Date: /

Linear Algebra HW3

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1. Find the projection of the vector **v** onto the subspace S.
   S=span{[0 0 -1 1]^T, [0 1 1 1]^T}
   v=[1 0 1 1]^T

$S = \text{span}\left(\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right)$, let $A = \begin{bmatrix}0&0\\0&1\\-1&1\\1&1\end{bmatrix}$ then $\text{Proj}_S \vec{v} = A(A^TA)^{-1}A^T\vec{v}$

$A^TA = \begin{bmatrix}0&0&-1&1\\0&1&1&1\end{bmatrix}\begin{bmatrix}0&0\\0&1\\-1&1\\1&1\end{bmatrix} = \begin{bmatrix}2&0\\0&3\end{bmatrix}$, $(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix}3&0\\0&2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{3}\end{bmatrix}$

$A(A^TA)^{-1}A^T = \begin{bmatrix}0&0\\0&1\\-1&1\\1&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0&0&-1&1\\0&1&1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&\frac{1}{3}\\-\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0&0&-1&1\\0&1&1&1\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\0&\frac{1}{3}&\frac{5}{6}&-\frac{1}{6}\\0&\frac{1}{3}&-\frac{1}{6}&\frac{5}{6}\end{bmatrix}$

so $\text{Proj}_S \vec{v} = \begin{bmatrix}0&0&0&0\\0&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\0&\frac{1}{3}&\frac{5}{6}&-\frac{1}{6}\\0&\frac{1}{3}&-\frac{1}{6}&\frac{5}{6}\end{bmatrix}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$ #

2. Find the least squares solution of the system **Ax=b**.
   A$_{3x2}$=[2 1;1 2;1 1], **b**=[2 0 -3]^T
   B$_{4x3}$==[1 0 1;1 1 1;0 1 1;1 1 0], **b**=[4 -1 0 1]^T

$A = \begin{bmatrix}2&1\\1&2\\1&1\end{bmatrix}$, $A^T = \begin{bmatrix}2&1&1\\1&2&1\end{bmatrix}$ $\quad A^TA\vec{x}^* = A^T\vec{b} \Rightarrow \begin{bmatrix}2&1&1\\1&2&1\end{bmatrix}\begin{bmatrix}2&1\\1&2\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2&1&1\\1&2&1\end{bmatrix}\begin{bmatrix}2\\0\\-3\end{bmatrix}$

$\Rightarrow \begin{bmatrix}6&5\\5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \vec{x}^* = \begin{bmatrix}1\\-1\end{bmatrix}$ #

$B = \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\\1&1&0\end{bmatrix}$, $B^T = \begin{bmatrix}1&1&0&1\\0&1&1&1\\1&1&1&0\end{bmatrix}$, $\begin{bmatrix}1&1&0&1\\0&1&1&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&1&0&1\\0&1&1&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}3&2&2\\2&3&2\\2&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\0\\3\end{bmatrix}$ $\left(\begin{bmatrix}3&2&2&4\\2&3&2&0\\2&2&3&3\end{bmatrix} \to \begin{bmatrix}1&0&-1&1\\0&1&-1&-3\\0&2&5&1\end{bmatrix} \to \begin{bmatrix}1&0&-1&1\\0&1&-1&-3\\0&0&1&1\end{bmatrix}\right) \Rightarrow \vec{x}^* = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ #

3. Find the least squares regression line (y=ax+b) for the data points: (-1, 1), (1, 0), (3 -3)

$y = f(x) = ax+b$

$f(-1) = -a+b = 1$
$f(1) = a+b = 0$
$f(3) = 3a+b = -3$

$\Rightarrow \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{\vec{b}}$

$A^TA = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\vec{x}^* = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{6} \end{bmatrix}$ #

4. Find the least squares regression quadratic polynomial $y = ax^2+bx+c$ for the data points: (-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)

$\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 5 \end{bmatrix}$

$A^TA = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$

$\Rightarrow \vec{x}^* = \begin{bmatrix} \frac{6}{5} \\ \frac{8}{5} \end{bmatrix}$ #

5. Find a unit vector orthogonal to both **u**=[1 -4 1]^T and **v**=[2 3 0]^T.

$\begin{bmatrix} 1 & -4 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 - 4x_2 + x_3 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = -\frac{2}{3}t \\ x_3 = -\frac{11}{3}t \end{cases}$

$\|(3t, -2t, -11t)\| = 1 \Rightarrow 134t^2 = 1 \Rightarrow t = \frac{1}{\sqrt{134}} \Rightarrow \left(\frac{3}{\sqrt{134}}, \frac{-2}{\sqrt{134}}, \frac{-11}{\sqrt{134}}\right)$ #

6. Find the area of a parallelogram that has **u**=[-3 4 1]^T and **v**=[0 -2 6]^T as adjacent sides.

$|\vec{u} \times \vec{v}| = \left|\begin{vmatrix} i & j & k \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}\right| = |26i + 18j + 6k| = 2\sqrt{169+81+9} = 2\sqrt{259}$ #

7. Apply the Gram Schmidt process to B={[0 3 4]^T, [1 0 0]^T, [1 1 0]^T} (according to this order).

$B = \left\{ \underset{v_1}{\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \right\}$

let $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, $B = \text{span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$

$\vec{y}_2 = \vec{v}_2 - \text{Proj}_{v_1}\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

let $\vec{u}_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $B = \text{span}(\vec{u}_1, \vec{u}_2, \vec{v}_3)$,

$\vec{y}_3 = \vec{v}_3 - \left((\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\right)$

$\vec{u}_3 = \frac{20}{25}\begin{bmatrix} 0 \\ \frac{16}{25} \\ -\frac{12}{25} \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{3}{5}\begin{bmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \frac{16}{25} \\ -\frac{12}{25} \end{bmatrix}$

$B = \text{span}\left\{ \begin{bmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{64}{125} \\ -\frac{48}{125} \end{bmatrix} \right\}$ #

8. **C**$_{3\times3}$=**QR** using the Gram Schmidt process.
C's 1st column is $[0\ 0\ 1]^T$, 2nd column is $[0\ 1\ 1]^T$, 3rd column is $[1\ 1\ 1]^T$.

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{q}_1 = 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} - \overset{\vec{q}_1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{q}_3 = \overset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} - \overset{\vec{q}_1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} - \overset{\vec{q}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \#$$

9. Find the determinant of
D$_{5\times5}$=[2 0 1 3 -2;-2 1 3 2 -1;1 0 -1 2 3;3 -1 2 4 -3;1 1 3 2 0]

$$\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 3 & -1 & -8 \\ 0 & -1 & 5 & -2 & -12 \\ 0 & 1 & 4 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 3 & -1 & 8 \\ 0 & 0 & 6 & 4 & -7 \\ 0 & 0 & 3 & -6 & -8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 3 & -1 & 8 \\ 0 & 0 & 0 & 6 & -23 \\ 0 & 0 & 0 & -5 & -16 \end{bmatrix} \quad \begin{bmatrix} 6 & -23 \\ -5 & -16 \end{bmatrix} \xrightarrow{\times\frac{1}{6}} \begin{bmatrix} 1 & -\frac{23}{6} \\ 0 & -\frac{211}{6} \end{bmatrix}$$

$$Det(D) = -1 \times \frac{3 \times \left(-\frac{211}{6}\right)}{\frac{1}{6}} = \underline{\underline{633}}$$

10. Find an equation of the plane passing through (0, 1, 0), (-1, 3, 2), (-2 0 1).

$$(0,1,0)-(-1,3,2) = (1,-2,-2)$$
$$(0,1,0)-(-2,0,1) = (2,1,-1)$$

$$X = \begin{vmatrix} 1 & -2 & -2 & 1 \\ 2 & 1 & -1 & 2 \end{vmatrix} = (4,-3,5)$$

$$4x - 3y + 5z = -3 \quad \#$$

11. Find the determinant of $\mathbf{E}_{n\times n}$ (The values on the diagonal are (1-n) and others are 1).

$$\begin{bmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 1 & 1 & 1 & \dots & 1-n \end{bmatrix}$$

12. Use the Cramer's rule to solve x, y and z.

$2x+3y+3z=3$

$6x+6y+12z=13$

$12x+9y-z=2$

$$\Delta = \begin{vmatrix} 2 & 3 & 3 \\ 6 & 6 & 12 \\ 12 & 9 & -1 \end{vmatrix} = \begin{matrix} 12+162+432 \\ -216-18-216 \end{matrix} = 168,$$

$$\Delta x = \begin{vmatrix} 3 & 3 & 3 \\ 13 & 6 & 12 \\ 2 & 9 & -1 \end{vmatrix} = \begin{matrix} -18+351+72 \\ -36+39-324 \end{matrix} = 84, \quad \Delta y = \begin{vmatrix} 2 & 3 & 3 \\ 6 & 13 & 12 \\ 12 & 2 & -1 \end{vmatrix} = \begin{matrix} -26+36+432 \\ -468+18-48 \end{matrix} = -56$$

$$\Delta z = \begin{vmatrix} 2 & 3 & 3 \\ 6 & 6 & 13 \\ 12 & 9 & 2 \end{vmatrix} = \begin{matrix} 24+162+468 \\ -216-36-234 \end{matrix} = 168, \quad x = \frac{84}{168}, \; y = \frac{-56}{168}, \; z = \frac{168}{168} \; ※$$

13. Let Dn be the determinant of the 1, 1, -1 tridiagonal matrix (nxn), that is

$$D_n = \det\begin{bmatrix} 1 & -1 & & & & \\ 1 & 1 & -1 & & & \\ & 1 & 1 & -1 & & \\ & & & \dots & & \\ & & & 1 & 1 & -1 \\ & & & & 1 & 1 \end{bmatrix}$$

Calculate D^{15}.