Solution Techniques

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Content

Agenda

Introduction

The Graphical Method is an excellent alternative for representing and solving LP models that have **two** decision variables. In the sequel, we will present a series of Linear Programming exercises that can be solved using the graphical method.

The Graphical Method also allows to visualize the optimization procedure explicitly and to understand the different terminologies associated with the solution of LP problem.

The Carpenter's Problem

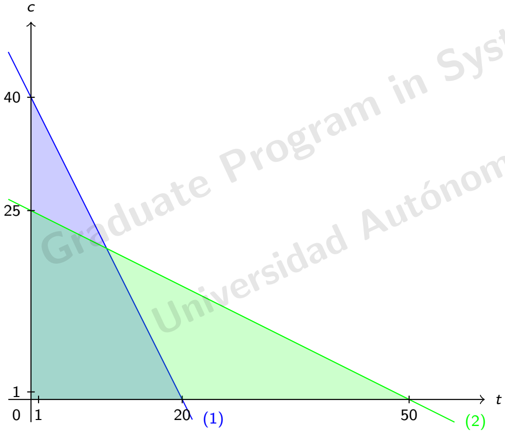
Suppose that a carpenter makes tables and chairs and sells all the tables and chairs he makes in a market. He does not have a steady income and he wishes to optimize this situation. So, the carpenter needs to determine how many tables and chairs he should make in order to maximize his net income. He knows that the income he receives per table sold is $500 and per chair is $300. The carpenter works 8 hours a day from Monday to Friday and takes 2 hours to make a table and 1 hour to make a chair. Also, each week he receives 50 units of raw material, of which he requires 1 unit for each table and 2 units for each chair he makes.

The model:
$$\begin{cases} Max & 500t & + & 300c & & \\ s.t. & 2t & + & c & \leq & 40 \\ & t & + & 2c & \leq & 50 \\ & t \geq 0, & c \geq 0 & & & \end{cases}$$

The Carpenter's Problem

Step 1

Plot one by one all constraints on a graph.

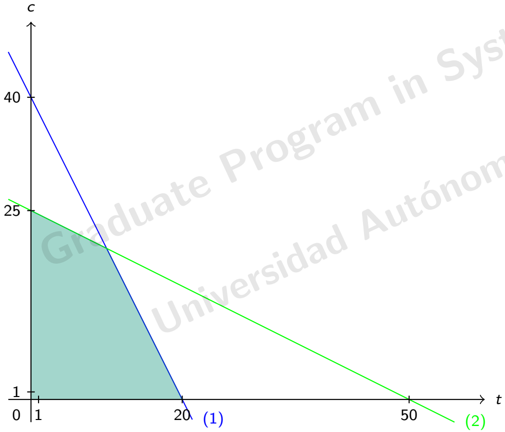


$$\begin{array}{llcll} (1) & 2t & + & c & \leq 40 \\ (2) & t & + & 2c & \leq 50 \\ & & t \geq 0, & c \geq 0 & \end{array}$$

The Carpenter's Problem

Step 2

Identify the **feasible region**.

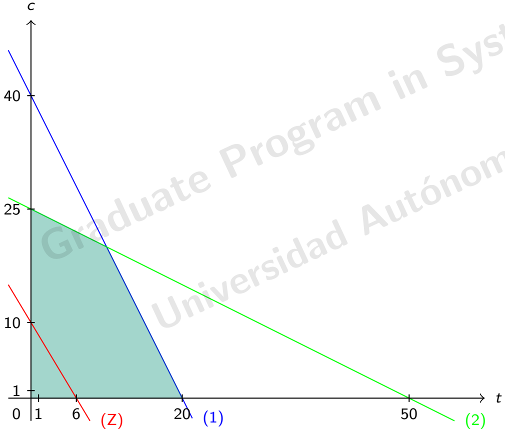


$$
\begin{array}{lrcrcr}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & & t \geq 0, & c \geq 0 & &
\end{array}
$$

The Carpenter's Problem

Step 3

Plot the objective function (**Z line**) with an arbitrary value of z.



$$
\begin{array}{lrcrcl}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & \multicolumn{5}{l}{t \geq 0,\ c \geq 0}
\end{array}
$$

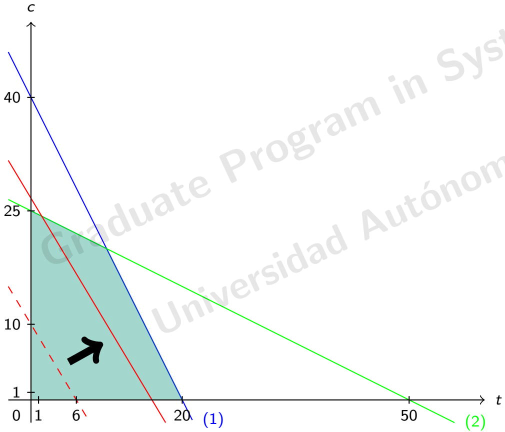
$$
(Z) \quad 500t + 300c = z
$$

$$
(z = 3000)
$$

The Carpenter's Problem

Step 4

Find the **Optimal Point** by moving the Z line.



$$\begin{array}{lcccccc} (1) & 2t & + & c & \leq & 40 \\ (2) & t & + & 2c & \leq & 50 \\ & & t \geq 0, & c \geq 0 & & \end{array}$$

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The Carpenter's Problem

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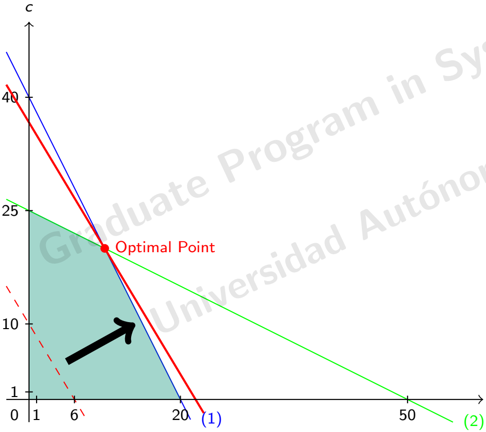

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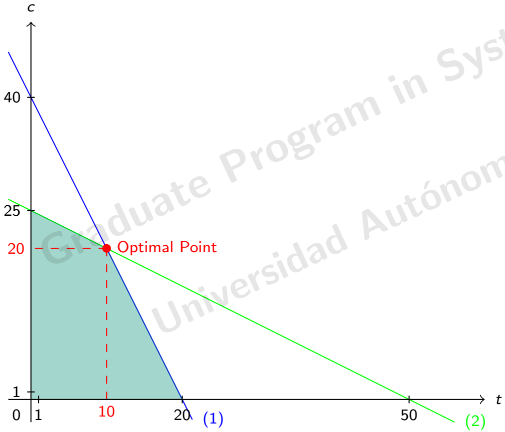
\begin{array}{lcccccc}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & & & t \geq 0, & c \geq 0 & &
\end{array}
$$

$$(Z) \quad 500t + 300c = z$$

The Carpenter's Problem

Step 5

Get the coordinates of the Optimal Point.



$$
\begin{array}{lrcrcr}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
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\end{array}
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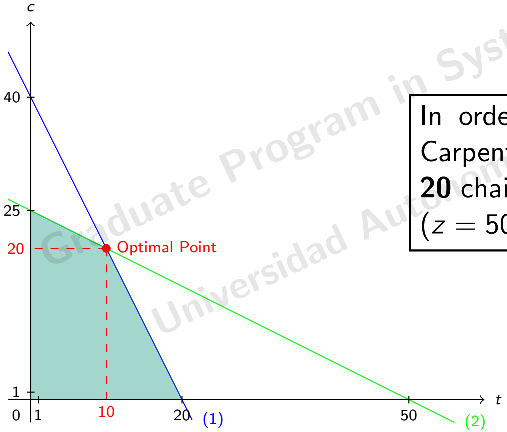
$$
(Z) \quad 500t + 300c = z
$$

The Carpenter's Problem

Step 6

Get the solution of the problem and conclude.

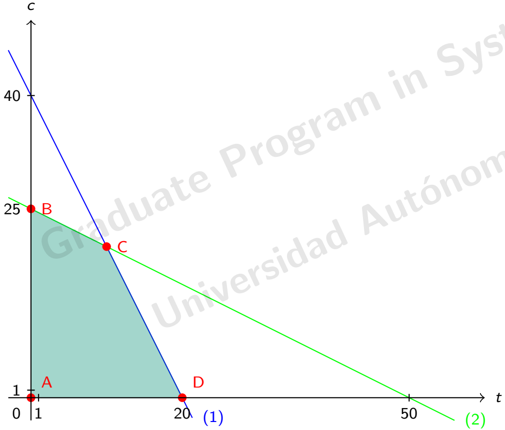


In order to optimize his profit, the Carpenter should make **10** tables and **20** chairs for a total profit of **$11000** ($z = 500 \times 10 + 300 \times 20 = 11000$)

The Carpenter's Problem

Step 3 (alternative)

Identify the **Extreme Points**.

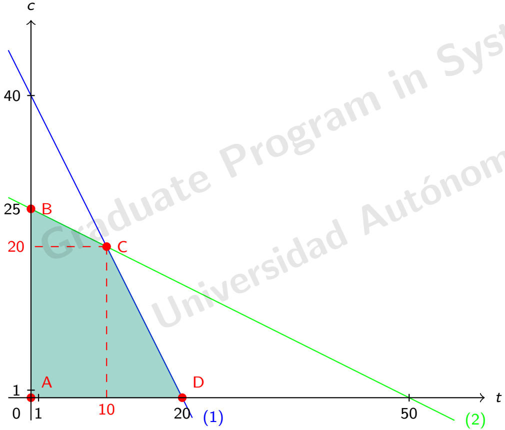


$$\begin{array}{lcccl} (1) & 2t & + & c & \leq 40 \\ (2) & t & + & 2c & \leq 50 \\ & & t \geq 0, & c \geq 0 & \end{array}$$

The Carpenter's Problem

Step 4 (alternative)

Get the coordinate of all **Extreme Points** and compute the value of z.



$$
\begin{array}{lcccccc}
(Z) & 500t & + & 300c & = & z \\
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
\end{array}
$$

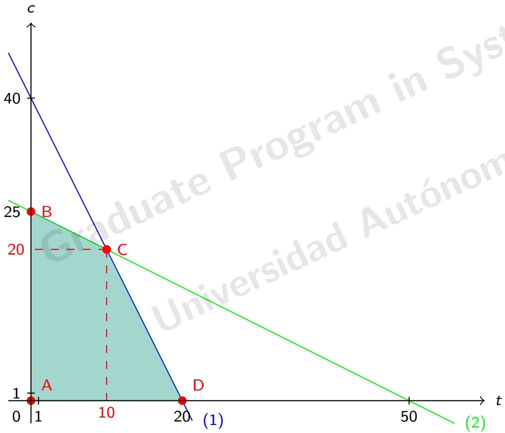
$$t \geq 0, \ c \geq 0$$

	t	c	z
A	0	0	0
B	0	25	7500
C	10	20	11000
D	20	0	10000

The Carpenter's Problem

Step 5 (alternative)

Identify the **Optimal Point**.



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\begin{array}{lccccc}
(Z) & 500t & + & 300c & = & z \\
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	t	c	z
A	0	0	0
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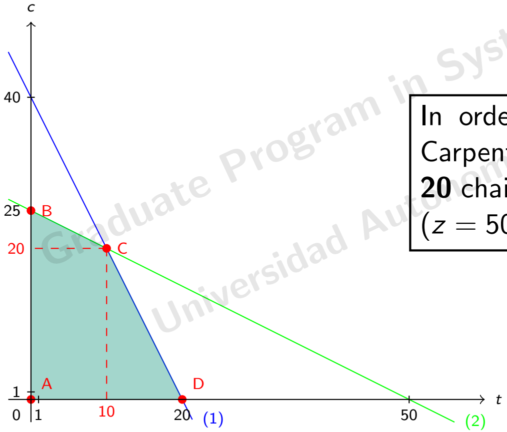
The Carpenter's Problem

Step 6

Get the solution of the problem and conclude.



In order to optimize his profit, the Carpenter should make **10** tables and **20** chairs for a total profit of **$11000** ($z = 500 \times 10 + 300 \times 20 = 11000$)

The Dorian's Problem

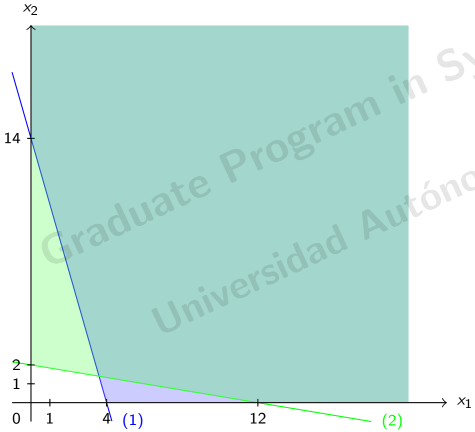
Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and high-income men. In order to reach these groups, Doran Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercials spots on two types of program: comedy shows and football games. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men. A 1-minute comedy show ad costs \$50000, and 1-minute football ad costs \$100000. Dorian would like the commercials to be seen by at least 28 million high-income women and at least 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum cost.

The model: $$\begin{cases} Min & 50x_1 & + & 100x_2 & & \\ s.t. & 7x_1 & + & 2x_2 & \geq & 28 \\ & 2x_1 & + & 12x_2 & \geq & 24 \\ & x_1 \geq 0, & x_2 \geq 0 & & & \end{cases}$$

The Dorian's Problem

Step 1

Plot one by one all constraints on a graph.

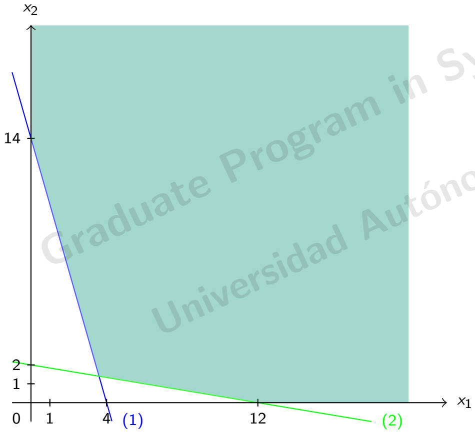


$$
\begin{array}{llcrcr}
(1) & 7x_1 & + & 2x_2 & \geq & 28 \\
(2) & 2x_1 & + & 12x_2 & \geq & 24 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$

The Dorian's Problem

Step 2

Identify the **feasible region**.



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\begin{array}{lrcrcr}
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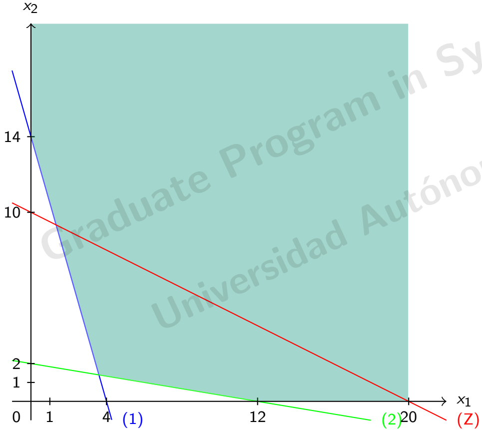
The Dorian's Problem

Step 3

Plot the objective function (**Z line**) with an arbitrary value of z.



$$\begin{array}{llllll} (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

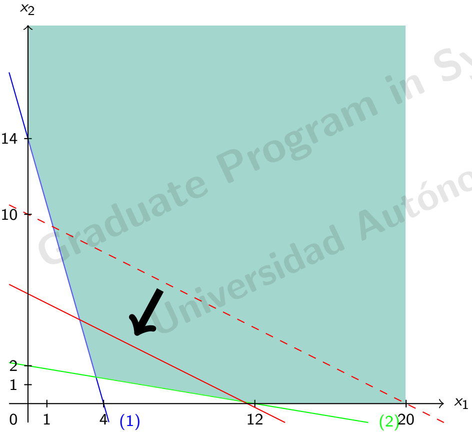
$$\begin{array}{llllll} (Z) & 50x_1 & + & 100x_2 & = & z \end{array}$$

$$(z = 1000)$$

The Dorian's Problem

Step 4

Find the **Optimal Point** by moving the Z line.



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\begin{array}{llcrcr}
(1) & 7x_1 & + & 2x_2 & \geq & 28 \\
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$$

The Dorian's Problem

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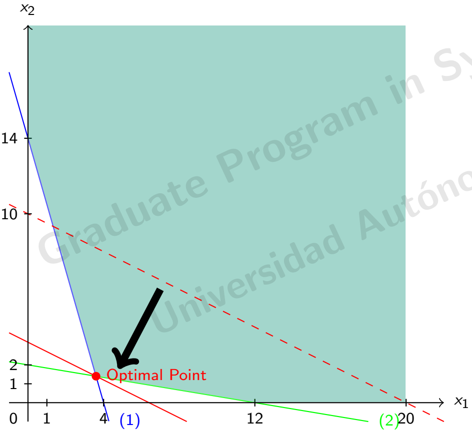

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\begin{array}{lcccccc}
(1) & 7x_1 & + & 2x_2 & \geq & 28 \\
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(Z) & 50x_1 & + & 100x_2 & = & z
\end{array}
$$

The Dorian's Problem

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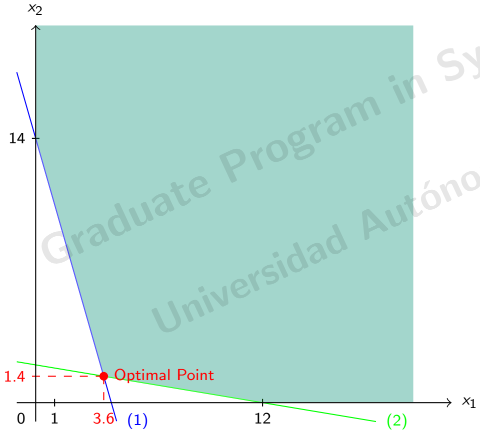
Get the coordinates of the Optimal Point.



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\begin{array}{lrcrcl}
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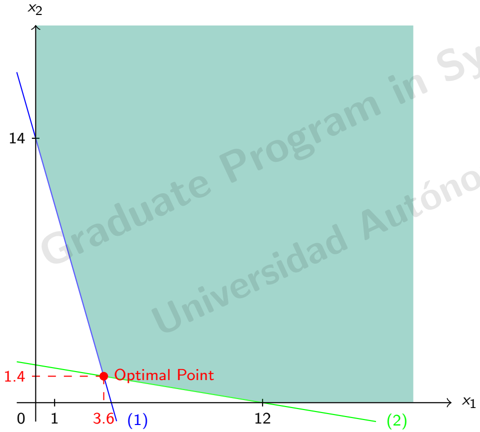
\end{array}
$$

$$
(Z) \quad 50x_1 + 100x_2 = z
$$

The Dorian's Problem

Step 6

Get the solution of the problem and conclude.

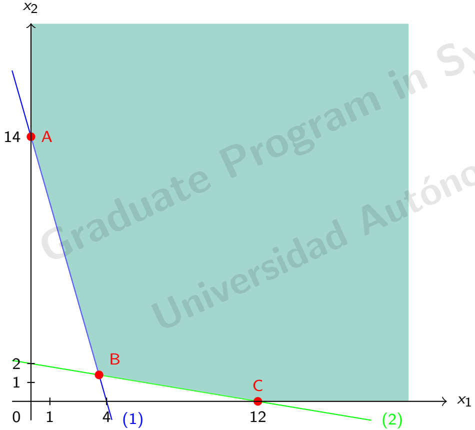


In order to minimize the costs, Dorian should purchase **3.6** ads on comedy shows and **1.4** ads on footbal games for a total cost of **$320,000** ($z = 50 \times 3.6 + 100 \times 1.4 = 320$)

The Dorian's Problem

Step 3 (alternative)

Identify the **Extreme Points**.



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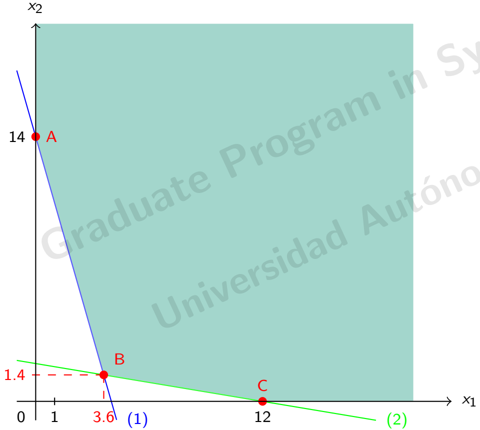
The Dorian's Problem

Step 4 (alternative)

Get the coordinates of all **Extreme Points** and compute the value of z.



$$
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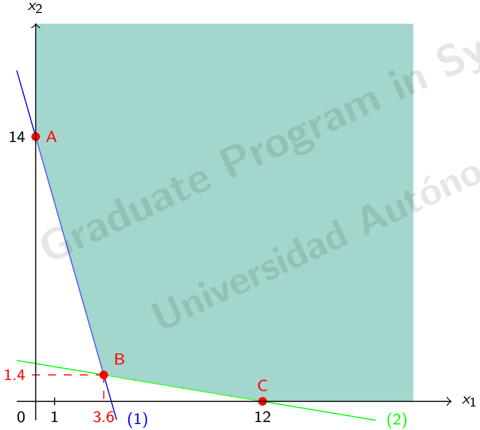
(1) & 7x_1 & + & 2x_2 & \geq & 28 \\
(2) & 2x_1 & + & 12x_2 & \geq & 24 \\
 & x_1 \geq 0, & x_2 \geq 0 & & &
\end{array}
$$

	x_1	x_2	z
A	0	14	1400
B	3.6	1.4	320
C	12	0	600

The Dorian's Problem

Step 5 (alternative)

Identify the **Optimal Point**.



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\begin{array}{lcccccc}
(Z) & 50x_1 & + & 100x_2 & = & z \\
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	x_1	x_2	z
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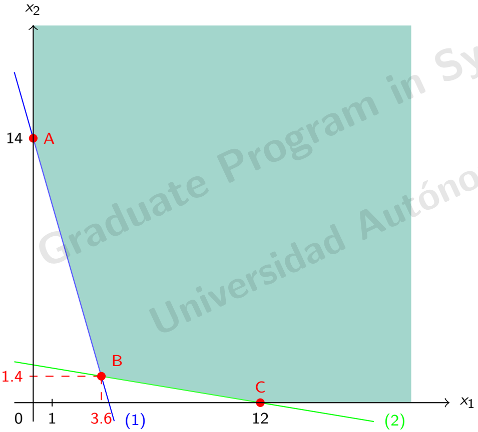
The Dorian's Problem

Step 6

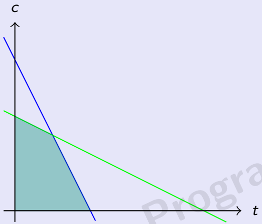
Get the solution of the problem and conclude.



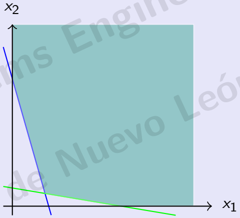
In order to minimize the costs, Dorian should purchase **3.6** ads on comedy shows and **1.4** ads on football games for a total cost of **\$320,000** ($z = 50 \times 3.6 + 100 \times 1.4 = 320$)

Types of Solution

Feasible solution: it satisfies all constraints for an LP.



The Carpenter's Problem



The Dorian's Problem

Optimal solution: it is the best of the feasible solutions for an LP.

Infeasible solution: it does not satisfy at least one constraint of the LP

Types of Solution

Feasible solution: it satisfies all constraints for an LP.

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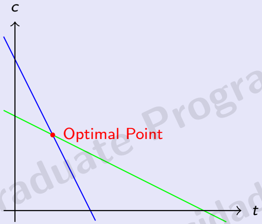
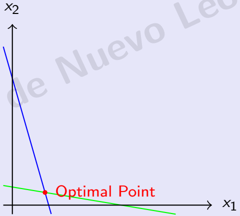


The Carpenter's Problem



The Dorian's Problem

Infeasible solution: it does not satisfy at least one constraint of the LP

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Feasible solution: it satisfies all constraints for an LP.

Optimal solution: it is the best of the feasible solutions for an LP.

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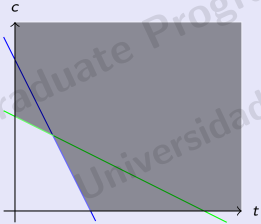

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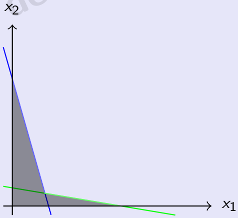


The Dorian's Problem

Types of LPs

Some LPs have a unique optimal solution.

Some LPs have multiple optimal solutions.

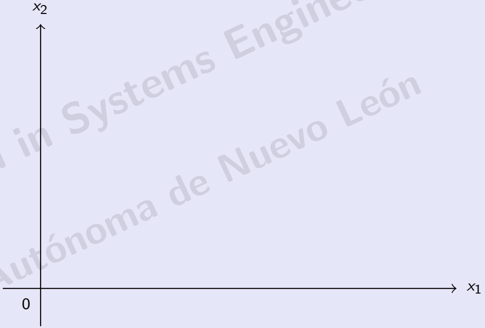
Some LPs have no feasible solutions (infeasible LPs).

Some LPs are unbounded.

Types of LPs

Some LPs have a unique optimal solution.

$$
\begin{array}{lllllll}
Max & 3x_1 & + & 1x_2 & & \\
s.t. & x_1 & + & 3x_2 & \leq & 9 \\
 & 5x_1 & + & 2x_2 & \leq & 12 \\
 & \multicolumn{5}{l}{x_1 \geq 0, \ x_2 \geq 0}
\end{array}
$$



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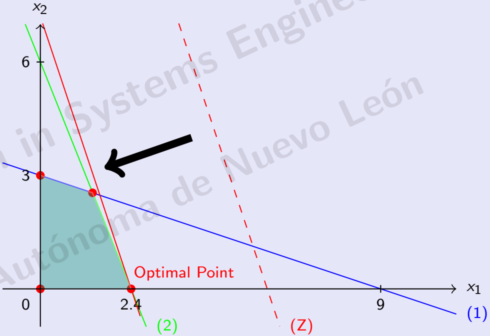

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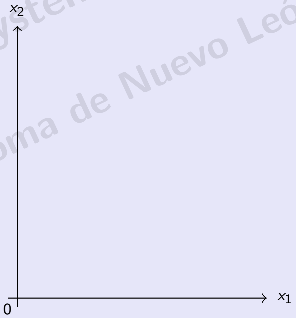
$$
\begin{array}{llcllll}
Max & 3x_1 & + & 2x_2 & & \\
s.t. & \frac{1}{4}x_1 & + & \frac{1}{6}x_2 & \leq & 9 \\
 & x_1 & + & x_2 & \leq & 50 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$



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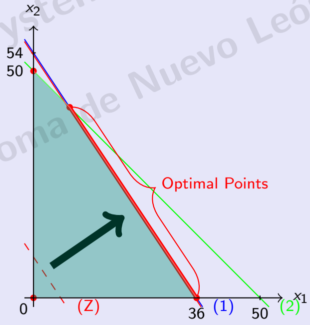

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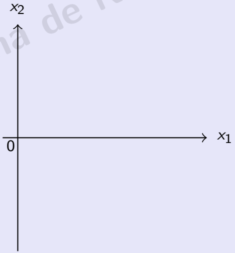
$$
\begin{array}{llllll}
Max & 5x_1 & + & 2x_2 & & \\
s.t. & x_1 & - & x_2 & \geq & 10 \\
 & 2x_1 & + & x_2 & \leq & 5 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$



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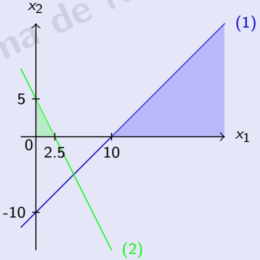

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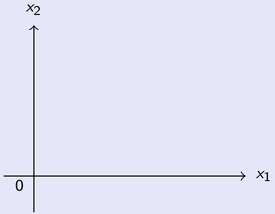
$$
\begin{array}{llcll}
Min & 2x_1 & - & x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & x_1 & + & 2x_2 & \geq & 6 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$



Types of LPs

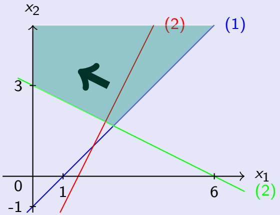
Some LPs have a unique optimal solution.

Some LPs have multiple optimal solutions.

Some LPs have no feasible solutions (infeasible LPs)

Some LPs are unbounded.

$$
\begin{array}{llcllll}
Min & 2x_1 & - & x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & x_1 & + & 2x_2 & \geq & 6 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$



Agenda

General steps of the Simplex method

Step 1. Convert the LP to standard form

Step 2. Build the initial tableau

Step 3. Determine whether the current feasible solution is optimal

Step 4. Identify the entering and the leaving variables

Step 5. Do the pivoting operation, then go to Step 3.

Step 1. Convert the LP to standard form

Original problem

$$
\begin{array}{llcllcl}
Max\ z = & 500t & + & 300c & & & \\
s.t. & 2t & + & c & \leq & 40 \\
 & t & + & 2c & \leq & 50 \\
 & \multicolumn{5}{l}{t \geq 0,\ c \geq 0}
\end{array}
$$

Standard form

All constraints are equations and the rhs and all variables are nonnegative.

We add for each constraint $\leq$ a slack variable s_i.

We subtract for each constraint $\geq$ a surplus (excess) variable e_i.

$$
\begin{array}{llclclclcl}
Max\ z = & 500t & + & 300c & & & & & & \\
s.t. & 2t & + & c & + & s_1 & & & = & 40 \\
 & t & + & 2c & & & + & s_2 & = & 50 \\
 & \multicolumn{9}{l}{t \geq 0,\ c \geq 0,\ s_1 \geq 0,\ s_2 \geq 0}
\end{array}
$$

Step 2. Build the initial tableau

Consider the standard form

$$
\begin{array}{llllll}
Max\ z = & 500t & + & 300c & & & & \\
s.t. & 2t & + & c & + & s_1 & & = 40 \\
 & t & + & 2c & & & + & s_2 = 50 \\
 & t \geq 0, & c \geq 0, & s_1 \geq 0, & s_2 \geq 0 & & &
\end{array}
$$

Convert the objective function to the row 0 format: $z - 500t - 300c = 0$

Initial tableau

B.V.	**t**	**c**	$\mathbf{s_1}$	$\mathbf{s_2}$	**rhs**
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

Step 3. Determine whether the current feasible solution is optimal

Current tableau

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

Notice that the current feasible solution is: $t = 0$, $c = 0$, $s_1 = 40$, $s_2 = 50$ and the value of the objective function is $z = 0$.

In case of maximization: If any coefficient in row z is negative, the current feasible solution is not optimal.

In case of minimization: If any coefficient in row z is positive, the current feasible solution is not optimal.

Step 4. Identify the entering and the leaving variables

If we maximize, the entering variable is the one with the smallest (negative) value in row z.
If we minimize, the entering variable is the one with the highest positive value in row z.

Current tableau

B.V.	**t**	**c**	$\mathbf{s_1}$	$\mathbf{s_2}$	**rhs**
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

↑
Input

In this case, the entering variable is t, with coefficient -500.

Step 4. Identify the entering and the leaving variables

The leaving variable is the winner of the ratio test. Hence, it is the one with the minimum value of $\dfrac{\text{rhs of row}}{\text{coefficient of entering variable in row}}$

Choosing the leaving variable

B.V.	t	c	s_1	s_2	rhs	ratio	
s_1	2	1	1	0	40	$\frac{40}{2}$	$\rightarrow$ *Output*
s_2	1	2	0	1	50	$\frac{50}{1}$	
z	-500	-300	0	0	**0**		

$\uparrow$

Input

The denominator in the ratio should be strictly positive (> 0). Otherwise, you cannot compute it.

Step 5. Do the pivoting operation

5.1 Update the row of the entering variable

Replace basic variable s_1 by t and divide each element of the row by the pivote of the row (in this case, divide by 2).

B.V.	**t**	**c**	$\mathbf{s_1}$	$\mathbf{s_2}$	**rhs**
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

$\rightarrow$

B.V.	**t**	**c**	$\mathbf{s_1}$	$\mathbf{s_2}$	**rhs**
t	1	$1/2$	$1/2$	0	20
s_2					
z					

Step 5. Do the pivoting operation

5.1 Update the remaining rows

Use the row of the entering variable to update the remaining rows.
To update row s_2 we must do the following:

$$[\text{Old row } s_2] - (\text{pivot in Old row } s_2) \times [\text{new pivot row } t]$$

$$[1\ 2\ 0\ 1\ 50] - 1 \times [1\ \frac{1}{2}\ \frac{1}{2}\ 0\ 20] = [0\ \frac{3}{2}\ -\frac{1}{2}\ 1\ 30]$$

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	−500	−300	0	0	**0**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	1/2	1/2	0	20
s_2	0	3/2	−1/2	1	30
z					

Step 5. Do the pivoting operation

5.1 Update the remaining rows

Use the row of the entering variable to update the remaining rows.
To update row z we must do the following:

$$[\text{Old row } z] - (\text{pivot in Old row } z) \times [\text{new pivot row } t]$$

$$[-500\ \ -300\ 0\ 0\ 0] - (-500) \times [1\ \frac{1}{2}\ \frac{1}{2}\ 0\ 20] = [0\ \ -50\ 250\ 0\ 10,000]$$

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	−500	−300	0	0	**0**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	1/2	1/2	0	20
s_2	0	3/2	−1/2	1	30
z	0	−50	250	0	**10,000**

Step 3. Determine whether the current feasible solution is optimal

Current tableau

B.V.	**t**	**c**	$\mathbf{s_1}$	$\mathbf{s_2}$	**rhs**
t	1	1/2	1/2	0	20
s_2	0	3/2	$-1/2$	1	30
z	0	-50	250	0	$\mathbf{10,000}$

In case of maximization: If any coefficient in row z is negative, the current feasible solution is not optimal.

In case of minimization: If any coefficient in row z is positive, the current feasible solution is not optimal.

Step 4. Identify the entering and the leaving variables

If we maximize, the entering variable is the one with the smallest (negative) value in row z.

If we minimize, the entering variable is the one with the highest positive value in row z.

Current tableau

B.V.	**t**	**c**	**s₁**	**s₂**	**rhs**
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	**10,000**

↑ *Input*

In this case, the entering variable is c, with coefficient -50.

Step 4. Identify the entering and the leaving variables

The leaving variable is the winner of the ratio test. Hence, it is the one with the minimum value of $\dfrac{\text{rhs of row}}{\text{coefficient of entering variable in row}}$

Choosing the leaving variable

B.V.	**t**	**c**	**s_1**	**s_2**	**rhs**	**ratio**	
t	1	1/2	1/2	0	20	$20/\frac{1}{2}$	
s_2	0	3/2	−1/2	1	30	$30/\frac{3}{2}$	$\rightarrow$ *Output*
z	0	−50	250	0	**10,000**		
		$\uparrow$ *Input*					

The denominator in the ratio should be strictly positive (> 0). Otherwise, you cannot compute it.

Step 5. Do the pivoting operation

5.1 Update the row of the entering variable

Replace basic variable s_2 by c and divide each element of the row by the pivote of the row (in this case, divide by $3/2$).

B.V.	**t**	**c**	**s_1**	**s_2**	**rhs**
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	$\mathbf{10,000}$

$\rightarrow$

B.V.	**t**	**c**	**s_1**	**s_2**	**rhs**
t					
c	0	1	$-1/3$	$2/3$	20
z					

Step 5. Do the pivoting operation

5.1 Update the remaining rows

Use the row of the entering variable to update the remaining rows.
To update row t we must do the following:

$$[\text{Old row } t] - (\text{pivot in Old row } t) \times [\text{new pivot row } c]$$

$$[1 \ \tfrac{1}{2} \ \tfrac{1}{2} \ 0 \ 20] - \frac{1}{2} \times [0 \ 1 \ -\tfrac{1}{3} \ \tfrac{2}{3} \ 20] = [1 \ 0 \ \tfrac{2}{3} \ -\tfrac{1}{3} \ 10]$$

B.V.	t	c	s_1	s_2	rhs
t	1	1/2	1/2	0	20
s_2	0	3/2	−1/2	1	30
z	0	−50	250	0	**10,000**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	0	2/3	−1/3	10
c	0	1	−1/3	2/3	20
z					

Step 5. Do the pivoting operation

5.1 Update the remaining rows

Use the row of the entering variable to update the remaining rows.

To update row z we must do the following:

$$[\text{Old row } z] - (\text{pivot in Old row } z) \times [\text{new pivot row } c]$$

$$[0\ \ -50\ 250\ 0\ 10,000] - (-50) \times [0\ 1\ \ -\tfrac{1}{3}\ \tfrac{2}{3}\ 20] = [0\ 0\ \tfrac{700}{3}\ \tfrac{100}{3}\ 11,000]$$

B.V.	t	c	s_1	s_2	rhs
t	1	1/2	1/2	0	20
s_2	0	3/2	$-1/2$	1	30
z	0	-50	250	0	**10,000**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	0	2/3	$-1/3$	10
c	0	1	$-1/3$	2/3	20
z	0	0	$\frac{700}{3}$	$\frac{100}{3}$	11,000

Step 3. Determine whether the current feasible solution is optimal

In case of maximization: If any coefficient in row z is negative, the current feasible solution is not optimal.

Current tableau

B.V.	t	c	s_1	s_2	rhs
t	1	0	$2/3$	$-1/3$	10
c	0	1	$-1/3$	$2/3$	20
z	0	0	$\frac{700}{3}$	$\frac{100}{3}$	$11,000$

Then, we have finished: The optimal solution has been found and the carpenter must produce 10 tables and 20 chairs to obtain a maximum profit equals to $11,000.

Agenda

General Steps of the Two-Phase Simplex method

Step 1. Convert the LP to standard form and add the artifical variables

Step 2. Replace the original objective function by $Min\ w =$ sum of the all artificial variables

Step 3. Solve the new optimization problem by Simplex (Phase I)

Step 4. Consider the original objective function

Step 5. Build the initial tableau by removing all nonbasic artificial variables from the last tableau in Phase I.

Step 6. Solve the updated problem with Simplex (Phase II)

The Dorian's optimization problem

The optimization model

$$
\begin{cases}
Min\ z = & 50x_1 & + & 100x_2 & & \\
s.t. & 7x_1 & + & 2x_2 & \geq & 28 \\
 & 2x_1 & + & 12x_2 & \geq & 24 \\
 & x_1 \geq 0, & x_2 \geq 0 & & &
\end{cases}
$$

Step 1. Convert the LP to standard form

Original problem

$$
\begin{array}{lrcrcl}
Min\ z = & 50x_1 & + & 100x_2 & & \\
s.t. & 7x_1 & + & 2x_2 & \geq & 28 \\
 & 2x_1 & + & 12x_2 & \geq & 24 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$

Standard form with artificial variables

We subtract for each constraint i of type $\geq$ a surplus (excess) variable e_i. If constraint i is an equality or $\geq$ constraint, we add an artificial variable a_i to constraint i.

$$
\begin{array}{lrcrcccccccl}
Min\ z = & 50x_1 & + & 100x_2 & & & & & & & & \\
s.t. & 7x_1 & + & 2x_2 & - & e_1 & & & + & a_1 & = & 28 \\
 & 2x_1 & + & 12x_2 & & & - & e_2 & + & a_2 & = & 24 \\
 & \multicolumn{11}{l}{x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0,\ a_1 \geq 0,\ a_2 \geq 0}
\end{array}
$$

Step 2. Replace the original objective function by $Min\ w =$ sum of the all artificial variables

As we added artificial variables a_1 and a_2, the new objective function is

$$Min\ w = a_1 + a_2$$

.

Updated problem

$$\begin{array}{llllllllllll}
Min\ w = & a_1 & + & a_2 & & & & & & & & \\
s.t. & 7x_1 & + & 2x_2 & - & e_1 & & & + & a_1 & & = 28 \\
 & 2x_1 & + & 12x_2 & & & - & e_2 & & & + a_2 & = 24 \\
 & \multicolumn{11}{l}{x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0,\ a_1 \geq 0,\ a_2 \geq 0}
\end{array}$$

Step 3. Solve the problem with Simplex (Phase I)

Consider the problem

$$\begin{array}{llllllllll} Min\ w = & a_1 & + & a_2 & & & & & & \\ s.t. & 7x_1 & + & 2x_2 & - e_1 & & + a_1 & & = & 28 \\ & 2x_1 & + & 12x_2 & & - e_2 & & + a_2 & = & 24 \\ & \multicolumn{9}{l}{x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0,\ a_1 \geq 0,\ a_2 \geq 0} \end{array}$$

Convert the objective function to the row 0 format: $w - a_1 - a_2 = 0$

Initial tableau

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**
a_1	7	2	−1	0	1	0	28
a_2	2	12	0	−1	0	1	24
w	0	0	0	0	−1	−1	**0**

Step 3. Solve the problem with Simplex (Phase I)

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	7	2	−1	0	1	0	28
a_2	2	12	0	−1	0	1	24
w	0	0	0	0	−1	−1	**0**

Notice that a_1 and a_2 are basic variables, so the coefficient of these variables in row w must be equals 0.

Update w in the initial tableau

$$
\begin{array}{rrrrrrrrl}
 & 7 & 2 & -1 & 0 & 1 & 0 & 28 & (Row\ a_1) \\
+ & 2 & 12 & 0 & -1 & 0 & 1 & 24 & (Row\ a_2) \\
 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & (Row\ w) \\
\hline
w = & 9 & 14 & -1 & -1 & 0 & 0 & 52 &
\end{array}
$$

Step 3. Solve the new optimization problem by Simplex (Phase I)

Updated Initial tableau

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	7	2	-1	0	1	0	28
a_2	2	12	0	-1	0	1	24
w	9	14	-1	-1	0	0	52

As you can observe, the coefficients of all basic variables in row w are equal to zero. The pivoting can start!

Step 3. Solve the new optimization problem by Simplex (Phase I)

Identify the entering and leaving variables.

If we maximize, the entering variable is the one with the smallest (negative) value in row w.

If we minimize, the entering variable is the one with the highest positive value in row w.

Choosing the entering and leaving variables

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**	*ratio*		
a_1	7	2	-1	0	1	0	28	$\frac{28}{2}$		
a_2	2	12	0	-1	0	1	24	$\frac{24}{12}$	$\rightarrow$	*Output*
w	9	14	-1	-1	0	0	52			
		$\uparrow$								
		Input								

Do the pivoting operation

Update the row of the entering variable

Replace basic variable a_2 by x_2 and divide each element of the row by the pivote of the row (in this case, divide by 12).

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**
a_1	7	2	-1	0	1	0	28
a_2	2	12	0	-1	0	1	24
w	9	14	-1	-1	0	0	52

$\rightarrow$

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**
a_1							
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2
w							

Do the pivoting operation

Update the other rows

After updating rows a_1 and w.

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	7	2	-1	0	1	0	28
a_2	2	12	0	-1	0	1	24
w	9	14	-1	-1	0	0	52

$\rightarrow$

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	$20/3$	0	-1	$1/6$	1	$-1/6$	24
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2
w	$20/3$	0	-1	$1/6$	0	$-7/6$	24

Choosing the entering and leaving variables

B.V.	$\boxed{\mathbf{x_1}}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**	*ratio*		
a_1	$20/3$	0	-1	$1/6$	1	$-1/6$	24	$24/\frac{20}{3}$	$\rightarrow$	*Output*
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2	$2/\frac{1}{6}$		
w	$20/3$	0	-1	$1/6$	0	$-7/6$	24			

$\uparrow$

Input

Do the pivoting operation

Update the row of the entering variable

Replace basic variable a_1 by x_1 and divide each element of the row by the pivote of the row (in this case, divide by $\frac{20}{3}$).

B.V.	x1	x2	e1	e2	a1	a2	rhs
a_1	20/3	0	−1	1/6	1	−1/6	24
x_2	1/6	1	0	−1/12	0	1/12	2
w	20/3	0	−1	1/6	0	−7/6	24

→

B.V.	x1	x2	e1	e2	a1	a2	rhs
x_1	1	0	−3/20	1/40	3/20	−1/40	18/5
x_2							
w							

Do the pivoting operation

Update the other rows

After updating rows x_1 and w.

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	$20/3$	0	-1	$1/6$	1	$-1/6$	24
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2
w	$20/3$	0	-1	$1/6$	0	$-7/6$	24

$\rightarrow$

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
x_1	1	0	$-3/20$	$1/40$	$3/20$	$-1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$-1/40$	$7/80$	$7/5$
w	0	0	0	0	-1	-1	0

Do the pivoting operation

Update the other rows

After updating rows x_1 and w.

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
a_1	$20/3$	0	-1	$1/6$	1	$-1/6$	24
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2
w	$20/3$	0	-1	$1/6$	0	$-7/6$	24

$\rightarrow$

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
x_1	1	0	$-3/20$	$1/40$	$3/20$	$-1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$-1/40$	$7/80$	$7/5$
w	0	0	0	0	-1	-1	0

Notice that the artificial variables have left the basis and w has reached its optimal value ($w = 0$). Hence, the Phase I of the method finishes here.

Phase II: Simplex Method

Step 4. Consider the original objective function

$$Min\ z = 50x_1 + 100x_2$$

Step 5. Build the initial tableau by removing all nonbasic artificial variables from the last tableau in Phase I.

B.V.	**x_1**	**x_2**	**e_1**	**e_2**	**a_1**	**a_2**	**rhs**
x_1	1	0	−3/20	1/40	3/20	−1/40	18/5
x_2	0	1	1/40	−7/80	−1/40	7/80	7/5
w	0	0	0	0	−1	−1	0

$\rightarrow$

B.V.	**x_1**	**x_2**	**e_1**	**e_2**	**rhs**
x_1	1	0	−3/20	1/40	18/5
x_2	0	1	1/40	−7/80	7/5
z	−50	−100	0	0	0

Step 6. Solve the updated problem with Simplex (Phase II)

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	**rhs**
x_1	1	0	$-3/20$	$1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$7/5$
z	-50	-100	0	0	0

Notice that x_1 and x_2 are basic variables, so the coefficient of these variables in row z must be equals 0.

Update z in the initial tableau

	1	0	$-3/20$	$1/40$	$18/5$	$(50 \times Row\ x_1)$
$+$	0	1	$1/40$	$-7/80$	$7/5$	$(100 \times Row\ x_2)$
	-50	-100	0	0	0	$(Row\ z)$
$z =$	0	0	-5	$-60/8$	320	

Step 6. Solve the updated problem with Simplex (Phase II)

Updated initial tableau

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	**rhs**
x_1	1	0	$-3/20$	$1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$7/5$
z	0	0	-5	$-60/8$	320

Identify the entering and leaving variables.

Notice that all coefficients of nonbasic variables in row z are negative. So, there is no entering variable, we have reach the optimal solution of the problem!.

The Two-Phase Simplex Method finishes here.