

Exercices (solution)

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1 Gradient dynamics for the anisotropic RGL equation

1. It can be shown easily that this equation derives from the functional

$$\mathcal{F}[A] = \int \left(\frac{1}{2} [|A(\mathbf{r}, t)|^2 - 1]^2 + \left| \left(\partial_x - \frac{i}{2} \partial_{yy} \right) A(\mathbf{r}, t) \right|^2 \right) d\mathbf{r}. \quad (1)$$

2. First, we observe that \mathcal{F} is bounded from below: $\mathcal{F}[A] \geq 0$. Second, we compute

$$\frac{d\mathcal{F}}{dt} = \int \left(\partial_t A \frac{\delta \mathcal{F}}{\delta A} + \partial_t A^* \frac{\delta \mathcal{F}}{\delta A^*} \right) d\mathbf{r} = -2 \int |\partial_t A|^2 d\mathbf{r} \leq 0; \quad (2)$$

hence, \mathcal{F} is a Lyapunov functional.

2 Multi-scale expansion of model A

3. We first write explicitly the equation for u :

$$\partial_t u = \Gamma(u) [\beta \nabla^2 u - v'(u)]. \quad (3)$$

To the first order in u , $v'(u) = uv''(0)$ and $\Gamma(u) = \Gamma(0)$, so that

$$\partial_t u = \Gamma(0) [\beta \nabla^2 u - v''(0)u]. \quad (4)$$

In Fourier space, it reads

$$\partial_t u = -\Gamma(0) [\beta q^2 + v''(0)] u. \quad (5)$$

The state $u = 0$ is stable for all wavevectors q if $v''(0) > 0$.

4. All the parameters $\Gamma(0)$, β and a can be absorbed by a rescaling of the variables, so we are left with

$$\partial_t u = \nabla^2 u + \epsilon u. \quad (6)$$

Rescaling lengths and time with $t = \bar{t}/\epsilon$, $\mathbf{r} = \bar{\mathbf{r}}/\epsilon^{1/2}$, so that $\partial_t = \epsilon \partial_{\bar{t}}$ and $\nabla = \epsilon^{1/2} \bar{\nabla}$, we get

$$\partial_{\bar{t}} u = \bar{\nabla}^2 u + u. \quad (7)$$

5. Expanding $v(u)$ to the next order, the equation becomes

$$\partial_t u = \nabla^2 u + \epsilon u - gu^3. \quad (8)$$

Rescaling t and \mathbf{r} as above and defining $u = \epsilon^\alpha \bar{u}$, we get

$$\epsilon^{1+\alpha} \partial_{\bar{t}} \bar{u} = \epsilon^{1+\alpha} \bar{\nabla}^2 \bar{u} + \epsilon^{1+\alpha} \bar{u} - \epsilon^{3\alpha} g \bar{u}^3. \quad (9)$$

The exponents should be equal, hence $\alpha = 1/2$. Note that a non-linear term of the form $u^2 \nabla^2 u$ would be irrelevant in the limit $\epsilon \rightarrow 0$.

3 Blow-up in the Fisher-Kolmogorov equation

We consider the Fisher-Kolmogorov equation for a real field $u(\mathbf{r}, t)$ over a domain Ω with volume V , with a Neumann boundary condition $\hat{\mathbf{n}}(\mathbf{r}) \cdot \nabla u(\mathbf{r}, t) = 0$ for $\mathbf{r} \in \partial\Omega$, where $\hat{\mathbf{n}}(\mathbf{r})$ is a unit vector normal to the boundary:

$$\partial_t u = u + u^2 + \nabla^2 u. \quad (10)$$

We define the mass

$$m(t) = \frac{1}{V} \int_{\Omega} u(\mathbf{r}, t) d\mathbf{r}. \quad (11)$$

6. Integrating the evolution equation for u , we get

$$\dot{m}(t) = m(t) + \frac{1}{V} \int_{\Omega} u(\mathbf{r}, t)^2 d\mathbf{r}. \quad (12)$$

The Laplacian integrates to zero due to the Neumann boundary condition. Introducing the scalar product

$$\langle f, g \rangle = \frac{1}{V} \int_{\Omega} f(\mathbf{r}) g(\mathbf{r}) d\mathbf{r}, \quad (13)$$

we see that the second term in the evolution of the mass can be written as

$$\frac{1}{V} \int_{\Omega} u(\mathbf{r}, t)^2 d\mathbf{r} = \|u\|^2 = \|u\|^2 \|1\|^2 \geq \langle u, 1 \rangle^2 = m(t)^2. \quad (14)$$

7. If $m(0) > 0$, then $m(t) \geq n(t)$ where $n(0) = m(0)$ and

$$\dot{n}(t) = n(t) + n(t)^2. \quad (15)$$

We now solve this equation. First, we define $p(t) = e^{-t} n(t)$, so that $\dot{p}(t) = e^{-t} [-n(t) + \dot{n}(t)] = e^{-t} n(t)^2 = e^t p(t)^2$. Integrating $\dot{p}(t)/p(t)^2 = e^t$ leads to $p(0)^{-1} - p(t)^{-1} = e^t - 1$, so that $p(t) = [1 + p(0)^{-1} - e^t]^{-1}$. Finally

$$n(t) = \frac{1}{[1 + p(0)^{-1}] e^{-t} - 1}. \quad (16)$$

Defining $t_0 = \log(1 + p(0)^{-1})$, it leads to

$$m(t) \geq n(t) = \frac{1}{e^{t_0-t} - 1}. \quad (17)$$

The lower bound diverges at $t = t_0$, hence the solution should diverge at last at t_0 .

Note that the lower bound is reached if $u(\mathbf{r}, t)$ is uniform.