Algorithm Homework 6

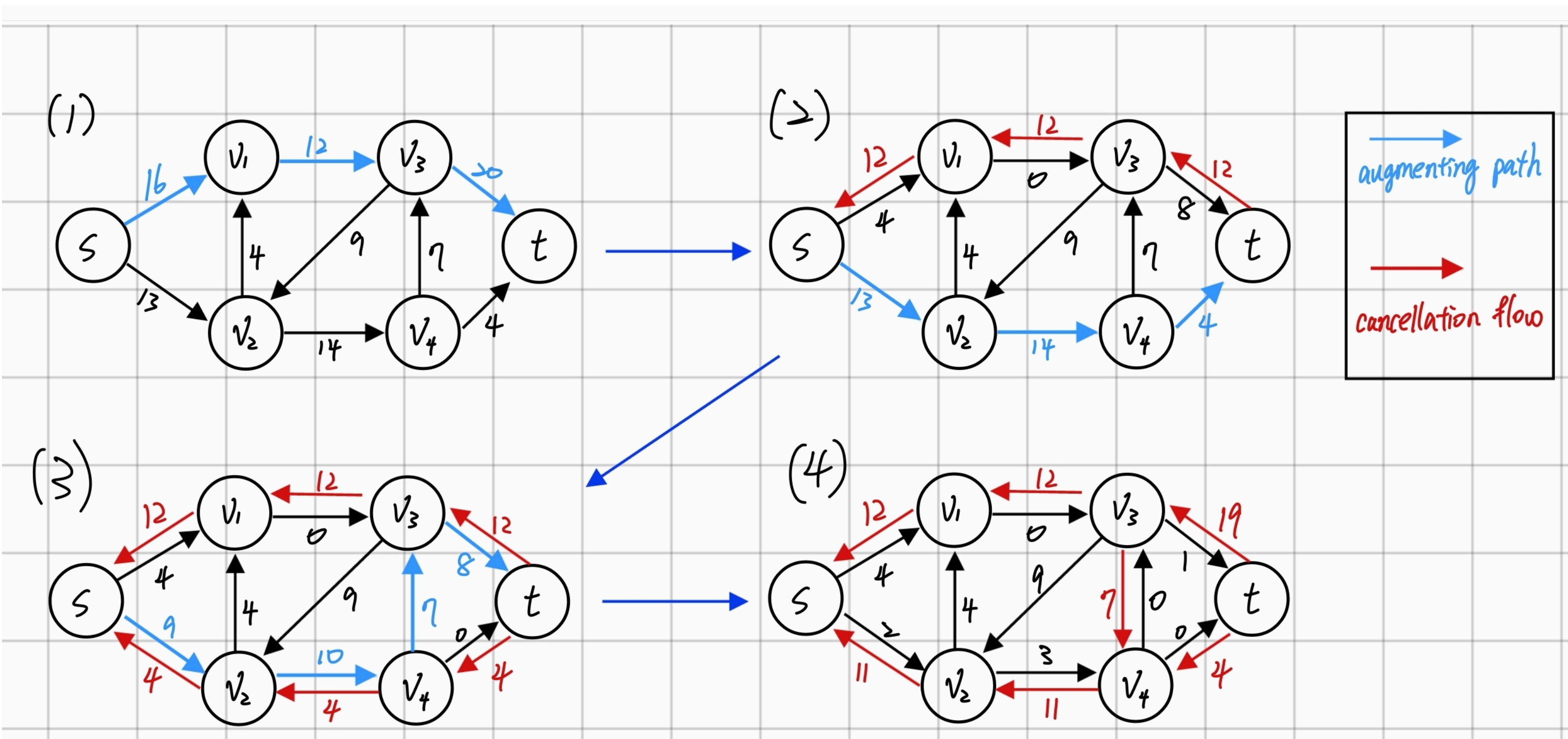
24.1-1

- The flow conservation property must hold at vertex x, which means the incoming flow to x is equal to the outgoing flow from x.
- so in the new graph G', the new edges (u, x)m (x, v) effectively behave like the original edge (u, v) in terms of capacity and flow.
- Therefore, any feasible flow in G can be transformed into a feasible flow in G' and vice versa, without changing the total amount of flow.

24.1-2

- Using flow conservation, a flow f in G can be converted to a flow f' in G by setting $f'(s, s_i) = \sum_{v \in V} f(s_i, v)$ and $f'(t_i, t) = \sum_{v \in V} f(v, t_i)$
- The value of the flows in G' is the sum of the flows from the supersource s to the sources S, which is the same as the sum of the flows form the sources S to the sinks T in G.
- Therefore, the maximum flow in G is equal to the maximum flow in G'.

24.2-3



24.2-5

- The flows must go through the original graph G, which has edges with finite compacities.
- Thus, any flow in the new network has a finite value if the edges of the original network with multiple sources and sinks have finite capacity.

24.2-8

1. **Capacity Constraint**: The capacity constraint is applied to the amount of flow that can pass through an edge, and this remains unchanged.

2. **Termination Condition**: The absence of edges into s means that once flow is sent out from s, it cannot be returned, but this doesn't affect the identification of valid paths from s to t

3. **Maximum Flow**: The absence of edges into s does not change the capacities of the edges crossing any cut. Thus, the maximum flow calculated is still accurate.

- The algorithm's key properties—flow conservation, capacity constraints, and termination condition—remain valid, ensuring the correct maximum flow is found.

24.3-3

- By definition, an augmenting path is a simple path $s \leadsto t$ in the residual network G'_f. The only edges involving s or t connect s to L and R to t.
- Thus any augmenting path must go $$s \to L \to R \to \cdots \to L \to R \to t$$ crossing back and forth between L and R at most as many times as it can do so without using a vertex twice. It contains s, t, and equal numbers of distinct vertices from L and R—at most $2 + 2 \cdot \min(|L|, |R|)$ vertices in all. The length of an augmenting path (i.e., its number of edges) is thus bounded above by $2 \cdot \min(|L|, |R|) + 1$.

Problems 24-1

Part (a)

We transform the original graph G into a new graph G' that only has edge capacities. The transformation involves splitting each vertex v with a capacity $c(v)$ into two vertices v_{in} and v_{out}, and connecting them with an edge of capacity $c(v)$.

1. **Vertex Splitting**:
   - For each vertex v in the original graph G:
     - Create two vertices v_{in} and v_{out} in the new graph G'.
     - Add an edge from v_{in} to v_{out} with capacity equal to the vertex capacity $c(v)$.

2. **Edge Transformation**:
   - For each edge (u, v) in the original graph G with capacity $c(u, v)$:
     - Add an edge from u_{out} to v_{in} in the new graph G' with capacity $c(u, v)$.

This transformation ensures that the flow through each vertex in the original graph G is subject to the vertex capacity constraint, as the flow through the corresponding edge (v_{in}, v_{out}) in the new graph G' is limited by the same capacity.

Part (b)

1. **Construct the Flow Network**:
    - Create a super source s and connect it to each of the m starting points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ with edges of infinite capacity.
    - Create a super sink t and connect all boundary vertices (vertices on the edge of the grid) to t with edges of capacity 1.
2. **Vertex Splitting**:
    - Split each vertex (i, j) (except for s and t) in the grid into two vertices $(i, j)_{in}$ and $(i, j)_{out}$ with an edge of capacity 1 between them.

3. **Edge Construction**:
   - For each vertex (i, j) in the grid, add edges from $(i, j)_{out}$ to the in-vertices of its neighboring vertices (up, down, left, right) with infinite capacity.
4. **Compute Maximum Flow**:
   - Run the Ford-Fulkerson algorithm or any of its implementations (e.g., Edmonds-Karp) to compute the maximum flow from s to t in the constructed flow network.
5. **Check the Solution**:
   - If the maximum flow value is equal to m, there exist m vertex-disjoint paths from the starting points to the boundary.
   - Otherwise, such paths do not exist.

Running Time Analysis:

- Constructing the flow network takes $O(n^2)$ time since there are $O(n^2)$ vertices and $O(n^2)$ edges in the grid.
- Splitting vertices and adding edges also takes $O(n^2)$ time.
- The Ford-Fulkerson algorithm with the Edmonds-Karp implementation runs in $O(VE^2)$ time, where $V = O(n^2)$ and $E = O(n^2)$.

Thus, the overall running time is $O(n^2)$

34.1-1

Definitions

1. **Optimization Problem: LONGEST-PATH-LENGTH**
    - Given an undirected graph $G = (V, E)$ and two vertices u and v, the problem is to find the number of edges in the longest simple path between u and v.
2. **Decision Problem: LONGEST-PATH**
    - Given an undirected graph $G = (V, E)$, vertices $u, v \in V$, and an integer $k \geq 0$, determine if there exists a simple path from u to v in G consisting of at least k edges.

Proof

If Part

If the decision problem LONGEST-PATH belongs to P, then we can solve the optimization problem LONGEST-PATH-LENGTH in polynomial time.

1. **Binary Search Approach**:
   - Use the decision problem LONGEST-PATH as a subroutine.
   - Perform a binary search on the number of edges to find the longest path.

Steps:

1. Initialize $low = 0$ and $high = |E|$.
2. While $low \leq high$:
   - Set $mid = \left\lfloor \frac{low+high}{2} \right\rfloor$.
   - Use the decision problem LONGEST-PATH to check if there is a path of length mid from u to v.
   - If such a path exists, set $low = mid + 1$.
   - Otherwise, set $high = mid - 1$.
3. The longest path length is $high$.

Since binary search runs in $O(\log |E|)$ iterations and each iteration involves solving the decision problem (which is polynomial), the overall time complexity is polynomial.

Only If Part

If the optimization problem LONGEST-PATH-LENGTH can be solved in polynomial time, then the decision problem LONGEST-PATH belongs to P.

1. **Reduction**:
   - Use the solution to the optimization problem LONGEST-PATH-LENGTH to solve the decision problem LONGEST-PATH.

Steps:

1. Solve the optimization problem LONGEST-PATH-LENGTH to find the longest path length L from u to v.
2. Compare L with k:
   - If $L \geq k$, return YES.
   - Otherwise, return NO.

Since solving the optimization problem takes polynomial time, the decision problem can also be solved in polynomial time using this approach.

Conclusion

The optimization problem LONGEST-PATH-LENGTH can be solved in polynomial time if and only if the decision problem LONGEST-PATH belongs to P.

34.1-5

Part 1: Constant Number of Polynomial-Time Subroutine Calls

1. **Polynomial-Time Subroutine**:
   - Let the subroutine $P(x)$ run in polynomial time $O(n^k)$, where n is the size of the input x and k is a constant.

2. **Algorithm with Constant Calls**:
   - Suppose the algorithm makes c calls to $P(x)$, where c is a constant.
   - Additionally, the algorithm performs some work that takes $O(n^m)$ time, where m is a constant.
3. **Total Running Time**:
   - The total time for the c calls to the polynomial-time subroutine is $c \cdot O(n^k)$.
   - The additional work takes $O(n^m)$ time.

4. **Combining the Times**:
   - The overall running time is $c \cdot O(n^k) + O(n^m)$.
   - Since c is a constant, $c \cdot O(n^k)$ is still $O(n^k)$.
5. **Polynomial Time**:
   - Therefore, the total running time is $O(n^k) + O(n^m) = O(n^{\max(k,m)})$, which is polynomial.

Part 2: Polynomial Number of Polynomial-Time Subroutine Calls

1. **Polynomial-Time Subroutine**:
   - Let the subroutine $P(x)$ run in polynomial time $O(n^k)$, where n is the size of the input x and k is a constant.
2. **Algorithm with Polynomial Calls**:
   - Suppose the algorithm makes $p(n)$ calls to $P(x)$, where $p(n)$ is a polynomial function of n.
3. **Total Running Time**:
   - The total time for the $p(n)$ calls to the polynomial-time subroutine is $p(n) \cdot O(n^k)$.

4. **Exponential Time**:
    - If $p(n)$ itself is a polynomial function, say $p(n) = n^m$, then the total running time is $n^m \cdot O(n^k) = O(n^{m+k})$, which is still polynomial.
    - However, if $p(n)$ grows exponentially, such as $p(n) = 2^n$, then the total running time becomes $2^n \cdot O(n^k) = O(2^n \cdot n^k)$, which is exponential.

Therefore, a polynomial number of calls to polynomial-time subroutines can result in an exponential-time algorithm if the number of calls grows exponentially with the input size.

34.2-1

Verification Algorithm

1. **Certificate**:
   - The certificate for GRAPH-ISOMORPHISM is a bijection π between the vertex sets of G_1 and G_2.
   - π is a permutation of the vertices such that $\pi : V(G_1) \rightarrow V(G_2)$ and it maps each vertex in G_1 to a unique vertex in G_2.

2. **Verification Process**:
   - Given the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and the certificate π:
     a. For each edge (u, v) in E_1:
        - Check if $(\pi(u), \pi(v)) \in E_2$.

     b. For each edge (u', v') in E_2:
        - Check if $(\pi^{-1}(u'), \pi^{-1}(v')) \in E_1$.

3. **Polynomial Time**:
    - The verification algorithm involves checking if each edge in G_1 maps to an edge in G_2 and vice versa.
    - The number of edges to be checked is $|E_1| + |E_2|$, and each check can be performed in constant time.
    - Therefore, the verification algorithm runs in $O(|E_1| + |E_2|)$, which is polynomial in the size of the input graphs.

Since we can verify the isomorphism of two graphs G_1 and G_2 in polynomial time given the bijection π, GRAPH-ISOMORPHISM is in NP.

34.2-3

1. **Algorithm to Detect Hamiltonian Cycle**:
    - Assume we have an algorithm $A(G)$ that runs in polynomial time and returns `true` if the graph G contains a Hamiltonian cycle and `false` otherwise.
2. **Constructing the Hamiltonian Cycle**:
    - Initialize an empty list to store the vertices of the Hamiltonian cycle.
    - Start from an arbitrary vertex v in the graph.
    - Recursively build the cycle by considering each vertex one by one.

Procedure:

1. **Find an Initial Hamiltonian Cycle**:
   - Use the algorithm $A(G)$ to check if G contains a Hamiltonian cycle. If not, return that no such cycle exists.
2. **Construct the Cycle**:
   - Start from an arbitrary vertex v and add it to the cycle list.
   - Remove v from the graph and use the algorithm $A(G - v)$ to find the next vertex in the cycle.
   - Repeat this process, each time removing the selected vertex and using A to find the next vertex in the cycle, until all vertices are added.

```
FindHamiltonianCycle(G):
    if not A(G):
        return "No Hamiltonian cycle exists"
    cycle = []
    v = any_vertex(G)  // Start with an arbitrary vertex
    cycle.append(v)
    G.remove(v)

    while len(cycle) < number_of_vertices_in_G:
        for each vertex u in G:
            if A(G - u):
                cycle.append(u)
                G.remove(u)
                break

    // Close the cycle by connecting the last vertex to the first
    cycle.append(cycle[0])

    return cycle
```

34.2-4

Closure Under Union

1. **Non-Deterministic Polynomial-Time Turing Machine (NPTM)**:
    - Since $L_1 \in \mathrm{NP}$, there exists an NPTM M_1 that decides L_1 in polynomial time.
    - Similarly, since $L_2 \in \mathrm{NP}$, there exists an NPTM M_2 that decides L_2 in polynomial time.
2. **Combined NPTM**:
    - Construct an NPTM M that, on input x:
        a. Non-deterministically choose either M_1 or M_2.
        b. Run the chosen machine on x.

3. **Acceptance**:
   - M accepts x if either M_1 or M_2 accepts x.

Since M runs in polynomial time, $L_1 \cup L_2 \in \mathrm{NP}$.

Closure Under Intersection

Let L_1 and L_2 be languages in NP. We want to show that $L_1 \cap L_2 \in \mathrm{NP}$.

1. **Non-Deterministic Polynomial-Time Turing Machine (NPTM)**:
   - Since $L_1 \in \mathrm{NP}$, there exists an NPTM M_1 that decides L_1 in polynomial time.
   - Similarly, since $L_2 \in \mathrm{NP}$, there exists an NPTM M_2 that decides L_2 in polynomial time.

2. **Combined NPTM**:
   - Construct an NPTM M that, on input x:
     a. Non-deterministically run both M_1 and M_2 on x
3. **Acceptance**:
   - M accepts x if both M_1 and M_2 accept x.

Since M runs in polynomial time, $L_1 \cap L_2 \in \mathrm{NP}$.

Closure Under Concatenation

Let L_1 and L_2 be languages in NP. We want to show that $L_1 \cdot L_2 \in \mathrm{NP}$.

1. **Non-Deterministic Polynomial-Time Turing Machine (NPTM)**:
    - Since $L_1 \in \mathrm{NP}$, there exists an NPTM M_1 that decides L_1 in polynomial time.
    - Similarly, since $L_2 \in \mathrm{NP}$, there exists an NPTM M_2 that decides L_2 in polynomial time.
2. **Combined NPTM**:
    - Construct an NPTM M that, on input x:
        a. Non-deterministically split x into $x = uv$.
        b. Run M_1 on u and M_2 on v.

3. **Acceptance**:
    - M accepts x if M_1 accepts u and M_2 accepts v.

Since M runs in polynomial time, $L_1 \cdot L_2 \in \mathrm{NP}$.

Closure Under Kleene Star

Let L be a language in NP. We want to show that $L^* \in \mathrm{NP}$.

1. **Non-Deterministic Polynomial-Time Turing Machine (NPTM)**:
    - Since $L \in \mathrm{NP}$, there exists an NPTM M that decides L in polynomial time.

2. **Combined NPTM**:
   - Construct an NPTM M' that, on input x:
     a. Non-deterministically split x into
        $x = x_1 x_2 \cdots x_k$
     b. For each x_i, run M on x_i.
3. **Acceptance**:
   - M' accepts x if M accepts each x_i.

Closure Under Complement

The class NP is not known to be closed under complement. This is equivalent to the open question of whether $\mathrm{NP} = \text{co-NP}$. If NP were closed under complement, it would imply $\mathrm{NP} = \text{co-NP}$, which is a major unsolved question in computer science.

34.3-2

1. **Polynomial-Time Reduction**:
   - A language L_1 is polynomial-time reducible to a language L_2 (denoted $L_1 \leq_P L_2$) if there exists a polynomial-time computable function f such that $x \in L_1 \iff f(x) \in L_2, \forall x$.

Proof

1. $L_1 \leq_P L_2$: There exists a polynomial-time computable function f_1 such that $x \in L_1 \iff f_1(x) \in L_2, \forall x$.
2. $L_2 \leq_P L_3$: There exists a polynomial-time computable function f_2 such that $y \in L_2 \iff f_2(y) \in L_3, \forall y$.

We need to show that $L_1 \leq_P L_3$.

Construction of the Reduction

1. Define a new function f such that $f(x) = f_2(f_1(x)), \forall x.$
   - Since f_1 is a polynomial-time computable function, $f_1(x)$ can be computed in polynomial time.
   - Since f_2 is a polynomial-time computable function, $f_2(f_1(x))$ can be computed in polynomial time after $f_1(x)$ is computed.
2. Therefore, $f(x)$ can be computed in polynomial time as it is the composition of two polynomial-time computable functions.

Verification

We need to verify that f satisfies the reduction condition:

1. $x \in L_1 \iff f(x) \in L_3, \forall x$
2. By the definitions of f_1 and f_2,
   $x \in L_1 \iff f_1(x) \in L_2 \iff f_2(f_1(x)) \in L_3$.
3. Thus, $x \in L_1 \iff f(x) \in L_3$.

Conclusion

Since f is polynomial-time computable and satisfies the reduction condition, $L_1 \leq_P L_3$. Therefore, the $\leq_P$ relation is transitive.

34.3-6

Part 1: $\emptyset$ and $\{0,1\}^*$ Are Not Complete for P

1. **Language $\emptyset$**:
   - $\forall L' \in P, L' \leq_P \emptyset$ is not true unless $L' = \emptyset$.
   - There is no polynomial-time computable function f that can reduce a non-empty language to $\emptyset$.
2. **Language $\{0,1\}^*$**:
   - $\forall L' \in P, L' \leq_P \{0,1\}^*$ is not true unless $L' = \{0,1\}^*$.
   - The language $\{0,1\}^*$ accepts all strings, so there is no meaningful reduction from a specific polynomial-time language to the universal language $\{0,1\}^*$.

Part 2: Any Other Language in P Is Complete for P

1. **Non-Trivial Language**:
   - Consider a non-trivial language $L \in P$ such that $L \neq \emptyset$ and $L \neq \{0, 1\}^*$.
   - For every language $L' \in P$, we need to show that $L' \leq_P L$.

2. **Reduction**:
   - Since both L and L' are in P, there exist polynomial-time algorithms to decide them.
   - We can construct a polynomial-time reduction from L' to L by mapping each input of L' to an input of L using a polynomial-time computable function.

Conclusion

The languages $\emptyset$ and $\{0, 1\}^*$ are the only languages in P that are not complete for P with respect to polynomial-time reductions because any non-trivial language in P can be used to polynomial-time reduce any other language in P.

34.4-1

Exponential Growth Example

Consider a binary tree structure circuit, where each internal node represents a logical AND gate and the leaves represent input variables. Let's denote the size of the circuit by n, which corresponds to the number of gates and input variables.

1. **Circuit Description**:
    - The circuit is a full binary tree of depth d.
    - The number of leaves (input variables) is 2^d.
    - The number of internal nodes (AND gates) is $2^d - 1$.
    - Therefore, the total size of the circuit n is approximately $2 \cdot 2^d$, which is $O(2^d)$.

2. **Conversion to Formula**:
   - To convert this circuit to a formula, start from the root and recursively replace each AND gate with its corresponding formula.
   - Each AND gate at depth k combines two subtrees, each of which is a formula.

Exponential Growth in Formula Size

- At the root level, the formula is a conjunction of two subformulas.
- Each of these subformulas is a conjunction of two more subformulas, and so on.

Size Analysis

1. **Depth of the Tree**:
   - The depth d of the tree is $\log_2(n)$.
2. **Formula Size**:
   - The size of the formula at the root level is the sum of the sizes of its two child subformulas plus the size of the AND operation.
   - Each AND operation at depth k contributes to a formula that is a conjunction of two subformulas of depth $k-1$.

3. **Exponential Growth**:
   - At each level, the formula size doubles.
   - For a tree of depth d, the formula size is $O(2^d)$.
   - Since $d = \log_2(n)$, the formula size is $O(2^{\log_2(n)}) = O(n)$.

Therefore, a circuit of size n can result in a formula of size $O(2^n)$ when converted using the method described, leading to exponential growth in the size of the formula.

Conclusion

The circuit that results in exponential growth when converted to a formula is a full binary tree with AND gates at each internal node. This structure ensures that the formula size grows exponentially with the size of the circuit.

34.4-3

To show that Professor Jagger's strategy does not yield a polynomial-time reduction from SAT to 3-CNF-SAT, we need to analyze the proposed steps and their computational complexity.

Analysis

1. **Forming a Truth Table**:
    - Given a boolean formula ϕ with n variables, the truth table will have 2^n rows.
    - Constructing this truth table requires evaluating ϕ for each of the 2^n possible assignments of the variables.
    - This step takes $O(2^n \cdot \text{poly}(n))$ time, where $\text{poly}(n)$ denotes a polynomial in n.

2. **Deriving a 3-DNF Formula**:
    - From the truth table, we derive a Disjunctive Normal Form (DNF) that represents $\neg\phi$.
    - This DNF formula might have up to 2^n terms (one for each row of the truth table where $\neg\phi$ is true).
    - Converting $\neg\phi$ to a 3-DNF may involve splitting terms to ensure each term has at most 3 literals, but the number of terms remains exponential.
3. **Applying DeMorgan's Laws to Obtain 3-CNF**:
    - Negating the 3-DNF formula and applying DeMorgan's laws to obtain a 3-CNF formula equivalent to ϕ will not change the exponential nature of the formula size.

Conclusion

- Each of these steps involves operations that take exponential time in the worst case.
- Therefore, the entire procedure takes $O(2^n \cdot \text{poly}(n))$ time, which is not polynomial in n.

Thus, Professor Jagger's strategy does not yield a polynomial-time reduction from SAT to 3-CNF-SAT.

34.4-5

A DNF formula is satisfiable if and only if at least one of its clauses is satisfiable. A clause is satisfiable if none of its literals contradict each other.

1. **Iterate Through Clauses**:
   - For each clause in the DNF formula, check if the clause is satisfiable.
2. **Check Each Clause**:
   - A clause is satisfiable if it does not contain both a variable and its negation.
3. **Return Result**:
   - If at least one clause is satisfiable, the entire DNF formula is satisfiable. Otherwise, it is not.

Complexity Analysis

- Let m be the number of clauses in the DNF formula, and let k be the maximum number of literals in any clause.
- Checking if a clause is satisfiable involves checking for contradictory literals, which can be done in $O(k)$ time.
- Thus, the total time to check all clauses is $O(m \cdot k)$, which is polynomial in the size of the DNF formula.

Conclusion

The problem of determining the satisfiability of boolean formulas in disjunctive normal form is polynomial-time solvable because it can be done in $O(m \cdot k)$ time, where m is the number of clauses and k is the maximum number of literals per clause.

34.5-1

The Problem is in NP

- Given graphs G_1 and G_2, and a proposed subgraph isomorphism f, we can verify whether f is a subgraph isomorphism in polynomial time.
- To verify f, check that f is a bijection between the vertices of G_1 and a subset of the vertices of G_2, and that edges are preserved. This verification can be done in polynomial time.

Thus, the subgraph-isomorphism problem is in NP.

The Problem is NP-hard

Reduction from the Clique Problem

1. **Clique Problem**: Given a graph G and an integer k, determine if there is a complete subgraph (clique) of k vertices in G.
2. **Reduction Construction**:
   - Let G be the graph in the Clique problem, and let k be the size of the clique we are looking for.
   - Construct G_1 as a complete graph K_k with k vertices.
   - Set $G_2 = G$.

3. **Correctness of the Reduction**:
   - If G contains a clique of size k, then K_k (which is G_1) is isomorphic to a subgraph of G (which is G_2).
   - If G_1 (which is K_k) is isomorphic to a subgraph of G_2 (which is G), then G contains a clique of size k.

This reduction is clearly polynomial in time, since constructing G_1 and G_2 and checking their properties are polynomial-time operations.

Conclusion

Since we have shown that the subgraph-isomorphism problem is in NP and that it is NP-hard, we conclude that the subgraph-isomorphism problem is NP-complete.

34.5-2

The Problem is in NP

- Given a matrix A, a vector b, and a proposed solution x where x is an n-vector with elements in $\{0, 1\}$, we can check whether $Ax \leq b$ in polynomial time.
- This involves performing the matrix-vector multiplication Ax and comparing the resulting vector to b, which can be done in polynomial time.

Reduction from 3-CNF-SAT

1. **3-CNF-SAT Problem**:
    - Given a boolean formula in 3-CNF (conjunctive normal form with exactly 3 literals per clause), determine if there is an assignment of truth values to the variables that makes the formula true.
2. **Constructing the Reduction**:
    - Let ϕ be a boolean formula in 3-CNF with n variables and m clauses.
    - Each variable in ϕ corresponds to a variable in the 0-1 integer programming problem.

3. **Matrix A and Vector b Construction**:

    - For each clause in ϕ, create an inequality.
    - If a clause is $(x_i \vee \neg x_j \vee x_k)$, create an inequality that ensures at least one of the literals in the clause is true.
    - Translate this clause to the inequality: $x_i + (1 - x_j) + x_k \geq 1$. This ensures that at least one of the literals in the clause is satisfied.
    - Convert this to a standard form by moving all terms to one side: $x_i - x_j + x_k \geq 0$.

4. **Matrix and Vector Representation**:

    - Each clause inequality is represented as a row in the matrix A and the corresponding entry in b.
    - Ensure that all clauses are represented in this way.

5. **Verification of the Reduction**:
   - If there is a satisfying assignment for ϕ, then there exists a vector x with elements in ({0, 1}) such that $Ax \leq b$.
   - If $Ax \leq b$ for some vector x with elements in ({0, 1}), then the corresponding assignment of boolean variables satisfies ϕ.

This reduction is polynomial in time because constructing A and b from the 3-CNF formula involves a polynomial number of operations.

Conclusion

Since we have shown that the 0-1 integer programming problem is in NP and that it is NP-hard, we conclude that the 0-1 integer programming problem is NP-complete.

34.5-4

When the target value t is expressed in unary, it means that the size of t is $O(t)$ rather than $O(\log t)$.

1. **Define the DP Table**:
   - Let $dp[i][j]$ be a boolean table where $dp[i][j]$ is true if there is a subset of $\{a_1, a_2, \ldots, a_i\}$ that sums to j, and false otherwise.
2. **Initialization**:
   - $dp[0][0] = \text{true}$ because a sum of 0 can always be achieved with an empty subset.
   - $dp[0][j] = \text{false}$ for all $j > 0$.

3. **Fill the DP Table**:
    - For each element a_i in the set S:
        - For each possible sum j from 0 to t:
            - If $j \geq a_i$, then $dp[i][j]$ is true if either $dp[i-1][j]$ is true (excluding a_i) or $dp[i-1][j-a_i]$ is true (including a_i).
            - Otherwise, $dp[i][j] = dp[i-1][j]$.
4. **Final Result**:
    - The answer to the subset-sum problem is $dp[n][t]$, where n is the number of elements in the set S.

Complexity Analysis

- The time complexity of filling the DP table is $O(n \cdot t)$ because there are n elements and each element can be part of sums from 0 to t.
- The space complexity is also $O(n \cdot t)$ for the DP table.

Since t is expressed in unary, t is $O(t)$ rather than $O(\log t)$. This means the time and space complexity are both polynomial in the size of the input.

Conclusion

By using a dynamic programming approach, we can solve the subset-sum problem in polynomial time when the target value t is expressed in unary.

34.5-6

The Problem is in NP

A problem is in NP if a proposed solution can be verified in polynomial time.

- Given a graph G and a proposed Hamiltonian path (a sequence of vertices), we can verify whether the path visits each vertex exactly once in polynomial time.
- This involves checking the sequence to ensure it includes each vertex of G exactly once and that each consecutive pair of vertices in the sequence are connected by an edge.
- Therefore, the Hamiltonian-path problem is in NP.

Reduction from Hamiltonian Cycle Problem

1. **Hamiltonian Cycle Problem**:
   - Given a graph G, determine if there exists a cycle that visits each vertex exactly once.
2. **Constructing the Reduction**:
   - Given a graph G for the Hamiltonian cycle problem, construct a new graph G' by adding a new vertex v and connecting v to all vertices in G.
   - The new graph G' will have $|V(G)| + 1$ vertices.

3. **Reduction Explanation**:
   - If G has a Hamiltonian cycle, then in G', there exists a Hamiltonian path that starts at v, traverses the cycle in G, and returns to v.
   - Conversely, if G' has a Hamiltonian path, the path must start or end at v. Removing v from this path results in a Hamiltonian cycle in G.

Conclusion

Since we have shown that the Hamiltonian-path problem is in NP and that it is NP-hard, we conclude that the Hamiltonian-path problem is NP-complete.

Problem 34-3: Graph Coloring

Part (a)

```
function isBipartite(graph): //O(V + E)
   n = number of vertices in graph
   color = array of size n initialized to -1  // no color

   function bfs(start):
      queue = empty queue
      queue.enqueue(start)
      color[start] = 0
      while not queue.isEmpty():
         u = queue.dequeue()
         for each v in graph.adjacent(u):
            if color[v] == -1:
               color[v] = 1 - color[u]
               queue.enqueue(v)
            else if color[v] == color[u]:
               return false
      return true

   for i from 0 to n - 1:
      if color[i] == -1:
         if not bfs(i):
            eturn false
   return true  // All components are bipartite
```

Part (b)

- **Polynomial Time Equivalence**:
  - If we can solve the graph k-colorability decision problem in polynomial time, we can determine the minimum number of colors needed by testing k from 1 to $|V|$ using binary search or sequentially.
  - Conversely, if we can solve the graph-coloring problem in polynomial time, we can use it directly to solve the decision problem for any k.

Thus, the decision problem is solvable in polynomial time if and only if the graph-coloring problem is solvable in polynomial time.

Part (c)

To show that if 3-COLOR is NP-complete, then the k-colorability decision problem is NP-complete, we need to show two things:

1. **3-COLOR is in NP**:
    - A 3-coloring of a graph can be verified in polynomial time by checking all edges to ensure no two adjacent vertices have the same color.
    - Therefore, 3-COLOR is in NP.

2. **Reduction from 3-CNF-SAT**:
   - Given a 3-CNF formula ϕ with m clauses on n variables $x_1, x_2, \ldots, x_n$, construct a graph $G = (V, E)$ such that G can be 3-colored if and only if ϕ is satisfiable.
   - The construction includes:
     - For each variable x_i, create vertices for x_i and $\neg x_i$.
     - Create a triangle with vertices TRUE, FALSE, and RED to ensure proper coloring.
     - For each clause $(x \vee y \vee z)$, create a clause gadget ensuring one of the literals is true.
   - This ensures a 3-colorable graph can represent a satisfiable formula, proving the reduction.

Part (d)

1. **Literal Coloring**:
    - In the graph, x_i and $\neg x_i$ form a triangle with the RED vertex.
    - In any 3-coloring, TRUE, FALSE, and RED are all distinct colors.
    - Hence, x_i and $\neg x_i$ must be colored TRUE and FALSE respectively (or vice versa).

2. **Truth Assignment to 3-Coloring**:
    - For any truth assignment, color the variable vertices accordingly:
        - If x_i is true, color x_i as TRUE and $\neg x_i$ as FALSE.
        - If x_i is false, color x_i as FALSE and $\neg x_i$ as TRUE.
    - This ensures the graph with literal edges is properly 3-colored.

Part (e)

1. **Clause Gadget Coloring**:
   - The clause gadget enforces that at least one literal must be true for the clause to be satisfied.
   - In the gadget, x, y, and z connect to the clause vertex in such a way that if all were FALSE, the gadget would not be 3-colorable.
2. **3-Colorability Condition**:
   - If at least one of x, y, or z is colored TRUE, the gadget can be colored properly.
   - This ensures the clause is satisfiable if and only if at least one literal is true.

Part (f)

1. **Combining Parts (c) to (e)**:
   - Use the reduction from 3-CNF-SAT to 3-COLOR as outlined.
   - Show that the constructed graph from a 3-CNF formula is 3-colorable if and only if the formula is satisfiable.
   - This completes the reduction proof and establishes NP-completeness of 3-COLOR.

Thus, 3-COLOR is NP-complete.