20220309

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Ex $\underbrace{2\sin(y^2)}_{M}dx+\underbrace{xy\cos(y^2)}_{N}dy=0$

$$\frac{\partial M}{\partial y}=2\cos(y^2)2y=4y\cos(y^2)$$

$$\frac{\partial N}{\partial x}=y\cos(y^2)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{3y\cos(y^2)}{xy\cos(y^2)}=\frac{3}{x}$$ depends on x only

$$\exp\left\{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx\right\}=\exp\left\{\int\frac{3}{x}dx\right\}=e^{3\ln x}=x^3$$

$$\underbrace{2x^3\sin(y^2)}_{M}dx+\underbrace{x^4y\cos(y^2)}_{N}dy=0$$

$$\frac{\partial u}{\partial x}=2x^3\sin(y^2)\Rightarrow u=\frac{1}{2}x^4\sin(y^2)+k(y)$$

$$\frac{\partial u}{\partial y}=N\Rightarrow\frac{1}{\cancel{2}}x^4\cos(y^2)\cancel{2}y+\frac{dk(y)}{dy}=x^4y\cos(y^2)$$

$$\Rightarrow k(y)=c^*$$

$\left(\frac{1}{2}\right)x^4\sin(y^2)=c$ is the solution.

另解 $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-3y\cos(y^2)}{2\sin(y^2)}$ depends on y only

$$\int-\frac{3}{2}y\frac{\cos(y^2)}{\sin(y^2)}dy=\int-\frac{3}{4}\frac{\cos(y^2)}{\sin(y^2)}d(y^2)$$

$$=-\frac{3}{4}\int\frac{1}{\sin(y^2)}d\sin(y^2)=-\frac{3}{4}\ln|\sin(y^2)|$$

$$e^{-\frac{3}{4}\ln|\sin(y^2)|}=(\sin(y^2))^{-\frac{3}{4}}$$

$$2\sin(y^2)(\sin(y^2))^{-\frac{3}{4}}dx+\underbrace{xy\cos(y^2)\sin(y^2)^{-\frac{3}{4}}}_{N}dy=0$$

$$\frac{\partial u}{\partial x}=2(\sin(y^2))^{\frac{1}{4}}\Rightarrow u=2x(\sin(y^2))^{\frac{1}{4}}+k(y)$$

$$\frac{\partial u}{\partial y}=\frac{x}{\ }\sin(y^2)^{-\frac{3}{4}}\cos(y^2)2y+\frac{dk(y)}{dy}=N$$

$$\frac{\partial u}{\partial y} = \frac{x}{2}\sin(y^2)^{-\frac{3}{4}}\cos(y^2)\,2y + \frac{dk(y)}{dy} = N$$

$$= xy\cos(y^2)\sin(y^2)^{-\frac{3}{4}}$$

$$\frac{dk(y)}{dy} = 0 \Rightarrow k(y) = C^*$$

$2x(\sin(y^2))^{\frac{1}{4}} = C$ is the solution.

Ex $\underbrace{2xy}_{M}\,dx + \underbrace{(4y+3x^2)}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = 6x$$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-4x}{4y+3x^2}$ depends on x & y

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{4x}{2xy} = \frac{2}{y}$ depends on y only

$$F(y) = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$$

$$\underbrace{2xy^3}_{M}\,dx + \underbrace{4y^3+3x^2y^2}_{N}\,dy = 0$$

$$\frac{\partial u}{\partial x} = 2xy^3 \Rightarrow u = x^2y^3 + k(y)$$

$$\frac{\partial u}{\partial y} = 3x^2y^2 + \frac{dk(y)}{dy} = N = 4y^3 + 3x^2y^2$$

$$\frac{dk(y)}{dy} = 4y^3 \Rightarrow k(y) = y^4$$

$x^2y^3 + y^4 = C$ is the solution.

另解 Guess the integrating factor $x^a y^b$

$$\underbrace{(2x^{a+1}y^{b+1})}_{M}dx + \underbrace{(4x^{a}y^{b+1} + 3x^{a+2}y^{b})}_{N}dy$$

$\frac{\partial M}{\partial y} = 2x^{a+1}(b+1)y^{b}$, $\frac{\partial N}{\partial x} = 4ax^{a-1}y^{b+1} + 3(a+2)x^{a+1}y^{b}$

$\frac{\partial}{\partial y} = 2x \ \ldots (b+1) \ldots, \quad \frac{\partial}{\partial x} = \ldots + 3(a+2)x^{a+1}y^{b}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \begin{array}{l} 4a = 0 \\ 2b+2 = 3a+6 \end{array} \Rightarrow \begin{cases} a = 0 \\ b = 2 \end{cases}$$

The integrating factor is y^2

Ex $(x^2y + y + 1)\,dx + x(1+x^2)\,dy = 0$

$$\frac{\partial M}{\partial y} = x^2 + 1 \neq \frac{\partial N}{\partial x} = 1 + 3x^2$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2x^2}{x + x^3} = \frac{-2x}{1+x^2}$$ depends on x only

$$F(x) = \exp\left\{\int \frac{-2x}{1+x^2}\,dx\right\} = \exp\left\{\frac{-1}{1+x^2}\,d(1+x^2)\right\}$$

$$= e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$$

$$\underbrace{\frac{x^2y + y + 1}{1+x^2}}_{M}\,dx + x\,dy = 0$$

$$\frac{\partial u}{\partial y} = x \Rightarrow u = xy + k(x)$$

$$\frac{\partial u}{\partial x} = y + \frac{dk(x)}{dx} = \underbrace{y + \frac{1}{1+x^2}}_{M}$$

$$\frac{dk(x)}{dx} = \frac{1}{1+x^2} \Rightarrow k(x) = \tan^{-1} x$$

$xy + \tan^{-1} x = C$ is the solution

Ex $\underbrace{(2y^2 - 9xy)}_{M}\,dx + \underbrace{(3xy - 6x^2)}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = 4y - 9x \qquad \frac{\partial N}{\partial x} = 3y - 12x$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y+3x}{3xy - 6x^2}$$ depends on x & y

$$\frac{1}{\ }\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-3x-y}{\ldots}$$ depends on x & y

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-3x-y}{2y^2-9xy} \text{ depends on } x \,\&\, y$$

Guess $F(x,y)=x^a y^b$

$$\underbrace{(2x^a y^{b+2}-9x^{a+1}y^{b+1})}_{M}dx+\underbrace{(3x^{a+1}y^{b+1}-6x^{a+2}y^b)}_{N}dy=0$$

$$\frac{\partial M}{\partial y}=2(b+2)y^{b+1}x^a-9(b+1)y^b x^{a+1}$$

$$\frac{\partial N}{\partial x}=3(a+1)x^a y^{b+1}-6(a+2)x^{a+1}y^b$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow \begin{matrix}2(2+b)=3(a+1)\\ 9(b+1)=6(a+2)\end{matrix} \Rightarrow \begin{matrix}a=1\\ b=1\end{matrix}$$

$F(x,y)=xy$

$$\underbrace{(2xy^3-9x^2y^2)}_{M}dx+\underbrace{(3x^2y^2-6x^3y)}_{N}dy=0$$

$$\frac{\partial u}{\partial x}=2xy^3-9x^2y^2 \Rightarrow u=x^2y^3-3x^3y^2+k(y)$$

$$\frac{\partial u}{\partial y}=3x^2y^2-6x^3y+\frac{dk(y)}{dy}=N=3x^2y^2-6x^3y$$

$$\frac{dk(y)}{dy}=0 \Rightarrow k(y)=c^*$$

$x^2y^3-3x^3y^2=C$ is the solution

First-Order Linear Differential Equation

A 1st order DE is said to be linear, if it can be written as $y'+p(x)y=r(x)$

(i) $r(x)=0 \Rightarrow$ DE is homogeneous

(ii) $r(x)\neq 0 \Rightarrow$ DE is non-homogeneous

(i) $y'+p(x)y=0$

$$\Rightarrow \frac{dy}{dx}=-p(x)y \Rightarrow \frac{dy}{y}=-p(x)dx$$

$$\Rightarrow \ln|y|=-\int p(x)dx+c^*$$

$$\Rightarrow \frac{dy}{dx} = -p(x)y \Rightarrow \frac{dy}{y} = -p\,dx$$

$$\Rightarrow \ln|y| = -\int p(x)\,dx + c^*$$

$$\Rightarrow y = ce^{-\int p(x)\,dx} \Rightarrow y = ce^{-h},\ h = \int p\,dx$$

* A homogeneous DE always has a (trivial) solution $y(x)=0$

(ii) $y' + p(x)y = r(x)$

$$\Rightarrow \frac{dy}{dx} + py = r \Rightarrow \underbrace{(py-r)}_{M}dx + \underbrace{1}_{N}dy = 0$$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(p)$ is a function of x

$$F(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} = e^{\int p\,dx}$$

$$e^{\int p\,dx}(y' + py) = e^{\int p\,dx} \cdot r$$

$$\left(e^{\int p\,dx} y\right)' = e^{\int p\,dx} \cdot r$$

$$\Rightarrow e^{\int p\,dx} y = \int e^{\int p\,dx} \cdot r\,dx + c$$

$$\Rightarrow y = ce^{-\int p\,dx} + e^{-\int p\,dx}\int e^{\int p\,dx} \cdot r\,dx$$

$$\boxed{y = ce^{-h} + e^{-h}\int e^{+h} r\,dx} \quad h = \int p\,dx$$

* The solution of the homogeneous DE is a special case of the solution of the non-homogeneous DE

另解 $\underbrace{e^{\int p\,dx}(py-r)}_{M}dx + \underbrace{e^{\int p\,dx}}_{N}dy = 0$

$$\frac{\partial u}{\partial y} = e^{\int p\,dx} \Rightarrow u = y e^{\int p\,dx} + k(x)$$

$$\frac{\partial u}{\partial x} = ye^{\int p\,dx} p + \frac{dk(x)}{dx} = M = e^{\int p\,dx} py - e^{\int p\,dx} r$$

$$\frac{dk(x)}{dx} = -e^{\int p\,dx} r \Rightarrow k(x) = -\int re^{\int p\,dx}\,dx$$

$$\frac{dk(x)}{dx} = -e^{\int p dx} r \Rightarrow k(x) = -\int r e^{\int p dx} dx$$

$$y e^{\int p dx} - \int \underline{r e^{\int p dx}} dx = C \quad \text{is the solution}$$

$$\Rightarrow y = c e^{-\int p dx} + e^{-\int p dx} \int e^{\int p dx} r \, dx \quad \#$$

(OR) $$\frac{\partial u}{\partial x} = e^{\int p dx}(py - r)$$

$$\Rightarrow u = \int e^h (py - r) dx + k(y), \quad \boxed{h = \int p dx}$$

$$= y \int e^h p dx - \int e^h r dx + k(y) \checkmark$$

$$\frac{\partial u}{\partial y} = \underline{\int e^h p dx} + \frac{dk(y)}{dy} = N = \underline{e^h}$$

$$\Rightarrow k(y) = \int \left(e^h - \int e^h p dx\right) dy$$

$$= y e^h - y \int e^h p dx$$

$$\cancel{y \int e^h p dx} - \int e^h r dx + y e^h - \cancel{y \int e^h p dx} = C$$

$$\Rightarrow y = \underline{c e^{-h}} + \underline{e^{-h}} \int \underline{e^h} r dx$$

Ex $y' - y = e^{2x}$

$y' + py = r \qquad p(x) = -1 \qquad r(x) = e^{2x}$

$y = ce^{-h} + e^{-h} \int e^h r dx \qquad h = \int p dx = -x$

$e^h = e^{-x} \qquad e^{-h} = e^x$

$$y = ce^x + e^x \int e^{-x} e^{2x} dx = ce^x + e^{2x}$$

另解 $\frac{dy}{dx} = e^{2x} + y$

$$\Rightarrow (e^{2x} + y) dx - dy = 0$$

$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 0 \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -1$

$$(e^{x} + y)\,dx - dy = 0$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 0 \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -1$$

$$F(x) = e^{-x}$$

$$(e^{x} + ye^{-x})\,dx - e^{-x}\,dy = 0$$

$$\frac{\partial u}{\partial y} = -e^{-x} \Rightarrow u = -y e^{-x} + k(x)$$

$$\frac{\partial u}{\partial x} = y e^{-x} + \frac{dk(x)}{dx} = e^{x} + y e^{-x}$$

$$k(x) = e^{x}$$

$-ye^{-x} + e^{x} = c$ is the solution

$$\Rightarrow y = e^{2x} - ce^{x}$$