Numerical Approximation of the Model

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First possible regularization of H by $C^2(\bar{\Omega})$ functions, as propsed in[27], is

$$H_{1,\epsilon}(z) = 1 \; if \; z > \epsilon \\ 0 \; if \; z < -\epsilon \\ \frac{1}{2}[1 + \frac{z}{\epsilon} + \frac{1}{\pi} sin(\frac{\pi z}{\epsilon})] \; if \; |z| \leq \epsilon$$

In this paper, we introduce and use in our experiments the following $C^\infty(\bar{\Omega})$ regularization of H

$$H_{2,\epsilon}(z) = \frac{1}{2}(1 + \frac{2}{\pi} arctan(\frac{z}{\epsilon}))$$

These distinct approximations and regularizations of the function $H \; and \; \delta_0 \; (\delta_\epsilon = H'_\epsilon)$ are presented in Fig. 3.

As $\epsilon \rightarrow 0$,both approximations converge to H and δ_0.

A difference is that $\delta_{1,\epsilon}$ has a small support, the interval $[-\epsilon, \epsilon\,]$, while $\delta_{2,\epsilon}$ is different of zero everywhere.

Because our energy is non convex (allowing therefore many local minima), the solution may depend on the initial curve.

With $H_{1,\epsilon} \; and \; \delta_{1,\epsilon}$ the algorithm sometimes computes a local minimizer of energy, while with $H_{2,\epsilon} \; and \; \delta_{2,\epsilon}$, the algorithm has the tendency to compute a global minimizer.

One of the reasons is that the Euler-Lagrange equation for ϕ acts only locally, on a few level curves around $\{\phi(x, y) = 0\}$ using $H_{1,\epsilon} and \; \delta_{1,\epsilon}$; but using $H_{2,\epsilon} \; and \; \delta_{2,\epsilon}$ the equation acts on all level curves.

In this way, in practice we can obtain a global minimizer, independently of the position of the initial curve; moreover, this allows to automatically detect interior contours(sevtion5).

We mention that, in order to extend the evolution to all level sets of ϕ, another possibility is to replace $\delta_0(\phi)$ by $|\nabla\phi|$.

In our paper, we work with $\delta_0(\phi)$, to remain close to the initial minimization problem.

The problem of extending the evolution to all level sets of ϕ was solved here using the approximation $\delta_{2,\epsilon}$ of δ_0 which is different of zero every where.

To discretize the equation in ϕ, we use a finite differences implicit scheme.

We recall first the usual notations: let h be the space step, Δt be the time step, and $(x_i, y_j) = (ih, jh)$ $(1 \leq i, j \leq M)$.

Let $\phi^n_{i,j} = \phi(n\Delta t, x_i, y_j)$ be an approximation of $\phi(t, x, y),\ with\ n \geq 0, \phi^0 = \phi_0$.

The finite differences are

$$\begin{aligned} &\Delta^x_- \phi_{i,j} = \phi_{i,j} - \phi_{i-1,j},\ \ \Delta^x_+ \phi_{i,j} = \phi_{i+1,j} - \phi_{i,j} \\ &\Delta^y_- \phi_{i,j} = \phi_{i,j} - \phi_{i,j-1},\ \ \Delta^y_+ \phi_{i,j} = \phi_{i,j} - \phi_{i,j+1} \end{aligned}$$

The algorithm is as follows: ϕ^n, we first compute $c_1(\phi^n)\ and\ c_2(\phi^n)$ using(6) and (7) respectively.

Then, we compute ϕ^{n+1} by the following discretization and linearization of (9) in ϕ

$$\frac{\phi^{n+1}_{i,j} - \phi^n_{i,j}}{\Delta t} =$$

$$\delta_h(\phi^n_{i,j})[\frac{\mu}{h^2}\Delta^x_- * (\frac{\Delta^x_+ \phi^{n+1}_{i,j}}{\sqrt{(\Delta^x_+ \phi^n_{i,j})^2/(h^2) + (\phi^n_{i,j+1} - \phi^n_{i,h-1})^2/(2h)^2}})$$

$$+\frac{\mu}{h^2}\Delta^y_- * (\frac{\Delta^y_+ \phi^{n+1}_{i,j}}{\sqrt{(\phi^n_{i+1,j} - \phi^n_{i-1,j})^2/(2h)^2 + (\Delta^y_+ \phi^n_{i,j})^2/(h^2)}}) - \nu - \lambda_1(u_{0,i,j} - c_1(\phi^n))^2 + \lambda_2(u_{0,i,j} - c_2(\phi^n))^2]$$

This linear system is solved by an iterative method, and for more details, werefer the reader to[1]/

When working with level set and Dirac delta functions, a standard procedure is to reinitialize ϕ to the signed distance function to its zero-level curve, as in [25] and [27].

This prevents the level set function to become too flat, or it can be seen as a rescaling and regularization.

For our algorithm, the reinitialization is optional.

On the other hand, it should not be too strong, because, as it was remarked by Fedkiw, it prevents interior contours from growing.

Only for a few numerical results we have applied the reinitialization, solving the follwing evlution equation[25]:

$$\begin{aligned} &\psi_\tau = sign(\phi(t))(1 - |\nabla\psi|) \\ &\psi(0, \bullet) = \psi(t, \bullet) \qquad\qquad (10) \end{aligned}$$

Where $\phi(t, \bullet)$ is our solution ϕ at time t.

Then the new $\phi(t, \bullet)$ will be ψ, such that ψ is obtained at the steady state of (10).

The solution $\psi(t, \bullet)$ of (10) will have the same zero-level set as $\phi(t, \bullet)$ and away from this set, $|\nabla \psi|$ will converge to 1

To discretize the equation(10), we use the scheme propsed in [22] and [25]

Algorithm

- Initialize ϕ^0 by ϕ_0, $n = 0$.

- Compute $c_1(\phi^n)$ and $c_2(\phi^n)$ by (6), (7).

- Solve the PDE from ϕ (9), to obtain ϕ^{n+1}.

- Reinitialize ϕ locally to the signed distance function to the curve. (this step is optional)

- Check whether the solution is stationary. If not, $n = n + 1$ and repeat.

We note that the use of a time-dependent PDE for ϕ is not crucial.

The stationary problem obtained directly from the minimization problem could also be solved numerically, using a similar finite differences scheme.