20220314

2022年3月14日 下午 03:05

Ex $xy' - 4y + 2x^2 + 4 = 0$

$\Rightarrow y' - \frac{4}{x}y = -2x - \frac{4}{x}$ $p = -\frac{4}{x}$ $r = -2x - \frac{4}{x}$

$y = ce^{-h} + e^{-h}\int e^{h} r\,dx$ $h = \int p\,dx = \int -\frac{4}{x}\,dx = -4\ln|x|$

$e^{-h} = x^4$ $e^{h} = x^{-4}$

$y = cx^4 + x^4\int x^{-4}\left(-2x - \frac{4}{x}\right)dx$

$= cx^4 + x^4\int -2x^{-3} - 4x^{-5}\,dx = cx^4 + x^4\left(x^{-2} + x^{-4}\right)$

$= cx^4 + x^2 + 1$

Ex $y' + 2y = e^{x}(3\sin 2x + 2\cos 2x)$

$p = 2$ $r = e^{x}(3\sin 2x + 2\cos 2x)$

$y = Ce^{-h} + e^{-h}\int e^{h} r\,dx$

$h = \int p\,dx = 2x$ $e^{-h} = e^{-2x}$ $e^{h} = e^{2x}$

$y = ce^{-2x} + e^{-2x}\underbrace{\int e^{3x}(3\sin 2x + 2\cos 2x)\,dx}$

$= ce^{-2x} + e^{x}\sin 2x$

$\int e^{3x}(3\sin 2x + 2\cos 2x)\,dx$

$= \int e^{3x}\,3\sin 2x\,dx + e^{3x}\,d\sin 2x\,dx$

$= \int e^{3x}\cancel{3\sin 2x\,dx} + e^{3x}\sin 2x$

$\cancel{-\int \sin 2x\,de^{3x}}$

$= e^{3x}\sin 2x$ ✓

另解

$\underbrace{\left(2y - e^{x}(3\sin 2x + 2\cos 2x)\right)}_{M}dx + \underbrace{dy}_{N} = 0$

$\frac{\partial M}{\partial y} = 2$ $\frac{\partial N}{\partial x} = 0$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2$ $F(x) = e^{2x}$

$\left(2e^{2x}y - e^{3x}(3\sin 2x + 2\cos 2x)\right)dx + e^{2x}dy = 0$

$(2e^{2x}y - e^{-3x}(3\sin 2x + 2\cos 2x))dx + e^{2x}dy = 0$

$$\frac{\partial u}{\partial y} = e^{2x} \qquad u = e^{2x}y + k(x)$$

$$\frac{\partial u}{\partial x} = 2e^{2x}y + \frac{dk(x)}{dx} = \underline{\underline{2e^{2x}y - e^{-3x}(3\sin 2x + 2\cos 2x)}}$$

$$k = e^{-3x}\sin 2x$$

$$e^{2x}y - e^{-3x}\sin 2x = c \Rightarrow y = ce^{-2x} + e^{x}\sin 2x$$

Certain non-linear 1st order DE can be transferred to linear

(1) Bernoulli equation

$$y' + p(x)y = r(x)y^a \quad (a=0 \text{ or } a=1 \text{ DE is linear})$$

Let $\boxed{u = y^{1-a}}$ ☆

$$u' = (1-a)y^{-a}y' = (1-a)\underline{y^{-a}(ry^a - py)}$$

$$= (1-a)(r - py^{1-a}) = (1-a)(r - pu)$$

$$\Rightarrow u' + (1-a)pu = r(1-a) \quad \text{linear !!}$$

$$(y' + p\,y = r)$$

Ex $xy' + y = x^2y^2$

$$\Rightarrow y' + x^{-1}y = xy^2$$

$$(y' + py = ry^a)$$

Let $u = y^{1-2} = y^{-1}$

$$u' = -y^{-2}y' = -y^{-2}(xy^2 - x^{-1}y)$$

$$= -x + x^{-1}u$$

$$\Rightarrow u' - x^{-1}u = -x$$

$h = \int p\,dx = \int -x^{-1}dx = -\ln x$

$$\Rightarrow u' - x^{-1}u = -x$$

$$u = ce^{-h} + e^{-h}\int e^{h} r\,dx \qquad h = \int p\,dx = \int -x^{-1}\,dx = -\ln x$$

$$e^{-h} = e^{\ln x} = x \quad e^{h} = x^{-1}$$

$$u = cx + x\int x^{-1}(-x)\,dx$$

$$= cx - x^2$$

$$y^{-1} = cx - x^2 \Rightarrow y = \frac{1}{cx - x^2}$$

Ex $y' + \frac{1}{x}y = 3x^2y^3 \quad (y' + py = ry^a,\ a = 3)$

Let $u = y^{1-3} = y^{-2}$

$$u' = -2y^{-3}y' = -2y^{-3}\left(3x^2y^3 - \frac{1}{x}y\right)$$

$$= -6x^2 + \frac{2}{x}u$$

$$\Rightarrow u' - \frac{2}{x}u = -6x^2$$

$$\Rightarrow u = ce^{-h} + e^{-h}\int e^{h} r\,dx \quad h = \int p\,dx = \int -\frac{2}{x}\,dx = -2\ln x$$

$$e^{-h} = x^2 \quad e^{h} = x^{-2}$$

$$\Rightarrow u = cx^2 + x^2\int x^{-2}(-6x^2)\,dx$$

$$\Rightarrow u = cx^2 - 6x^3 \Rightarrow y = u^{-\frac{1}{2}}$$

$$y = \frac{1}{\sqrt{cx^2 - 6x^3}}$$