20220221

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Reduction to Separable Form

Sometimes it is necessary to introduce new variables or transformation to make a DE separable.

* DE of the form $y' = g(\frac{y}{x})$

Let $\frac{y}{x} = u(x) \Rightarrow y = xu \Rightarrow y' = u + xu'$

$y' = g(\frac{y}{x}) = g(u) = x\frac{du}{dx} + u$

$\Rightarrow \frac{du}{g(u)-u} = \frac{dx}{x}$ a separable form!

Ex $xy' = \frac{y^2}{x} + y \Rightarrow y' = (\frac{y}{x})^2 + (\frac{y}{x})$ $(= g(\frac{y}{x}))$

Let $\boxed{\frac{y}{x} = u} \Rightarrow y = ux \Rightarrow y' = u'x + u$

$u'x + u = u^2 + u \Rightarrow \frac{du}{dx}x = u^2 \Rightarrow \int u^{-2}du = \int x^{-1}dx + c$

$\Rightarrow -u^{-1} = \ln|x| + c$

$\Rightarrow u = \frac{-1}{\ln|x|+c} \Rightarrow \frac{y}{x} = \frac{-1}{\ln|x|+c}$

$\Rightarrow y = \frac{-x}{\ln|x|+c}$

Verify $xy' = x \cdot \frac{-(\ln|x|+c) + \frac{1}{x}\cdot x}{(\ln|x|+c)^2}$

$\frac{y^2}{x} + y = \frac{x}{(\ln|x|+c)^2} - \frac{x}{\ln|x|+c} = \frac{x - x(\ln|x|+c)}{(\ln|x|+c)^2}$

Ex $x^3y' = x^2y - 2y^3$

$\Rightarrow y' = \frac{y}{x} - 2(\frac{y}{x})^3$

Let $\frac{y}{x} = u \Rightarrow y = ux \Rightarrow y' = u'x + u$

$u'x + u = u - 2u^3 \Rightarrow \frac{du}{dx}x = -2u^3 \Rightarrow \int -\frac{1}{2}u^{-3}du = \int \frac{1}{x}dx + c$

$\Rightarrow \frac{u^{-2}}{4} = \ln|x| + c \Rightarrow \frac{1}{4}\frac{x^2}{y^2} = \ln|x| + c$

Verify $\frac{1}{4}x^2y^{-2} = \ln|x| + c$

Verify $\frac{1}{4}x^2y^{-2} = \ln|x| + c$

$\Rightarrow \frac{1}{2}xy^{-2} - \frac{1}{2}x^2y^{-3}y' = \frac{1}{x}$

$\Rightarrow \frac{1}{2}x^2y - \frac{1}{2}x^3y' = y^3 \Rightarrow x^2y - 2y^3 = x^3y'$

Ex $2xyy' = y^2 - x^2 \Rightarrow y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{1}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}$

Let $\frac{y}{x} = u \Rightarrow y' = u'x + u$

$u'x + u = \frac{1}{2}u - \frac{1}{2u} \Rightarrow \frac{2u}{u^2+1}du = -\frac{1}{x}dx$

$\left(\int \frac{f'(x)}{f(x)}dx = \int \frac{1}{f(x)}\,df(x) = \ln|f(x)| + c\right)$

$\Rightarrow \underline{\ln(u^2+1) = -\ln|x| + c^*} \Rightarrow e^{\ln(u^2+1)} = e^{\ln\frac{1}{|x|}}e^{c^*}$

$\Rightarrow \left(\frac{y}{x}\right)^2 + 1 = \frac{c}{x} \Rightarrow x^2 + y^2 = cx \Rightarrow \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$

以$(\frac{c}{2}, 0)$為圓心

$\frac{c}{2}$為半徑的圓

$(\sin x)' = \cos x$

$e^{jx} = \cos x + j\sin x$ $\qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$

$e^{-jx} = \cos x - j\sin x$ $\qquad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$