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Chapter 1 First-Order Differential Equations

1.1 Basic Concepts

Definition of a Differential Equation

A differential equation is an equation involving an unknown function and one or more of its derivatives.

Ex $\dfrac{d^2y}{dx^2} = xy$ $y = f(x)$ or $y(x)$

independent variable

$\dfrac{\partial^2 z(x,y)}{\partial x^2} + \dfrac{\partial^2 z(x,y)}{\partial y^2} = 1$ dependent variable

Types of Differential Equations

(i) O.D.E. One independent variable One dependent variable

ordinary

Ex $y' = \cos x$ ✓ $y' = \dfrac{dy}{dx}$ Differential form

$y'' + 4y = 0$

$x^2 y^{(3)} y' + 2y' = x^2 y^2$ $y'' = \dfrac{d}{dx}\dfrac{d}{dx}(y) = \dfrac{d^2y}{dx^2}$

Derivative form

(ii) PDE MORE THAN ONE I.V. One D.V.

partial

Ex $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 1$

(iii) System of DE ONE I.V. MORE THAN ONE DV

Ex $y_1' = 2y_1 - 4y_2$ $y_1 = c_1 4e^{x} + c_2 e^{-2x}$

$y_2' = y_1 - 3y_2$ $y_2 = c_1 e^{x} + c_2 e^{-2x}$

((iv) System of PDE RARE)

Importance of DE

Since the derivatives are involved, the DE describes "the rate of change of a certain thing".

Physical system → Mathematical Modeling → Mathematical solution → Physical Interpretation

工技!

Ex: (i) Motion of fluid

工科!

Ex: (i) Motion of fluid

(ii) Motion of mechanical systems

(iii) Flow of current in electrical circuit

(iv) Dissipation of heat in solid object.

(v) Seismic waves

(vi) Population dynamics

- A population of backteria grows at a rate equal (in proportional) to the present population.

$y' = \cos x$
$y = \sin x + c$
$y' = \cos x$

$$y' = (k)y \qquad \frac{dy}{dx} = (k)y$$

$$y = ce^{(k)x} \quad (y' = c(k)e^{x})$$

A differential equation that describes a physical process is often called a mathematical model.

Order of a D.E.

The order of its highest derivative.

$$(y'')^2 + 3x = (y')^3 \qquad \text{order} = 2$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 1 \qquad \text{order} = 2$$

At the beginning, we will cosider "1st order" ODE.

General Form of 1st order DE

$y' = f(x, y)$ explicit form Ex: $y' = 3y$

$F(x, y, y') = 0$ implicit form Ex $xy' - 3y = 0$

Solution of a DE

The relationship between variables which are FREE of derivatives and satisfies the DE.

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Ex $y = e^{3x}$ is a solution of $y' = 3y$

$y = x^3$ $\quad xy' - 3y = 0$

$y = \varphi(x)$ explicit solution (explicitly defined sol.)

$x^2y^3 + 2x + 2e^{4y} = k$ is a solution of

$$y' = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$$

$H(x,y)=0$ implicit solution (implicitly defined sol)

Verify $\frac{d}{dx}(x^2y^3+2x+2e^{4y}) = \frac{d}{dx}k$

$$\Rightarrow 2xy^3 + 3x^2y^2(y') + 2 + 8e^{4y}(y') = 0$$

$$\Rightarrow y' = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$$

$x^2+y^2=1$ is an implicit sol. of $yy' = -x$

verify $2x + 2yy' = 0 \Rightarrow yy' = -x$

General Solution

A general solution of an n-th order DE is one sol. involving "n" (essential) arbitrary constants
任意

$y = x^2 + c_1x + c_2$ is a GS. of $y'' = 2$

$(y' = 2x + c_1 \quad y'' = 2)$

$y = \sin x + c$ is a GS of $y' = \cos x$

Particular Solution

A solution obtained from the GS by assigning specific values to the arbitrary constants

$y = x^2 - 3x + 2$ is a particular sol. of $y'' = 2$

Integral Curves (Solution Curves)

A graph of a solution of 1st order DE

A graph of a solution of 1st order DE

Ex ✓ $y = 2 + ce^{-x}$ is a GS of $y' + y = 2$

$y = e^{-\frac{x^2}{2}} \int_0^x 2e^{t^2/2} dt + ce^{-\frac{x^2}{2}}$ is a GS. of $y' + xy = 2$

Initial Value Problem (IVP)

A DE together with an initial condition

$y' = f(x, y)$ $y(x_0) = k_0$ where x_0, k_0 are given values.

Ex $xy' = 3y$, $y(-4) = 16$

$\Rightarrow y = cx^3$, $c = -\frac{1}{4}$ $y = \frac{-x^3}{4}$

Singular Solution

A solution that cannot be obtained from the G.S.

Ex $y = cx - c^2$ is the GS of $y = xy' - (y')^2$

BUT $y = \frac{x^2}{4}$ is also a (singular) solution

1.3 Separable Equations

For a 1st order DE

$$\frac{dy}{dx} = f(x, y)$$

If we can write it as

$g(y)\,dy = h(x)\,dx$, (or $f(x, y) = \frac{h(x)}{g(y)}$)

then this DE is a separable DE.

Assume that $h(x)$, $g(y)$ are continuous, then we have

$$\int g(y)\,dy = \int h(x)\,dx + c$$

Ex $y' = ky \Rightarrow \frac{dy}{dx} = ky \Rightarrow \int \frac{1}{y}\,dy = \int k\,dx + c^*$

$\Rightarrow \ln|y| = kx + c^*$

$\Rightarrow |y| = e^{kx + c^*}$

$$\Rightarrow \ln|y| = kx + c^*$$
$$\Rightarrow |y| = e^{kx+c^*}$$
$$\Rightarrow y = \pm e^{c^*} e^{kx}$$
$$= Ce^{kx}$$

Ex $y' = y^2 e^{-x}$

$$\Rightarrow \frac{dy}{dx} = y^2 e^{-x} \Rightarrow \int y^{-2}\,dy = \int e^{-x}\,dx + c$$
$$\Rightarrow -y^{-1} = -e^{-x} + c \Rightarrow y = \frac{1}{e^{-x} - c} \quad \checkmark$$

$+$

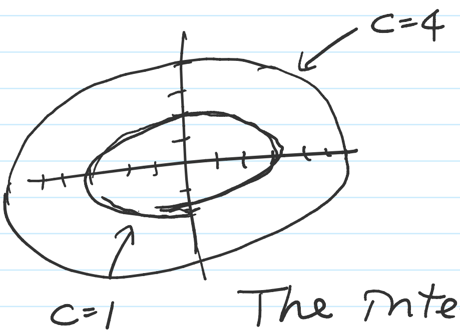
($y=0$ is a singular sol.)

Ex $x^2 y' = 1 + y$

$$\Rightarrow \int \frac{1}{1+y}\,dy = \int \frac{1}{x^2}\,dx + c^*$$
$$\Rightarrow \ln|1+y| = -x^{-1} + c^*$$
$$\Rightarrow |1+y| = e^{-x^{-1}+c^*} \Rightarrow 1+y = ce^{-\frac{1}{x}} \Rightarrow y = ce^{-\frac{1}{x}} - 1$$

Ex $9yy' + 4x = 0 \Rightarrow \int 9y\,dy = \int -4x\,dx + c^*$

$$\Rightarrow \frac{9}{2}y^2 + 2x^2 = c^*$$
$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = \frac{c^*}{18} = C$$



The integral curves form a family of ellipses

Ex $y' = 1 + y^2 \Rightarrow \frac{dy}{dx} = 1 + y^2$

$$\Rightarrow \int \frac{1}{1+y^2}\,dy = \int dx + c$$

Let $y = \tan\theta \qquad y' = \sec^2\theta\,d\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{d}{d\theta}\tan\theta = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta}$

$$\Rightarrow \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta = x + c$$

$$\frac{d}{d\theta}\tan\theta = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

$$\Rightarrow \int \frac{1}{1+\tan^2\theta}\sec^2\theta \, d\theta = x + c$$

$$\Rightarrow \int \frac{\sec^2\theta}{\sec^2\theta} d\theta = x + c$$

$$\Rightarrow \theta = x + c$$

$$\Rightarrow \tan^{-1} y = x + c$$

$$\Rightarrow y = \tan(x+c)$$

Ex $(y - y^2) + x\frac{dy}{dx} = 0$

$$\Rightarrow x\frac{dy}{dx} = y^2 - y \Rightarrow \int \frac{1}{y^2-y} dy = \int \frac{1}{x} dx + c^*$$

$$\Rightarrow \int \frac{1}{y-1} - \frac{1}{y} \, dy = \ln|x| + c^*$$

$$\Rightarrow \ln|y-1| - \ln|y| = \ln|x| + c^* \quad \longleftarrow \text{sol.}$$

$$\Rightarrow e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|x|} e^{c^*}$$

$$\Rightarrow \frac{y-1}{y} = cx \Rightarrow 1 - \frac{1}{y} = cx \Rightarrow \frac{1}{1-cx} = y$$

Ex $y' = -\frac{y}{x}$ $\quad y(1) = 1$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x} + c \Rightarrow \ln|y| = -\ln|x| + c^*$$

$$\Rightarrow y = \frac{c}{x}$$

$y(1) = 1 \Rightarrow c = 1$ $\quad y = \frac{1}{x}$ is the solution.