

Localized buckling of a floating sheet

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We study the buckling of a sheet placed at a liquid air interface and submitted to a uniaxial compression. It has been observed that uniform wrinkles form above the compression threshold, but that the buckling profile localizes as the compression is increased [1] (Fig. 1). Here, we analyse the buckling pattern beyond the linear instability using a multiple lengthscales expansion, following Ref. [2]. We note that an exact solution to the same problem has been found [3], which describes the buckling pattern as a soliton.

The shape of the sheet is described by the height function $h(s)$, where s is the curvilinear coordinate along the sheet. The dimensionless energy functional is

$$E[h(s)] = \int_{-L/2}^{L/2} e(h(s), h'(s), h''(s)) ds \quad (1)$$

where L is the length of the sheet, which is taken to infinity when convenient, and the energy density is

$$e(h, h', h'') = \frac{1}{2} \frac{h''^2}{1 - h'^2} - P \left(1 - \sqrt{1 - h'^2} \right) + \frac{h^2}{2} \sqrt{1 - h'^2}. \quad (2)$$

The first term is the bending energy, the second is the potential energy associated to the compression with “pressure” P , which is the control parameter, and the last is the gravitational energy of the deformed liquid interface.

1. * Perform the linear stability analysis of the flat interface.

Above the threshold, we define the small parameter $\epsilon = P_c - P$, where P is the critical pressure and we use the fact that the pressure decreases above threshold. We look for a solution of the form $h(s) = \epsilon^\alpha H(\epsilon^\beta s) \cos(k_c s)$, where α and β are to be determined, and $H(S)$ describes the envelope of the profile.

2. ** Expand the energy to the lowest order in ϵ and average it with respect to the “fast oscillations” ($\langle \cos(k_c s)^2 \rangle = 1/2 \dots$). Show that it can be written under the form

$$E[H(S)] = A \int \left[\frac{1}{2} H'(S)^2 - V(H(S)) \right] dS, \quad (3)$$

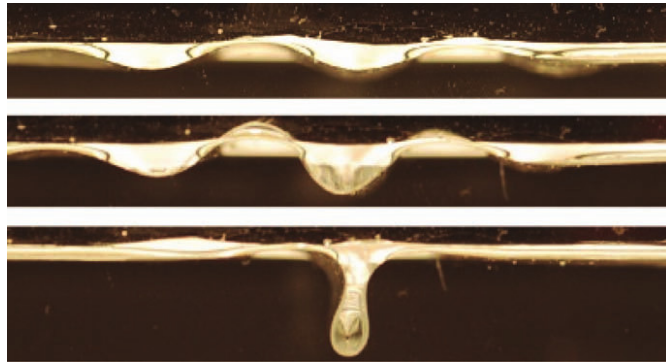


Figure 1: Side view of a compressed elastic sheet resting on a liquid [1]. The compression length increases from top to bottom.

where A is a constant and $V(H) = -\frac{1}{8}H^2 + \frac{1}{32}H^4$.

3. * Interpret this energy and the localized solutions that it can produce. Show that it has a solution of the form $H(S) = a/\cosh(bS)$. Write down the final form of the profile $h(s)$ and discuss it.

References

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