工程數學 HW4

Deadline: 6/6 17:00

(1~4) Find the general solution of the following systems. For each solution, you should show all real bases.

1. $\begin{cases} y_1' = y_1 - 2y_2 \\ y_2' = 3y_1 - 4y_2 \end{cases}$ $\quad C_1\begin{bmatrix}1\\1\end{bmatrix}e^{-t} + C_2\begin{bmatrix}2\\3\end{bmatrix}e^{-2t}$

2. $\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = 2y_1 + y_2 - y_3 \\ y_3' = -8y_1 - 5y_2 - 3y_3 \end{cases}$ $\quad C_1\begin{bmatrix}-3\\4\\2\end{bmatrix}e^{-t} + C_2\begin{bmatrix}-4\\5\\7\end{bmatrix}e^{-2t} + C_3\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{2t}$

3. $\begin{cases} y'_1 = 4y_1 + 3y_2 + t \\ y'_2 = -2y_1 - y_2 - 2t \end{cases}$ $\quad C_1\begin{bmatrix}1\\-1\end{bmatrix}e^{t} + C_2\begin{bmatrix}3\\-2\end{bmatrix}e^{2t} + \begin{bmatrix}-\frac{5}{2}t - \frac{13}{4}\\ 3t + \frac{7}{2}\end{bmatrix}$

4. $\begin{cases} y_1' = 10y_1 - 6y_2 + 10 - 10t - 10t^2 \\ y_2' = 6y_1 - 10y_2 + 4 - 20t - 6t^2 \end{cases}$ $\quad C_1\begin{bmatrix}3\\1\end{bmatrix}e^{8t} + C_2\begin{bmatrix}1\\3\end{bmatrix}e^{-8t} + \begin{bmatrix}t^2-1\\2t\end{bmatrix}$

5. Find the inverse transforms of the following function by convolution.

$F(s) = \dfrac{1}{(S-2)(S^2+1)}$ $\quad \frac{1}{5}\left(e^{2t} - \cos t - 2\sin t\right)$

6. Use Laplace transform to find $y(t)$.

$y'' - y' = e^t \cos t, y(0) = 0, y'(0) = 0$ $\quad \frac{e^t}{2}(\sin t - \cos t) + \frac{1}{2}$

7. Solve the integral equation by Laplace Transforms.

$f(t) = \sin 2t + \int_0^t f(\tau)\sin 2(t-\tau)d\tau$ $\quad \sqrt{2}\sin(\sqrt{2}t)$

8. Find the Laplace transform of the given function.

$f(t) = \begin{cases} 0, if\ 0 < t < 2 \\ t-3, if\ 2 < t < 3 \\ -1, if\ 3 < t \end{cases}$ $\quad e^{-2s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$

9. $f(t) = t\cos 2t, F(s) = ?$ $\quad \dfrac{s^2-4}{(s^2+4)^2}$

10. $F(s) = \frac{2S^2-6S+7}{S^3-4S^2+7S}, f(t) = ?$ $\quad 1 + e^{2t}\cos\sqrt{3}t$

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Engineering Math HW4

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1. $\begin{cases} y_1' = y_1 - 2y_2 \\ y_2' = 3y_1 - 4y_2 \end{cases}$

$$y_1'' = y_1' - 2y_2' = y_1' - 6y_1 + 8y_2$$

$$= y_1' - 6y_1 + 4y_1 - 4y_1' = -3y_1' - 2y_1$$

$$\Rightarrow y_1'' + 3y_1' + 2y_1 = 0$$

$$y_1 = c_1 e^{-t} + c_2 e^{-2t}, \quad y_1' = -c_1 e - 2c_2 e$$

$$2y_2 = c_1 e^{-t} + c_2 e^{-2t} + c_1 e^{-t} + 2c_2 e^{-2t}$$

$$\Rightarrow y_2 = c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t}$$

Ans. : $\begin{cases} y_1 = c_1 e^{-t} + 2c_2 e^{-2t} \\ y_2 = c_1 e^{-t} + 3c_2 e^{-2t} \end{cases}$

2. $\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = 2y_1 + y_2 - y_3 \\ y_3' = -8y_1 - 5y_2 - 3y_3 \end{cases}$

$$\vec{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \vec{y}$$

, let $\vec{y} = \vec{v} e^{\lambda t}$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda+1)(\lambda-2)(\lambda+2) = 0 \Rightarrow \lambda = -1, -2, 2$$

$$\lambda = -1, \quad \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

$$\lambda = -2, \quad \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$$

$$\lambda = 2, \quad \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\boxed{\vec{y} = c_1 \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} e^{2t}}$$

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3. $\vec{y}' = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t$, let $\vec{y}_H = \vec{v} e^{\lambda t}$

$\begin{vmatrix} 4-\lambda & 3 \\ -2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1 \vee 2$

$\lambda = 1$, $\begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2$, $\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\vec{y}_H = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t}$

let $\vec{y}_P = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$, $\vec{y}_P' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\langle i \rangle \begin{cases} 0 = 4a_1 + 3a_2 + 1 \\ 0 = -2a_1 - a_2 - 2 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{5}{2} \\ a_2 = 3 \end{cases}$

$\langle ii \rangle \begin{cases} a_1 = -\frac{5}{2} = 4b_1 + 3b_2 \\ a_2 = 3 = -2b_1 - b_2 \end{cases} \Rightarrow \begin{cases} b_1 = -\frac{13}{4} \\ b_2 = \frac{7}{2} \end{cases}$

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{2t} + \begin{bmatrix} -\frac{5}{2}t - \frac{13}{4} \\ 3t + \frac{7}{2} \end{bmatrix}$$

4. $\vec{y}' = \begin{bmatrix} 10 & -6 \\ 6 & -10 \end{bmatrix} \vec{y} + \begin{bmatrix} -10t^2 - 10t + 10 \\ -6t^2 - 20t + 4 \end{bmatrix}$, $\vec{y}_H = \vec{v} e^{\lambda t} \Rightarrow \begin{vmatrix} 10-\lambda & -6 \\ 6 & -10-\lambda \end{vmatrix} = 0$, $\lambda = \pm 8$

$\lambda = 8$, $\begin{bmatrix} 2 & -6 \\ 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = -8$, $\begin{bmatrix} 18 & -6 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Rightarrow \vec{y}_H = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{8t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-8t}$

$\vec{y}_P = \begin{bmatrix} a_1 t^2 + b_1 t + c_1 \\ a_2 t^2 + b_2 t + c_2 \end{bmatrix}$, $\vec{y}_P' = \begin{bmatrix} 2a_1 t + b_1 \\ 2a_2 t + b_2 \end{bmatrix}$

$\partial_2 : \begin{cases} 0 = 10a_1 - 6a_2 - 10 \\ 0 = 6a_1 - 10a_2 - 6 \end{cases} \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \end{cases}$

$\partial_1 : \begin{cases} 2a_1 = 10b_1 - 6b_2 - 10 = 2 \\ 2a_2 = 6b_1 - 10b_2 - 6 = 0 \end{cases} \Rightarrow \begin{cases} b_1 = 0 \\ b_2 = -2 \end{cases}$

$\partial_0 , \begin{cases} b_1 = 10c_1 - 6c_2 + 10 = 0 \\ b_2 = 6c_1 - 10c_2 + 4 = -2 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 0 \end{cases}$

$\Rightarrow \vec{y}_P = \begin{bmatrix} t^2 - 1 \\ -2t \end{bmatrix}$

$$\vec{y} = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{8t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-8t} + \begin{bmatrix} t^2 - 1 \\ 2t \end{bmatrix}$$ #

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5. $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\frac{1}{(s^2+1)}\right\} = e^{2t} * \sin t$, $\int_0^t \sin\tau\, e^{2(t-\tau)}\,d\tau = e^{2t}\underbrace{\int_0^t \sin\tau\, e^{-2\tau}\,d\tau}_{A}$

$A = -\cos\tau\, e^{-2\tau} - 2\sin\tau\, e^{-2\tau} - 4A \Rightarrow A = \frac{-e^{-2\tau}}{5}(\cos\tau + 2\sin\tau)$

$A\Big|_0^t = \frac{-e^{-2t}}{5}(\cos t + 2\sin t) + \frac{1}{5} \Rightarrow e^{2t}A = \mathcal{L}^{-1}\ \frac{1}{5}(e^{2t} - \cos t - 2\sin t)$ ※

6. $\mathcal{L}y'' - \mathcal{L}y' = \frac{s-1}{(s-1)^2+1} \Rightarrow s(s-1)\mathcal{L}y = \frac{s-1}{(s-1)^2+1}$, $\mathcal{L}y = \frac{1}{s}\cdot\frac{1}{(s-1)^2+1}$

$\mathcal{L}^{-1}\{\mathcal{L}y\} = \underbrace{\int_0^t e^{\tau}\sin\tau\,d\tau}_{A}$ $A = -\cos\tau\, e^{\tau} + \sin\tau\, e^{\tau} - A$

$\Rightarrow A = \frac{e^{\tau}}{2}(\sin\tau - \cos\tau)$

$A\Big|_0^t = \frac{e^t}{2}(\cos t - \sin t) - \frac{1}{2}\cdot(-1) = \frac{e^t}{2}(\cos t - \sin t) + \frac{1}{2}$ ※

7. $\mathcal{L}f = \mathcal{L}\sin 2t + \mathcal{L}f\cdot\mathcal{L}\sin 2t \Rightarrow \mathcal{L}f = \frac{\mathcal{L}\sin 2t}{1-\mathcal{L}\sin 2t} = \frac{\left(\frac{2}{s^2+2^2}\right)}{\left(\frac{s^2+2^2-2}{s^2+2^2}\right)} = \frac{2}{s^2+2}$

$\mathcal{L}f = 2\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+\sqrt{2}^2} \Rightarrow f = \sqrt{2}\sin(\sqrt{2}t)$ ※

8. $f = (t-3)\left(u(t-2) - u(t-3)\right) - u(t-3)$

$= (t-2-1)\,u(t-2) - (t-3)\,u(t-3) - u(t-3)$

$= (t-2)\,u(t-2) - (t-3)\,u(t-3) - u(t-2) - u(t-3)$

$\mathcal{L}\{f\} = e^{-2s}\frac{1}{s^2} - e^{-3s}\frac{1}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$ ※

9. $t \Leftrightarrow (-D_s) \Rightarrow \mathcal{L}\{t\cos 2t\} = -D_s\,\mathcal{L}\cos 2t$

$-D_s\frac{s}{s^2+4} = \frac{s^2+4-s(2s)}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$ ※

10. $F(s) = \frac{2s^2-6s+7}{s(s^2-4s+7)} = \frac{a}{s} + \frac{bs+c}{s^2-4s+7} \Rightarrow a(s^2-4s+7) + bs^2 + cs = 2s^2-6s+7$

$\Rightarrow \begin{cases} a+b=2 \\ -4a+c=-6 \\ 7a=7 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=1 \\ c=-2 \end{cases}$, $F(s) = \frac{1}{s} + \frac{s-2}{s^2-4s+7} = \frac{1}{s} + \frac{s-2}{(s-2)^2+\sqrt{3}^2}$

$\Rightarrow f(t) = 1 + e^{2t}\cos\sqrt{3}t$ ※