20220302

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1.4 Exact Differential Equation

正合

- For a 1st order DE

$y' = f(x,y) \quad F(x\ y\ y') = 0$ $\qquad Mdx + Ndy = 0$

$$y' = \frac{-M(x,y)}{N(x,y)} \Rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} \Rightarrow M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

- $u(x,y)$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad \left(du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \right)$$

Ex $u(x,y) = x + x^2y^3$

$$\frac{du}{dx} = \underbrace{1 + 2xy^3}_{\frac{\partial u}{\partial x}} + \underbrace{3x^2y^2}_{\frac{\partial u}{\partial y}}\frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = 1 + 2xy^3$$

$$\frac{\partial u}{\partial y} = 3x^2y^2$$

- For a 1st order DE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

We say that this DE is "exact" if there exists a function $u(x,y)$ such that

$$\frac{\partial u(x,y)}{\partial x} = M(x,y) \qquad \frac{\partial u(x,y)}{\partial y} = N(x,y)$$

We then plug in these two terms into $\underline{M} + \underline{N}\frac{dy}{dx} = 0$

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{du(x,y)}{dx} = 0 \Rightarrow u(x,y) = C$$

From this, we see that if a 1st order DE is exact with $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$

Then the solution of this DE is $u(x,y) = C$.

Questions: ① How to know a DE is exact?

② How to find such a $u(x,y)$?

Proposition (Test of exactness) $\qquad Mdx + Ndy = 0$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ is exact iff } \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

$\Longleftrightarrow$

Ex $\underbrace{2y}_{M} + \underbrace{2x}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}$ Exact!

$\Rightarrow$ Prof: If a 1st order DE is exact, we have $u(x,y)$

$\Rightarrow$ Prof: If a 1st order DE is exact, we have $u(x,y)$ ($\overset{M}{}$ $\overset{N}{}$)

$$\frac{\partial u}{\partial x} = M \qquad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$u(x,y) = x + x^2y^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(1+2xy^3) = 6xy^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2$$

($\Leftarrow$) If M, N satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M + N\frac{dy}{dx} = 0$ is exact

Prof: To show this, we have to find a function $u(x,y)$ that satisfies $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$

$$\frac{\partial u(x,y)}{\partial x} = M(x,y) \Rightarrow u(x,y) = \int M(x,y)\,dx + k(y) \quad \text{\#}$$

$$\frac{\partial u(x,y)}{\partial y} = N(x,y) \Rightarrow \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + \frac{dk(y)}{dy} = N(x,y)$$

$$\Rightarrow \frac{dk(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)$$

If there exists any "x", then $k(y)$ doesn't exist.

$$\frac{\partial}{\partial x}\left(\frac{dk(y)}{dy}\right) = \frac{\partial N(x,y)}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$= \frac{\partial N(x,y)}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)\,dx$$

$$= \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} = 0 \quad \star$$

So $\frac{dk(y)}{dy}$ depends on y only and $k(y)$ is obtainable.

$u(x,y)$ satisfying $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$ can thus be found from $u(x,y) = \int M\,dx + k(y)$

For a DE $\quad M\,dx + N\,dy = 0$

Step 1 Test for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

Step 2 TP use $\frac{\partial u}{\partial x} = M$ $\quad \frac{\partial u}{\partial y} = N$

Step1 test for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?

Step2 If yes $\frac{\partial u}{\partial x} = M \quad \frac{\partial u}{\partial y} = N$

Integrate M with respect to x to obtain u
(N wrt y)

$$\Rightarrow u(x,y) = \int M\,dx + \underline{k(y)}$$

$$\left(u(x,y) = \int N\,dy + \underline{k(x)} \right)$$

Step3 Partial differentiate u wrt y (x) and compare with N (M) to find k function

$$\Rightarrow \frac{\partial u}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(\int M\,dx + k(y)\right) = N \quad \text{solve } k(y)$$

$$\left(\frac{\partial u}{\partial x} = M \Rightarrow \frac{\partial}{\partial x}\left(\int N\,dy + k(x)\right) = M \quad \text{solve } k(x) \right)$$

$u(x,y) = C$ is the solution

(Step4 Check/verify)

Ex $\underbrace{(x^3 + 3xy^2)}_{M}\,dx + \underbrace{(3x^2y + y^3)}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 6xy \;=\; \frac{\partial N}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int x^3 + 3xy^2\,dx + k(y)$$

$$= \frac{x^4}{4} + \frac{3}{2}x^2y^2 + k(y)$$

$$\frac{\partial u}{\partial y} = N \Rightarrow 3x^2y + \frac{dk(y)}{dy} = N = 3x^2y + y^3$$

$$\Rightarrow k(y) = \frac{y^4}{4} \cancel{(+c^*)}$$

$$u(x,y) = \frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4}$$

The sol. of this DE is $\underline{x^4 + 6x^2y^2 + y^4 = C}$

(Check $4x^3 + 12xy^2 + 12x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$

$\Rightarrow (4x^3 + 12xy^2)\,dx + (12x^2y + 4y^3)\,dy = 0$)

(Check $4x^3+12xy^2+12x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}=0$
$\Rightarrow (4x^3+12xy^2)dx+(12x^2y+4y^3)dy=0$ #)

另解 $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$

$\Rightarrow \left(1+3\left(\frac{y}{x}\right)^2\right)dx+\left(3\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^3\right)dy=0$

Let $\frac{y}{x}=u \Rightarrow y=ux \Rightarrow y'=u'x+u$

$(1+3u^2)+(3u+u^3)\frac{dy}{dx}=0$

$\Rightarrow 1+3u^2+(3u+u^3)\left(\frac{du}{dx}x+u\right)=0$

$\Rightarrow -\frac{1}{x}dx=\frac{3u+u^3}{u^4+6u^2+1}du$

$\Rightarrow -\ln|x|+c^*=\frac{1}{4}\ln|u^4+6u^2+1|$

$\Rightarrow \ln|u^4+6u^2+1|=\underbrace{-\ln x^4}+4c^*$

$\Rightarrow \ln|(u^4+6u^2+1)x^4|=4c^*$

$\Rightarrow y^4+6x^2y^2+x^4=C$

Ex $\underbrace{\cos(x+y)}_{M}dx+\underbrace{(3y^2+2y+\cos(x+y))}_{N}dy=0$

$\frac{\partial M}{\partial y}=-\sin(x+y)=\frac{\partial N}{\partial x}=-\sin(x+y)$

$\frac{\partial u}{\partial x}=\cos(x+y)\Rightarrow u=\sin(x+y)+k(y)$

$\frac{\partial u}{\partial y}=\cancel{\cos(x+y)}+\frac{dk(y)}{dy}=\underbrace{3y^2+2y+\cancel{\cos(x+y)}}_{N}$

$\Rightarrow k(y)=y^3+y^2$

$\sin(x+y)+y^3+y^2=C$ is the solution

Ex $y'=-\frac{2xy^3+2}{3x^2y^2+8e^{4y}}\Rightarrow \underbrace{(2xy^3+2)}_{M}dx+\underbrace{(3x^2y^2+8e^{4y})}_{N}dy=0$

$\frac{\partial M}{\partial y}=6xy^2 \overset{☺}{=} \frac{\partial N}{\partial x}=6xy^2$

$$\frac{\partial M}{\partial y} = 6xy^2 \overset{☺}{=} \frac{\partial N}{\partial x} = 6xy^2$$

$$\frac{\partial u}{\partial x} = 2xy^3 + 2 \Rightarrow u = x^2y^3 + 2x + k(y)$$

$$\frac{\partial u}{\partial y} = N \Rightarrow 3x^2y^2 + \underline{\frac{dk(y)}{dy}} = N = 3x^2y^2 + \underline{8e^{4y}}$$

$$\Rightarrow k(y) = 2e^{4y}$$

$x^2y^3 + 2x + 2e^{4y} = C$ is the solution.

Ex $(\sin x \cosh(y))\,dx - (\cos x \sinh(y))\,dy = 0 \quad y(0) = 3$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \begin{matrix} (\cosh(x))' = \sinh(x) \\ (\sinh(x))' = \cosh(x) \end{matrix}$$

$$\frac{\partial M}{\partial y} = \sin x \sinh(y) \overset{☺}{=} \frac{\partial N}{\partial x} = \sin x \sinh(y)$$

$$\frac{\partial u}{\partial x} = \sin x \cosh(y) \Rightarrow u = -\cos x \cosh(y) + k(y)$$

$$\frac{\partial u}{\partial y} = -\cos x \sinh(y) + \underline{\underline{\frac{dk(y)}{dy}}} = -\cos x \sinh(y)$$

$$\frac{dk(y)}{dy} = 0 \Rightarrow k(y) = c^*$$

$$\underline{-\cos x \cosh(y) + c^* = c^{**}} \Rightarrow \cos x \cosh(y) = C$$

is the general solution.

$$y(0) = 3 \Rightarrow \cos(\underline{0}) \cosh(\underline{\underline{y(0)}}) = C$$

$$\Rightarrow C = \cosh(3)$$

The solution of this IVP is $\cos x \cosh y = \cosh(3) = 10.07$