Discrete Mathematics Homework 3

Section 5-4

10 . Give a recursive algorithm for finding the maximum of a finite set of integers, making use of the fact that the maximum of n integers is the larger of the last integer in the list and the maximum of the first $n-1$ integers in the list.

> ***Let S_n be a finite set of n integers, and a_i be the ith integer of the set.***
> ***Basis Step:*** $\max(a,b) = \begin{cases} a \text{ , if } a \geq b \\ b \text{ , if } b > a \end{cases}$
> ***Recursive Step:*** $\max(S_n) = \max(S_{n>1}, a_n)$

44 . Use a merge sort to sort $4, 3, 2, 5, 1, 8, 7, 6$ into increasing order. Show all the steps used by the algorithm.

> $[4, 3, 2, 5, 1, 8, 7, 6]$
> $[4, 3, 2, 5][1, 8, 7, 6]$
> $[4, 3][2, 5][1, 8][7, 6]$
> $[3, 4][2, 5][1, 8][6, 7]$
> $[2, 3, 4, 5][1, 6, 7, 8]$
> $[1, 2, 3, 4, 5, 6, 7, 8]$

Section 6-1

8 . How many different three-letter initials with none of the letters repeated can people have?

> $26 \times 25 \times 24 \times 2 = 31200$

14 . How many bit strings of length n, where n is a positive integer, start and end with 1s?

> $\begin{cases} 2^{n-2} & , n > 1 \\ 1 & , n = 1 \end{cases}$

16 . How many strings are there of four lowercase letters that have the letter x in them?

> $4 \times 26^3 = 70304$

24 . How many positive integers between 1000 and 9999 inclusive
(a) are divisible by 9?
(b) are even?
(c) have distinct digits?
(d) are not divisible by 3?
(e) are divisible by 5 or 7?
(f) are not divisible by either 5 or 7?
(g) are divisible by 5 but not by 7?
(h) are divisible by 5 and 7?

> *(a)* 1000

> *(b)* 4500

> *(c)* 4536

> *(d)* 6000

> *(e)* 2829

> *(f)* 6171

> *(g)* 1543

> *(h)* 257

32 . How many strings of eight uppercase English leters are there
(a) if letters can be repeated?
(b) if no letter can be repeated?
(c) that strat with X, if letters can be repeated?
(d) that strat with X, if no letter can be repeated?
(e) that start and end with X, if letters can be repeated?
(f) that start with the letters BO (in that order), if lettes can be repeated?
(g) that start and end with the letters BO (in that order), if lettes can be repeated?

> *(a)*
>
> > $26^8 = 208827064576$

> *(b)*
>
> > $26 \times 25 \times \cdots \times 19 = 62990928000$

> *(c)*

> > $26^7 = 8031810176$

> *(d)*
>
> > $25 \times 24 \times \cdots \times 19 = 2422728000$

> *(e)*
>
> > $26^6 = 308915776$

> *(f)*
>
> > $26^6 = 308915776$

> *(g)*
>
> > $26^4 = 456976$

38 . How many partial functions (see Section 2.3) are there from a set with five elements to sets with each of these number of elements?
(a) 1
(b) 2
(c) 5
(d) 9

> *(a)*
>
> > $2^5 = 32$

> *(b)*
>
> > $3^5 = 243$

> *(c)*
>
> > $6^5 = 7776$

> *(d)*
>
> > $10^5 = 100,000$

46 . In how many ways can a photographer at a wedding arrange 6 **people in a row from a group of** 10 **people, where the bride and the groom are among these** 10 **people, if**
(a) the bride must be in the picture?
(b) both the bride and groom must be in the picture?

(c) exactly one of the bride and the groom is in the picture?

> *(a)*
>
> > $\binom{9}{5} \times 6! = 90720$

> *(b)*
>
> > $\binom{8}{4} \times 6! = 50400$

> *(c)*
>
> > $\binom{8}{5} \times 6! \times 2 = 80640$

58 . The International Telecommunications Union (ITU) specifies that a telephone number must consist of a country code with between 1 and 3 digits, except that the code 0 is not available for use as a country code, followed by a number with at most 15 digits. How many available possible telephone numbers are there that satisfy these restrictions?

> $(9 + 9^2 + 9^3) \times \sum_{k=1}^{15} 10^k = 909,999,999,999,999,090$

Section 6-2

4 . A bowl contains 10 red balls and 10 blue balls.A woman selects balls at random without looking at them.
(a) How many balls must she select to be sure of having at least three balls of the same color?
(b) How many balls must she select to be sure of having at least three blue balls?

> *(a)*
>
> > 5

> *(b)*
>
> > 13

12 . How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \ mod \ 5 = a_2 \ mod \ 5$ and $b_1 \ mod \ 5 = b_2 \ mod \ 5$?

> $5 \times 5 + 1 = 26$

16 . How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

> 5

18 . Suppose that there are nine students in a discrete mathematics class at a small college.
(a) Show that the class must have at least five male students or at least five female students.
(b) Show that the class must have at least three male students or at least seven female students.

> *(a)*
>
> > ***If the number of male students is less than five, then the number of female students will be greater than or equal to five.***

> *(b)*
>
> > ***Again, if the number of male students in the class is less then three, then the rest of them will be female students, which is more than or equal to seven people.***

22 . Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

> ***Assume the opposite, that is for all position i, the length of the longest following increasing or decreasing subsequence is less than or equal to 10. That gives us $10 \times 10 = 100$ posibilities of combinations, and there's 101 positions for them. By Pigeonhole principle, there must be at least 2 positions with the same length of longest increasing or decreasing subsequences. But notice that the two people in that two positions have different heights, so they must not have the same length of the longest following increasing or decreasing subsequence, because if the former is taller than the other, then his length of the longest following decreasing subsequence will be one more then the other's. This is a contradiction. So there must be 11 people in the order they are standing in the line with the heights that are either increasing or decreasing if there are 101 people of different heights standing in a line.***

34 . Assuming that no one has more than $1,000,000$ hairs on the head of any person and that the population of New York City was $8,008,278$ in 2010, show there had to be at least nine people in NewYork City in 2010 with the same number of hairs on their heads.

> $8,008,278$ ***People are distributed to $1,000,001$ groups corresponding to the number of their hairs on their heads ranging from 0 to $1,000,000$. There must have $\lceil 8,008,278/1,000,001 \rceil = 9$ people in the same group, that is, having the same amount of hairs.***

38 . Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.

> *Let C_i, P_i denote the ith printer, computer. Firstly, we connect (C_i, P_i) , $i = 1, 2, 3, 4$, then we connect each of the rest computers to all the four printers, in which case we'll have $4 + 4 \times 4 = 20$ cables.*
> *Now, if some of the cables are unplugged, les's say (C_i, P_j). We know that i can't be $1, 2, 3, 4$, because that will make C_i connected to no printer. If we unplugged (C_i, P_j) , $i = 5, 6, 7, 8$, $j = 1, 2, 3, 4$, then the choose of C_i and C_j for the rest of js can't access four different printers.*

Section 6-3

12 . How many bit strings of length 12 contain
(a) exactly three 1s?
(b) at most three 1s?
(c) at least three 1s?
(d) an equal number of 0s and 1s?

> *(a)*
>
> > $\frac{12!}{3! \times 9!} = 220$

> *(b)*
>
> > $\frac{12!}{3! \times 9!} + \frac{12!}{2! \times 10!} + \frac{12!}{1! \times 11!} + 1 = 220 + 66 + 12 + 1 = 299$

> *(c)*
>
> > $12! - 299 = 479001301$

> *(d)*
>
> > $\frac{12!}{6! \times 6!} = 924$

18 . A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
(a) are there in total?
(b) contain exactly three heads?
(c) contain at least three heads?
(d) contain the same number of heads and tails?

> *(a)*
>
> > $2^8 = 256$

> *(b)*

> > $\frac{8!}{3!\times 5!} = 56$

> *(c)*
>
> > $256 - \frac{8!}{2!\times 6!} - \frac{8!}{1!\times 7!} - 1 = 256 - 28 - 8 - 1 = 219$

> *(d)*
>
> > $\frac{8!}{4!\times 4!} \times 2 = 140$

22 . How many permutations of the letters "ABCDEFGH" contain

(a) the string "ED"?

(b) the string "CDE"?

(c) the strings "BA" and "FGH"?

(d) the strings "AB", "DE", and "GH"?

(e) the strings "CAB" and "BED"?

(f) the strings "BCA" and "ABF"?

> *(a)*
>
> > $7! = 5040$

> *(b)*
>
> > $6! = 720$

> *(c)*
>
> > $5! = 120$

> *(d)*
>
> > $5! = 120$

> *(e)*
>
> > $3! = 6$

> *(f)*
>
> > 0

26 . Thirteen people on a softball team show up for a game.
(a) How many ways are there to choose 10 players to take the field?
(b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
(c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

> *(a)*
>
> > $\binom{13}{10} = \frac{13!}{10! \times 3!} = 286$

> *(b)*
>
> > $\binom{13}{10} \times 10! = \frac{13!}{3!} = 1037836800$

> *(c)*
>
> > $\binom{13}{10} - 1 = 285$

30 . Seven women and nine men are on the faculty in the mathematics department at a school.
(a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
(b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

> *(a)*
>
> > $\binom{12}{5} - \binom{9}{5} = 666$

> *(b)*
>
> > $\binom{12}{5} - \binom{9}{5} - \binom{7}{5} = 645$

42 . Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

> $f(r, n) = \frac{(r-1)!}{2(r-n)!}$

Section 6-4

8 . What is the coefficient of $x^8 y^9$ in the expansion of $(3x + 2y)^{17}$?

> $\frac{17!}{8! \times 9!} \times 3^8 \times 2^9 = 81662929920$

20 . Suppose that k and n are integers with $1 \leq k < n$. Prove the hexagon identity $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$, which relates terms in Pascal's triangle that form a hexagon.

> $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{n!}{(k+1)!(n-k-1)!} \times \frac{(n+1)!}{k!(n-k+1)!}$
>
> $= \frac{(n-1)!n!(n+1)!}{(k-1)!k!(k+1)!(n-k-1)!(n-k)!(n-k+1)!} = \frac{(n-1)!}{k!(nk-1)!} \times \frac{n!}{(k-1)!(n-k+1)!} \times \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$

22 . Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and$k$ are nonnegative integers with $r \leq n$ and $k \leq r$,

(a) using a combinatorial argument.

(b) using an argument based on the formula for the number of r-combinations of a set with n elements.

> *(a)*
>
> > $\binom{n}{r}\binom{r}{k}$: *Choose a subset A of r elements from the set of n elements. Then choose k elements from A.*
> > $\binom{n}{k}\binom{n-k}{r-k}$: *Choose k elements from the set of n elenemts, and then choose $r-k$ elements form the rest $n-k$ elements to form the set A.*

> *(b)*
>
> > $\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$
> > $\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}$

28 . Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(a) using a combinatorial argument.

(b) by algebraic manipulation.

> *(a)*
>
> > *To choose 2 elements from the set of $2n$ elements, the right hand side describes the process that we can first spread the set in half to get subsets $A: \{1, 2, \ldots, n\}, B: \{n+1, n+2, \ldots, 2n\}$ each with n elements. then those two chosen elements are either both from A or both from B, or one from A and the other from B.*
> > $\begin{cases} \binom{n}{2} & \text{both from set A} \\ \binom{n}{2} & \text{both from set B} \\ n \times n & \text{from different sets} \end{cases} = 2\binom{n}{2} + n^2$

> *(b)*
>
> > $\binom{2n}{2} = \frac{2n \times (2n-1)}{2} = 2n^2 - n$
> > $2\binom{n}{2} + n^2 = 2 \times \frac{n(n-1)}{2} + n^2 = 2n^2 - n$

38 . Give a combinatorial proof that if n is a positive integer then $\sum\limits_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$.

[Hint: Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$.]

> $\sum\limits_{k=0}^{n} k^2\binom{n}{k}$: ***Choose a subset A of k elements form the set of n elements, then choose arbitrary element from A twice.***
> $n(n-1)2^{n-2} + n2^{n-1}$: ***The two terms of it represent two cases: the chosen two elements are the same, and are different. The first term is saying that choose two different elements, and then choose any subset of the set of the rest $(n-2)$ elements to form A, and the second one is saying that choose it, and then choose any subset of the set of the rest $(n-1)$ elements to form A.***

Section 6-5

10 . A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose
(a) a dozen croissants?
(b) three dozen croissants?
(c) two dozen croissants with at least two of each kind?
(d) two dozen croissants with no more than two broccoli croissants?
(e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
(f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?

> *(a)*
>
> > $\binom{17}{5} = \frac{17!}{5!\times 12!} = 6188$

> *(b)*
>
> > $\binom{41}{5} = \frac{41!}{5!\times 36!} = 749398$

> *(c)*
>
> > $\binom{17}{5} = \frac{17!}{5!\times 12!} = 6188$

> *(d)*
>
> > $\binom{28}{4} + \binom{27}{4} + \binom{26}{4} = 52975$

> *(e)*
>
> > $\binom{20}{4} = 4845$

> *(f)*
>
> > $\binom{19}{4} + \binom{18}{4} + \binom{17}{4} + \binom{16}{4} = 11136$

16 . How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$, where x_i , $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that
(a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
(b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?
(c) $x_1 \leq 5$?
(d) $x_1 < 8$ and $x_2 > 8$?

> *(a)*
>
> > $\binom{22}{5} = 26334$

> *(b)*
>
> > $\binom{13}{5} = 1287$

> *(c)*
>
> > $\sum_{k=28}^{33} \binom{k}{4} = \binom{34}{5} - \binom{28}{5} = 179976$

> *(d)*
>
> > $\sum_{k=17}^{24} \binom{k}{4} = \binom{25}{5} - \binom{17}{5} = 46942$

20 . How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11$, where x_1, x_2, and x_3 are nonnegative integers? [Hint: Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

> $\binom{14}{3} = 364$

24 . How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

> $\binom{15}{1}\binom{14}{2}\binom{12}{3}\binom{9}{4}\binom{5}{5} = 37837800$

34 . How many strings with five or more characters can be formed from the letters in "SEERESS"?

$$\begin{cases} 5: \begin{cases} 2S3E: \frac{5!}{2!\times 3!} = 10 \\ 2E3S: \frac{5!}{2!\times 3!} = 10 \\ 1S1R3E: \frac{5!}{3!} = 20 \\ 1E1R3S: \frac{5!}{3!} = 20 \end{cases} \\ \\ 6: \begin{cases} 3E3S: \frac{6!}{3!\times 3!} = 20 \\ 2S3E1R: \frac{6!}{2!\times 3!} = 60 \\ 2E3S1R: \frac{6!}{2!\times 3!} = 60 \end{cases} \\ \\ 7: 3E3S1R = \frac{7!}{3!\times 3!} = 140 \end{cases}$$

Total: 340

44 . In how many ways can a dozen books be placed on four distinguishable shelves
(a) if the books are indistinguishable copies of the same title?
(b) if no two books are the same, and the positions of the books on the shelves matter? [Hint: Break this into 12asks, placing each book separately. Start with the sequence $1, 2, 3, 4$ to represent the shelves. Represent the books by $b_i, i = 1, 2, \ldots, 12$. Place b_1 to the right of one of the terms in $1, 2, 3, 4$. Then successively place $b_2, b_3, \ldots,$ and b_{12}.]

> *(a)*
>
> > $\binom{15}{3} = \frac{15!}{3!\times 12!} = 455$

> *(b)*
>
> > $\frac{15!}{3!} = 217945728000$

54 . How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

> $$\begin{aligned} 5 &= 5+0+0 \\ &= 4+1+0 \\ &= 3+2+0 \\ &= 3+1+1 \\ &= 2+2+1 \end{aligned}$$

58 . How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if
(a) both the balls and boxes are labeled?
(b) the balls are labeled, but the boxes are unlabeled?
(c) the balls are unlabeled, but the boxes are labeled?
(d) both the balls and boxes are unlabeled?

> *(a)*

> > $\binom{7}{5} \times 5! = 21 \times 120 = 2520$

> *(b)*
>
> > 1

> *(c)*
>
> > $\binom{7}{5} = 21$

> *(d)*
>
> > 1

Section 6.6

8 . Use Algorithm 2 to list all the subsets of the set $\{1,2,3,4\}$.

> $$\begin{cases} 0000: & \emptyset \\ 0001: & \{4\} \\ 0010: & \{3\} \\ 0011: & \{3,4\} \\ 0100: & \{2\} \\ 0101: & \{2,4\} \\ 0110: & \{2,3\} \\ 0111: & \{2,3,4\} \\ 1000: & \{1\} \\ 1001: & \{1,4\} \\ 1010: & \{1,3\} \\ 1011: & \{1,3,4\} \\ 1100: & \{1,2\} \\ 1101: & \{1,2,4\} \\ 1110: & \{1,2,3\} \\ 1111: & \{1,2,3,4\} \end{cases}$$

14 . Find the Canter digits $a_1, a_2, \ldots, a_{n-1}$ that correspond to these permutations.
(a) 246531
(b) 12345
(c) 654321

> *(a)*
>
> > $(1,1,2,2,3)$

> *(b)*
>
> > $(0,0,0,0)$

> *(c)*

> > $(1, 2, 3, 4, 5)$

Section 8.1

4 . A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

> $a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}, \forall n \geq 100$

8 .
(a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.
(b) What are the initial conditions?
(c) How many bit strings of length seven contain three consecutive 0s?

> *(a)*
>
> > $a_n = 2a_{n-1} + 2^{n-3}, \forall n \geq 3$

> *(b)*
>
> > $a_0 = 0, a_1 = 0, a_2 = 0$

> *(c)*
>
> > 31

12 .
(a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.
(b) What are the initial conditions?
(c) In many ways can this person climb a flight of eight stairs?

> *(a)*
>
> > $a_n = a_{n-1} + a_{n-2} + a_{n-3}, \forall n \geq 4$

> *(b)*
>
> > $a_1 = 1, a_2 = 2, a_3 = 3$

> *(c)*

> > 81

26 .
(a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]
(b) What are the initial conditions for the recurrence relation in part (a)?
(c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

> *(a)*
>
> > $a_n = a_{n-1} + 2a_{n-2}, \forall n \geq 3$

> *(b)*
>
> > $a_1 = 1, a_2 = 2$

> *(c)*
>
> > 2584

Section 8.2

4 . Solve these recurrence relations together with the initial conditions given.
(a) $a_n = a_{n-1} + 6a_{n-2}$ **for** $n \geq 2, a_0 = 3, a_1 = 6$
(b) $a_n = 7a_{n-1} - 10a_{n-2}$ **for** $n \geq 2, a_0 = 2, a_1 = 1$
(c) $a_n = 6a_{n-1} - 8a_{n-2}$ **for** $n \geq 2, a_0 = 4, a_1 = 10$
(d) $a_n = 2a_{n-1} - a_{n-2}$ **for** $n \geq 2, a_0 = 4, a_1 = 1$
(e) $a_n = a_{n-2}$ **for** $n \geq 2, a_0 = 5, a_1 = -1$
(f) $a_n = -6a_{n-1} - 9a_{n-2}$ **for** $n \geq 2, a_0 = 3, a_1 = -3$
(g) $a_{n+2} = -4a_{n+1} + 5a_n$ **for** $n \geq 0, a_0 = 2, a_1 = 8$

> *(a)*
>
> > $r^2 - r - 6 = 0 \to r = 3, -2 \to a_i = a(3)^i + b(-2)^i$
> > $\begin{cases} a_0 = a + b = 3 \\ a_1 = 3a - 2b = 6 \end{cases} \to (a, b) = (\frac{12}{5}, \frac{3}{5}) \to a_n = \frac{12}{5} \cdot (3)^n + \frac{3}{5} \cdot (-2)^n$

> *(b)*
>
> > $r^2 - 7r - 10 = 0 \to r = 2, 5 \to a_i = a(2)^i + b(5)^i$
> > $\begin{cases} a_0 = a + b = 2 \\ a_1 = 2a + 5b = 1 \end{cases} \to (a, b) = (3, -1) \to a_n = 3 \cdot (2)^n - (5)^n$

> (c)
>
> > $r^2-6r+8=0 \to r=2,4 \to a_i=a(2)^i+b(4)^i$
> > $\begin{cases} a_0=a+b=4 \\ a_1=2a+4b=10 \end{cases} \to (a,b)=(3,1) \to a_n=3\cdot(2)^n+(4)^n$

> (d)
>
> > $r^2-2r+1=0 \to r=1 \to a_i=a(1)^i+ib(1)^i$
> > $\begin{cases} a_0=a=4 \\ a_1=a+b=1 \end{cases} \to (a,b)=(4,-3) \to a_n=4-3n$

> (e)
>
> > $r^2-1=0 \to r=-1,1 \to a_i=a(1)^i+b(-1)^i$
> > $\begin{cases} a_0=a+b=5 \\ a_1=a-b=-1 \end{cases} \to (a,b)=(2,3) \to a_n=2\cdot(1)^n+3(-1)^n$

> (f)
>
> > $r^2+6r+9=0 \to r=-3 \to a_i=a(-3)^i+ib(-3)^i$
> > $\begin{cases} a_0=a=3 \\ a_1=-3a-3b=-3 \end{cases} \to (a,b)=(3,-2) \to a_n=3\cdot(-3)^n-2n(-3)^n$

> (g)
>
> > $r^2+4r-5=0 \to r=-5,1 \to a_i=a(-5)^i+b(1)^i$
> > $\begin{cases} a_0=a+b=2 \\ a_1=-5a+b=8 \end{cases} \to (a,b)=(-1,3) \to a_n=-1\cdot(-5)^n+3(1)^n$

8 . A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.
(a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.
(b) Find L_n if $100,000$ lobsters were caught in year 1 and $300,000$ were caught in year 2.

> (a)
>
> > $L_n=\frac{1}{2}l_{n-1}+\frac{1}{2}L_{n-2}, \forall n>2$

> (b)
>
> > $2r^2-r-1=0 \to r=-\frac{1}{2},1 \to a_i=a(-\frac{1}{2})^i+b(1)^i$
> > $\begin{cases} a_0=a+b=100000 \\ a_1=-\frac{1}{2}a+b=300000 \end{cases} \to (a,b)=(-\frac{400000}{3},\frac{700000}{3}) \to a_n=-\frac{400000}{3}\cdot(-\frac{1}{2})^n+\frac{700000}{3}\cdot$

10 . Prove Theorem 2.

> $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
> *assume that* $a_n = kr^n + mnr^n$
> r *is the multiple root of* $f(x) = r^2 - c_1 r - c_2 = 0$
> $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
> $= c_1(kr^{n-1} + m(n-1)r^{n-1}) + c_2(kr^{n-2} + m(n-2)r^{n-2})$
> $= k(c_1 r^{n-1} + c_2 r^{n-2}) + m(c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2}) \cdots (1)$
> $f(r) = 0 \rightarrow r^{n-2} f(r) = 0$
> $r^n = c_1 r^{n-1} + c_2 r^{n-2} \cdots (2)$
> *because* r *is the multiple root of* $f(x)$
> $f'(r) = 0 \wedge f(r) = 0 \rightarrow q(x) = (x^{n-2} f(x))', q(r) = 0$
> $q(r) = nr^{n-1} - c_1(n-1)r^{n-2} - c_2(n-2)r^{n-3} = 0$
> $nr^n = c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2} \cdots (3)$
> *by* $(1), (2), (3)$
> $k(c_1 r^{n-1} + c_2 r^{n-2}) + m(c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2}) = kr^n + mnr^n$
> *so* $a_n = kr^n + mnr^n$

14 . Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3,\ a_1 = 2, a_2 = 6$, and $a_3 = 8$.

> $r^4 - 5r^2 + 4 = 0 \rightarrow r = 1, -1, 2, -2 \rightarrow a_i = a(1)^i + b(-1)^i + c(2)^i + d(-2)^i$
> $$\begin{cases} a_0 = a + b + c + d = 3 \\ a_1 = a - b + 2c - 2d = 2 \\ a_2 = a + b + 4c + 4d = 6 \\ a_3 = a - b + 8c - 8d = 8 \end{cases} \rightarrow (a, b, c, d) = (1, 1, 2, 0) \rightarrow a_n = 1(1)^n + 1(-1)^n + 2(2)^n$$

24 . Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.
(a) Show that $a_n = n2^n$ is a solution of this recurrence relation.
(b) Use Theorem 5 to find all solutions of this recurrence relation.
(c) Find the solution with $a_0 = 2$.

> *(a)*
>
> > $a_{n-1} = (n-1)2^{n-1} \rightarrow 2a_{n-1} + 2^n = 2(n-1)2^{n-1} + 2^n = n2^n = a_n$

> *(b)*
>
> > $a_n = a_n^{(h)} + a_n^{(p)}$
> > $a_n^{(h)} = r^n$
> > $r = 2, a_n^{(h)} = \alpha 2^n$
> > $a_n^{(p)} = n2^n$
> > $2(n-1)2^{n-1} + 2^n = (n-1)2^n + 2^n = n2^n$
> > $a_n = \alpha 2^n + n2^n$

> *(c)*
>
> > $a_0 = \alpha = 2$
> > $a_n = 2^{n+1} + n2^n$

30 .

(a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.

(b) Find the solution of this recurrence relation with $a_1 = 56$ **and** $a_2 = 278$.

> *(a)*
>
> > $a_n = a_n^{(h)} + a_n^{(p)}$
> > $let\ a_n^{(h)} = r^n$
> > $r^2 + 5r + 6 = 0$
> > $r = -2, -3$
> > $a_n^{(h)} = i(-2)^n + j(-3)^n$
> > $a_n^{(p)} = c4^n$
> > $c4^n = -5c4^{n-1} - 6c4^{n-2} + 42 \cdot 4^n$
> > $42c = 42 \cdot 16$
> > $c = 16$
> > $a_n^{(p)} = 4^{n+2}$
> > $a_n = i(-2)^n + j(-3)^n + 4^{n+2}$

> *(b)*
>
> > $a_1 = -2i - 3j + 64 = 56$
> > $a_2 = 4i + 9j + 256 = 278$
> > $i = 1, j = 2$
> > $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$