$

$A: 90 \cdot 45 \cdot 45$

§13-8. Lagrange multipliers 習題

Example 1.

Find the point $p=(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin

i.e. Find the minimum of $f(x,y,z)=x^2+y^2+z^2$ subject to $\underbrace{2x+y-z-5}_{g(x,y,z)}=0$.

Sol:

$\sqrt{x^2+y^2+z^2}$ min

$\nabla f=(2x,2y,2z)$

$\nabla g=(2,1,-1)$

$$\begin{cases} 2x=2\lambda \\ 2y=\lambda \\ 2z=-\lambda \\ 2x+y-z-5=0 \end{cases}$$

$x=\lambda$

$y=\frac{1}{2}\lambda$

$z=-\frac{1}{2}\lambda$

$\lambda=\frac{5}{3}$

A: $\left(\frac{5}{3}, \frac{5}{6}, \frac{-5}{6}\right)$

Example 2.

Find the minimum of $f(x,y,z)=x^2+y^2+z^2$ subject to $\underbrace{x^2-z^2-1}_{g(x,y,z)}=0$.

Sol:

$\nabla f=(2x,2y,2z)$

$\nabla g=(2x,0,-2z)$

$$\begin{cases} x=\lambda x \\ y=0 \\ z=-\lambda z \\ x^2-z^2=1 \end{cases}$$

$x=0$ ✗

$x\neq 0$ $\lambda=1$ $z=0$

$x=\pm 1$

A: $f(\pm 1,0,0)=1$ min

The method of Lagrange multipliers.

f, g are diff., $g(P_0)=0$, $\nabla g(P_0)\neq 0$

If f has a local extrema at P_0 subject to $g=0$.

$\Rightarrow (\nabla f)_{P_0}=\lambda\cdot(\nabla g)_{P_0}$ for some $\lambda\neq 0$.

Example 3.

Find the greatest and smallest values that $f(x,y)=xy$ takes on the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$.

Sol: $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$.

$\nabla f=(y,x)$

$\nabla g=(\frac{1}{4}x, y)$

$$\begin{cases} y=\frac{1}{4}x\lambda \\ x=y\lambda \\ \frac{x^2}{8}+\frac{y^2}{2}=1 \end{cases}$$

$y=\frac{1}{4}\lambda^2 y$

$(\frac{1}{4}\lambda^2-1)y=0$

$y=0$ 不合

$\lambda=\pm 2$

$\lambda=2$ 時 $x=2y$ $y^2=1$ $y=\pm 1$ $x=\pm 2$

$F(\pm 2, \pm 1) = 2$ max

$f(\mp 2, \pm 1) = -2$ min

$\lambda = 2 \quad x = -2y \quad y^2 = 1 \quad y = \pm 1 \quad x = \mp 2$

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Example 4

Find max. and min. of $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$

$\nabla f = (3, 4)$

$\nabla g = (2x, 2y)$

$3 = 2x\lambda$

$4 = 2y\lambda$

$x^2 + y^2 = 1$

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$$

$$\frac{25}{4\lambda^2} = 1$$

$$\lambda^2 = \frac{25}{4} \quad \lambda = \pm\frac{25}{2}$$

$+ \quad x = \frac{3}{5}, y = \frac{4}{5}$

$- \quad x = \frac{-3}{5}, y = \frac{-4}{5}$

$f\left(\frac{3}{5}, \frac{4}{5}\right) = 5$ max

$f\left(\frac{-3}{5}, \frac{-4}{5}\right) = -5$ min