Theorem

The range R(n) associated with n independent random variables with an exponential distribution of parameter λ, satisfies:

$$\mathrm{E}[R_n] = \frac{1}{\lambda} \cdot \sum_{k=1}^{n-1} \frac{1}{k},$$

$$\mathrm{Var}[R_n] = \frac{1}{\lambda^2} \cdot \sum_{k=1}^{n-1} \frac{1}{k^2}.$$

Before proving the theorem, note that the first formula is well known, only the second one is new. The standard proof for the expectation is not considered simple: it is based on computing the expectation for the maximum (see here) and the fact that the minimum also has an exponential distribution with known expectation (see here). Our proof is simpler, and it also covers the variance.

Proof

The general distribution of the range is known for any distribution, see here. The range is defined as

$$R_n = \max(X_1, \cdots, X_n) - \min(X_1, \cdots, X_n).$$

In the case of the exponential distribution, the range computed on *n* random variables has the following density (see here page 3):

$$f_{R_n}(x) = (n-1)\lambda e^{-\lambda x}(1-e^{-\lambda x})^{n-2}.$$

With a simple change of variable, the *k*-th moment of the range is equal to

$$\mathrm{E}[R_n^k] = \int_0^\infty x^k f_{R_n}(x)dx = \frac{n-1}{\lambda^k}|J_n(k)|,$$

$$J_n(k) = \int_0^1 \log^k(x) \cdot (1-x)^{n-2}dx.$$

Using WolframAlpha (see here and here) one obtains

$$J_n(1) = -\frac{H_{n-1}}{n-1} \text{ and } J_n(2) = \frac{1}{n-1} \cdot \left(H_{n-1}^2 - \psi_1(n) + \frac{\pi^2}{6}\right).$$

Thus,

$$\mathrm{E}[R_n] = \frac{H_{n-1}}{\lambda} \text{ and } \mathrm{Var}[R_n] = \mathrm{E}[R_n^2] - \mathrm{E}^2[R_n] = \frac{1}{\lambda^2}\left(\frac{\pi^2}{6} - \psi_1(n)\right).$$

The two symbols $H(n\text{-}1)$ and $\psi_1(n)$ represent the harmonic numbers and the Trigamma function, respectively. To complete the proof, use the fact that

$$H_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}, \quad \frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}, \quad \psi_1(n) = \sum_{k=n}^{\infty} \frac{1}{k^2}.$$

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