Homework1

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1. Geometry of linear equations:
   (1) Describe the intersection of the three "planes" u+v+w+z=8, u+w+z=6 and u+w=1, all in 4-dimensional space. Is it a 0-D point, a 1-D line or a 2-D plane? What is the intersection if the fourth "plane" u=2 is included?
   (2) For the plane ax+by+cz=1 in a 3-D space, what vector is perpendicular to this plane and why?
   (3) For the equations x+y=4, 2x-2y=4, draw the row picture (two intersection lines) and the column picture (combination of two columns equal to the column vector $[4,4]^T$.

2. Gaussian elimination
   (1) Find the pivots and solve the system:

$$\begin{aligned} 2u - v &= 0 \\ -u + 2v - w &= 0 \\ -v + 2w - z &= 0 \\ -w + 2z &= 5 \end{aligned}$$

   (2) Consider the system

$$\begin{aligned} x + ay &= 4 \\ ax + 9y &= b \end{aligned}$$

   (a) For which values of a does the system have a unique solution?
   (b) Find those pairs of values (a,b) for which the system has more than one solution.

3. Matrix multiplication
   (1) $\underline{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (2) $\underline{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (3) $\underline{C} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

   Find the powers $\mathbf{A}^2$, $\mathbf{A}^3$,...., $\mathbf{B}^2$, $\mathbf{B}^3$..., $\mathbf{C}^2$, $\mathbf{C}^3$,...

   (2) $\underline{A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Verify that $\underline{A}(\theta_1)\underline{A}(\theta_2) = \underline{A}(\theta_1 + \theta_2)$

   (3) More general than multiplication by columns is block multiplication. If matrices are separated into blocks (submatrices) and their shapes make block multiplication possible, then it is allowed.

$$\underline{A}\,\underline{B} = \left[\begin{array}{cc|c} X & X & X \\ X & X & X \\ \hline X & X & X \end{array}\right] \left[\begin{array}{cc|c} X & X & X \\ X & X & X \\ \hline X & X & X \end{array}\right]$$

   (a) Replace those X's by numbers (1~9 different values in each matrix) and confirm that block multiplication works.
   (b) Give two more examples (with X's) if **A** is 3 by 4 and **B** is 4 by 2.

4. Matrices in triangular and echelon form
   (1) Factor **A** = **LU**
   (a) $\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$ (b) $\underline{A} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$

   (2) Solve **LUx**=**b** without multiplying **LU**

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

   (3) Find **PA**=**LDU** for
   (a) $\underline{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $\underline{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

(a) $\underline{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ (b) $\underline{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

5. Inverses and transposes

   (1) Use the Gauss-Jordan method to invert

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (2) Show that for any square matrix **B**, **A**=**B**+**B**^{T} is always symmetric, and **K**=**B**-**B**^{T} is always skew-symmetric (**K**^{T}=-**K**). Find these matrices when $\underline{B} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and write B as the sum of a symmetric matrix and a skew symmetric matrix.

   (3) Compute **LDL**^{T} of $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

6. Find the general solution of the following **Ax**=**b** by finding $\mathbf{x_n}$ and $\mathbf{x_p}$.

   (1) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 1 & 1 & 4 & -1 \\ 2 & 5 & 9 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$

7. Find the dimensions and the bases for the four subspaces associated with **A**

   (1) $\underline{A} = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$ (2) $\underline{A} = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ (3) $\underline{A} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 1 & 3 & -2 & 0 \\ 3 & 8 & -7 & -2 \\ 2 & 1 & -9 & -10 \end{bmatrix}$