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1. Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

a. $a_n = \frac{\sin n}{n}$

sol: $-1 \le \sin n \le 1 \Longrightarrow \frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$ (by sandwich thm.)

$\therefore \{a_n\} = \left\{\frac{\sin n}{n}\right\}$ converges to 0.

b. $a_n = \frac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}}$

sol: $\frac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}} = \frac{e^{n} - 2}{e^{n} - e^{2n}}$ lead lead

$\Rightarrow \lim_{n\to\infty} a_n \doteq \lim_{n\to\infty} \frac{e^n}{e^{2n}} = 0 \quad \therefore$ it converges to 0

2. Assume following sequence converges and find its limit.

sol: $a_n = 2 + \frac{1}{a_{n-1}}$

$\{a_n\}$ converges $\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} \overset{\text{let}}{=\!=} L$

$L = 2 + \frac{1}{L} \Rightarrow L^2 - 2L - 1 = 0 \Rightarrow L = 1 \pm \sqrt{2}$

$\because a_n > 0, \forall n \in \mathbb{N} \quad \therefore L = \underline{1+\sqrt{2}}$

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3. Determine if the series converges or diverges. If a series converges, find its sum.

$$\sum_{n=1}^{\infty}\left(\ln\sqrt{n+1}-\ln\sqrt{n}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\ln\sqrt{k+1}-\ln\sqrt{k}\right)$$

$$=\lim_{n\to\infty}\left(\cancel{\ln\sqrt{2}}-\ln\sqrt{1}\right)+\left(\cancel{\ln\sqrt{3}}-\cancel{\ln\sqrt{2}}\right)+\cdots+\left(\cancel{\ln\sqrt{n}}-\cancel{\ln\sqrt{n-1}}\right)+\left(\ln\sqrt{n+1}-\cancel{\ln\sqrt{n}}\right)$$

$$=\lim_{n\to\infty}\ln\sqrt{n+1}-\cancel{\ln\sqrt{1}}=\lim_{n\to\infty}\ln\sqrt{n+1}=\infty$$

so the series diverges.

4. Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum

a. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}-\frac{n+2}{n+3}\right)$

~~sol~~: $S_n=\left(\frac{1}{2}-\cancel{\frac{3}{4}}\right)+\left(\frac{2}{3}-\cancel{\frac{4}{5}}\right)+\left(\cancel{\frac{3}{4}}-\cancel{\frac{5}{6}}\right)+\cdots$

$$\left(\cancel{\frac{n-3}{n-2}}-\cancel{\frac{n-1}{n}}\right)+\left(\cancel{\frac{n-2}{n-1}}-\cancel{\frac{n}{n+1}}\right)+\left(\cancel{\frac{n-1}{n}}-\frac{n+1}{n+2}\right)+\left(\cancel{\frac{n}{n+1}}-\frac{n+2}{n+3}\right)$$

$$=\frac{1}{2}+\frac{2}{3}-\frac{n+1}{n+2}-\frac{n+2}{n+3}$$

$$\lim_{n\to\infty}S_n=\frac{1}{2}+\frac{2}{3}-1-1=\frac{7}{6}-2=\underline{\underline{-\frac{5}{6}}}$$

b. $\sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n}+\sin\frac{\pi}{n}\right)$

sol: $\lim_{n\to\infty}\left(\cos\frac{\pi}{n}+\sin\frac{\pi}{n}\right)$ D.N.E. because it oscillates.

so the series diverges

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5. Use the Integral Test to determine if the following series converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n+4}}$$

sol: $n \int_1^{\infty} (n+4)^{-\frac{1}{2}} = n \cdot \lim_{n \to \infty} \left[2(n+4)^{\frac{1}{2}} \right]_1^n = \infty$

$\because \frac{n}{\sqrt{n+4}}$ is continous, positive, and decreasing.

$\therefore \sum_{n=1}^{\infty} \frac{n}{\sqrt{n+4}}$ diverges (by Integral Test)

6. Which of the following series converge, and which diverge? Give reasons for your answers.

a. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

sol: $\lim_{n \to \infty} n \sin \frac{1}{n} = \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{t \to 0} \frac{\sin t}{t} = 1$

therefore $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ diverges

b. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

sol: $\lim_{n \to \infty} \frac{e^n}{1+e^{2n}} = 0$ $\therefore$ it converges

7. Which of the series converge, and which diverge? Use any method, and give reasons for your answers.

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a. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$

sol: $0 \le \frac{\cos^2 n}{n^{\frac{3}{2}}} \le \frac{1}{n^{\frac{3}{2}}}$,

because $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a p-series with $p>1$, it converges.

since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$ converges.

b. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

sol: let $a_n = \frac{5^n}{\sqrt{n}\,4^n}$, $b_n = \frac{1}{\sqrt{n}}$

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