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Fact: $\int \frac{1}{x^2+a^2}\,dx$

Let $x = a\tan\theta$

$\int \frac{a\sec^2\theta}{a^2\sec^2\theta}\,d\theta = \frac{\theta}{a} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Ex $\frac{dy}{dx} = \left(2+\frac{y}{x}\right)^2$

Let $\frac{y}{x} = u \Rightarrow y' = u'x + u$

$u'x + u = (2+u)^2$

$\Rightarrow \frac{du}{dx}x = u^2+3u+4 \Rightarrow \int \frac{1}{u^2+3u+4}\,du = \int \frac{1}{x}\,dx + c^*$

$\Rightarrow \int \frac{1}{(u+\frac{3}{2})^2+(\sqrt{\frac{7}{4}})^2}\,du = \ln|x| + c^*$

$\Rightarrow \frac{2}{\sqrt{7}}\tan^{-1}\frac{2}{\sqrt{7}}(u+\frac{3}{2}) = \ln|x| + c^*$ ✓

$\Rightarrow \tan^{-1}\frac{2}{\sqrt{7}}\left(\frac{y}{x}+\frac{3}{2}\right) = \frac{\sqrt{7}}{2}\ln|x| + C$

$\Rightarrow y = \left(\frac{\sqrt{7}}{2}\tan\left(\frac{\sqrt{7}}{2}\ln|x|+c\right) - \frac{3}{2}\right)x$

$$\boxed{\begin{array}{l}(\ln x)' = \frac{1}{x} \\ \times \int (\ln x)\,dx \\ = x\ln x - x\end{array}}$$

Ex $y' = \frac{y}{x} + \frac{2x^3\cos(x^2)}{y}$ tricky!

Let $\frac{y}{x} = u \Rightarrow y' = u'x + u$

$u'x + \cancel{u} = \cancel{u} + \frac{2x^2\cos(x^2)}{u}$

$\Rightarrow \frac{du}{dx} = \frac{2x\cos(x^2)}{u} \Rightarrow \int u\,du = \int 2x\cos(x^2)\,dx + C$

$\Rightarrow \frac{u^2}{2} = \int d\sin(x^2)$　$u\,dv$

$\int \cos(x^2)\,dx^2$

$\sin(x^2) + C$

$\Rightarrow \frac{y^2}{2x^2} = \sin(x^2) + c$

$uv = \int \frac{d}{dx}(uv)\,dx$

$= \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$

$= \int u\,dv + \int v\,du$

Ex $\int xe^x\,dx$

$= \int x\,de^x$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x$ ✓

* Transformation $v = ax + by (+d)$

$(2x-4y+5)y' + x - 2y + 3 = 0$

Let $x - 2y = v \Rightarrow v' = 1 - 2y' \Rightarrow y' = \frac{1-v'}{2}$

$(2v+5)\frac{1-v'}{2} + v + 3 = 0$

$\Rightarrow 4v + 11 = 2v+5\frac{dv}{dx} \Rightarrow \int \frac{2v+5}{4v+11}\,dv = \int dx + c^*$

$\Rightarrow \frac{1}{2}\int \frac{4v+10}{4v+11}\,dv = x + c^*$

$\Rightarrow \frac{1}{2}\int 1 - \frac{1}{4v+11}\,dv = x + c^*$

$\Rightarrow \frac{v}{2} - \frac{1}{8}\ln|4v+11| = x + c^*$

$\Rightarrow \frac{x-2y}{2} - \frac{1}{8}\ln|4x-8y+11| = x + c^*$

$\Rightarrow 4x+8y+\ln|4x-8y+11| = C$

$$\Rightarrow 4x+8y+\ln|4x-8y+11| = C$$

* Consider the DE

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+h}\right) \quad \text{where } a.b.c,d,e\ h \text{ are constants}$$

$$= f\left(\frac{a+b\left(\frac{y}{x}\right)+\frac{c}{x}}{d+e\left(\frac{y}{x}\right)+\frac{h}{x}}\right)$$

when $c=h=0$ $\quad \frac{dy}{dx} = f\left(\frac{a+b\left(\frac{y}{x}\right)}{d+e\left(\frac{y}{x}\right)}\right) = g\left(\frac{y}{x}\right)$

when $c\neq 0$, $h\neq 0$, We need to to change the variables to make it become separable.

Let $X = x-\underline{\underline{\alpha}}$ $\quad Y = y-\underline{\underline{\beta}}$ ✓

$\Rightarrow x = X+\alpha \quad \Rightarrow y = Y+\beta$

$$\frac{dy}{dx} = \frac{dY}{dX} = f\left(\frac{a(X+\alpha)+b(Y+\beta)+c}{d(X+\alpha)+e(Y+\beta)+h}\right)$$

$$= f\left(\frac{aX+bY+a\alpha+b\beta+c}{dX+eY+d\alpha+e\beta+h}\right)$$

If there exist α & β s.t. $\begin{array}{l} a\alpha+b\beta+c=0 \\ d\alpha+e\beta+h=0 \end{array}$

then the DE becomes $\frac{dY}{dX} = f\left(\frac{a+b\left(\frac{Y}{X}\right)}{d+e\left(\frac{Y}{X}\right)}\right)$

How to make sure α. β exist?

Recall $\begin{array}{l} a\alpha+b\beta=-c \\ d\alpha+e\beta=-h \end{array}$ $\qquad \alpha = \frac{\begin{vmatrix} -c & b \\ -h & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad \beta = \frac{\begin{vmatrix} a & -c \\ d & -h \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$

Cramer's Rule

(1) $\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae-bd \neq 0$ $\quad \alpha.\beta$ 唯一解

✓(2) $ae-bd=0$, $\begin{vmatrix} -c & b \\ -h & e \end{vmatrix} = \begin{vmatrix} a & -c \\ d & -h \end{vmatrix} = 0$

$\alpha.\beta$ 無限多解

✓(3) Else $\alpha.\beta$ 無解

Case 1 $\quad ae-bd\neq 0$

Ex $\quad \frac{dy}{dx} = \frac{y-3}{x+y-1}$

Let $X = x-\alpha \quad Y = y-\beta$

$\frac{dy}{dx} = \frac{dY}{dX} = \frac{Y+\beta-3}{X+Y+\alpha+\beta-1}$ $\qquad \begin{cases} \beta-3=0 \\ \alpha+\beta-1=0 \end{cases} \Rightarrow \begin{cases} \alpha=-2 \\ \beta=3 \end{cases}$

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{Y+\beta-3}{X+Y+\alpha+\beta-1} \qquad \begin{cases}\beta-3=0\\ \alpha+\beta-1=0\end{cases} \Rightarrow \begin{cases}\alpha=-2\\ \beta=3\end{cases}$$

$$\frac{dY}{dX}=\frac{Y}{X+Y}$$

Let $\frac{Y}{X}=u \Rightarrow \frac{dY}{dX}=u+X\frac{du}{dX}$

$$u+X\frac{du}{dX}=\frac{u}{1+u} \Rightarrow X\frac{du}{dX}=\frac{u}{1+u}-u=\frac{-u^2}{1+u}$$

$$\Rightarrow -\frac{1+u}{u^2}\,du=\frac{1}{X}\,dX \Rightarrow \int-(u^{-2}+u^{-1})\,du=\int X^{-1}\,dX+C$$

$$\Rightarrow u^{-1}-\ln|u|=\ln|X|+C$$

$$\Rightarrow \frac{X}{Y}-\ln\left|\frac{Y}{X}\right|=\ln|X|+C$$

$$\Rightarrow \frac{x+2}{y-3}-\ln\left|\frac{y-3}{x+2}\right|=\ln|x+2|+C$$

Ex $\frac{dy}{dx}=\frac{2x-5y-9}{-4x+y+9}$

Let $X=x-\alpha \quad Y=y-\beta \qquad \begin{matrix}2\alpha-5\beta-9=0\\ -4\alpha+\beta+9=0\end{matrix} \Rightarrow \begin{cases}\alpha=2\\ \beta=-1\end{cases}$

$$\frac{dY}{dX}=\frac{2X-5Y+(2\alpha-5\beta-9)}{-4X+Y(-4\alpha+\beta+9)}$$

$$\frac{dY}{dX}=\frac{2-5\left(\frac{Y}{X}\right)}{-4+\left(\frac{Y}{X}\right)} \qquad \text{Let } \frac{Y}{X}=u \Rightarrow \frac{dY}{dX}=\frac{du}{dX}X+u$$

$$\frac{du}{dX}X+u=\frac{2-5u}{-4+u} \Rightarrow \frac{du}{dX}=\frac{1}{X}\,\frac{2-5u+4u-u^2}{-4+u}$$

$$\Rightarrow \int\frac{4-u}{u^2+u-2}\,du=\int\frac{1}{X}\,dX+C^* \qquad \frac{4-u}{u^2+u-2}=\frac{A}{u+2}+\frac{B}{u-1}$$

$A=-2 \quad B=1$

$$\Rightarrow \int\left(\frac{-2}{u+2}+\frac{1}{u-1}\right)du=\ln|X|+C^*$$

$$\Rightarrow -2\ln|u+2|+\ln|u-1|=\ln|X|+C^*$$

$$\Rightarrow (u+2)^{-2}(u-1)=CX$$

$$\Rightarrow \left(\frac{Y}{X}+2\right)^{-2}\left(\frac{Y}{X}-1\right)=CX$$

$$\Rightarrow \left(\frac{y+1}{x-2}+2\right)^{-2}\left(\frac{y+1}{x-2}-1\right)=C(x-2)$$

$$\Rightarrow \frac{(x-2)^2}{(2x+y-3)^2}\,\frac{-x+y+3}{x-2}=C(x-2) \Rightarrow -x+y+3=C(2x+y-3)^2$$

Ex $\frac{dy}{dx}=\left(\frac{2x+y-1}{x-2}\right)^2 \qquad$ Let $X=x-\alpha \quad Y=y-\beta$

Ex $\frac{dy}{dx} = \left(\frac{\quad}{x-2}\right)$ Let $X = x-\alpha$ $Y = y-\beta$

$$\frac{dY}{dX} = \left(\frac{2X+Y+2\alpha+\beta-1}{X+\alpha-2}\right)^2 \quad \begin{cases} 2\alpha+\beta-1=0 \\ \alpha-2=0 \end{cases} \Rightarrow \begin{cases} \alpha=2 \\ \beta=-3 \end{cases} \Rightarrow \begin{cases} X = x-2 \\ Y = y+3 \end{cases}$$

$$\frac{dY}{dX} = \left(2+\frac{Y}{X}\right)^2$$

Let $\frac{Y}{X} = u$ $\quad u'X + u = (2+u)^2$

$$\Rightarrow \int \frac{1}{(u+\frac{3}{2})^2+\frac{7}{4}}\,du = \ln|X| + c^*$$

$$\Rightarrow Y = \left(\frac{\sqrt{7}}{2}\tan\left(\frac{\sqrt{7}}{2}\ln|X| + c\right) - \frac{3}{2}\right)X$$

$$\Rightarrow y = Y-3 = \left(\frac{\sqrt{7}}{2}\tan\left(\frac{\sqrt{7}}{2}\ln|x-2| + c\right) - \frac{3}{2}\right)(x-2) - 3$$

case 2 & 3 $\quad ae-bd=0 \quad$ We have seen this before.

Ex $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y+4}$

Let $2x+y = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 2$

$$\frac{dv}{dx} - 2 = \frac{v-1}{2v-4} \Rightarrow \frac{dv}{dx} = \frac{5v-9}{2v-4} \Rightarrow \frac{2v-4}{5v-9}\,dv = dx$$

$$\Rightarrow \frac{2}{5}\int \frac{10v-20}{10v-18}\,dv = x + C$$

$$\Rightarrow \frac{2}{5}\int 1 - \frac{2}{10v-18}\,dv = x + C$$

$$\Rightarrow \frac{2}{5}v + \frac{2}{25}\int \frac{1}{5v-9}\,d(5v-9) = x + C$$

$$\Rightarrow \frac{2}{5}v + \frac{2}{25}\ln|5v-9| = x + C$$

$$\Rightarrow \frac{2}{5}(2x+y) + \frac{2}{25}\ln|10x+5y-9| = x + C$$