CALCULUS

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Contents

1 Notations

1.1 Sets

Symbol	Meaning
$\in$ $\notin$	set membership ex: "$a \in A$" means a is an element of the set A and "$a \notin B$" means a is not an element of set B
$\cup$	union of sets ex: $A \cup B$ defines the set of elements that belongs to A **or** B
$\cap$	intersection of sets ex: $A \cap B$ defines the set of elements that belongs to A **and** B
$\subset$	subset ex: $A \subset B$ means all elements of A are also elements of B
$\backslash$	minus operator for sets ex: $A \backslash B$ defines the set that contains all elements of A that are not elements of B
$]a, b[$ or (a, b)	open interval ex: $x \in]3, 5[$ (or $x \in (3, 5)$) means $3 < x < 5$.
$\emptyset$	the empty set, i.e. the set that has no elements.
$\mathbb{R}$	the set of all real numbers
$\mathbb{N}$	the set of natural numbers (i.e. 0,1,2,3,...)
$\mathbb{Z}$	the set of all integer numbers (i.e. ...,-3,-2,-1,0,1,2,3,...)
$\mathbb{R}^*$	the set of all real numbers different of zero ($\mathbb{R}^* = \mathbb{R} \setminus \{0\}$)
$\mathbb{R}_+$	the set of all positive real numbers
$\mathbb{R}_-$	the set of all negative real numbers
$\overline{\mathbb{R}}$	the extended real number line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, such that $\forall e \in \mathbb{R},\ -\infty \leq e \leq +\infty$

1.2 Quantifiers

Symbol	Meaning
$\forall$	universal quantification ex: "$\forall x \in \mathbb{R}$" means for all element x of $\mathbb{R}$
$\exists$	existential quantification ex: "$\exists x \in \mathbb{R}$" means it exists at least one element x of $\mathbb{R}$
$\exists!$	uniqueness quantification ex: "$\exists! x \in \mathbb{R}$" means it exists only one element x of $\mathbb{R}$

$\implies$	implication ex: "$x = 5 \implies x^2 = 25$" is true, but "$x^2 = 25 \implies x = 25$" is false since x can also take the value -5.
$\Leftrightarrow$	equivalence ex: $x^2 = 25 \Leftrightarrow \begin{cases} x = 5 \text{ or} \\ x = -5 \end{cases}$

2 Functions

2.1 Definitions

Definition 1: *We call an **ordered pair** a pair of objects (a, b) written in a certain order.*

In this definition, the objects a and b can be of the same nature, such as numbers, vectors, matrices, etc. or have different nature. The ordered pair (a, b) is different from the ordered pair (b, a), if $a \neq b$, because the order, in which the objects appear, is significant. In contrast, the unordered pair $\{a, b\}$ is the same as the unordered pair $\{b, a\}$.

Example 1 (Ordered pairs)**:**

- $(5, 25) \neq (25, 5)$
- $\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}, 5\right) \neq \left(5, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$
- $(b, 98) \neq (98, b)$

Definition 2: *We call a **relation** $\mathcal{R}$ a set of ordered pairs. Besides, we will say that $\mathcal{R}$ defines a relation between the set X and Y if $\forall a \in X$, $\exists b \in Y$ such that $(a, b) \in \mathcal{R}$*

A relation defines a mapping between two sets of objects X and Y, that is to say it associates every elements of the first set X to at least one element of the second set Y.For instance, the relation $\mathcal{R} = \{(0, 4), (0, 3), (2, 5), (5, 7)\}$ defines a relation between $X = \{0, 2, 5\}$ and $Y = \{3, 4, 5, 7, 8\}$. The fact that there is no object in X is mapped to $b = 8$ does not contradict the definition. However, this relation $\mathcal{R}$ does not define a relation between $X = \{0, 1, 2, 5\}$ and $Y = \{3, 4, 5, 7, 8\}$ because $a = 1$ is not mapped to an object in Y.

Example 2 (Relations)**:**

- $\{(5, 25),\ (4, 16),\ (6, 36), (1, 1),\ (0, 0),\ (9, 81),\ (4, 10)\}$ is a relation between the set $\{0,\ 1,\ 4,\ 5,\ 6\}$ and the set $\{0,\ 1,\ 10,\ 16,\ 25,\ 36,\ 81\}$
- $\left\{\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}, 5\right), \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}, 4\sqrt{2}\right), \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \sqrt{17}\right)\right\}$ is a relation between the set $\left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ and the set $\{5,\ 4\sqrt{2},\ \sqrt{17}\}$
- $\{(b, 98), (a, 97), (e, 101), (g, 103)\}$ is a relation between the set $\{a,\ b,\ e,\ g\}$ and the set $\{97,\ 98,\ 101,\ 103\}$

Definition 3: *A* ***function*** *is a relation between two sets X and Y that associates every object of the set X to exactly one object of the set Y.*

In other words, if f is a function, then $\forall (a,b) \in f,\ \nexists (c,d) \in f$ such that $a = c$ and $b \neq d$.

Example 3 (Functions)**:**

- $\{(5,25),\ (4,16),\ (6,36),(1,1),\ (0,0),\ (9,81),\ (4,10)\}$ is not a function because 4 is associated with 16 and 10. However, $\{(5,25),\ (4,16),\ (6,36),(1,1),\ (0,0),\ (9,81)\}$ is a function.
- $\left\{\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}, 5\right), \left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}, 4\sqrt{2}\right), \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \sqrt{17}\right)\right\}$ is a function.
- $\{(b,98),(a,97),(e,101),(g,103)\}$ is a relation between the set $\{a,\ b,\ e,\ g\}$ and the set $\{97,\ 98,\ 101,\ 103\}$ is a function.

The following notations are usually used to define a function:

Symbol	Meaning
$f : X \to Y$	it defines a function f that maps the elements of the set X to an element of the set Y ex: $f : \mathbb{N} \to \mathbb{R}$
$f : x \mapsto y$	it defines a function f that maps x to y ex: $f : x \mapsto \sqrt{x}$
$f(x)$	it is the image of x returned by the function f ("f of x") ex: if $f : x \mapsto x^3$, then $f(4) = 4^3 = 64$
Y^X	the set of all function from X to Y

Example 4 (Functions Notation)**:**

- $\left\{\begin{array}{llll} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array}\right.$
- $\left\{\begin{array}{llll} f: & \mathbb{R}^2 & \to & \mathbb{R} \\ & x & \mapsto & ||x|| \end{array}\right.$
- $\left\{\begin{array}{llll} f: & \mathbb{R}^2 & \to & \mathbb{R}^{\mathbb{R}} \\ & (a,b) & \mapsto & (g : x \mapsto ax + b) \end{array}\right.$

Exercise 1: Which of the following relations define a function:

- $\{(x,y) \in \mathbb{R}^2 \mid y = 4x + 2\}$
- $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$
- $\{(x,y) \in \mathbb{R}^2 \mid y^2 = 4x + 2\}$
- $\{(x,y) \in \mathbb{R}^2 \mid y^3 + 4x^2 = 16\}$

2.2 Domain and range of a function

Definition 4: *The **domain** of a function $f : X \to Y$ is the set of objects $\mathcal{D} = X$.*

Indeed,by definition of a function $f : X \to Y$, every objects in X is mapped to an object in Y. It is common to omit X when defining a function, in this case, the domain of definition will be assumed to be the set X that contains all the objects mapped to an object in Y by the function. For instance, consider the function $g : x \mapsto \sqrt{x}$, where the domain of definition is omitted, hence it is assumed that $X = \mathbb{R}_+$. Now, if we define the function $\left\{ \begin{array}{llll} h: & [5,10] & \to & \mathbb{R} \\ & x & \mapsto & \sqrt{x} \end{array} \right.$, its domain of definition is $X = [5,10]$ and we have $g \neq h$ since they are defined on different domains. Finding the domain of a function f consists in searching all the values of $x \in X$ that makes the function "works".

Example 5: Consider the function $f : (x,y) \mapsto \sqrt{x-y}$. The function f "works" only for $(x,y) \in \mathbb{R}^2$ such that $x - y \geq 0$. In this case, we can write the domain of f as $X = \{(x,y) \in \mathbb{R}^2 \mid x \geq y\}$.

Example 6: Consider the function $f : x \mapsto \frac{1}{\sin(x)}$. The function f is defined for $x \in \mathbb{R}$ such that $\sin(x) \neq 0$, hence we can write the domain of f as $X = \{x \in \mathbb{R} \mid sin(x) \neq 0\}$. In this case, we can try to have a more explicit definition of X since we know when the function sinus takes the value 0, hence we can rewrite the domain as $X = \{x \in \mathbb{R} \mid x \neq k\pi,\ k \in \mathbb{Z}\}$.

Exercise 2: Determine the domain of the following functions:

- $f : x \mapsto \sqrt{5 - |x|}$
- $g : x \mapsto \frac{\sqrt{3-x^2}}{\ln(5x)}$
- $h : (x,y) \mapsto \ln(x^2 - y^2)$

Definition 5: *The **range** of a function $f : X \to Y$ is the set of objects $R \subset Y$ such that for every object $y \in R$, there exists an object $x \in X$ mapped to y.*

The range R of a function $f : X \to Y$ is also called the **image** of the function and it is sometimes denoted as $f(X)$. This definition can be written with the quantifiers as follows:

$$y \in R \implies \exists x \in X \textit{ such that } f(x) = y$$

Finding the range of a function is not in general an easy task. For functions such $f : X \to Y$ where $Y \subset \mathbb{R}$, it generally involves the study of the extrema of the function. For instance, the range of the cosine function is $[-1,1]$ but the range of $\left\{ \begin{array}{llll} g: & [-\frac{\pi}{2}, \frac{\pi}{2}] & \to & \mathbb{R} \\ & x & \mapsto & \cos(x) \end{array} \right.$ is $[0,1]$.

2.3 Composition of Functions

Definition 6: *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. We called the composite function of f and g the function* $\left\{ \begin{array}{llll} g \circ f: & X & \to & Y \\ & x & \mapsto & f(g(x)) \end{array} \right.$

We read $g \circ f$ as "g composed with f" or "g of f". The operator "$\circ$" is always <u>associative</u> (i.e. $(g \circ f) \circ h = g \circ (f \circ h) = g \circ f \circ h$) and sometimes <u>commutative</u> (i.e. $g \circ f \neq f \circ g$ in the general case). Besides, if g can be composed with f, that does not imply f can be composed with g.

Exercise 3: Determine, when possible, the composite functions $g \circ f$ and $f \circ g$ and their domain of definition:

- $f = \{(0,1),(2,3),(4,4),(6,5),(8,6),(9,9)\}$ and $g : x \mapsto (x^2,\ x^2)$
- $f : x \mapsto \sqrt{x}$ and $g : x \mapsto x^2$
- $f : x \mapsto (2x,\ 3x+2)$ and $g : (x,y) \mapsto \ln(xy)$
- $f : x \mapsto |x|$ and $g : x \mapsto x^2$

2.4 Bijection, Injection, and Surjection

Definition 7: *Consider a function $f : X \to Y$, with $X \subset \mathcal{D}$ and $\mathcal{D}$ the domain of f, then:*

- *f is* ***injective*** *if:* $\forall (x,x') \in X^2,\ f(x) = f(x') \implies x = x'$;
- *f is* ***surjective*** *if:* $\forall y \in Y,\ \exists x \in X,$ *such that* $f(x) = y$;
- *f is* ***bijective*** *if f is injective and surjective.*

The fact that a function $f : X \to Y$ is injective and/or surjective is closely link to the definition of the sets X and Y. One sufficient condition to prove that f is not surjective is $Y \not\subset f(X)$.

Example 7: Consider the function $\left\{\begin{array}{llll} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array}\right.$

This function is not injective, since $f(5) = f(-5)$ but $5 \neq -5$. It is also not surjective, since $y = -3$ has no preimage.

However, we can define the function $\left\{\begin{array}{llll} g: & \mathbb{R} & \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 \end{array}\right.$ that is surjective but not injective, and also the functions $\left\{\begin{array}{llll} h_+: & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 \end{array}\right.$ and $\left\{\begin{array}{llll} h_-: & \mathbb{R}_- & \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 \end{array}\right.$ that are both injective and surjective.

Proposition 1: *If the function $f : X \to Y$ is bijective then $\forall y \in Y,\ \exists! x \in X,$ such that $f(x) = y$.*

Proof. Let $f : X \to Y$ be a bijective function and $y \in Y$. Since f is surjective, then there exists $x \in X$ such that $f(x) = y$.
We will prove the uniqueness of x by contradiction: Suppose there exists $x' \in X$, $x \neq x'$, such that $f(x) = f(x')$. Since f is injective, it implies that $x = x'$ which is a contradiction.
We can then conclude that $\exists! x \in X,$ *such that* $f(x) = y$.

$\square$

Proposition 2: *A function $f : X \to Y$ is bijective if and only there exists* ***an inverse function*** *$f^{-1} : Y \mapsto X$ such that $\forall (x,y) \in X \times Y,\ f(x) = y \Leftrightarrow f^{-1}(y) = x$.*

Exercise 4: Prove that the following functions are a bijection:

- $\left\{\begin{array}{llll} f: & [1,+\infty[& \to & \mathbb{R}_+ \\ & x & \mapsto & x^2 - 2x + 1 \end{array}\right.$
- $\left\{\begin{array}{llll} f: & \mathbb{R}^* & \to & \mathbb{R} \setminus \{1\} \\ & x & \mapsto & \frac{1}{x} + 1 \end{array}\right.$

- $\left\{\begin{array}{llll} f: & \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ & (x,y) & \mapsto & (x-2, 2y-5) \end{array}\right.$

Exercise 5: Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two bijective functions:

1) Prove that $g \circ f$ is also bijective
2) Give an example

2.5 Functions of a real variable

In this section, we will focus on functions of a real variable, their link to functions of several real variables, and their main properties. These properties are mainly derived from the fact that the set $\mathbb{R}$ has a relation of order (the operators "$<$" and "$>$").

Definition 8: ***A function of a real variable*** *is a function $f : X \rightarrow Y$ where $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$.*

Example 8: Some examples of functions of a real variable:

- $x \mapsto x^2 + 2$
- $x \mapsto \sin^2(x^2)$
- $x \mapsto \frac{x+2}{\sqrt{x}}$

Definition 9: ***A function of several real variables*** *is a function $f : X \rightarrow Y$ where $X \subset \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $Y \subset \mathbb{R}$.*

Example 9: Some examples of functions of several real variables:

- $(x,y) \mapsto x^2 + 2y$
- $(x,y,z) \mapsto \sqrt{x^2+y^2+z^2}$
- $(x,y,t) \mapsto \frac{xt+2}{\sqrt{y}}$

Definition 10: *A* ***piecewise function*** *is a function defined on a sequence of distinct intervals $I_1, ..., I_n$, $n > 1$, by a sequence of functions $f_1, ..., f_n$, respectively.*

Example 10: Some examples of piecewise functions:

- $x \mapsto \left\{\begin{array}{ll} x, & for\ x \geq 0 \\ -x, & otherwise \end{array}\right.$ (The absolute value function)
- $x \mapsto \left\{\begin{array}{ll} e^{\frac{1}{3}\ln(x)}, & for\ x > 0, \\ 0, & for\ x = 0, \\ -e^{\frac{1}{3}\ln(-x)}, & for\ x < 0 \end{array}\right.$ (The basic cubic function)
- $x \mapsto \left\{\begin{array}{ll} 1, & for\ x > 0, \\ 0, & otherwise \end{array}\right.$ (The unit step function)

Definition 11: *Given a function of a real variable $f : X \rightarrow Y$, its* ***graph*** *is the set*

$$\mathcal{G} = \{(x, f(x)) \mid x \in X\}.$$

In the rest of the section, we will consider only functions of a real variables. The following properties that we will see are specific to this class of functions, however, some of them may be extended to other type of functions. .

2.5.1 Increasing and Decreasing Functions

Definition 12: *Consider a function f defined on an interval I, then:*

- *f is **increasing** on I if: $\forall (x, x') \in I^2,\ x < x' \implies f(x) \leq f(x')$;*
- *f is **strictly increasing** on I if: $\forall (x, x') \in I^2,\ x < x' \implies f(x) < f(x')$;*
- *f is **decreasing** on I if: $\forall (x, x') \in I^2,\ x < x' \implies f(x) \geq f(x')$;*
- *f is **strictly decreasing** on I if: $\forall (x, x') \in I^2,\ x < x' \implies f(x) > f(x')$;*

Definition 13: *A **(strictly) monotonic function** (or a (strict) monotone function) is a function which is either entirely (strictly) increasing or (strictly) decreasing on its domain of definition.*

Proposition 3: *If the function f is strictly monotonic then f is injective.*

Proof. Let $f : X \to Y$ be a strictly monotonic function. We will use a proof by contradiction. Suppose there exists $(x_1, x_2) \in X^2$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.
Without lost of generality, assume that $x_1 < x_2$. Since f is strictly monotonic, then $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. In both cases, we infer $f(x_1) \neq f(x_2)$ which contradicts our hypothesis. Hence we conclude that it does not exist $(x_1, x_2) \in X^2$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. So, if $f(x_1) = f(x_2)$ then $x_1 = x_2$ and f is injective.

$\square$

Exercise 6: Prove that the following functions are monotone:

- $\left\{ \begin{array}{rrcl} f : & \mathbb{R}_+ & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array} \right.$
- $\left\{ \begin{array}{rrcl} g : & \mathbb{R}_+ & \to & \mathbb{R} \\ & x & \mapsto & \ln(x) \end{array} \right.$
- $\left\{ \begin{array}{rrcl} g : & [-1, +\infty[& \to & \mathbb{R}_+ \\ & x & \mapsto & \sqrt{x+1} \end{array} \right.$

Exercise 7: Let f and g be two increasing functions. Prove the following:

1) $f + g$ is increasing
2) $-f$ is decreasing
3) $g \circ f$ is increasing

2.5.2 Even and Odd Functions

Definition 14: *Consider a function $f : X \to Y$, then:*

- *f is **even** if: $\forall x \in X,\ -x \in X$ and $f(-x) = f(x)$;*
- *f is **odd** if: $\forall x \in X,\ -x \in X$ and $f(-x) = -f(x)$;*

Note that evenness and oddness definition can be extended to functions of several real variables and to complex valued functions. Geometrically, an even function is symmetric about the y-axis and an odd functions is symmetric about the origin.

Exercise 8: Prove that the following functions are even, odd, or neither:

- $f : x \mapsto 2^{x+3}$
- $g : x \mapsto x^3 + \frac{1}{x}$
- $h : x \mapsto \sqrt{4 - x^2}$
- $k : x \mapsto \frac{x^3 - x^2}{x-1}$

Proposition 4: *Let f and g be two functions:*

1. *If f and g are evens, then $f + g$, $f - g$, $f \times g$, and $\frac{f}{g}$ are evens;*
2. *If f and g are odds, then $f + g$ and $f - g$ are odds and $f \times g$ and $\frac{f}{g}$ are evens;*
3. *If f is even and g is odd, then $f \times g$ and $\frac{f}{g}$ are odds.*
3. *If f is odd, then $|f|$ is even.*

Proof. (of 1.)
Let f and g be two even functions both defined on a domain $\mathcal{D}$.
Let $x \in \mathcal{D}$. Since f (or g) is even then $-x \in \mathcal{D}$ and we have:

- $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x)$, thus $f + g$ is even;
- $(f - g)(-x) = f(-x) - g(-x) = f(x) - g(x)$, thus $f - g$ is even;
- $(f \times g)(-x) = f(-x) \times g(-x) = f(x) \times g(x)$, thus $f \times g$ is even.

If $g(x) = g(-x) \neq 0$, we have $(\frac{f}{g})(-x) = \frac{f(-x}{g(-x)} = \frac{f(x)}{g(x)}$, thus $\frac{f}{g}$ is even. □

Proof. (of 2. and 3.) The proof is very similar to the proof of 1. and let as an exercise. □

Proposition 5 (Even-Odd Decomposition)**:** *Let $f : X \to Y$ be a function and set $f_e : x \mapsto \frac{f(x)+f(-x)}{2}$ and $f_d : x \mapsto \frac{f(x)-f(-x)}{2}$. Then f_e is even and f_d is odd and we have $f = f_e + f_d$*

In this context, the hyperbolic cosine and the hyperbolic sine can be seen as the even part and the odd part, respectively, of the exponential function. Besides, we can note that, if $f = g + h$ with g even and h odd, then we have $g = f_e$ and $h = f_d$.

2.5.3 Periodic Functions

Periodicity of a functions is relevant when studying the behavior of a function. They allow to limit the study to a restricted intervals instead of considering the whole domain of definition.

Definition 15: *A function $f : X \to Y$ is periodic with period T, $T > 0$, when:*

$$\forall x \in X,\ f(x + T) = f(x).$$

A common example of periodic functions are the trigonometric functions. For instance, the sine function has a period of $T = 2\pi$, but also of 4π, 6π. etc. Hence, we refers in this case, 2π as the least period. An other example of periodic functions are the constant functions with any period $T \in \mathbb{R}_+^*$.

Exercise 9: Show that the following functions are periodic and find their least period:

- $f : x \mapsto 3\sin(3x + 2)$
- $g : x \mapsto |cos(x)|$

- $h : x \mapsto \begin{cases} 1, & for\ 10n \leq x < 10n+5,\ n \in \mathbb{Z}, \\ 0, & otherwise \end{cases}$
- $k : x \mapsto \cos(2x) + \sin(3x)$

Exercise 10: Let f and g be two functions.

- Prove if f is periodic, then $g \circ f$ is also periodic;
- Give an example of a function g and a periodic function f such that $f \circ g$ is not periodic.

Proposition 6: *If f is periodic, then f is not injective.*

Proof. Let f be a periodic function with period T, defined on a domain $\mathcal{D}$. Let x be an element of $\mathcal{D}$ and set $x' = x + T$. Then, we have $x \neq x'$, since $T > 0$, and $f(x) = f(x')$ by definition of periodic functions. This prove that f is not injective. □

This proposition tells us that a periodic function is not bijective and, by consequence, it does not have an inverse. However, one can note that the inverse trigonometric functions exist. Indeed, to be able to define these inverse functions, the domain of the trigonometric functions have been limited to one period where the function is bijective.
For instance, sine is not bijective on $\mathbb{R}$, but $\begin{cases} f : & [-\pi, \pi] & \to & [-1, 1] \\ & x & \mapsto & sin(x) \end{cases}$ is bijective and its inverse is $\begin{cases} f^{-1} : & [-1, 1] & \to & [-\pi, \pi] \\ & x & \mapsto & \arcsin(x) \end{cases}$. Note also that the choice of the domain is very important to have a bijective function, the function $\begin{cases} g : & [0, 2\pi] & \to & [-1, 1] \\ & x & \mapsto & sin(x) \end{cases}$ is not bijective even so $[0, 2\pi]$ has a radius of one period as $[-\pi, \pi]$.

3 Limits of functions of a real variable

3.1 Introduction

The notions of limits may have been seen during high-school. In this chapter, we will give their precise definitions as well as demonstrate their main properties.

Definition 16: ***A real function*** *or a* ***function of a real variable*** *is a function $f : \mathcal{D} \to \mathbb{R}$ which domain $\mathcal{D}$ is included in $\mathbb{R}$ ($\mathcal{D} \subset \mathbb{R}$).*

3.2 Neighborhood of a point

Studying the limit of a function f in a point a means studying the behavior of $f(x)$ when x is in the neighborhood (or very close) of a. In this section we will define a neighborhood of a point $a \in \mathbb{R}$ and then we will define the neighborhood of $+\infty$ and $-\infty$.

Definition 17: *A* ***neighborhood*** *of a point a in $\mathbb{R}$ is any open interval such as*

$$V_a(\alpha) =]a - \alpha,\ a + \alpha[,\ \textit{with}\ \alpha \in \mathbb{R}_+^*$$

that can be equivalently written as $V_a(\alpha) = \{x \in \mathbb{R}\ /\ |x - a| < \alpha\}$.

Hence, from this definition, one has to understand that a point $a \in \mathbb{R}$ has many neighborhoods (an infinite number) and each of them is characterized by the associated value of $\alpha \in \mathbb{R}_+^*$. From this definition, we can derive some properties that will be used later in the chapter.

Proposition 7: *If V and V' are two neighborhoods of $a \in \mathbb{R}$ then $V \cap V'$ is also a neighborhood of a.*

Proof: Let V and V' be two neighborhoods of $a \in \mathbb{R}$. By definition, there exists $\alpha \in \mathbb{R}_+^*$ and $\alpha' \in \mathbb{R}_+^*$ such that $V =]a-\alpha,\ a+\alpha[$ and $V' =]a-\alpha',\ a+\alpha'[$. Then, we can deduced:

$$\begin{aligned} x \in V \cap V' &\iff \begin{cases} a-\alpha < x < a+\alpha \quad \textit{and} \\ a-\alpha' < x < a+\alpha' \end{cases} \\ &\iff \begin{cases} a-\alpha'' < x < a+\alpha'' \quad \textit{with} \\ \alpha'' = \min\{\alpha,\ \alpha'\} \end{cases} \\ &\iff \begin{cases} x \in V_a(\alpha'') \qquad\qquad \textit{with} \\ \alpha'' = \min\{\alpha,\ \alpha'\} \end{cases} \end{aligned}$$

Hence, $V \cap V' = V_a(\alpha'')$, with $\alpha'' = \min\{\alpha,\ \alpha'\}$, then $V \cap V'$ is a neighborhood of a.

Proposition 8: *If a and b are distinct elements of $\mathbb{R}$, there exists a neighborhood V_a of a and a neighborhood V_b of b that have no element in common.*

Proof: Let a and b be two distinct elements of $\mathbb{R}$ (i.e. $a \neq b$). Suppose, without loss of generality, that $a < b$. We set $\alpha = \frac{b-a}{2}$. Hence $V_a(\alpha)$ and $V_b(\alpha)$ does not have any point in common ($V_a(\alpha) \cap V_b(\alpha) = \emptyset$). We will prove it by contradiction.
Assume $V_a(\alpha) \cap V_b(\alpha) \neq \emptyset$. Let $x \in V_a(\alpha) \cap V_b(\alpha)$ (x exists according to our hypothesis). Therefore:

$$\begin{aligned} |b-a| &= |b-x+x-a| \\ &\leq |a-x| + |x-b| \ \textit{(triangular inequality)} \\ &< \alpha + \alpha \ (\textit{since } x \in V_a(\alpha) \textit{ and } x \in V_b(\alpha)) \\ &< |b-a| \ \textit{which results in a contradiction.} \end{aligned}$$

Then, our hypothesis $V_a(\alpha) \cap V_b(\alpha) \neq \emptyset$ is false, which implies $V_a(\alpha) \cap V_b(\alpha) = \emptyset$.

Exercise 11: Let $I =]u, v[$ be an open interval of $\mathbb{R}$. Show that, for all $a \in I$, there exists a neighborhood V_a of a that is contained in I.

Exercise 12: Let $V_a(\alpha)$, $\alpha \in \mathbb{R}_+^*$, be a neighborhood of $a \in \mathbb{R}$ and let $V_b(\beta)$, $\beta \in \mathbb{R}_+^*$, be a neighborhood of $b \in \mathbb{R}$. Show that if $x \in V_a(\alpha)$ and $y \in V_b(\beta)$, then $(x+y) \in V_{a+b}(\alpha+\beta)$.

Definition 18: *A **neighborhood** of $+\infty$ is any interval such as*

$$V_{+\infty}(A) =]A,\ +\infty[,\ \textit{with } A \in \mathbb{R}_+^*$$

that can be equivalently written as $V_{+\infty}(A) = \{x \in \mathbb{R}\ /\ x > A\}$.
*In the same manner, a **neighborhood** of $-\infty$ is any interval such as*

$$V_{-\infty}(A) =]-\infty,\ A[,\ \textit{with } A \in \mathbb{R}_+^*$$

that can be equivalently written as $V_{-\infty}(A) = \{x \in \mathbb{R}\ /\ x < A\}$.

Exercise 13: Show that, for all real number a, there exists a neighborhood of a and a neighborhood of $+\infty$ that have no points in common.

3.3 Limit of a function at a point of $\mathbb{R}$

3.3.1 Finite limit at a point of $\mathbb{R}$

In this section, we will consider a function $f : \mathcal{D} \to \mathbb{R}$ and a real a, where $\mathcal{D}$ is the domain of definition of f. We suppose that any neighborhood of a contains at least one point of $\mathcal{D}$. That means two possibilities: f is defined at a (i.e. $a \in \mathcal{D}$) or f is not defined at a (i.e. $a \notin \mathcal{D}$) but only in "a neighborhood of a". For instance, if $\mathcal{D}$ is an interval and a does not belong to $\mathcal{D}$, this implies that a is one of the extremities of $\mathcal{D}$.

Definition 19: *Considering a real number l, we say that $f(x)$ approaches l as x approaches a or the limit of f as x approaches a is l if, for all neighborhood V_l of l, there exists a neighborhood V_a of a such that:*

$$x \in V_a \cap \mathcal{D} \implies f(x) \in V_l.$$

Which can be written with the quantifiers:

$$\forall \epsilon > 0,\ \exists \alpha > 0,\ \textit{such that}\ \forall x \in \mathcal{D},\ (|x-a| < \alpha \Longrightarrow\ |f(x) - l| < \epsilon).$$

In this case, we will say that f as a finite limit at a and this limit is l.

If f has a limit at a, then this limit is well defined and will be noted as $\lim\limits_{x \to a} f(x) = l$ or $\lim\limits_{a} f = l$ (We can also write "$f(x) \to l$ as $x \to a$"). To be clear, we cannot write $\lim\limits_{x \to a} f(x) = l$ as long as we have not proved that the limit exists. That is why, in all exercise on limits, you will be asked to study the limit of function, not to calculate it.

Proposition 9: *If the function f has a finite limit at a, this limit is unique.*

Proof: We will prove it by contradiction. Assume that f approaches l and also l', two distinct real numbers, as x approaches a. As we saw in proposition 8, there exists a neighborhood V_l and a neighborhood $V_{l'}$ that do not have any point in common (i.e. $V_l \cap V_{l'} \neq \emptyset$). Since f approaches l as x approaches a, by definition, there exists $\alpha > 0$ such that, for $x \in \mathcal{D}$, if $|x - a| < \alpha$ then $f(x) \in V_l$. Likewise, since f approaches l' as x approaches a, by definition, there exists $\alpha' > 0$ such that, for $x \in \mathcal{D}$, if $|x - a| < \alpha'$ then $f(x) \in V_{l'}$.
Let $x \in \mathcal{D}$ such that $|x - a| < \min\{\alpha, \alpha'\}$. Then, we have $f(x) \in V_l$ and $f(x) \in V_{l'}$ (which is equivalent saying $f(x) \in V_l \cap V_{l'}$), but this is impossible because $V_l \cap V_{l'} = \emptyset$ according to our hypothesis. Hence, the limit of f at a, if it exists, is necessarily unique.

Proposition 10: *If the function f is defined at point a (i.e. $a \in \mathcal{D}$) and has a finite limit at this point, this limit is necessarily $f(a)$.*

Proof: We will reason by contradiction. Assume that f has l as limit at $a \in \mathcal{D}$ and $l \neq f(a)$. Set $\epsilon = \frac{|f(a)-l|}{4}$. Then $\epsilon > 0$ and $f(a) \notin]l - \epsilon, l + \epsilon[$. Yet, by definition of the limit, there exists a real number $\alpha > 0$ such that for all $x \in \mathcal{D}$, if $|x - a| < \alpha$ then $|f(x) - l| < \epsilon$. Since f is defined at a, it applies in particular at $x = a$, hence we have $f(a) \in]l - \epsilon, l + \epsilon[$ which results in a contradiction.

Exercise 14: Let f be the identity function from $\mathbb{R}$ to $\mathbb{R}$ that is to say the application that satisfies for all $x \in \mathbb{R},\ f(x) = x$. Show that f has $l = a$ as limit at any point a of $\mathbb{R}$.

Exercise 15: Show that if f is a constant function on $\mathbb{R}$, it admits a finite limit at any point of $\mathbb{R}$.

Exercise 16: Let f be the function defined on $\mathbb{R}$ by $\forall x \in \mathbb{R}_+,\ f(x) = \sqrt{x}$.

1. Show that f has 0 as limit at 0.
2. Show that f has a limit at any point $a > 0$ and find this limit.
Tip: For all $a > 0$ and $x > 0$, show the inequality $|\sqrt{a} - \sqrt{x}| \leq \left(\frac{|a-x|}{\sqrt{a}}\right)$ by using the following equation $|\sqrt{a} - \sqrt{x}|.|\sqrt{a} + \sqrt{x}| = |a - x|$.

3.3.2 Infinite limit at a point of $\mathbb{R}$

Definition 20: *We say that $f(x)$ approaches $+\infty$ as x approaches a or f has $+\infty$ as limit at a if, for all neighborhood $V_{+\infty}$ of $+\infty$, there exists a neighborhood V_a of a such that:*

$$x \in V_a \cap \mathcal{D} \implies f(x) \in V_{+\infty}.$$

Which can be written with the quantifiers:

$$\forall A \in \mathbb{R},\ \exists \alpha > 0,\ \textit{such that } \forall x \in \mathcal{D},\ (|x - a| < \alpha \Longrightarrow f(x) > A).$$

In this case, we will use the notation $\lim\limits_{x \to a} f(x) = +\infty$ *or* $\lim\limits_{a} f = +\infty$*, or "$f(x) \to +\infty$ as $x \to a$".*

Similarly, we define the limit $-\infty$ of a function at a point of $\mathbb{R}$:

Definition 21: *We say that $f(x)$ approaches $-\infty$ as x approaches a or f has $-\infty$ as limit at a if, for all neighborhood $V_{-\infty}$ of $-\infty$, there exists a neighborhood V_a of a such that:*

$$x \in V_a \cap \mathcal{D} \implies f(x) \in V_{-\infty}.$$

Which can be written with the quantifiers:

$$\forall A \in \mathbb{R},\ \exists \alpha > 0,\ \textit{such that } \forall x \in \mathcal{D},\ (|x - a| < \alpha \Longrightarrow f(x) < A).$$

In this case, we will use the notation $\lim\limits_{x \to a} f(x) = -\infty$ *or* $\lim\limits_{a} f = -\infty$*, or "$f(x) \to -\infty$ as $x \to a$".*

Definition 22: *When a function f has $\pm\infty$ as limit at $a \in \mathbb{R}$, we say that f have a* ***vertical asymptote****.*

Exercise 17: Show that a function f cannot have a finite limit at a and approach at the same time $+\infty$ or $-\infty$ at a.

Exercise 18: Show that if a function f is defined at $a \in \mathcal{D}$,then f cannot approach $+\infty$ or $-\infty$ as x approaches a.

Exercise 19: Show that a function f cannot approach $+\infty$ and $-\infty$ at the same time.

Exercise 20: Show that the function $f : x \mapsto \frac{1}{x^2}$ has $+\infty$ as limit at 0.

3.3.3 Continuity of a function

Definition 23: *If the function f is defined at point $a \in \mathbb{R}$ and has a limit at this point, we say that the function f is* ***continuous*** *at point a.*

From 18, we can deduce that this limit at a is necessarily finite and, from proposition 10, this limit is equal to $f(a)$.

Remarque: If f has a finite limit $l \in \mathbb{R}$ at a, but either f is not defined at a or $f(a) \neq l$, then we say that f has a **removable discontinuity** at a. In this case, the function defined as:

$$g : x \mapsto \begin{cases} l \ if \ x = a \\ f(x) \ otherwise \end{cases}$$

is continuous at a.

Theorem 1 (Intermediate value theorem)**:** *If f is continuous on a closed interval $[a, b]$ and u is a point between $f(a)$ and $f(b)$, then there exist $c \in [a, b]$ such that $f(c) = u$.*

In other words if f is continuous on $[a, b]$, then it must takes all the values between $f(a)$ and $f(b)$.

Proof. Let f be a continuous function on $[a, b]$.
The cases $u = f(a)$ or $u = f(b)$ are obvious.
Without loss of generality, assume $f(a) < u < f(b)$ (The case $f(b) < u < f(a)$ has a similar proof).
Set $S = \{x \in [a, b] \mid f(x) < u\}$ and let c be the smallest number greater or equal to every $x \in S$ (c is called the supremum of S. Note that c exists because $S \neq \emptyset$, since $a \in S$, and S is bounded above by b. Note also that $c \in [a, b]$.We will prove that $f(c) = u$ by contradiction.

- Assume $f(c) > u$. Then if we set $\epsilon = f(c) - u > 0$, given$f$ is continuous at c, then there exists $\alpha > 0$ such that
$$|x - c| < \alpha \Rightarrow |f(x) - f(c)| < \epsilon$$
Therefore, for $x \in]c - \alpha, c]$, $f(x) > f(c) - \epsilon = u$. However, since c is the supremum of S, there exists $c^* \in]c - \alpha, c[$ such that $f(c^*) < u$ (otherwise, c cannot be the supremum of S because $c - \alpha$ would be an upper bound of S), which yields to a contradiction. We infer that $f(c) \leq u$.

- Assume $f(c) < u$. Then if we set $\epsilon = u - f(c) > 0$, given$f$ is continuous at c, then there exists $\alpha > 0$ such that
$$|x - c| < \alpha \Rightarrow |f(x) - f(c)| < \epsilon$$
Therefore, for $x \in [c, c + \alpha[$, $f(x) < f(c) + \epsilon = u$. Then, there exists $c^{**} \in]c, c + \alpha[$ such that $f(c^{**}) < u$, i.e $c^{**} \in S$, which yields to a contradiction because c is the supremum of S. We infer that $f(c) \geq u$.

Hence, $f(c) \leq u$ and $f(c) \geq u$, which implies $f(c) = u$.

□

Theorem 2 (The extreme value theorem)**:** *If f is continuous on a closed interval $[a, b]$, then f is bounded and reach its bounds, i.e. there exists $c \in [a, b]$ and $d \in [a, b]$ such that $\forall x \in [a, b]$, $f(c) \leq f(x) \leq f(d)$.*

Proof. Let f be a continuous function on $[a, b]$.
Since f is continuous on $[a, b]$, it must be bounded (we will admit this results). Let show that f reaches its upper bound (the proof for the lower bound is similar).
Let denote by M the least upper bound of f on $[a, b]$ (i.e. $\nexists M'$ such that $f(x) \leq M' < M$ for $x \in [a, b]$).
Assume f does not reach its bound, then $\forall x \in [a, b]$, $f(x) < M$.

Set $g : x \mapsto \frac{1}{M-f(x)}$. Note that g is defined and continuous on $[a,b]$, and g is strictly positive on $[a,b]$. Hence g is bounded on $[a,b]$ and there exists $K > 0$ such that, for $x \in [a,b]$, $g(x) \leq K$. Therefore:

$$\forall x \in [a,b], \; \frac{1}{M - f(x)} \leq K \;\Rightarrow\; f(x) \leq M - \frac{1}{K}$$

Then, we found $M' = M - \frac{1}{K}$ such that $f(x) \leq M' < M$ for $x \in [a,b]$, which is impossible since M is the least upper bound of f on $[a,b]$.
We infer that f must attain its upper bound and there exists $c \in [a,b]$ such that $f(c) = M$. □

3.4 Limit of a function at $+\infty$ or $-\infty$

In this section, we consider a function $f : \mathcal{D} \to \mathbb{R}$ and we now assume that any neighborhood of $+\infty$ contains a point of $\mathcal{D}$. This condition is satisfied, for instance, if $\mathcal{D}$ is an interval such as $]b, +\infty[$, which is the case most of the time.

Definition 24: *We say that f has $l \in \mathbb{R}$ as limit at $+\infty$ if, for all neighborhood V_l of l, there exists a neighborhood $V_{+\infty}$ of $+\infty$ such that:*

$$x \in V_{+\infty} \cap \mathcal{D} \implies f(x) \in V_l.$$

Which can be written with the quantifiers:

$$\forall \epsilon \in \mathbb{R}_+^*, \; \exists B \in \mathbb{R}, \; \textit{such that } \forall x \in \mathcal{D}, \; x > B \Longrightarrow \; |f(x) - l| < \epsilon).$$

In this case, we will use the notation $\lim\limits_{x \to +\infty} f(x) = l$ *or* $\lim\limits_{+\infty} f = l$, *or "$f(x) \to l$ as $x \to +\infty$".*

Exercise 21: Show that if f has $l \in \mathbb{R}$ as limit at $+\infty$, then for all $M > 0$, there exist a neighborhood of $+\infty$ where f is bounded by M, that is to say: $\forall M > 0$, $\exists B \in \mathbb{R}$ such that $\forall x \in \mathcal{D}$, $x \in]B, +\infty[\implies |f(x)| < M$.

Definition 25: *We say that f has $+\infty$ as limit at $+\infty$ if, for all neighborhood $V_{+\infty}$ of $+\infty$, there exist a neighborhood $W_{+\infty}$ of $+\infty$ such that:*

$$x \in W_{+\infty} \cap \mathcal{D} \implies f(x) \in V_{+\infty}.$$

Which can be written with the quantifiers:

$$\forall A \in \mathbb{R}, \; \exists B \in \mathbb{R}, \; \textit{such that } \forall x \in \mathcal{D}, \; x > B \Longrightarrow \; f(x) > A).$$

In this case, we will use the notation $\lim\limits_{x \to +\infty} f(x) = +\infty$ *or* $\lim\limits_{+\infty} f = +\infty$, *or "$f(x) \to +\infty$ as $x \to +\infty$".*

Definition 26: *We say that f has $-\infty$ as limit at $+\infty$ if, for all neighborhood $V_{-\infty}$ of $-\infty$, there exist a neighborhood $V_{+\infty}$ of $+\infty$ such that:*

$$x \in V_{+\infty} \cap \mathcal{D} \implies f(x) \in V_{-\infty}.$$

Which can be written with the quantifiers:

$$\forall A \in \mathbb{R}, \; \exists B \in \mathbb{R}, \; \textit{such that } \forall x \in \mathcal{D}, \; x > B \Longrightarrow \; f(x) < A).$$

In this case, we will use the notation $\lim\limits_{x \to +\infty} f(x) = -\infty$ *or* $\lim\limits_{+\infty} f = -\infty$, *or"$f(x) \to -\infty$ as $x \to +\infty$".*

For a function $g : \mathcal{D}' \to \mathbb{R}$ such that all neighborhood of $-\infty$ contains a point of $\mathcal{D}'$ (for instance, $\mathcal{D}'$ contains an interval such as $]-\infty,\ B[$), the limits of g at $-\infty$ are defined in a similar manner.

Definition 27: *When a function f has $l \in \mathbb{R}$ as limit at $\pm\infty$, we say that f have a **horizontal asymptote**.*

Proposition 11: *A function f cannot has at $+\infty$ (resp. at $-\infty$) two distinct limits, wether they are finite or infinite.*

Exercise 22: Prove the proposition 11.

Exercise 23: Write, using the quantifiers, the definition of the following limits: $\lim\limits_{-\infty} f = l$ ($l \in \mathbb{R}$), $\lim\limits_{-\infty} f = +\infty$, and $\lim\limits_{-\infty} f = -\infty$.

3.5 The set $\overline{\mathbb{R}}$

In order to have a uniform definition for the finite or infinite limits of a function at a point of $\mathbb{R}$ or at $\pm\infty$, we define a new set containing the set of real numbers and two other elements, $+\infty$ and $-\infty$.

Definition 28: *We call **extended real number line** the set:*

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}.$$

To have totally ordered set $\overline{\mathbb{R}}$, we define the following order of relation which extends the one in $\mathbb{R}$:

$$\forall c \in \mathbb{R},\ \ -\infty < c < +\infty.$$

This set $\overline{\mathbb{R}}$ allows to state in a synthetic manner the definition of limits:

Definition 29: *Let a and l be two elements (finite or infinite) of $\overline{\mathbb{R}}$ and $f : \mathcal{D} \to \mathbb{R}$ a function such that all neighborhood of a intersects with $\mathcal{D}$. Then the function f has l as limit at a if, for all neighborhood W_l of l, there exists a neighborhood V_a of a such that:*

$$x \in V_a \cap \mathcal{D} \implies f(x) \in W_l.$$

Note: The elements $+\infty$ and $-\infty$ are not numbers. They should be used only to write limits.

Similarly, we can state the propositions on the uniqueness of a limit:

Proposition 12: *If a function f has a (finite or infinite) limit at $a \in \overline{\mathbb{R}}$, this limit is unique.*

3.6 Limits and relation of order

In this section f, g, and h are functions defined on $\mathcal{D}$ and a is an element (finite or infinite) of $\overline{\mathbb{R}}$ such taht all neighborhood of a intersects with $\mathcal{D}$.

Proposition 13 (Preservation of Inequalities for Limits of Functions): *If f has $l \in \mathbb{R}$ as limit at $a \in \overline{\mathbb{R}}$ and if there exists a real number r such that, for all $x \in \mathcal{D}$, we have $r \leq f(x)$ (resp. $f(x) \leq r$) then $r \leq l$ (resp. $l \leq r$).*

Proof: Let f be a function that has $l \in \mathbb{R}$ as limit at $a \in \overline{\mathbb{R}}$ and $r \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $r \leq f(x)$. We will prove it by contradiction, hence we assume that $l < r$. Set $\epsilon = r - l > 0$ (since $l < r$ from our hypothesis). Since $\lim\limits_{a} f = l$, by definition, there exists a neighborhood V_a of a such that $x \in V_a \cap \mathcal{D} \implies |f(x) - l| < \epsilon$.
Let V_a be such a neighborhood and $x \in V_a \cap \mathcal{D}$, therefore

$$\begin{aligned} |f(x) - l| < \epsilon &\iff l - \epsilon < f(x) < l + \epsilon \\ &\iff l - (r - l) < f(x) < l + (r - l) \\ &\iff 2.l - r < f(x) < r \end{aligned}$$

This results in a contradiction because $r \leq f(x)$. Our hypothesis is then wrong and we have necessarily $r \leq l$.

Be aware that a strict inequality is not preserved since equality can occur in the general case. For instance, if $\mathcal{D} = \mathbb{R}_+$ and $f(x) = \frac{x}{x+1}$, we have $f(x) < 1$ for all $x > 0$ but $\lim\limits_{+\infty} f = 1$.
Remind that "$f \leq g$ sur $\mathcal{D}$" means that the function f is lower than or equal to g at all point of its domain of definition, that is to say "$f(x) \leq g(x)$ for all $x \in \mathcal{D}$".

Proposition 14 (**The squeeze theorem**)**:** *Let f, g, and h be three functions defined on $\mathcal{D}$, and $a \in \overline{\mathbb{R}}$:*

1. *If $f \leq g \leq h$ on $\mathcal{D}$ and $\lim\limits_{a} f = \lim\limits_{a} h = l$, $l \in \mathbb{R}$, then $\lim\limits_{a} g = l$.*

2. *If $f \leq g$ on $\mathcal{D}$ and $\lim\limits_{a} f = +\infty$, then $\lim\limits_{a} g = +\infty$.*

3. *If $g \leq h$ on $\mathcal{D}$ and $\lim\limits_{a} h = -\infty$, then $\lim\limits_{a} g = -\infty$.*

Proof (**of 1.**)**:** Let f, g, and h be three functions defined on $\mathcal{D}$, and $a \in \overline{\mathbb{R}}$ such that $f \leq g \leq h$ on $\mathcal{D}$, $\lim\limits_{a} f = \lim\limits_{a} h = l$, with $l \in \mathbb{R}$.
Let $\epsilon \in \mathbb{R}_+^*$. Then, by definition, there exists a neighborhood V_a of a such that

$$x \in V_a \cap \mathcal{D} \implies l - \epsilon < f(x) < l + \epsilon.$$

Also, there exists a neighborhood V'_a of a such that

$$x \in V'_a \cap \mathcal{D} \implies l - \epsilon < h(x) < l + \epsilon.$$

Let $W = V_a \cap V'_a$, according to proposition 7, W is also a neighborhood of a, hence:

$$\begin{aligned} x \in W \cap \mathcal{D} &\implies \begin{cases} l - \epsilon < f(x) < l + \epsilon & et \\ l - \epsilon < h(x) < l + \epsilon \end{cases} \\ &\implies \begin{cases} l - \epsilon < f(x) \leq g(x) & and \\ g(x) \leq h(x) < l + \epsilon \end{cases} \\ &\implies l - \epsilon < f(x) \leq g(x) \leq h(x) < l + \epsilon \\ &\implies l - \epsilon < g(x) < l + \epsilon \end{aligned}$$

Therefore, we prove that for $\epsilon \in \mathbb{R}_+^*$, there exists a neighborhood W ($W = V_a \cap V'_a$) such that

$$x \in W \cap \mathcal{D} \implies l - \epsilon < g(x) < l + \epsilon.$$

Hence, we conclude that g has a limit at $a$and this limit is l.

Exercise 24: Prove points 2. and 3. of proposition 14.

Exercise 25: Let $f : x \mapsto \frac{sin(x)}{x}$. Show that $|f|$ has a limit at $+\infty$ and this limit is 0. Infer that f has also 0 has limit at $+\infty$.

3.7 Some classic limits

$$\lim_{x\to 0}\sin(x)=0 \qquad \lim_{x\to 0}\cos(x)=1$$
$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}$$
$$\lim_{x\to 0}\ln(x)=-\infty \qquad \lim_{x\to +\infty}\ln(x)=+\infty$$
$$\lim_{x\to -\infty}e^x=0 \qquad \lim_{x\to +\infty}e^x=+\infty$$

For any real number $\alpha>0$ we also have:

$$\lim_{x\to 0}x^\alpha\ln(x)=0 \qquad \lim_{x\to +\infty}\frac{\ln(x)}{x^\alpha}=0$$
$$\lim_{x\to -\infty}|x|^\alpha e^x=0 \qquad \lim_{x\to +\infty}\frac{e^x}{x^\alpha}=+\infty$$

In the sequel, we will admit that the trigonometric, logarithm, exponential, and power functions are continuous at any point a of their respective domain of definition. (Their limit at a is then their value at a by definition 23).

3.8 Limit of a sum, product, quotient

To state in a synthetic manner the theorems on limits of $f+g$, $f\times g$, ..., we will extend the addition and multiplication operations of real number to the set $\overline{\mathbb{R}}$. We will take care that these operations are not always defined between two elements of $\overline{\mathbb{R}}$.
Addition: For any real number c, we set:

- $c+(+\infty)=(+\infty)+c=c-(-\infty)=(+\infty)$
- $(+\infty)+(+\infty)=(+\infty)$
- $c+(-\infty)=(-\infty)+c=c-(+\infty)=(-\infty)$
- $(-\infty)+(-\infty)=(-\infty)$

However, the following operations are **not defined**:

- $(+\infty)+(-\infty)$
- $(-\infty)+(+\infty)$
- $(+\infty)-(+\infty)$
- $(-\infty)-(-\infty)$

Multiplication: we set:

- $(+\infty)\times(+\infty)=(+\infty)$
- $(-\infty)\times(-\infty)=(+\infty)$
- $(+\infty)\times(-\infty)=(-\infty)\times(+\infty)=(-\infty)$

For any real number $c>0$:

- $c\times(+\infty)=(+\infty)\times c=(+\infty)$
- $c\times(-\infty)=(-\infty)\times c=(-\infty)$

For any real number $c < 0$:

- $c \times (+\infty) = (+\infty) \times c = (-\infty)$
- $c \times (-\infty) = (-\infty) \times c = (+\infty)$

However, the following operations are **not defined**:

- $(+\infty) \times 0$
- $0 \times (+\infty)$
- $(-\infty) \times 0$
- $0 \times (-\infty)$

Theorem 3: *If the functions f and g have a limit (finite or infinite) at $a \in \overline{\mathbb{R}}$ and if we set $\lim\limits_a f = l$ and $\lim\limits_a g = m$ with $l \in \overline{\mathbb{R}}$ and $m \in \overline{\mathbb{R}}$, then:*

1. *If $l + m$ is defined, then*
$$\lim_a (f+g) = l + m$$
If $l + m$ is not defined, we say that we have an indeterminate form.

2. *If $l \times m$ is defined, then*
$$\lim_a (f \times g) = l \times m$$
If $l \times m$ is not defined, we say that we have an indeterminate form.

3. *In particular, with $g : x \mapsto \lambda$, $\lambda \in \mathbb{R}$, then*
$$\lim_a (\lambda f) = \lambda l$$

Proof ((de 1.))**:**
Let f and g be two function that have a limit (finite or infinite) at $a \in \overline{\mathbb{R}}$ and we set $\lim\limits_a f = l$ and $\lim\limits_a g = m$ with $l \in \overline{\mathbb{R}}$ and $m \in \overline{\mathbb{R}}$. We set $\mathcal{D}_f$ and $\mathcal{D}_g$ the domain of definition of f and g, respectively.
Case 1: Assume that $l \in \mathbb{R}$ and $m \in \mathbb{R}$
We will show that $(f+g)$has a limit at a and that this limit is $(l+m)$.
Let $\epsilon > 0$, and we set $\epsilon' = \frac{\epsilon}{2} (> 0)$.
Since $\lim\limits_a f = l$, by definition, there exists a neighborhood V_a of a such that $\forall x \in \mathcal{D}_f$:

$$(x \in V_a \Longrightarrow |f(x) - l| < \epsilon')$$

Since $\lim\limits_a g = m$, by definition, there exists a neighborhood V'_a of a such that $\forall x \in \mathcal{D}_g$:

$$(x \in V'_a \Longrightarrow |g(x) - m| < \epsilon')$$

Set $W_a = V_a \cap V'_a$, by proposition, W_a is also a neighborhood of a and, hence, $W_a \neq \emptyset$, then we have $\forall x \in \mathcal{D}_f \cap \mathcal{D}_g$:

$$
\begin{aligned}
x \in W_a &\Longrightarrow \begin{cases} x \in V_a, \text{ and} \\ x \in V'_a \end{cases} \\
&\Longrightarrow \begin{cases} |f(x) - l| < \epsilon', \text{ and} \\ |g(x) - m| < \epsilon' \end{cases} \\
&\Longrightarrow \begin{cases} -\epsilon' < f(x) - l < \epsilon', \text{ and} \\ -\epsilon' < g(x) - m < \epsilon' \end{cases} \\
&\Longrightarrow -2.\epsilon' < (f(x) - l) + (g(x) - m) < 2.\epsilon' \\
&\Longrightarrow |(f(x) + g(x)) - (l + m)| < 2.\epsilon' = \epsilon
\end{aligned}
$$

Therefore, we showed that for all $\epsilon > 0$, there exists a neighborhood of a ($W_a = V_a \cap V'_a$) such that for all $x \in \mathcal{D}_f \cap \mathcal{D}_g = \mathcal{D}_{f+g}$, if $x \in W_a$, then $|(f(x) + g(x)) - (l + m)| < \epsilon$. Hence $(f + g)$ has a limit at a and this limit is $(l + m)$.

Case 2: Assume that $l \in \mathbb{R}$ **and** $m = +\infty$
We will show that $(f + g)$ has a limit at a and this limit is $+\infty$.
Let $A \in \mathbb{R}$ and $\epsilon > 0$, and we set $A' = A - l + \epsilon$.
Since $\lim_a f = l$, by definition, there exists a neighborhood V_a of a such that $\forall x \in \mathcal{D}_f$:

$$(x \in V_a \Longrightarrow |f(x) - l| < \epsilon)$$

Since $\lim_a g = m$, by definition, there exists a neighborhood V'_a de a such that $\forall x \in \mathcal{D}_g$:

$$(x \in V'_a \Longrightarrow g(x) > A')$$

Set $W_a = V_a \cap V'_a$, by proposition, W_a is also a neighborhood of a and, hence, $W_a \neq \emptyset$, then we have $\forall x \in \mathcal{D}_f \cap \mathcal{D}_g$:

$$
\begin{aligned}
x \in W_a &\Longrightarrow \begin{cases} x \in V_a, \text{ and} \\ x \in V'_a \end{cases} \\
&\Longrightarrow \begin{cases} |f(x) - l| < \epsilon, \text{ and} \\ g(x) > A' \end{cases} \\
&\Longrightarrow \begin{cases} l - \epsilon < f(x) < l + \epsilon, \text{ et} \\ A' < g(x) \end{cases} \\
&\Longrightarrow A' + l - \epsilon < f(x) + g(x) \\
&\Longrightarrow A < f(x) + g(x)
\end{aligned}
$$

Therefore, we showed that for all $A \in \mathbb{R}$, there exists a neighborhood of a ($W_a = V_a \cap V'_a$) such that for all $x \in \mathcal{D}_f \cap \mathcal{D}_g = \mathcal{D}_{f+g}$, if $x \in W_a$, then $(f(x) + g(x) > A$. Hence $(f + g)$ has a limit at a and this limit is $+\infty$.

In a similar manner, we can prove the rest of the cases, that is to say ($l \in \mathbb{R}$ and $m = -\infty$), ($l = m = +\infty$) and ($l = m = -\infty$). Try to do it as an exercise to practice.

Proof ((of 2.))**:**
Let f and g be two function that have a limit (finite or infinite) at $a \in \overline{\mathbb{R}}$ and we set $\lim_a f = l$ and $\lim_a g = m$ with $l \in \overline{\mathbb{R}}$ and $m \in \overline{\mathbb{R}}$. We set $\mathcal{D}_f$ and $\mathcal{D}_g$ the domain of definition of f and g, respectively.
Case 1: Assume that $l \in \mathbb{R}$ **and** $m \in \mathbb{R}$
We will show that $(f \times g)$has a limit at a and that this limit is $(l \times m)$.
Let $\epsilon > 0$, and set $\epsilon' = \frac{-(|l|+|m|)+\sqrt{(|l|+|m|)^2+4.\epsilon}}{2}$ (> 0). ϵ' is a root of the polynomial $\epsilon'^2 + (|l| + |m|)\epsilon' - \epsilon$

Since $\lim_a f = l$, by definition, there exists a neighborhood V_a of a such that $\forall x \in \mathcal{D}_f$:

$$(x \in V_a \Longrightarrow |f(x) - l| < \epsilon')$$

Since $\lim_a g = m$,by definition, there exists a neighborhood V'_a of a such that $\forall x \in \mathcal{D}_g$:

$$(x \in V'_a \Longrightarrow |g(x) - m| < \epsilon')$$

We set $W_a = V_a \cap V'_a$, by proposition, W_a is also a neighborhood of a and, hence, $W_a \neq \emptyset$, then we have $\forall x \in \mathcal{D}_f \cap \mathcal{D}_g$:

$$\begin{array}{rcl} |f(x) \times g(x) - l \times m| & = & |(f(x) - l + l) \times (g(x) - m + m) - l \times m| \\ & = & |(f(x) - l) \times m + (g(x) - m) \times l + (f(x) - l) \times (g(x) - m)| \\ & \leq & |(f(x) - l)| \times |m| + |(g(x) - m)| \times |l| + |(f(x) - l)| \times |(g(x) - m)| \\ & < & \epsilon' \times |m| + \epsilon' \times |l| + \epsilon'^2 \\ & < & \epsilon'^2 + (|l| + |m|) \times \epsilon' = \epsilon \end{array}$$

Therefore, we showed that for all $\epsilon > 0$, there exists a neighborhood of a ($W_a = V_a \cap V'_a$) such that for all $x \in \mathcal{D}_f \cap \mathcal{D}_g = \mathcal{D}_{f \times g}$, if $x \in W_a$, then $|(f(x) \times g(x)) - (l.m)| < \epsilon$. Hence $(f \times g)$ has a limit at a and this limit is $l \times m$.

Case 2: Assume that $l \in \mathbb{R}_+^*$ ($l > 0$) and $m = +\infty$
We will show that $(f \times g)$has a limit at a and that this limit is $(m = +\infty)$.
Let $A \in \mathbb{R}$ and $\epsilon > 0$ such that $\epsilon < l$ (ϵ exists because $l > 0$).
Set $B = \frac{|A|}{l-\epsilon}$, hence we have $A > 0$,
Since $\lim_a f = l$, by definition, there exists a neighborhood V_a of a such that $\forall x \in \mathcal{D}_f$:

$$(x \in V_a \Longrightarrow |f(x) - l| < \epsilon')$$

Since $\lim_a g = m = +\infty$, by definition, there exists a neighborhood V'_a of a such that $\forall x \in \mathcal{D}_g$:

$$(x \in V'_a \Longrightarrow g(x) > B)$$

We set $W_a = V_a \cap V'_a$, by proposition, W_a is also a neighborhood of a and, hence, $W_a \neq \emptyset$, then we have $\forall x \in \mathcal{D}_f \cap \mathcal{D}_g$:

$$\begin{array}{rcl} f(x) \times g(x) & > & (l - \epsilon) \times B \ car \ f(x) > 0, \quad g(x) > 0, (l - \epsilon) > 0 \ et \ A > 0 \\ & > & |A| \\ & > & A \end{array}$$

Therefore, we showed that for all $A \in \mathbb{R}$, there exists a neighborhood of a ($W_a = V_a \cap V'_a$) such that for all $x \in \mathcal{D}_f \cap \mathcal{D}_g = \mathcal{D}_{f+g}$, if $x \in W_a$, then $(f(x) \times g(x))| > A$. Hence $(f \times g)$ has a limit at a and this limit is $m = +\infty$.

In a similar manner, we can prove the rest of the cases, that is to say ($l \in \mathbb{R}_-^*$ and $m = -\infty$), ($l \in \mathbb{R}_-^*$ and $m = +\infty$), ($l \in \mathbb{R}_+^*$ and $m = -\infty$), ($l = m = +\infty$), ($l = m = -\infty$) and ($l = +\infty$ and $m = -\infty$). Try to do it as an exercise to practice.

Remarque: The theorem 3 is directly extended by induction to the limit of a sum or a product of more than two functions.

Theorem 4: *We keep the previous notation but we assume for all $x \in \mathcal{D}_g$, $g(x) \neq 0$.*

1. *If $m \in \mathbb{R}^*$ and $l \in \overline{\mathbb{R}}$, then* $\lim_a \frac{f}{g} = \frac{l}{m}$

2. *If $m = +\infty$ or $m = -\infty$ and if $l \in \mathbb{R}$, then* $\lim_a \frac{f}{g} = 0$

3. *If $m = 0$ and $g(x) > 0$ for all x in a neighborhood of a: we say in this case that g aaproaches 0 from the right (or positive side) and we write $\lim_a g = 0^+$. Then, if $l \in \mathbb{R}_+^*$ or if $l = +\infty$, we have $\lim_a \frac{f}{g} = +\infty$. If $l \in \mathbb{R}_-^*$ or if $l = -\infty$, we have $\lim_a \frac{f}{g} = -\infty$. We have a similar results (mutatis mutandis) when $\lim_a g = 0^-$.*

4. *If the quotient $\frac{l}{m}$ take one of the following form*

$$\frac{0}{0}, \qquad \frac{+\infty}{+\infty}, \qquad \frac{+\infty}{-\infty}, \qquad \frac{-\infty}{+\infty}, \qquad \frac{-\infty}{-\infty}$$

   there is no general results (we say that we have an undetermined form).

Proof: Similar to the proof of theorem 3. Try to do it as an exercise.

Remarque: We can infer from the previous propositions that a polynomial function, i.e. function of the form $P : x \mapsto c_0 + \ldots + c_n x^n$ where $c_0 + \ldots + c_n$ are fixed real numbers, has $P(a)$ as limit at any point $a \in \mathbb{R}$. Indeed, we have $\lim_{x \to a} x = a$. By product, we get for all $k \in \mathbb{N}^*$ $\lim_{x \to a} x^k = a^k$, then $\lim_{x \to a} c_k x^k = c_k a^k$, and finally $\lim_{x \to a} P(x) = P(a)$ by sum. Since $\lim_{x \to +\infty} x = +\infty$, we have $\lim_{x \to +\infty} x^k = +\infty$ for all $k \in \mathbb{N}^*$ by product of limits, which allows us to get the limit of P at $+\infty$ after factoring the term with the highest degree.
What can be said of $\lim_{x \to -\infty} x^k$?

Remarque: Since a polynomial function has a limit at any point of $\mathbb{R}$, form the previous propositions, we can infer that all rational function (i.e. the quotient of the form $f = \frac{P}{Q}$ of two polynomial functions) has a limit at any point of its domain of definition.Hence, if $a \in \{x \in \mathbb{R}; Q(x) \neq 0\}$, then $\lim_a f = \frac{P(a)}{Q(a)}$.

Exercise 26: Let f and g be two functions defined on $\mathbb{R}$ by $f(x) = x^2$ and $g(x) = \sin(x)$. Study the limits of f, g, and $f \times g$ at any point of $\mathbb{R}$ and at $+\infty$.
Hint: The polynomial function $f : x \mapsto x^2$ and the function $g : x \mapsto \sin(x)$ are continuous at any point $a \in \mathbb{R}$. Then we have $\lim_a f = a^2$ and $\lim_a g = \sin(a)$ for all $a \in \mathbb{R}$. Besides, $\lim_{+\infty} f = +\infty$ and g does not have a limit at $+\infty$.

Theorem 5: ***Case of bounded functions***

1. *If $\lim_a f = 0$ and if there exists a real number M and a neighborhood of a such that $|g(x)| \leq M$ for all x in this neighborhood (i.e. if g is bounded in a neighborhood of a), then*

$$\lim_a (f \times g) = 0$$

2. *If $\lim_a f = +\infty$ and if there exists a real number $A > 0$ such that $g(x) \geq A$ for all x in a neighborhood of a, then*

$$\lim_a (f \times g) = +\infty$$

3. *If* $\lim_a f = +\infty$ *and if there exists a real number* M *such that* $g(x) \geq M$ *for all* x *in a neighborhood of* a *(i.e. if* g *has a lower bound in a neighborhood of* a*), then*

$$\lim_a (f+g) = +\infty$$

4. *If* $\lim_a f = -\infty$ *and if there exists a real number* M *suc that* $g(x) \leq M$ *for all* x *in a neighborhood of* a *i.e. if* g *has a upper bound in a neighborhood of* a*), then*

$$\lim_a (f+g) = -\infty$$

Proof: Left as an exercise.

Exercise 27: Study the limit of the following functions : $\lim\limits_{x\to+\infty} \dfrac{1+\sin(x)}{x.(2+\cos^2(x))}$ and $\lim\limits_{x\to+\infty} \dfrac{1}{1+|x|}.\sin(x)$

3.9 Limits of composite functions

Notations: We consider a function $f : \mathcal{D} \longrightarrow \mathbb{R}$ that has $b \in \overline{\mathbb{R}}$ as limit at $a \in \overline{\mathbb{R}}$. We also consider a non-empty subset E of $\mathbb{R}$ and a function $g : E \longrightarrow \mathbb{R}$ that has $l \in \overline{\mathbb{R}}$ as limit at b (it infers that any neighborhood of b intersects with E). Besides, we consider $f(\mathcal{D}) \subseteq E$: the composite function $g \circ f$ is then defined on $\mathcal{D}$.

Proposition 15 ((Composition of limits)): *Using the notations introduced above, assume that* $\lim_a f = b$ *and that* $\lim_b g = l$*. Then* $g \circ f$ *has* l *as limit at* a*.*

Proof: Using the notations introduced above, assume that $\lim_a f = b$ and that $\lim_b g = l$.
We will show that $g \circ f$ has l as limit at a.
Let V_l be a neighborhood of l.
Since $\lim_b g = l$, by definition, there exists a neighborhood V_b of b such that for all $x \in E$, if $x \in V_b$, then $g(x) \in V_l$.
Since $\lim_a f = b$ and V_b is a neighborhood of b, there exists a neighborhood V_a of a such that for all $x \in \mathcal{D}$, if $x \in V_a$, then $f(x) \in V_b$.
Let $x \in V_a \cap \mathcal{D}$, then:

$$x \in V_a \implies f(x) \in V_b \implies g(f(x)) \in V_l \implies g \circ f(x) \in V_l.$$

Therefore, we showed that for all neighborhood V_l of l, there exists a neighborhood V_a of a such that for all $x \in \mathcal{D}$, if $x \in V_a$, then $g \circ f(x) \in V_l$. Hence, $g \circ f$ has a limit at a and this limit is l.

Exercise 28: Study the limits of the following functions:

- Limit of $f : x \mapsto \sin\left(\frac{1}{x}\right)$ at $+\infty$;
- Limit of $f : x \mapsto \cos\left(\frac{1}{x}\right)$ at $+\infty$;
- Limit of $f : x \mapsto \cos\left(\sin\left(\frac{1}{x}\right)\right)$ at $+\infty$;
- Limit of $f : x \mapsto \sqrt{\sin\left(\frac{1}{x}\right)}$ at $+\infty$;

3.10 Left hand and right hand limits at a point of $\mathbb{R}$

To defined the left hand and right hand limits, we will first define the left hand and the right hand neighborhoods of a point in $\mathbb{R}$.

Definition 30: *We call **right hand neighborhood** of a point a of $\mathbb{R}$ any open interval, such as*

$$V_{a^+}(\alpha) =]a,\ a+\alpha[,\ \textit{with } \alpha \in \mathbb{R}_+^*$$

which can be also written $V_{a^+}(\alpha) = \{x \in \mathbb{R} \ / \ 0 < x - a < \alpha\}$.
*Also, we call **left hand neighborhood** of a point a of $\mathbb{R}$ any open interval, such as*

$$V_{a^-}(\alpha) =]a-\alpha,\ a[,\ \textit{with } \alpha \in \mathbb{R}_+^*$$

which can be also written $V_{a^-}(\alpha) = \{x \in \mathbb{R} \ / \ -\alpha < x - a < 0\}$.

Definition 31: *Let $f : \mathcal{D} \longrightarrow \mathbb{R}$ and a real number a such that, for all $\eta > 0$, we have $]a,\ a+\eta[\cap\mathcal{D} \neq \emptyset$. Let l be an element of $\mathbb{R}$, finite or infinite. We say that f has l as **right hand limit** at a if, for all neighborhood W of l, there exist a right hand neighborhood V_{a^+} of a such that for all $x \in \mathcal{D}$,*

$$x \in V_{a^+} \cap \mathcal{D} \implies f(x) \in W.$$

If f has $l \in \overline{\mathbb{R}}$ as right hand limit at $a \in \mathbb{R}$, it will be denoted $\lim\limits_{a^+} f = l$ or $\lim\limits_{x \longrightarrow a^+} f(x) = l$.

Definition 32: *Let $f : \mathcal{D} \longrightarrow \mathbb{R}$ and a real number a such that, for all $\eta > 0$, we have $]a-\eta,\ a[\cap\mathcal{D} \neq \emptyset$. Let l be an element of $\mathbb{R}$, finite or infinite. We say that f has l as **left hand limit** at a if, for all neighborhood W of l, there exist a left hand neighborhood V_{a^-} de a such that for all$x \in \mathcal{D}$,*

$$x \in V_{a^-} \cap \mathcal{D} \implies f(x) \in W.$$

If f has $l \in \overline{\mathbb{R}}$ as left hand limit at $a \in \mathbb{R}$, it will be denoted $\lim\limits_{a^-} f = l$ or $\lim\limits_{x \longrightarrow a^-} f(x) = l$.

Proposition 16: *A function f has a limit at a if and only if it has a left hand limit and a right hand limit a and if these limits are equals.*

Proof: We have to prove here both implications $\Longrightarrow$ and $\Longleftarrow$.

$\Longrightarrow$ Let f be a function that has $l \in \overline{\mathbb{R}}$ as limit at $a \in \mathbb{R}$.

Let V_l a neighborhood of l, then there exists $\alpha > 0$ such that for all $x \in \mathcal{D}$: $x \in]a-\alpha, a+\alpha[\implies f(x) \in V_l$.

Let $V_{a^-}(\alpha) =]a-\alpha, a[$ be a left hand neighborhood of a and $V_{a^+}(\alpha) =]a, a+\alpha[$ a right hand neighborhood of a. Hence $V_{a^-}(\alpha) \subset]a-\alpha, a+\alpha[$ and $V_{a^+}(\alpha) \subset]a-\alpha, a+\alpha[$.

Therefore, if $x \in V_{a^-}(\alpha) \cap \mathcal{D}$, then $f(x) \in V_l$, hence f has a left hand limit at a and this limit is l.

Similarly, if $x \in V_{a^+}(\alpha) \cap \mathcal{D}$, then $f(x) \in V_l$, hence f has a right hand limit at a and this limit is l.

$\Longleftarrow$ Let f be a function that has $l \in \overline{\mathbb{R}}$ as right hand limit and as left hand limit at $a \in \mathbb{R}$.

Let V_l be a neighborhood of l.

Therefore, there exists $\alpha > 0$ such that for all $x \in \mathcal{D}$: $x \in]a-\alpha, a[\implies f(x) \in V_l$.

Also,there exists $\alpha' > 0$ such that for all $x \in \mathcal{D}$: $x \in]a, a+\alpha'[\implies f(x) \in V_l$.

Let $\alpha'' = \min\{\alpha, \alpha'\}$ and set $V_a(\alpha'') =]a-\alpha'', a+\alpha''[$ a neighborhood of a.

Then we have the following cases:

- If $a \notin \mathcal{D}$, then for all $x \in \mathcal{D}$: $x \in V_a(\alpha'') \implies f(x) \in V_l$
- If $a \in \mathcal{D}$, then $l = f(a)$, and for all $x \in \mathcal{D}$: $x \in V_a(\alpha'') \implies f(x) \in V_l$, in particular $f(a) = l \in V_l$.

Hence f has a limit at a (and this limit is $l = \lim\limits_{a^-} f = \lim\limits_{a^+} f$).

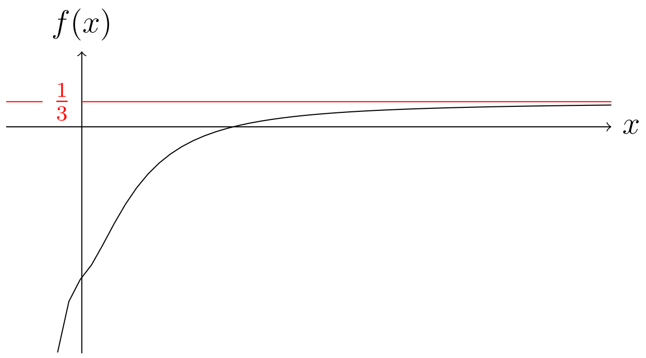
3.11 Solved exercises

Exercise 29: Study the limit of the function $f : x \mapsto \frac{x^2-4}{3x^2+x+2}$ at $+\infty$.

Solution: We have an undetermined form such as $\frac{+\infty}{+\infty}$.
Let factor the numerator and the denominator by the terms with the highest degree.
Let $x \in \mathbb{R}^*$, then:

$$\frac{x^2-4}{3x^2+x+2} = \frac{x^2(1-\frac{4}{x^2})}{3x^2(1+\frac{1}{3x}+\frac{2}{3x^2})} = \frac{(1-\frac{4}{x^2})}{3(1+\frac{1}{3x}+\frac{2}{3x^2})}$$

We have $\lim\limits_{x \longrightarrow +\infty} \left(1 - \dfrac{4}{x^2}\right) = 1$ and $\lim\limits_{x \longrightarrow +\infty} 3.\left(1 + \dfrac{1}{3x} + \dfrac{2}{3x^2}\right) = 3$.
Hence the function $f : x \mapsto \frac{x^2-4}{3x^2+x+2}$ has a limit at $+\infty$, and $\lim\limits_{+\infty} f = \dfrac{1}{3}$ (Limit of a quotient).
Note: f has a horizontal asymptote of equation $y = \frac{1}{3}$.



Exercise 30: Study the limit of the function $f : x \mapsto \frac{\sqrt{x+1}-2}{x-3}$ at 3.

Solution: We have an undetermined form such as $\frac{0}{0}$.
Let $x \in [-1, 3[\cup]3, +\infty[$, then:

$$\begin{aligned} \frac{\sqrt{x+1}-2}{x-3} &= \frac{(\sqrt{x+1}-2)}{(x-3)} \times \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)} &= \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} \\ &= \frac{(x-3)}{(x-3)(\sqrt{x+1}+2)} &= \frac{1}{(\sqrt{x+1}+2)} \end{aligned}$$

We have $\lim\limits_{x \longrightarrow 3} (\sqrt{x+1}+2) = \sqrt{4}+2 = 4$, hence the function $f : x \mapsto \frac{\sqrt{x+1}-2}{x-3}$ has a limit at 3, and $\lim\limits_{3} f = \dfrac{1}{4}$ (Limit of a quotient).
Note: f has a removable discontinuity at $x = 3$.

Exercise 31: Study the limit of the function $f : x \mapsto \frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}$ at 3.

Solution: We have an undetermined form such as $\frac{0}{0}$.
Let $x \in [-1, 3[\cup]3, +\infty[$, then:

$$\begin{aligned}\frac{\sqrt{7-x}-2}{1-\sqrt{4-x}} &= \frac{\sqrt{7-x}-2}{1-\sqrt{4-x}} \times \frac{\sqrt{7-x}+2}{\sqrt{7-x}+2} \times \frac{1+\sqrt{4-x}}{1+\sqrt{4-x}} &&= \frac{(7-x)-4}{1-(4-x)} \times \frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}\\ &= \frac{(3-x)}{(x-3)} \times \frac{1+\sqrt{4-x}}{\sqrt{7-x}+2} &&= -\frac{1+\sqrt{4-x}}{\sqrt{7-x}+2}\end{aligned}$$

We have $\lim\limits_{x\longrightarrow 3}\left(-\dfrac{1+\sqrt{4-x}}{\sqrt{7-x}+2}\right) = -\dfrac{1+\sqrt{1}}{\sqrt{4}+2} = -\dfrac{1}{2}$, hence the function $f : x \mapsto \frac{\sqrt{7-x}-2}{1-\sqrt{4-x}}$ has a limit at 3, and $\lim\limits_{3} f = -\dfrac{1}{2}$.
Note: f has a removable discontinuity at $x = 3$.

Exercise 32: Study the limit of the function $f : x \mapsto \frac{2x^2-4}{\sqrt{4x^4-5x^2+1}}$ at $+\infty$.

Solution: We have an undetermined form such as $\frac{+\infty}{+\infty}$.
Let factor the numerator and the denominator by the terms with the highest degree.
Let $x \in \mathbb{R}$ such that $4x^4 - 5x^2 + 1 > 0$, then:

$$\begin{aligned}\frac{2x^2-4}{\sqrt{4x^4-5x^2+1}} &= \frac{2x^2(1-\frac{1}{2x^2})}{\sqrt{x^4(1-\frac{5}{4x^2}+\frac{1}{4x^4})}} = \frac{2x^2(1-\frac{1}{2x^2})}{x^2\sqrt{1-\frac{5}{4x^2}+\frac{1}{4x^4}}}\\ &= \frac{2(1-\frac{1}{2x^2})}{\sqrt{1-\frac{5}{4x^2}+\frac{1}{4x^4}}}\end{aligned}$$

We have $\lim\limits_{x\longrightarrow +\infty} 2\left(1-\dfrac{1}{2x^2}\right) = 2$ and $\lim\limits_{x\longrightarrow +\infty} \sqrt{1-\dfrac{5}{4x^2}+\dfrac{1}{4x^4}} = \sqrt{1} = 1$, hence the function $f : x \mapsto \frac{2x^2-4}{\sqrt{4x^4-5x^2+1}}$ has a limit at $+\infty$, and $\lim\limits_{+\infty} f = \dfrac{2}{1} = 2$ (Limit of a quotient).
Note: f has a horizontal asymptote of equation $y = 2$.

Exercise 33: Study the limit of the function $f : x \mapsto \sqrt[3]{x+1}$ at $+\infty$.

Solution: We have, $\forall x \in \mathbb{R}_+^*$ and $\forall n \in \mathbb{R}$, $\sqrt[n]{x} = x^{\frac{1}{n}} = e^{n\ln(x)}$.
Hence for $x \in]-1, +\infty[$, we get $\sqrt[3]{x+1} = e^{3\ln(x+1)}$.
Since $\lim\limits_{x\longrightarrow +\infty} 3\ln(x+1) = +\infty$ and $\lim\limits_{x\longrightarrow +\infty} e^x = +\infty$, then the function $f : x \mapsto \sqrt[3]{x+1}$ has a limit at $-\infty$, and $\lim\limits_{+\infty} f = +\infty$ (Limit of a composite function).

Exercise 34: Study the limit of the function $f : x \mapsto \frac{|x-1|}{x-1}$ at 1.

Solution: We have an undetermined form such as $\frac{0}{0}$.
We cannot simplify this function with the absolute values $|\ |$. We have to consider the following two cases:

- If $x < 1$, then $|x-1| = -(x-1)$ and $f(x) = \frac{-(x-1)}{x-1} = -1$
- If $x > 1$, then $|x-1| = (x-1)$ and $f(x) = \frac{(x-1)}{x-1} = 1$

Hence, we will study the left hand limit and the right hand limits of this function:

- $\lim\limits_{x\longrightarrow 1^-} \dfrac{|x-1|}{x-1} = -1$;
- $\lim\limits_{x\longrightarrow 1^+} \dfrac{|x-1|}{x-1} = 1$.

Therefore $\lim\limits_{1^-} f \neq \lim\limits_{1^-} f$ and f **does not have a limit** at 1.

Exercise 35: Study the limit of the function $f : x \mapsto \frac{x}{\sqrt{x^2-4}}$ at -2^-.

Solution: The function f is defined on $]-\infty,-2[\cup]2,+\infty[$ and for $x<-2$, $f(x)<0$. Therefore, $\lim\limits_{x\longrightarrow -2^-}\sqrt{x^2-4}=0^+$.
Hence the function $f: x\mapsto \frac{x}{\sqrt{x^2-4}}$ has a limit at -2^-, and $\lim\limits_{-2^-} f=-\infty$ (Limit of a quotient).
Note: f has a vertical asymptote of equation $x=2$.

Exercise 36: Study the limit of the function $f: x\mapsto \frac{4x^2+2x-1}{x-2}$ at 2.

Solution: We have $\lim\limits_{x\longrightarrow 2^-} x-2=0^-$ and $\lim\limits_{x\longrightarrow 2^+} x-2=0^+$. Besides, $\lim\limits_{x\longrightarrow 2}(4x^2+2x-1)=19$.
Hence $\lim\limits_{2^-} f=-\infty$ and $\lim\limits_{2^+} f=+\infty$ (Limit of a quotient). Therefore $\lim\limits_{2-} f\neq \lim\limits_{2^+} f$ and f **does not have a limit** at 2.

4 Derivative of functions of a real variable

Notations: In all this chapter, I is an interval of $\mathbb{R}$ that cannot be reduced at one point. Unless otherwise specified, a is a point of I and $f, g, ...$ are functions defined on I with values in $\mathbb{R}$. In particular, unless otherwise specified, these functions are defined at a.

4.1 Derivative at a point

Definition 33: *We say that function f is differentiable at a, if the function $\tau_{f,a}: x\mapsto \frac{f(x)-f(a)}{x-a}$ has a finite limit at a. If this limit exist, it will be denoted $f'(a)$.*
$f'(a)$ is the derivative of f at a.

Remarque: The function $\tau_{f,a}: x\mapsto \frac{f(x)-f(a)}{x-a}$ is also called the **rate of change** of f at a. As we can see on Figure 1, $\tau_{f,a}(x)$ is the slope of the green line, and when x approaches a, $\tau_{f,a}(x)$ approaches the slope of the tangent line to the graph at a (the red line).

If f is differentiable at a, we will write:

$$\lim_{x\longrightarrow a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

With the following change in the variable $x=a+h$, we can also write:

$$\lim_{h\longrightarrow 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

Definition 34: *We say that the function f is differentiable on I, if it is differentiable at any point $a\in I$. In this case, it will be denoted $f': x\mapsto f'(x)$ the function of I in $\mathbb{R}$ which maps all $x\in I$ to its derivative at x.*

Exercise 37: If f is a constant function, show that f is differentiable at any point of $\mathbb{R}$. Infer the function f'.

Exercise 38: Let f be the identity function of $\mathbb{R}$ in $\mathbb{R}$, show that f is differentiable at any point of $\mathbb{R}$. Infer the function f'.

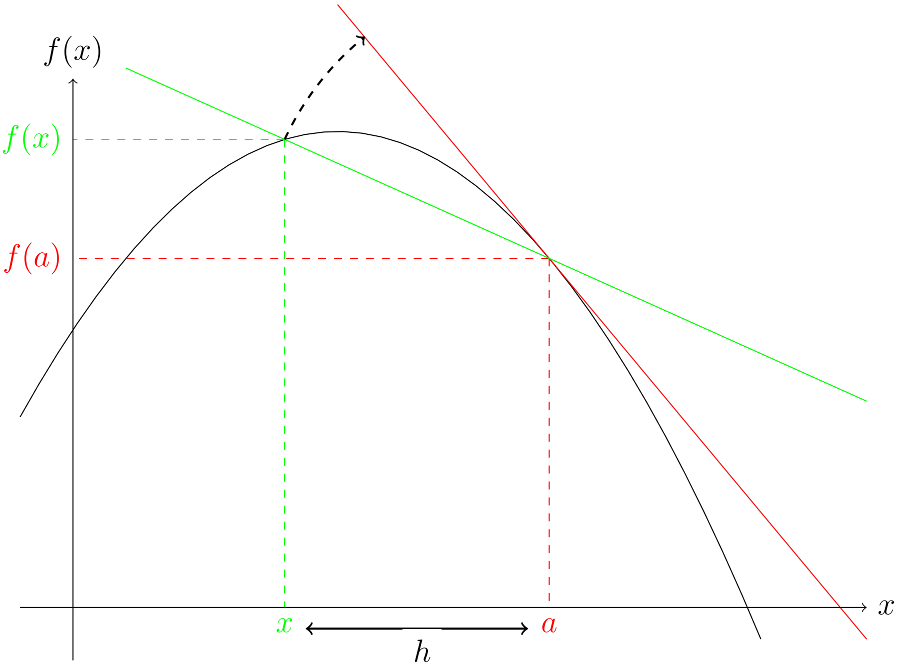


Figure 1: Interprètation géométrique de la dérivée d'une fonction

Exercise 39: Let $f : x \mapsto x^2$, show that f is differentiable at any point of $\mathbb{R}$. Infer the function f'.

Exercise 40: Let $f : x \mapsto \sqrt{x}$, show that f is differentiable at any point of $a > 0$ but not at 0. Infer the function f'.

Exercise 41: Let $f : x \mapsto |x|$ of $\mathbb{R}$ in $\mathbb{R}_+$, show that f is differentiable at any point of $\mathbb{R}^*$ but not at 0. Infer the function f'.

Remarque: When f is differentiable on an interval I, its derivative is generally denoted f'. This notation is due to Newton.
Another notation used for the derivative of a function f is $\frac{df}{dx}$. This notation, intriduced by Leibniz (contemporary and rival of Newton), remind that the derivative is the limit of the ratio of the increment of the function and the corresponding increment of the variable. But, be careful, this notation does not mean that the derivative of f is the ration of the differential of f and the one of x. You have to interpret it as **a symbol, not a quotient**.

We will assume in the sequel the derivative of the following classical functions:

$$\begin{array}{lcll} \sin'(x) & = & \cos(x), & for\ all\ x \in \mathbb{R}; \\ \cos'(x) & = & -\sin(x), & for\ all\ x \in \mathbb{R}; \\ \ln'(x) & = & \frac{1}{x}, & for\ all\ x > 0; \\ \exp'(x) & = & \exp(x), & for\ all\ x \in \mathbb{R}; \end{array}$$

4.2 Differentiability and continuity

Proposition 17: *If f is differentiable at a, then it is continuous at a.*

Proof: Let f be a function defined on I and differentiable at $a \in I$.
We will show that f has a limit and that this limit is $f(a)$.
Let $x \neq a$, then:

$$f(x) = f(a) + (x-a)\frac{f(x) - f(a)}{x - a} = f(a) + (x-a)\tau_{f,a}(x).$$

Now we will study the limit of f:

$$\left.\begin{array}{l}\lim\limits_{x\longrightarrow a}(x-a)=0\\ \lim\limits_{x\longrightarrow a}\tau_{f,a}(x)=f'(a)\end{array}\right\}\Longrightarrow \lim_{x\longrightarrow a}(x-a)\tau_{f,a}(x)=0$$

Hence f has a limit at a and this limit is $f(a)$. Therefore f is continuous at a.

4.3 The tangent function

Definition 35: *If f is differentiable at a, the affine function $x \mapsto f(a)+(x-a)f'(a)$ is **the tangent function** of f at a.*

When f is a function differentiable at a, the equation of the tangent at a of the graph representing this function is $y=f(a)+(x-a)f'(a)$.

Exercise 42: Write the equation of the tangent of the graph of the function cosine at the points 0, $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π.

4.4 Left hand and right hand derivative

The derivative at a is defined with the limit at a of $\tau_{f,a} : x \mapsto \frac{f(x)-f(a)}{x-a}$. Hence the left hand limit (resp. right hand limit) of $\tau_{f,a}$ is associated to the left hand derivative (resp. right hand derivative) of f at a.

Definition 36: *If $I' = I\cap]-\infty, a[$ is non-empty, we say that f is left hand differentiable at a, if $\tau_{f,a}$ has a left hand limit at a.*
Similarly, if $I'' = I\cap]a,+\infty[$ is non-empty, we say that f is right hand differentiable at a, if $\tau_{f,a}$ has a right hand limit at a.

Remarque: We will denote by $f'_-(a)$ (resp. $f'_+(a)$) the left hand derivative (resp. right hand derivative) of f at a when it exists.

Proposition 18: *Let a be an interior point of I. If f is left hand and right hand differentiable and if $f'_-(a)=f'_+(a)$, then f is differentiable at a and $f'(a)=f'_-(a)=f'_+(a)$.*

Proof: This is a direct consequence of proposition 16.

Exercise 43: Study the differentiability of function $f : x \mapsto |\sin(x)|$.

Definition 37: *If f is left hand and right hand differentiable and if $f'_-(a) \neq f'_+(a)$, the point $A(a, f(a))$ of the graph of f is said **angular point**.*

4.5 Algebraic operations

Proposition 19: *Let f and g be two function differentiable at a. Then:*

1. *$(f+g)$ is differentiable at a and $(f+g)'(a)=f'(a)+g'(a)$.*
2. *$(f\times g)$ is differentiable at a and $(f\times g)'(a)=f'(a)g(a)+f(a)g'(a)$.*
3. *For all $\lambda\in\mathbb{R}$, (λf) is differentiable at a and $(\lambda f)'(a)=\lambda f'(a)$.*
4. *If $f(x)\neq 0$ for all $x\in I$, the function $\frac{1}{f}$ is differentiable at a and $\left(\frac{1}{f}\right)'(a)=-\frac{f'(a)}{f^2(a)}$.*

4. *If $g(x) \neq 0$ for all $x \in I$, the function $\frac{f}{g}$ is differentiable at a and* $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$.

Proof (1.)**:** Let f and g be two functions defined on I and differentiable at $a \in I$.
Let $x \in I$, $x \neq a$, then:

$$\begin{array}{rcl} \tau_{(f+g),a}(x) & = & \frac{(f+g)(x)-(f+g)(a)}{x-a} \quad = \quad \frac{f(x)+g(x)-f(a)+g(a)}{x-a} \\ & = & \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \end{array}$$

Yet, $\lim\limits_{x\longrightarrow a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$ since f is differentiable at a, and $\lim\limits_{x\longrightarrow a} \dfrac{g(x)-g(a)}{x-a} = g'(a)$ since g is differentiable at a.
Therefore the function $\tau_{(f+g),a}$ has a limit at $a$and this limit is $f'(a)+g'(a)$.
Hence, by definition, $(f+g)$ is differentiable at a and $(f+g)'(a) = f'(a)+g'(a)$.

Proof (2.)**:** Let f and g be two functions defined on I and differentiable at $a \in I$.
Let $x \in I$, $x \neq a$, then:

$$\begin{array}{rcl} \tau_{(f\times g),a}(x) & = & \frac{(f\times g)(x)-(f\times g)(a)}{x-a} \\ & = & \frac{(f(x)+f(a)-f(a))g(x)-f(a)g(a)}{x-a} \\ & = & \frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a} \\ & = & \tau_{f,a}(x)g(x) + f(a)\tau_{g,a}(x) \end{array}$$

Since g is differentiable at a, g is continuous at a and then $\lim\limits_{a} g = g(a)$.
Therefore, we have:

$$\left.\begin{array}{l} \lim\limits_{x\longrightarrow a} g(x) = g(a) \\ \lim\limits_{x\longrightarrow a} \tau_{f,a}(x) = f'(a) \end{array}\right\} \Longrightarrow \lim\limits_{x\longrightarrow a} \tau_{f,a}(x)g(x) = f'(a) \times g(a)$$

Besides $\lim\limits_{x\longrightarrow a} f(a)\tau_{g,a}(x) = f(a)g'(a)$.
Then $\tau_{(f\times g),a}$ has a limit at a and this limit is $f'(a)g(a)+f(a)g'(a)$. It infers that $(f\times g)$ is differentiable at a and its derivative is $f'(a)g(a)+f(a)g'(a)$.

Proof (3.)**:** Direct inference from 2.

Proof (4.)**:** Let f be a function defined on I, such that $f(x) \neq 0$ for all $x \in I$, and differentiable at $a \in I$.
Let $x \in I$, $x \neq a$, then:

$$\begin{array}{rcl} \tau_{\frac{1}{f},a}(x) & = & \frac{\frac{1}{f(x)}-\frac{1}{f(a)}}{x-a} \quad = \quad \frac{\frac{f(a)-f(x)}{f(x)f(a)}}{x-a} \\ & = & \frac{1}{f(x)f(a)}.\frac{f(a)-f(x)}{x-a} \quad = \quad -\frac{1}{f(x)f(a)}.\frac{f(x)-f(a)}{x-a} \\ & = & -\frac{1}{f(x)f(a)}.\tau_{f,a}(x) \end{array}$$

Since f is differentiable at a, f is continuous at a and then $\lim\limits_{a} f = f(a) \neq 0$.
We can then compute the limits:

$$\left.\begin{array}{l} \lim\limits_{x\longrightarrow a} \dfrac{1}{f(x)f(a)} = \dfrac{1}{f^2(a)} \\ \lim\limits_{x\longrightarrow a} \tau_{f,a}(x) = f'(a) \end{array}\right\} \Longrightarrow \lim\limits_{x\longrightarrow a} \tau_{\frac{1}{f},a}(x) = -\frac{f'(a)}{f^2(a)}$$

Therefore $\frac{1}{f}$ is differentiable at a and its derivative is $-\frac{f'(a)}{f^2(a)}$.

Proof (4.)**:** We write $\frac{f}{g} = f \times \frac{1}{g}$ and we use the results of points 2 and 4.

Exercise 44: Study the differentiability of the function $f : x \mapsto \tan(x)$ on $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$.

4.6 Derivative of a composite function

Proposition 20: *Let I and J be two intervals of $\mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions. Assume that $f(I) \subseteq J$, that f is differentiable at $a \in I$, and that g is differentiable at $b = f(a) \in J$. Then $g \circ f$ is differentiable at a and:*

$$(g \circ f)'(a) = (g' \circ f)(a) f'(a) = g'(f(a)) f'(a).$$

Proof: Let I and J be two intervals of $\mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions. Assume that $f(I) \subseteq J$, that f is differentiable at $a \in I$, and that g is differentiable at $b = f(a) \in J$.
Let $x \in I$, then:

$$\begin{aligned}
\tau_{g \circ f, a}(x) &= \tfrac{(g \circ f)(x) - (g \circ f)(a)}{x - a} \\
&= \tfrac{(g(f(x)) - (g(f(a))}{x - a} \\
&= \tfrac{(g(f(x)) - (g(f(a))}{f(x) - f(a)} \times \tfrac{(f(x) - f(a)}{x - a} \\
&= \tau_{g, f(a)}(f(x)) \times \tau_{f,a}(x).
\end{aligned}$$

Sine f is differentiable at a, then f is continuous at a and we have $\lim_a f = f(a)$ and $\lim_a \tau_{f,a} = f'(a)$.
Besides, since g is differentiable at $f(a)$ then $\lim_{x \longrightarrow a} \tau_{g, f(a)}(x) = g'(f(a))$.
Therefore, by composition of limits, $\lim_{x \longrightarrow a} (\tau_{g,f(a)} \circ f)(x) = (g \circ f)(a) = g'(f(a))$.
Hence $\tau_{g \circ f, a}$ has a limit at a and this limit is $g'(f(a))f'(a)$. Consequently $(g \circ f)$ is differentiable at a and its derivative is $(g \circ f)'(a) = (g' \circ f)(a) f'(a) = g'(f(a)) f'(a)$.

Exercise 45: Let $\alpha \in \mathbb{R}$, study the differentiability of $f_\alpha : x \longrightarrow x^\alpha = e^{\alpha . \ln(x)}$ defined on $\mathbb{R}_+^*$.

Exercise 46: Study the differentiability of $f : x \longrightarrow x^2 . \sin(\frac{1}{x})$ defined on $\mathbb{R}_+^*$.

Exercise 47: Study the differentiability of the following functions:

1. $x \mapsto \sqrt{1 + \sin^2(x)}$
2. $x \mapsto \sqrt{x^2 + \sin^2(x^2)}$
3. $x \mapsto \ln(|sin(x)|)$
4. $x \mapsto \frac{e^{x^2} - 1}{e^x + 1}$
5. $x \mapsto \frac{1}{\ln(x^2 + x + 1)}$

4.7 Successive derivatives

Assume f is differentiable on I. Hence we can define its derivative function $f' : I \longrightarrow \mathbb{R}$ which maps all point $a \in I$ to its derivative $f'(a)$. We can then wonder whether f' is also differentiable on I. It it is the case, we will write $f'' = (f')'$ the second derivative of f (note that f' is the first derivative). In the general case, we will denote by $f^{(n)}$, $n \in \mathbb{N}$, the $n - th$ derivative of f, if it exists. Hence, we will use the convention $f^{(0)} = f$ (and $f' = f^{(1)}$).
The Leibniz notation for the $n - th$ of a function $f : x \mapsto f(x)$ is $\frac{d^n f}{dx^n}$.

Exercise 48: Compute, if it exists, the second derivative of $f : x \mapsto \frac{1}{1+x}$.

4.8 Solved exercises

Exercise 49: Let $f : x \mapsto \sin(x)$, show that f is differentiable at any point $a \in \mathbb{R}$. Infer the function f'.

Solution: The domain of definition of f is $\mathcal{D} = \mathbb{R}$.
Let $a \in \mathbb{R}$ and $x \neq a$, then:

$$\begin{array}{rcl} \tau_{f,a}(x) & = & \frac{f(x)-f(a)}{x-a} \\ & = & \frac{sin(x)-sin(a)}{x-a} \end{array}$$

Yet $sin(x) - sin(a) = 2.\sin\left(\frac{x-a}{2}\right).\cos\left(\frac{x+a}{2}\right)$, then:

$$\begin{array}{rcl} \tau_{f,a}(x) & = & 2.\frac{\sin\left(\frac{x-a}{2}\right).\cos\left(\frac{x+a}{2}\right)}{x-a} \\ & = & \frac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}}.\cos\left(\frac{x+a}{2}\right) \end{array}$$

Hence we can compute the limits:

$$\left.\begin{array}{l} \left.\begin{array}{l} \lim\limits_{x\longrightarrow a} \dfrac{x-a}{2} = 0 \\ \lim\limits_{x\longrightarrow 0} \dfrac{\sin(x)}{x} = 1 \end{array}\right\} \Longrightarrow \lim\limits_{x\longrightarrow a} \dfrac{\sin\left(\frac{x-a}{2}\right)}{\frac{x-a}{2}} = 1 \\ \left.\begin{array}{l} \lim\limits_{x\longrightarrow a} \dfrac{x+a}{2} = a \\ \lim\limits_{x\longrightarrow a} \cos(x) = \cos(a) \end{array}\right\} \Longrightarrow \lim\limits_{x\longrightarrow a} \cos\left(\dfrac{x+a}{2}\right) = \cos(a) \end{array}\right\} \Longrightarrow \lim\limits_{x\longrightarrow a} \tau_{f,a}(x) = \cos(a)$$

Therefore the function sin is differentiable at any point of $a \in \mathbb{R}$ and we have $\sin' = \cos$ (or $f' : x \mapsto \cos(x)$).

Exercise 50: Study the differentiability of $f : x \mapsto \frac{1}{(e^{3x}-3)^3}$.

Solution: The domain of definition of f is $\mathcal{D} = \mathbb{R} - \left\{\frac{\ln(3)}{3}\right\}$.
Set $g : x \mapsto x^3$ and $h : x \mapsto e^{3x} - 3$. Hence $f = \frac{1}{goh}$.
The function h is differentiable on $\mathbb{R}$ and $f(\mathbb{R}) \subseteq \mathbb{R}$, and g is differentiable on $f(\mathbb{R})$ (since g is differentiable on $\mathbb{R}$). Then $g \circ h$ is differentiable on $\mathbb{R}$ and therefore also on $\mathcal{D} \subset \mathbb{R}$.
Hence f is differentiable on $\mathcal{D}$ and $f' = -\frac{(goh)'}{(goh)^2}$.
Let $x \in \mathcal{D}$, then:

$$\begin{array}{rcl} (g \circ h)'(x) & = & h'(x)(g' \circ h)(x) \\ & = & \left(3e^{3x}\right)\left(3f^2(x)\right) \\ & = & \left(3e^{3x}\right)\left(3(e^{3x}-3)^2\right) \\ & = & 9e^{3x}(e^{3x}-3)^2 \end{array}$$

Then:

$$\begin{array}{rcl} f'(x) & = & -\frac{(goh)'(x)}{(goh)^2(x)} \\ & = & -\frac{9e^{3x}(e^{3x}-3)^2}{\left((e^{3x}-3)^3\right)^2} \\ & = & -\frac{9e^{3x}(e^{3x}-3)^2}{(e^{3x}-3)^6} \\ & = & -\frac{9e^{3x}}{(e^{3x}-3)^4} \end{array}$$

Finally we can conclude that $f' : x \mapsto -\frac{9e^{3x}}{(e^{3x}-3)^4}$.

Exercise 51: Study the differentiability of $f : x \mapsto \frac{\sinh(x^2)}{\cosh(x^2)}$.

Solution: Set $g : x \mapsto \sinh(x^2) = \frac{e^{x^2}-e^{-x^2}}{2}$ and $h : x \mapsto \cosh(x^2) = \frac{e^{x^2}+e^{-x^2}}{2}$.
The function h does not vanish on $\mathbb{R}$ then f is defined on $\mathcal{D} = \mathbb{R}$.
The functions g and h are differentiable on $\mathcal{D}$ (and $h(x) \neq 0$ for all $x \in \mathcal{D}$), therefore the function $f = \frac{g}{h}$ is differentiable on $\mathcal{D}$ and we have $f' = \frac{g' \times h - g \times h'}{h^2}$.
Let $x \in \mathcal{D}$, then:

$$\begin{aligned} g'(x) &= \frac{2xe^{x^2}-(-2xe^{-x^2})}{2} \\ &= 2x\frac{e^{x^2}+e^{-x^2}}{2} \\ &= 2x\cosh(x^2) \end{aligned}$$

Also,

$$\begin{aligned} h'(x) &= \frac{2xe^{x^2}+(-2xe^{-x^2})}{2} \\ &= 2x\frac{e^{x^2}-e^{-x^2}}{2} \\ &= 2x\sinh(x^2) \end{aligned}$$

Hence,

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)} \\ &= \frac{(2x\cosh(x^2))(\cosh(x^2))-(\sinh(x^2))(2x\sinh(x^2))}{\cosh^2(x^2)} \\ &= 2x\frac{\cosh^2(x^2)-\sinh^2(x^2)}{\cosh^2(x^2)} \\ &= \frac{2x}{\cosh^2(x^2)} \ \textit{since} \ \cosh^2(x^2)-\sinh^2(x^2) = 1 \end{aligned}$$

We can now conclude that $f' : x \mapsto \frac{2x}{\cosh^2(x^2)} = 2x\,\mathrm{sech}^2(x^2)$.

Exercise 52: Let $f : x \mapsto \sqrt{8x-3}$. Compute, if it exists, f''

Solution: By composition of function, f is differentiable on $\mathcal{D} =]\frac{3}{8}, +\infty[$.
Let $x \in \mathcal{D}$, then:

$$\begin{aligned} f'(x) &= \frac{8x}{2.\sqrt{8x-3}} \\ &= \frac{4x}{\sqrt{8x-3}} \end{aligned}$$

Then, f' is differentiable on $\mathcal{D}$ (quotient of functions that are differentiable on $\mathcal{D}$).
Let $x \in \mathcal{D}$, then:

$$f''(x) = \frac{4\sqrt{8x-3}-4x\frac{4x}{\sqrt{8x-3}}}{(\sqrt{8x-3})^2}$$

5 Derivative and properties of transcendental functions

5.1 Logarithm functions

Naperian Logarithm	$f : x \mapsto \ln(x)$ $\mathcal{D} = \mathbb{R}_+^*$	$f' : x \mapsto \frac{1}{x}$
General Logarithm	$f : x \mapsto \log_a(x) = \frac{\ln(x)}{\ln(a)}$, $a > 0, \ \mathcal{D} = \mathbb{R}_+^*$	$f' : x \mapsto \frac{1}{\ln(a).x}$

Property: Let $a \in \mathbb{R}_+^*$, $b \in \mathbb{R}_+^*$ and $n \in \mathbb{R}$,

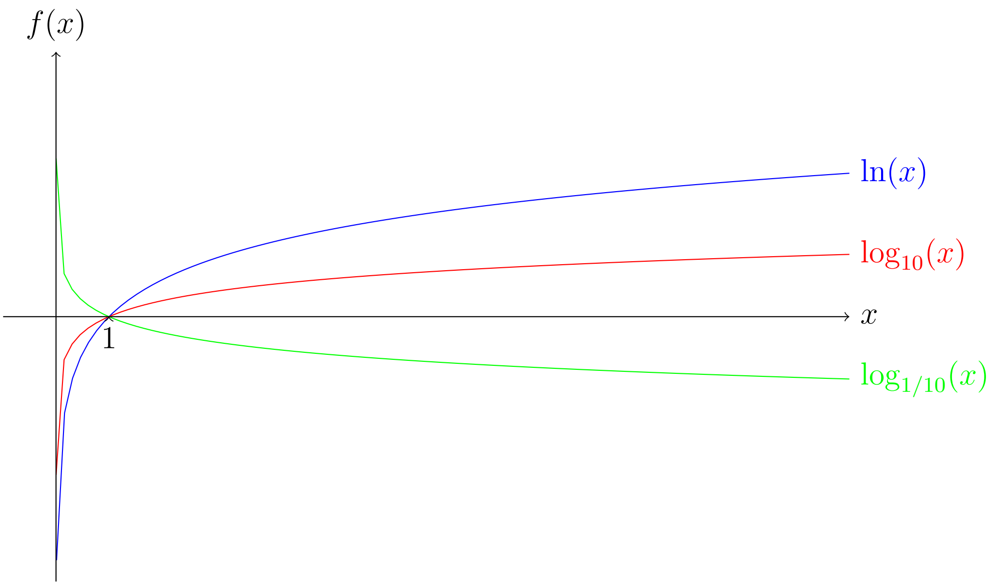


Figure 2: Logarithm functions

- $\ln(1) = 0$;
- $\ln(e) = 1$;
- $\ln(ab) = \ln(a) + \ln(b)$;
- $\ln(a^n) = n\ln(a)$;
- $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$;
- $e^{\ln(a)} = a$;

5.2 Exponential functions

Natural exponential	$f : x \mapsto e^x$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto e^x$
General exponential	$f : x \mapsto a^x = e^{x\ln(a)}$, $a > 0,\ \mathcal{D} = \mathbb{R}$	$f' : x \mapsto \ln(a)a^x$

Property: Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $n \in \mathbb{R}$,

- $e^0 = 1$;
- $e^{ab} = (e^a)^b$;
- $e^{a+b} = e^a e^b$;
- $\ln(e^a) = a$;

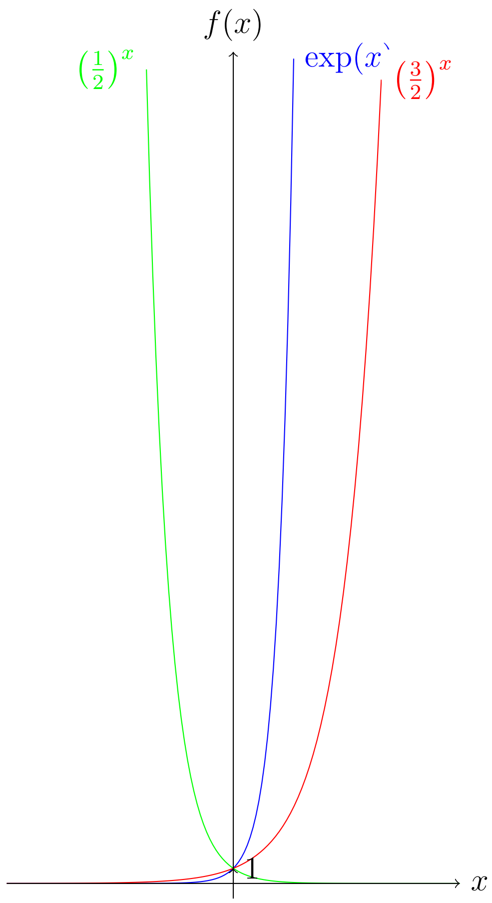


Figure 3: Exponential functions

5.3 trigonometric functions

Cosine	$f : x \mapsto \cos(x)$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto -\sin(x)$
Sine	$f : x \mapsto \sin(x)$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \cos(x)$
Tangent	$f : x \mapsto \tan(x) = \frac{\sin(x)}{\cos(x)}$ $\mathcal{D} = \mathbb{R} - \{\frac{\pi}{2} + k.\pi,\ k \in \mathbb{Z}\}$	$f' : x \mapsto \frac{1}{\cos^2(x)} = \sec^2(x)$
Cotangent	$f : x \mapsto \cot(x) = \frac{\cos(x)}{\sin(x)}$ $\mathcal{D} = \mathbb{R} - \{k.\pi,\ k \in \mathbb{Z}\}$	$f' : x \mapsto -\frac{1}{\sin^2(x)} = -\csc^2(x)$
Secant	$f : x \mapsto \sec(x) = \frac{1}{\cos(x)}$ $\mathcal{D} = \mathbb{R} - \{\frac{\pi}{2} + k.\pi,\ k \in \mathbb{Z}\}$	$f' : x \mapsto \frac{\sin(x)}{\cos^2(x)} = \sec(x).\tan(x)$
Cosecant	$f : x \mapsto \csc(x) = \frac{1}{\sin(x)}$ $\mathcal{D} = \mathbb{R} - \{k.\pi,\ k \in \mathbb{Z}\}$	$f' : x \mapsto -\frac{\cos(x)}{\sin^2(x)} = -\csc(x).\cot(x)$

Property: Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

- $\cos^2(a) + \sin^2(a) = 1$;
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$;
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$;
- $\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$;
- $\sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$;

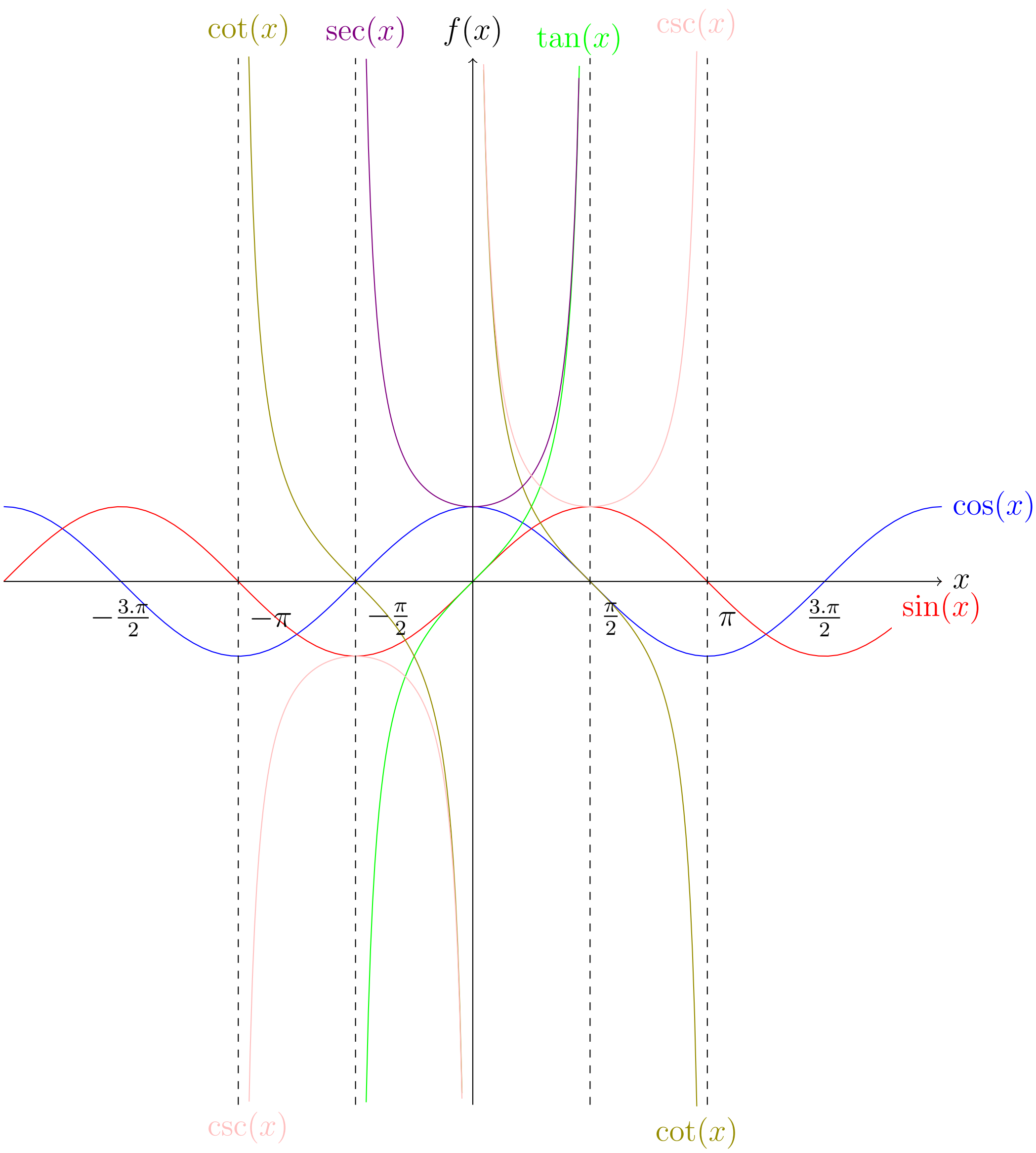


Figure 4: Trigonometric functions

- $\sin(a)\cos(b) = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$;

5.4 Inverse trigonometric functions

Arccosine	$f : x \mapsto \arccos(x)$ $\mathcal{D} = [-1, 1]$	$f' : x \mapsto -\frac{1}{\sqrt{1-x^2}}$ $\mathcal{D} =]-1, 1[$
Arcsine	$f : x \mapsto \arcsin(x)$ $\mathcal{D} = [-1, 1]$	$f' : x \mapsto \frac{1}{\sqrt{1-x^2}}$ $\mathcal{D} =]-1, 1[$
Arctangent	$f : x \mapsto \arctan(x)$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \frac{1}{1+x^2}$
Arccotangent	$f : x \mapsto \mathrm{arccot}(x)$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto -\frac{1}{1+x^2}$
Arcsecant	$f : x \mapsto \mathrm{arcsec}(x)$ $\mathcal{D} =]-\infty, -1] \cup [1, +\infty[$	$f' : x \mapsto \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $\mathcal{D} =]-\infty, -1[\cup]1, +\infty[$
Arccosecant	$f : x \mapsto \mathrm{arccsc}(x)$ $\mathcal{D} =]-\infty, -1] \cup [1, +\infty[$	$f' : x \mapsto -\frac{1}{\lvert x\rvert.\sqrt{x^2-1}}$ $\mathcal{D} =]-\infty, -1[\cup]1, +\infty[$

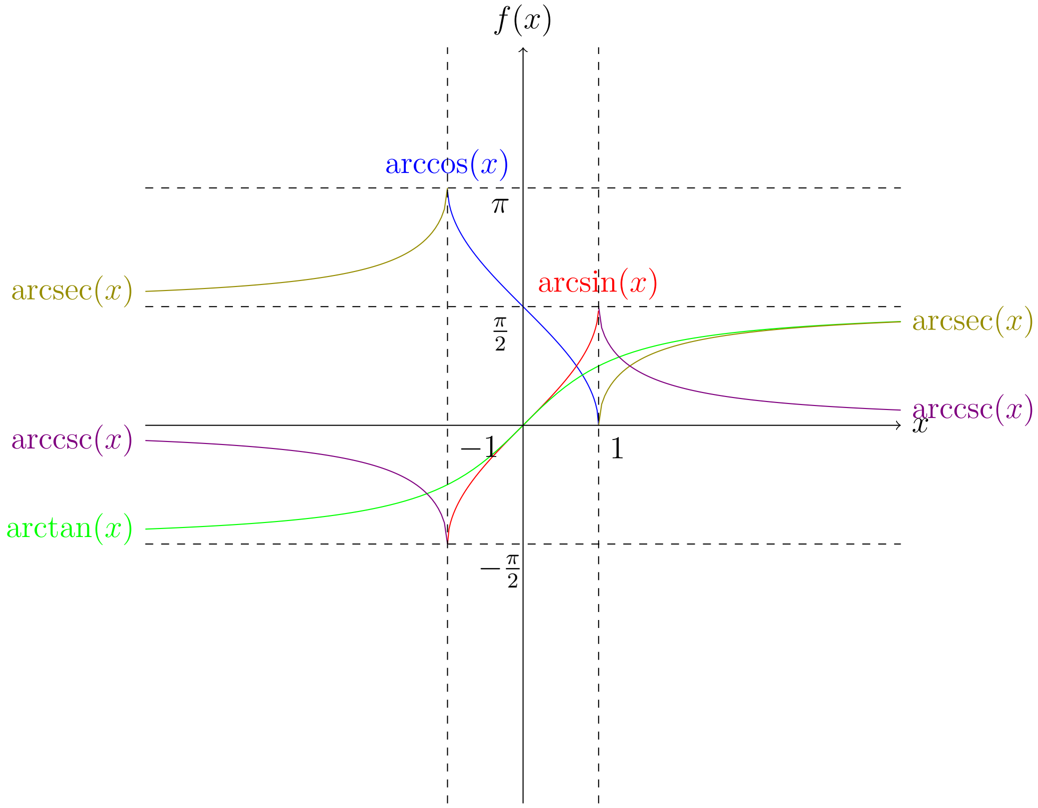


Figure 5: Inverse trigonometric functions

Property:

- $\forall x \in [-1, 1], \ \sin(\arcsin(x)) = x;$
- $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \ \arcsin(\sin(x)) = x;$
- $\forall x \in \mathbb{R}, \ \tan(\arctan(x)) = x;$
- $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \ \arctan(\tan(x)) = x;$
- $\forall x \in]-\infty, -1] \cup [1, +\infty[, \ \text{sec}(\text{arcsec}(x)) = x;$
- $\forall x \in]0, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \pi[, \ \text{arcsec}(sec(x)) = x;$

5.5 Hyperbolic functions

Hyperbolic Cosine	$f : x \mapsto \cosh(x) = \frac{e^x+e^{-x}}{2}$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \sinh(x)$
Hyperbolic Sine	$f : x \mapsto \sinh(x) = \frac{e^x-e^{-x}}{2}$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \cosh(x)$
Hyperbolic Tangent	$f : x \mapsto \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \operatorname{sech}^2(x)$
Hyperbolic Cotangent	$f : x \mapsto \coth(x) = \frac{1}{\tanh(x)}$ $\mathcal{D} = \mathbb{R}^*$	$f' : x \mapsto -\operatorname{csch}^2(x)$
Hyperbolic Secant	$f : x \mapsto \operatorname{sech}(x) = \frac{1}{\cosh(x)}$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \operatorname{sech}(x)\tanh(x)$
Hyperbolic Cosecant	$f : x \mapsto \operatorname{csch}(x) = \frac{1}{\sinh(x)}$ $\mathcal{D} = \mathbb{R}^*$	$f' : x \mapsto -\operatorname{csch}(x)\coth(x)$

Property: Let $a \in \mathbb{R}$,

- $\cosh^2(a) - \sinh^2(a) = 1$;
- $\tanh^2(a) + \operatorname{sech}^2(a) = 1$;
- $\coth^2(a) - \operatorname{csch}^2(a) = 1 \ (a \neq 0)$;
- $\cosh(2a) = \cosh^2(a) + \sinh^2(a)$;
- $\sinh(2a) = 2\sinh(a)\cosh(a)$;
- $\sinh^2(a) = \frac{\cosh(2a)-1}{2}$;
- $\cosh^2(a) = \frac{\cosh(2a)+1}{2}$;

5.6 Inverse hyperbolic functions

Inverse Hyperbolic Cosine	$f : x \mapsto \cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ $\mathcal{D} = [1, +\infty[$	$f' : x \mapsto \frac{1}{\sqrt{x^2-1}}$ $\mathcal{D} =]1, +\infty[$
Inverse Hyperbolic Sine	$f : x \mapsto \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$ $\mathcal{D} = \mathbb{R}$	$f' : x \mapsto \frac{1}{\sqrt{x^2+1}}$
Inverse Hyperbolic Tangent	$f : x \mapsto \tanh^{-1}(x) = \frac{1}{2}.\ln\left(\frac{1+x}{1-x}\right)$ $\mathcal{D} =]-1, 1[$	$f' : x \mapsto \frac{1}{1-x^2}$
Inverse Hyperbolic Cotangent	$f : x \mapsto \coth^{-1}(x) = \ln\left(\frac{x+1}{x-1}\right)$ $\mathcal{D} =]-\infty, -1[\cup]1, +\infty[$	$f' : x \mapsto \frac{1}{1-x^2}$
Inverse Hyperbolic Secant	$f : x \mapsto \operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$ $\mathcal{D} =]0, 1]$	$f' : x \mapsto -\frac{1}{x\sqrt{1-x^2}}$
Inverse Hyperbolic Cosecant	$f : x \mapsto \operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$ $\mathcal{D} = \mathbb{R}^*$	$f' : x \mapsto -\frac{1}{\lvert x\rvert\sqrt{1+x^2}}$

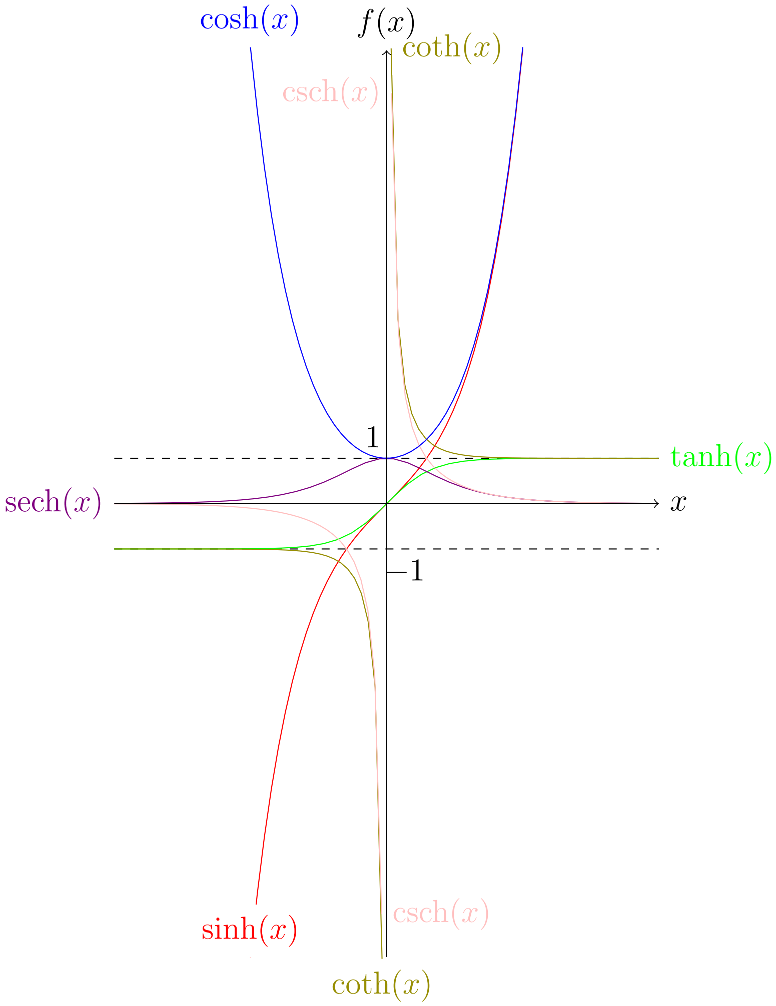


Figure 6: Fonctions hyperboliques

6 Differentiation techniques

6.1 Implicit differentiation

6.1.1 Implicit functions

The functions $f : x \mapsto f(x)$ that we have seen so far are all defined explicitely, i.e. we had their direct expression in function of x. However, in some cases, f can be defined as the solution of an equation. For instance, if we consider the function of two variables $\mathcal{F} : (x, y) \mapsto y - 3x^2 + 2$, then $\mathcal{F}(x, y) = 0 \iff y = 3x^2 + 2$. Indeed, the equation $\mathcal{F}(x, y) = 0$ hide, in some way, the definition of the function $f : x \mapsto 3x^2 + 2$, then we will say that f is defined **implicitly** by $\mathcal{F}(x, y) = 0$.

In general, it is no possible to transform an implicit expression $\mathcal{F}(x, y) = 0$ to an explicit one. $y = f(x)$. Often it is possible, but only locally, in the neighborhood of a particular solution of $\mathcal{F}(x, y) = 0$.

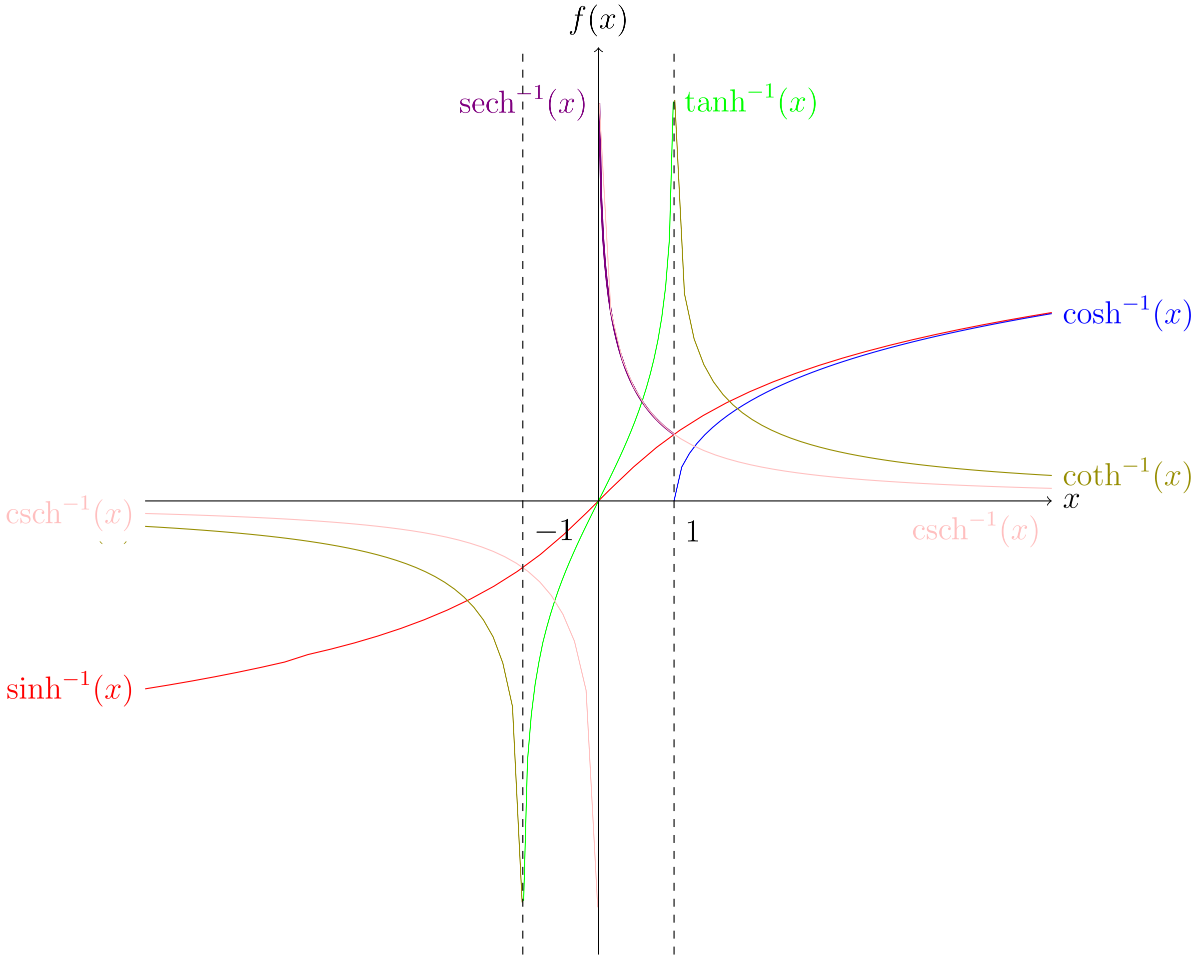


Figure 7: Inverse hyperbolic functions

Assume, for instance, the function $\mathcal{F} : (x, y) \mapsto x(x^2 + y^2) - 10(x^2 - y^2)$ and try to define the explicit expression of f defined implicitly by $\mathcal{F}(x, y) = 0$. Let $(x, y) \in \mathbb{R}^2$, then we have:

$$\begin{aligned}\mathcal{F}(x,y)=0 &\iff x(x^2+y^2)-10(x^2-y^2)=0\\ &\iff (x+10)y^2=(10-x)x^2\\ &\iff y^2=\tfrac{10-x}{10+x}x^2\end{aligned}$$

In this case, we cannot find f. Indeed, for some values of x, y has 2 values (for example, if $x = 2$ then $y = -\sqrt{6}$ or $y = \sqrt{6}$). This is a sufficient condition to show that there is no function f that satisfies $\mathcal{F} : (x, y) = 0$ with $(x, y) \in \mathbb{R}^2$.
However, if we assume that $y \geq 0$, we get $y = \sqrt{\frac{10-x}{10+x}x^2} = |x|\sqrt{\frac{10-x}{10+x}}$, with $x \in]-10, 10]$. Hence for $(x, y) \in (]-10, 10] \times \mathbb{R}_+)$ (which means that $x \in]-10, 10]$ and $y \in \mathbb{R}_+$), $\mathcal{F}(x, y) = 0$ defines implicitly $f_+ : x \mapsto |x|\sqrt{\frac{10-x}{10+x}}$. Also, for $(x, y) \in (]-10, 10] \times \mathbb{R}_-)$ (which means that $x \in]-10, 10[$ and $y \in \mathbb{R}_-$), $\mathcal{F}(x, y) = 0$ defines implicitly the function $f_- : x \mapsto -|x|\sqrt{\frac{10-x}{10+x}}$. See Figure 8 for a graphical representation of these functions.

Definition 38: *Let $\mathcal{F}$ be a function of two variables of $I \subseteq \mathbb{R}^2$ to $\mathbb{R}$. If there exists a function f such that for all $(x, y) \in I$, $\mathcal{F}(x, y) = 0 \iff y = f(x)$, then we say that f is an* ***implicit*** *function, or that f is defined implicitly.*

Remarque: When f is differentiable, it is commun to use the notation $y' = \frac{dy}{dx} = D_x y = f'(x)$ for the derivative of f at a point $x \in \mathbb{R}$.

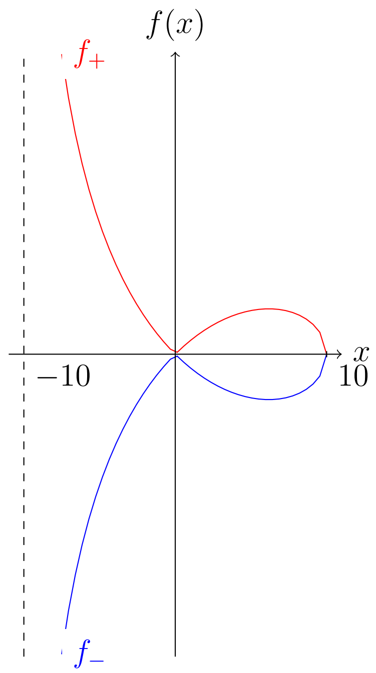


Figure 8: Graphical representation of $x(x^2+y^2)-10(x^2-y^2)=0$

Exercise 53: Let $\mathcal{F}:(x,y)\mapsto 3x^3-2xy^2-2$. Find the implicit function defined by $\mathcal{F}(x,y)=0$. *Hint: Consider the cases* $y\leq 0$ *and* $y\geq 0$.

6.1.2 Differentiation of implicit functions

The implicit differentiation allows to find the expression of $y'=\frac{dy}{dx}$ from the equation $\mathcal{F}(x,y)=0$. This is particularly useful when it is impossible or difficult to get the associated explicit function.
For instance, consider the function $\mathcal{F}:(x,y)\mapsto y^2+x^2+2xy$. If we assume y is a function of x, then the derivative of $\mathcal{F}$ in function of x is:

$$\begin{aligned}\tfrac{d}{dx}(y^2+x^2+2.x.y) &= \tfrac{d}{dx}(y^2)+\tfrac{d}{dx}(x^2)+\tfrac{d}{dx}(2xy)\\ &= 2y'y+2x+(2y+2xy')\\ &= 2y'(y+x)+2(x+y)\end{aligned}$$

Exercise 54: If y is a function of x, differentiate the following expressions:

$$a)\ x^3 \qquad b)\ y^3 \qquad c)\ x^3y^3$$

Consider an equation such as $\mathcal{F}(x,y)=\mathcal{G}(x,y)$. To compute y', we will follow the following steps:

1. Differentiate the two members of the equation: $\frac{d\mathcal{F}}{dx}(x,y)=\frac{d\mathcal{G}}{dx}(x,y)$;
2. Isolate $\frac{dy}{dx}$ of the equation obtained after the first step.

For instance, set $\mathcal{F}:(x,y)\mapsto y^2+x^2+2xy$ and $\mathcal{G}:(x,y)\mapsto 2$, that give the equation $y^2+x^2+2xy=2$. If we differentiate both side of the equation in function of x, we get:

$$\begin{aligned}2y'(y+x)+2(x+y)=0 &\implies y'=-\tfrac{x+y}{x+y}\\ &\implies y'=-1\end{aligned}$$

Exercise 55: Find $\frac{dy}{dx}$ from the following equations:

$$a)\ x^2+y^2=4 \qquad b)\ x^4+y\ln(x)=4y^3-5 \qquad c)\ y^3-2xy-3x^2=1 \qquad d)\sin(y)=x$$

6.2 Logarithmic differentiation

6.2.1 Definition

The logarithmic derivative is a technique allowing to facilitate the calculation of derivative. It is particularly useful in the case of exponential functions or functions composed of quotient and/or product of functions.

Definition 39: *Let f be a function differentiable in $x \in \mathbb{R}$ such that $f(x) \neq 0$, we call the logarithmic derivative of f at the point x, the function $\mathcal{L} : f \mapsto \frac{f'(x)}{f(x)}$. It is the derivative of the function $x \mapsto \ln(|f(x)|)$.*

Take for example the function $f : x \mapsto (x^2+1)^x$. This function is defined on $\mathbb{R}$ and is derivable on $\mathbb{R}$. Let's apply the classical derivation rules (those we have seen so far) in order to calculate f'. For $x \in \mathbb{R}$, we have $f(x) = (x^2+1)^x = e^{x\ln(x^2+1)}$, which gives:

$$\begin{array}{rcl} f'(x) & = & \left(\ln(x^2+1) + x \times 2x\frac{1}{x^2+1}\right) e^{x\ln(x^2+1)} \\ & = & \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right) e^{x\ln(x^2+1)} \\ & = & \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right) f(x) \end{array}$$

Now let's apply the logarithmic derivation in order to calculate f', which is possible because f does not vanish on $\mathbb{R}$. Let $x \in \mathbb{R}$, we first apply the logarithm:

$$\begin{array}{lcl} f(x) & = & (x^2+1)^x \\ \ln(|f(x)|) & = & \ln(|(x^2+1)^x|) \\ \ln(f(x)) & = & x\ln(x^2+1) \text{ since } f(x) > 0 \end{array}$$

Then we compute the derivatives in both sides:

$$\begin{array}{rcl} \frac{f'(x)}{f(x)} & = & \left(\ln(x^2+1) + x \times 2x\frac{1}{x^2+1}\right) \\ f'(x) & = & \left(\ln(x^2+1) + \frac{2x^2}{x^2+1}\right) f(x) \end{array}$$

Exercise 56: Study the differentiability of $f : x \mapsto x^{\cos(x)}$. If it is differentiable, calculate its derivative using classical rules, and then the logarithmic derivative.

Exercise 57: Same exercise with $f : x \mapsto \frac{5x-6}{\sqrt{2x-1}}$.

6.2.2 Some properties of the logarithmic differentiation

Proposition 21: *Let f and g be two functions which can be differentiated over an interval I and which do not vanish over I. So:*

1. $\mathcal{L}(f \times g) = \mathcal{L}(f) + \mathcal{L}(g)$;
2. $\mathcal{L}\left(\frac{f}{g}\right) = \mathcal{L}(f) - \mathcal{L}(g)$;
3. $\mathcal{L}\left(f^\lambda\right) = \lambda\mathcal{L}(f)$ *with* $\lambda \in \mathbb{R}$.

Proof: Proved directly using the definition of $\mathcal{L}$. Try to do it as an exercise.

Exercise 58: Study the differentiability of the following functions and calculate their derivative using the logarithmic derivative.

$$\begin{array}{ll} a) f : x \mapsto (4x-1)^{2x} & b) f : x \mapsto (\sinh(2x))^{3x} \\ c) f : x \mapsto x^{\tan(x)} & d) f : x \mapsto x^3 x^x \end{array}$$

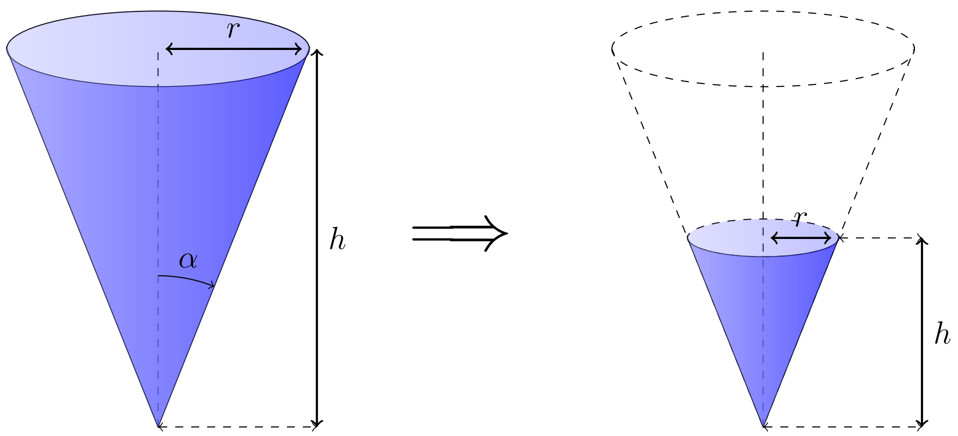


Figure 9: Conical tank filled with water

7 Differentiation and rate of change

We saw in chapter 4 that the derivative of a function f at a point $a \in D$ is defined as the slope of the tangent at a to the curve representative of f. However, the derivative can also be interpreted as a rate of change.

7.1 Example 1: Conical tank

Consider a conically shaped reservoir with radius r and filled to the height h with water. The volume of water is therefore defined as $V = \frac{\pi}{3}r^2h$ (see Figure 9). When this reservoir is emptied of its water, the volume V, the radius r and the height h depend on time. We can therefore determine the rate of change of the volume as a function of time by differentiation of $V = \frac{\pi}{3}r^2h$, which gives:

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

The rate of change of V as a function of time therefore depends on the rate of change of r and of h. For example, if $\frac{dr}{dt} = -0.05\ cm/s$ and $\frac{dh}{dt} = -0.20\ cm/s$ with $r = 5\ cm$ and $h = 20\ cm$, then $\frac{dV}{dt} \approx -15.70\ cm^3/s$.
We can notice that in this case, we have $h\tan(\alpha) = r$ and α being constant, by differentiation we get:

$$\tan(\alpha)\tfrac{dh}{dt} = \tfrac{dr}{dt} \iff \tfrac{r}{h}\tfrac{dh}{dt} = \tfrac{dr}{dt}$$

Then:

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} = \pi\tan^2(\alpha)h^2\frac{dh}{dt}$$

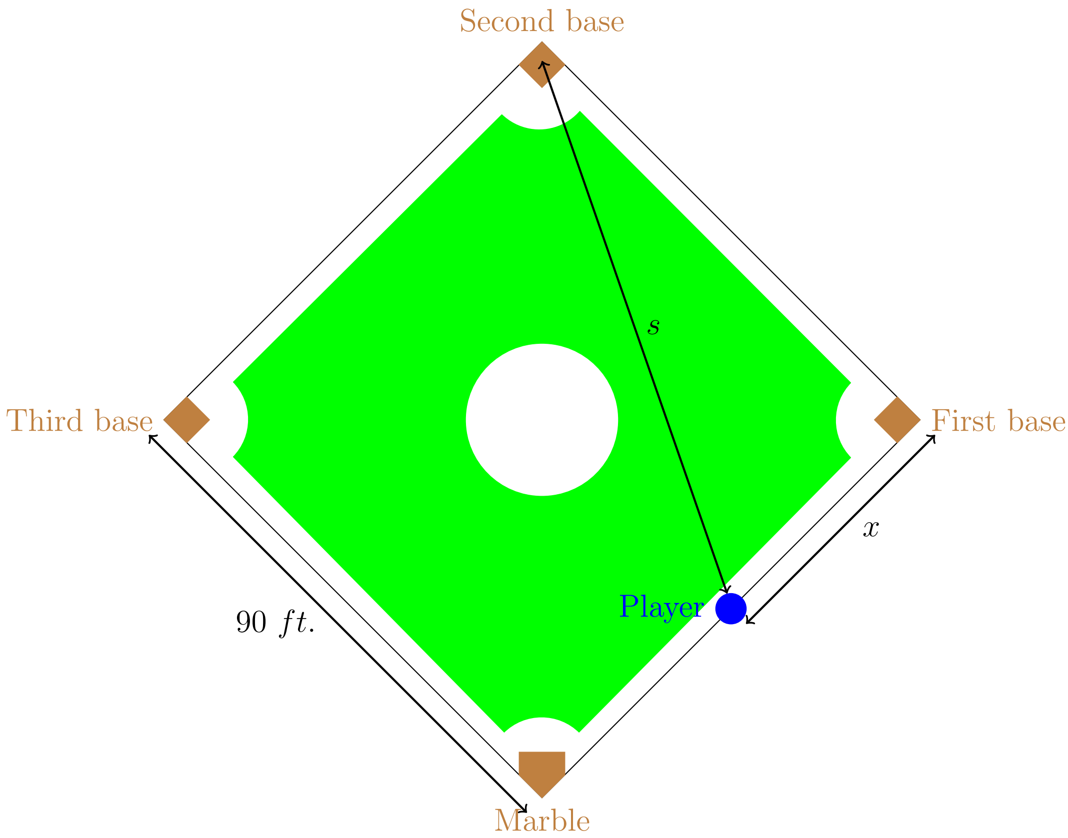


Figure 10: Baseball field

7.2 Example 2: Baseball

During a game of baseball, a player is running from home plate to first base at a speed of 20 ft/s. The question that arises is: How fast does the distance of the player vary from second base when he is halfway to first base?
As shown in Figure 10, we must formulate the relationship between the distance s separating the player from the second base, and the distance x separating the player from the first base. For that, it is enough to use the Pythagorean theorem, which gives:

$$s^2 = x^2 + 90^2$$

Let's differentiate with respect to time:

$$\begin{aligned} 2s\tfrac{ds}{dt} = 2x\tfrac{dx}{dt} \quad &\Longleftrightarrow \quad \tfrac{ds}{dt} = \tfrac{x}{s}\tfrac{dx}{dt} \\ &\Longleftrightarrow \quad \tfrac{ds}{dt} = \tfrac{x}{\sqrt{x^2+90^2}}\tfrac{dx}{dt} \end{aligned}$$

We have $\frac{dx}{dt} = 20\ ft/s$ and $x = \frac{90}{2} = 45$, then: $\frac{ds}{dt} \approx 8.94\ ft/s$.

7.3 Example 3: Moving piston

We consider a moving piston as described in Figure 11. The crankshaft of length $r = 10\ cm$ rotates clockwise at a speed of 4 rev/min. It is connected to a rod of length $L = 30\ cm$. We try to determine the speed of the piston when $\theta = \frac{\pi}{2}$.
We will therefore formulate θ as a function of r, L and x. If we apply the Pythagorean theorem to the red triangle (see Figure 11), we get:

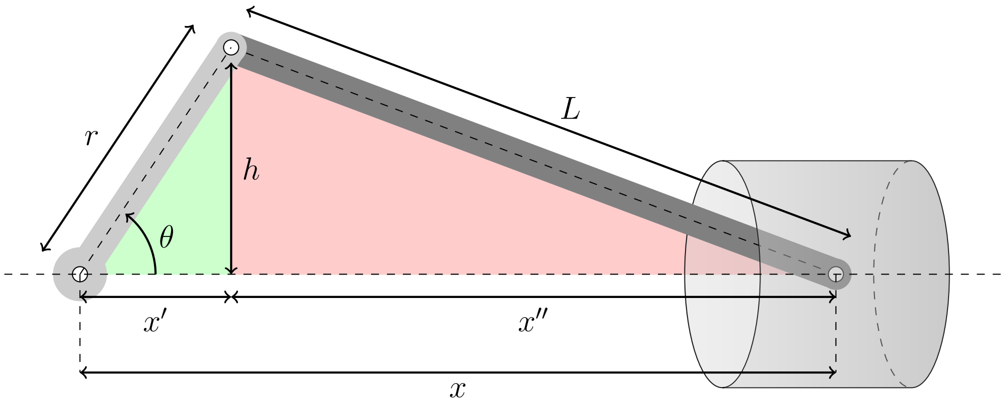


Figure 11: Moving piston

$$\begin{aligned} L^2 &= h^2 + (x'')^2 \\ &= r^2\sin^2(\theta) + (x - x')^2 \\ &= r^2\sin^2(\theta) + (x - r\cos(\theta))^2 \\ &= r^2 + x^2 - 2xr\cos(\theta) \end{aligned}$$

Let's differentiate with respect to t:

$$\begin{aligned} 0 &= 2x\tfrac{dx}{dt} - 2r\left(\cos(\theta)\tfrac{dx}{dt} - x\sin(\theta)\tfrac{d\theta}{dt}\right) \\ 0 &= (x - r\cos(\theta))\tfrac{dx}{dt} + rx\sin(\theta)\tfrac{d\theta}{dt} \end{aligned}$$

Then, we have:

$$\tfrac{dx}{dt} = -\tfrac{rx\sin(\theta)}{x - r\cos(\theta)}\tfrac{d\theta}{dt}$$

With $\frac{d\theta}{dt} = 4\ rev/min \approx 25.13\ rad/min$, $r = 10\ cm$, $L = 30\ cm$, and $\theta = \frac{\pi}{2}$, we get:

$$\begin{aligned} \tfrac{dx}{dt} &= -\tfrac{10x}{x}\tfrac{d\theta}{dt} \\ \tfrac{dx}{dt} &= -10\tfrac{d\theta}{dt} \\ \tfrac{dx}{dt} &= -251.23\ cm/min \end{aligned}$$

We can note that:

$$\begin{aligned} x &= x' + x'' \\ x &= r\cos(\theta) + \sqrt{L^2 - h^2} \\ x &= r\cos(\theta) + \sqrt{L^2 - r^2\sin^2(\theta)} \end{aligned}$$

Hence:

$$\begin{aligned} \tfrac{dx}{dt} &= -\tfrac{r(r\cos(\theta) + \sqrt{L^2 - r^2\sin^2(\theta)})\sin(\theta)}{r\cos(\theta) + \sqrt{L^2 - r^2\sin^2(\theta)} - r\cos(\theta)}\tfrac{d\theta}{dt} \\ \tfrac{dx}{dt} &= -\tfrac{r\sin(\theta)(r\cos(\theta) + \sqrt{L^2 - r^2\sin^2(\theta)})}{\sqrt{L^2 - r^2\sin^2(\theta)}}\tfrac{d\theta}{dt} \\ \tfrac{dx}{dt} &= -r\sin(\theta)\left(\tfrac{r\cos(\theta)}{\sqrt{L^2 - r^2\sin^2(\theta))}} + 1\right)\cdot\tfrac{d\theta}{dt} \end{aligned}$$

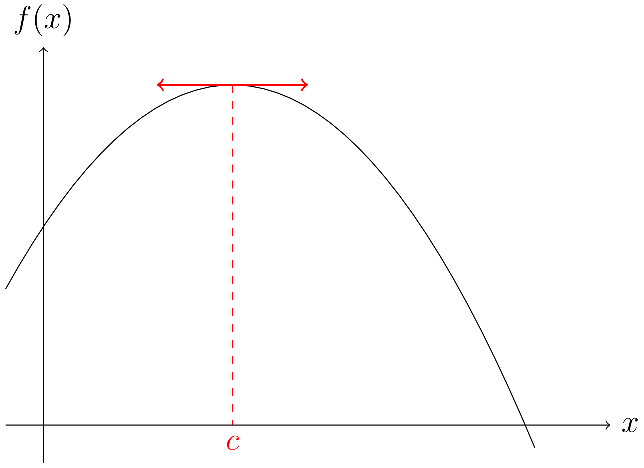


Figure 12: Local extremum of a function

8 Study of functions

8.1 Extrema of a differentiable function

Definition 40: *Let f be a function of I in $\mathbb{R}$ and let $c \in I$.*
*We say that f admits a **local maximum** at the point c if there exists a neighborhood V_c of c such that:*

$$\forall x \in V_c \cup I,\ f(x) \leq f(c).$$

*We say that f admits a **local minimum** at the point c if there exists a neighborhood V_c of c such that:*

$$\forall x \in V_c \cup I,\ f(x) \geq f(c).$$

Proposition 22: *If f is differentiable on $I =]a,\ b[$ and admits at a point c of I a local extremum then $f'(c) = 0$ (see Figure 12).*

Proof: Let f be a differentiable function on $I =]a,\ b[$ and which has at a point c of I a local maximum.
As f has a maximum in c, then, by definition, there exists $\alpha > 0$ such that, for all $x \in]c-\alpha,\ c+\alpha,$ $f(x) \leq f(c)$.
Moreover f is differentiable in I and in particular in c, then $\lim\limits_{c} \tau_{f,c} = f'(c)$
For $x \in]c,\ c+\alpha[$, $f(x)-f(c) \leq 0$ and therefore $\tau_{f,c}(x) = \frac{f(x)-f(c)}{x-c} \leq 0$. Hence $\lim\limits_{c^+} \tau_{f,c} = f'(c) \leq 0$.
For $x \in]c-\ alpha,\ c[$, $f(x) - f(c) \leq 0$ and therefore $\tau_{f,c}(x) = \frac{f(x)-f(c)}{x-c} \geq 0$. Hence $\lim\limits_{c^-} \tau_{f,c} = f'(c) \geq 0$.
We infer that $f'(c) = 0$ (since $f'(c)$ must be both negative and positive).
Similar proof if f has a local minimum in c (Do it as an exercise).

Remarque:

a) The fact that c is a point in the open interval $]a, b[$ is crucial. Thus, the identity $f : x \mapsto x$ over $I = [0, 1]$ is strictly increasing. It has a local minimum in 0 and a local maximum in 1 (which are the minimum and the maximum of f on I). And yet $f'(0)$ and $f'(1)$ are not zero!

b) The derivative of a function can be zero at a point without an extremum at this point. For example, the function $x \mapsto x^3$ has a zero derivative at 0 and has neither a maximum nor a minimum at this point.

c) There can be local extrema for a non-derivable function! Thus, the absolute value $x \mapsto |x|$ has a minimum at 0, point where it is not differentiable.

Theorem 6 (Rolle's theorem)**:** *Let f be a continuous function on $[a,\ b]$ and differentiable on $]a,\ b[$ such that $f(a) = f(b)$. Then there exists $c \in]a,\ b[$ such that $f'(c) = 0$.*

This theorem will be admitted because its proof requires results which are not seen in this course. If f is the constant function, the result is obvious. Otherwise, since f is continuous over $[a,\ b]$, f is bounded over $[a,\ b]$ and reaches its bounds. This means that there exists $m \in [a,\ b]$ and $M \in [a,\ b]$ such that $\forall x \in [a,\ b]$, $f(m) \leq f(x) \leq f(M)$ ($f(m) \neq f(M)$ because f is not a constant function). As $f(a) = f(b)$, then we have the following cases:

1. if $m = a$ or $m = b$, then $M \in]a,\ b[$, hence we have $c = M$;

2. if $M = a$ or $M = b$, then $m \in]a,\ b[$, hence we have $c = m$;

3. $m \in]a,\ b[$ and $M \in]a,\ b[$, hence we have $c = m$ or $c = M$.

So in all cases f admits a local extremum at a point c of $]a,\ b[$ and is differentiable on $]a,\ b[$, hence, according to the proposition 22, $f'(c) = 0$.

Theorem 7 (Mean value theorem)**:** *Let f be a continuous function on the segment $[a, b]$ and differentiable in the open interval $]a, b[$. There is an element $c \in]a, b[$ such that $f(b) - f(a) = (b - a)f'(c)$.*

The mean value theorem states that, if f is a continuous function on the segment $[a, b]$ and differentiable in the open interval $]a, b[$, there exists a point $c \in]a, b[$ such that the tangent at c to the graph representing f is parallel to the line passing through point $(a, f(a))$ and $(b, f(b))$ (see Figure 13).

Exercise 59 (Proof of the Mean value theorem)**:**
Let f be a continuous function on the segment $[a, b]$ and differentiable in the open interval $]a, b[$. We set $g : x \mapsto f(x) - x\frac{f(b)-f(a)}{b-a}$.

1. Show that $g(a) = g(b)$.

2. Study the differentiability of g and calculate its derivative if it exists.

3. What can we deduce from g? (Rolle's theorem).

4. What can we then conclude about f?

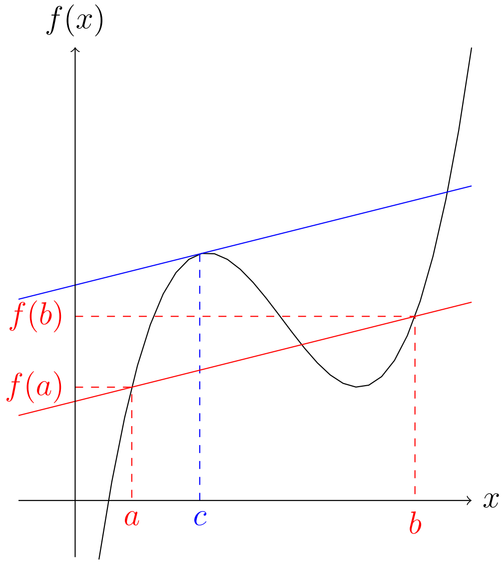


Figure 13: Mean value theorem

8.2 Derivative and monotonicity of functions

The Rolle's theorem gives us a way to study the monotonicity of differentiable functions.

Proposition 23 (Monotonic condition)**:** *Let f be a function derivable from I in $\mathbb{R}$, then:*

1. *f is increasing if and only if $f'(x) \geq 0$ for any x of I;*
2. *f is decreasing if and only if $f'(x) \leq 0$ for all x of I;*
3. *f is constant if and only if $f'(x) = 0$ for all x of I.*

Proof (of 1.)**:** Let f be a function differentiable from I in $\mathbb{R}$.

$\Longrightarrow$ Suppose that f is increasing on I.

Let $x \in I$ and $a \in I$ such that $x \neq a$, then $\tau_{f,a}(x) = \frac{f(x)-f(a)}{x-a} \geq 0$, because f is increasing over I.

As f is differentiable on I, we get:

$$\lim_{x \longrightarrow a} \tau_{f,a}(x) = f'(a) \geq 0$$

So, for any point $a \in I$, $f'(a) \geq 0$.

$\Longleftarrow$ Suppose that $f'(x) \geq 0$ for all x of I.

Let $x \in I$ and $x' \in I$ such that $x < x'$. The function f is differentiable, therefore continuous, over $[x,\ x'] \subseteq I$ and is differentiable over $]x,\ x'[\subseteq I$. Hence, According to the Rolle's theorem, there exists $c \in]x,\ x'[$ such that $f(x') - f(x) = (x' - x)f'(c)$.

However, by hypothesis, $f'(c) \geq 0$ and $x' - x > 0$, so $f(x') - f(x) \geq 0$.

We have therefore shown that $\forall (x,\ x') \in I^2$, $x < x' \implies f(x) \leq f(x')$, so f is increasing over I.

Proof (of 2.)**:** Similar to 1.

Proof (of 3.)**:** Consequence of 1. and 2. since a constant function is both increasing and decreasing.

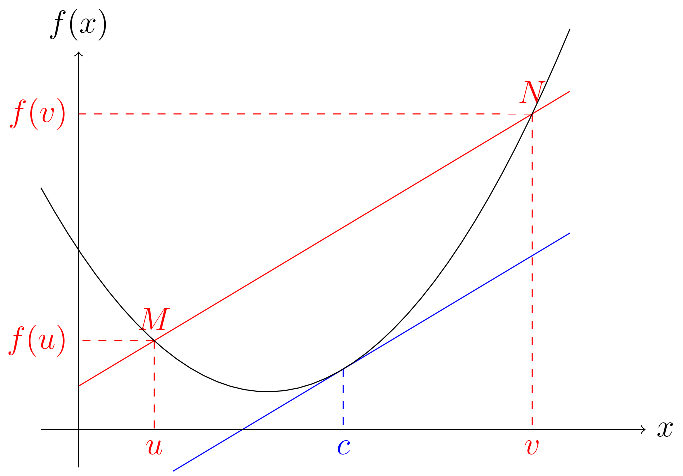


Figure 14: Convex function

Proposition 24 (Strict monotonic condition)**:** *Let f be a function differentiable from I in $\mathbb{R}$.*

1. *If $f'(x) > 0$ for all x of I, then f is strictly increasing on I;*
1. *If $f'(x) < 0$ for all x of I, then f is strictly decreasing on I;*

Proof: Similar proof of proposition 23. Do it as an exercise.

We should take care that the condition "$f'(x) > 0$ for any x of I" is not a necessary condition for f to be strictly increasing over I. Thus the application $f : x \mapsto x^3$ is strictly increasing over $\mathbb{R}$ although $f'(0) = 0$.

Exercise 60: Study the monotonicity of $f : x \mapsto x + \sin(x)$ on $\mathbb{R}$.

8.3 Convex functions

Definition 41: *Let f be a function defined on an interval I of $\mathbb{R}$ and $\mathcal{C}_f$ be its representative curve in a plane related to a coordinate system. We say that f is* ***convex*** *(resp.* ***concave****) on I if, for any pair of distinct points $M = (u, f(u))$ and $N = (v, f(v))$ of $\mathcal{C}_f$, the segment of line joining these points is located above (resp. below) the curve $\mathcal{C}_f$.*

Proposition 25: *Let f be a differentiable function on an interval I of $\mathbb{R}$. If the derivative f' of f is an increasing (resp. decreasing) function, then f is a convex (resp. concave) function.*

Proof: Let f be a differentiable function on an interval I of $\mathbb{R}$. Suppose that f' is increasing over I.
Let $(u,\ v) \in I^2$ such that $u < v$, and let M and N be the points of $\mathcal{C}_f$ of abscissa u and v. The line MN has the equation:

$$y = f(u) + m(x - u) \text{ with } m = \frac{f(v) - f(u)}{v - u}.$$

Let us show that the function $g : x \mapsto f(x) - (f(u) + m(x - u))$ is negative on the interval $[u,\ v]$.
We have $g(u) = g(v)$, and g is continuous over $[a,\ b]$ and derivable over $]a,\ b[$. So, according to Rolle's theorem, there exists $c \in]a,\ b[$ such that $g'(c) = 0$.
Now $g' = f' - m$ is increasing over I, so g' is negative over $[u,\ c]$ and positive over $[c,\ v]$. This implies that g is decreasing on $[u,\ c]$ and increasing on $[c,\ v]$. As it vanishes in u and v, it is therefore negative on $[u,\ v]$.

Proposition 26: *Assume f is twice differentiable on I. Then, if $f''(x) \geq 0$ (resp. $f''(x) \leq 0$) for all $x \in I$, the function f is convex (resp. concave) .*

Proof: If f'' is positive, then f' is increasing and, hence, f is convex.

Remarque: At a point where the second derivative f'' vanishes and changes sign, the concavity of the curve $\mathcal{C}_f$ is reversed: we say that $\mathcal{C}_f$ has an **inflection point** .

Exercise 61: Study the convexity of the following function on their domain of definition:

1. $f : x \mapsto e^x$;
2. $f : x \mapsto \ln(1 + x^2)$;
3. $f : x \mapsto \ln(\ln(x))$.

8.4 Study of functions

The study of a function often involves the study of its monotony and its graphic representation. Current calculation software provides useful assistance for this, but they still cannot replace a desk study. The plan for such a study may be like this:

- Start by determining the domain of definition $\mathcal{D}$ of f, the intervals on which f is continuous, those on which it is differentiable. Possibly determine the field of study (use of parity, periodicity, ...).
- Study (with or without the derivative) the variations of f. Determine the limits of f at the limits of its domain of definition. Study infinite branches, in particular asymptotes.
- To refine the plot of the curve, determine the extrema of f, the angular points, the intervals on which f is convex, the inflection points, ...

Exercise 62: Study the following functions:

1. $f : x \mapsto \frac{1}{x^4}$
2. $f : x \mapsto \sqrt{x^2 - 9}$
3. $f : x \mapsto 2.x + \sqrt{x^2 - 1}$

8.5 Solved exercises

Exercise 63: Study the function $f : x \mapsto 2x^3 + x^2 - 20x - 3$.

Solution: The domain of definition of f is $\mathcal{D} = \mathbb{R}$.
Limits:

Obviously, we have: $\lim_{-\infty} f = -\infty$ and $\lim_{+\infty} f = +\infty$.

Monotony:

f is differentiable on $\mathbb{R}$ and for any $x \in \mathbb{R}$, we have $f'(x) = 6(x+2)\left(x - \frac{5}{3}\right)$. We can then study the sign of f':

x	$-\infty$ $\quad$ -2	-2 $\quad$ $\frac{5}{3}$	$\frac{5}{3}$ $\quad$ $+\infty$
$x+2$	$-$	$+$	$+$
$x - \frac{5}{3}$	$-$	$-$	$+$
f'	$+$	$-$	$+$

We infer that f is increasing on $]-\infty,\ -2]$ and $[\frac{5}{3},\ +\infty[$ and decreasing on $[-2,\ \frac{5}{3}]$.

Besides, the curve representative of f has vertical tangents at the points of abscissa -2 (local maximum) and $\frac{5}{3}$ (local minimum).

Convexity:

f' is differentiable on $\mathbb{R}$ and, for any $x \in \mathbb{R}$, we have $f''(x) = 12x + 2 = 12\left(x + \frac{1}{6}\right)$.

x	$-\infty$ $\quad$ $-\frac{1}{6}$	$-\frac{1}{6}$ $\quad$ $+\infty$
f''	$-$	$+$

We infer that f is concave on $]-\infty,\ -\frac{1}{6}]$ and convex on $[-\frac{1}{6},\ +\infty[$ and presents an inflection point at the point of abscissa $-\frac{1}{6}$.

Particular points:

Let's find the points where f vanishes. We can notice that $f(3) = 0$, and the Euclidean division gives us: $\forall x \in \mathbb{R},\ f(x) = (x-3).(2x^2 + 7x + 1)$. Then we get:

$$\forall x \in \mathbb{R},\ f(x) = 2(x-3)\left(x - \frac{-7+\sqrt{41}}{4}\right)\left(x - \frac{-7-\sqrt{41}}{4}\right).$$

Hence f vanishes at 3, $\frac{-7+\sqrt{41}}{4}$, and $\frac{-7-\sqrt{41}}{4}$.

Monotony table:

x	$-\infty$		-2		$-\frac{1}{6}$		$\frac{5}{3}$		$+\infty$
f'		$+$		$-$		$-$		$+$	
f	$-\infty$	$\nearrow$	25	$\searrow$	$\frac{19}{54}$	$\searrow$	$-\frac{658}{27}$	$\nearrow$	$+\infty$

Graphical representation: see Figure 15

Exercise 64: Study the function $f : x \mapsto \arccos(x)$.

Solution: The domain of definition of f is $\mathcal{D} = [-1,\ 1]$.
Limits:

The function f is continuous on $[-1,\ 1]$, then $\lim\limits_{-1} f = \pi$ and $\lim\limits_{1} f = 0$.

Monotony:

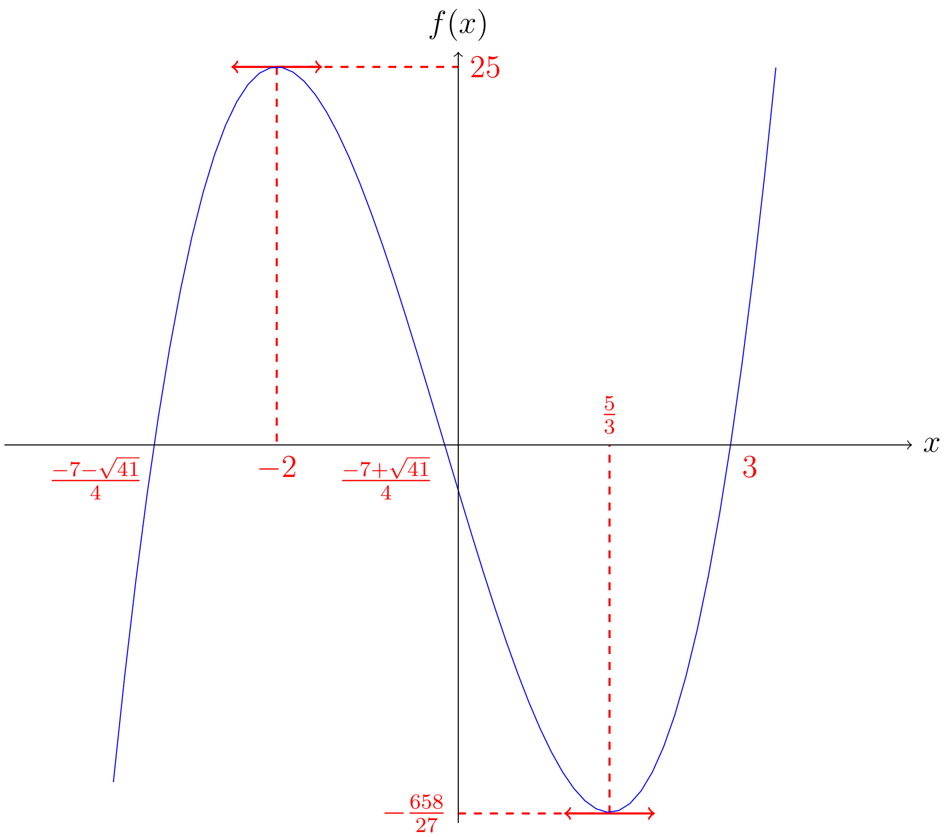


Figure 15: Graphical representation of $f : x \mapsto 2x^3 + x^2 - 20x - 3$

f is differentiable on $]-1,\ 1[$ and for all $x \in \mathbb{R}$, we have $f'(x) = -\frac{1}{\sqrt{1-x^2}} < 0$.

We infer that f is strictly decreasing on $]-1,\ 1[$.

We can also note that $\lim\limits_{-1} f' = -\infty$ and $\lim\limits_{1} f' = -\infty$. Hence f has a vertical asymptote at -1 and 1.

Convexity:

f' is differentiable on $]-1,\ 1[$ and for all $x \in]-1,\ 1[$, we have $f''(x) = -\frac{x}{(1-x^2).\sqrt{1-x^2}}$.

x	-1 $\quad$ 0	0 $\quad$ 1
f''	$+$	$-$

We infer that f is convex on $]-1,\ 0]$ and concave on $[0,\ 1[$, besides f has an inflexion point at point of abscissa 0.

Monotony table:

x	-1		0		1
f'		$+$		$-$	
f	π	$\nearrow$	$\frac{\pi}{2}$	$\nearrow$	0

Graphical presentation: see Figure 16

Exercise 65: Study the function $f : x \mapsto \frac{x^2-2.x+4}{x-2}$.

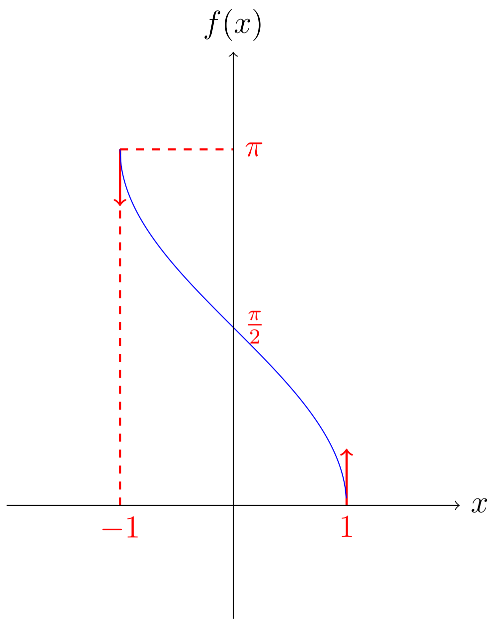


Figure 16: Graphical presentation of $f : x \mapsto \arccos(x)$

Solution: The domain of definition of f is $\mathcal{D} =]-\infty,\ 2[\cup]2,\ +\infty[$.

Limits:

We have:

- $\lim\limits_{-\infty} f = -\infty$;
- $\lim\limits_{2^-} f = -\infty$;
- $\lim\limits_{2^+} f = +\infty$;
- $\lim\limits_{+\infty} f = +\infty$.

Monotony:

f is differentiable on $]-\infty,\ 2[\cup]2,\ +\infty[$ and for all $x \in \mathbb{R}$, we have $f'(x) = \frac{x.(x-4)}{(x-2)^2}$.

x	$-\infty$	0	2	2	4	$+\infty$
x	$-$	$+$		$+$	$+$	
$x-4$	$-$	$-$		$-$	$+$	
$(x-2)^2$	$+$	$+$		$+$	$+$	
$f'(x)$	$+$	$-$		$-$	$+$	

Therefore f is increasing on $]-\infty,\ 0]$ and on $[4,\ +\infty[$, and f is decreasing on $[0,\ 2]$ and on $[2,\ 4]$. Besides, f has a vertical tangent at 0 and 4.

Convexity:

f' is differentiable on $]-\infty,\ 2[\cup]2,\ +\infty[$ and for all $x \in \mathbb{R}$, we have $f''(x) = \frac{8}{(x-2)^3}$.

Hence f is concave on $]-\infty,\ 2[$ and convex on $]2,\ +\infty[$.

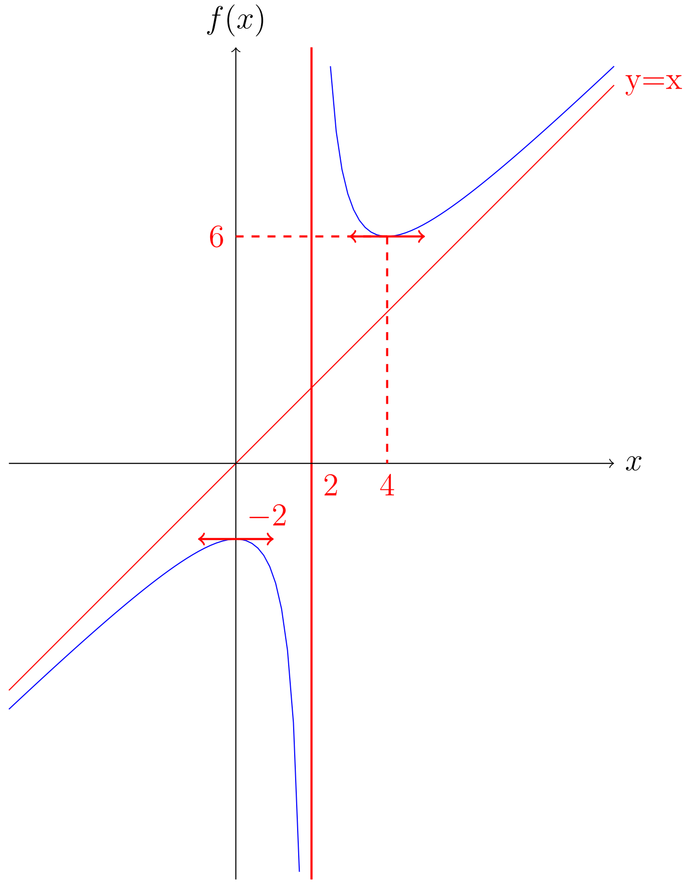


Figure 17: Graphical representation of $f : x \mapsto \frac{x^2-2.x+4}{x-2}$

Monotony table:(voir Figure 17)

x	$-\infty$		0		2	2		4		$+\infty$
f'		$+$		$-$			$-$		$+$	
f	$-\infty$	$\nearrow$	-2	$\searrow$	$-\infty$	$+\infty$	$\searrow$	6	$\nearrow$	$+\infty$

Remarque: We can note that the euclidean division gives $\forall x \in \mathcal{D}$, $f(x) = x + \frac{4}{x-2}$.
If we set $g : x \mapsto x$, we get $\forall x \in \mathcal{D}$, $f(x) - g(x) = \frac{4}{x-2}$.
Then $\lim\limits_{+\infty} f - g = 0$ and $\lim\limits_{-\infty} f - g = 0$. Hence, when x approaches infinity, $f(x)$approaches $g(x)$.
We say that the line of equation $y = x$ is asymptote to f at $+\infty$ and at $-\infty$.
Graphical representation: see Figure 17

9 Application to optimization

9.1 Example 1: Maximizing a volume

A company wants to make an open top box with a square base and an area of 108 square centimeters (see Figure 18). We try to determine the dimensions that give us the biggest possible volume.

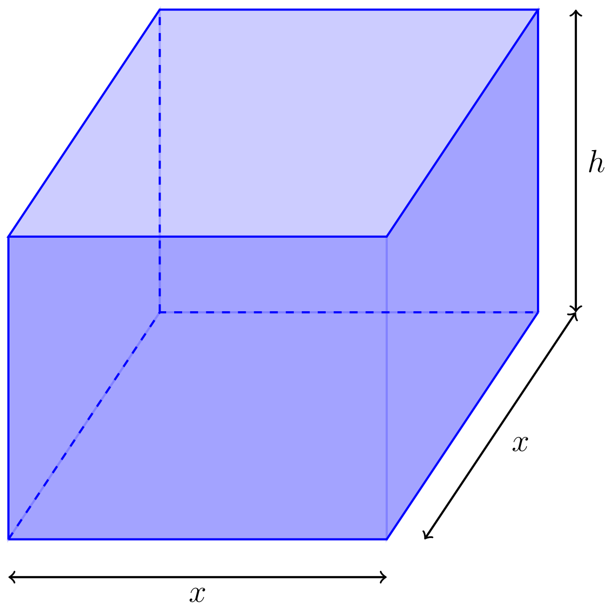


Figure 18: Box with a square base

We denote by x the length of one side of the square base and h the height of the box (in cm.). The total volume of the box is given by the equation: $V = x^2 h$
This first formula is important because it links the quantity to be optimized (the volume V) with the dimensions of the box (x and h). However, here the volume is expressed as a function of 2 variables (x and h) and it is preferable to have a function with only one variable in order to optimize the volume. For that, we will try to exploit the information of the problem.
We want the area S to be equal to 108 cm^2, which gives: $S = x^2 + 4xh = 108$.
Hence: $h = \frac{108-x^2}{4x}$.
We then obtain: $V = x^2 \left(\frac{108-x^2}{4x}\right) = 27x - \frac{x^3}{4}$.
We will now study the extrema of $f : x \mapsto 27x - \frac{x^3}{4}$:

f is defined and differentiable on $\mathbb{R}$ and we have $f' : x \mapsto 27 - \frac{3}{4}x^2 = -\frac{3}{4}(x-6)(x+6)$.

Monotony table of f:

x	$-\infty$		-6		6		$+\infty$
f'		$-$		$+$		$-$	
f	$+\infty$	$\searrow$	-108	$\nearrow$	108	$\searrow$	$-\infty$

Hence f has a local maximum at 6 and a local minimum at -6.

In our case, the fact that $x \leq 0$ does not make sense because a distance cannot be negative. We can also notice that to have a positive volume, we must have $x \leq \sqrt{108}$. So $x = 6$ cm is a valid solution for our problem. The dimensions of the box are therefore $x = 6$ cm, $h = 3$ cm for a volume of $V = 108$ cm^3.

9.2 Example 2: Minimizing a distance

We look for the points of the curve of equation $y = 4 - x^2$ which are closest to the coordinate point $(0,\ 2)$ (see Figure 19).

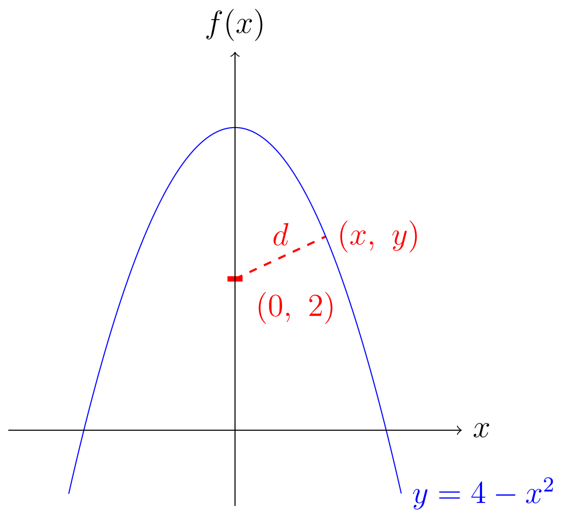


Figure 19: Graph of $y = 4 - x^2$

Consider a point of coordinates $(x,\ y)$ belonging to the curve of equation $y = 4 - x^2$. The distance separating it from the point $(0,\ 2)$ is given by the following formula:

$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

The distance d is here expressed as a function with two variables (x and y), so we will try to reduce it to a single variable by exploiting the data of the problem. Indeed, as the point $(x,\ y)$ belongs to the curve of equation $y = 4 - x^2$, we then have:

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

In this case, minimizing d amounts to minimizing the function $f : x \mapsto x^4 - 3x^2 + 4$. We will therefore study this function:

> f is defined and differentiable on $\mathbb{R}$ and we have $f' : x \mapsto 2x(2x^2 - 3)$.
>
> Monotony table of f:

x	$-\infty$		$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$		$+\infty$
f'		$-$		$+$		$-$		$+$	
f	$+\infty$	$\searrow$	$\frac{7}{4}$	$\nearrow$	4	$\searrow$	$\frac{7}{4}$	$\nearrow$	$+\infty$

So f reaches a minimum of $\frac{5}{2}$ in $x = -\sqrt{\frac{3}{2}}$ and in $x = \sqrt{\frac{3}{2}}$. We infer that the points of the curve closest to $(0,\ 2)$ are the points of coordinates $(-\sqrt{\frac{3}{2}},\ \frac{5}{2})$ and $(\sqrt{\frac{3}{2}},\ \frac{5}{2})$ and the minimum distance is $\sqrt{\frac{7}{4}}$.

9.3 Example 3: Minimizing a surface

A rectangular page should contain 24 square inches of printing area. The top and bottom margins of the page are 1.5 inches, and the right and left margins are 1 inch. What should the page dimensions be in order to use the smallest amount of paper possible?
We denote by x the length of the printing area and y its width in inches.

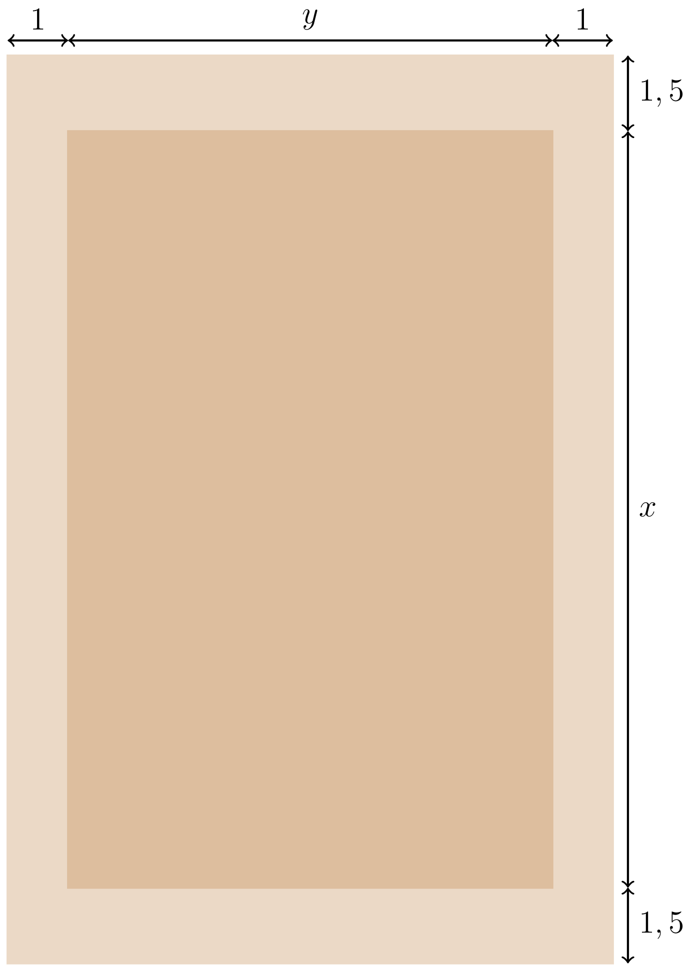


Figure 20: Paper page with margins

The quantity of paper used is directly related to the surface of the page: $S = (x+3)(y+2)$ (see Figure 20). We therefore seek to minimize S.
However, we know that the page must contain 24 square inches of printing, which gives: $xy = 24$.
Hence:

$$S = (x+3)(\frac{24}{x}+2) = 30 + 2x + \frac{72}{x}$$

.
Let's study the function $f : x \mapsto 30 + 2x + \frac{72}{x}$:

f is defined and differentiable on $\mathbb{R}^*$ and we have $f' : x \mapsto 2 - \frac{72}{x^2} = \frac{2.x^2-72}{x^2}$.

Monotony table of f:

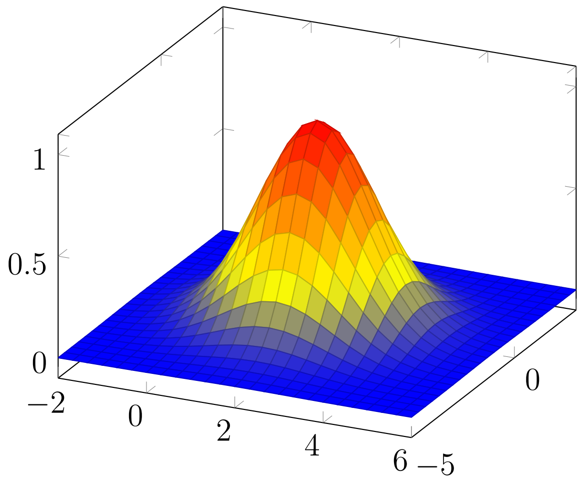
x	$-\infty$		-6		0	0		6		$+\infty$
f'		$+$		$-$			$-$		$+$	
f	$-\infty$	$\nearrow$	6	$\searrow$	$-\infty$	$+\infty$	$\searrow$	54	$\nearrow$	$+\infty$

Here, we are only interested in the case where x is positive (because a length cannot be negative). In this case, f has a minimum in $x = 6$ inches which corresponds to $y = 4$ inches.

In order to minimize the amount of paper, the page should be the following dimensions: 9 inches (= 6 + 3) in length and 6 inches (= 4 + 2) in width.

10 Function of multiple variables

The aim of this chapter is to generalize the notion of derivative of a function of a real variable with real values from the theory of differential calculus applied to functions of several variables. The fundamental idea of this theory is to approach an “arbitrary” function (of several real variables here) by a linear function in the neighborhood of a point.
Remember that YOU CANNOT DIVIDE BY A VECTOR! Now, in $\mathbb{R}$, the definition of the derivative involves the ratio $\frac{f(x)-f(a)}{x-a}$. It therefore implies being able to divide by $(x-a)$. But in $\mathbb{R}^n$, $n > 1$, it doesn't make sense because division by vector is undefined. What to do then if we cannot define the derivative of a function $D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$? This is the whole goal of this chapter: to introduce a generalized notion of the derivative.
In the following, when not specified, n is a natural number, and 0_n is the zero vector of $\mathbb{R}^n$.



10.1 Norms and distances in $\mathbb{R}^n$

Definition 42 (Norm)**:** *Let $E \subseteq \mathbb{R}^n$. We call a norm on E an application $\mathcal{N} : E \longrightarrow \mathbb{R}_+$ which satisfies:*

- *Separation:* $\forall x \in E$, $\mathcal{N}(x) = 0 \Longleftrightarrow x = 0_n$,
- *Homogeneity:* $\forall \lambda \in \mathbb{R}$ *and* $\forall x \in E$, $\mathcal{N}(\lambda.x) = |\lambda|.\mathcal{N}(x)$,
- *Triangle inequality:* $\forall (x,\ y) \in E^2$, $\mathcal{N}(x+y) \leq \mathcal{N}(x) + \mathcal{N}(y)$.

The norm of $x \in E$ is generally noted $||x||$. Likewise, it is common to use the notation $||.||$ for the application $\mathcal{N}$.

Example 11: Examples of classic norms on $\mathbb{R}^n$:

- Manhattan Norm: $\forall x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ ||x||_1 = \sum_{i=1}^{n} |x_i|$;

- Euclidean Norm: $\forall x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ ||x||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$;
- Norm $p \in \mathbb{N}^*$: $\forall x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$;
- Infinite Norm: $\forall x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ ||x||_\infty = \max_{1 \leq i \leq n} |x_i|$.

Definition 43 (Distance induced by a norm)**:** *Let $E \subseteq \mathbb{R}^n$ and $\mathcal{N}$ a norm on E. We call a distance induced by $\mathcal{N}$ the application:*

$$\begin{array}{rccl} d: & E \times E & \rightarrow & \mathbb{R}_+ \\ & (x,\ y) & \mapsto & \mathcal{N}(x-y) = ||x-y|| \end{array}$$

Remarque: Note that all norm induced a distance, but not all distances are resulting from a norm.

10.2 Open ball, closed ball, and sphere

Definition 44: *Let $E \subseteq \mathbb{R}^n$ and $||.||$ a norm on E. Let $a \in E$ and $r \in \mathbb{R}_+$, then:*

- $\overline{B}_{||.||}(a, r) = \{x \in E\ /\ ||x-a|| \leq r\}$ *is called a* ***closed ball*** *centered at a and radius r;*
- $B_{||.||}(a, r) = \{x \in E\ /\ ||x-a|| < r\}$ *is called a* ***open ball*** *centered at a and radius r;*
- $S_{||.||}(a, r) = \{x \in E\ /\ ||x-a|| = r\}$ *is called a* ***sphere*** *centered at a and radius r.*

If we place ourselves in the particular case where $n = 1$, we can notice that the definition of an open ball is equivalent to that of a neighborhood. This is normal, because with the notion of ball, we extend the notion of neighborhood in a vector space. This will also allow us to extend to define the limit of a function of several variables.
It will also be understood that the definition of a ball depends on the standard used. The figure 21 represents different ball with center 0 and radius 1 with the norm $||.||_p$ with different values of p.

Definition 45: *We say that a function $f : E \rightarrow \mathbb{R}$, $E \subseteq \mathbb{R}^n$, is defined at the* ***neighborhood*** *of a point a, if there exists an open ball $B_{||.||}(a, r)$ with center a and radius $r > 0$ such that $B_{||.||}(a, r) \subseteq E$.*

10.3 Limits

Definition 46: *We say that a function $f : E \rightarrow \mathbb{R}$, $E \subseteq \mathbb{R}^n$, has $l \in \mathbb{R}$ as limit at $a \in \mathbb{R}^n$, if for any open ball $B_{||.||}(l)$ (or neighborhood) of center l, there exists an open ball $B_{||.||}(a)$ of center a such that:*

$$x \in E \cap B_{||.||}(a) \Longrightarrow f(x) \in B_{||.||}(l)$$

Which is equivalent to saying that:

$$\forall \epsilon > 0,\ \exists \alpha > 0 \text{ such that } \forall x \in E,\ ||x-a|| < \alpha \implies |f(x) - l| < \epsilon.$$

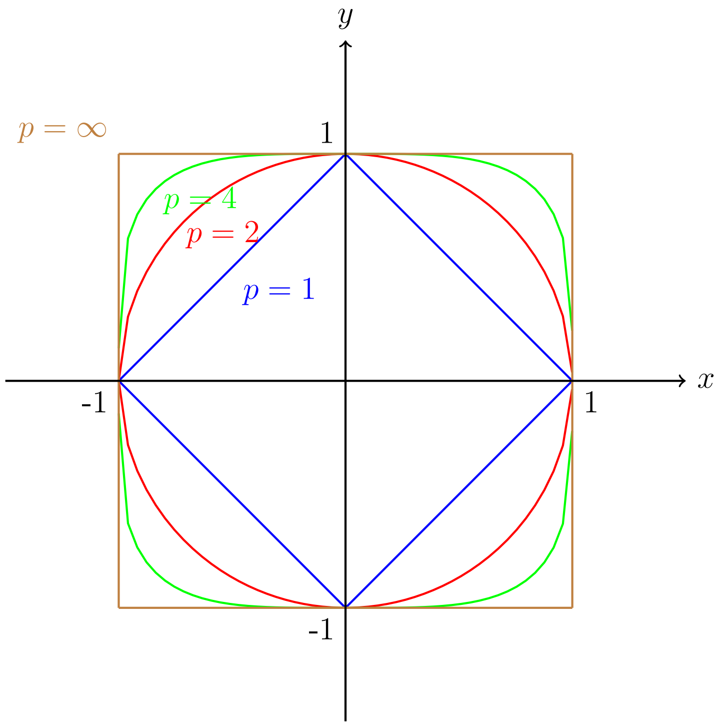


Figure 21: Graphical representation of balls with center 0 and radius 1 with the norm $||.||_p$

As can be seen, the definition of limits for multi-variable functions, with the notion of ball, is similar to that of functions from $\mathbb{R}$ to $\mathbb{R}$. Therefore, all the properties on the limits seen previously remain valid in the case of functions of several variables.

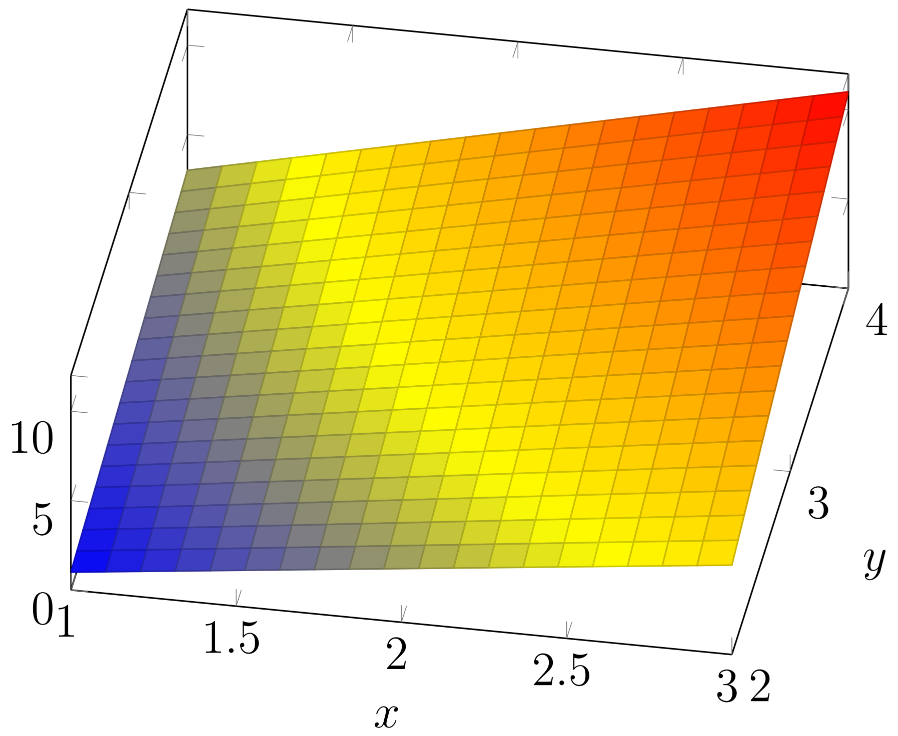
Definition 47: *Let f be a function defined on $E \subseteq \mathbb{R}^n$ and $a \in E$. We say that f is continuous at a if it has a limit at a. We then have in this case: $\lim\limits_{x \to a} f(x) = f(a)$.*

Example 12: Study the limit of $f : (x, y) \mapsto xy - 1$ at $(2, 3)$.

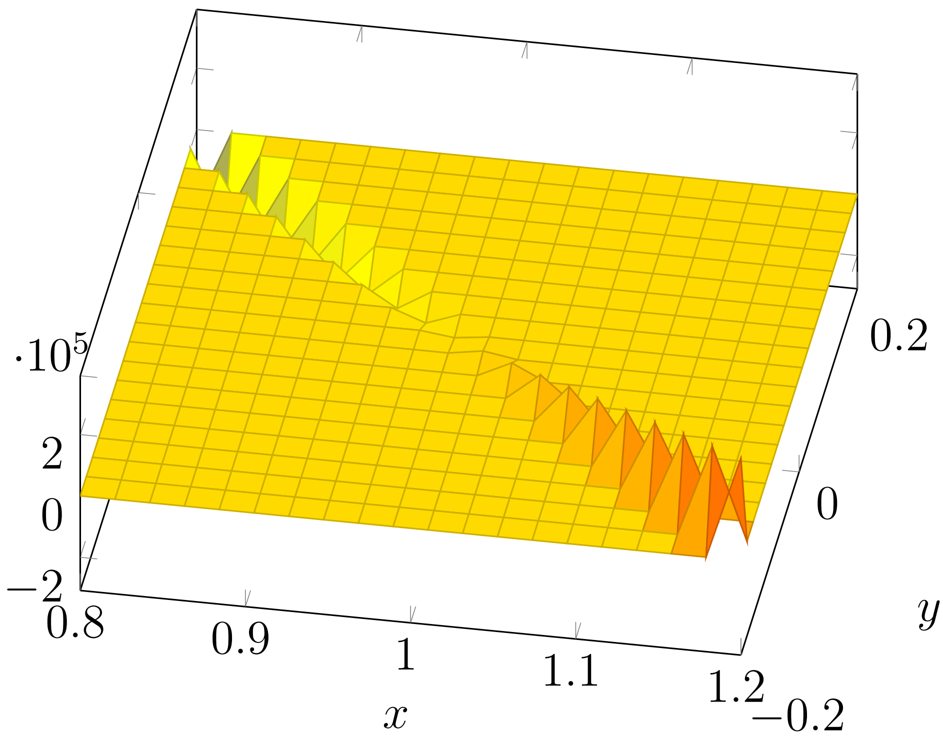
This function is defined on $\mathcal{D} = \mathbb{R}^2$, it is also continuous on$\mathcal{D}$.

We have $(2, 3) \in \mathcal{D}$ and $f(2, 3) = 5$.

Therefore we conclude that $\lim\limits_{(2.3)} f = 5$.



Example 13: Study the limit of $f : (x, y) \mapsto \frac{y}{x+y-1}$ at $(1, 0)$.
This function is defined on $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \;/\; x + y - 1 \neq 0\}$, and continuous on $\mathcal{D}$.
Note that f is not defined at $(1, 0)$.
Let's fix $x = 1$, in this case: $\forall y \in \mathbb{R}^*,\ f(1, y) = \frac{y}{y} = 1$ and $\lim\limits_{y\to 0} f(1, y) = 1$.
Now let's fix $y = 0$, then: $\forall x \in \mathbb{R} - \{1\},\ f(x, 0) = 0$ and $\lim\limits_{x\to 1} f(x, 0) = 0$.
Since we get different values, the uniqueness of the limit allows us to conclude that f has no limit in $(1, 0)$.

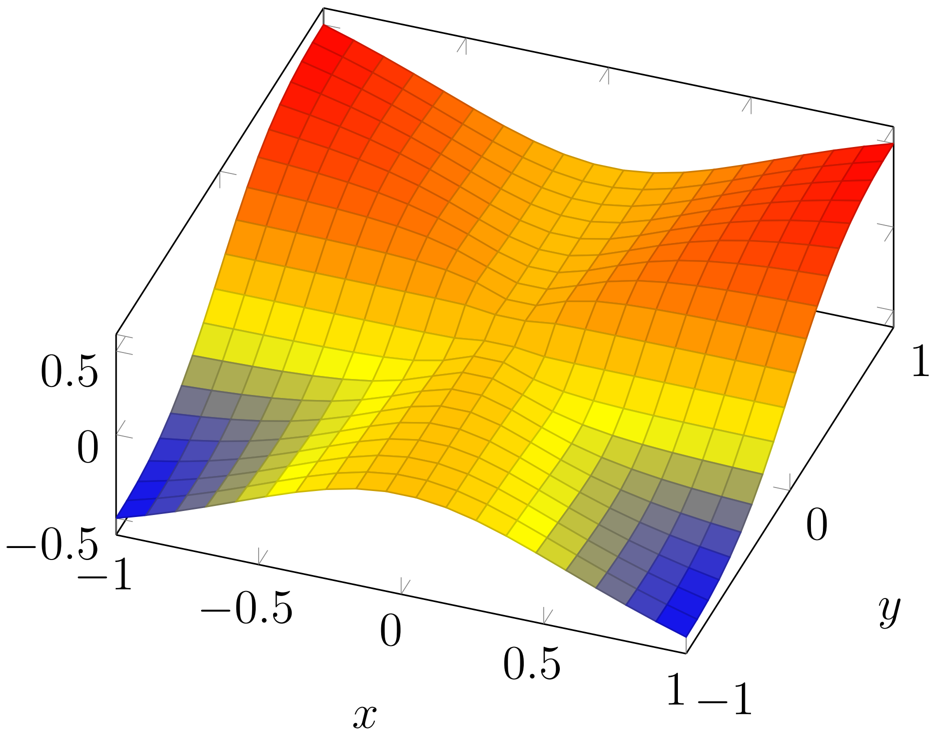


Example 14: Study the limit of $f : (x, y) \mapsto \frac{x^2 y}{x^2+y^2}$ at $(0, 0)$.
This function is defined on $\mathcal{D} = \{x \in \mathbb{R}^2 \;/\; x^2 + y^2 \neq 0\} = \mathbb{R}^2 - \{(0, 0)\}$.

The function f is not defined in $(0,0)$.
Let's set $x=0$, in this case: $\forall y \in \mathbb{R}^*,\ f(0,y)=0$ and $\lim_{y\to 0} f(0,y)=0$.
Let's set $y=0$, in this case: $\forall x \in \mathbb{R}^*,\ f(x,0)=0$ and $\lim_{x\to 0} f(x,0)=0$.
In this case the two limits are equal, however this **is not sufficient** to conclude that f admits a limit in $(0,0)$. For that, we will try to bound $|f(x,y)-0|$, $(x,y)\in\mathcal{D}$.
Let $(x,y)\in D$, we then have:

$$\begin{array}{rcl} |f(x,y)-0| & = & \left|\frac{x^2y}{x^2+y^2}\right| \\ & = & \frac{x^2|y|}{x^2+y^2} \\ & \leq & \frac{x^2|y|}{x^2} \\ & \leq & |y| \end{array}$$

Yet $\lim_{(x,y)\to(0,0)} |y|=0$, therefore f has a limit at $(0,0)$ and this limit is 0: $\lim_{(0,0)} f=0$.



10.4 Differential of a function

Definition 48: *We say that a function $f: E\subseteq\mathbb{R}^n \to F$ is differentiable at $a\in E$ if there exists a continuous linear application $u: E\to F$ such that:*

$$\lim_{h\to 0} \frac{||f(a+h)-f(a)-u(h)||}{||h||}=0.$$

In this case, u is the differential of f at a and it will be denoted df_a.

Reminder: An application $u: E\to F$ is linear if $\forall\lambda\in\mathbb{R}$ and $\forall(x,y)\in E^2$:

$$u(x+\lambda y)=u(x)+\lambda u(y)$$

The notion of differential extends that of differentiability of functions from $\mathbb{R}$ to $\mathbb{R}$. Let $f: E\subseteq\mathbb{R}\to\mathbb{R}$, differentiable on E, then f is differentiable over E and $\forall a\in E$, $df_a: h\mapsto f'(a)h$.

Indeed, if f is differentiable at $a \in E$, there exists, by definition, a linear function $u : E \to \mathbb{R}$ such that: $\lim\limits_{h \to 0} \dfrac{||f(a+h) - f(a) - u(h)||}{||h||} = 0$. Yet $\forall h \neq 0$:

$$\begin{aligned}
\frac{||f(a+h)-f(a)-u(h)||}{||h||} &= \frac{|f(a+h)-f(a)-u(h)|}{|h|} \\
&= \left|\frac{f(a+h)-f(a)-u(h)}{h}\right| \\
&= \left|\frac{f(a+h)-f(a)}{h} - \frac{u(h)}{h}\right| \\
&= \left|\frac{f(a+h)-f(a)}{h} - u\left(\frac{h}{h}\right)\right| \\
&= \left|\frac{f(a+h)-f(a)}{h} - u(1)\right|
\end{aligned}$$

We infer that $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = u(1)$, and that f is differentiable at a. If we set $u : h \mapsto df_a(h) = f'(a)h$, we have indeed u linear and $u(1) = f'(a)$.
By defining the function $dx : x \mapsto x$, we can then write: $df_a = f'(a)dx$. We then notice that if we replace df_a by $y - f(a)$ and dx by $x - a$, we obtain $y - f(a) = f'(a)(x - a)$, which represents the equation of the tangent to the representative curve of f at the point of abscissa a.

Proposition 27: *If $f : E \subseteq \mathbb{R}^n \to F$ is differentiable at $a \in E$, its differential df_a is unique.*

Proof: Assume a function $f : E \subseteq \mathbb{R}^n \to F$ differentiable at $a \in E$ and which admits two differentials u and v at a.
Let $h \in E^*$, then:

$$\begin{aligned}
\frac{||u(h)-v(h)||}{||h||} &= \frac{||-(f(a+h)-f(a)-u(h))+(f(a+h)-f(a)-v(h))||}{||h||} \\
&\leq \frac{||(f(a+h)-f(a)-u(h)||}{||h||} + \frac{||(f(a+h)-f(a)-v(h))||}{||h||} \text{ (triangular inequality)}
\end{aligned}$$

However, by definition, $\lim\limits_{h \to 0} \dfrac{||f(a+h) - f(a) - u(h)||}{||h||} = 0$ and $\lim\limits_{h \to 0} \dfrac{||f(a+h) - f(a) - v(h)||}{||h||} = 0$, so:

$$\lim_{h \to 0} \frac{||u(h) - v(h)||}{||h||} = 0$$

In particular, for $x \in E^*$ and $t \in \mathbb{R}^*$, we have:

$$\lim_{t \to 0} \frac{||u(tx) - v(tx)||}{||tx||} = \frac{||u(x) - v(x)||}{||x||} = 0$$

Hence $||u(x) - v(x)|| = 0$, therefore we infer that $u(x) = v(x)$ (Separation property of the norm). As this is true for all $x \in E^*$ and also for $x = 0_n$, then $u = v$.

Exercise 66: Show that $f : (x, y) \mapsto x^2 + y^2$ is differentiable at any point $(a_1, a_2) \in \mathbb{R}^2$ and that $df_{(a_1,a_2)} : (h_1, h_2) \mapsto 2a_1h_1 + 2a_2h_2$.

Definition 49: *If $f : E \subseteq \mathbb{R}^n \to F$ is differentiable at any point $a \in E$, we say that f is differentiable on E and* ***the differential*** *of f is the application $df : a \mapsto df_a$.*

Note that, df_a is also an application. Indeed, for $x \in E$, $df(a)(x) = df_a(x)$.

Proposition 28: *Let f and g be two functions differentiable at $a \in E$, then:*

- *$\lambda f + g$, $\lambda \in \mathbb{R}$, is differentiable and $d(\lambda f + g)(a) = \lambda df(a) + dg(a)$*
- *$f \times g$ is differentiable and $d(f \times g)(a) = f(a) \times dg(a) + df(a) \times g(a)$*

Proposition 29: *Let $f : E \to F$ and $g : F \to G$ be two functions. If f is differentiable at $a \in E$ and if g is differentiable at $b = f(a) \in F$, then $g \circ f$ is also differentiable at a and:*

$$d(g \circ f)(a) = dg(f(a)) \circ df(a)$$

10.5 Directional derivative

Definition 50: *We say that f has a* ***partial derivative at*** *a* ***along the direction defined by the vector*** *v, if the function $\tau_{f,a,v} : t \mapsto \frac{f(a+tv)-f(a)}{t}$ has a limit at 0. In this case we will denote by $f'_v(a)$ this limit:*

$$\lim_{t\to 0} \frac{f(a+tv)-f(a)}{t} = f'_v(a)$$

When $||v|| = 1$, $f'_v(a)$ is called the ***directional derivative*** *at a along the vector v.*

The directional derivative gives us the rate of change of f in a direction v. If for example, f represents a temperature of a surface, $f'_v(a)$ represents the rate of change of the temperature if by placing at the point a we move in the direction of v .

Example 15: Consider the function $f : (x, y) \mapsto x^3 + y^2$ and study its partial derivative along $v = (3, 4)$.
We have $\mathcal{D} = \mathbb{R}^2$.
Let $a = (a_1, a_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}^*$, then:

$$\begin{array}{rcl} \tau_{f,a,v}(t) & = & \frac{f(a+tv)-f(a)}{t} \\ & = & \frac{(a_1+3t)^3+(a_2+4t)^2-a_1^3-a_2^2}{t} \\ & = & \frac{9a_1^2t+27a_1t^2+27t^3+8a_2t+16t^2}{t} \\ & = & 9a_1^2 + 27a_1t + 27t^2 + 8a_2 + 16t \end{array}$$

Hence $\lim\limits_{t\to 0} \tau_{f,a,v}(t) = 9a_1^2 + 8a_2$. We infer that f has a partial derivative along $v = (3, 4)$ and $f'_v : a \mapsto 9a_1^2 + 8a_2$.
Similarly, if we set $u = \frac{v}{||v||}$, with $||v|| = \sqrt{3^2 + 4^2} = 5$, we get $f'_u : a \mapsto \frac{1}{5}(9a_1^2 + 8a_2)$. In this case f'_u is the directional derivative along u.

Proposition 30 (Differential and Partial Derivative)**:** *If f is differentiable at a, it has a partial derivative at a along any vector $v \neq 0$ and we have:*

$$f'_v(a) = df(a)(v)$$

Proof: If f is differentiable at a, then $\lim\limits_{h\to 0} \dfrac{||f(a+h) - f(a) - df(a)(h)||}{||h||} = 0$. In particular, if we set $h = tv$, $\lim\limits_{t\to 0} \dfrac{||f(a+tv) - f(a) - df(a)(tv)||}{||tv||} = 0$.
Now, for $t \neq 0$, we have:

$$\begin{array}{rcl} \frac{||f(a+tv)-f(a)-df(a)(tv)||}{||tv||} & = & \frac{1}{||v||}\left\| \frac{f(a+tv)-f(a)}{|t|} - \frac{t}{|t|}df(a)(v) \right\| \\ & = & \frac{1}{||v||}\left\| \frac{f(a+tv)-f(a)}{t} - df(a)(v) \right\| \\ & = & \frac{1}{||v||}\|\tau_{f,a,v}(t) - df(a)(v)\| \end{array}$$

Then $\lim\limits_{t\to 0} \|\tau_{f,a,v}(t) - df(a)(v)\| = 0$ and $\tau_{f,a,v}$ has a limit at 0 and this limit is $f'_v(a) = df(a)(v)$.

Remarque: The converse of the preceding proposition is false. A function can has partial derivatives along all directions without being differentiable or even continuous (whereas for the functions of a real variable, differentiability involves continuity).

Example 16: Consider the function $f : (x, y) \mapsto \begin{cases} \frac{xy^2}{x^2+y^2} & if\ (x, y) \neq (0, 0) \\ 0 & if\ (x, y) = (0, 0) \end{cases}$.

We showed with example 14 that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0$, hence f is continuous at 0.
Study the partial derivatives along $v = (v_2, v_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ at $a = (0,0)$.
Let $t \in \mathbb{R}^*$, then:

$$\tau_{f,a,v}(t) = \frac{f(tv_1, tv_2) - f(0,0)}{t} = \frac{v_1 v_2^2}{v_1^2 + v_2^2}.$$

Hence $\lim\limits_{t\to 0} \tau_{f,a,v}(t) = \dfrac{v_1 v_2^2}{v_1^2+v_2^2} = f'_v(0,0)$. The function f has partial derivatives along any direction at $(0,0)$.
If f is differentiable at $(0,0)$ then $df(0,0)(v) = f'_v(0,0)$, but $df(0,0) : (v_1, v_2) \mapsto \frac{v_1 v_2^2}{v_1^2+v_2^2}$ is not linear, hence f is not differentiable at $(0,0)$.

10.6 Partial derivative

In the particular case where $E = \mathbb{R}^n$, denoting by $(e_i)_{1\leq i\leq n}$ its canonical basis, the derivatives following the basis vectors e_i, when they exist, are called partial derivatives. Precisely, we give the following definition.

Definition 51: *Let $f : E \subseteq \mathbb{R}^n \to F$ and $a \in E$. We say that f has a partial derivative at a with respect to the variable x_i, $i \in \{1, ..., n\}$ if it has a partial derivative at a along the base vector e_i. We then denote by $\frac{\partial f}{\partial x_i}(a)$ or $f'_{x_i}(a)$ this derivative.*

Then we have:

$$\frac{\partial f}{\partial x_i}(a) = f'_{e_i}(a) = \lim_{t\to 0} \frac{f(a+te_i) - f(a)}{t}$$

Which is equivalent saying $\frac{\partial f}{\partial x_i}(a)$ is the derivative at a_i of the function $x_i \mapsto f(a_1, ..., a_{i-1}, x_1, a_{i+1}, ..., a_n)$.

Proposition 31: *Let $f : E \subseteq \mathbb{R}^n \to F$ and $a \in E$. If f is differentiable at a, then it has partial derivatives at a with respect to each of the variables x_i, $i \in \{1, ..., n\}$ and we have :*

$$\forall h \in \mathbb{R}^n,\ df(a)(h) = f'_h(a) = \sum_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial x_i}(a)$$

Proof: Indeed, we have $\frac{\partial f}{\partial x_i}(a) = f'_{e_i}(a) = df(a)(e_i)$.
But $h = \sum\limits_{i=1}^{n} h_i e_i$, which gives:

$$\begin{aligned} df(a)(h) &= \sum_{i=1}^{n} h_i df(a)(e_i) \\ &= \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a). \end{aligned}$$

Remarque: If we set $dx_i : x \mapsto x_i$, $i \in \{1, ..., n\}$, then:

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) dx_i$$

Hence, we can note that if we set $df(a) = y - f(a)$ and $dx_i = x_i - a_i$, then we have:

$$y - f(a) = \sum_{i=1}^{n} (x_i - a) \frac{\partial f}{\partial x_i}(a)$$

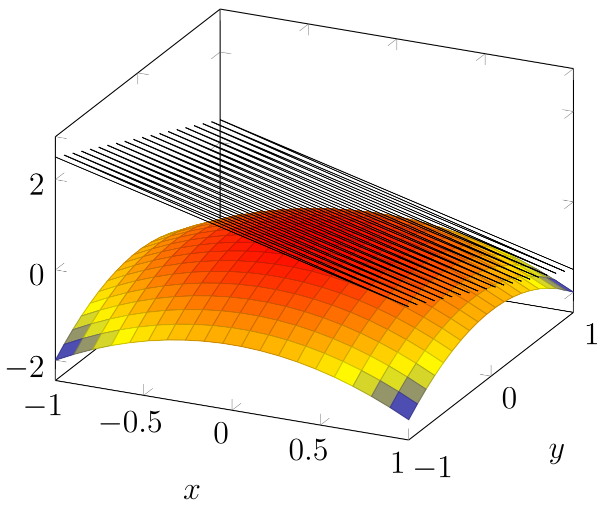


Figure 22: Tangent plane

This is the equation of the plane tangent to the curve representative of f at the point of abscissa a (see Figure 22). We then find the analogue of what has been seen in the case of functions with only one variable ($n = 1$).

Proposition 32: *Let $f : E \subseteq \mathbb{R}^n \to F$ and $a \in E$. If f is differentiable at a, then it has partial derivatives at a along any vector $v \neq 0$ and we have:*

$$\forall a \in \mathbb{R}^n,\ \ f'_v(a) = df(a)(v) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(a)$$

Proof: Direct induction from propositions 30 and 31.

Theorem 8: *Let $f : E \subseteq \mathbb{R}^n \to F$ to be a function having partial derivatives with respect to all the variables at any point of E. If these partial derivatives are continuous at a point $a \in E$, f is then differentiable at a.*

Exercise 67: Study the partial derivatives and the differentiability of the following functions:

- $f : (x, y, z) \mapsto xyz$;
- $f : (x, y) \mapsto \cos(x - y)$;
- $f : (x, y) \mapsto \ln(x^2 + y^2)$.

Exercise 68: Let $X : t \mapsto X(t)$, $Y : t \mapsto Y(t)$ and $\mathcal{F} : (x, y) \mapsto \mathcal{F}(x, y)$, differentiable on their respective domain of definition. We set $f : t \mapsto \mathcal{F}(X(t), Y(t))$.
Find $f' = \frac{df}{dt}$ with respect to $\frac{dX}{dt}$, $\frac{dY}{dt}$, $\frac{\partial \mathcal{F}}{dx}$ and $\frac{\partial \mathcal{F}}{dy}$ using the differential of a composite function.

10.7 Implicit function

Consider the function $\mathcal{F} : \mathbb{R}^3 \to \mathbb{R}$ differentiable on $E \subseteq \mathbb{R}^3$. Suppose that the equation $\mathcal{F}(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$ and try to determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
As $\mathcal{F}$ is differentiable, we have, for $(x, y, z) \in E$ and $h = (h_1, h_2, h_3) \in \mathbb{R}^3$:

$$d\mathcal{F}(x, y, z)(h_1, h_2, h_3) = h_1 \frac{\partial \mathcal{F}}{\partial x}(x, y, z) + h_2 \frac{\partial \mathcal{F}}{\partial y}(x, y, z) + h_3 \frac{\partial \mathcal{F}}{\partial z}(x, y, z) = 0$$

Then set $g : (x, y) \mapsto (x, y, f(x, y))$, which gives for $(x, y, z) \in \mathcal{D}_g$ and $h = (h_1, h_2) \in \mathbb{R}^2$:

$$d(\mathcal{F} \circ g)(x, y)(h_1, h_2) \quad = \quad \left(d\mathcal{F}(g(x, y)) \circ dg(x, y)\right)(h_1, h_2)$$

But, $dg(x, y) : (h_1, h_2) \mapsto \Big(h_1,\ h_2,\ \frac{\partial f}{\partial x}(x, y) \cdot h_1 + \frac{\partial f}{\partial y}(x, y) \cdot h_2\Big)$.
Hence, with $z = f(x, y)$:

$$\begin{aligned} d(\mathcal{F} \circ g)(x, y)(h_1, h_2) &= h_1 \tfrac{\partial \mathcal{F}}{\partial x}(x, y, z) + h_2 \tfrac{\partial \mathcal{F}}{\partial y}(x, y, z) + \\ &\qquad + \tfrac{\partial \mathcal{F}}{\partial z}(x, y, z) \left(h_1 \tfrac{\partial f}{\partial x}(x, y) + h_2 \tfrac{\partial f}{\partial y}(x, y)\right) \\ &= h_1 \left(\tfrac{\partial \mathcal{F}}{\partial x}(x, y, z) + \tfrac{\partial \mathcal{F}}{\partial z}(x, y, z) \tfrac{\partial f}{\partial x}(x, y)\right) + \\ &\qquad + h_2 \left(\tfrac{\partial \mathcal{F}}{\partial y}(x, y, z) + \tfrac{\partial \mathcal{F}}{\partial z}(x, y, z) \tfrac{\partial f}{\partial y}(x, y)\right) \end{aligned}$$

Since $d(\mathcal{F} \circ g)(x, y)(h_1, h_2) = 0$ for all $(h_1, h_2) \in \mathbb{R}^2$, we have necessarily:

- $\frac{\partial \mathcal{F}}{\partial x}(x, y, z) + \frac{\partial \mathcal{F}}{\partial z}(x, y, z) \frac{\partial f}{\partial x}(x, y) = 0$ and
- $\frac{\partial \mathcal{F}}{\partial y}(x, y, z) + \frac{\partial \mathcal{F}}{\partial z}(x, y, z) \frac{\partial f}{\partial y}(x, y) = 0$

Then we get if $\mathcal{F}'_z(x, y, z) \neq 0$:

- $\frac{\partial z}{\partial x}(x, y) = \frac{\partial f}{\partial x}(x, y) = -\frac{\mathcal{F}'_x(x,y,z)}{\mathcal{F}'_z(x,y,z)}$
- $\frac{\partial z}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x, y) = -\frac{\mathcal{F}'_y(x,y,z)}{\mathcal{F}'_z(x,y,z)}$

In a similar manner, we can infer the following equation if $\mathcal{F}'_y(x, y, z) \neq 0$:

- $\frac{\partial y}{\partial x}(x, y) = -\frac{\mathcal{F}'_x(x,y,z)}{\mathcal{F}'_y(x,y,z)}$
- $\frac{\partial y}{\partial z}(x, y) = -\frac{\mathcal{F}'_z(x,y,z)}{\mathcal{F}'_y(x,y,z)}$

Exercise 69: Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ with z defined implicitely with the following equations:

- $xy^2 + z^3 + \sin(xyz) = 0$;
- $x^2 + y^2 + z^2 - 5 = 0$;
- $e^x + \ln(y - z) = 0$.

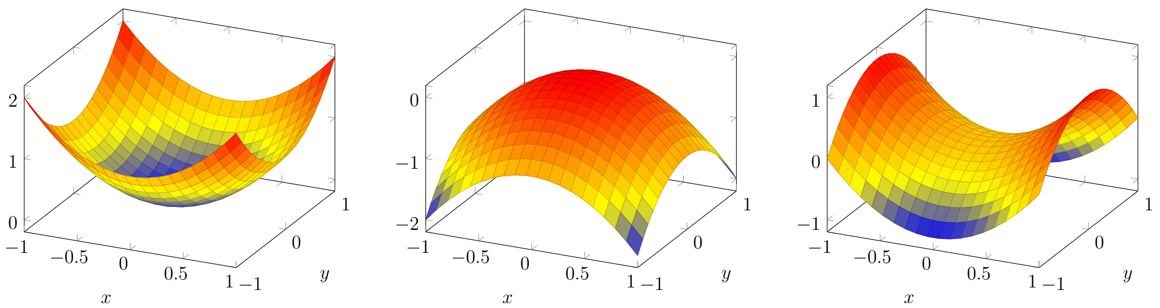
10.8 The gradient and critical points

Definition 52: *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function which has partial derivatives, we call* ***gradient****, the function:*

$$\begin{aligned} grad(f) :\ E \subseteq \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto (grad(f))(x) = \left(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x)\right)^t \end{aligned}$$

We also use the notation ∇f for the gradient of f.

Remarque: We can note that $df(a)(v) =< \nabla f(a) | v >$ (scalar product).
The gradient indicates the direction of the greatest slope (or rate of change) of the curve representative of f.



Local Minimum Local Maximum Saddle Point

Figure 23: Different natures of a critical point

Definition 53: *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function which has partial derivatives, then $a \in E$ is a* ***critical point*** *if $\nabla f(a) = 0$.*

In the case of the functions of two variables, there is a test to determine the nature of a critical point a. Indeed, if the second derivatives exist, we st:

$$D(a) = \frac{\partial^2 f}{\partial x^2}(a)\frac{\partial^2 f}{\partial y^2}(a) - \left(\frac{\partial^2 f}{\partial x \partial y}(a)\right)^2$$

Then we have (see Figure 23):

- Si $D(a) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a) > 0$, then $f(a)$ is a local minimum;
- Si $D(a) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a) < 0$, then $f(a)$ is a local maximum;
- Si $D(a) < 0$, then f has a saddle point at a;
- Si $D(a) = 0$, a has an undetermined nature.

Exercise 70: Find the critical point(s) and their nature of the following functions:

- $f : (x, y) \mapsto 2x^2 + y^2 - xy - 7y$
- $f : (x, y) \mapsto x^3 + y^3 - 3xy$
- $f : (x, y) \mapsto (x^2 + y^2)e^{-x}$
- $f : (x, y) \mapsto x \sin(y)$

10.9 Lagrangian Relaxation

Lagrange relaxation is a widely used method for solving optimization problems. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ with $E \subseteq \mathbb{R}^n$, we then try to solve the following problem:

$$(P) \begin{cases} \min & f(x) \\ s.t. & g(x) = 0 \end{cases}$$

The equation $g(x) = 0$ is a constraint on the values that $x \in E$ can take and among these possible values we seek the one(s) which minimize(s) f ($f(x)$ as small as possible). This problem is generally not easy to solve.

Lagrange's method consists in solving the problem:

$$(L):\quad \min_{x\in E,\ \lambda\in\mathbb{R}} f(x)-\lambda g(x).$$

Let's show that problem (L) is equivalent to problem (P).
We set $\mathcal{F}:(x,\lambda)\mapsto f(x)-\lambda g(x)$, then we seek $\min\limits_{x\in E,\ \lambda\in\mathbb{R}} \mathcal{F}:(x,\lambda)$, which is equivalent to finding the critical points of $\mathcal{F}$. Then we have, for $x\in E$ and $\lambda\in\mathbb{R}$:

$$\nabla\mathcal{F}(x,\lambda)=\begin{pmatrix}\nabla f(x)-\lambda\nabla g(x)\\ g(x)\end{pmatrix}$$

The equation $\nabla\mathcal{F}(x,\lambda)=0$ is equivalent at solving:

$$\begin{cases}\nabla f(x)=\lambda\nabla g(x)\\ g(x)=0\end{cases}$$

Hence, at a critical point (x^*,λ^*) of $\mathcal{F}$ we have $g(x^*)=0$ and $\mathcal{F}(x^*,\lambda^*)=f(x^*)$. We then conclude that solving(P) is equivalent to solving (L).

Example 17: Consider the following problem:

$$(P)\begin{cases}\min & x^2+3y^2-4x+5\\ s.t. & x^2+y^2\leq\ 5\end{cases}$$

We set $f:(x,y)\mapsto x^2+3y^2-4x+5$ and $g:(x,y)\mapsto x^2+y^2-5$.
<u>1st case:</u> $\begin{cases}\min & f(x,y)\\ s.t. & g(x,y)=0\end{cases}$
We will use the Lagrangian method. Hence, we start by solving the following system of equations:

$$\begin{cases}\nabla f(x) &=& \lambda\nabla g(x)\\ g(x,y) &=& 0\end{cases}\iff\begin{cases}2x-4 &=& 2\lambda x\\ 6y &=& 2\lambda y\\ x^2+y^2 &=& 5\end{cases}$$
$$\iff\begin{cases}(1-\lambda)x &=& 2\\ (3-\lambda)y &=& 0\\ x^2+y^2 &=& 5\end{cases}$$
$$\iff\begin{cases}x=\frac{2}{1-\lambda}\ \textit{and}\ \lambda\neq 1\\ \lambda=3\ \textit{and/or}\ y=0\\ x^2+y^2=5\end{cases}$$
$$\iff (x,y,\lambda)\in\left\{(-1,2,3),\ (-1,-2,3),\ \left(\sqrt{5},0,-\tfrac{2-\sqrt{5}}{\sqrt{5}}\right),\ \left(-\sqrt{5},0,-\tfrac{2+\sqrt{5}}{\sqrt{5}}\right)\right\}$$

Therefore, we get the following four points $a=(-1,2)$, $b=(-1,-2)$, $c=(\sqrt{5},0)$, and $d=(-\sqrt{5},0)$.
Which gives:

Point	x	y	$f(x,y)$
a	-1	2	22
b	-1	-2	22
c	$\sqrt{5}$	0	$10-4\sqrt{5}$
d	$-\sqrt{5}$	0	$10-4\sqrt{5}$

then the minimum is reached at c and at d with $f(d) = f(c) = 10 - 4\sqrt{5}$.

2nd case: $\begin{cases} \min & f(x,y) \\ s.t. & g(x,y) < 0 \end{cases}$

We will then study the critical points of f on the domain $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \ / \ g(x,y) < 0\}$:

$$\nabla f(x,y) = 0 \iff \begin{cases} 2x-4 &= 0 \\ 6y &= 0 \end{cases} \iff \begin{cases} x &= 2 \\ y &= 0 \end{cases}$$

Therefore we get the critical point $e = (2,0)$, and we have indeed $g(e) = 4 - 5 < 0$ then $e \in \mathcal{D}$. Then we compute $D = \frac{\partial^2 f}{\partial x^2}(e)\frac{\partial^2 f}{\partial y^2}(e) - \left(\frac{\partial^2 f}{\partial x \partial y}(e)\right)^2 = 0$. The test of the second derivative does not allow us to conclude

However we can note that for all $(x,y) \in \mathbb{R}^2$,

$$f(x,y) = (x-2)^2 + 3y^2 + 1 \geq f(e) = 1.$$

Then f has a local minimum at e.

Conclusion: We have $f(e) < f(d) = f(c)$, then the solution of the problem (P) is $x = 2$ and $y = 0$ with $f(2,0) = 1$.

11 Integration

11.1 Antiderivative

Definition 54: *Let f be a function defined on an interval $I \subset \mathbb{R}$. The function F is called an **antiderivative** of f on I if $\forall x \in I, \ F'(x) = f(x)$.*

Note that, in the definition, F is "an" antiderivative of f. Indeed, if F is an antiderivative of f, then for all constant $C \in \mathbb{R}$, the function $x :\mapsto F(x) + C$ is also an antiderivative of f.

Proposition 33: *Let F be an antiderivative of f on I.*
Then G is an antiderivative of f on I if and only if there exists $C \in \mathbb{R}$ such that $G(x) = F(x)+C$ for all $x \in I$.

Proof. Let F be an antiderivative of f on I.

$(\Rightarrow)$

Let G be an antiderivative of f on an interval I. We will show by contradiction that the function $h : x \mapsto G(x) - F(x)$ is a constant function on I.

Assume that h is not a constant function on I. Hence, there exists $(a,b) \in I^2$, $a < b$, such that $h(a) \neq h(b)$.

By definition, h is continuous on $[a,b]$ and differentiable on $]a,b[$, and, according to the Mean-Value theorem, there exists $c \in]a,b[$ such that $h'(c) = \frac{h(b)-h(a)}{b-a}$. This yields to a contradiction since $h'(c) = 0$ and $h(b) \neq h(a)$, therefore h is a constant function.

$(\Leftarrow)$

Let $G : x \mapsto= F(x)+C$ with $C \in \mathbb{R}$. G is differentiable on I and we have $G'(x) = F'(x) = h(x)$, for all $x \in I$. Then G is an antiderivative of f on I.

$\square$

Definition 55 (Indefinite Integral)**:** *Let f be a function defined on an interval $I \subset \mathbb{R}$ and let F be an antiderivative of f on I. The **indefinite integral** (or general derivative) of f on I is:*

$$\int f(x)\ dx = F(x) + C,\ \ x \in I,\ \ C \in \mathbb{R}.$$

Solving an indefinite integral is called **integration** or **anti-differentiation**. By definition of the indefinite integral, we have the following obvious properties using the same notation of the previous definition:

$$\int F'(x)\ dx = F(x) + C \text{ and } \frac{d}{dx}\left(\int f(x)\ dx\right) = f(x)$$

Besides, by definition F is differentiable, hence continuous, on I. Also, on some intervals, a function may not have any antiderivative. We will see later that if a function is continuous, it can be anti-differentiated. However this does not imply that only continuous functions are integrable.

- $\int 0\ dx = C$ on $\mathbb{R}$
- For $n \neq -1$, $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$ on intervals where $x \mapsto x^n$ is defined
- $\int \frac{1}{x}\ dx = \ln(|x|) + C$ on $]-\infty, 0[$ and $]0, +\infty[$
- $\int \sin(x)\ dx = -\cos(x) + C$ on $\mathbb{R}$
- $\int \frac{1}{1+x^2}\ dx = \arctan(x) + C$ on $\mathbb{R}$
- $\int \cosh(x)\ dx = \sinh(x) + C$ on $\mathbb{R}$
- For $k \in \mathbb{R}$, $\int k\ dx = kx + C$ on $\mathbb{R}$
- For $a > 0$ and $a \neq 1$, $\int a^x\ dx = \frac{a^x}{\ln(a)} + C$ on $\mathbb{R}$
- $\int \cos(x)\ dx = \sin(x) + C$ on $\mathbb{R}$
- $\int \sec^2(x)\ dx = \tan(x) + C$ on $]-\pi/2 + n\pi, \pi/2 + n\pi[$, $n \in \mathbb{Z}$
- $\int \frac{1}{\sqrt{1+x^2}}\ dx = \arcsin(x) + C$ on $]-1, 1[$
- $\int \sinh(x)\ dx = \cosh(x) + C$ on $\mathbb{R}$

Table 4: Basic integration rules

The Table 4 gives some basic integration rules obtained from the differentiation of basic functions. Note that it is easy to prove that a function F is an antiderivative of f on an interval I using differentiation. The difficult part is to find one antiderivative, since with one you can derive all of the antiderivatives.

Exercise 71: Validate the following antiderivatives, with $C \in \mathbb{R}$:

- $\int \tan(x)\ dx = -\ln(\cos(x)) + C$ on $]-\pi/2, \pi/2[$
- $\int \frac{1}{\sqrt{x^2-1}}\ dx = \ln(x + \sqrt{x^2-1}) + C$ on $]1, +\infty[$

Proposition 34 (Antiderivative properties)**:** *Let f and g be two functions defined on I. Assume that F is an antiderivative of f on I and G is an antiderivative of g on I, then, with $C \in \mathbb{R}$:*

- *$\forall k \in \mathbb{R}$, $\int kf(x)\ dx = k\int f(x)\ dx = kF(x) + C$;*
- *$\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx = F(x) + G(x) + C$;*

- $\int (f(x) - g(x))\ dx = \int f(x)\ dx - \int g(x)\ dx = F(x) - G(x) + C$.

Note that there is no particular properties for the multiplication and the quotient of two functions. The properties describe in proposition 34 can be prove directly by differentiation. In the sequel,when solving an indefinite integral, the interval I will not always be specified. In this case, one has to find the anti-derivatives together with the interval or intervals on which they are valid.

Exercise 72: Solve the following indefinite integrals:

- $\int x^3 + 2x^2\ dx$
- $\int \frac{x}{\sqrt[3]{x}}\ dx$
- $\int \frac{x^3+3\sqrt{x}+2}{x}\ dx$
- $\int (\sqrt{x} - 2)^2\ dx$
- $\int \tan^2(x)\ dx$

Exercise 73: Show that the *sign* function defined as $sign : x \mapsto \begin{cases} -1 & if\ x < 0 \\ 0 & if\ x = 0 \\ 1 & if\ x > 0 \end{cases}$ has no antiderivative on $[-5, 5]$.

Exercise 74: Show that the function $f : x \mapsto |x|$ has an antiderivative on $[-1, 1]$.

Definition 56: *We called a* ***particular solution*** *of an indefinite integral* $\int f(x)\ dx$*, the antiderivative* F *of* f *that satisfies an known* ***initial condition*** *such as* $F(x_0) = y_0$ *for* $(x_0, y_0) \in \mathbb{R}^2$.

For instance, let's find the particular solution F of $\int x^2\ dx$ that satisfies the initial condition $F(1) = 5$. The solution of $\int x^2\ dx$ is, for $x \in \mathbb{R}$:

$$\int x^2\ dx = \frac{1}{3}x^3 + C,\ with\ C \in \mathbb{R}$$

Hence, we are looking for a particular value of C such that $F : x \mapsto \frac{1}{3}x^3 + C$ and $F(1) = 5$. We infer the particular solution $F : x \mapsto \frac{1}{3}x^3 + \frac{14}{3}$.

11.2 Riemann sum

Definition 57: *We called a* ***partition*** *of a closed interval* $[a, b]$*, a set* $P = \{x_0, x_1, ..., x_n\}$*,* $n \in \mathbb{N}^*$*, of* $n + 1$ *points such that:*

$$a = x_0 < x_1 < ... < x_n = b$$

A partition $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, divides an interval $[a, b]$ in n sub-intervals

$$[x_0, x_1],\ [x_1, x_2],\ ..., [x_{n-1}, x_n]$$

In this case, we will denote by $\Delta x_i = x_i - x_{i-1}$, for $i \in \{1, ..., n\}$, the length of the $i - th$ interval $[x_{i-1}, x_i]$, and the maximum length of these intervals, defined as $\Delta x = \max\limits_{i \in \{1,...,n\}} \Delta x_i$, will be called **the norm** of the partition.

If the length of all partition intervals are equal, we will say that $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, is a **regular partition** of a closed interval $[a, b]$, and we have for $i \in \{1, ..., n\}$:

$$\Delta x = \Delta x_i = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

Example 18:

- $\{1, 3, 4, 5\}$ and $\{1, 2, 4, 5\}$ are two partitions of $[1, 5]$
- $\{1, 3, 5\}$ and $\{1, 2, 3, 4, 5\}$ are two regular partitions of $[1, 5]$
- $\{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$ is a partition of $[0, 1]$
- $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ is a regular partition of $[0, 1]$

Definition 58: *Let f be a function defined on an interval $[a, b]$ and $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, a partition of $[a, b]$. Then the **Riemann sum** of the function f with the partition P for the intermediate points $c_i \in [x_{i-1}, x_i]$, $i \in \{1, ..., n\}$ is:*

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + ... + f(c_n)(x_n - x_{n-1})$$

If the partition P is regular, then, for $i \in \{1, ..., n\}$, $\Delta x_i = \Delta x = \frac{b-a}{n}$ and:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f(c_i)\Delta x = (b-a)\sum_{i=1}^{n} \frac{f(c_i)}{n}$$

Geometrically, the Riemann sum is an approximation of the net area bounded by the graph of $y = f(x)$ and the x-axis. The smaller is Δx for the partition P, the more accurate is the approximation. In particular, when the partition P is regular, we will consider:

- The left Riemann sum where $c_i = x_{i-1}$, for $i \in \{1, ..., n\}$:

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x$$

- The right Riemann sum where $c_i = x_i$, for $i \in \{1, ..., n\}$:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

- The midpoint Riemann sum where $c_i = \frac{x_{i-1}+x_i}{2}$, for $i \in \{1, ..., n\}$:

$$\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x = \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)\Delta x$$

Definition 59: *Let f be a function defined on a closed interval $[a, b]$. We say that f is **(Riemann-)integrable** on $[a, b]$ if the limit of the Riemann sum, when $\Delta x = \max\limits_{i\in\{1,...,n\}} \Delta x_i$ approaches 0, <u>exists</u>, over all partition $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, of $[a, b]$ and intermediate points $c_i \in [x_{i-1}, x_1]$, $i \in \{1, ..., n\}$, and is <u>finite</u>.*
*In this case, this limit is called the **definite integral** of f from a to b and is denoted by:*

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\ dx$$

By definition of finite limit, $\lim\limits_{\Delta x \to 0} \sum\limits_{i=1}^{n} f(c_i)\Delta x_i = l$, with $l \in \mathbb{R}$, means for each $\epsilon > 0$, there exists an $\alpha > 0$ such that for every partition with $\Delta x < \alpha$ we have $\left| \sum\limits_{i=1}^{n} f(c_i)\Delta x_i - l \right| < \epsilon$.

Table 5 gives some basic summation formulas that may be useful when evaluating a Riemann sum. These formulas can be proved by recursion.

• For $c \in \mathbb{R}$, $\sum\limits_{i=1}^{n} c = nc$	• $\sum\limits_{i=1}^{n} i = \dfrac{n(n+1)}{2}$
• $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$	• $\sum\limits_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

Table 5: Some basic summation formulas

Example 19: Consider the function $f : x \mapsto x^2$ and the regular partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$, $n \in \mathbb{N}^*$, of the interval $[0, 1]$. Hence, for this partition we have $x_i = \frac{i}{n}$, $i \in \{0, ..., n\}$, $\Delta x = \frac{1}{n}$, and we note that when n approaches $+\infty$, Δx approaches 0. The right Riemann sum of f with the partition P is:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} f(0 + i\Delta x)\Delta x \\
&= \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) \\
&= \sum_{i=1}^{n} \frac{1}{n} \left(\frac{i}{n}\right)^2 \\
&= \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \frac{n(n+1)(2n+1)}{6n^3}
\end{aligned}$$

Therefore, $\lim\limits_{\Delta x \to 0} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n \to +\infty} \dfrac{n(n+1)(2n+1)}{6n^3} = \dfrac{1}{3}$.

Similarly, the left Riemann sum of f with the partition P is:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i-1})\Delta x &= \sum_{i=1}^{n} f(0+(i-1)\Delta x)\Delta x \\ &= \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i-1}{n}\right) \\ &= \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i-1}{n}\right)^2 \\ &= \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2 \\ &= \frac{1}{n^3}\sum_{i=0}^{n-1} i^2 \\ &= \frac{1}{n^3}\left(\sum_{i=0}^{n} i^2 - n^2\right) \\ &= \frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n}\end{aligned}$$

This yields to the same limit than the right Riemann sum:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\left(\frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n}\right) = \frac{1}{3}$$

Then, we infer:

$$\int_0^1 x^2 \, dx = \frac{1}{3}$$

In example 19, we use two different types of intermediate points in the right Riemann sum and the left Riemann sum. Indeed the limit of the Riemann sum when Δx approaches 0 is independent of the choice of the intermediate point. The left, right, or midpoint Riemann sum are generally used when studying a definite integral. Besides, we use a regular partition that partitions the interval $[a, b]$ in n sub-intervals. Such partition has the properties when n approaches $+\infty$, Δx approaches 0. The partition used to find the definite integral does not really matter and the proof is out of the scope of this book. Regular partitions are often used but other partitions can be used according to the function under study.

Example 20: Consider the function $f : x \mapsto \sqrt{x}$ and the partition $P = \left\{0, \frac{1}{n^2}, \frac{4}{n^2}, ..., 1\right\}$, $n \in \mathbb{N}^*$, of the interval $[0, 1]$. Hence, this partition is not regular and we have, for $i \in \{0, ..., n\}$, $x_i = \frac{i^2}{n^2}$ and, for $i > 0$, $\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{2i-1}{n^2}$, and we note that when n approaches $+\infty$, $\Delta x = \max_{i\in\{1,...,n\}} \Delta x_i = \dfrac{2n-1}{n^2}$ approaches 0. The right Riemann sum of f with the partition P

is:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x_i &= \sum_{i=1}^{n} \frac{2i-1}{n^2}\sqrt{\frac{i^2}{n^2}} \\ &= \frac{1}{n^3}\sum_{i=1}^{n}(2i^2-i) \\ &= \frac{1}{n^3}\left(2\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i\right) \\ &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}\right) \\ &= \frac{n(n+1)(4n-1)}{6n^3}\end{aligned}$$

Therefore, $\lim\limits_{\Delta x \to 0} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to+\infty} \dfrac{n(n+1)(4n-1)}{6n^3} = \dfrac{2}{3}$.

Then, we infer

$$\int_0^1 \sqrt{x}\ dx = \frac{2}{3}$$

Exercise 75: Study the definite integral of $f : x \mapsto \sqrt{x}$ on the interval $[0, 1]$ using the left Riemann sum.

Exercise 76: Study the definite integral of the following functions on the interval $[0, 1]$:

- $f : x \mapsto c,\ c \in \mathbb{R}$;
- $g : x \mapsto x$;
- $h : x \mapsto 2x + 1$.

Theorem 9 (Continuity and Integrability)**:** *If f is defined on $[a, b]$ and piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$.*

A piecewise continuous function on an interval I is a function that is continuous on I, except at a finite number of points in I. A direct consequence of this theorem is all continuous functions are integrable. Note that it is a sufficient condition for integrability, but not necessary.

Proposition 35 (Properties of the definite integral)**:** *Let f and g be two functions integrable on $[a, b]$, then:*

1) $\int_a^a f(x)\ dx = 0$;
2) $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$;
3) *for* $c \in \mathbb{R}$, $f + cg$ *is integrable and* $\int_a^b (f(x) + cg(x))\ dx = \int_a^b f(x)\ dx + c\int_a^b g(x)\ dx$;
4) *for* $c \in [a, b]$, $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$;
5) *If* $\forall x \in [a, b]$, $f(x) \leq g(x)$, *then* $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.

We will not provided a proof of these properties. Most of them are a direct consequence of the definition of the definite integral. Also, a formal proof requires some results on interval partitioning that are out of the scope of this book. If you think of the integrale as the net area between the graph of a function and the x-axis, these properties come naturally.

Theorem 10 (The fundamental theorem of calculus)**:** *If f is a continuous function on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then:*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Proof. Let f be a continuous function on $[a,b]$, hence f is integrable on $[a,b]$.
Set $A : t \mapsto \displaystyle\int_a^t f(x)\ dx$. The function A is defined on $[a,b]$ and we will show that A is differentiable on $[a,b]$ using the definition.
Let $t \in [a,b]$ and set $\tau_{A,t} : h \mapsto \frac{A(t+h)-A(t)}{h}$. Then for $h \neq 0$ such that $t+h \in [a,b]$, we have:

$$\begin{aligned} \tau_{A,t}(h) &= \frac{\int_a^{t+h} f(x)\ dx - \int_a^t f(x)\ dx}{h} \\ &= \frac{\int_a^{t+h} f(x)\ dx + \int_t^a f(x)\ dx}{h} \\ &= \frac{1}{h}\int_t^{t+h} f(x)\ dx \end{aligned}$$

Now, we have to prove that the limit of $\tau_{A,t}(h)$ when h approaches 0 is $f(t)$.
Let $\epsilon > 0$. Given f is continuous at t, there exists $\alpha > 0$ such that for $x \in [a,b]$, $|x-t| < \alpha$ implies $|f(x) - f(t)| < \epsilon$. Then, let $|h| < \alpha$.
If $h > 0$, for $x \in [t, t+h]$, we have $|x - t| < h$ which gives:

$$\begin{aligned} |f(x) - f(t)| < \epsilon \ &\Rightarrow f(t) - \epsilon < f(x) < f(t) + \epsilon \\ &\Rightarrow \int_t^{t+h} (f(t) - \epsilon)\ dx < \int_t^{t+h} f(x)\ dx < \int_t^{t+h} (f(t) + \epsilon)\ dx \\ &\Rightarrow (f(t) - \epsilon)h < \int_t^{t+h} f(x)\ dx < (f(t) + \epsilon)h \\ &\Rightarrow (f(t) - \epsilon) < \frac{1}{h}\int_t^{t+h} f(x)\ dx < (f(t) + \epsilon) \\ &\Rightarrow \left| \frac{1}{h}\int_t^{t+h} f(x)\ dx - f(t) \right| < \epsilon \end{aligned}$$

If $h < 0$, we have the same result, by taking $x \in [t+h, t]$.
Therefore, we showed $\forall \epsilon > 0, \exists \alpha > 0$, such that $|h| < \alpha \ \Rightarrow\ |\tau_{A,t}(h) - f(t)| < \epsilon$, that is to say $\lim\limits_{h \to 0} \tau_{A,t}(h) = f(t)$.
We conclude that A is differentiable at $t \in [a,b]$ and $A'(t) = f(t)$.
Therefore, if F is an antiderivative of f on $[a,b]$, there exists $C \in \mathbb{R}$ such that $A(x) = F(x) + C$ for $x \in [a,b]$. However, we can note that $A(b) = 0$ which gives $C = -F(b)$, thus $A(x) = F(x) - F(b)$, for $x \in [a,b]$, and in particular for $x = a$:

$$A(b) = \int_a^b f(x)\ dx = F(b) - F(a).$$

□

Note that $A : x \mapsto \int_a^x f(t)\ dt$ is <u>the</u> antiderivative of f that vanishes at $x = a$. Besides, the following notation is used:

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

As a direct consequence of the fundamental theorem of calculus, if f is continuous on an interval I and $a \in I$, then the function $F : x \mapsto \displaystyle\int_a^x f(t)\ dt$ is defined and differentiable on I, and we have $F' : x \mapsto f(x)$.

Example 21: Let evaluate $\displaystyle\int_1^4 x^2\ dx$. Since $f : x \mapsto x^2$ is continuous on $[1, 4]$, it is integrable on $[1, 4]$. The function $F : x \mapsto \frac{1}{3}x^3$ is an antiderivative of f, hence:

$$\int_1^4 x^2\ dx = \frac{1}{3}x^3\Big|_1^4 = \frac{1}{3}(4^3 - 1^3) = \frac{63}{3}.$$

Example 22: Let evaluate $\displaystyle\int_{-1}^2 |x^3|\ dx$. Since $f : x \mapsto |x^3|$ is continuous on $[-1, 2]$, it is integrable on $[-1, 2]$. Hence:

$$\begin{aligned}\textstyle\int_{-1}^2 |x^3|\ dx &= \textstyle\int_{-1}^0 |x^3|\ dx + \int_0^2 |x^3|\ dx = \int_{-1}^0 (-x^3)\ dx + \int_0^2 x^3\ dx \\ &= \left(-\tfrac{1}{4}x^4\right)\Big|_{-1}^0 + \tfrac{1}{4}x^4\Big|_0^2 = \tfrac{17}{4}.\end{aligned}$$

We can also note that $F : x \mapsto \begin{cases} -\frac{1}{4}x^4 & if\ x \leq 0 \\ \frac{1}{4}x^4 & if\ x > 0 \end{cases}$ is an antiderivative of f on $[-1, 2]$, therefore

$$\textstyle\int_{-1}^2 |x^3|\ dx = F(x)\Big|_{-1}^2 = \tfrac{17}{4}.$$

Example 23: Let evaluate $\displaystyle\int_{-5}^5 sign(x)\ dx$. The *sign* function is piecewise continuous on $[-5, 5]$, so it is integrable on $[-5, 5]$. Remember that the *sign* function has no antiderivative on $[-5, 5]$, however it has one on $[-5, 0]$ and $[0, 5]$. Hence, we have:

$$\begin{aligned}\int_{-5}^5 sign(x)\ dx &= \int_{-5}^0 sign(x)\ dx + \int_0^5 sign(x)\ dx = \int_{-5}^0 (-1)\ dx + \int_0^5 1\ dx \\ &= (-x)\Big|_{-5}^0 + x\Big|_0^5 = 0.\end{aligned}$$

Exercise 77: Evaluate the following definite integrale:

- $\displaystyle\int_0^2 |x - 1|\ dx$
- $\displaystyle\int_0^\pi \sin(x)\ dx$
- $\displaystyle\int_1^5 \frac{1}{x}\ dx$
- $\displaystyle\int_0^\pi x\cos(x^2)\ dx$

Exercise 78: Given the definition of the logarithm function as $\ln(x) = \displaystyle\int_1^x \frac{1}{t}\ dt$, for $x \in]0, +\infty[$, prove that $\ln(xy) = \ln(x) + \ln(y)$, for $(x, y) \in (\mathbb{R}_+^*)^2$.
Hint: Considering y as a constant, differentiate then anti-differentiate the following function $f : x \mapsto \ln(xy) - \ln(x)$

Theorem 11 (Integral Mean Value Theorem)**:**
If f is continuous on $[a, b]$, then there exists a point $c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx.$$

Proof. Let f be a continuous function on $[a, b]$, hence f is integrable on $[a, b]$.
By the extreme value theorem, there exists $d \in [a, b]$ and $e \in [a, b]$ such that:

$$\forall x \in [a, b],\ f(d) \leq f(x) \leq f(e).$$

Therefore:

$$\int_a^b f(d)\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b f(e)\ dx$$

$$(b-a)f(d) \leq \int_a^b f(x)\ dx \leq (b-a)f(e)$$

$$f(d) \leq \frac{1}{b-a}\int_a^b f(x)\ dx \leq f(e)$$

By the intermediate value theorem, there exists $c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

□

The value of $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\ dx$ in the mean value theorem is called the **average value** of f on $[a, b]$.

Exercise 79: For each definite integrals of exercise 77 find the value of c that satisfies the integral mean value theorem.

Proposition 36: *Some other useful properties of the definite integral:*

1) *If f is odd and integrable on $[-a, a]$ $(a > 0)$, then* $\displaystyle\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$*;*

2) *If f is even and integrable on $[-a, a]$ $(a > 0$, then* $\displaystyle\int_{-a}^a f(x)\ dx = 0$*;*

3) *If f is T-periodic and integrable on $[0, T]$, then, for $n \in \mathbb{N}$,* $\displaystyle\int_0^{nT} f(x)\ dx = n\int_0^T f(x)\ dx$*.*

Exercise 80 (The vertical motion problem)**:** A ball is thrown upward at an initial height of 10 meters at an initial speed of 5 m/s.

1) What is the maximum height reached by the ball?
2) What should be the initial velocity for the ball to reached a maximum height of 100 meters?

Exercise 81 (Rate of change)**:** A population of bacteria grow at a rate of the form $k\sqrt{t}$, where t is the time in days and k a parameter yet to determine. The initial population size is 1000.

1) If $k = 2$ what is the size of the population of bacteria after 5 days?

2) If the population size has grown to 1500 after 1 days, determine k and the size of the population after 5 days?

Exercise 82 (Average value)**:** The sale of a product between the years 2010 and 2020 is given by the equation $\sqrt{t-2010}+100$, where t is the year.

1) What is the average sale between 2010 and 2020?

2) In which year will the average sale reach 110?

Exercise 83 (Motion Problem)**:** The horizontal velocity in meter per second of a particle is $v(t) = (t-5)(t-10)$, where t is the time in second.

1) What is the displacement of the particle between $t = 0$ and $t = 10$?

2) What is the total distance traveled by the particle between $t = 0$ and $t = 10$?

11.3 Integration Techniques

11.3.1 Direct Integration

Example 24 (Integration by tables)**:** Let find the indefinite integral of $f : x \mapsto x^2 + \frac{1}{x} - \sin(x)$. For $x \in \mathbb{R}_+^*$, we have:

$$\begin{aligned}\int f(x)\ dx &= \int x^2 + \frac{1}{x} - \sin(x)\ dx\\ &= \int x^2\ dx + \int \frac{1}{x}\ dx - \int \sin(x)\ dx\\ &= \frac{x^3}{3} + \ln(x) + \cos(x) + C,\ C \in \mathbb{R}.\end{aligned}$$

Example 25 (Multiplication before Integration)**:** Let find the indefinite integral of $f : x \mapsto (x+1)(x^2-1)$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}\int f(x)\ dx &= \int (x+1)(x^2-1)\ dx\\ &= \int x^3 + x^2 - x - 1\ dx\\ &= \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} - x + C,\ C \in \mathbb{R}.\end{aligned}$$

Example 26 (Division before Integration)**:** Let find the indefinite integral of $f : x \mapsto \frac{x^2+2x+\sqrt[3]{x}}{\sqrt{x}}$. For $x \in \mathbb{R}_+^*$, we have:

$$\begin{aligned}\int f(x)\ dx &= \int \frac{x^2+2x+\sqrt[3]{x}}{\sqrt{x}}\ dx\\ &= \int x^{3/4} + 2\sqrt{x} + x^{1/6}\ dx\\ &= \frac{4}{7}x^{7/4} + \frac{4}{3}x^{3/2} + \frac{6}{7}x^{7/6} + C,\ C \in \mathbb{R}\end{aligned}$$

Exercise 84: Solve the following indefinite integral:

- $\int (x^2 + e^{2x} - 5)\ dx$

- $\displaystyle\int (e^{-x}+3)(2e^{x}+5)\ dx$
- $\displaystyle\int \frac{5+2x\sinh(x)}{x}\ dx$
- $\displaystyle\int \frac{5+2x\cos^2(x)+3\sin(x)}{\cos^2(x)}\ dx$

Example 27 (Making some adjustments)**:** Let find the indefinite integral of $f : x \mapsto \frac{x^2}{(x^2+1)}$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}\int f(x)\ dx &= \int \frac{x^2}{(x^2+1)}\ dx\\ &= \int \frac{x^2+1-1}{(x^2+1)}\ dx\\ &= \int \left(1-\frac{1}{x^2+1}\right)\ dx\\ &= \int\ dx - \int \frac{1}{x^2+1})\ dx\\ &= x-\arctan(x)+C,\ C\in\mathbb{R}\end{aligned}$$

Example 28 (Making some adjustments)**:** Let find the indefinite integral of $f : x \mapsto x\sqrt{3x+1}$. For $x \in [-1,+\infty[$, we have:

$$\begin{aligned}\int f(x)\ dx &= \int x\sqrt{3x+2}\ dx\\ &= \frac{1}{3}\int 3x\sqrt{3x+2}\ dx\\ &= \frac{1}{3}\int (3x+2-2)(3x+2)^{1/2}\ dx\\ &= \frac{1}{3}\int \left((3x+2)^{3/2}-2(3x+2)^{1/2}\right)\ dx\\ &= \frac{1}{3}\left(\frac{2}{5}\frac{1}{3}(3x+2)^{5/2}-2\frac{2}{3}\frac{1}{3}(3x+2)^{3/2}\right)+C,\ C\in\mathbb{R}\\ &= \frac{2}{9}\left(\frac{1}{5}(3x+2)^{5/2}-\frac{2}{3}(3x+2)^{3/2}\right)+C\end{aligned}$$

11.3.2 Powers of trigonometric function

In this section, we will learn how to integrate functions of the form "$\sin^m(x)\cos^n(x)$". Table 6 gives some basic trigonometric identities that will be used in this section.

- $\cos^2(x)+\sin^2(x)=1$
- $\cos^2(x)=\frac{1+\cos(2x)}{2}$
- $\sin^2(x)=\frac{1-\cos(2x)}{2}$
- $\sec^2(x)=1+\tan^2(x)$

Table 6: Some useful trigonometric identities

Example 29: Let find the indefinite integral of $f : x \mapsto \sin^3(x)$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}
\int f(x)\ dx &= \int \sin^3(x)\ dx \\
&= \int (1-\cos^2(x))\sin(x)\ dx \\
&= \int \sin(x)\ dx - \int \sin(x)\cos^2(x)\ dx \\
&= -\cos(x) + \frac{1}{3}\cos^3(x) + C,\ C \in \mathbb{R}
\end{aligned}$$

Example 30: Let find the indefinite integral of $f : x \mapsto \cos^4(x)$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}
\int f(x)\ dx &= \int \cos^4(x)\ dx \\
&= \int \left(\frac{1+\cos(2x)}{2}\right)^2\ dx \\
&= \frac{1}{4}\int (1+2\cos(2x)+\cos^2(2x))\ dx \\
&= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1+\cos(4x)}{2}\right)\ dx \\
&= \frac{1}{4}\left(x+\sin(2x)+\frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\right) + C,\ C \in \mathbb{R} \\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

Exercise 85: Solve the following definite integrals:

- $\displaystyle\int_0^{\pi/2} \cos(x)\sin^5(x)\ dx$
- $\displaystyle\int_0^{\pi/4} \cos^5(x)\ dx$
- $\displaystyle\int_0^{\pi} \cos^3(x)\ dx$

Example 31: Let find the indefinite integral of $f : x \mapsto \cos^4(x)\sin^3(x)$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}
\int f(x)\ dx &= \int \cos^4(x)\sin^3(x)\ dx \\
&= \int \cos^4(x)(1-\cos^2(x))\sin(x)\ dx \\
&= \int (\cos^4(x)\sin(x) - \cos^6(x)\sin(x))\ dx \\
&= -\frac{1}{5}\cos^5(x) + \frac{1}{7}\cos^7(x) + C,\ C \in \mathbb{R}
\end{aligned}$$

Example 32: Let find the indefinite integral of $f : x \mapsto \cos^2(x)\sin^2(x)$. For $x \in \mathbb{R}$, we have:

$$\begin{aligned}
\int f(x)\ dx &= \int \cos^2(x)\sin^2(x)\ dx \\
&= \int \cos^2(x)(1-\cos^2(x))\ dx \\
&= \int (\cos^2(x) - \cos^4(x))\ dx \\
&= \int \left(\frac{1+\cos(2x)}{2} - \cos^4(x)\right)\ dx \\
&= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) - \left(\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right) + C,\ C \in \mathbb{R} \\
&= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C,\ C \in \mathbb{R}
\end{aligned}$$

Exercise 86: Solve the following definite integrals:

- $\displaystyle\int_0^{\pi/2} \cos^3(x)\sin^5(x)\ dx$
- $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos^5(x)}{\sin^2(x)}\ dx$
- $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos^3(x)}{\sqrt{\sin(x)}}\ dx$
- $\displaystyle\int_0^{\pi/4} \sec^4(x)\ dx$

11.3.3 Integration by parts

Theorem 12: *If u and v are two functions continuously differentiable on an interval I, then, for $x \in I$:*

$$\int u'(x)v(x)\ dx = u(x)v(x) - \int u(x)v'(x)\ dx$$

Proof. Let u and v be two functions continuously differentiable on an interval I.
Given uv is differentiable on I and $(uv)' = u'v + vu'$, uv is an antiderivative of $u'v + vu'$ and we get:

$$\int (u'(x)v(x) + v(x)u'(x))\ dx = \int u'(x)v(x)\ dx + \int u(x)v'(x)\ dx = u(x)v(x) + C,\ C \in \mathbb{R}.$$

□

Note that, given $dv = v'(x)dx$ and $du = u'(x)dx$, then we can rewrite the integration by part formula as:

$$\int v\ du = uv - \int u\ dv.$$

which can be extended directly to the definite integrals on an interval $[a, b]$:

$$\int_a^b v\ du = uv\Big|_a^b - \int_a^b u\ dv.$$

Example 33: Let evaluate the indefinite integral $\displaystyle\int x\sin(x)\ dx$.
We apply the integration by part with:

$$\begin{array}{lll} v = x & \to & v' = 1 \\ u' = \sin(x) & \to & u = -\cos(x) \end{array}$$

Therefore,

$$\begin{aligned} \int x\sin(x) &= -x\cos(x) - \int \cos(x)\ dx \\ &= -x\cos(x) + \sin(x) + C,\ C \in \mathbb{R} \end{aligned}$$

Example 34: Let evaluate the definite integral $\displaystyle\int_0^1 (x^2+2x)e^{3x}\ dx$.
We apply the integration by part with:

$$\begin{array}{lll} v = x^2+2x & \to & v' = 2x+2 \\ u' = e^{3x} & \to & u = \frac{1}{3}e^{3x} \end{array}$$

Therefore,

$$\begin{aligned} \int_0^1 (x^2+2x)e^{3x} &= \left.\frac{(x^2+2x)e^{3x}}{3}\right|_0^1 - \frac{1}{3}\int_0^1 (2x+2)e^{3x}\ dx \\ &= e^3 - \frac{1}{3}\int_0^1 (2x+2)e^{3x}\ dx \end{aligned}$$

Once again, we apply the integration by part with:

$$\begin{array}{lll} v = 2x+2 & \to & v' = 2 \\ u' = e^{3x} & \to & u = \frac{1}{3}e^{3x} \end{array}$$

Then,

$$\begin{aligned} \int_0^1 (x^2+2x)e^{3x} &= e^3 - \frac{1}{3}\int_0^1 (2x+2)e^{3x}\ dx \\ &= e^3 - \frac{1}{3}\left[\left.\frac{(2x+2)e^{3x}}{3}\right|_0^1 - \frac{2}{3}\int_0^1 e^{3x}\right] \\ &= e^3 - \frac{1}{9}\left[4e^3 - 2 - \left.\frac{2}{3}e^{3x}\right|_0^1\right] \\ &= \frac{14}{3}e^3 + \frac{4}{27} \end{aligned}$$

Example 35: Let evaluate the indefinite integral $\displaystyle\int (x^3+3x+2)\ln(2x)\ dx$.
We apply the integration by part with:

$$\begin{array}{lll} v = \ln(2x)x & \to & v' = \frac{1}{x} \\ u' = x^3+3x+2 & \to & u = \frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x \end{array}$$

Therefore,

$$\begin{aligned} \int (x^3+3x+2)\ln(2x) &= \left(\frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x\right)\ln(2x) - \int \frac{\frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x}{x}\ dx \\ &= \left(\frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x\right)\ln(2x) - \frac{1}{16}x^4 - \frac{3}{4}x^2 - 2x + C,\ C \in \mathbb{R} \end{aligned}$$

Exercise 87: Solve the following definite integrals:

- $\displaystyle\int_1^{10} x^2 \ln(x)\ dx$
- $\displaystyle\int_0^{\pi} e^{2x} \sin(x)\ dx$
- $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{e^x}\ dx$
- $\displaystyle\int_0^{1} x \arccos(x)\ dx$

Exercise 88: Solve the following indefinite integrals:

- $\displaystyle\int \sqrt{4-x^2}\ dx$
- $\displaystyle\int \frac{\arccos(x)}{\sqrt{1+x}}\ dx$
- $\displaystyle\int \ln(x^2+2)\ dx$
- $\displaystyle\int \ln^2(x)\ dx$
- $\displaystyle\int x^3 e^{3x}\ dx$

11.3.4 Integration by substitution

Theorem 13 (Indefinite integral of composite functions)**:**
Let f be a function defined on an interval I and F be an antiderivative of f on I.
Let g be a function differentiable on an interval J such that $g(J) \subseteq I$.
Then for $x \in J$, with $u = g(x)$. we have:

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du = F(g(x)) + C,\ \ C \in \mathbb{R}.$$

Proof. Let f be a function defined on an interval I and F be an antiderivative of f on I. Let g be a function differentiable on an interval J such that $g(J) \subseteq I$.
Given $\displaystyle\int f(u)\ du = F(u) + C$, $C \in \mathbb{R}$ and $u \in I$, if we set $u = g(x)$, with $x \in J$, we get:

$$\frac{d}{dx}(F(u) + C) = \frac{d}{dx}(F(g(x))) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Therefore,

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

□

Example 36: Let find the indefinite integral of $f : x \mapsto x\sqrt{x^2+1}$. For $x \in \mathbb{R}$, let $u = x^2 + 1$, then:

$$du = 2xdx \quad \Rightarrow xdx = \tfrac{1}{2}du$$

Therefore,

$$\begin{aligned}
\int f(x)\ dx &= \int x\sqrt{x^2+1}\ dx \\
&= \int \frac{1}{2}\sqrt{u}\ du \\
&= \frac{1}{2}\frac{2}{3}u^{3/2} + C,\ C \in \mathbb{R} \\
&= \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

Example 37: Let find the indefinite integral of $f : x \mapsto \frac{2x}{x^2-1}$. For $x \notin \{-1, 1\}$, let $u = x^2 + 1$, then:

$$du = 2xdx$$

Therefore,

$$\begin{aligned}
\int f(x)\ dx &= \int \frac{2x}{x^2-1}\ dx \\
&= \int \frac{1}{u}\ du \\
&= \ln(|u|) + C,\ C \in \mathbb{R} \\
&= \ln(|x^2+1|) + C
\end{aligned}$$

Example 38: Let find the indefinite integral of $f : x \mapsto \cot(x)$. For $x \neq n\pi$, $n \in \mathbb{Z}$, let $u = \sin(x)$, then:

$$du = \cos(x)dx$$

Therefore,

$$\begin{aligned}
\int f(x)\ dx &= \int \cot(x)\ dx \\
&= \int \frac{\cos(x)}{\sin(x)}\ dx \\
&= \int \frac{1}{u}\ du \\
&= \ln(|u|) + C,\ C \in \mathbb{R} \\
&= \ln(|\cos(x)|) + C
\end{aligned}$$

Exercise 89: Solve the following indefinite integral:

- $\displaystyle\int \frac{e^x}{e^{2x}+1}\ dx$
- $\displaystyle\int \frac{\ln(x)}{x}\ dx$
- $\displaystyle\int \frac{3^{1/x}}{x^2}\ dx$
- $\displaystyle\int \frac{x^5}{x^3+4}\ dx$

Exercise 90: Solve the following indefinite integral:

- $\displaystyle\int x\sin^3(x^2)\cos(x^2)\ dx$
- $\displaystyle\int \frac{\sin^3(x)\cos(x)}{\sin^2(x)+1}\ dx$

Theorem 14 (Definite integral of composite functions)**:**
Let f be a function defined on an interval I and let g be a function continuously differentiable on an interval $J = [a, b]$ such that $g(J) \subseteq I$.
Then for $x \in J$, with $u = g(x)$, we have:

$$\int_a^b g(f(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Proof. Let f be a function defined on an interval I. Let g be a function continuously differentiable on an interval $J = [a, b]$ such that $g(J) \subseteq I$.
Given f is continuous on I, it has an antiderivative F on I.
Besides, given g' is continuous on $[a, b]$, the function $x \mapsto f(g(x))g'(x)$ is also continuous on $[a, b]$ and, hence, integrable on $[a, b]$.
By the theorem on the indefinite integral of composite function, we have:

$$\forall x \in [a, b], \int f(g(x))g'(x)\ dx = F(g(x) + C,\ C \in \mathbb{R}.$$

Therefore, by the fundamental theorem of calculus, we get:

$$\int_a^b f(g(x))g'(x)\ dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\ du.$$

$\square$

Example 39: Let find the definite integral of $f : x \mapsto \frac{\sqrt{2\ln(x)+3}}{x}$ on $[1, e]$. For $x \in [1, e]$, let $u = 2\ln(x) + 3$, then:

$$du = \tfrac{2}{x} dx$$

Therefore,

$$\begin{aligned} \int_1^e f(x)\ dx &= \int_1^e \frac{\sqrt{2\ln(x) + 3}}{x}\ dx \\ &= \frac{1}{2}\int_{u(1)}^{u(e)} \sqrt{u}\ du \\ &= \frac{1}{2}\int_0^1 \sqrt{u}\ du \\ &= \frac{1}{3}u^{(3/2)}\Big|_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

Exercise 91: Solve the following definite integrals:

- $\displaystyle\int_0^{\pi/2} 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x) + 5}\ dx$
- $\displaystyle\int_0^1 x^2\sqrt{1 - x}\ dx$
- $\displaystyle\int_0^{\ln(2)} \frac{1}{e^x + 1}\ dx$
- $\displaystyle\int_{-\pi/2}^{\pi/2)} \sin^3(x)\cos(x)\ dx$

11.3.5 Trigonometric substitution

Trigonometric substitution is used to evaluate integrals involving expressions such as:

$$\sqrt{a^2-u^2} \quad \sqrt{a^2+u^2} \quad \sqrt{u^2-a^2}$$

In these case, we may apply the following substitutions, for $a>0$:

$$\begin{array}{lll} \sqrt{a^2-u^2} & \Rightarrow & u=a\sin(\alpha),\ \alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \sqrt{a^2+u^2} & \Rightarrow & u=a\tan(\alpha),\ \alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \sqrt{u^2-a^2} & \Rightarrow & u=a\sec(\alpha), \begin{cases} \alpha\in\left[0,\frac{\pi}{2}\right], & if\ u>a; \\ \alpha\in\left[\frac{\pi}{2},\pi\right], & if\ u<-a. \end{cases} \end{array}$$

Exercise 92: Solve the following definite integrals:

- $\displaystyle\int_2^{2\sqrt{2}} \frac{x^2}{\sqrt{x^2-2}}\,dx$
- $\displaystyle\int_{3/2}^{\sqrt{3}} \frac{\sqrt{3-x^2}}{x^3}\,dx$
- $\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{x^2+4}}\,dx$

Exercise 93: Solve the following indefinite integrals:

- $\displaystyle\int \frac{1}{x^2\sqrt{8-2x^2}}\,dx$
- $\displaystyle\int \frac{1}{\sqrt{1+4x^2}}\,dx$
- $\displaystyle\int \frac{1}{(x^2-4)^{3/2}}\,dx$

11.3.6 Partial fraction

The method of partial fractions consists in decomposing rational functions of the form $\frac{N}{D}$, where N and P are two polynomials, into simpler ones allowing the use of basic integration techniques. In this purpose, the we will use the partial fraction decomposition method.

Theorem 15 (Partial Fraction Decomposition)**:** *Let N and D be two non-zero polynomials such that $D=\prod_{i=1}^{k} P_i^{n_i}$, where P_i, $i\in\{1,2,...,k\}$, is an irreducible polynomial. Then there exists polynomials B and A_{ij}, $i\in\{1,2,...,k\}$, $k\in\{1,2,...,n_i\}$, such that deg $A_{ij}<$ deg P_i and:*

$$\frac{N}{D}=B+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{A_{ij}}{P_i^j}$$

If deg $N<$ deg D, then $B=0$.

Example 40: Consider the rational expression expression $Q(x)=\dfrac{x^4+2x^3+6x^2+6x+3}{x^3+2x^2+x+2}$. Since the degree of the numerator is greater than the one of the denominator, we will first perform the division to rewrite this expression:

$$\begin{aligned} Q(x) &= \frac{x^4+2x^3+6x^2+6x+3}{x^3+2x^2+x+2} \\ &= \frac{x(x^3+2x^2+x+2)+5x^2+4x+3}{x^3+2x^2+x+2} \\ &= x + \frac{5x^2+4x+3}{x^3+2x^2+x+2} \end{aligned}$$

We can note that $x=-2$ is a root of x^3+2x^2+x+2, then we can factorize the denominator, i.e. $x^3+2x^2+x+2=(x^2+1)(x+2)$. Hence, there exists reals a, b, and c such that:

$$\frac{5x^2+4x+3}{x^3+2x^2+x+2} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

By multiplying both side by $(x^2+1)(x+2)$ we get:

$$\begin{aligned} 5x^2+4x+3 &= (ax+b)(x+2)+c(x^2+1) \\ &= (a+c)x^2+(2a+b)x+(2b+c) \end{aligned}$$

Given that the coefficients of the polynomials must be equal, we get the following system of equation:

$$\begin{cases} a+c=5 \\ 2a+b=4 \\ 2b+c=3 \end{cases} \implies \begin{cases} a=2 \\ b=0 \\ c=3 \end{cases}$$

It follows that: $Q(x) = x + \dfrac{2x}{x^2+1} + \dfrac{3}{x+2}$.
Therefore, for $x \neq -2$, $\displaystyle\int Q(x)\ dx = \frac{1}{2}x^2 + \ln(x^2+1) + 3\ln(|x+2|) + C$, $C \in \mathbb{R}$.

Example 41: Let evaluate the definite integral $\displaystyle\int \frac{2x+1}{x^2(x^2+x+1)}\ dx$. Note that x^2+x+1 is an irreducible polynomials, hence there exists reals a, b, c, and d such that:

$$\frac{2x+1}{x^2(x^2+x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx+d}{x^2+x+1}$$

We multiply both side by $x^2(x^2+x+1)$ we get:

$$\begin{aligned} 2x+1 &= ax(x^2+x+1)+b(x^2+x+1)+(cx+d)x^2 \\ &= (a+c)x^3+(a+b+d)x^2+(a+b)x+b \end{aligned}$$

Equating the coefficients gives the system of equations:

$$\begin{cases} a+c=0 \\ a+b+d=0 \\ a+b=2 \\ b=1 \end{cases} \implies \begin{cases} a=1 \\ b=1 \\ c=-1 \\ d=-2 \end{cases}$$

Then we get, for $x \neq 0$:

$$\begin{aligned}
\int \frac{2x+1}{x^2(x^2+x+1)}\,dx &= \int \left(\frac{1}{x}+\frac{1}{x^2}-\frac{x-2}{x^2+x+1}\right)\,dx \\
&= \ln|x| - \frac{1}{x} - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx \\
&= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\
&= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+x+1) - 2\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx \\
&= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)
\end{aligned}$$

Exercise 94: Evaluate the following definite integrals:

- $\displaystyle\int_0^1 \frac{x^3+2}{x^2-4}\,dx$
- $\displaystyle\int_0^1 \frac{x^2-x}{x^2+2x+1}\,dx$
- $\displaystyle\int_1^2 \frac{x+4}{x(x^2+4)}\,dx$
- $\displaystyle\int_{-1}^1 \frac{x^2+6x+4}{x^4+8x^2+16}\,dx$
- $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos(x)}{\sin^2(x)-\sin(x)}\,dx$
- $\displaystyle\int_1^5 \frac{e^x}{e^{2x}-e^x-2}\,dx$

11.4 Applications

11.4.1 Area of a Region Between Two Curves

Proposition 37: *Let f and g be two continuous functions on an interval $[a,b]$ such that, for all $x \in [a,b]$, $f(x) \geq g(x)$. The area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is:*

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Figure 24 gives an illustration of the previous definition. It is important to check, before applying the formula, that $f \geq g$ on the interval of integration.

Example 42: Let find the area A of the region bounded by the graphs of $f : x \mapsto 1$ and $g : x \mapsto \frac{1}{1+e^x}$ on the interval $I = [-1, 1]$. We note that f and g are both continuous on I and, for $x \in I$, $f(x) \geq g(x)$, therefore we have:

$$A = \int_{-1}^1 [f(x) - g(x)]\,dx = \int_{-1}^1 \frac{e^x}{1+e^x}\,dx$$

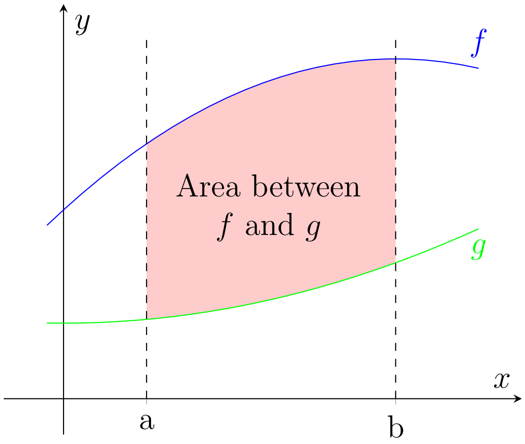


Figure 24: Area of a Region Between Two Curves

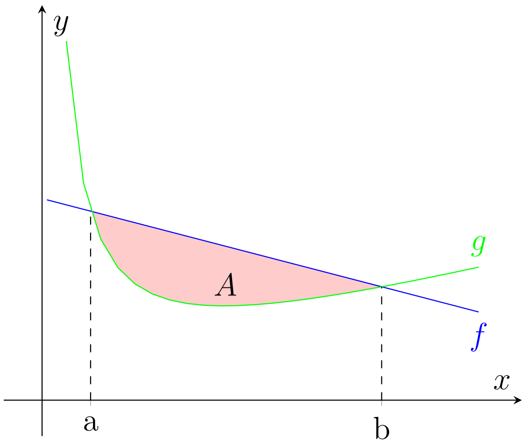


Figure 25: Region Between the Curves of Example 43

Using the substitution $u = e^x$ ($dx = \frac{du}{u}$), we get:

$$\begin{aligned} A &= \int_{1/e}^{e} \frac{1}{1+u}\, du &= \ln(1+u)\Big|_{1/e}^{e} \\ &= 1 \end{aligned}$$

Example 43: Le find the area A of the region bounded by the graphs of $f : x \mapsto -x + 16$ and $g : x \mapsto \frac{x^2+14}{x}$. In this case, no interval is specified and so it has to be determined. A look at the graphs of these functions (see Figure 25) show that that they intersect at two points, a and b, that bound the area A between the curves. .
Solving the equation $f(x) = g(x)$ gives $a = 1$ and $b = 7$. We note that f and g are both defined and continuous on $[1, 7]$ and, for $x \in [1, 7]$, $f(x) \geq g(x)$. Therefore, we get:

$$\begin{aligned} A &= \int_1^7 [f(x) - g(x)]\ du &= -2\int_1^7 \frac{x^2 - 8x + 7}{x}\, du \\ &= -2\left(\tfrac{1}{2}x^2 - 8x + 7\ln(x)\right)\Big|_1^7 \\ &= 2\,(24 - 7\ln(7)) \end{aligned}$$

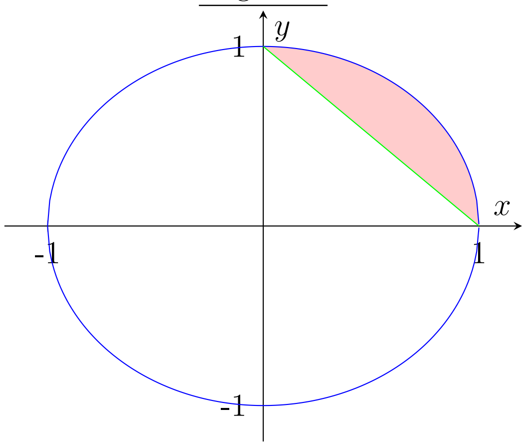
Exercise 95: Draw and determine the area of the curves defined by the following equations:

- $y = x^2$ and $y = x + 2$;
- $y = \sin(x)$ and $y = 4\sin(x)$ on $[0, \pi]$;

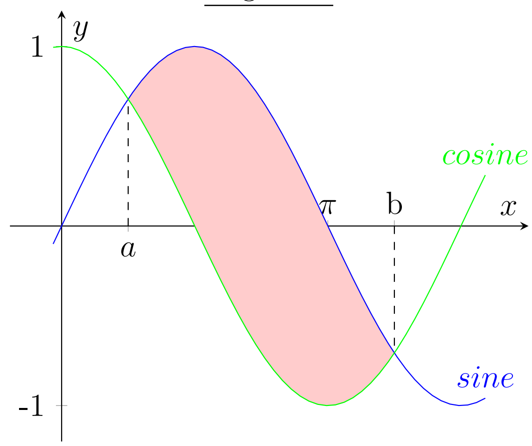
- $y = 4x^3 - 8x^2 - 3x + 14$ and $y = x + 6$;
- $x = y^2 - 1$ and $x = -y^2 + 1$;
- $\frac{y^2}{a} + \frac{x^2}{b} = 1$, $a > 0$ and $b > 0$.

Exercise 96: Determine the area of the region represented in the following graphs:

Region 1:



Region 2:



11.4.2 Volume of a Solid of Revolution

In this section we will learn how to compute the volume of a solid obtained by revolving a curve around an axis. To compute the volume of such solids, we will consider them as a pile of disks and we will sum their discrete volume to get the total volume. Each of these disk has a radius R whose values depend of the position of the disk within the solid as shown in Figure 26. The volume of these solid can be obtained using the following formula:

Volume of the solid obtained by revolving the curve $y = f(x)$ with f defined, continuous, and monotone on $[a, b]$	
around the $y - axis$	around the $x - axis$
$V = \pi \left\lvert \int_{f(a)}^{f(b)} [R(y)]^2 \, dy \right\rvert$	$V = \pi \int_a^b [R(x)]^2 \, dx$
$R(y) = f^{-1}(y)$	$R(x) = f(x)$

Example 44: Let compute the volume of the solid obtained by revolving the curve of equation $y = \ln(x)$, $1 \le x \le e$, around the $x - axis$. An illustration of the solid is given in Figure 27. The volume of this solid is:

$$V = \pi \int_1^e [R(x)]^2 \, dx = \pi \int_1^e [\ln(x)]^2 \, dx$$

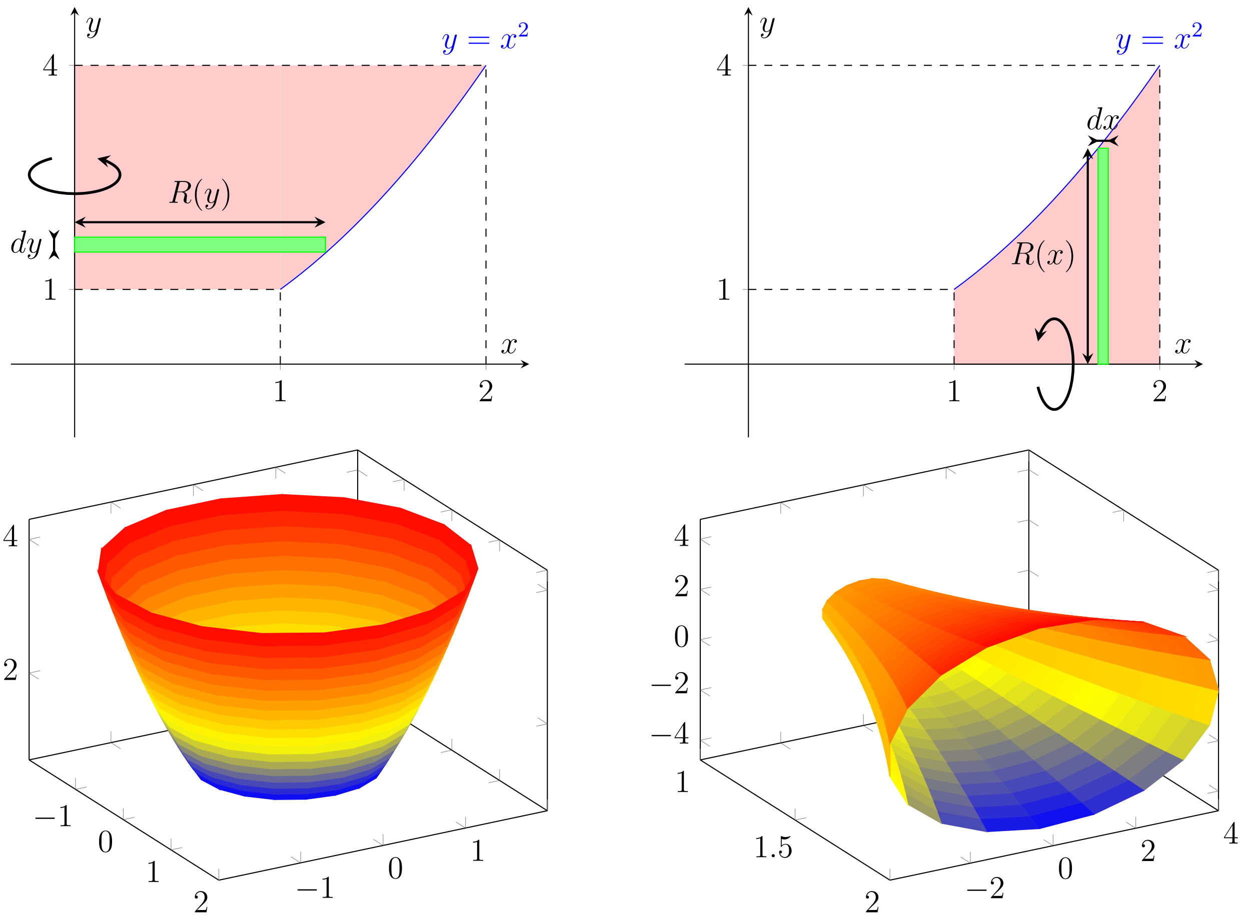
We use integration by parts with:

$$\begin{array}{lll} v = [\ln(x)]^2 & \to & v' = \frac{2}{x}\ln(x) \\ u' = 1 & \to & u = x \end{array}$$

We get:

$$V = \pi \left(x \ln^2(x) \Big|_1^e - 2 \int_1^e \ln(x) \, dx \right) = \pi(e - 2).$$

Exercise 97: For each of the following cases, find the volume of the solid obtained by revolving the curve :



Solid obtained by revolving the curve of equation $y = x^2$, with $1 \leq x \leq 2$, around the y-axis.

Solid obtained by revolving the curve of equation $y = x^2$, with $1 \leq x \leq 2$, around the x-axis.

Figure 26: Examples of Solid of Revolution

- $y = 4x - x^3$, $0 \leq x \leq 2$, around the $x - axis$
- $y = e^{x-1}$, $1 \leq x \leq 2$, around the $y - axis$
- $\left\{ \begin{array}{ll} y = x^2 - 1, & for\ \ -1 \leq y \leq 0 \\ y = \sqrt{1 - x^2} & for\ 0 \leq y \leq 1 \end{array} \right.$, around the $y - axis$

This approach can be also used to find the volume of solid of revolution with holes in their center. These solids are generally obtained by revolving a region bounded by two (or more) curves around an axis. In this case, instead of a plain disk, we have a disk with a hole in its center similar to a washer. As one can see on Figure 28, the volume of the washer is $V = \pi\left(R^2 - r^2\right) d$.

Example 45: Let find the volume of the solid obtained by revolving the region bounded by the graphs of $f : x \mapsto (x-1)^2$ and $g : x \mapsto \sqrt{x-1}$ around the $y-axis$. The solution of the equation $f(x) = g(x)$ tells us that the two graphs intersects at $x = 1$ and $x = 2$. A representation of this solid is shown in Figure 29 where the green rectangle is a cut of the washer shape.
Using the same notation used in Figure 29, we get:

$$V = \pi \int_0^1 \left([R(y)]^2 - [r(y)]^2\right)\ dx.$$

Given that $\left\{ \begin{array}{l} R(y) = f^{-1}(y) = \sqrt{y} + 1 \\ r(y) = g^{-1}(y) = y^2 + 1 \end{array} \right.$, we infer:

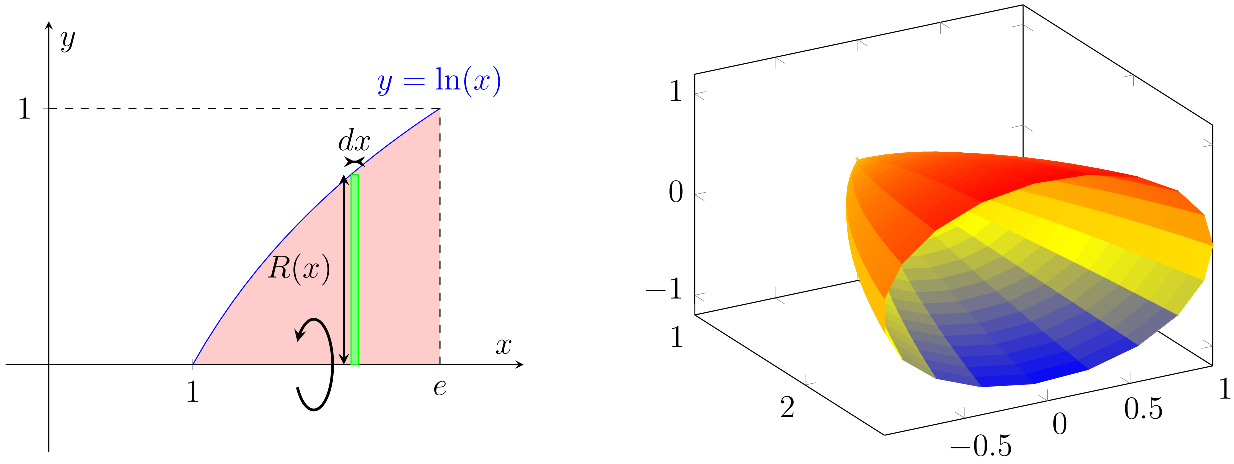


Figure 27: Solid obtained by revolving the curve $y = \ln(x)$, $1 \leq x \leq e$, around the $x - axis$

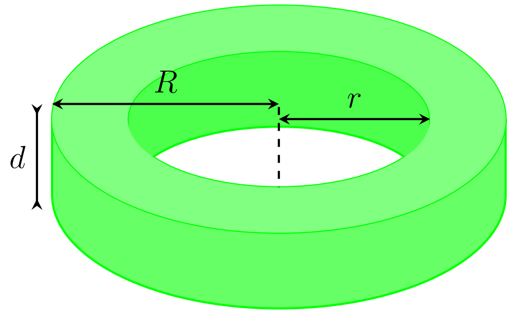


Figure 28: Washer shape

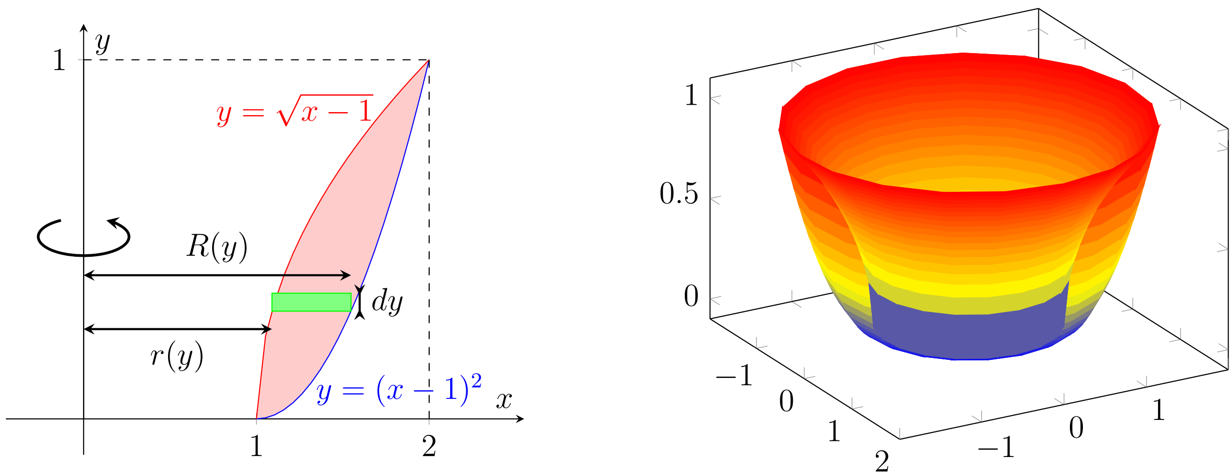


Figure 29: Solid obtained by revolving the region bounded by two curves

$$
\begin{aligned}
V &= \pi \int_0^1 \left((\sqrt{y}+1)^2 - \left(y^2+1\right)^2 \right) \, dx \\
&= \pi \int_0^1 \left(-y^4 - 2y^2 + y + 2\sqrt{y} \right) \, dx \\
&= \pi \left(-\frac{1}{5}y^5 - \frac{2}{3}y^3 + \frac{1}{2}y^2 + \frac{4}{3}y^{3/2} \right) \Big|_0^1 \\
&= \frac{29\pi}{30}.
\end{aligned}
$$

Exercise 98: For each of the following cases, find the volume of the solid obtained by revolving the following region around the indicated axis:

- $y = \sqrt{x-1}$ and $y = (x-1)^2$, around the $x - axis$
- $y = \frac{1}{2}(x-1)^2 + 1$ and $y = \frac{1}{2}(x+1)$, around the $x - axis$
- $y = 2\sqrt{1-x^2}$, $y = -x^2+1$, and $y = 0$, around the $y - axis$

11.4.3 Arc Length and Surfaces of Revolution

Definite integrals can be also used to find the length of an arc from a curve and the surface of a solid of revolution.We will consider only smooth curves that are defined by continuously differentiable functions.

Proposition 38: *Let f be a continuously differentiable function on an interval $[a, b]$. The **arc length** of f between a and b is:*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Proof. Let f be a continuously differentiable function on an interval $[a, b]$ and $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, be a partition of $[a, b]$.For $i \in \{1, ..., n\}$, we set $\Delta x_i = x_i - x_{i-1}$, $y_i = f(x_i)$, and $\Delta y_i = y_i - y_{i-1}$. The length L of the arc defined by the curve of f can be approximate by summing the length of the n segments determined by the partition P (see Figure 30:

$$\begin{aligned} L &\approx \sum_{i=1}^{n} \sqrt{[\Delta x_i]^2 + [\Delta y_i]^2} \\ &\approx \sum_{i=1}^{n} \left(\sqrt{1 + \left[\frac{\Delta y_i}{\Delta x_i}\right]^2} \right) \Delta x_i \end{aligned}$$

Given f is differentiable on $[x_{i-1}, x_i]$, according to the Mean Value Theorem, there exists $c_i \in [x_{i-1}, x_i]$, such that $\dfrac{\Delta y_i}{\Delta x_i} = f'(c_i)$. Then we get:

$$L \approx \sum_{i=1}^{n} \left(\sqrt{1 + [f'(c_i)]^2} \right) \Delta x_i$$

The smaller is $\Delta x = \max\limits_{1 \le i \le n} \Delta x_i$, the better is the approximation, hence:

$$L = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(\sqrt{1 + [f'(c_i)]^2} \right) \Delta x_i$$

Since f' is continuous on $[a, b]$, $x \mapsto \sqrt{1 + [f'(x)]^2}$ is also continuous on $[a, b]$, we infer:

$$L = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(\sqrt{1 + [f'(c_i)]^2} \right) \Delta x_i = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

□

Exercise 99: Find the length of the following arcs:

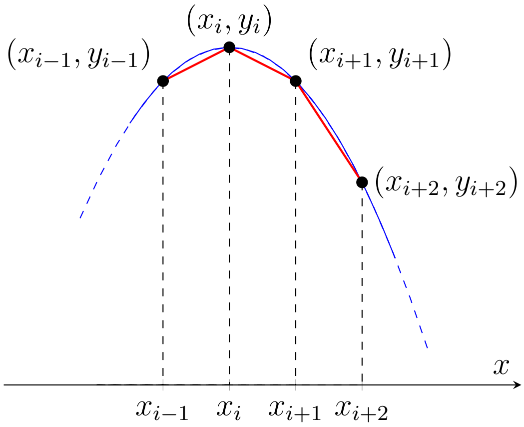


Figure 30: Arc length approximation from a partition

- The arc length of the graph $y = x + 1$ on the interval $[0, 4]$;
- The arc length of the graph $y = x^{3/2}$ on the interval $[0, 1]$;
- The arc length from $(0, 1)$ to $(1, 0)$ along the graph $y = \sqrt{1 - x^2}$.
- The arc length from $(0, 0)$ to $\left(\frac{\pi}{4}, -\frac{1}{2}\ln(2)\right)$ along the graph $y = \ln(\cos(x))$.

11.4.4 Work Done by a Variable Force

The work done by a constant force F on an object that moves form point a to point b is $W = F\Delta x$, where $\Delta x = b - a$ is the displacement. When the force F is variable, i.e. it depends of the position x of the object, then we have to sum all the infinitesimal contributions to the work on infinitesimal displacements dx.

Proposition 39: *The* ***work done*** *by a continuously variable force F on an object that moves from a point a to a point b is:*

$$W = \int_a^b F(x)\ dx$$

Proof. Let F be a continuous variable force applied to an object moving from a point a to a point b and $P = \{x_0, x_1, ..., x_n\}$, $n \in \mathbb{N}^*$, be a partition of $[a, b]$.For $i \in \{1, ..., n\}$, we set $\Delta x_i = x_i - x_{i-1}$. The work W done by F on the moving object can be approximate by summing the work done on each point c_i, $i \in \{1, ..., n\}$, where c_i is a point arbitrarily chosen in $[x_{i-1}, x_i]$:

$$W \approx \sum_{i=1}^{n} F(c_i)\Delta x_i$$

The smaller is $\Delta x = \max\limits_{1 \le i \le n} \Delta x_i$, the better is the approximation, hence:

$$W = \lim_{\Delta x \to 0} F(c_i)\Delta x_i$$

Since F is continuous on $[a, b]$, we infer:

$$W = \lim_{\Delta x \to 0} F(c_i)\Delta x_i = \int_a^b F(x)\ dx$$

□

Exercise 100: A force of $150N$ is required to compress a spring $3cm$. Determine the spring constant from the Hooke Law ($F = kx$) and compute the work done by compressing the spring an other $3cm$.

Exercise 101: A bucket weighing $2kg$ is filled with $10l.$ of water. The bucket is leaking at constant rate and is lifted $20m$ at a constant speed. The bucket is emptied just as it reaches the top. Compute the work done to lift the bucket.

Exercise 102: A conical water tank of height $10m$ and diameter $5m$ at its top is half filled with water (water weight is $1000kg/m^3$). Compute the work done to empty the tank from its top.

11.5 Multiple Integration

In this section we will see several application of integration involving functions with multiple variables. For instance, assume f is a function from $\mathbb{R}^2$ to $\mathbb{R}$. Then we may integrate f:

- with respect to x: $\displaystyle\int_{a(y)}^{b(y)} f(x,y)\ dx$
  In this case, y is considered constant and the bound of integration, $a(y)$ and $b(y)$, can be constant or functions of y;
- with respect to y: $\displaystyle\int_{c(x)}^{d(x)} f(x,y)\ dy$
  In this case, x is considered constant and the bound of integration, $c(x)$ and $d(x)$, can be constant or functions of x;
- as an iterated integral: $\displaystyle\int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dx\ dy$ or $\displaystyle\int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dy\ dx$
  In this case, the outside limit of integration must be constant and we integrate first the inner integral.

For indefinite integrals of the form $\displaystyle\int f(x,y)\ dx$ or $\displaystyle\int f(x,y)\ dy$ the resulting constant of integration C is a function of y and x respectively.

Example 46: Consider the function $f : (x,y) \mapsto \frac{1}{xy} + y^2$. This function is defined on $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and is also continuous on $\mathcal{D}$. Let integrate f with respect to x:

$$\begin{aligned}\int f(x,y)\ dx &= \int \left(\frac{1}{xy} + y^2\right)\ dx \\ &= \frac{1}{y}\ln(|xy|) + xy^2 + C(y)\end{aligned}$$

Note that the constant of integration C is now a function of y. Now, let integrate f with respect to y:

$$\begin{aligned}\int f(x,y)\ dy &= \int \left(\frac{1}{xy} + y^2\right)\ dy \\ &= \frac{1}{x}\ln(|xy|) + \frac{1}{3}y^3 + C(x)\end{aligned}$$

In this case C is a function of x.

Exercise 103: Evaluate the following integrals:

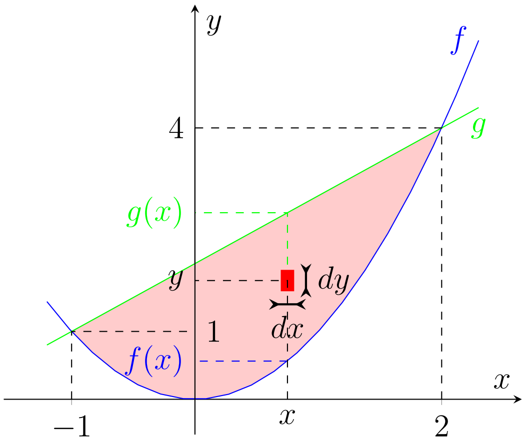


Figure 31: Area between two curves of Example 47

- $\displaystyle\int e^{xy}\sin(x+2y)\ dx$
- $\displaystyle\int \frac{1}{\sqrt{x^2-y^2}}\ dy$
- $\displaystyle\int_0^1 \frac{x+y}{x^2-xy^2+2xy-2y^3}\ dx$
- $\displaystyle\int_1^5\int_0^{\pi/y} \sin^2(xy)\ dx\ dy$

We saw in section 11.4.1 how to find the area between two curves. This can be also obtained using iterated integrals. We can interpret $dA = dx\ dy$ has the area of an infinitesimal rectangle of dimension $dx \times dy$, hence we have to sum all these rectangles within the region to get the total area.

Example 47: Let find the area of the region between the graph of $f : x \mapsto x^2$ and $g : x \mapsto x+2$. Solving the equation $f(x) = g(x)$ tells us that the curves intersect at $x = -1$ and $x = 2$. As sown on Figure 31, at a position (x, y) we define an infinitesimally small rectangle of dimension $dx \times dy$. For a position x, we have $f(x) \le y \le g(x)$, and the value of x range from -1 to 2. Hence, we can compute the area of the region as follows:

$$\begin{aligned}
A &= \int\int_R dA,\ with\ R = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,2],\ f(x) \le y \le g(x)\} \\
&= \int_{-1}^{2}\int_{f(x)}^{g(x)} dy\ dx \\
&= \int_{-1}^{2} [g(x) - f(x)]\ dx\ (Same\ formula\ than\ Proposition\ 37) \\
&= \int_{-1}^{2} (x + 2 - x^2)\ dx \\
&= \frac{9}{2}
\end{aligned}$$

Also, the surface and the volume of a solid of revolution can be obtained through iterated integral. The surface is obtained by summing the area of infinitesimal rectangles on the surface of the solid. The volume can be computed by summing all the infinitesimal cubes inside of the solid. Since the solid is obtained by revolution, it is convenient to introduce the angle θ of

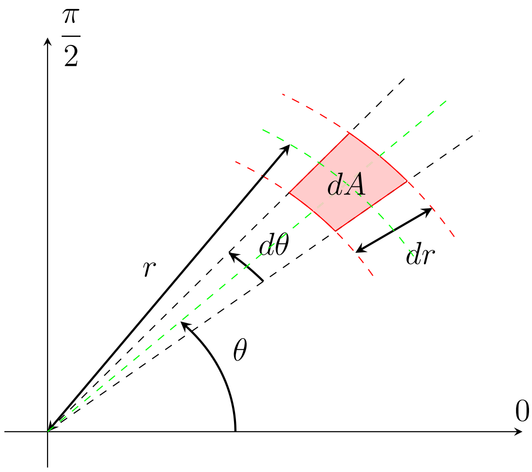


Figure 32: Infinitesimal polar sector

rotation to describe the solid in polar coordinates. Figure 32 show an infinitesimal polar sector at position (r, θ) in polar coordinate. The area of this sector is $dA = rd\theta dr$.

Example 48: Let find the surface and the volume of the solid of revolution obtained by revolving the graph of $f : x \mapsto x^2$, $1 \leq x \leq 2$ around the $y - axis$. In Figure 33, we defined an infinitesimal small rectangle on the surface of the solid in red color at a position x. The area of this rectangle is $dA = xd\theta dl$, where $xd\theta$ is the length of the arc obtained by rotation of an angle $d\theta$ a point at distance x and $dl = \sqrt{dx^2 + dy^2}$. Since $y = x^2$, then $dy = 2xdx$ and therefore $dA = x\sqrt{1+4x^2}dxd\theta$. We have $\theta \in [0, 2\pi]$ and $x \in [1, 2]$, hence we get:

$$
\begin{aligned}
S &= \int\int_R dA, \; with \; R = [0, 2\pi] \times [1, 2] \\
&= \int_0^{2\pi}\int_1^2 x\sqrt{1+4x^2}\; dx\; d\theta \\
&= \int_0^{2\pi} \frac{1}{12}(1+4x^2)^{3/2}\Big|_1^2 \; d\theta \\
&= \frac{1}{12}\int_0^{2\pi} [17\sqrt{17} - 5\sqrt{5}]\; d\theta \\
&= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\; \theta\;\Big|_0^{2\pi} \\
&= \frac{17\sqrt{17} - 5\sqrt{5}}{6}\pi
\end{aligned}
$$

To get the total surface, we have also to add the lower and upper disks of surface π and 4π, respectively.

Now, for the volume, we define an infinitesimal cube at position (x, y, θ) whose volume is

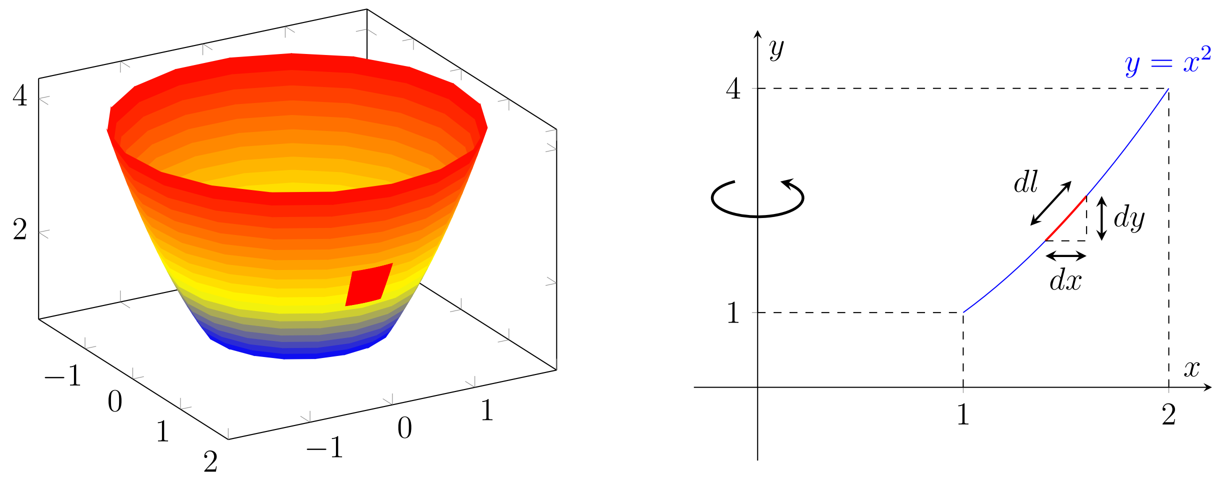


Figure 33: Surface of a solid of revolution from Example 48

$dV = xd\theta dxdy$. We note that $\theta \in [0, 2\pi]$, $1 \leq y \leq 4$, and $0 \leq x \leq \sqrt{y}$, hence we get:

$$
\begin{aligned}
V &= \int\int\int_R dV, \; with \; R = \{(x, y, \theta) \in \mathbb{R}^2 \times [0, 2\pi] \mid 1 \leq y \leq 4, \; 0 \leq x \leq \sqrt{y}\} \\
&= \int_0^{2\pi} \int_1^4 \int_0^{\sqrt{y}} x \; dx \; dy \; d\theta \\
&= \int_0^{2\pi} \int_1^4 \frac{x^2}{2} \Big|_0^{\sqrt{y}} dy \; d\theta \\
&= \int_0^{2\pi} \int_1^4 \frac{y}{2} \; dy \; d\theta \\
&= \int_0^{2\pi} \frac{y^2}{4} \Big|_1^4 d\theta \\
&= \int_0^{2\pi} \frac{15}{4} \; d\theta \\
&= \frac{15}{4}\theta \Big|_0^{2\pi} \\
&= \frac{15}{2}\pi
\end{aligned}
$$

Exercise 104: Find the surface and the volume of the following solid using iterated integrals:

- A disk of radius 5 and height 1;
- The solid obtained by revolving the graph of $f : x \mapsto \sqrt{x}$, $0 \leq x \leq 1$, around the $x-axis$;
- The solid defined by $\frac{x^2}{2} + y^2 = 1$ with $0 \leq z \leq 1$;
- A sphere of radius 1.