Operational Research

Dr. Vincent A. L. Boyer

Exercise 1

Suppose that a carpenter makes tables and chairs and sells all the tables and chairs he makes in a market. He does not have a steady income and he wishes to optimize this situation. So, the carpenter needs to determine how many tables and chairs he should make in order to maximize his net income. He knows that the income he receives per table sold is \$500 and per chair is \$300. The carpenter works 8 hours a day from Monday to Friday and takes 2 hours to make a table and 1 hour to make a chair. Also, each week he receives 50 units of raw material, of which he requires 1 unit for each table and 2 units for each chair he makes.

Exercise 2

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and high-income men. In order to reach these groups, Doran Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercials spots on two types of program: comedy shows and football games. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men. A 1-minute comedy show ad costs \$50000, and 1-minute football ad costs \$100000. Dorian would like the commercials to be seen by at least 28 million high-income women and at least 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum cost.

Exercise 3

An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, 40 trucks per day could be painted. If the paint shop were only painting cars, 60 cars per day could be painted. If the body shop were only producing cars, it could process 50 cars per day. If the body shop were only producing trucks, it could process 50 trucks per day. Each truck contributes \$300 to profit, and each car contributes \$200 to profit. Use linear programming to find the daily production schedule that will maximize the company's profit.

Exercise 4

A farmer seeks to determine the amount of food to give to his chickens in order to satisfy the minimum nutritional standards. The diet should contains 4 types of ingredients:

- type A: at least 0.4kg;
- type B: at least 0.6kg;
- type C: at least 2kg;
- type D: at least 1.7kg.

Two kinds of foods are available on the market, type M and N with a respective cost of \$8/kg and \$4/kg. The composition of each type of food is given in the following table:

	A	B	C	D
M	100g.	-	100g.	200g.
N	-	100g.	200g.	100g.

The farmer would like to spend as less money as possible.

Exercise 5

The Whitt Window Company is a company with only three employees which makes two different kinds of hand-crafted windows: a wood-framed and an aluminum-framed window. They earn \$60 profit for each wood-framed window and \$30 profit for each aluminum-framed window. Doug makes the wood frames, and can make 6 per day. Linda makes the aluminum frames, and can make 4 per day. Bob forms and cuts the glass, and can make 48 square feet of glass per day. Each wood-framed window uses 6 square feet of glass and each aluminum-framed window uses 8 square feet of glass. The company wishes to determine how many windows of each type to produce per day to maximize total profit. Formulate a linear programming model for this problem.

Exercise 6

A farmer is wondering how many acres of corn and rice he should plant for this year. One acre of rice produces 25 brushels of rice and requires 10 hours of work per week. One acre of corn produces 10 brushels of corn and requires 4 hours of work per week. The selling price is \$4 per brushel of rice and \$3 per brushel of corn. 7 acres for farming and 40 hours of work per week are available. At least 30 brushels of corn should be produced this year. Model this problem as a linear programming problem, solve it with the graphical method and discuss about the results.

Exercise 7

Draw a graphic representation of the following problems and determine for each of them if the solution is unique, multiple, unbounded or infeasible:

a)
$$\begin{array}{rl} max & x_1 + x_2 \\ s.t. & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

b)
$$\begin{array}{rl} max & 4x_1 + x_2 \\ s.t. & 8x_1 + 2x_2 \leq 16 \\ & 5x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

c)
$$\begin{array}{rl} max & -x_1 + 3x_2 \\ s.t. & x_1 - x_2 \leq 4 \\ & x_1 + 2x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

d)
$$\begin{array}{rl} max & 30x_1 + 100x_2 \\ s.t. & x_1 + 3x_2 \leq 7 \\ & 4x_1 + 10x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{array}$$

Exercise 8

Use the graphical method to find all optimal solutions for the following model:

$$\begin{array}{rl} Max & 500x_1 + 300x_2 \\ s.t. & 15x_1 + 5x_2 \leq 300 \\ & 10x_1 + 6x_2 \leq 240 \\ & 8x_1 + 12x_2 \leq 450 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{rl} Max & 5x_1 + 7x_2 \\ s.t. & 2x_1 - x_2 \leq -1 \\ & -x_1 + 2x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{array}$$

Exercise 9

This is your lucky day. You have just won a \$10,000 prize. You are setting aside \$4,000 for taxes and partying expenses, but you have decided to invest the other \$6,000. Upon hearing this news, two different friends have offered you an opportunity to become a partner in two different entrepreneurial ventures, one planned by each friend. In both cases, this investment would involve expending some of your time next summer as well as putting up cash. Becoming a full partner in the first friend's venture would require an investment of \$5,000 and 400 hours, and your estimated profit (ignoring the value of your time) would be \$4,500. The corresponding figures for the second friend's venture are \$4,000 and 500 hours, with an estimated profit to you of \$4,500. However, both friends are flexible and would allow you to come in at any fraction of a full partnership you would like. If you choose a fraction of a full partnership, all the above figures given for a full partnership (money investment, time investment, and your profit) would be multiplied by this same fraction. Because you were looking for an interesting summer job anyway (maximum of 600 hours), you have decided to participate in one or both friends'ventures in whichever combination would maximize your total estimated profit. Formulate a linear programming model for this problem and solve it graphically.

Exercise 10

A manufacturer should produce 1200 laptops and 1800 desktops for a customer. Two different plants are available which can produce laptops and desktops. In one day, the plant 1 can produce 200 laptops and 200 desktops. In one day, the plant 2 can produce 100 laptops and 300 desktops. Model a problem as a linear problem in order to help this factory to deliver the order as soon as possible (i.e. in a minimum number of days). Solve it with a graphic representation and discuss about the solution.

Exercise 11

Write the following problems in standard form:

a)
$$\begin{array}{rl} min & -5x_1 - 4x_2 - 3x_3 \\ s.t. & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 8 \\ & 3x_1 + 4x_2 + 2x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

b)
$$\begin{array}{rl} max & 2x_1 - x_2 - 5x_3 \\ s.t. & x_1 + x_2 + x_3 \geq 6 \\ & 2x_1 - x_2 + 3x_3 \leq 5 \\ & x_1 + 2x_2 - 3x_3 \leq -10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

c)
$$\begin{array}{rl} min & -x_1 + x_2 + x_3 \\ s.t. & x_1 \geq 10 \\ & x_1 \leq 20 \\ & x_1 + x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

d)
$$\begin{array}{rl} max & 6x_1 + 8x_2 + 5x_3 \\ s.t. & 2x_1 + x_2 + 4x_3 \geq 10 \\ & x_1 + 5x_2 + 3x_3 \leq -9 \\ & x_1 - x_2 - 3x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Exercise 12

Solve the following problems with the Simplex:

a)
$$\begin{array}{rl} max\ z = & 2x_1 + 3x_2 \\ s.t. & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

b)
$$\begin{array}{rl} max\ z = & 2x_1 - x_2 + x_3 \\ s.t. & 3x_1 + x_2 + x_3 \leq 60 \\ & x_1 - x_2 + 2x_3 \leq 10 \\ & x_1 + x_2 - x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Exercise 13

A company manufactures two curtains models A and B, which generate a profit of \$8 and \$4 respectively. Each curtain of model A requires 4 hours of labor and 3 units of fabric. Each curtain of model B requires 3 hours and 5 units of fabric. The company has 48 hours and 60 units of fabric available. At most 9 curtains of model A must be produced. How many curtains of each model must be manufactured to maximize the profit? Propose a model for this problem and solve it with the graphical and the simplex method.

Exercise 14

The WorldLight Company produces two light fixtures (products 1 and 2) that require both metal frame parts and electrical components. Management wants to determine how many units of each product to produce so as to maximize profit. For each unit of product 1, 1 unit of frame parts and 2 units of electrical components are required. For each unit of product 2, 3 units of frame parts and 2 units of electrical components are required. The company has 200 units of frame parts and 300 units of electrical components. Each unit of product 1 gives a profit of \$1, and each unit of product 2, up to 60 units, gives a profit of \$2. Any excess over 60 units of product 2 brings no profit, so such an excess has been ruled out. Formulate a linear programming model for this problem and solve it with the graphical and the simplex method.

Exercise 15

Giapetto Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead cost by \$10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor.
Each week, Giapetto can obtain all the needed material but only 100 finishing hours and 80 carpentry hours are available. Demand for trains is unlimited, but at most 40 soldiers are sold each week. Giapetto wishes to maximize weekly profit (revenues - costs).
Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit. Solve it with the graphical approach and with the simplex algorithm. Discuss about the results.

Exercise 16

We have \$21,000 to invest in the stock market. Our friends have recommended two types of actions. The type A has an annual interest of 7% and Type B has an annual interest of 9%. We decided to invest a maximum of \$13,000 in type A and at least \$6,000 in the type B. We want our investment in the type B to be at most twice the investment in A. What has to be the distribution of investment to get the maximum annual interest?
Propose a model for this problem and solve it with the graphical and the simplex method.

Exercise 17

A diabetic has to take in his daily food two kinds of components, called A and B. He needs to take at least 70 units of A and 120 units of B. The doctor suggests two types of food compounds as follows:

- Food 1: 2 units of A and 3 units of B;
- Food 2: 1 units of A and 2 units of B.

If the price of Food 1 is \$25 and the one of Food 2 is \$14.5. What is the optimal distribution of food that provides the lowest cost?
Propose a model for this problem and solve it with the graphical and the simplex method.

Exercise 18

A small petroleum company owns two refineries. Refinery 1 costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 200 barrels of medium-grade oil, and 150 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$30,000 per day to operate, and it can produce 300 barrels of high-grade oil, 250 barrels of medium-grade oil, and 400 barrels of low-grade oil each day. The company has orders totaling 35,000 barrels of high-grade oil, 30,000 barrels of medium-grade oil, and 40,000 barrels of low-grade oil. How many days should the company run each refinery to minimize its costs and still meet its orders? Propose a model for this problem and solve it with the graphical and the simplex method.

Exercise 19

The Tefla Corporation manufactures tables and chairs. A table requires 1 hour of labor and 9 sq ft of wood, and a chair requires 1 hour of labor and 5 sq ft of wood. Currently, 6 hours of labor and 45 sq ft of wood are available. Each table contributes \$8 to profit, and each chair contributes \$5 to profit. Formulate and solve an IP to maximize Telfa's profit.

Exercise 20

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and high-income men. In order to reach these groups, Doran Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercials spots on two types of program: comedy shows and football games. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men. A 1-minute comedy show ad costs \$50000, and 1-minute football ad costs \$100000. Dorian would like the commercials to be seen by at least 28 million high-income women and at least 24 million high-income men. Formulate and solve an IP to minimize Dorian's cost.

Exercise 21

Use Branch-and-Bound to solve the following IPs:

a)
$$\begin{aligned} max \quad & 5x_1 + 2x_2 \\ s.t. \quad & 3x_1 + x_2 \leq 12 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \ integer \end{aligned}$$

b)
$$\begin{aligned} max \quad & 2x_1 + 3x_2 \\ s.t. \quad & x_1 + x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \ integer \end{aligned}$$

Exercise 22

Solve the following problems:

a)
$$\begin{aligned} Max \quad & x_1 + x_2 \\ s.t. \quad & 2x_1 + 5x_2 \leq 16 \\ & 6x_1 + 5x_2 \leq 27 \\ & x_1, x_2 \geq 0 \ and \ integer \end{aligned}$$

b)
$$\begin{aligned} Min \quad & 5x_1 + 4x_2 \\ s.t. \quad & 3x_1 + 2x_2 \geq 5 \\ & 2x_1 + 3x_2 \geq 7 \\ & x_1, x_2 \geq 0 \ and \ integer \end{aligned}$$

Exercise 23

Stocko is considering 4 investments. Investments 1 will yield a net present value (NPV) of \$16000; Investments 2, a NPV of \$22,000; Investments 3, a NPV of \$12000; and Investments 4, a NPV of \$8000. Each investment needs a certain cash outflow at the present time: investment 1, \$5000; investment 2, \$7000; investment 3, \$4000; investment 4, \$3000. At present, \$14000 is available for investment.

1) Formulate an IP whose solution will tell Stockco how to maximize the NPV obtained from investments 1-4.

2) Modify the Stockco formulation to account for each of the following constraints:

(a) Stockco can invest in at most 2 investments.

(b) If Stockco invests in investment 2, they must also invest in investment 1.

(c) If Stockco invests in investment 2, they cannot invest in investment 4.

Exercise 24

Gandhi Cloth Company is capable of manufacturing 3 types of clothing: shirts, shorts and pants. The manufacture of each type of clothing requires that Ghandhi has the appropriate type of machinery available. The machinery needed to manufacture each type of clothing must be rented at the following rates:

- shirt machinery, \$200 per week;
- shorts machinery, \$150 per week;

- pants machinery, \$100 per week.

The manufacture of each type of clothing also requires the amount of cloth and labor given in the table below. Each week 150 hours of labor and 160 sq yd of cloth are available. The variable unit cost and selling price for each type of clothing are shown in the following table. Formulate an IP whose solution will maximize Gandhi's weekly profits.

	LABOR (Hours)	CLOTH (Square Yards)	SALES PRICE	VARIABLE COST
Shirt	3	4	\$12	\$6
Shorts	2	3	\$8	\$4
Pants	6	4	\$15	\$8

Exercise 25

Dorian Auto is considering manufacturing 3 types of autos: compact, midsize, and large. The resources required for, and the profit yielded by, each type of car are shown in the table below. At present, 6000 tons of steel and 60000 hours of labor are available. In order for production of a type of car to be economically feasible, at least 1000 cars of that type must be produced. Formulate an IP to maximize Dorian's profit.

	COMPACT	MIDSIZE	LARGE
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit yielded	\$2000	\$3000	\$4000

Exercise 26

There are 6 cities in Kilroy Country. The country must determine where to build fire stations. The country wants to build the minimum number of stations needed to ensure that at least one fire station is within 15 minutes (driving time) from each city. The time (in minutes) required to drive between two cities in Kilroy Country are shown in the table below. Formulate an IP that will tell Kilroy how many fire stations should be build and where they should be located.

	TO					
FROM	City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2	10	0	25	35	20	10
City 3	20	25	0	15	30	20
City 4	30	35	15	0	15	25
City 5	30	20	30	15	0	14
City 6	20	10	20	25	14	0

Exercise 27

Powerco has three electric power plants that supply the power needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1, 35 million; plant 2, 50 million; plant 3, 40 million. The peak power demands in these cities which occur at the same time (2 p.m.), are as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million.
The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to minimize the cost of meeting each city's peak power demand.

	TO				
FROM	City 1	City 2	City 3	City 4	SUPPLY*
Plant 1	\$8	\$6	\$10	\$9	35
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	40
DEMAND*	45	20	30	30	

*Million kwh

Exercise 28

Coach Jeff is trying to choose the starting lineup for the basketball team. The team consists of seven players who have been rated (on a scale of 1=poor to 3=excellent) according to their ball-handling, shooting, rebounding, and defensive abilities. The position that each player is allowed to play and the player's abilities are listed in the table below.

The five-player starting lineup must satisfy the following restrictions:

1. At least 4 members of the starting team must be able to play guard, at least 2 members must be able to play forward, and at least 1 member of the starting team must be able to play center.
2. The average ball-handling, shooting, and rebounding level of the starting lineup must be at least 2.
3. If player 3 starts, the 6 cannot start.
4. If player 1 starts, the players 4 and 5 must both starts.
5. Either player 2 or player 3 must starts.

Given these constraints, coach Jeff wants to maximize the total defensive ability of the starting team. Formulate an IP that will help coach Jeff chooses his starting team.

PLAYER	POSITION	BALL-HANDLING	SHOOTING	REBOUNDING	DEFENCE
1	G	3	3	1	3
2	C	2	1	3	2
3	G-F	2	3	2	2
4	F-C	1	3	3	1
5	G-F	1	3	1	2
6	F-C	3	1	2	3
7	G-F	3	2	2	1

Exercise 29

A company supplies goods to three customers who require 30 units each one. The company has two warehouses. Warehouse 1 has 40 units available and warehouse 2 has 30 units available. The costs of shipping 1 unit from warehouses to customers are shown in the table below. There is a penalty for unmet demand. For each unit of customer 1's demand that is unmet, a penalty of $90 is incurred; for each unit of customer 2's unmet demand, a penalty of $80 is incurred; and for each unit of customer 3's unmet demand, a penalty of $110 is incurred.

(a) Formulate a balanced transportation problem to minimize the sum of shortage and shipping costs.

(b) Propose a feasible solution.

(c) Suppose that extra units could be purchased and shipped to either warehouse for a total cost of $100 per unit and that all customers demand must be met. Formulate a balanced transportation problem to minimize the sum of purchasing and shipping costs.

(e) Propose a feasible for this new model.

	TO		
FROM	Customer 1	Customer 2	Customer 3
Warehouse 1	$15	$35	$25
Warehouse 2	$10	$50	$40

Exercise 30

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons of water per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is $20; at city 2, the penalty is $22; and at city 3, the penalty is $23. The cost of transporting 1 million gallons of water from each reservoir to each city are shown in the table below.

FROM	TO City 1	City 2	City 3	SUPPLY*
Reservoir 1	\$8	\$6	\$10	50
Reservoir 2	\$9	\$12	\$13	50
DEMAND*	40	40	40	

*Million gallons per day

(a) Formulate an LP that can be used to minimize the sum of shortage and transportation costs.
(b) Find two feasible solutions by using the northwest corner and the minimum cost methods.

Exercise 31

Consider the following linear model and its corresponding optimal tableau:

$$
\begin{array}{llllllll}
Max\ z = & 4x_1 & + & x_2 & + & 5x_3 & & \\
s.t. & x_1 & + & x_2 & + & x_3 & \leq & 4 \\
 & 2x_1 & + & x_2 & + & 3x_3 & \leq & 10 \\
 & 3x_1 & + & x_2 & + & 4x_3 & \leq & 16 \\
 & x_1, x_2, x_3 \geq & & & & & &
\end{array}
$$

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	**rhs**
x_1	1	2	0	3	-1	0	2
x_3	0	-1	1	-2	1	0	2
s_3	0	-1	0	-1	-1	1	2
z	0	2	0	2	1	0	18

(a) What is the optimal solution of this problem?

(b) Analyze how the following discrete changes affect this optimal solution. For each of the changes, determine the optimal solution to the new linear model.

- rhs: $b = \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} \rightarrow b' = \begin{pmatrix} 5 \\ 10 \\ 16 \end{pmatrix}$
- rhs: $b = \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} \rightarrow b' = \begin{pmatrix} 4 \\ 10 \\ 18 \end{pmatrix}$
- objective: $c^t = \begin{pmatrix} 4 & 1 & 5 \end{pmatrix} \rightarrow c'^t = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}$
- objective: $c^t = \begin{pmatrix} 4 & 1 & 5 \end{pmatrix} \rightarrow c'^t = \begin{pmatrix} 4 & 1 & 7 \end{pmatrix}$