Duality in Linear Programming: Theory and Applications

A Duality in Linear Programming

Duality is a central concept in linear programming that associates every optimization problem (the *primal*) with another related problem (the *dual*). The dual problem provides bounds on the optimal value of the primal and offers deep insights into the structure and sensitivity of the original problem. Understanding duality enables us to derive optimality conditions, interpret shadow prices, and design efficient algorithms. In this section, we introduce the theory of duality, its mathematical formulation, and its significance in both theory and applications.

A.1 Mathematical Formulation

Given the primal LP in standard form:

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{max} \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

where $y \in \mathbb{R}^m$ are the dual variables.

Example: Primal and Dual Formulation

Consider the following primal problem:

$$\begin{aligned} \text{min} \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 \\ & x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The corresponding dual problem is:

$$\begin{aligned} \text{max} \quad & y_1 + 2y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \leq 3 \\ & y_1 + 2y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Example: Mixed Constraints in Primal and Dual

Consider the following primal problem:

$$\begin{aligned} \text{min} \quad & 2x_1 + 5x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 4 \quad \text{(Constraint 1)} \\ & 2x_1 - x_2 + 3x_3 \geq 6 \quad \text{(Constraint 2)} \\ & x_2 + x_3 = 3 \quad \text{(Constraint 3)} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

First we rewritte the primal to have only $\leq$ constraints:

$$\begin{aligned} \text{min} \quad & 2x_1 + 5x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 4 \\ & -2x_1 + x_2 - 3x_3 \leq -6 \\ & x_2 + x_3 \leq 3 \\ & -x_2 - x_3 \leq -3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

This yields the dual problem:

$$\begin{aligned} \text{max} \quad & 4y_1 - 6y_2 + 3y_3 - 3y_4 \\ \text{s.t.} \quad & y_1 - 2y_2 \leq 2 \quad \text{(for } x_1\text{)} \\ & 2y_1 + y_2 + y_3 - y_4 \leq 5 \quad \text{(for } x_2\text{)} \\ & y_1 - 3y_2 + y_3 - y_4 \leq 3 \quad \text{(for } x_3\text{)} \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

If set $y_1' = y_1$, $y_2' = -y_2$, $y_3' = y_3 - y_4$, we can rewrite the dual as:

$$\begin{aligned} \text{max} \quad & 4y_1' - 6y_2' + 3y_3' \\ \text{s.t.} \quad & y_1' + 2y_2' \leq 2 \quad \text{(for } x_1\text{)} \\ & 2y_1' + y_2' + y_3' \leq 5 \quad \text{(for } x_2\text{)} \\ & y_1' + 3y_2' + y_3' \leq 3 \quad \text{(for } x_3\text{)} \\ & y_1' \geq 0, \quad y_2' \leq 0, \quad y_3' \text{ free} \end{aligned}$$

In the example above, the primal has mixed constraint types: one $\leq$, one $\geq$, and one $=$. The dual formulation reflects these types:

- $\leq$ primal: dual variable ≥ 0
- $\geq$ primal: dual variable ≤ 0
- $=$ primal: dual variable free

This example shows how to systematically construct the dual when the primal has mixed constraint types. The dual variables' signs reflect the direction of the original constraints, and equality constraints in the primal correspond to free dual variables. This flexibility is essential in modeling real-world problems with diverse requirements.

Practice Exercise:

Write the dual of the following LPs and solve both the primal and dual problems using the solver of your choice:

(a)

$$\begin{aligned} \text{min} \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 - x_2 \le 3 \\ & 2x_1 + x_2 \ge 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(b)

$$\begin{aligned} \text{max} \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = 5 \\ & x_1 - x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(c)

$$\begin{aligned} \text{min} \quad & 4x_1 + x_2 + 2x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \ge 6 \\ & 2x_1 - x_2 = 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

(d)

$$\begin{aligned} \text{max} \quad & 5x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \le 7 \\ & x_1 - 2x_2 \ge 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

A.2 Interpretation of Dual Variables

The dual variables y_i can be interpreted as the shadow prices of the constraints in the primal problem. They represent the rate of change in the optimal value of the primal objective function with respect to a unit increase in the right-hand side of the i-th constraint. If $y_i > 0$, it indicates that increasing the right-hand side of the i-th constraint will improve the objective value of the primal problem.

A.3 Duality Gap

The duality gap is the difference between the optimal values of the primal and dual problems. For feasible solutions x for the primal and y for the dual, the duality gap is defined as:

$$\text{duality gap} = c^T x - b^T y$$

The weak duality theorem guarantees that this gap is non-negative for any feasible solutions, while strong duality states that if both the primal and dual have optimal solutions, the duality gap is zero.

A.4 Weak Duality Theorem

The weak duality theorem states that for any feasible solution x of the primal and y of the dual, the objective value of the dual is less than or equal to that of the primal:

Theorem 1. *For any feasible x for the primal and y for the dual, $b^T y \le c^T x$.*

Proof. Since $Ax \ge b$ and $y \ge 0$, $y^T Ax \ge y^T b$. Also, $A^T y \le c$ and $x \ge 0$ imply $y^T Ax \le c^T x$. Thus, $b^T y \le y^T Ax \le c^T x$. □

A.5 Strong Duality Theorem

The strong duality theorem is a fundamental result in linear programming that asserts, under mild conditions, the optimal values of the primal and dual problems are equal. This result not only guarantees that solving either problem yields the same optimal objective value, but also underpins the correctness of the simplex method and the interpretation of dual variables as shadow prices. Strong duality enables sensitivity analysis and forms the basis for advanced algorithms such as decomposition and column generation.

Theorem 2. *If the primal (or dual) has an optimal solution, so does the dual (or primal), and the optimal values are equal: $c^T x^* = b^T y^*$.*

Proof. (Proof via the simplex method or Farkas' Lemma; see advanced texts for full details.) □

B Complementary Slackness

Complementary slackness provides a set of conditions that must be satisfied at optimality for both the primal and dual problems. These conditions link the feasibility of the primal and dual solutions to their optimality.

Theorem 3 (Complementary Slackness)**.** *Let x^* and y^* be optimal solutions to the primal and dual. The complementary slackness conditions in matrix form are:*

$$(y^*)^T(Ax^* - b) = 0$$
$$(x^*)^T(c - A^Ty^*) = 0$$

These conditions provide necessary and sufficient conditions for optimality.

Proof. ($\Rightarrow$ Necessity) Suppose x^* and y^* are optimal solutions.
By weak duality:

$$b^Ty^* \le c^Tx^*$$

But since both are optimal:

$$c^Tx^* = b^Ty^*$$

Now define the slack vectors:

$$s = Ax^* - b \ge 0, \quad t = c - A^Ty^* \ge 0$$

We write:

$$c^Tx^* = x^{*T}A^Ty^* + x^{*T}t = y^{*T}Ax^* + x^{*T}t = y^{*T}b + y^{*T}s + x^{*T}t$$

Since $c^Tx^* = b^Ty^*$, it follows that:

$$y^{*T}s + x^{*T}t = 0$$

But $y^* \ge 0$, $s \ge 0$, and $x^* \ge 0$, $t \ge 0$. Hence, this sum being zero implies:

$$y_i^*s_i = 0 \quad \text{and} \quad x_j^*t_j = 0 \quad \text{for all } i, j$$

Thus, the complementary slackness conditions hold.
($\Leftarrow$ Sufficiency) Now suppose the complementary slackness conditions hold for feasible x^* and y^*. Then:

$$y^{*T}(Ax^* - b) = 0, \quad x^{*T}(c - A^Ty^*) = 0$$

As before, we use:

$$c^Tx^* = y^{*T}Ax^* + x^{*T}t = y^{*T}b + y^{*T}s + x^{*T}t = b^Ty^* + 0 + 0 = b^Ty^*$$

Since x^* and y^* are feasible and achieve equal objective values, both are optimal. □

Practice Exercise:

Practice Exercise: Complementary Slackness Given the primal LP:

$$\begin{aligned} \text{min} \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \ge 4 \\ & 3x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Find the dual problem and write the complementary slackness conditions at the optimal solution. Use a solver to find the optimal solutions for both the primal and dual problems, and check the complementary slackness conditions.

C Primal-Dual Relationships and Solution Recovery

The primal-dual relationship is a powerful aspect of linear programming that allows us to recover optimal solutions from each other. The primal and dual solutions are linked through the optimal simplex tableau, which provides a structured way to interpret the solutions and their relationships.

Recovering Solutions: Let B be the basis matrix corresponding to an optimal basic feasible solution x^*. Then the optimal dual solution is:

$$y^* = (B^T)^{-1} c_B$$

This corresponds to the *shadow prices*, i.e., the marginal value of relaxing each right-hand side constraint.
At optimality, by complementary slackness:

$$x^* \geq 0, \quad c - A^T y^* \geq 0, \quad (x^*)^T (c - A^T y^*) = 0$$

Conversely, if y^* is known and satisfies dual feasibility, one can reconstruct x^* using the active constraints and the structure of the tableau.

Reduced Costs from Dual Variables: The reduced cost of a (non-basic) primal variable x_j is given by:

$$\bar{c}_j = c_j - a_j^T y^*$$

where a_j is the j-th column of A, and y^* is the dual optimal solution. These reduced costs indicate how the objective value would change if the non-basic variable were introduced into the basis, and guide pivoting decisions in the simplex algorithm.
Note: This formula for reduced costs is fundamental in column generation algorithms. It determines whether introducing a new variable (column) can improve the objective, guiding the selection of columns to add to the restricted master problem.