# Mathematical Modeling

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# Content


1. Introduction to Linear Programming
2. Examples of Optimization Problems
3. Modeling

# Agenda

1 Introduction to Linear Programming

2 Examples of Optimization Problems

3 Modeling

# Main components of an optimization model



- Decision variables: these are used to described the decisions to be made
- Objective function: it is a function of the decision variables, it establishes what the decision maker wants to minimize (cost) or maximize (profits)
- Constraints: these represent the limitations, conditions or requirements of the problem (example, vehicle capacity, demand to be satisfied, manpower, etc.)

# Definitions



- A function $f(x_1, x_2, ..., x_n)$ of $x_1, x_2, ..., x_n$ is a **linear function** if and only if for some set of constants $c_1, c_2, ..., c_n$, $f(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 + ... + c_nx_n$.
  - For example, $f(x_1, x_2) = 2x_1 + x_2$ is a linear function of $x_1$ and $x_2$, but $f(x_1, x_2) = {x^2}_1x_2$ is not a linear function of $x_1$ and $x_2$.
- For any linear function $f(x_1, x_2, ..., x_n)$ and any number $b$, the inequalities $f(x_1, x_2, ..., x_n) \leq b$ and $f(x_1, x_2, ..., x_n) \geq b$ are **linear inequalities**.

# Definitions



A **linear programming problem (LP)** is an optimization problem for which we do the following:

1. We attempt to maximize (or minimize) a *linear function* of the decision variables. This function that is to be maximized or minimized is called *objective function*
2. The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality
3. A sign restriction is associated with each decision variable. For any variable $x_i$, the sign restriction specifies either that $x_i$ must be nonnegative ($x_i \geq 0$) or that $x_i$ may be unrestricted in sign.

# Definitions

The fact that the objective function for an LP must be a linear function has two implications:

1. **Proportionality assumption:** The contribution to the objective function for each decision variable is proportional to the value of the decision variable
2. **Additivity assumption:** The contribution to the objective function for any variable in independent of the values of the other decision variables.

Analogously, the fact tha each LP constraint must be a linear inequality or linear equation has two implications:

1. **Proportionality assumption:** The contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable
2. **Additivity assumption:** The contribution of a variable to the left-hand side of each constraint is independent of the values of the other variables.

# Definitions

1. The **feasible region** for an LP is the set if all points satisfying all the LP's constraints and all the LP's sign restrictions.
2. For a maximization problem, an **optimal solution** to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.

# Types of solutions

1. Feasible solution: it satisfies all constraints for an LP.
2. Optimal solution: it is the best of the feasible solutions for an LP, according to the objective function.
3. Infeasible solution: it does not satisfy at least one constraint of the LP.

# Types of LPs

1. Some LPs have a unique optimal solution
2. Some LPs have an infinite number of optimal solutions (alternative or multiple optimal solutions)
3. Some LPs have no feasible solutions (infeasible LPs)
4. Some LPs are unbounded: there are points in the feasible region with arbitrarily large (in a maximization problem) objective function values.

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# Facility Location Problem

The facility location problem consists of a set of potential facility sites where a facility can be opened, and a set of customers that must be serviced. The goal is to pick a subset of facilities to open, in order to minimize the operative cost (transportation cost, cost of opening/building a facility, distance to customers, ...).

- Decision: Facility Localization
- Objective: Minimize the operative cost
- Constraints:
  - All customers must be served
  - Location constraint
  - Capacity of the facility

# Fleet Sizing Problem

The fleet sizing problem consists of a set of vehicles and a set of customers that must be serviced. The goal is to pick a subset of the vehicles (fleet size), in order to minimize the operative cost (transportation cost, leasing cost, customer satisfaction, ...).

- Decision: Number of vehicles
- Objective: Minimize the operative cost
- Constraints:
  - All customers must be served
  - The set of possible routes
  - Shift length of the Driver
  - Vehicle capacity

# Shift Scheduling Problem

The shift scheduling problem consists of a set of employees and a set of activities to be fulfilled during a planning horizon (days/week/month). The goal is to assign the activities to the employees, in order to minimize the operative cost (cost of contracting an employee, over or under covering an activity, inactive employees, ...).

- Decision: Shift (Sequence of activities) of an employee
- Objective: Minimize the operative cost
- Constraints:
  - All the activities should be covered
  - Employees should have some breaks
  - Shift length
  - Employee's skill

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# General Linear Programming Problem

## Objective

$$Min \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

## Constraints

$$\begin{array}{lcccccccc}
s.t. & a_{11}x_1 & + & a_{12}x_2 & + \dots + & a_{1n}x_n & (\leq, =, \geq) & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + \dots + & a_{2n}x_n & (\leq, =, \geq) & b_2 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + \dots + & a_{mn}x_n & (\leq, =, \geq) & b_m
\end{array}$$

## Nature of the variables

$$x_i \geq 0, \ \forall i \in \{1, \dots, n\}$$

# The carpenter's problem

Suppose that a carpenter makes tables and chairs and sells all the tables and chairs he makes in a market.

He does not have a steady income and he wishes to optimize this situation.

So, the carpenter needs to determine how many tables and chairs he should make in order to maximize his net income. He knows that the income he receives per table sold is \$500 and per chair sold is \$300.

The carpenter works 8 hours a day from Monday to Friday and takes 2 hours to make a table and 1 hour to make a chair. Also, each week he receives 50 units of raw material, of which he requires 1 unit for each table and 2 units for each chair he makes.



# How can we fomulate this problem?

# The carpenter's problem



## Decision

Number of tables and chairs to be produced

## Objective

Maximize the net income

## Constraints

- The total working time should not exceed the available hours for working
- At most 50 units of raw material must be used to produce the tables and chairs.

# The carpenter's problem

## Decision variables

- Let $t$ be the number of tables to be produced.
- Let $c$ be the number of chairs to be produced.

## Objective

- Max $z = 500t + 300c$

## Constraints

- The total working time should not exceed the available hours for working:

  $$2t + c \leq 40$$

- At most 50 units of raw material must be used to produce the tables and chairs:

  $$t + 2c \leq 50$$

- Nature of the variables: $t \geq 0, \ c \geq 0$

# The carpenter's problem

## The optimization model

$$\begin{cases} Max & 500t & + & 300c & = & z \\ s.t. & 2t & + & c & \leq & 40 \\ & t & + & 2c & \leq & 50 \\ & t \geq 0, \ c \geq 0 & & & & \end{cases}$$

# Dorian's problem

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and high-income men. In order to reach these groups, Doran Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercials spots on two types of program: comedy shows and football games. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men. A 1-minute comedy show ad costs $50000, and 1-minute football ad costs $100000. Dorian would like the commercials to be seen by at least 28 million high-income women and at least 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum cost.

# Dorian's problem

### Decision

Number of 1-minute ads on comedy shows and football game to purchase.

### Objective

Minimize the cost (in thousands of dollars).

### Constraints

- At least 28 million high-income women must see the ads.
- At least 24 million high-income men must see the ads.

# Dorian's problem

## Decision variables

- $x_1$: number of 1-minute comedy ads to purchase.
- $x_2$: number of 1-minute football ads to purchase.

## Objective

- Min $z = 50x_1 + 100x_2$ (in thousands of dollars)

## Constraints

- At least 28 million high-income women must see the ads:
  $$7x_1 + 2x_2 \geq 28$$
- At least 24 million high-income men must see the ads:
  $$2x_1 + 12x_2 \geq 24$$
- Nature of the variables: $x_1 \geq 0, \; x_2 \geq 0$

# The Dorian's optimization problem



## The optimization model

$$
\begin{cases}
Min & 50x_1 & + & 100x_2 & & \\
s.t. & 7x_1 & + & 2x_2 & \geq & 28 \\
 & 2x_1 & + & 12x_2 & \geq & 24 \\
 & x_1 \geq 0, \; x_2 \geq 0 & & & &
\end{cases}
$$