Euler lagrange equations for nested functions

The two energies (11) and (13) derive their invariance from the fact that ϕ is evaluated in coordinates relative to its own location and scale.

In a knowledge-driven segmentation process, one can maximize the similarity of the evolving shape encoded by ϕ and the template shape ϕ_0 by locally minimizing one of the two shape energies.

For the sake of differentiability, we will approximate the Heaviside function H by smoothed approximations H_δ as was done in the work of Chan

The associated shape gradient is particularly interesting since the energies (11) and (13) exhibit a nested dependence on ϕ via the moments $\mu_\phi \ and \ \sigma_\phi$

In the following, we will detail the computation of the corresponding Gateaux derivatives for the two invariant energies introduced above.

shape derivative of the translation invariant distance

The gradient of energy(11) with respect to ϕ in direction of an arbitrary deviation $\tilde{\phi}$ is given by the Gateaux derivative

$$\frac{\backslash\mathrm{part}E}{\backslash\mathrm{part}\phi}\Big|_{\tilde{\phi}} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(E(\phi+\epsilon\tilde{\phi})-E(\phi)) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega (H(\phi+\epsilon\tilde{\phi})(x+\mu_{\phi+\epsilon\tilde{\phi}})-H\phi_0(x))^2-(H(\phi)(x+\mu_\phi)-H\phi_0(x))^2dx \tag{15}$$

with the short-hand notation $\delta\phi \equiv \delta(\phi)$, the effect of the shape variation on the center of gravity is given by:

$$\mu_{\phi+\epsilon\tilde{\phi}} = \int xh(\phi+\epsilon\tilde{\phi})dx = \frac{\int x(H\phi+\epsilon\tilde{\phi}\ \delta\phi)dx}{\int(H\phi+\epsilon\tilde{\phi}\ \delta\phi)dx} = \mu_\phi + \frac{\epsilon}{\int H\phi dx}\int(x-\mu_\phi)\tilde{\phi}\ \delta\phi\ dx + \mathcal{O}(\epsilon^2) \tag{16}$$

Inserting (16) into (15) and further linearization in ϵ leads to a directional shape derivative of the form:

$$\frac{\backslash\mathrm{part}E}{\backslash\mathrm{part}\phi}\Big|_{\tilde{\phi}} = 2\int(H\phi(\tilde{x})-H\phi_0(x)\delta\phi(\bar{x})\Big[\tilde{\phi}(\bar{x})+\nabla\phi(\bar{x})\frac{1}{\int H\phi\ dx'}\times\int(x'-\mu_\phi)\tilde{\phi}(x')\delta\phi(x')dx'\Big]dx$$

Where $\bar{x} = x + \mu_\phi$ denotes the coordinates upon centering.

We therefore deduce that the shape gradient for the translation-invariant energy(11) is given by:

$$\frac{\backslash\mathrm{part}E}{\backslash\mathrm{part}\phi} = 2\delta\phi(x)\Big(H\phi(x)-H\phi_0(x-\mu_\phi)\Big)+\frac{(x-\mu_\phi)^\top}{\int H\phi\ dx}\times\int\Big(H\phi(x')-H\phi_0(x'-\mu_\phi)\delta\phi(x')\nabla\phi(x')dx'\Big) \tag{18}$$

REMARK

1. As for the image-driven flow in(3), the entire expression in (18) is weighted by the $\delta-$ function which stems from the fact that the function E in (11) only depends on $H\phi$
2. In a gradient descent evolution, the first of the two terms in (18) will draw $H\phi$ to the template $H\phi_0$, transported to the local coordinate frame associated with ϕ
3. The second term in (18) results from the ϕ- dependency of μ_ϕ in(11)

   It compensates for shape deformations which merely lead to a translation of the center of gravity μ_ϕ

   Not surprisingly, this second term contains an integral over the entire domain because the center of gravity is an integral quantity

   Figure 2 demonstrates that when applied as a shape prior in a segmentation process, this additional term tends to facilitate the translation of the evolving shape.

   While the boundary evolution represented in the top row was obtained using the first term of gradient(18) only, the contour flow shown in the bottom row exploits the full shape gradient

The additional term speeds up the convergence (140회 진행후 얻어짐) and generates a more accurate segmentation.

shape derivative of the translation and scale invariant distance

The above computation of a translation invariant shape gradient can be extended to the functional(1)

An infinitesimal variation of the level set function ϕ in direction $\tilde{\phi}$ affects the scale σ_ϕ defined in(14) as follows:

$$\sigma_{\phi+\epsilon\tilde{\phi}} = (\int (x-\mu_{\phi+\epsilon\tilde{\phi}})^2 h(\phi+\epsilon\tilde{\phi})dx)^{\frac{1}{2}} = \sigma_\phi + \frac{\epsilon}{2\sigma_\phi \int H\phi\, dx} \times \int ((x-\mu_\phi)^2 - \sigma_\phi^2)\tilde{\phi}\,\delta\,\phi\, dx + \mathcal{O}(\epsilon^2)$$

This expression is inserted into the definition (15) of the shape gradient for the shape energy(13).

Further linearization in ϵ gives a directional shape derivative of the form:

$$\frac{\part E}{\part \phi}\Big|_{\tilde{\phi}} = 2\int (H\phi(\bar{x}) - H\phi_0(x))\delta\phi(\bar{x}) \times \big(\tilde{\phi}(\bar{x}) + \nabla\phi(\bar{x})[\frac{1}{\int H\phi\, dx'} \times \int (x'-\mu_\phi)\tilde{\phi}\,\delta\,\phi\, dx' + \frac{x}{2\sigma_\phi \int H\phi\, dx'} \times \int ((x'-\mu_\phi)^2 - \sigma_\phi^2)\tilde{\delta}\phi\, dx']\big)$$

Where $\bar{x} = \sigma_\phi x + \mu_\phi$ Since this directional shape gradient corresponds to a projection fof the full shape gradient onto the respective direction $\tilde{\phi}$

$$\frac{\part E}{\part \phi}\Big|_{\tilde{\phi}} = \int \tilde{\phi}(x)\frac{\part E}{\part \phi}(x)dx$$

we need to rearrange the integrals in order to read off the shape gradient:

$$\frac{\part E}{\part \phi} = \frac{2}{\sigma_\phi}\delta\phi(x)(\frac{1}{\sigma_\phi}(H\phi(x) - H\phi_0(Tx)) + \frac{(Tx)^\top}{\int H\phi dx} \times \int (H\phi(x') - H\phi_0(Tx'))\delta\phi(x')\nabla\phi(x')dx' + \frac{(Tx)^2-1}{2\int H\phi dx}\int (H\phi(x') - H\phi_0(Tx')) \times$$

Where $Tx \equiv \frac{x-\mu_\phi}{\sigma_\phi}$ denotes the transformation into the local coordinate frame associated with ϕ

The three terms in the shape gradient(2) can be interpreted as follows:

1. The first term draws the contour toward the boundary of the familiar shape represented by ϕ_0, transported to the intrinsic coordinate frame of the evolving function ϕ
2. The second term results from the ϕ-dependency of μ_ϕ

   It compensates for deformations which merely result in a shift of the center of gravity.
3. The third term stems from the ϕ-dependency of σ_ϕ

   analogous to the second term, it compensates for variations of ϕ which merely lead to changes in the scale σ_ϕ

To demonstrate the scale-invariant property of the shape energy(13), we applied the segmentation scheme to an image of a partially occluded human silhouette, observed at three different scales.

Figure 3 show the contour evolutions generated by minimizing the total energy(13), where ϕ_0 is the level set function associated with a normalized(centered and rescaled) version of the silhouette of interest.

The results demonstrate that for the same (fixed) set of parameters, the shape prior enables the reconstruction of the familiar silhouette at arbitrary location and scale.

For a visualization of the intrinsic alignment process, we also plotted the evolving contour in the normalized coordinate frame(left)

In these normalized coordinates the contour converges to essentially the same solution in all three cases.