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1. Which of the sequences  $\{a_n\}$  converge, and which diverge? Find the limit of each convergent sequence.

a.  $a_n = \frac{\sin n}{n}$

sol:  $-1 \leq \sin n \leq 1 \Rightarrow \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

$\lim_{n \rightarrow \infty} \frac{-1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$  (by sandwich thm.)

$\therefore \{a_n\} = \left\{ \frac{\sin n}{n} \right\}$  converges to 0.

b.  $a_n = \frac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}}$

sol:  $\frac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}} = \frac{e^n - 2}{e^n - e^{2n}}$  (lead)

$\Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{e^n}{e^{2n}} = 0 \therefore$  it converges to 0

2. Assume following sequence converges and find its limit.

sol:  $a_n = 2 + \frac{1}{a_{n+1}}$

$\{a_n\}$  converges  $\Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} \stackrel{\text{let}}{=} L$

$L = 2 + \frac{1}{L} \Rightarrow L^2 - 2L - 1 = 0 \Rightarrow L = 1 \pm \sqrt{2}$

$\therefore a_n > 0, \forall n \in \mathbb{N} \therefore L = \underline{1 + \sqrt{2}}$



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3. Determine if the series converges or diverges.  
If a series converges, find its sum.

$$\sum_{n=1}^{\infty} (\ln \sqrt{n+1} - \ln \sqrt{n})$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (\ln \sqrt{k+1} - \ln \sqrt{k})$$

$$= \lim_{n \rightarrow \infty} (\cancel{\ln \sqrt{2}} - \cancel{\ln \sqrt{1}}) + (\cancel{\ln \sqrt{3}} - \cancel{\ln \sqrt{2}}) + \dots + (\cancel{\ln \sqrt{n}} - \cancel{\ln \sqrt{n-1}}) + (\cancel{\ln \sqrt{n+1}} - \cancel{\ln \sqrt{n}})$$

$$= \lim_{n \rightarrow \infty} \ln \sqrt{n+1} - \ln \sqrt{1} = \lim_{n \rightarrow \infty} \ln \sqrt{n+1} = \infty$$

so the series diverges.

4. Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

a.  $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} - \frac{n+2}{n+3} \right)$

sol:  $S_n = \left( \frac{1}{2} - \frac{3}{4} \right) + \left( \frac{2}{3} - \frac{4}{5} \right) + \left( \frac{3}{4} - \frac{5}{6} \right) + \dots$

$$\left( \frac{n-3}{n-2} - \frac{n-1}{n} \right) + \left( \frac{n-2}{n-1} - \frac{n}{n+1} \right) + \left( \frac{n-1}{n} - \frac{n+1}{n+2} \right) + \left( \frac{n}{n+1} - \frac{n+2}{n+3} \right)$$

$$= \frac{1}{2} + \frac{2}{3} - \frac{n+1}{n+2} - \frac{n+2}{n+3}$$

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{2} + \frac{2}{3} - 1 - 1 = \frac{7}{6} - 2 = -\frac{5}{6}$$

b.  $\sum_{n=1}^{\infty} \left( \cos \frac{\pi}{n} + \sin \frac{\pi}{n} \right)$

sol:  $\lim_{n \rightarrow \infty} \left( \cos \frac{\pi}{n} + \sin \frac{\pi}{n} \right)$  D.N.E. because it oscillates.

so the series diverges



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5. Use the Integral Test to determine if the following series converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

$$\text{sol: } 1 \int_1^{\infty} (n+4)^{-\frac{1}{2}} = 1 \cdot \lim_{n \rightarrow \infty} \left[ 2(n+4)^{\frac{1}{2}} \right]_1^n = \infty$$

$\therefore \frac{1}{\sqrt{n+4}}$  is continuous, positive, and decreasing.

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$  diverges (by Integral Test)

6. Which of the following series converge, and which diverge? Give reasons for your answers.

a.  $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

$$\text{sol: } \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$$

therefore  $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$  diverges

b.  $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

$$\text{sol: } \lim_{n \rightarrow \infty} \frac{e^n}{1+e^{2n}} = 0 \quad \therefore \text{it converges}$$

7. Which of the series converge, and which diverge? Use any method, and give reasons for your answers.

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a.  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$

sol:  $0 \leq \frac{\cos^2 n}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}},$

because  $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$  is a p-series with  $p > 1$ , it converges.

since  $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$  converges,  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$  converges.

b.  $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n} 4^n}$

sol: let  $a_n = \frac{5^n}{\sqrt{n} 4^n}$ ,  $b_n = \frac{1}{\sqrt{n}}$