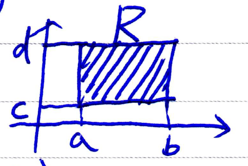
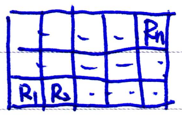
Chapter 15. Multiple Integrals

§15-1. Double and iterated integrals over rectangles.

Notations:

a rectangle $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$
$\equiv [a,b] \times [c,d]$.





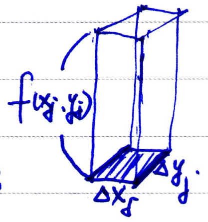
a partition $P = \{R_1, \cdots, R_n\} \in \mathcal{P}([a,b]\times[c,d])$.

Let ΔA_j = area of R_j, $\forall j = 1, \cdots, n$.
$= \Delta x_j \Delta y_j$.

$\|P\| = \max\{\Delta x_1, \cdots, \Delta x_n, \Delta y_1, \cdots, \Delta y_n\}$.

If $\|P\| \to 0 \Rightarrow n \to \infty$, $\Delta A_j \to 0$, $\forall j$.

Now, $f(x,y) : [a,b]\times[c,d] \to \mathbb{R}$, conti. positive



Let $S_n = \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j$, where $(x_j, y_j) \in R_j$, $\forall j$, Riemann sum.

$$\lim_{\|P\|\to 0} \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\Delta A_j$$

$$\equiv \iint_R f(x,y)\,dA \equiv \iint_R f(x,y)\,dx\,dy \quad \text{if the limit exists.}$$

(the double integral of f over R)

In this case, f is integrable over R.

Note

(1) f is conti. on $R \Rightarrow f$ is integrable

(2) $\#\{(x,y)\in R \mid f \text{ is not conti. at } (x,y)\} < \infty \Rightarrow f$ is integrable.

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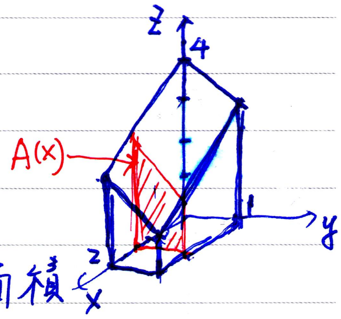
(3) $\iint_R f(x,y)\,dA$ = Volume.

Iterated integrals.



The plane $z = 4 - x - y$ over $R = [0,2]\times[0,1]$

For each $x \in [0,2]$, let $A(x) = \int_0^1 4-x-y\,dy$ 截面積
$= 4 - x - \frac{1}{2} = \frac{7}{2} - x$

$\Rightarrow$ Volume $= \int_0^2 A(x)\,dx = \int_0^2 (\frac{7}{2} - x)\,dx = 7 - 2 = 5 \equiv \int_0^2\int_0^1 (4-x-y)\,dy\,dx$

Similarly, for each $y \in [0,1]$, let $A(y) = \int_0^2 4-x-y\,dx = 8 - 2 - 2y = 6 - 2y$

$\Rightarrow$ Volume $= \int_0^1 A(y)\,dy = \int_0^1 6-2y\,dy = 6 - 1 = 5 \equiv \int_0^1\int_0^2 (4-x-y)\,dx\,dy$.

$f(x,y) : [a,b]\times[c,d] \to \mathbb{R}$ is conti.

$\int_a^b\int_c^d f(x,y)\,dy\,dx$, $\int_c^d\int_a^b f(x,y)\,dx\,dy$: iterated integral of f over $[a,b]\times[c,d]$.

Fubini's theorem (First form).

$R=[a,b]\times[c,d]$, $f: R\to\mathbb{R}$ is conti. on R.

$$\Rightarrow \iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

Example 1.

$R=[0,2]\times[-1,1]$. Find $\iint_R (100-6x^2y)\,dA$.

Sol:

$$\int_{-1}^1\int_0^2 100-6x^2y\ dx\,dy$$

$$=\int_{-1}^1 100x-2x^3y\Big|_0^2\,dy = \int_{-1}^1 200-16y\ dy$$

$$= 200y-8y^2\Big|_{-1}^1 = 400$$

Example 2.

Find the volume of the region bdd above by $z=10+x^2+3y^2$ and below by $R=[0,1]\times[0,2]$

Sol:

$$\int_0^2\int_0^1 10+x^2+3y^2\ dx\,dy = \int_0^2 10+\tfrac{1}{3}+3y^2\,dy$$

$$=\int_0^2 10x+\tfrac{1}{3}x^3+3y^2x\Big|_0^1\,dy = 10y+\tfrac{1}{3}y+y^3\Big|_0^2 = 20+\tfrac{2}{3}+8=\tfrac{86}{3}$$

pf of Fubini's Thm:

Let $P_{mn}=\{R_{11},\cdots,R_{mn}\}\in\mathcal{P}(R)$ s.t. $\Delta A_{ij} = \underbrace{\frac{b-a}{n}}_{\Delta x}\times\underbrace{\frac{d-c}{n}}_{\Delta y}$

$\because f$ is integrable over R.

$$\Rightarrow \iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{j=1}^m\sum_{i=1}^n f(x_i,y_j)\Delta x\Delta y = \lim_{m\to\infty}\sum_{j=1}^m\Big(\lim_{n\to\infty}\sum_{i=1}^n f(x_i,y_j)\Delta x\Big)\Delta y$$

$$=\lim_{m\to\infty}\sum_{j=1}^m\Big(\underbrace{\int_a^b f(x,y_j)\,dx}_{g(y_j)}\Big)\Delta y = \int_c^d g(y)\,dy = \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

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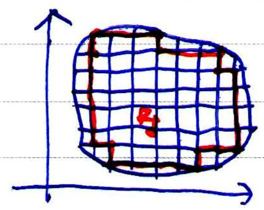
§15-2. Double integrals over general regions.



$R \subseteq \mathbb{R}^2$, a closed bdd region

$P = \{R_1, \cdots, R_n\} \in \mathcal{P}(R)$, a partition, where $R_j \subseteq R$ is a rectangle, $\forall j$.

ΔA_j = area R_j, $(x_j, y_j) \in R_j$, $\forall j = 1, \cdots, n$.

$S_n = \sum_{j=1}^{n} f(x_j, y_j) \Delta A_j$ is a Riemann sum.

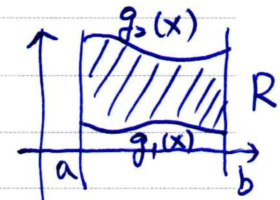
$$\lim_{\|P\| \to 0} \sum_{j=1}^{n} f(x_j, y_j) \Delta A_j \equiv \iint_R f(x,y)\, dA$$

is the double integral of $f(x,y)$ over R.

Volumes

$f(x,y) : R \to \mathbb{R}$, positive, conti,

Case 1: $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

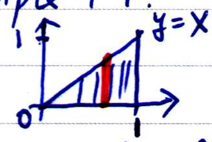


For each $x \in [a,b]$,

The cross-sectional area $A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy$

$$\Rightarrow V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Example 1-1.

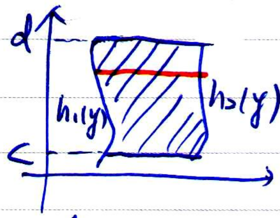
R : 

$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$.

$f(x,y) = 3 - x - y$.

$$\Rightarrow V = \int_0^1 \int_0^x 3 - x - y\, dy\, dx = \int_0^1 3x - x^2 - \frac{x^2}{2}\, dx = \frac{3}{2}x^2 - \frac{1}{2}x^3 \Big|_0^1 = 1$$

Case 2.



$f(x,y) : R \to \mathbb{R}$, positive, conti.

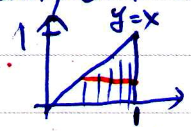
$R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

For each $y \in [c,d]$

The cross-sectional area $A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx$

$$\Rightarrow V = \int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example 1-2.

R : 

$R = \{(x,y) : 0 \le y \le 1,\ y \le x \le 1\}$

$f(x,y) = 3 - x - y$

$\Rightarrow V =$

$$\int_0^1 \int_0^x 3-x-y \; dy\, dx = \int_0^1 3y - xy - \tfrac{1}{2}y^2 \Big|_0^x dx$$
$$= \int_0^1 3x - x^2 - \tfrac{1}{2}x^2 \, dx = \tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{6}x^3 \Big|_0^1$$
$$= \tfrac{3}{2} - \tfrac{1}{3} - \tfrac{1}{6} = \tfrac{6}{6}$$
$$= 1$$

Theorem (Fubini's theorem. Ver. 2).

$f(x,y) : R \to \mathbb{R}$ is conti.

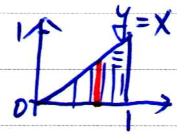
1. If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, g_1, g_2, are conti. on $[a,b]$.
$$\Rightarrow \iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

2. If $R = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, h_1, h_2 conti. on $[c,d]$.
$$\Rightarrow \iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Note.

"dxdy" or "dydx"? 容易積的先積.

Example 2.

Calculate $\displaystyle\iint_R \frac{\sin x}{x}\,dA$, where R : 

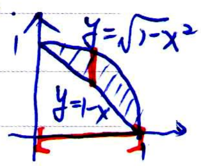
Sol:

$$\int_0^1 \int_0^x \frac{\sin x}{x} \; dy\, dx$$
$$= \int_0^1 \frac{\sin x}{x} y \Big|_0^x dx = \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = -(\cos 1 - 1)$$
$$= 1 - \cos 1$$

Note.

Step 1: Sketch the region and label the bdry curves.

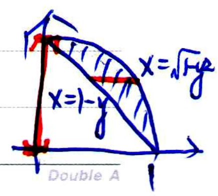
Case 1: 先積 y i.e. $\int\left(\int f\,dy\right)dx$



Step2: Take a vertical line to find the y-limits

Step3: Find the x-limits.

Case 2: 先積 x i.e. $\int\left(\int f\,dx\right)dy$



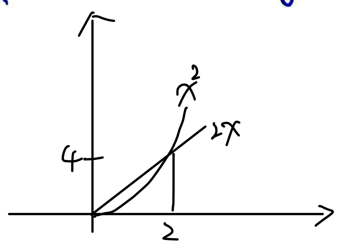
Step2: Take a horizontal line to find the x-limit

Step3: Find the y-limits

Example 3. $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$

Sketch the region, and write the order of integration reversed.

Sol:



$$\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} 4x+2 \; dx\,dy$$

$$\int_0^4 2x^2+2x\Big|_{\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 2\left(y-\tfrac{1}{4}y^2\right)+2\sqrt{y}-y\;dy$$

$$= \int_0^4 -\tfrac{1}{2}y^2+y+2\sqrt{y}\;dy =$$

Properties of double integrals:

$f(x,y),\ g(x,y): R \to \mathbb{R}$, conti., where $R \subseteq \mathbb{R}^2$ is a bdd region.

1. $\iint_R c\cdot f(x,y)\,dA = c\cdot\iint_R f(x,y)\,dA$, $\forall c\in\mathbb{R}$

2. $\iint_R (f(x,y)\pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$.

3. If $f\geq 0 \Rightarrow \iint_R f(x,y)\,dA \geq 0$

   If $f\geq g \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

4. If $R=R_1\cup R_2$, $R_1\cap R_2=\phi$

   $\Rightarrow \iint_R f(x,y)\,dA = \int_{R_1} f(x,y)\,dA + \int_{R_2} f(x,y)\,dA$.

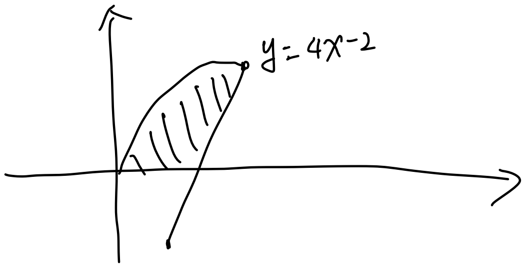
pf.

By def. (Riemann sum). i.e. $\sum_{j=1}^{n} c\cdot f(x_j,y_j)\Delta A_j = c\cdot\sum_{j=1}^{n} f(x_j,y_j)\Delta A_j$.

Example 4.

Find the volume of the solid that lies beneath the surface $z=16-x^2-y^2$ and above the region R bdd by $y=2\sqrt{x}$, $y=4x-2$ and the x-axis.

Sol: $2\sqrt{x}=4x-2$ $\quad x=4x^2-4x+1$ $\quad 4x^2-5x+1=0$ $\quad (4x-1)(x-1)=0$ $\quad x=1 \vee \frac{1}{4}$ $\quad (1,2)$ $\quad (\frac{1}{4},-1)$



$$\int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} 16-x^2-y^2 \; dx\,dy$$

$$\int_0^2 \left(16x-\tfrac{1}{3}x^3-y^2x\right)\Big|_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy = \int_0^2$$

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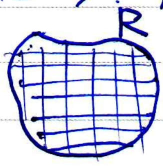
Note: If f 有正有負，積出來不是体積，會抵消.

15-3. Area by double integration.



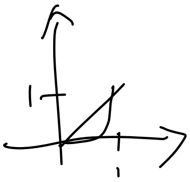
$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

If $f(x,y) \equiv 1 \Rightarrow S_n = \sum_{k=1}^{n} \Delta A_k$.

$$\Rightarrow \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A_k = A = \iint_R 1\, dA$$

Def.
The area of a closed bdd plane region R is $\iint_R dA$.

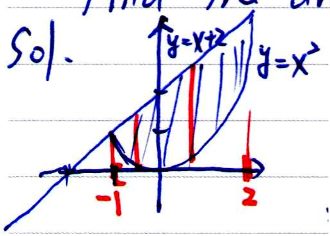
Example 1.
Find the area of R bdd by $y=x$, $y=x^2$ in the first quadrant.
Sol:



$$\int_0^1 \int_{x^2}^{x} 1 \, dy\, dx$$

$$\int_0^1 x - x^2 \, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{6}$$

Example 2.
Find the area of R enclosed by $y=x^2$ and $y=x+2$.
Sol.

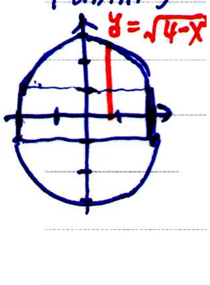


$x^2 = x+2 \quad (x-2)(x+1)=0 \quad x = 2, -1$

$$\int_{-1}^{2} \int_{x^2}^{x+2} 1 \, dy\, dx$$

$$= \int_{-1}^{2} x+2-x^2 \, dx = \frac{1}{2}x^2+2x-\frac{1}{3}x^3\Big|_{-1}^{2} = \frac{3}{2}+6-\frac{1}{3}\times 9 =$$

Example 3.
Find the area of $R = \{(x,y) : -2 \le x \le 2,\ -1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2}\}$
Sol:
Fubini's :



$$\int_{-2}^{2} \int_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}} 1 \, dy\, dx$$

$$= 2\int_{-2}^{2} 1+\sqrt{4-x^2} \, dx = 2\times 4 + 2\times 2\pi = 8+4\pi$$

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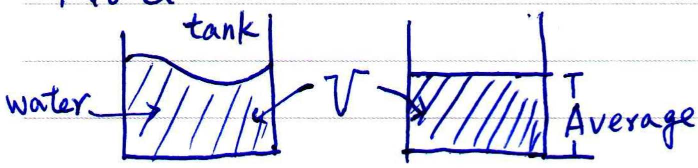
Average value

Def.

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA$$

Note



$$V = \iint_R f(x,y)\,dA$$
$$= \text{area} \times \text{Average value of } f.$$

f is the temperature of a plate $\Rightarrow$ average temp.

f = distane from (x,y) to P_0 $\Rightarrow$ average distance

Example 4.

Find the average value of $f(x,y) = x\cos xy$ over $R = [0,\pi]\times[0,1]$

Sol:

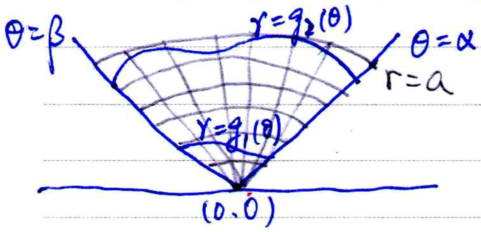
$$\frac{1}{\pi}\int_0^{\pi}\int_0^1 x\cos xy\,dy\,dx$$

$$\frac{1}{\pi}\int_0^{\pi} \sin xy\Big|_0^1\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin x\,dx$$

$$= \frac{1}{\pi}\,-\cos x\Big|_0^{\pi} = \frac{2}{\pi}$$

§15-4. Double integrals in polar form



$R=\{(r,\theta): g_1(\theta)\le r\le g_2(\theta),\ \alpha\le\theta\le\beta\}$

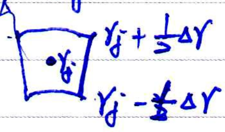
$\subseteq\{(r,\theta): 0\le r\le a,\ \alpha\le\theta\le\beta\}\equiv Q$

$\Delta r=\dfrac{a}{m}$, $\Delta\theta=\dfrac{\beta-\alpha}{m'}$

$P=\{R_1,\cdots,R_n\}$, a partition of R, where R_j is a polar rectangle, $\forall j$

ΔA_j = area of R_j, $\forall j$

$S_n=\sum_{j=1}^{n} f(r_j,\theta_j)\,\Delta A_j$, since R_j :



$\Rightarrow \Delta A_j=\dfrac{\Delta\theta}{2}\cdot\left((r_j+\tfrac{1}{2}\Delta r)^2-(r_j-\tfrac{1}{2}\Delta r)^2\right)=r_j\,\Delta r\,\Delta\theta$

$\Rightarrow S_n=\sum_{j=1}^{n} f(r_j,\theta_j)\,r_j\,\Delta r\,\Delta\theta$

$\Rightarrow \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(r_j,\theta_j)\,r_j\,\Delta r\,\Delta\theta \equiv \iint_R f(r,\theta)\,r\,dr\,d\theta$.

By Fubini's theorem

$\Rightarrow \iint_R f(r,\theta)\,r\,dr\,d\theta=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,\underbrace{\boxed{r\,dr\,d\theta}}_{dA}$ r 不要漏

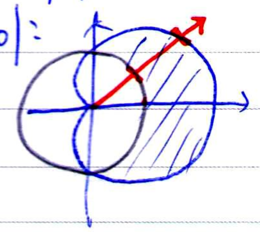
Note.

Step 1: Sketch the bdd region and bdry curves.
Step 2: Find r-limits
Step 3: Find θ-limits.

Example 1.

R lies inside $r=1+\cos\theta$ and outside $r=1$.
Find the limits (範圍) of integration of $f(r,\theta)$ over R.

Sol:



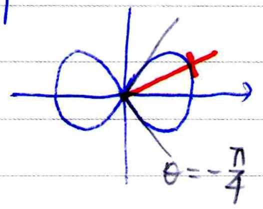
$\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta} 1\cdot r\ \ dr\,d\theta$

Note.

A = area of $R=\iint_R r\,dr\,d\theta$.

Example 2.
Find the area enclosed by $r^2 = 4\cos 2\theta$

Sol:



$\because r^2 > 0 \quad \therefore -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$$4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} 1\cdot r\, dr\, d\theta$$

$$2\int_0^{\frac{\pi}{4}} r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta = 2\int_0^{\frac{\pi}{4}} 4\cos 2\theta\, d\theta$$

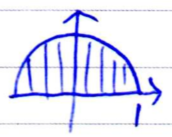
$$= 4 \sin 2\theta \Big|_0^{\frac{\pi}{4}} = 4$$

Note:
$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow \iint_R f(x,y)\, dy\, d\theta = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$,

where G denotes the same region describle in polar coordinates.

Example 3.
Find $\iint_R e^{x^2+y^2} dy\, dx$, where R bdd by the x-axis and $y = \sqrt{1-x^2}$

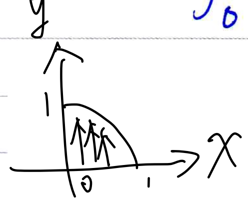
Sol:
R: 

$$\int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta$$

$$\int_0^{\pi} \frac{1}{2} e^{r^2} \Big|_0^1 d\theta = \int_0^{\pi} \frac{1}{2}e - \frac{1}{2}\, d\theta = \frac{1}{2}e\theta - \frac{1}{2}\theta \Big|_0^{\pi}$$

$$= \frac{1}{2}e\pi - \frac{1}{2}\pi$$

Example 4.
Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

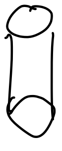
Sol: 

$$\int_0^{\frac{\pi}{2}}\int_0^1 r^3\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta$$

$$\frac{1}{4}\theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

Example 5.
Find the volume of the solid region bdd above by $z = 9 - x^2 - y^2$ and below by the unit circle in xy-plane.

Sol:



$$\int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta \qquad \int_0^{2\pi} \frac{9}{2}r^2 - \frac{1}{4}r^4 \Big|_0^1 d\theta$$

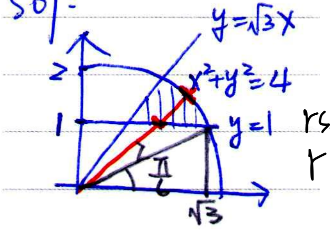
$$= \int_0^{2\pi} \frac{17}{4}\, d\theta = \frac{17}{2}\pi$$

Example 6.

Find the area of R in xy-plane enclosed by $x^2+y^2=4$, above $y=1$, and below $y=\sqrt{3}x$.

$r^2=4$

Sol:



$r\sin\theta=1$

$r=\csc\theta$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc\theta}^{2} r\, dr\, d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} r^2 \Big|_{\csc\theta}^{2} d\theta$$

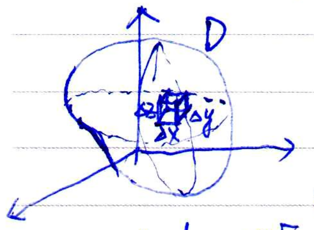
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2 - \frac{\csc^2\theta}{2}\, d\theta$$

$$= 2\theta + \frac{1}{2}\cot\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$\frac{1}{\sqrt{3}} - \sqrt{3}$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{3}$$

§15-5. Triple integrals in rectangular coordinates.



$D \subseteq \mathbb{R}^3$, a closed bdd region.

$F(x,y,z): D \to \mathbb{R}$, conti.

Let $P = \{D_1, D_2, \cdots, D_n\} \in \mathcal{P}(D)$, a partition of D

where $D_j \subseteq D$, rectangular cell, $\forall j$.

Let ΔV_k = volume of $D_k = \Delta x_k \Delta y_k \Delta z_k$, $(x_k, y_k, z_k) \in D_k$, $\forall k$.

$\Rightarrow S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$, a Riemann sum

Let $\|P\| = \max\{\Delta x_1, \cdots, \Delta x_n, \Delta y_1, \cdots, \Delta y_n, \Delta z_1, \cdots, \Delta z_n\}$.

If $\lim_{\|P\|\to 0} S_n$ exists, we say that F is integrable over D, and write

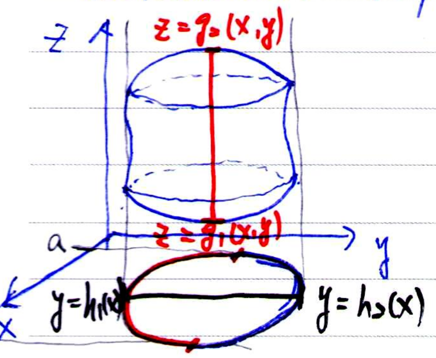
$$\iiint_D F(x,y,z)\,dV \left(\equiv \iiint_D F(x,y,z)\,dx\,dy\,dz\right) = \lim_{\|P\|\to 0} S_n .$$

(triple integral of F over D)

Def.

The volume of $D \equiv V = \iiint_D 1\,dV$

Calculation: (by Fubini's thm).



Step 1: sketch

Step 2: Find z-limits: $g_1(x,y) \le z \le g_2(x,y)$

Step 3: Find y-limits: $h_1(x) \le y \le h_2(x)$

Step 4: Find x-limits: $a \le x \le b$.

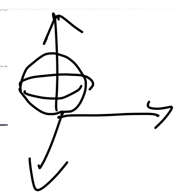
$$\Rightarrow \iiint_D F(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx.$$

Note: You might choose a different order of integration.

Example 1.

Find the volume of D enclosed by $z = x^2+3y^2$ and $z = 8-x^2-y^2$

Sol:



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} \int_{x^2+3y^2}^{8-x^2-y^2} 1 \quad dz\ dx\ dy$$

$$x^2+3y^2=8-x^2-y^2$$
$$x^2+2y^2=4$$

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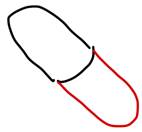
$$8-2x^2-4y^2$$

$$\int_{-\sqrt{2}}^{\sqrt{2}} 8x-\frac{2}{3}x^3-4y^2x \Big|_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dy$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}} 16\sqrt{4-2y^2}-\frac{4}{3}\left(\sqrt{4-2y^2}\right)^3-8y^2\sqrt{4-2y^2}\ dy$$

$8\sqrt{2}\pi$



Example 2.

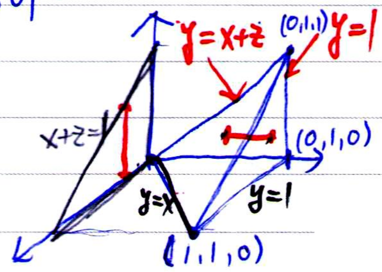
The tetrahedron D has vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,1,1)$.

$F(x,y,z): D\to\mathbb{R}$.

(1) Use the order $dy\,dz\,dx$ to set up the limits of $\iiint_D F(x,y,z)\,dV$.

(2) If $F\equiv 1$. Use the order $dz\,dy\,dx$ (dydzdx) to integrate $\iiint_D F(x,y,z)\,dV$.

Sol:



$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} F(x,y,z)\,dy\,dz\,dx$$

$$\int_0^1\int_0^{1-x} 1-x-z\,dz\,dx$$

$$\int_0^1 z-xz-\frac{1}{2}z^2\Big|_0^{1-x}dx$$

$$\int_0^1 1-x-x(1-x)-\frac{1}{2}(1-x)^2\,dx$$

$$\int_0^1 (1-x)\left(1-x-\frac{1}{2}(1-x)\right)dx$$

$$\int_0^1 (1-x)^2\left(1-\frac{1}{2}\right)dx=\frac{1}{6}$$

$$\begin{vmatrix} 1 & 0 \\ 1 & 1\end{vmatrix} \quad \begin{vmatrix} 1 & 1 \\ 0 & 1\end{vmatrix}$$

$(1,-1,1)$

$x-y+z=0$

Average value:

Average value of F over D $= \dfrac{1}{\text{volume of } D} \iiint_D F(x,y,z)\,dV$. 偏移

Ex:

F : temperature at $(x,y,z) \in D$ $\Rightarrow$ the average temp. of the solid D.

$F = \sqrt{x^2+y^2+z^2}$, $(x,y,z) \in D$ $\Rightarrow$ average distance of point in D from the origin.

Example 3.

$F(x,y,z) = xyz$, $D = \{(x,y,z) : 0 \le x, y, z \le 2\}$, a cubical region.

Find the average value of F over D.

Sol:

$$\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz \, dz\,dy\,dx = \frac{1}{8}\int_0^2\int_0^2 \frac{xy}{2} z^2 \Big|_0^2 \, dy\,dx$$

$$= 1$$

Note.

1. $\iiint_D (cf \pm g)\,dV = c\iiint_D f\,dV \pm \iiint_D g\,dV$

2. If $f \ge 0 \Rightarrow \iiint_D f\,dV \ge 0$

   If $f \ge g \Rightarrow \iiint_D f\,dV \ge \iiint_D g\,dV$

3. $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$

   $\Rightarrow \iiint_D f\,dV = \iiint_{D_1} f\,dV + \iiint_{D_2} f\,dV$.

§15-7. Triple integrals in cylindrical and spherical coordinates

Cyclindrical coordinate:

Recall:

A point $P_0=(x_0,y_0,z_0)\in\mathbb{R}^3$, where (x_0,y_0,z_0) is the xyz-coord. of P_0

The point $(x_0,y_0)\in\mathbb{R}^2$ has its polar coord. (r_0,θ_0)

Then (r_0,θ_0,z_0) is called the cyclindrical coord. of P_0.

Example. $r_0 \geq 0$

$P_0=(1,\sqrt{3},5)$ in xyz-coord.

$\Rightarrow (1,\sqrt{3})$ has the polar coord. $(2,\frac{\pi}{3})$ $\left(\because \begin{matrix} 1=2\cdot\cos\frac{\pi}{3} \\ \sqrt{3}=2\cdot\sin\frac{\pi}{3}\end{matrix}\right)$

$\Rightarrow (2,\frac{\pi}{3},5)$ is the cyclindrical coord. of P_0

Definition

The cyclindrical coordinate (r,θ,z) represent a point P in which $r\geq 0$, (r,θ) is the polar coordinate of the projection of P on xy-plane, and z is the vertical coordinate.

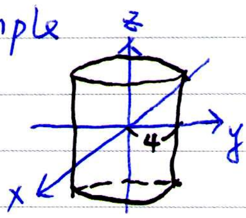
Note

Relations of (x,y,z) and (r,θ,z):

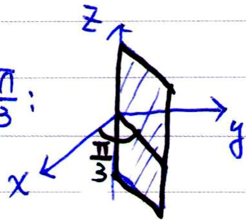
$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,\quad r^2=x^2+y^2,\quad \tan\theta=\frac{y}{x}$$

Example

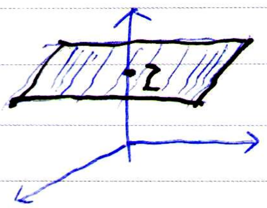
$r=4$:



$\theta=\frac{\pi}{3}$:



$z=2$



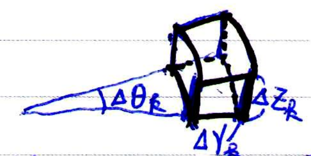
Triple integral:

$$\iiint_R f\,dV = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(r_k,\theta_k,z_k)\Delta V_k$$

$$=\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(r_k,\theta_k,z_k)\Delta z_k\, r_k\Delta r_k\Delta\theta_k$$

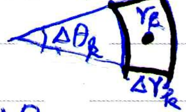
$$=\iiint_D f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

ΔV_k:



$\Delta V_k = \Delta A_k\cdot\Delta z_k$

ΔA_k:



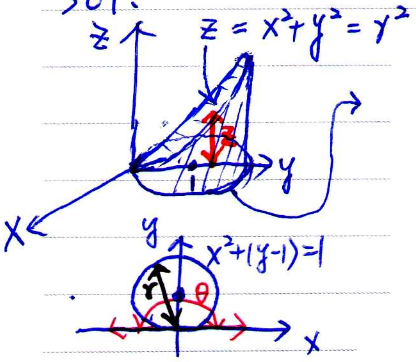
$\Delta A_k = r_k\Delta r_k\Delta\theta_k$

Example 1.

The region D is bdd by $z=0$, $x^2+(y-1)^2=1$ and $z=x^2+y^2$.

Find the limits of integration in cyclindrical coord. for $\iiint_D f\,dV$.

Sol:



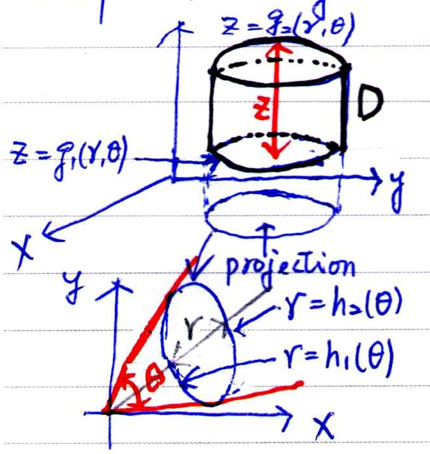
$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

$$x^2+y^2-2y+1=1$$

$$r^2=2r\sin\theta$$

$$r=2\sin\theta$$

Steps of integration:



1. Sketch the region D
2. Find z-limits : $g_1(r,\theta)\le z\le g_2(r,\theta)$.
3. Skech the projection of D on xy-plane
4. Find r-limits : $h_1(\theta)\le r\le h_2(\theta)$
5. Find θ-limits : $\alpha\le\theta\le\beta$

$$\Rightarrow \iiint_D f(r,\theta,z)\,dV=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

$$r=2$$

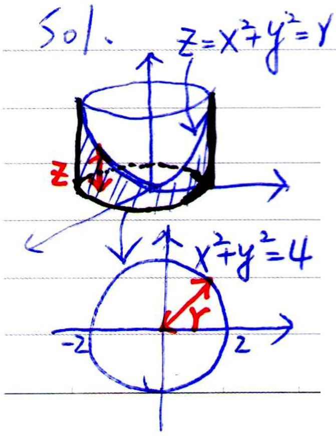
$$r^2=4$$

Example 2.

The solid D is enclosed by $x^2+y^2=4$, $z=x^2+y^2$ and xy-plane.

Find the volume of D and $\iiint_D z\,dV$.

Sol.



$$\int_0^{2\pi}\int_0^2\int_0^{r^2} 1\, r\,dz\; r\,dr\,d\theta = 8\pi$$

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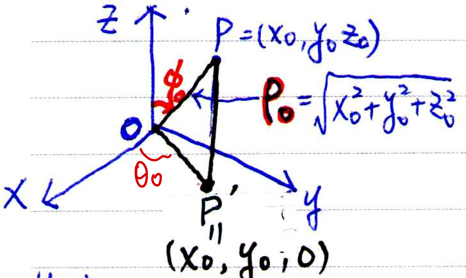
$$\int_0^{2\pi}\int_0^2\int_0^{r^2}(z)\,r\,dz\,dr\,d\theta$$

Spherical coordinates:

Recall:

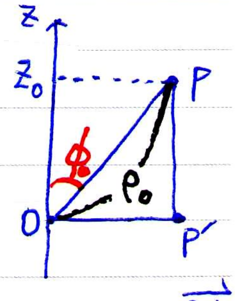


A point $P=(x_0,y_0,z_0)\in\mathbb{R}^3$

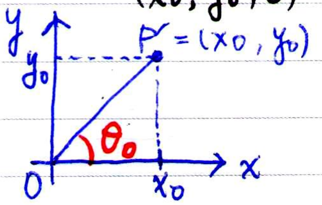
Let $\rho_0=\sqrt{x_0^2+y_0^2+z_0^2}$

$\Rightarrow$ $z_0=\rho_0\cos\phi_0$, where

ϕ_0 = the angle from positive z-axis to $\overrightarrow{OP}$



Let $P'=(x_0,y_0,0)$ be the projection of P on xy-plane



$\Rightarrow$ $\overrightarrow{OP'}$ be the projection of $\overrightarrow{OP}$ on xy-plane, and $|\overrightarrow{OP'}|=\rho_0\sin\phi_0$

Let θ_0 = the angle from positive x-axis to $\overrightarrow{OP'}$

$\Rightarrow$ $x_0=|\overrightarrow{OP'}|\cos\theta_0=\rho_0\sin\phi_0\cos\theta_0$

$y_0=|\overrightarrow{OP'}|\sin\theta_0=\rho_0\sin\phi_0\sin\theta_0$

Then (ρ_0,ϕ_0,θ_0) is called the spherical coordinate of the point P.

Definition:

The spherical coordinate (ρ,ϕ,θ) represent a point P in which

(1) ρ = the distance from P to the origin O $(\rho\geq 0)$

(2) ϕ = the angle from positive z-axis to $\overrightarrow{OP}$ $(0\leq\phi_0\leq\pi)$

(3) θ = the angle from positive x-axis to $\overrightarrow{OP'}$ on xy-plane where P' is the projection of P on xy-plane.

Relations of xyz-coord. to spherical coord.:

$$\begin{cases} z=\rho\cos\phi\,, \quad r=\rho\sin\phi \\ x=r\cos\theta=\rho\sin\phi\cos\theta\,, \quad y=r\sin\theta=\rho\sin\phi\sin\theta \\ \rho=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2} \end{cases}$$

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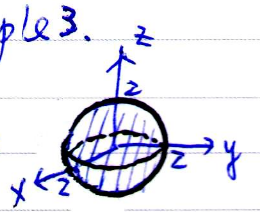
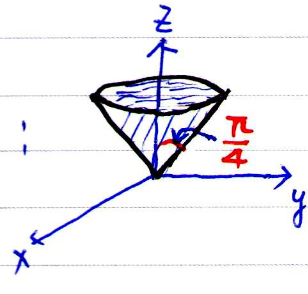
Example 3.

$\rho=2$:



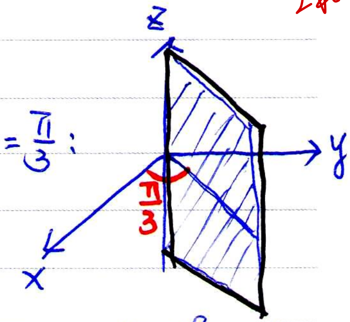
Sphere of radius 2.

$\phi=\frac{\pi}{4}$:



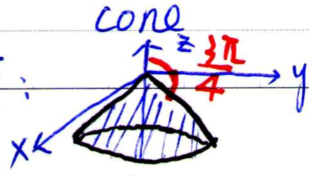
cone

$\theta=\frac{\pi}{3}$:



half-plane

$\phi=\frac{3\pi}{4}$:

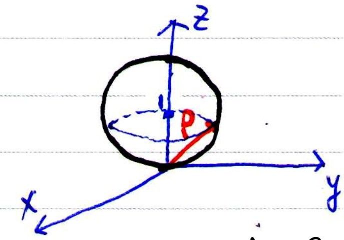


Example 4.

Find a spherical coordinate equation for

(a) the sphere $x^2+y^2+(z-1)^2=1$,

(b) the cone $z=\sqrt{x^2+y^2}$.



Sol:

$r = \rho\sin\phi$

$\underbrace{x^2+y^2+z^2}-2z=0$

$\rho^2 = 2\rho\cos\phi \quad 0 \le \phi \le \frac{\pi}{2}$

$\rho = 2\cos\phi$

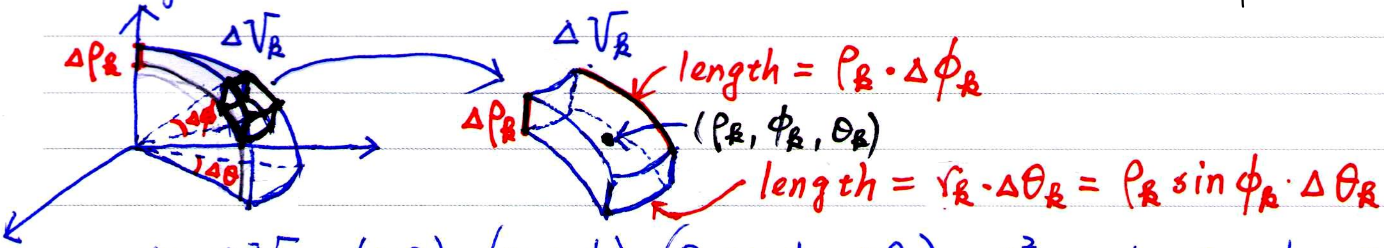
$z^2 = x^2+y^2$

$(\rho\cos\phi)^2 = (\rho\sin\phi)^2$

$\tan^2\phi = 1 \qquad z>0 \quad 0 \le \phi \le \frac{\pi}{2}$

$\phi = \frac{\pi}{4}$

Integral:



$\Rightarrow \Delta V_k = (\Delta\rho_k)\cdot(\rho_k\Delta\phi_k)\cdot(\rho_k\sin\phi_k\cdot\Delta\theta_k) = \rho_k^2\sin\phi_k\,\Delta\rho_k\Delta\phi_k\Delta\theta_k.$

$\Rightarrow \iiint_D f\,dV = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\rho_k,\phi_k,\theta_k)\,\Delta V_k = \lim_{n\to\infty}\sum_{k=1}^{n} f(\rho_k,\phi_k,\theta_k)\,\rho_k^2\sin\phi_k\,\Delta\rho_k\Delta\phi_k\Delta\theta_k$

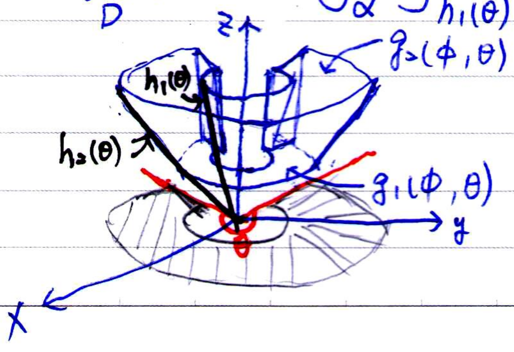
$= \iiint_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$

Steps of integration:

Sketch the region D → find ρ-limits → find φ-limits → find θ-limits.

$g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta) \qquad h_1(\theta) \le \phi \le h_2(\theta) \qquad \alpha \le \theta \le \beta$

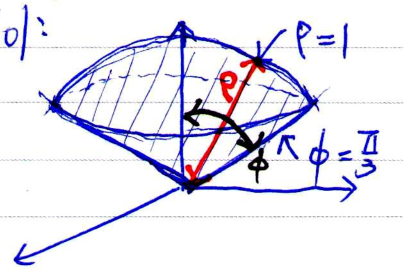
$\Rightarrow \iiint_D f\,dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$



Example 5.

Find the volume of D cut from the solid sphere $\rho \le 1$ by $\phi = \frac{\pi}{3}$.

Sol:



$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}} \frac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} -\cos\phi \Big|_0^{\frac{\pi}{3}} d\theta = \frac{1}{6}\times 2\pi = \frac{\pi}{3}$$

Example 6.

The region D is as in Example 5, $f(x,y,z) = x^2+y^2$. Find $\iiint_D f \, dV$.

r^2

$r = \rho\sin\phi$

Sol:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2 \sin^2\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$u = \cos\phi \qquad du = -\sin\phi \, d\phi$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{3}} \sin^2\phi \, \sin\phi \, d\phi \, d\theta$$

$(1-\cos^2\phi)(-du)$

$$= \frac{1}{5}\int_0^{2\pi}\int_{\frac{1}{2}}^1 1-u^2 \, du \, d\theta$$

$\frac{1}{3}u^3$

$\frac{1}{2} - (\frac{1}{3} - \frac{1}{24}) = \frac{1}{2} - \frac{7}{24} = \frac{5}{24}$

Note

Cyclindrical to xyz	Spherical to xyz	Spherical to Cyclindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

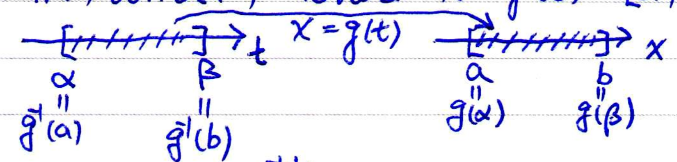
$\Rightarrow dV = dx\,dy\,dz = r\,dz\,dr\,d\theta = \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$.

§15-8. Substitutions in multiple integrals

Recall:

One variable:

$f(x): [a,b] \to \mathbb{R}$, conti., where $x = g(t): [\alpha, \beta] \to [a,b]$ is 1-1, diff. ($\alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$, $a = g(\alpha)$, $b = g(\beta)$)

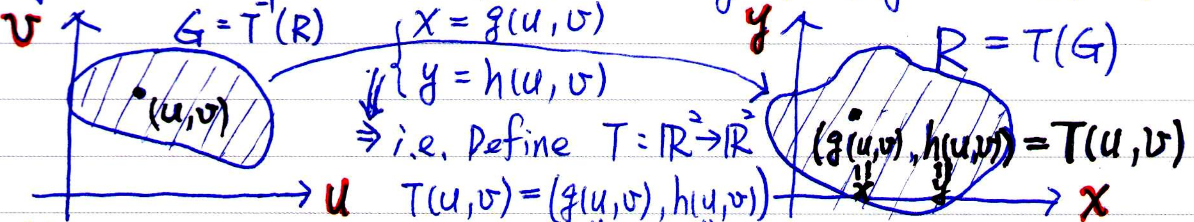


$$\Rightarrow \int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\,dt$$

$$\text{or } \int_\alpha^\beta f(g(t))\,g'(t)\,dt = \int_{g(\alpha)}^{g(\beta)} f(x)\,dx$$

Two variables:

$f(x,y): R \to \mathbb{R}$, conti., where $x = g(u,v)$, $y = h(u,v)$, $(u,v) \in G$



$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$

$\Rightarrow$ i.e. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(u,v) = (g(u,v), h(u,v))$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_G f(\underset{x}{g(u,v)}, \underset{y}{h(u,v)}) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot du\,dv,$$

where $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \dfrac{\partial x}{\partial u} \cdot \dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v} \cdot \dfrac{\partial y}{\partial u}$ $\left(\equiv J(u,v) \right)$ is called the Jacobian (determinant) of $x = g(u,v)$, $y = h(u,v)$.

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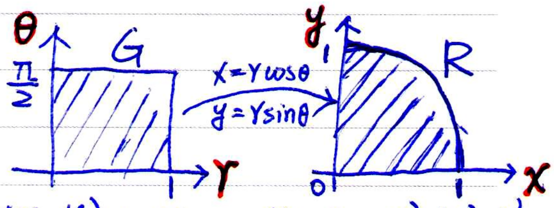
Note: $R = T(G)$, $G = T^{-1}(R)$.

Example 1.



$G = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1\}$

$x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow R = \{(x,y) : x \ge 0, y \ge 0, x^2 + y^2 \le 1\}$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \Rightarrow \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = |r| = r \quad (\because r \ge 0)$$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta) \cdot r \cdot dr\,d\theta$$

$$\left(\left(\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx \qquad \int_0^{\frac{\pi}{2}} \int_0^1 f(r\cos\theta, r\sin\theta) \cdot r \cdot dr\,d\theta. \right)\right)$$

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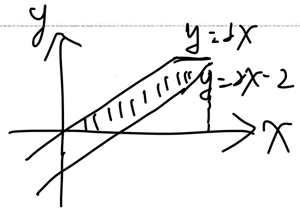
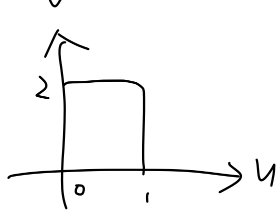
Example 2.
Evaluate $\int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$ by the transformation

$u=\frac{2x-y}{2}$, $v=\frac{y}{2}$.

Sol: $0\le y\le 4$
$\frac{y}{2} \le x\le \frac{y}{2}+1$



$G=\{(u,v)\mid 0\le u\le 1 , 0\le v\le 2\}$



$y=2v$
$x=u+v$

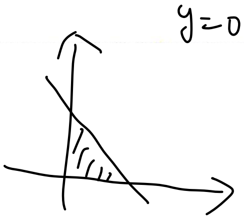
$$\int_0^2\int_0^1 u\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
$$\int_0^2 u^2\Big|_0^1 dv = 2$$

$$\left|\begin{vmatrix}1&1\\0&2\end{vmatrix}\right|=2$$

Example 3.
Evaluate $\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

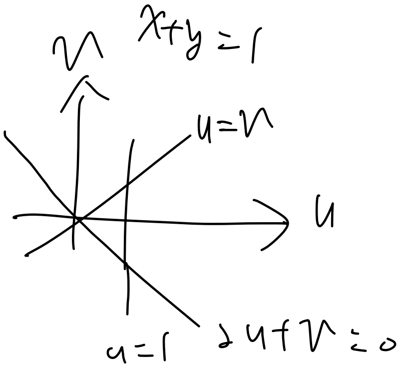
$u=1$
$x=0 \Rightarrow u=v$
$y=0 \Rightarrow 2u+v=0$

Sol. Let $u=x+y$, $v=y-2x$
$x=\frac{u-v}{3}$ $y=\frac{v+2u}{3}$



$x+y=1$

$\{(u,v)\mid$



要絕對值

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}\frac13 & -\frac13\\ \frac23 & \frac13\end{vmatrix} = \frac19+\frac29=\frac13$$

u^3+8u^3

$$\int_0^1\int_{-2u}^{u}\sqrt{u}\,v^2\frac13\,dv\,du$$

$$\int_0^1 \frac19 v^3\sqrt{u}\Big|_{-2u}^{u} du = \frac19\int_0^1 9u^3\sqrt{u}\,du = \int_0^1 u^{\frac72}\,du$$

$= \frac{2}{9}u^{\frac{9}{2}}\Big|_0^1$

$= \frac{2}{9}$

Example 4.
Evaluate $\displaystyle\int_1^2\int_{\frac{1}{y}}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dx\,dy.$

Sol.

$u=\sqrt{\frac{y}{x}}\qquad v=\sqrt{xy}$

$\frac{v}{u}=x\qquad uv=y\qquad \begin{vmatrix}-vu^{-2} & \frac{1}{u}\\ v & u\end{vmatrix} = -vu^{-1}-vu^{-1}=-2vu^{-1}\ \xrightarrow{\text{abs}}\ 2vu^{-1}$

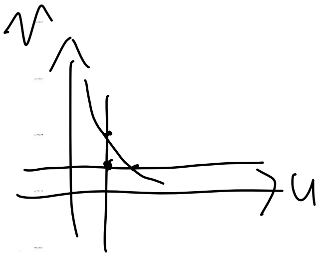
$t=v\quad dt=dv$

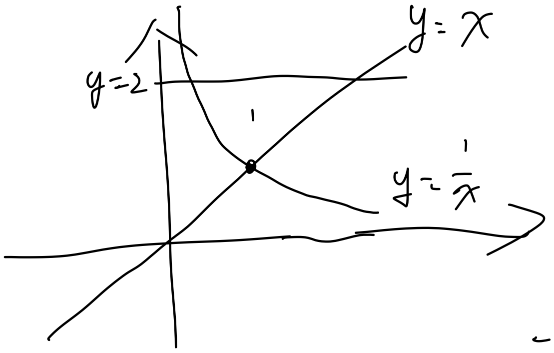
$dk=e^{v}dv\quad k=e^{v}$

$u=1$

$v=1$

$uv=2$





$\displaystyle\int_1^2\int_1^{\frac{2}{v}} ue^{v}\cdot 2vu^{-1}\,du\,dv$

$\displaystyle\int_1^2\int_1^{\frac{2}{v}} 2ve^{v}\,du\,dv$

$\displaystyle\int_1^2 2ve^{v}\left(\frac{2}{v}-1\right)dv$

$\displaystyle=\int_1^2 4e^{v}dv-\int_1^2 2ve^{v}dv$

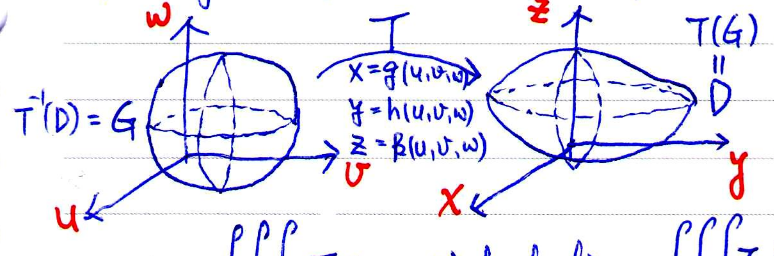
$=4e^2-4e-2(ve^{v}-e^{v}\big|_1^2)$

$=4e^2-4e-2(2e^2-e-e^2+e)$

$=2e^2-4e$

Three variables:

$F(x,y,z):D\to\mathbb{R}$, conti., where $x=g(u,v,w)$, $y=h(u,v,w)$, $z=k(u,v,w)$, $\forall (u,v,w)\in G$



i.e. $T:\mathbb{R}^3\to\mathbb{R}^3$

$T(u,v,w)=(\underset{x}{\underset{\shortparallel}{g(u,v,w)}},\ \underset{y}{\underset{\shortparallel}{h(u,v,w)}},\ \underset{z}{\underset{\shortparallel}{k(u,v,w)}})$

$$\Rightarrow \iiint_D F(x,y,z)\,dx\,dy\,dz=\iiint_G F(g(u,v,w),h(u,v,w),k(u,v,w))\cdot\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw,$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}.$

For example,

if $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}\Rightarrow\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|=\left|\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}\right|=|r|=r\quad(\because r\geq 0)$

$$\Rightarrow \iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F(r\cos\theta, r\sin\theta, z)\cdot r\cdot dr\,d\theta\,dz$$

If $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \Rightarrow \left|\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \left|\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}\right|$

$$= \left|\cos\phi\cdot\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}\right|$$

$$= |\rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi| = |\rho^2\sin\phi| = \rho^2\sin\phi \quad (\because 0\le\phi\le\pi)$$

$$\Rightarrow \iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example 5.

Evaluate $\displaystyle\int_0^3\int_0^4\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$ by the transformation

$u = \dfrac{2x-y}{2}$, $v = \dfrac{y}{2}$, $w = \dfrac{z}{3}$.

Sol:

$0\le u\le 1$ $\quad 0\le v\le 2$ $\quad 0\le w\le 1$

$x = u+v$ $\quad y = 2v$ $\quad z = 3w$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \quad 3\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 6$$

$$\int_0^1\int_0^2\int_0^1 (u+w)\cdot 6\;du\,dv\,dw$$

Appendix:

Two variables:

Question: Why does the equation

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv \text{ hold?}$$

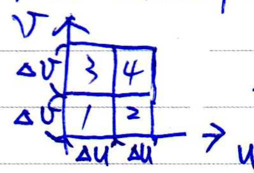
We know that $\iint_R f(x,y)\,dx\,dy = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$

If we can show that $\Delta A_k = \left|\begin{vmatrix} g_u(u_k,v_k) & g_v(u_k,v_k) \\ h_u(u_k,v_k) & h_v(u_k,v_k)\end{vmatrix}\cdot\Delta u\Delta v\right|, \forall k$

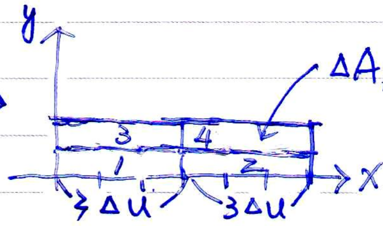
then $\iint_R f(x,y)\,dx\,dy = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(g(u_k,v_k),h(u_k,v_k))\cdot\left|\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_k,v_k)}\right|\cdot\Delta u\Delta v$

$$= \iint_G f(g(u,v),h(u,v))\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\cdot du\,dv.$$

For example:

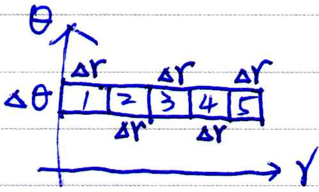


$x = 3u$, $y = \frac{1}{3}v$

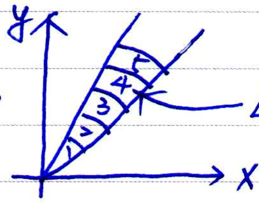


$\Delta A_k = 3\Delta u\cdot\frac{1}{3}\Delta v = \frac{3}{3}\Delta u\cdot\Delta v$

$= \begin{vmatrix} 3 & 0 \\ 0 & \frac{1}{3}\end{vmatrix}\cdot\Delta u\cdot\Delta v$



$x = r\cos\theta$, $y = r\sin\theta$



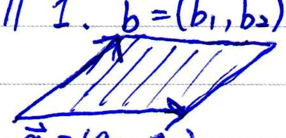
$\Delta A_k = r_k\cdot\Delta r\cdot\Delta\theta$

$= \begin{vmatrix} \cos\theta_k & -r_k\sin\theta_k \\ \sin\theta_k & r_k\cos\theta_k\end{vmatrix}\cdot\Delta r\cdot\Delta\theta$

Now,

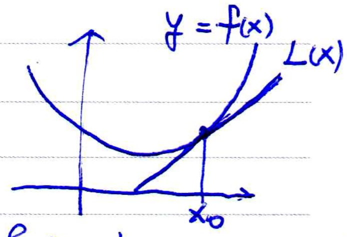
Check: $\Delta A_k = \left|\left.\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right|_{(u_k,v_k)}\cdot\Delta u\Delta v\right|, \quad \forall k.$

Recall 1. $\vec{b} = (b_1,b_2)$



$\vec{a} = (a_1,a_2)$

Area $= \left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2\end{vmatrix}\right|$



Recall 2.

If $y = f(x): a,b \to \mathbb{R}$, diff., $x_0 \in (a,b)$.

We use the linearization $L(x)$ to estimate $f(x)$ when x near x_0, where $L(x) = f(x_0) + f'(x_0)(x-x_0)$.

$\Rightarrow f(x) - f(x_0) \approx L(x) - f(x_0) = f'(x_0)(x-x_0)$

$\overset{\text{MVT}}{\|}$

$f'(c)(x-x_0)$, where c between x and x_0

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Recall 3.

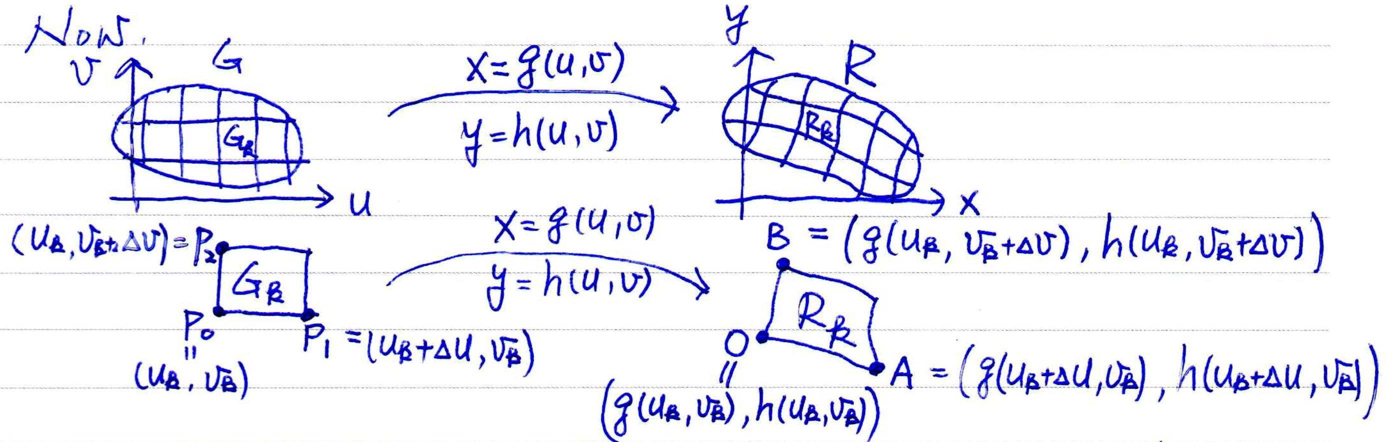
If $y = f(u,v) : G \to \mathbb{R}$ is diff. and $(u_0, v_0) \in G$ ($G \subseteq \mathbb{R}^2$, open)

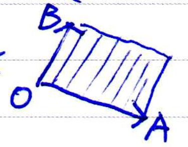
We use the linearization $L(u,v)$ to estimate $f(u,v)$ when (u,v) near (u_0,v_0)

where $L(u,v) = f(u_0,v_0) + f_u(u_0,v_0)\cdot(u-u_0) + f_v(u_0,v_0)(v-v_0)$.

$\Rightarrow f(u,v) - f(u_0,v_0) \approx L(u,v) - f(u_0,v_0) = f_u(u_0,v_0)(u-u_0) + f_v(u_0,v_0)(v-v_0)$.

Now.



$\Rightarrow \Delta A_k$ = area of $R_k \approx$  $= \left| \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right|$, where $\vec{OA} = (a_1, a_2)$, $\vec{OB} = (b_1, b_2)$

$$\left.\begin{aligned} a_1 &= g(u_k+\Delta u, v_k) - g(u_k, v_k) \\ &\approx g_u(u_k,v_k)\cdot\Delta u + g_v(u_k,v_k)\cdot 0 \end{aligned}\right\} \text{by Recall 3.}$$
$$= g_u(u_k,v_k)\cdot\Delta u$$

$$\left.\begin{aligned} a_2 &= h(u_k+\Delta u, v_k) - h(u_k, v_k) \\ &\approx h_u(u_k,v_k)\cdot\Delta u + h_v(u_k,v_k)\cdot 0 \end{aligned}\right\} \text{by Recall 3.}$$
$$= h_u(u_k,v_k)\cdot\Delta u$$

$$\left.\begin{aligned} b_1 &= g(u_k, v_k+\Delta v) - g(u_k, v_k) \\ &\approx g_u(u_k,v_k)\cdot 0 + g_v(u_k,v_k)\cdot\Delta v \end{aligned}\right\} \text{by Recall 3.}$$
$$= g_v(u_k,v_k)\cdot\Delta v$$

$$\left.\begin{aligned} b_2 &= h(u_k, v_k+\Delta v) - h(u_k, v_k) \\ &\approx h_u(u_k,v_k)\cdot 0 + h_v(u_k,v_k)\cdot\Delta v \end{aligned}\right\} \text{by Recall 3.}$$
$$= h_v(u_k,v_k)\cdot\Delta v$$

$$\Rightarrow \Delta A_k = \left| \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right| = \left| \begin{vmatrix} g_u(u_k,v_k)\Delta u & g_v(u_k,v_k)\Delta v \\ h_u(u_k,v_k)\Delta u & h_v(u_k,v_k)\Delta v \end{vmatrix} \right|$$
$$= \left| \left. \frac{\partial(x,y)}{\partial(u,v)} \right|_{(u_k,v_k)} \right| \cdot \Delta u \Delta v.$$

Three variables:

$F(x,y,z): D \to \mathbb{R}$, conti., where $x=g(u,v,w)$, $y=h(u,v,w)$, $z=k(u,v,w)$.

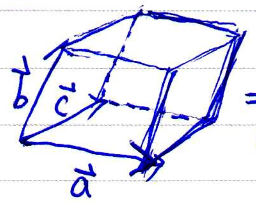
$$\Rightarrow \iiint_D F(x,y,z)\,dx\,dy\,dz = \lim_{\|P\|\to 0} \sum_{j=1}^{n} F(x_j,y_j,z_j)\cdot \Delta V_j$$

We will show that $\Delta V_j = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)}\Big|_{(u_j,v_j,w_j)} \right| \cdot \Delta u \Delta v \Delta w$

$$\Rightarrow \iiint_D F(x,y,z)\,dx\,dy\,dz = \lim_{\|P\|\to 0} \sum_{j=1}^{n} F(g(u_j,v_j,w_j), h(u_j,v_j,w_j), k(u_j,v_j,w_j))\cdot |J(u_j,v_j,w_j)|\,\Delta u\Delta v\Delta w$$

$$= \iiint_G F(g(u,v,w), h(u,v,w), k(u,v,w))\cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Recall 1.

$\begin{cases} \vec{a} = (a_1, a_2, a_3) \\ \vec{b} = (b_1, b_2, b_3) \\ \vec{c} = (c_1, c_2, c_3) \end{cases} \Rightarrow$ volume of 

$= |\vec{a}\cdot(\vec{b}\times\vec{c})| = \left| \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \right|$

Recall 2.

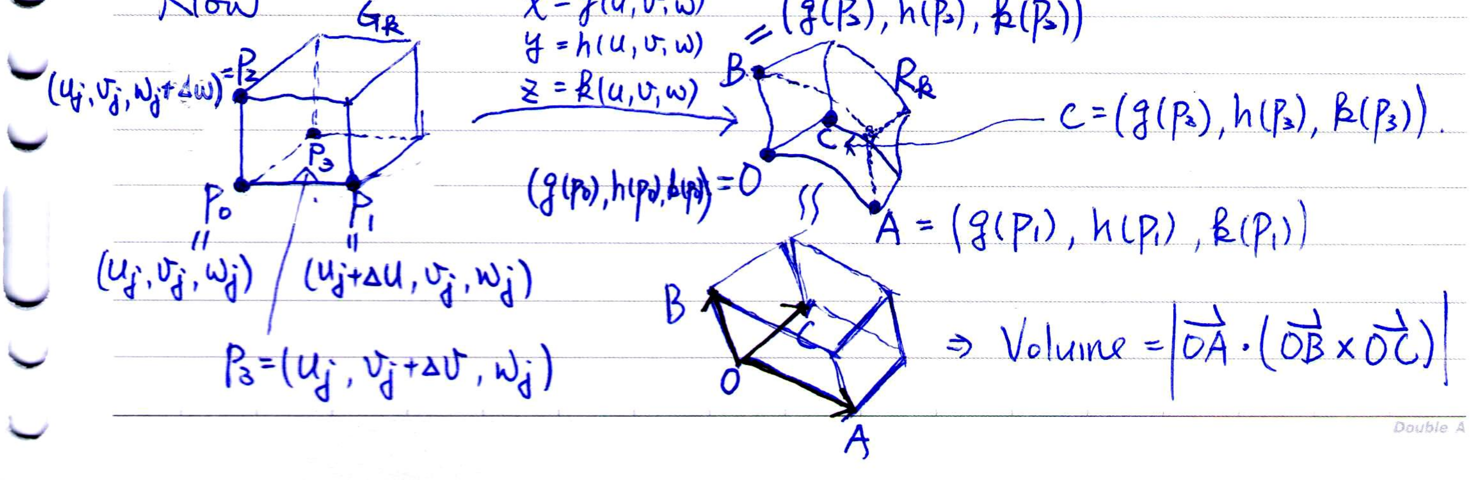
$f(u,v,w): G \to \mathbb{R}$ is diff., $(u_0,v_0,w_0)\in G$ (with $G \subseteq \mathbb{R}^3$, open)

We use the linearization $L(u,v,w)$ to estimate $f(u,v,w)$ when (u,v,w) near (u_0,v_0,w_0), where

$$L(x,y,z) = f(u_0,v_0,w_0) + f_u(u_0,v_0,w_0)\cdot(u-u_0) + f_v(u_0,v_0,w_0)\cdot(v-v_0) + f_w(u_0,v_0,w_0)\cdot(w-w_0)$$

$$\Rightarrow f(u,v,w) - f(u_0,v_0,w_0) \approx L(u,v,w) - f(u_0,v_0,w_0)$$
$$= f_u(u_0,v_0,w_0)(u-u_0) + f_v(u_0,v_0,w_0)(v-v_0) + f_w(u_0,v_0,w_0)(w-w_0).$$

Now



$\Rightarrow$ Volume $= |\vec{OA}\cdot(\vec{OB}\times\vec{OC})|$

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$$\overrightarrow{OA} = \left(g(P_1)-g(P_0),\ h(P_1)-h(P_0),\ k(P_1)-k(P_0)\right)$$

$$g(P_1)-g(P_0) = g(u_j+\Delta u, v_j, w_j) - g(u_j, v_j, w_j) \approx g_u(u_j, v_j, w_j)\cdot\Delta u$$

$$h(P_1)-h(P_0) = h(u_j+\Delta u, v_j, w_j) - h(u_j, v_j, w_j) \approx h_u(u_j, v_j, w_j)\cdot\Delta u$$

$$k(P_1)-k(P_0) = k(u_j+\Delta u, v_j, w_j) - k(u_j, v_j, w_j) \approx k_u(u_j, v_j, w_j)\cdot\Delta u$$

$$\overrightarrow{OB} = \left(g(P_2)-g(P_0),\ h(P_2)-h(P_0),\ k(P_2)-k(P_0)\right)$$

$$g(P_2)-g(P_0) = g(u_j, v_j+\Delta v, w_j) - g(u_j, v_j, w_j) \approx g_v(u_j, v_j, w_j)\cdot\Delta v$$

$$h(P_2)-h(P_0) = h(u_j, v_j+\Delta v, w_j) - h(u_j, v_j, w_j) \approx h_v(u_j, v_j, w_j)\cdot\Delta v$$

$$k(P_2)-k(P_0) = k(u_j, v_j+\Delta v, w_j) - k(u_j, v_j, w_j) \approx k_v(u_j, v_j, w_j)\cdot\Delta v$$

$$\overrightarrow{OC} = \left(g(P_3)-g(P_0),\ h(P_3)-h(P_0),\ k(P_3)-k(P_0)\right)$$

$$g(P_3)-g(P_0) = g(u_j, v_j, w_j+\Delta w) - g(u_j, v_j, w_j) \approx g_w(u_j, v_j, w_j)\cdot\Delta w$$

$$h(P_3)-h(P) = h(u_j, v_j, w_j+\Delta w) - h(u_j, v_j, w_j) \approx h_w(u_j, v_j, w_j)\cdot\Delta w$$

$$k(P_3)-k(P_0) = k(u_j, v_j, w_j+\Delta w) - k(u_j, v_j, w_j) \approx k_w(u_j, v_j, w_j)\cdot\Delta w$$

$$\Delta V_j \approx \left|\overrightarrow{OA}\cdot(\overrightarrow{OB}\times\overrightarrow{OC})\right| = \left|\begin{vmatrix} g_u(u_j,v_j,w_j)\Delta u & g_v(u_j,v_j,w_j)\cdot\Delta v & g_w(u_j,v_j,w_j)\Delta w \\ h_u(u_j,v_j,w_j)\Delta u & h_v(u_j,v_j,w_j)\Delta v & h_w(u_j,v_j,w_j)\Delta w \\ k_u(u_j,v_j,w_j)\Delta u & k_v(u_j,v_j,w_j)\Delta v & k_w(u_j,v_j,w_j)\Delta w \end{vmatrix}\right|$$

$$= \left|J(u_j, v_j, w_j)\right|\cdot\Delta u\,\Delta v\,\Delta w.$$

$$\Rightarrow \sum_{j=1}^{n} F(x_j, y_j, z_j)\,\Delta V_j = \sum_{j=1}^{n} F\left(g(u_j,v_j,w_j), h(u_j,v_j,w_j), k(u_j,v_j,w_j)\right)\cdot\left|J(u_j,v_j,w_j)\right|\Delta u\Delta v\Delta w$$

Let $\|P\|\to 0 \Rightarrow \iiint_R F(x,y,z)\,dx\,dy\,dz$

$$= \iiint_G F\left(g(u,v,w), h(u,v,w), k(u,v,w)\right)\cdot\left|J(u,v,w)\right|\,du\,dv\,dw.$$

*