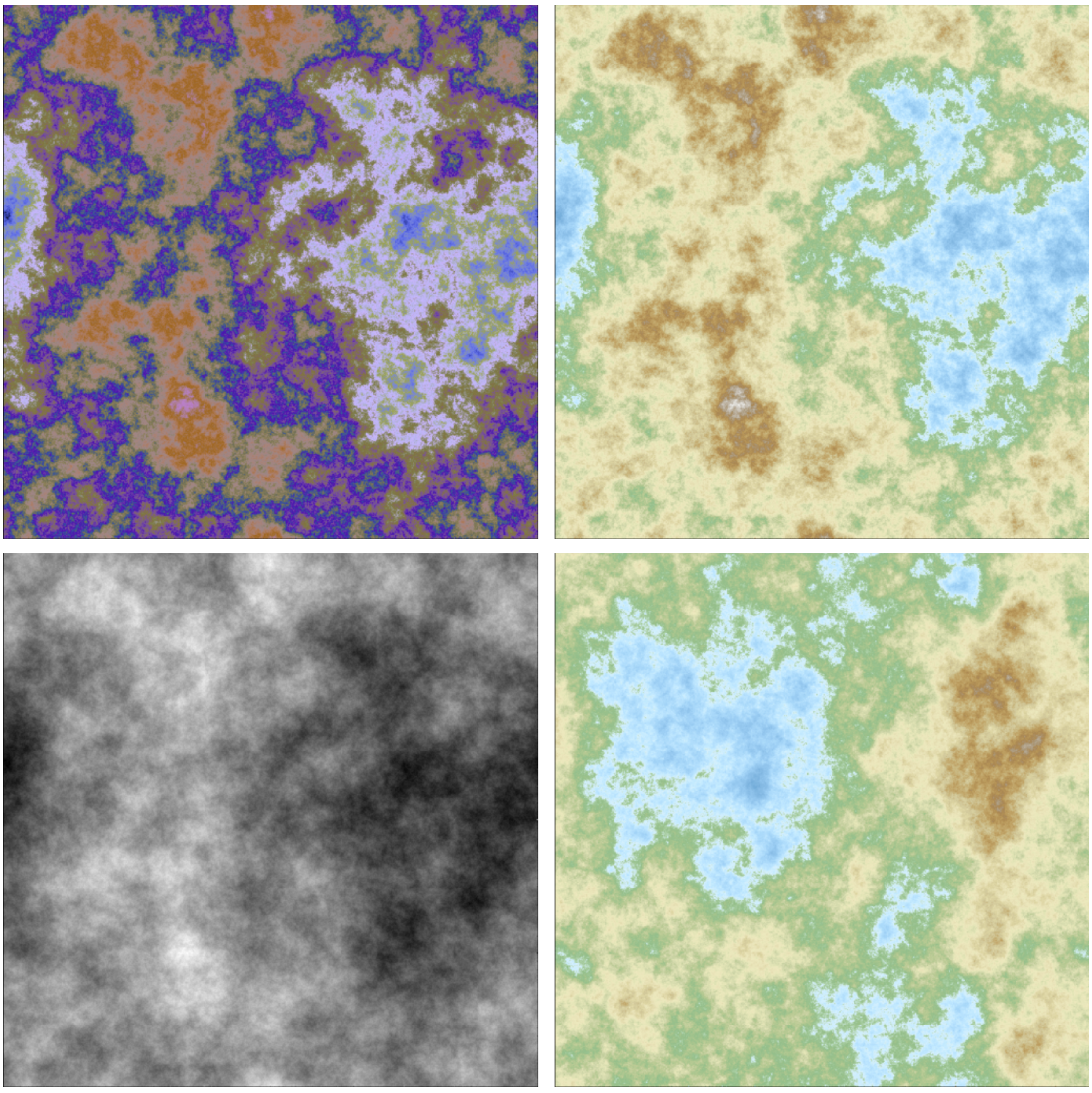
Synthetic Data and Generative AI



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Preface

This book covers the foundations of machine learning, with modern approaches to solving complex problems and the systematic generation and use of synthetic data. Emphasis is on scalability, automation, testing, optimizing, and interpretability (explainable AI). For instance, regression techniques – including logistic and Lasso – are presented as a single method, without using advanced linear algebra. There is no need to learn 50 versions when one does it all and more. Confidence regions and prediction intervals are built using parametric bootstrap, without statistical models or probability distributions. Models (including generative models and mixtures) are mostly used to create rich synthetic data to test and benchmark various methods.

Topics covered include clustering and classification, GPU machine learning, ensemble methods including an original boosting technique, elements of graph modeling, deep neural networks, auto-regressive and non-periodic time series, Brownian motions and related processes, simulations, interpolation, random numbers, natural language processing (smart crawling, taxonomy creation and structuring unstructured data), computer vision (shapes generation and recognition), curve fitting, cross-validation, goodness-of-fit metrics, feature selection, curve fitting, gradient methods, optimization techniques and numerical stability.

Several chapters focus on synthetic data, agent-based modeling and GIS applications: fractal-like terrain generation with the diamond-square algorithm, disaggregation of of ocean tides time series, geospatial interpolation of temperatures in the Chicago area, and synthetic star clusters evolving over time and bound by gravity. The latter provides great insights to explore the past and future of our universe or studying collision graphs. It also allows you to explore alternative universes, for instance with negative masses. Chapters 15 and 16 are more advanced and may be skipped in introductory classes. The former focuses on point process applications, while the later focuses on applications a machine learning methods to discover new insights in a famous mathematical conjecture: the Riemann Hypothesis. Section 17.7.2 illustrates the use of copulas to produce synthetic data, applied to a well-known insurance dataset.

Methods are accompanied by enterprise-grade Python code, replicable datasets and visualizations, including data animations (gifs, videos, even sound done in Python). The code uses various data structures and library functions sometimes with advanced options. It constitutes a solid introduction to scientific programming. The code, datasets, spreadsheets and data visualizations are also on GitHub, spread across the following repositories: Machine Learning, Point Processes, Visualizations, and Experimental Math. Chapters are mostly independent from each other, allowing you to read in random order. A glossary, index and numerous cross-references make the navigation easy and unify all the chapters.

The style is very compact, getting down to the point quickly, and suitable to business professionals. Jargon and arcane theories are absent, replaced by simple English to facilitate the reading by non-experts, and to help you discover topics usually made inaccessible to beginners. While state-of-the-art research is presented in all chapters, the prerequisites to read this book are minimal: an analytic professional background, or a first course in calculus and linear algebra. The original presentation avoids all unnecessary math and statistics, yet without eliminating advanced topics. Finally, this book is the main reference for my course on intuitive machine learning. For details about the classes, see here.

About the Author

Vincent Granville is a pioneering data scientist and machine learning expert, co-founder of Data Science Central (acquired by a publicly traded company in 2020), founder of MLTechniques.com, former VC-funded executive, author and patent owner. Vincent's past corporate experience includes Visa, Wells Fargo, eBay, NBC, Microsoft, and CNET.

Vincent is also a former post-doc at Cambridge University, and the National Institute of Statistical Sciences (NISS). He published in *Journal of Number Theory*, *Journal of the Royal Statistical Society* (Series B), and *IEEE Transactions on Pattern Analysis and Machine Intelligence*. He is also the author of multiple books, available here. He lives in Washington state, and enjoys doing research on stochastic processes, dynamical systems, experimental math and probabilistic number theory.

Contents

List of Figures

List of Tables

Glossary

Autoregressive process	Auto-correlated time series, as described in section 3.4. Time-continuous versions include Gaussian processes and Brownian motions, while random walks are a discrete example; two-dimensional versions exist. These processes are essentially integrated white noise. See pages 49, 97, 151
Binning	Feature binning consists of aggregating the values of a feature into a small number of bins, to avoid overfitting and reduce the number of nodes in methods such as naive Bayes, neural networks, or decision trees. Binning can be applied to two or more features simultaneously. I discuss optimum binning in this book. See pages 37, 73, 247
Boosted model	Blending of several models to get the best of each one, also referred to as ensemble methods. The concept is illustrated with hidden decision trees in this book. Other popular examples are gradient boosting and AdaBoost. See pages 36, 260
Bootstrapping	A data-driven, model-free technique to estimate parameter values, to optimize goodness-of-fit metrics. Related to resampling in the context of cross-validation. In this book, I discuss parametric bootstrap on synthetic data that mimics the actual observations. See pages 15, 96, 208, 260
Confidence Region	A confidence region of level γ is a 2D set of minimum area covering a proportion γ of the mass of a bivariate probability distribution. It is a 2D generalization of confidence intervals. In this book, I also discuss dual confidence regions – the analogous of credible regions in Bayesian inference. See pages 12, 15, 18, 20, 29, 205, 206, 242, 245
Cross-validation	Standard procedure used in bootstrapping, and to test and validate a model, by splitting your data into training and validation sets. Parameters are estimated based on training set data. An alternative to cross-validation is testing your model on synthetic data with known response. See pages 15, 37, 93, 99, 183, 247, 260
Decision trees	A simple, intuitive non-linear modeling techniques used in classification problems. It can handle missing and categorical data, as well as a large number of features, but requires appropriate feature binning. Typically one blends multiple binary trees each with a few nodes, to boost performance. See pages 36, 37, 39, 41, 260, 261
Dimension reduction	A technique to reduce the number of features in your dataset while minimizing the loss in predictive power. The most well known are principal component analysis and feature selection to maximize goodness-of-fit metrics. See pages 12, 16, 261, 262
Empirical distribution	Cumulative frequency histogram attached to a statistic (for instance, nearest neighbor distances), and based on observations. When the number of observations tends to infinity and the bin sizes tend to zero, this step function tends to the theoretical cumulative distribution function of the statistic in question. See pages 16, 96, 120, 129, 192, 195, 201, 207, 212, 214, 226, 254
Ensemble methods	A technique consisting of blending multiple models together, such as many decision trees with logistic regression, to get the best of each method and outperform each method taken separately. Examples include boosting, bagging, and AdaBoost. In this book, I discuss hidden decision trees. See pages 36, 83, 260
Explainable AI	Automated machine learning techniques that are easy to interpret are referred to as interpretable machine learning or explainable artificial intelligence. As much as possible, the methods discussed in this book belong to that category. The goal is to design black-box systems less likely to generate unexpected results with unintended consequences. See pages 13, 35, 69, 74, 83, 90, 155, 171, 209, 253

Feature selection — Features – as opposed to the model response – are also called independent variables or predictors. Feature selection, akin to dimensionality reduction, aims at finding the minimum subset of variables with enough predictive power. It is also used to eliminate redundant features and find causality (typically using hierarchical Bayesian models), as opposed to mere correlations. Sometimes, two features have poor predictive power when taken separately, but provide improved predictions when combined together. See pages 12, 15, 37, 94, 97, 238, 246, 260, 262

Generative model — Bayesian Gaussian mixtures (GMM) combined with kernel density estimation and the EM algorithm is a classic modeling tool. In this book, I used m-interlacings instead. Generative adversarial networks (GAN) work as follows: the generator creates new observations and the discriminator tests whether the new observations are statistically indistinguishable from training set data. When this goal is achieved, the new observations is your synthetic data. In this book, new observations are generated with parametric bootstrap instead. See pages 35, 52, 99, 166, 167, 169, 176, 183, 262

Goodness-of-fit — A model fitting criterion or metric to assess how a model or sub-model fits to a dataset, or to measure its predictive power on a validation set. Examples include R-squared, Chi-squared, Kolmogorov-Smirnov, error rate such as false positives and other metrics discussed in this book. See pages 15, 56, 93, 94, 247, 260, 262

Gradient methods — Iterative optimization techniques to find the minimum of maximum of a function, such as the maximum likelihood. When there are numerous local minima or maxima, use swarm optimization. Gradient methods (for instance, stochastic gradient descent or Newton's method) assume that the function is differentiable. If not, other techniques such as Monte Carlo simulations or the fixed-point algorithm can be used. Constrained optimization involves using Lagrange multipliers. See pages 15, 31, 55, 89

Graph structures — Graphs are found in decision trees, in neural networks (connections between neurons), in nearest neighbors methods (NN graphs), in hierarchical Bayesian models, and more. See pages 70, 74, 184, 250, 251

Hyperparameter — An hyperparameter is used to control the learning process: for instance, the dimension, the number of features, parameters, layers (neural networks) or clusters (clustering problem), or the width of a filtering window in image processing. By contrast, the values of other parameters (typically node weights in neural networks or regression coefficients) are derived via training. See pages 29, 56, 70, 75, 101, 170, 261

Link function — A link function maps a nonlinear relationship to a linear one so that a linear model can be fit, and then mapped back to the original form using the inverse function. For instance, the logit link function is used in logistic regression. Generalizations include quantile functions and inverse sigmoids in neural network to work with additive (linear) parameters. See pages 13, 16, 261

Logistic regression — A generalized linear regression method where the binary response (fraud/non-fraud or cancer/non-cancer) is modeled as a probability via the logistic link function. Alternatives to the iterative maximum likelihood solution are discussed in this book. See pages 16, 33, 36, 40, 260, 261

Neural network — A blackbox system used for predictions, optimization, or pattern recognition especially in computer vision. It consists of layers, neurons in each layer, link functions to model non-linear interactions, parameters (weights associated to the connections between neurons) and hyperparameters. Networks with several layers are called deep neural networks. Also, neurons are sometimes called nodes. See pages 69, 73, 75, 83, 101, 260, 261

NLP — Natural language processing is a set of techniques to deal with unstructured text data, such as emails, automated customer support, or webpages downloaded with a crawler. The example discussed in section 17.5 deals with creating a keyword taxonomy based on parsing Google search results pages. See pages 36, 249

Numerical stability — This issue occurring in unstable optimization problems typically with multiple minima or maxima, is frequently overlooked and leads to poor predictions or high volatility. It is sometimes referred to as ill-conditioned problems. I explain how to fix it in several examples in this book, for instance in section 3.4.2. Not to be confused with numerical precision. See pages 12, 14, 59

Overfitting — Using too many unstable parameters resulting in excellent performance on the training set, but poor performance on future data or on the validation set. It typically occurs with numerically unstable procedures such as regression (especially polynomial regression) when the training set is not large enough, or in the presence of wide data (more features than observations) when using a method not suited to this situation. The opposite is underfitting. See pages 15, 92, 101, 254, 255, 260, 262

Predictive power — A metric to assess the goodness-of-fit or performance of a model or subset of features, for instance in the context of dimensionality reduction or feature selection. Typical metrics include R-squared, or confusion matrices in classification. See pages 38, 40, 44, 246, 248, 253, 261

R-squared — A goodness-of-fit metric to assess the predictive power of a model, measured on a validation set. Alternatives include adjusted R-squared, mean absolute error and other metrics discussed in this book. See pages 12, 15, 35, 56, 90, 93, 95, 97, 104, 261, 262

Random number — Pseudo-random numbers are sequences of binary digits, usually grouped into blocks, satisfying properties of independent Bernoulli trials. In this book, the concept is formally defined, and strong pseudo-number generators are built and used in computer-intensive simulations. See pages 29, 128, 135, 252

Regression methods — I discuss a unified approach to all regression problems in chapter 1. Traditional techniques include linear, logistic, Bayesian, polynomial and Lasso regression (to deal with numerical instability and overfitting), solved using optimization techniques, maximum likelihood methods, linear algebra (eigenvalues and singular value decomposition) or stepwise procedures. See pages 12, 13, 15, 16, 19, 27, 36, 40, 46, 50, 52, 56, 89, 95, 101, 108, 261, 262

Supervised learning — Techniques dealing with labeled data (classification) or when the response is known (regression). The opposite is unsupervised learning, for instance clustering problems. In-between, you have semi-supervised learning and reinforcement learning (favoring good decisions). The technique described in chapter 1 fits into unsupervised regression. Adversarial learning is testing your model against extreme cases intended to make it fail, to build better models. See pages 262

Synthetic data — Artificial data simulated using a generative model, typically a mixture model, to enrich existing datasets and improve the quality of training sets. Called augmented data when blended with real data. See pages 12, 13, 15, 17, 27, 29, 33, 35, 48, 52, 55, 69, 70, 75, 88, 94, 105, 112, 118, 128, 141, 147, 155, 169, 176, 183, 243, 252, 254, 260

Tensor — Matrix generalization with three of more dimensions. A matrix is a two-dimensional tensor. A triple summation with three indices is represented by a three-dimensional tensor, while a double summation involves a standard matrix. See pages 69, 74

Training set — Dataset used to train your model in supervised learning. Typically, a portion of the training set is used to train the model, the other part is used as validation set. See pages 13, 15, 17, 20, 29, 36, 40, 56, 72, 88, 95, 101, 105, 183, 247, 260, 262

Validation set — A portion of your training set, typically 20%, used to measure the actual performance of your predictive algorithm outside the training set. In cross-validation and bootstrapping, the training and validation sets are split into multiple subsets to get a better sense of variations in the predictions. See pages 15, 27, 41, 56, 93, 101, 183, 247, 254, 255, 260, 261, 262

Bibliography

[1] Weighted percentiles using numpy. *Forum discussion*, 2020. StackOverflow [Link]. 101

[2] Jan Ackmann et al. Machine-learned preconditioners for linear solvers in geophysical fluid flows. *Preprint*, pages 1–19, 2020. arXiv:2010.02866 [Link]. 93

[3] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley, fourth edition, 2016. [Link]. 201

[4] José M. Amigó, Roberto Dale, and Piergiulio Tempesta. A generalized permutation entropy for random processes. *Preprint*, pages 1–9, 2012. arXiv:2003.13728 [Link]. 212

[5] Luc Anselin. *Point Pattern Analysis: Nearest Neighbor Statistics*. The Center for Spatial Data Science, University of Chicago, 2016. Slide presentation [Link]. 199

[6] Adrian Baddeley. Spatial point processes and their applications. In Weil W., editor, *Stochastic Geometry. Lecture Notes in Mathematics*, pages 1–75. Springer, Berlin, 2007. [Link]. 198

[7] David Bailey and Richard Crandall. Random generators and normal numbers. *Experimental Mathematics*, 11, 2002. Project Euclid [Link]. 144

[8] N. Balakrishnan and C.R. Rao (Editors). *Order Statistics: Theory and Methods*. North-Holland, 1998. 201, 215

[9] Christopher Beckham and Christopher Pal. A step towards procedural terrain generation with GANs. *Preprint*, pages 1–5, 2017. arXiv:1707.03383 [Link]. 168

[10] Rabi Bhattacharya and Edward Waymire. *Random Walk, Brownian Motion, and Martingales*. Springer, 2021. 146

[11] Barbara Bogacka. *Lecture Notes on Time Series*. 2008. Queen Mary University of London [Link]. 49

[12] B. Bollobas and P. Erdös. Cliques in random graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 80(3):419–427, 1976. [Link]. 202

[13] Miklos Bona. *Combinatorics of Permutations*. Routledge, second edition, 2012. 212

[14] Peter Borwein, Stephen K. Choi, and Michael Coons. Completely multiplicative functions taking values in $\{-1, 1\}$. *Transactions of the American Mathematical Society*, 362(12):6279–6291, 2010. [Link]. 220

[15] Peter Borwein and Michael Coons. Transcendence of power series for some number theoretic functions. *Proceedings of the American Mathematical Society*, 137(4):1303–1305, 2009. [Link]. 222

[16] Oliver Bröker and Marcus J. Groteb. Sparse approximate inverse smoothers for geometric and algebraic multigrid. *Applied Numerical Mathematics*, 41(1):61–80, 2002. 90

[17] H. M. Bui and M. B. Milinovich. Gaps between zeros of the Riemann zeta-function. *Quarterly Journal of Mathematics*, 69(2):402–423, 2018. [Link]. 232

[18] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. Clustering and percolation of point processes. *Preprint*, pages 1–20, 2013. Project Euclid [Link]. 198

[19] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. On comparison of clustering properties of point processes. *Preprint*, pages 1–26, 2013. arXiv:1111.6017 [Link]. 198

[20] Bartłomiej Błaszczyszyn and Dhandapani Yogeshwaran. Clustering comparison of point processes with applications to random geometric models. *Preprint*, pages 1–44, 2014. arXiv:1212.5285 [Link]. 198

[21] Oliver Chikumbo and Vincent Granville. Optimal clustering and cluster identity in understanding high-dimensional data spaces with tightly distributed points. *Machine Learning and Knowledge Extraction*, 1(2):715–744, 2019. 253

[22] Keith Conrad. *L-functions and the Riemann Hypothesis*. 2018. 2018 CTNT Summer School [Link]. 131, 217, 220, 225

[23] Noel Cressie. *Statistic for Spatial Data*. Wiley, revised edition, 2015. 198

[24] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer, second edition, 2002. Volume 1 – Elementary Theory and Methods. 150

[25] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer, second edition, 2014. Volume 2 – General Theory and Structure. 150

[26] Tilman M. Davies and Martin L. Hazelton. Assessing minimum contrast parameter estimation for spatial and spatiotemporal log-Gaussian Cox processes. *Statistica Neerlandica*, 67(4):355–389, 2013. 244

[27] Marc Deisenroth, A. Faisal, and Cheng Soon Ong. *Mathematics for Machine Learning*. Cambridge University Press, 2020. [Link]. 53

[28] Harold G. Diamond and Wen-Bin Zhang. *Beurling Generalized Numbers*. American Mathematical Society, 2016. Mathematical Surveys and Monographs, Volume 213 [Link]. 132, 228

[29] D.J.Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes – Volume I: Elementary Theory and Methods*. Springer, second edition, 2013. 199

[30] D.J.Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes – Volume II: General Theory and Structure*. Springer, second edition, 2014. 199

[31] David Coupier (Editor). *Stochastic Geometry: Modern Research Frontiers*. Wiley, 2019. 209

[32] Ding-Geng Chen (Editor), Jianguo Sun (Editor), and Karl E. Peace (Editor). *Interval-Censored Time-to-Event Data: Methods and Applications*. Chapman and Hall/CRC, 2012. 200

[33] Khaled Emam, Lucy Mosquera, and Richard Hoptroff. *Practical Synthetic Data Generation*. O'Reilly, 2020. 99

[34] Paul Erdős and Alfréd Rényi. On the evolution of random graphs. In *Publication of the Mathematical Institute of the Hungarian Academy of Sciences*, volume 5, pages 17–61, 1960. [Link]. 202

[35] Achim Zeileis et al. Colorspace: A toolbox for manipulating and assessing colors and palettes. *Preprint*, pages 1–45, 2019. arXiv:1903.06490 [Link] [R Library]. 168

[36] Arash Farahmand. *Math 55 Lecture Notes*. 2021. University of Berkeley [Link]. 48, 53

[37] W. Feller. On the Kolmogorov-Smirnov limit theorems for empirical distributions. *Annals of Mathematical Statistics*, 19(2):177–189, 1948. [Link]. 201, 208

[38] Nikos Frantzikinakis. Ergodicity of the Liouville system implies the Chowla conjecture. *Preprint*, pages 1–41, 2016. arXiv [Link]. 222

[39] P. M. Gauthier. Approximating the Riemann zeta-function by polynomials with restricted zeros. *Canadian Mathematical Bulletin*, 62(3):475–478, 2018. [Link]. 232

[40] P. A. Van Der Geest. The binomial distribution with dependent Bernoulli trials. *Journal of Statistical Computation and Simulation*, pages 141–154, 2004. [Link]. 147

[41] Stamatia Giannarou and Tania Stathaki. Shape signature matching for object identification invariant to image transformations and occlusion. 2007. ResearchGate [Link]. 84

[42] Minas Gjoka, Emily Smith, and Carter Butts. Estimating clique composition and size distributions from sampled network data. *Preprint*, pages 1–9, 2013. arXiv:1308.3297 [Link]. 202

[43] B.V. Gnedenko and A. N. Kolmogorov. *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, 1954. 151

[44] Manuel González-Navarrete and Rodrigo Lambert. Non-markovian random walks with memory lapses. *Preprint*, pages 1–14, 2018. arXiv [Link]. 146

[45] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016. [Link]. 53

[46] Vincent Granville. Estimation of the intensity of a Poisson point process by means of nearest neighbor distances. *Statistica Neerlandica*, 52(2):112–124, 1998. [Link]. 199

[47] Vincent Granville. *Applied Stochastic Processes, Chaos Modeling, and Probabilistic Properties of Numeration Systems*. MLTechniques.com, 2018. [Link]. 132

[48] Vincent Granville. *Stochastic Processes and Simulations: A Machine Learning Perspective*. MLTechniques.com, 2022. [Link]. 51, 59, 151, 161, 192, 193, 194, 196, 200, 202, 228, 232, 246

[49] Vincent Granville, Mirko Krivanek, and Jean-Paul Rasson. Simulated annealing: A proof of convergence. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 16:652–656, 1996. 72

[50] Vincent Granville and Richard L Smith. Disaggregation of rainfall time series via Gibbs sampling. *NISS Technical Report*, pages 1–21, 1996. [Link]. 107

[51] Kristen Grauman. Shape matching. 2008. University of Texas, Austin [Link]. 87

[52] Hui Guo et al. Eyes tell all: Irregular pupil shapes reveal gan-generated faces. *Preprint*, pages 1–7, 2021. arXiv:2109.00162 [Link]. 254

[53] Aurélien Géron. *Hands-On Machine Learning with Scikit-Learn, Keras & TensorFlow*. O'Reilly, third edition, 2023. 35

[54] Radim Halir and Jan Flusser. Numerically stable direct least squares fitting of ellipses. *Preprint*, pages 1–8, 1998. [Link]. 17, 19

[55] Peter Hall. *Introduction to the theory of coverage processes*. Wiley, 1988. 209

[56] Adam J. Harper. Moments of random multiplicative functions, II: High moments. *Algebra and Number Theory*, 13(10):2277–2321, 2019. [Link]. 128, 228

[57] Adam J. Harper. Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos. *Forum of Mathematics, Pi*, 8:1–95, 2020. [Link]. 128, 130, 228

[58] Adam J. Harper. Almost sure large fluctuations of random multiplicative functions. *Preprint*, pages 1–38, 2021. arXiv [Link]. 130, 222, 228

[59] K. Hartmann, J. Krois, and B. Waske. *Statistics and Geospatial Data Analysis*. Freie Universität Berlin, 2018. E-Learning Project SOGA [Link]. 195

[60] D. R. Heath-Brown. Primes represented by $x^3 + 2y^3$. *Acta Mathematica*, 186:1–84, 2001. [Link]. 224

[61] T. W. Hilberdink and M. L. Lapidus. Beurling Zeta functions, generalised primes, and fractal membranes. *Preprint*, pages 1–31, 2004. arXiv [Link]. 131, 132, 228

[62] Christian Hill. *Learning Scientific Programming with Python*. Cambridge University Press, 2016. [Link]. 19

[63] Robert V. Hogg, Joseph W. McKean, and Allen T. Craig. *Introduction to Mathematical Statistics*. Pearson, eighth edition, 2016. [Link]. 53

[64] Zhiqiu Hu and Rong-Cai Yang. A new distribution-free approach to constructing the confidence region for multiple parameters. *PLOS One*, pages 1–13, 2013. [Link]. 243

[65] Peter Humphries. The distribution of weighted sums of the Liouville function and Pólya's conjecture. *Preprint*, pages 1–33, 2011. arXiv [Link]. 229

[66] Timothy D. Johnson. Introduction to spatial point processes. *Preprint*, page 2008. NeuroImaging Statistics Oxford (NISOx) group [Link][Mirror]. 199

[67] Chigozie Kelechi. Towards efficiency in the residual and parametric bootstrap techniques. *American Journal of Theoretical and Applied Statistics*, 5(5), 2016. [Link]. 97

[68] Denis Kojevnikov, Vadim Marmer, and Kyungchul Song. Limit theorems for network dependent random variables. *Journal of Econometrics*, 222(2):419–427, 2021. [Link]. 199

[69] Samuel Kotz, Tomasz Kozubowski, and Krzystof Podgorski. *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*. Springer, 2001. 213

[70] Faraj Lagum. *Stochastic Geometry-Based Tools for Spatial Modeling and Planning of Future Cellular Networks*. PhD thesis, Carleton University, 2018. [Link]. 198

[71] Günther Last and Mathew Penrose. *Lectures on the Poisson Process*. Cambridge University Press, 2017. 198

[72] Yuk-Kam Lau, Gerald Tenenbaum, and Jie Wu. On mean values of random multiplicative functions. *Proceedings of the American Mathematical Society*, 142(2):409–420, 2013. [Link]. 128, 130

[73] Gary R. Lawlor. A l'Hospital's rule for multivariable functions. *Preprint*, pages 1–13, 2013. arXiv:1209.0363 [Link]. 113

[74] Jing Lei et al. Distribution-free predictive inference for regression. *Journal of the American Statistical Association*, 113:1094–1111, 2018. [Link]. 97

[75] G. Last M.A. Klatt and D. Yogeshwaran. Hyperuniform and rigid stable matchings. *Random Structures and Algorithms*, 2:439–473, 2020. [Link][PowerPoint]. 198

[76] Jorge Mateu, Frederic P Schoenberg, and David M Diez. On distances between point patterns and their applications. *Preprint*, pages 1–29, 2010. [Link]. 199

[77] Natarajan Meghanathan. Distribution of maximal clique size of the vertices for theoretical small-world networks and real-world networks. *Preprint*, pages 1–20, 2015. arXiv:1508.01668 [Link]. 202

[78] Masahiro Mine. Probability density functions attached to random Euler products for automorphic L-functions. *Preprint*, pages 1–38, 2020. arXiv [Link]. 228, 229

[79] Christoph Molnar. *Interpretable Machine Learning*. ChristophMolnar.com, 2022. [Link]. 97, 253

[80] Marc-Andreas Muendler. Linear difference equations and autoregressive processes. 2000. University of Berkeley [Link]. 49

[81] V. Kumar Murty. Seminar on Fermat's last theorem. In *Canadian Mathematical Society – Conference Proceedings*, volume 17, Toronto, Canada, 1995. [Link]. 225

[82] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge University Press, 2010. Cambridge Series in Statistical and Probabilistic Mathematics, Volume 30 [Link]. 146, 150

[83] Jesper Møller. Introduction to spatial point processes and simulation-based inference. In *International Center for Pure and Applied Mathematics (Lecture Notes)*, Lomé, Togo, 2018. [Link][Mirror]. 199, 212, 244

[84] Jesper Møller and Rasmus P. Waagepetersen. *An Introduction to Simulation-Based Inference for Spatial Point Processes*. Springer, 2003. 199

[85] Jesper Møller and Rasmus P. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. CRC Press, 2007. 199

[86] S. Ghosh N., Miyoshi, and T. Shirai. Disordered complex networks: energy optimal lattices and persistent homology. *Preprint*, pages 1–44, 2020. arXiv:2009.08811. 192

[87] Saralees Nadarajah. A modified Bessel distribution of the second kind. *Statistica*, 67(4):405–413, 2007. [Link]. 213

[88] Hasan Nasab, Mahdi Tavana, and Mohsen Yousefu. A new heuristic algorithm for the planar minimum covering circle problem. *Production and Manufacturing Research*, pages 142–155, 2014. [Link]. 209

[89] Guillermo Navas-Palencia. Optimal binning: mathematical programming formulation. *Preprint*, pages 1–21, 2020. arXiv:2001.08025 [Link]. 37

[90] Nathan Ng. Large gaps between the zeros of the Riemann zeta function. *Journal of Number Theory*, 128(3):509–556, 2007. [Link]. 232

[91] Yosihiko Ogata. Cluster analysis of spatial point patterns: posterior distribution of parents inferred from offspring. *Japanese Journal of Statistics and Data Science*, 3:367–390, 2020. 198

[92] Fred Park. Shape descriptor / feature extraction techniques. 2011. UCI iCAMP 2011 [Link]. 84

[93] Yuval Peres and Allan Sly. Rigidity and tolerance for perturbed lattices. *Preprint*, pages 1–20, 2020. arXiv:1409.4490 [Link]. 192, 198

[94] Carl Rasmussen and Christopher Williams. *Gaussian Processes for Machine Learning*. MIT Press, 2006. [Link]. 52

[95] Alfred R.Osborne. Multidimensional Fourier series. *International Geophysics*, 97:115–145, 2010. [Link]. 120

[96] Kamron Saniee. A simple expression for multivariate Lagrange interpolation. *SIAM Undergraduate Research Online*, 2007. SIURO [Link]. 103

[97] Mahesh Shivanand and all. Fitting random regression models with Legendre polynomial and B-spline to model the lactation curve for Indian dairy goat of semi-arid tropic. *Journal of Animal Breeding and Genetics*, pages 414–422, 2022. [Link]. 120

[98] Karl Sigman. Notes on the Poisson process. New York NY, 2009. IEOR 6711: Columbia University course [Link]. 198

[99] Joshua Snoke et al. General and specific utility measures for synthetic data. *Journal of the Royal Statistical Society Series A*, 181:663–688, 2018. arXiv:1604.06651 [Link]. 35

[100] Luuk Spreeuwers. *Image Filtering with Neural Networks: Applications and Performance Evaluation*. PhD thesis, University of Twente, 1992. 73

[101] J. Michael Steele. Le Cam's inequality and Poisson approximations. *The American Mathematical Monthly*, 101(1):48–54, 1994. [link]. 161

[102] Dietrich Stoyan, Wilfrid S. Kendall, Sung Nok Chiu, and Joseph Mecke. *Stochastic Geometry and Its Applications*. Wiley, 2013. 209

[103] E.C. Titchmarsh and D.R. Heath-Brown. *The Theory of the Riemann Zeta-Function*. Oxford Science Publications, second edition, 1987. 58, 131, 217

[104] Chris Tofallis. Fitting equations to data with the perfect correlation relationship. *Preprint*, pages 1–11, 2015. Hertfordshire Business School Working Paper[Link]. 13

[105] D. Umbach and K.N. Jones. A few methods for fitting circles to data. *IEEE Transactions on Instrumentation and Measurement*, 52(6):1881–1885, 2003. [Link]. 14, 17

[106] D. A. Vaccari and H. K. Wang. Multivariate polynomial regression for identification of chaotic time series. *Mathematical and Computer Modelling of Dynamical Systems*, 13(4):1–19, 2007. [Link]. 17

[107] Remco van der Hofstad. *Random Graphs and Complex Networks*. Cambridge University Press, 2016. [Link]. 201

[108] Yu Vizilter and Sergey Zheltov. Geometrical correlation and matching of 2D image shapes. 2012. ResearchGate [Link]. 86

[109] Fengyun Wang and all. Bivariate Fourier-series-based prediction of surface residual stress fields using stresses of partial points. *Mathematics and Mechanics of Solids*, 2018. [Link]. 120

[110] Luyao Wang and Hai Cheng. Pseudo-random number generator based on logistic chaotic system. *Entropy*, 21, 2019. [Link]. 144

[111] Mingguang Wu, Yanjie Sun, and Yaqian Li. Adaptive transfer of color from images to maps and visualizations. *Cartography and Geographic Information Science*, pages 289–312, 2021. [Link]. 168

[112] Lan Wua, Yongcheng Qi, and Jingping Yang. Asymptotics for dependent Bernoulli random variables. *Statistics and Probability Letters*, pages 455–463, 2012. [Link]. 146

[113] Oren Yakir. Recovering the lattice from its random perturbations. *Preprint*, pages 1–18, 2020. arXiv:2002.01508 [Link]. 198

[114] Ruqiang Yan, Yongbin Liub, and Robert Gao. Permutation entropy: A nonlinear statistical measure for status characterization of rotary machines. *Mechanical Systems and Signal Processing*, 29:474–484, 2012. 212

[115] Shaohong Yan, Aimin Yang, et al. Explicit algorithm to the inverse of Vandermonde matrix. In *2009 International Conference on Test and Measurement*, 2009. IEEE [Link]. 47

[116] D. Yogeshwaran. Geometry and topology of the boolean model on a stationary point processes : A brief survey. *Preprint*, pages 1–13, 2018. Researchgate [Link]. 199

[117] Tonglin Zhang. A Kolmogorov-Smirnov type test for independence between marks and points of marked point processes. *Electronic Journal of Statistics*, 8(2):2557–2584, 2014. 194

Index