20220307

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$$\frac{dy}{dx} = f(x,y) = \frac{-M(x,y)}{N(x,y)} \Rightarrow M + N\frac{dy}{dx} = 0 \quad (Mdx + Ndy = 0)$$

$u(x,y)$

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}}$$

(i) Differential

$$du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

(ii) Chain rule

$x = x(t) \quad y = y(t)$

$$\frac{du(x,y)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$t \to x$

$$\frac{du(x,y)}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = M \quad \frac{\partial u}{\partial y} = N$$

$$M + N\frac{dy}{dx} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{du}{dx} = 0 \Rightarrow u = C$$

$$Mdx + Ndy = 0 \qquad \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}}$$

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int M dx + \underline{k(y)} \qquad u = C$$

$$\frac{\partial u}{\partial y} = N$$

Ex $\underbrace{\cos x - 2xy}_{M} + \underbrace{(e^y - x^2)}_{N}\frac{dy}{dx} = 0$

$$\frac{\partial M}{\partial y} = -2x \underset{\text{exact}}{=} \frac{\partial N}{\partial x} = -2x$$

$$\frac{\partial u}{\partial y} = e^y - x^2 \Rightarrow u = e^y - x^2 y + k(x)$$

$$\frac{\partial u}{\partial x} = -2xy + k'(x) = M = \cos x - 2xy$$

$$\frac{dk(x)}{dx} = \cos x \qquad k(x) = \sin x$$

$$\frac{dk(x)}{dx} = \cos x \qquad k(x) = \sin x$$

$e^y - x^2 y + \sin x = \underline{C}$ is the general solution

<u>IVP</u> $y(\underline{1}) = 4$

$$e^4 - 4 + \sin(1) = \boxed{C}$$

$e^y - x^2 y + \sin x = e^4 - 4 + \sin(1)$ is the solution of this IVP

Reduction to Exact Form: Integrating Factors

Consider the following DE $\underbrace{2y}_{M} + \underbrace{x}_{N} \frac{dy}{dx} = 0$

$$\frac{\partial M}{\partial y} = 2 \neq \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial u}{\partial x} = M \Rightarrow u = \int 2y\,dx + k(y) = 2xy + k(y)$$

$$\frac{\partial u}{\partial y} = N \Rightarrow 2x + \frac{dk(y)}{dy} = x \Rightarrow \frac{dk(y)}{dy} = -x$$

$$\Rightarrow k(y) = -xy \text{ (不合)}$$

However if we multiply x to this DE

$$\underbrace{2xy}_{M} + \underbrace{x^2}_{N} \frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = 2x \overset{\text{exact!}}{=} \frac{\partial N}{\partial x} = 2x$$

Then this DE becomes exact and we call "x" an "integrating factor."

The "Integrating factor" is not unique.

Ex $-y\,dx + x\,dy = 0 \qquad \frac{\partial M}{\partial y} = -1 \neq \frac{\partial N}{\partial x} = 1$

Multiply $\frac{1}{x^2}$ $\qquad \frac{-y}{x^2}dx + \frac{1}{x}dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{-1}{x^2} = \frac{\partial N}{\partial x} = -x^{-2}$

$$\frac{\partial u}{\partial y} = x^{-1} \Rightarrow u = x^{-1}y + k(x)$$

$$\frac{\partial u}{\partial x} = -x^{-2}y + \underbrace{\frac{dk(x)}{dx}}_{} = \frac{-y}{x^2} \qquad k(x) = C^*$$

$\frac{y}{x} = C$ is the solution.

Multiply $\frac{1}{y^2}$ $\qquad \frac{x}{y} = C$

Multiply $\frac{1}{y^2}$ $\quad \frac{x}{y} = C$

Multiply $\frac{1}{xy}$ $\quad \ln\left(\frac{y}{x}\right) = C$

Multiply $\frac{1}{x^2+y^2}$ $\quad \tan^{-1}\frac{x}{y} = C$

Question: How to find the integrating factor?

"Inspection" or "trial & error"

How to "trial & error"?

$$FM\,dx + FN\,dy = 0$$

$$\frac{\partial(FM)}{\partial y} = \frac{\partial(FN)}{\partial x} = F\frac{\partial M}{\partial y} + M\frac{\partial F}{\partial y} = F\frac{\partial N}{\partial x} + N\frac{\partial F}{\partial x} \quad \underline{(*)}$$

Case 1 If $F(x,y) = F(x)$

$$(*) \Rightarrow F\frac{\partial M}{\partial y} = F\frac{\partial N}{\partial x} + N\frac{dF}{dx}$$

$$\Rightarrow F\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{dF}{dx}$$

$$\Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = \frac{1}{F}\,dF$$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ depends on x only

then this DE is separable.

$$\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx = \int \frac{1}{F}\,dF = \ln F$$

$$\Rightarrow F = \exp\left\{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\underline{dx}\right\}$$

Case 2 If $F(x,y) = F(y)$

$$(*)\ F\frac{\partial M}{\partial y} + M\frac{dF}{dy} = F\frac{\partial N}{\partial x}$$

$$\Rightarrow M\frac{dF}{dy} = F\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\Rightarrow \frac{dF}{F} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy$$

$$\Rightarrow \frac{dF}{F} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy$$

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ depends on y only

$$F = \exp\left\{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\right\} \checkmark$$

case 3 If the DE involves only integer powers of x & y, then $F(x,y) = x^a y^b$ may be a good choice (and solve a & b by substituting F into $(*)$)

(case 4 Try e^{ax+by}, $x^a e^{by}$, $e^{ax} y^b$ and so on)