École Normale Supérieure de Lyon – Université Claude Bernard Lyon I

Physique Nonlinéaire et Instabilités

Blood pressure waves

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1 Blood pressure waves

We study the non-linear propagation of blood pressure waves. We combine the elasticity of the artery, mass conservation and the Euler equation of fluid mechanics to derive coupled partial differential equations for the section of the artery, the blood pressure and the fluid velocity. We then linearize these equations to determine the dispersion relation of linear waves. Finally, we perform a weakly non-linear expansion and show that at the first non-linear level the perturbations satisfy a Korteveg-de Vries equation.

We model an artery as a cylinder of constant radius r_0, thickness h, mass density ρ_0 and Young's modulus E. We model the blood at an incompressible fluid with mass density ρ and a negligible viscosity. We consider that the radius of the cylinder $r(x,t)$, or equivalently its section $A(x,t) = \pi r(x,t)^2$, the pressure in the fluid $p(x,t)$ and the velocity of the fluid along the artery, $v(x,t)$, are functions of the position along the artery x and the time t.

1. Write the equation of conservation of the fluid volume that relates the section $A(x,t)$ and the fluid velocity $v(x,t)$.

2. Write the Euler equation relating the velocity $v(x,t)$ and pressure $p(x,t)$.

3. * The artery is submitted to elastic forces and to the blood pressure. Apply the Newton's second law of motion to an element of the artery to obtain a relation between the section $A(x,t)$ and the pressure $p(x,t)$. Justify that we can assume that $pr_0 \ll Eh$ and use this approximation to linearize the final relation and obtain

$$\partial_{tt} A(x,t) = \frac{2\pi r_0}{\rho_0 h} p(x,t) - \frac{\pi E}{\rho_0 A_0}[A(x,t) - A_0]. \tag{1}$$

4. * Define the dimensionless variables $\bar{x}$, $\bar{t}$, $\bar{A}$, $\bar{p}$ and $\bar{v}$ to obtain the dimensionless equations

$$\partial_{\bar{t}\bar{t}} \bar{A} = 1 - \bar{A} + \bar{p}, \tag{2}$$

$$\partial_{\bar{t}} \bar{v} + \bar{v}\partial_{\bar{x}} \bar{v} = -\partial_{\bar{x}} \bar{p}, \tag{3}$$

$$\partial_{\bar{t}} \bar{A} + \partial_{\bar{x}} (\bar{v}\bar{A}) = 0. \tag{4}$$

What is the characteristic length of these equations? In the following, we use these equations and drop the bars.

5. What is the rest state (A_0, p_0, v_0) of the system? Linearize the evolution equation for a small perturbation $\epsilon(A_1, p_1, v_1)$ and determine the dispersion relation of the system. Show that in the long wavelength limit, the waves propagate with phase speed is $v_\phi = 1$ and $A_1 = p_1 = v_1$.

6. * To go to the next order, we place ourselves in a rescaled and moving reference frame $y = \epsilon^\chi (x - t)$, $s = \epsilon^\tau t$. Expand the equations to the order ϵ. Under what condition on χ and τ is $A_1 = p_1 = v_1$ a solution? We assume that this condition is fulfilled in the following.

7. * Expand the equations to the next non-trivial order in ϵ. Determine χ and τ. Eliminate the terms A_2, p_2 and v_2 to get a non-linear equation for A_1. What equation is it?

2 Dispersion in optical fibers

In this part, we study the dispersion of a wave packet in an optical fiber. The propagation of a signal in optical fibers is given, under some approximations, by the nonlinear Schrödinger equation:

$$\partial_x A + v_g^{-1}\partial_t A + \frac{\mathrm{i}q''}{2}\partial_{tt} A - \mathrm{i}q\frac{n_2}{n}|A|^2 A = 0. \tag{5}$$

8. What is the origin of the nonlinear term? We focus on the linear propagation in the following and take $n_2 = 0$. Explain the origin of the other terms, given that q'' is the second derivative of the wavevector with respect to the pulsation ω.

9. Show that the second term in Eq. (5) can be eliminated by a change of variables.

10. * We consider a wave packet described by $A(0,t) = \exp(-t^2/[2t_0^2])$. Determine $A(x,t)$. You can use that, for $a \in \mathbb{C}$ with $\Re(a) > 0$,

$$\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2} + \mathrm{i}kx\right) \mathrm{d}x = \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{k^2}{2a}\right). \tag{6}$$

11. Discuss the resulting expression. In particular, how do the height and width of the packet evolve with x? How does its shape change?