Table 2.1 Fourier Transform Properties

Property	Function	Fourier Transform	
	$f(t)$	$\hat{f}(\omega)$	
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$	**(2.15)**
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$	**(2.16)**
Multiplication	$f_1(t)\, f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$	**(2.17)**
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$	**(2.18)**
Modulation	$e^{i\xi t}f(t)$	$\hat{f}(\omega-\xi)$	**(2.19)**
Scaling	$f(t/s)$	$\lvert s\rvert \hat{f}(s\,\omega)$	**(2.20)**
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p \hat{f}(\omega)$	**(2.21)**
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$	**(2.22)**
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$	**(2.23)**
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$	**(2.24)**

Table 2.1 summarizes important Fourier transform properties that are often used in calculations. Most of the formulas are proved with a change of variable in the Fourier integral.

2.2.2 Fourier Transform in $\mathbf{L}^2(\mathbb{R})$

The Fourier transform of the indicator function $f = \mathbf{1}_{[-1,1]}$ is

$$\hat{f}(\omega) = \int_{-1}^{1} e^{-i\omega t}\, dt = \frac{2\sin\omega}{\omega}.$$

This function is not integrable because f is not continuous, but its square is integrable. The inverse Fourier transform, Theorem 2.1, thus does not apply. This motivates the extension of the Fourier transform to the space $\mathbf{L}^2(\mathbb{R})$ of functions f with a finite energy $\int_{-\infty}^{+\infty} |f(t)|^2\, dt < +\infty$. By working in the Hilbert space $\mathbf{L}^2(\mathbb{R})$, we also have access to all the facilities provided by the existence of an inner product. The inner product of $f \in \mathbf{L}^2(\mathbb{R})$ and $g \in \mathbf{L}^2(\mathbb{R})$ is

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t)\, g^*(t)\, dt,$$

and the resulting norm in $\mathbf{L}^2(\mathbb{R})$ is

$$\|f\|^2 = \langle f, f\rangle = \int_{-\infty}^{+\infty} |f(t)|^2\, dt.$$