École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Modulational instability with the Nonlinear Schrödinger Equation

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In this tutorial, we study the stability of the plane wave solutions of the Nonlinear Schrödinger Equation (NLS), which we write as

$$\mathrm{i}\partial_t A = P\partial_{xx}A + Q|A|^2 A, \tag{1}$$

for the complex valued function $A(x,t)$. This equation is a simplified version of the nonpolynomial Schrödinger equation, which describes the evolution of cigar-shaped Bose-Einstein condensates [1].

1. When can bound states exist in the Schrödinger equation in a localized potential $V(x)$? Explain why the case $PQ>0$ is called "focusing" and the case $PQ<0$ is called "defocusing".

We now write $A(x,t)=\rho(x,t)\mathrm{e}^{\mathrm{i}\theta(x,t)}$, where $\rho(x,t)$ and $\theta(x,t)$ are the real amplitude and phase of $A(x,t)$.

2. * Write the coupled equations for $\partial_t\rho$ and $\partial_t\theta$.

3. Show the existence of a two-parameter family of solutions with constant amplitude, $\rho(x,t)=\rho_0$ and describe these solutions.

4. * Determine the stability of the plane wave solutions by introducing a small perturbation $\rho_1(x,t)$ and $\theta_1(x,t)$. You can look for plane wave solutions: $a(x,t)=\bar{a}\mathrm{e}^{\sigma t+\mathrm{i}px}$, where $a\in\{\rho_1,\theta_1\}$.

5. What changes if the cubic term $Q|A|^2A$ is replaced by $f(|A|^2)A$, where $f(u)$ is an arbitrary function?

References

[1] P. J. Everitt, M. A. Sooriyabandara, M. Guasoni, P. B. Wigley, C. H. Wei, G. D. McDonald, K. S. Hardman, P. Manju, J. D. Close, C. C. N. Kuhn, S. S. Szigeti, Y. S. Kivshar, and N. P. Robins. Observation of a modulational instability in bose-einstein condensates. *Phys. Rev. A*, 96:041601, Oct 2017.