Key concepts of Statistical Physics
List of questions

1. Foundations of Statistical Physics
   - In what type of situation does one use statistical physics? Why?
   - What is the basic idea of statistical physics?
   - What is the ergodic hypothesis? Can you explicitly state this hypothesis?
   - What is its main practical consequence?
2. Hamiltonian Dynamics
   - Write down the Hamilton equation for a system of N particles.
   - Show that the energy is conserved.
   - What does it imply regarding the dynamics in phase space?
   - What is the Liouville equation? What does it express?
3. Random variables
   - What are the moment and the cumulant generating functions of a random variable?
   - What is the entropy of a probability density
   - Given the joint distribution of two random variables X_1 and X_2, what is the marginal distribution of X_1, what is the conditional distribution of X_2, knowing X_1?
   - Give the expression of the Gaussian distribution with average μ and standard deviation σ.
   - What is the law of large number?
   - What is the central limit theorem?
   - Define the Legendre transform of a strictly convex function.
   - What is a large deviation principle?
   - What is the Gärtner-Ellis theorem? What is it useful for? When is it clear that it is useless?
   - Let the X_i's be integer variables obeying a Poisson distribution $\rho(x) = \frac{\lambda^x e^{-\lambda}}{x!}$. One can show that $\langle X \rangle = \lambda$ and the variance $var(X) = \lambda$. The cumulant generating function is $K_X(k) = \lambda \left(e^k - 1\right)$. What is the large deviation function of $S_N := \frac{1}{N} \sum X_i$?
4. Fundamentals of Equilibrium Statistical Physics
   - Using the Gärtner-Ellis theorem, compute the large deviation function of a system of N independent spins taking the value -1 or 1 with equal probability.
   - Describe the micro-canonical equilibrium situation.
   - What is the probability of a micro-canonical configuration? Why?
   - Describe the canonical equilibrium situation.
   - What is the probability of a canonical equilibrium configuration? How is the celebrated β defined?
   - How is the canonical partition function defined?
   - Derive the distribution of energy in the canonical case.
   - Deduce that in the thermodynamic limit the average energy is also the most probable energy.
   - How are related the mean energy and the energy fluctuations?

- What is the free energy of a system in equilibrium with a thermostat?
- How is the free energy related to the large deviation function of the energy ?
- Derive the mean energy of a system of N independent spins in the canonical setting.

5. Equilibrium Liquids
   - Derive the equation of state of an ideal gas in the canonical setting.
   - Show that for any liquid the partition function factorizes between a kinetic part, an ideal configurational part and an interaction configuration part.
   - What is the Mayer function ?
   - What is the pair correlation function? How can it be measured? What is it useful for?

6. Phase transitions
   - Define the fully connected spin model and compute the distribution of the magnetization in the presence of an external field. Sketch its shape in zero field, and in finite field at different temperature. What is remarkable?
   - Draw and describe the phase diagram of the Ising model in 2 dimension.
   - What is a critical point and what is so specific in the vicinity of such a point?
   - What are the so-called critical exponent? What do they describe?
   - Considering the 2D Ising model, describe the Curie-Weiss mean field approximation
   - How would you compute the critical exponent from the knowledge of the large deviation of the magnetization?

7. Let M be the matrix that enters the master equation of a discrete system in continuous time,
$$\dot{P}_i(t) = \sum_j M_{ij} P_j(t). \tag{1}$$
Which property of M ensures the conservation of the total probability, $\sum_i P_i(t) = 1$? If the eigenvalues of M are real, what properties do they satisfy? What is the dimension of the eigenspace associated to the eigenvalue 0 if the system is ergodic, i.e., if the space of states is connected?

8. We consider a system satisfying a master equation. What is the difference between a stationnary state and an equilibrium state? Give the detailed balance condition for transition rates entering the master equation.

9. Give the detailed balance condition of a discrete system and the hypothesis on which it rests. Define a dynamics that satisfies the detailed balance where the transition rate between arbitrary states A and B, $k_{A\to B}$, only depends on the energy of the state A, E_A.

10. The Langevin equation reads
$$m\ddot{x}(t) = -\lambda\dot{x}(t) + f(x(t), t) + a\eta(t). \tag{2}$$
Interpret the different terms and give the average and the correlation of the noise $\eta(t)$. Introduce a characteristic time τ above which we can consider that the motion is over-damped, and write the corresponding equation.

11. Without external force, the over-damped Langevin equation reads
$$\dot{x}(t) = \sqrt{2D}\eta(t), \tag{3}$$
the correlation of the noise being given by $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. Compute the mean square displacement $\langle [x(t) - x(0)]^2\rangle$. Does it correspond to ballistic or diffusive motion?

12. The Fokker-Planck equation for the probability density $p(\boldsymbol{r}, t)$ reads
$$\frac{\partial p}{\partial t}(\boldsymbol{r}, t) = -\nabla \cdot \left[\frac{1}{\lambda}\boldsymbol{f}(\boldsymbol{r}, t)p(\boldsymbol{r}, t) - D\nabla p(\boldsymbol{r}, t)\right]. \tag{4}$$
Give the idea of the proof of this equation and how the different terms are obtained. What do the coefficients λ et D represent?

13. The Fokker-Planck equation for the probability density $p(\boldsymbol{r}, t)$ reads

$$\frac{\partial p}{\partial t}(\boldsymbol{r}, t) = -\nabla \cdot \left[\frac{1}{\lambda}\boldsymbol{f}(\boldsymbol{r}, t)p(\boldsymbol{r}, t) - D\nabla p(\boldsymbol{r}, t)\right]. \tag{5}$$

Interpret the two terms on the right hand side; what do the coefficients λ and D represent? Prove the Einstein relation which relates λ and D, using the equilibrium distribution of the Fokker-Planck equation.

14. Give the definition of the free energy of a statistical state P_i of a discrete system, and $p(x)$ of a continuous system. Show that the free energy of a system following the Fokker-Planck equation, given below, decreases with time (here we set $k_B = 1$);

$$\frac{\partial p}{\partial t}(\boldsymbol{r}, t) = -\frac{1}{\lambda}\nabla \cdot \left[\boldsymbol{f}(\boldsymbol{r}, t)p(\boldsymbol{r}, t) - T\nabla p(\boldsymbol{r}, t)\right]. \tag{6}$$

15. Is the Fokker-Planck equation a particular master equation? For a master equation defined by the matrix M, which satisfies detailed balance, the spectrum of M is real; what are its properties? From the spectrum, how do we know if the system is connected ? How can we identify metastable states?

16. Give the definition of the totally asymmetric simple exclusion process (TASEP). Does this system satisfy detailed balance; are its stationnary states equilibrium states? What approximation should we make to determine its stationnary state and why?

17. A stationnary state of the totally asymmetric simple exclusion process (TASEP) $\rho(x)$ associated to the particle current J satisfies $\rho'(x) = 2\rho(x)[1 - \rho(x)] - 2J$, with $\rho(0) = \alpha$ and $\rho(L) = 1 - \beta$. What values the current J can take in a large system? Draw the phase diagram as a function of α and β.

18. We consider a self-affine interface $h(r, t)$, meaning that if we define

$$h(r, t) = b^{-\alpha}h_b(br, b^\kappa t) \tag{7}$$

for $b > 0$, then h and h_b have the same statistical properties. The roughness of the interface is defined by $w(r, t) = \sqrt{\langle [h(r, t) - h(0, t)]^2 \rangle}$. Show that the roughness of the interface can be written with a single function $f(u)$, and discuss the short time and long time behaviors of the roughness as a function of the behavior of f when $u \to 0$ and $u \to \infty$.

19. We consider the Edwards-Wilkinson interface model,

$$\partial_t h(\boldsymbol{r}, t) = \nu\nabla^2 h(\boldsymbol{r}, t) + \eta(\boldsymbol{r}, t). \tag{8}$$

We look for a self-affine solution, meaning that if $h(\boldsymbol{r}, t)$ is a solution, then h_b defined by $h(\boldsymbol{r}, t) = b^{-\alpha}h_b(b\boldsymbol{r}, b^\kappa t)$ is also a solution. We know that $\eta(b\boldsymbol{r}, b^\kappa t) = b^{\frac{d+\kappa}{2}}\eta(\boldsymbol{r}, t)$, where d is the dimension of space. Determine α and κ as a function of d.